FIXED POINT INDICES AND EXISTENCE THEOREMS FOR SEMILINEAR EQUATIONS IN CONES

by

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Contents

PREFACE

This thesis is submitted in accordance with the degree of Doctor of Philosophy in the University of Glasgow.

All results of this thesis are the original work of the author except where indicated within the text.

I should like to express my very deep and sincere gratitude to my supervisor, Professor J.R.L. Webb, F.R.S.E., for suggesting this research topic and for his limitless patience, invaluable assistance and constant encouragement, without which this thesis would not have been possible; also to my parents for their generous financial and moral support; and to my wife, whom I owe more than words can express.

ABSTRACT

The purpose of this thesis is to develop fixed point indices for A-proper semilinear operators defined on cones in Banach spaces and use the results to obtain existence theorems to semilinear equations. We consider semilinear equations of the form $Lx = Nx$ where L is a linear Fredholm operator of index zero, N a nonlinear operator such that $L - N$ is A-proper at zero relative to a projection scheme Γ_L.

Chapter 1 is an introduction to basic concepts used throughout the thesis, including: Banach spaces, linear operators, A-proper maps, Fredholm operators of index zero, and the definition and properties of the generalised degree for A-proper maps.

In Chapter 2, we define a fixed point index for A-proper maps on cones in terms of the generalised degree and derive the basic properties of this index. We then extend the definition to include unbounded sets.

A more general fixed point index than that of Chapter 2 is developed in Chapter 3 for A-proper maps based on limits of a finite dimensionally defined index. Properties of the index are given and a definition for unbounded sets is provided.

Chapter 4 extends the Lan-Webb fixed point index for weakly inward A-proper at 0 maps to semilinear operators. This index is also extended to include unbounded sets.

Existence theorems of positive and non-negative solutions to semilinear equations on cones are established in Chapter 5 using the fixed point indices of Chapters 2, 3, and 4.

Finally, in Chapter 6, we apply some of the existence theorems of Chapter 5 to several differential and integral equations. We prove the existence of: a positive solution to a Picard boundary value problem; a non-negative solution to a periodic boundary value problem; and, a non-negative solution to a Volterra integral equation.

Introduction

This thesis is concerned with the development and application of a fixed point index for semilinear equations in cones.

Semilinear equations, also referred to as alternative problems, are operator equations of the form

$$Lx = Nx \tag{0.1}$$

acting on certain topological vector spaces X and Y where L is a linear mapping and N nonlinear. They are abstract formulations of differential, integral or integro-differential equations which arise naturally in various areas of science and engineering. Initial investigations of these equations are attributed to Lyapunov [33] in the study of integral equations related to a problem in fluid dynamics and Schmidt [50] for theoretical research in nonlinear integral equations. The method they employed, now called the Lyapunov-Schmidt method, involved applying certain projections $P : X \to X$ where $\operatorname{im} P = \ker L$ (in contemporary notation) and $Q : Y \to Y$ where $\ker Q = \operatorname{im} L$. Then the spaces X and Y can be represented as $X = \ker L \oplus \ker P$, $Y = \operatorname{im} L \oplus \operatorname{im} Q$ and every $x \in X$ can be expressed as $x = x_0 + x_1$ with $x_0 = Px \in \ker L$ and $x_1 = (I - P)x \in \ker P$. Thus $Lx = Nx$ becomes equivalent to

$$Lx = (I - Q)\,Nx, \;\; QNx = 0$$

or

$$x - Px = L_1^{-1}\,(I - Q)\,Nx, \;\; QNx = 0 \tag{0.2}$$

where $L_1^{-1}(I-Q): Y \to \operatorname{dom} L \cap \ker P$. Whence we obtain the system

$$x_1 = L_1^{-1}(I-Q)\,N(x_0+x_1)$$

and

$$QN(x_0+x_1) = 0.$$

For a fixed x_0 the first equation becomes a fixed point problem

$$x_1 = Sx_1, \tag{0.3}$$

for the operator $Sx_1 \equiv L_1^{-1}(I-Q)\,N(x_0+x_1)$ and, under certain conditions on S, may be solved using suitable fixed point theorems such as those of Banach, Schauder, Sadovskii *etc., cf.* [58] for a detailed account. If (0.3) has a unique solution $x_1 = T(x_0)$, the solution of (0.1) reduces to solving the second equation of the system, $QN(x_0+T(x_0)) = 0$ for $x_0 \in \ker L$. This last equation is finite dimensional if L is a Fredholm operator, *q.v.* Section 1.4, a condition we shall impose throughout this thesis.

The class of Fredholm operators are of considerable importance in functional analysis; as aptly stated by Zeidler [58]: "The entire development of linear analysis in this century is intimately related to the concept of the linear Fredholm operator". These operators generalise certain properties of linear functions in $\mathbb{R}^n$ to operators on Banach spaces. The modern theory is based upon the results of I. Fredholm [17] who established the celebrated "Fredholm alternatives" for the solvability of a class of integral equations of the second type with regular kernel. In so doing, he showed that the operators were, in modern terminology, Fredholm of index zero. That is, the integral operators of his investigations were bounded linear operators whose kernels and coimages were of the same finite dimension.

Fredholm operators of nonzero index were later discovered but because the correspondence in dimension of the afore mentioned subspaces is lost, the transformation of $L-N$ to $I-T$ cannot be made and, consequently, degree theoretic arguments are not directly applicable. We therefore confine our attention to Fredholm operators of index zero and refer to Deimling [13] for a discussion of operators of nonzero index.

The existence of solutions to many problems in nonlinear analysis cannot be determined by purely analytic means and other techniques must be employed. A frequently useful and now fairly general topological method of proving the existence of solutions to equations is by use of a topological degree and related fixed point index. The basic procedure in such an argument is to first formulate the problem in terms of a map for which a topological degree is defined: *e.g.* compact, contractive, condensing *etc.*; then show the degree of the map over a specified set is nonzero (usually employing the homotopy property of the degree). The existence property of the degree then implies the equation has a solution.

The concept of a topological mapping degree was introduced by Brouwer [3] around 1910 for a continuous map defined on a euclidean simplicial complex. He used this degree to prove that a continuous mapping of a sphere in $\mathbb{R}^n$ into itself has a fixed point. Like much else in mathematics, the idea was not completely new and for $\mathbb{C}$ can be traced back to the "winding number" or index of a plane closed curve surrounding some point. It is defined in terms of a Cauchy integral and gives an "algebraic count" (*i.e.,* counting $+1$ for each positively oriented revolution and -1 for each negatively oriented revolution) of the number of windings a curve makes about that point. Let G be a simply connected region in $\mathbb{C}$, $f : G \to \mathbb{C}$ be analytic and let γ be a closed C^1 curve in G. Kronecker [27], in what is now called the Kronecker existence principle, observed that if $f(z) \neq 0$ on γ, and the winding number of $f(\gamma)$ is not zero then f has a zero in G_0, the region enclosed by γ. The winding number was also seen to possess another useful property; that of homotopy invariance. Together, these two properties form the basis of most applications of degree theory in nonlinear analysis. More details may be found in Zeidler [58].

The extension of the Brouwer degree to compact maps in infinite dimensional Banach spaces was made by Leray and Schauder in 1934 [31]. Since a large number of differential, integral and integro-differential equations can be formulated in terms of compact operators on infinite dimensional Banach spaces, the Leray-Schauder degree has had wide and extensive use. More recent developments in degree theory include Nussbaum's degree for condensing maps, *q.v.*, Section 1.2 for a definition, and the "coincidence degree" of Mawhin which extends the Leray-Schauder degree to certain semilinear maps.

Another relatively recent degree theory and the one which is of particular importance to our results is that for A-proper maps, *q.v.*, Section 1.3, developed by Browder and Petryshyn [4] in 1968. The class of A-proper maps was introduced by Petryshyn [39] in 1968 and shown, under certain conditions and projection schemes, to include: compact perturbations of the identity, β-Lipschitz, monotone, and accretive type operators. Thus, results obtained for this class have some generality. An additional advantage of the A-proper degree is that not only existence of a solution may be inferred if the degree is nonzero but also, by nature of the theory, constructive solvability of the equation is obtained.

In determining the existence of non-negative solutions to equations the notion of a cone proves useful. That is, a closed convex subset K of a Banach space X satisfying $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$. Elementary examples are $K = \mathbb{R}^+$ and the subset of non-negative functions in $C[0,1]$. These cones have nonempty interior in their respective spaces; however, many other cones of interest have empty interior such as the subset of non-negative functions in $L^p[0,1]$. A somewhat problematic consequence of this is that topological degree theory, in its strict sense, cannot be applied directly to mappings defined on relatively open subsets of a cone with empty interior (an assumption in defining a degree is that $Tx \neq x$ on $\partial\Omega$, but $\partial\Omega = \Omega$ if Ω has empty interior). A concept closely related to the mapping degree but of broader definition is the fixed point index of a map. Essentially, like the degree, it is an algebraic count of the number of solutions to the equation $Tx = x$ in a given set, *i.e.*, the number of points left fixed by T. In 1928, Hopf [23] defined a fixed point index for a continuous map on a combinatorial manifold (a generalisation of polyhedra) and used it to considerably simplify the proof of the Lefschetz Fixed Point Theorem. He used a homeomorphism to map disjoint neighbourhoods of the fixed points of a continuous function onto balls in $\mathbb{R}^n$, where the Brouwer degree is defined. For each fixed point, he then defined the fixed point index of the continuous function to be the value of the Brouwer degree on the corresponding ball in $\mathbb{R}^n$. Since this connection was made, it has become common practice to define fixed point indices in terms of a topological degree that is not directly applicable for a given situation but where properties of a degree are desired.

In [36], Nussbaum extended the definition of the fixed point index to condensing maps over closed convex subsets of a Banach space using a retraction argument. Since these sets include cones, he was able to use this index to prove some cone compression and expansion type existence theorems in [37] and obtain results for various nonlinear functional differential equations. He also pointed out that, in a particular form, this index is equal to the Leray-Schauder degree and hence an extension of it.

Furthering these ideas, Amann, in his survey article [2], mentioned a fixed point index for compact operators mapping a retract into itself. He used a retraction argument to modify the Leray-Schauder degree so that the degree, and equivalently the index, could be determined over closed convex sets.

As this extension of the Leray-Schauder degree is illustrative of the techniques we shall employ later using A-proper and Brouwer degrees, we provide a derivation of this fixed point index according to Amann [2]. We mention that all topological notions such as open, closed, boundary, *etc.* refer to the relative topology of K as a subspace of X.

Let K be a retract of a Banach space X with retraction r, $f : \overline{\Omega} \to K$ a compact map with $\Omega \subset K$ relatively open and assume $f(x) \neq x$ on $\partial\Omega$. Then we may define the fixed point index of f over Ω with respect to K by

$$i(f,\Omega,K) = i\left(fr, r^{-1}(\Omega), X\right) = \deg\left(I - fr, r^{-1}(\Omega), 0\right)$$

the Leray-Schauder degree for identity minus compact maps.

Some of the important properties of this index are given in the next theorem, *cf.* Amann [2] for a proof.

Theorem 0.0.1 *Let K be a retract of a Banach space X, $\Omega \subset K$ an open set and $f : \overline{\Omega} \to K$ a compact map such that $f(x) \neq x$ on $\partial\Omega$. Then there exists an integer $i(f,\Omega,K)$ satisfying the following conditions:*
(i) (Normalisation) for every constant map f mapping $\overline{\Omega}$ into Ω, $i(f,\Omega,K) = 1$
(ii) (Additivity) for every pair of disjoint open subsets Ω_1, Ω_2 of Ω such that f has no fixed points on $\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$,
$i(f,\Omega,K) = i(f,\Omega_1,K) + i(f,\Omega_2,K)$ where $i(f,\Omega_n,K) = i(f\mid_{\bar{\Omega}_n},\Omega_n,K)$ for $n = 1,2$
(iii) (Homotopy invariance) for every compact interval $[a,b] \subset \mathbb{R}$ and every compact map $h : [a,b] \times \overline{\Omega} \to K$ such that $h(\lambda,x) \neq x$ for $(\lambda,x) \in [a,b] \times \partial\Omega$, $i(h(\lambda,\cdot),\Omega,K)$ is well defined and independent of $\lambda \in [a,b]$.

The modern approach to index theory is rather axiomatic in that once an index is defined, it is then shown to satisfy various properties of the classical index. The ones we verify in this thesis are those most common and useful to the analyst, *viz.*, existence, normalisation, additivity and homotopy. There are other properties that we don't discuss since we don't use them in our existence theorems, such as: commutativity, excision, and permanence; but may be proved following similar arguments used in proving the other properties (often appealing to the equivalent properties of the underlying degree).

For more detailed-including historical-accounts of these topics, see: Gaines and Mawhin [18], Zeidler [58] for semilinear equations; Alexandrov and Hopf [1], Brown [6], Deimling [13], Dieudonné [14], and Zeidler [58] for topological degree and fixed point index theory.

This thesis is organised into six chapters: the first is preparatory and concerns basic concepts; the second, third and fourth develop fixed point indices for A-proper maps and are ordered roughly according to increasing generality of application; the fifth establishes existence theorems; and the sixth is on applications.

In Chapter 1, we introduce the basic ideas necessary in the development of our theory. We begin with a review of Banach spaces and linear operators then proceed to a discussion of A-proper maps, Fredholm operators of index zero and conclude with the definition and properties of the generalised degree for A-proper maps.

Our first result, the content of Chapter 2, establishes a new fixed point index for A-proper (at 0) maps defined on cones. Our definition of this index places some requirements on the retraction ρ mapping the Banach space X to the cone K, *viz.*, $\|\rho x - x\| \leq 2\,\mathrm{dist}\,(x, K)$ and $\rho(X_n) \subseteq X_n$. The first condition seems to be satisfied by most retractions used in practice; however, the second is more restrictive and disqualifies some common retractions and projection schemes. Examples of retractions and projection schemes that satisfy and fail these conditions are provided in the introduction to this chapter. Another requirement of this index is that the A-proper map T must be defined on all of the cone K. This condition is used to prove the index is independent of the retraction chosen in the definition. Despite these limitations, an attractive feature of this index is that it is defined in terms of the generalised degree for A-proper maps and does not require any reduction to finite dimensional degree arguments. Assuming the above conditions on the retraction $\rho : X \to K$, we define in Section 2.2 the fixed point index as follows.

Let $\Omega \subset K$ be open bounded and K a cone in a Banach space X. Assume $I - T$ is A-proper at 0 relative to a projection scheme Γ, $T : K \to K$ and $Tx \neq x$ on $\partial\Omega$. Then we define the fixed point index of T over Ω relative to K as

$$\mathrm{ind}_K(T, \Omega) = \mathrm{Deg}\left(I - T\rho, \rho^{-1}(\Omega), 0\right)$$

where the degree is that for A-proper maps defined by Browder and Petryshyn [4]. We show that the index is well defined, independent of the retraction chosen provided it satisfies the stated conditions and has most of the properties of the classical fixed point index.

In Section 2.3, we show that the semilinear equation $Lx - Nx = w$ where L is Fredholm of index zero, N nonlinear and $L - N$ A-proper at 0 can be converted to the form $(I - T)y = w$ and thereby extend the above index to semilinear operators. We conclude the chapter with the definition of the index on unbounded sets U. This is done in the usual way of taking an open bounded set $V \subset U$ such that $(I - T)^{-1}(0) \subset V$ and defining

$$\text{ind}_K(T, U) = \text{ind}_K(T, V)$$

which is a consequence of the additivity over domains and excision properties of the index. We show that this definition extends to semilinear operators $L - N$.

In Chapter 3, we develop an index for A-proper at 0 maps without the restrictions on the retraction ρ and the domain of T mentioned above; thus obtaining an index of greater application than that of Chapter 2. The method used is similar to that of Fitzpatrick and Petryshyn [16] where we first define a finite dimensional index and then obtain the infinite dimensional version through a limiting process. We add that the finite dimensional index also plays a part in defining the fixed point index for weakly inward A-proper at 0 maps in Chapter 4. As in Chapter 2, we first define the index for A-proper at 0 maps $T : \overline{\Omega_K} \to K \subset X$ ($\Omega_K = \Omega \cap K$), then establish the index for maps $L - N : \text{dom}\, L \cap \overline{\Omega_K} \to Y$ and end with the definition for unbounded sets.

Chapter 4 extends the Lan-Webb [30] fixed point index for weakly inward A-proper at 0 maps to semilinear operators. With weakly inward maps, the previous requirement of Chapters 2 and 3, that the operators map cones to cones is relaxed. Weakly inward operators map closed convex sets K to so called weakly inward sets that contain K

(precise definitions will be found in Chapter 4). We show that the concept of a weakly inward map may be extended to semilinear maps and define a fixed point index for them. We also extend the Lan-Webb index to unbounded sets.

Existence theorems for semilinear equations in cones are established in Chapter 5 using the fixed point indices of the preceding chapters. In Section 5.2, we extend a general continuation theorem of Mawhin [34] and an existence result of Petryshyn [42] to cones. We then establish two corollaries of practical interest related to results of Cesari [9], Mawhin [34], Petryshyn [42], and Webb [54]. We also obtain an existence theorem for positive solutions and one for weakening *a priori* bound requirements. These theorems extend results of Webb [54]. Theorem 5.2.13 extends to semilinear maps a result of Petryshyn [41] which in turn extends results of Gatica and Smith [19], Nussbaum [37], and several others, *cf.* [41] for a list.

Section 5.3 involves existence theorems on quasinormal cones where we extend to semilinear equations many of the results established by Lafferriere and Petryshyn [28] for P_γ -compact cone maps. The idea of quasinormality was introduced by Petryshyn in [43] where it proved to be useful in studying the existence of positive eigenvectors and fixed points of noncompact maps. Included in this section are several norm type cone compression and expansion theorems.

We obtain existence theorems for weakly inward A-proper maps in Section 5.4 which extend results of Lan and Webb [30]. We use a variation of the Leray-Schauder boundary condition in Theorem 5.4.4 to prove the existence of a solution to $Lx = Nx$. We also provide conditions which imply the index is 0 and, in conjunction with Theorem 5.4.4, obtain a result that gives a positive solution to a semilinear equation. Our last theorem in this chapter gives conditions that ensure the existence of at least two positive solutions but under the rather restrictive hypothesis; that $(N + J^{-1}P)(K)$ be bounded for a cone K.

The final chapter, Chapter 6, is on applications of the existence theorems from Chapter 5 to differential and integral equations. In our first application, we prove the existence of a positive solution to the second order boundary value problem

$$-x''(t) = f(t, x(t), x'(t), x''(t)) \text{ where } x(0) = x(1) = 0. \tag{0.4}$$

We convert the equation to an operator equation of the form $L - N$ and use Theorem 5.3.8 to obtain a positive solution to (0.4) in the cone

$$K = \left\{x \in C^2[0,1] : -x''(t) \geq 0,\ x(0) = x(1) = 0\right\}.$$

Our second result determines a non-negative solution to the second order periodic boundary value problem

$$-x''(t) = f(t, x(t), x'(t)) \text{ where } x(0) = x(1) \text{ and } x'(0) = x'(1). \tag{0.5}$$

After converting (0.5) to an operator equation, we apply Corollary 5.2.7 to obtain a non-negative solution in the Banach space $X = \{x \in C^2[0,1] : x(0) = x(1),\ x'(0) = x'(1)\}$.

Lastly, using our weakly inward results, we prove the existence of a non-negative solution to the Volterra equation

$$y(t) = \int_0^t k(t, s, x(s))\, ds,\ t \in J = [0,1]$$

where k and y are $\mathbb{R}^n$-valued. The problem, as we formulate it, is similar to one mentioned by Deimling [13] where he obtains a solution in the cone of non-negative functions in $C[0,a]$ using a theorem valid only for cones with nonempty interior. Our index theory applies to cones with empty and nonempty interior thus enabling us to obtain a solution in the cone of non-negative *a.e.* functions of $L^2[0,1]$.

Chapter 1

PRELIMINARY TOPICS

1.1 Introduction

We present the basic and relevant concepts used throughout this thesis concerning Banach spaces, A-proper maps, Fredholm maps of index zero, and topological degree theory. Standard references for this material are Taylor and Lay [52], Yosida [56] for Banach spaces; Petryshyn [47] for A-proper maps; Deimling [13], Dunford and Schwartz [15], Taylor and Lay [52] for Fredholm operators of index zero; and Cronin [10], Deimling [13], Lloyd [32] for topological degree theory.

1.2 Banach spaces, linear operators, cones

In the sequel, X and Y will denote Banach spaces with norms $\|\|_X$ and $\|\|_Y$ respectively. When there is no ambiguity, we shall simply write $\|\|$ to denote both of these norms. A Banach space is said to be *separable* if it has a countable dense subset. We note that every Banach space with a Schauder basis (see remark 1.3.3 below) is separable, the converse being false as proved by P. Enflo. The *dual space* X^* of a Banach space X is the vector space of all bounded linear functionals $x^* : X \to \mathbb{R}$ with $x^*(x) = (x, x^*)$ being the value of x^* at x. A Banach space is said to be *reflexive* if the canonical embedding $F : X \to X^{**}$ defined by $(x^*, Fx) = x^*(x)$ on X^* is surjective.

We define several Banach spaces of particular interest in applications to differential and integral equations.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $\overline{\Omega}$ its closure. $C(\overline{\Omega})$ denotes the space of continuous functions $f : \overline{\Omega} \to \mathbb{R}$ with norm $\|f\| = \max_{x \in \overline{\Omega}} |f(x)|$. $C^k(\Omega)$, $k \geq 1$, denotes the space of k times continuously differentiable functions on Ω and $C^k(\overline{\Omega})$ denotes those $f \in C^k(\Omega)$ all of whose partial derivatives of order $\leq k$ have continuous extensions to $\overline{\Omega}$. This space is endowed with the norm $\|f\| = \sum_{i=0}^k \max_{x \in \overline{\Omega}} |D^i f(x)|$; here $D^0 f = f$ and $D^i f$ stands for all partial derivatives of f of order i. When $\overline{\Omega} = [a, b] \subset \mathbb{R}$, $C^k(\overline{\Omega})$ is written $C^k[a, b]$.

For $1 \leq p < \infty$, $L^p(\Omega)$ is the space of (equivalence classes of) functions whose p-th power is Lebesgue integrable. When endowed with the norm

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

$L^p(\Omega)$ forms a Banach space. $L^\infty(\Omega)$ is the space of essentially bounded functions with norm

$$\|f\|_\infty = \text{ess} \sup_{x \in \Omega} |f(x)|.$$

For $1 \leq p < \infty$, $k \in \mathbb{N}$, $W^{k,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$ such that the distributional derivatives, $D^s f$ of order $|s| = \sum_{i=1}^n |s_i| \leq k$, belong to $L^p(\Omega)$. The norm is defined as

$$\|f\|_{k,p} = \left(\sum_{|s| \leq k} \int_\Omega |D^s f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We define a *cone* K in a Banach space X as a closed convex subset of X such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap \{-K\} = 0$. The set $K = \{x(t) \in C[0, 1] : x(t) \geq 0\}$ is a common example in applications. $x \leq y$ iff $y - x \in K$ defines a partial ordering on X. For $0 \leq x \leq y$, the norm on X is called *monotone* if $\|x\| \leq \|y\|$ and *semimonotone* if $\|x\| \leq \gamma \|y\|$ for $\gamma > 0$. A cone is said to be *normal* if $\|\|$ on X is semimonotone with respect to $K \subset X$. This is equivalent to $\|x+y\| \geq \gamma \|y\|$ for all $x, y \in K$ and some $\gamma \in (0,1]$, *cf.* Lafferriere and Petryshyn [28].

The terms map, transformation, and operator will be used synonymously. Maps will be denoted by the symbols T, L, N and I, the last representing the identity operator $Ix = x$. A *linear operator* L satisfies $L(\alpha x + \beta y) = \alpha Lx + \beta Ly$ for $x, y \in \operatorname{dom} L$ and $\alpha, \beta \in \mathbb{R}$. A linear operator $L : X \to Y$ is *bounded* if there exists $M \in \mathbb{R}$ such that $\|Lx\|_Y \leq M \|x\|_X$ for every $x \in \operatorname{dom} L$. The *norm* of a bounded linear operator is defined as

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

A set $\Omega \subset X$ is said to be *compact* if every open covering of Ω has a finite subcovering. Equivalently, $\Omega \subset X$ is compact iff (if and only if) every sequence $\{x_n\} \subset \Omega$ has a convergent subsequence with a limit in Ω. A set Ω is said to be *relatively compact* if $\overline{\Omega}$ is compact.

A linear operator $L : X \to Y$ is said to be *compact* if $\overline{L\Omega}$ is compact for every bounded $\Omega \subset X$. Equivalently, a linear operator $L : X \to Y$ is compact iff $\{Lx_n\}$ has a convergent subsequence for every bounded sequence $\{x_n\}$.

Let $K \subseteq X$ and $U \subseteq K$, then U is called relatively open in K if there exists an open set $V \subset X$ such that $U = K \cap V$. Relatively closed sets are defined analogously. The boundary of U relative to K, denoted $\partial_K U$, is the relative closure of U minus the relative interior of U.

H is called a *homeomorphism* iff H is bijective and both H and H^{-1} are continuous.

A *projection* is a linear operator P from X onto a subspace X_n which satisfies $P^2 x = Px$, for $x \in X$ and $Px = x$ for $x \in X_n$. A set $\Omega \subset X$ is said to be a *retract* of X if there exists a continuous map $R : X \to \Omega$ such that $Rx = x$ for every $x \in \Omega$. The map R is called a *retraction*. An important consequence of the following theorem by Dugundji is that every closed convex set in a normed linear space is a retract. This fact plays a crucial role in defining a fixed point index on cones.

Theorem 1.2.1 *(Dugundji [13]) Let X and Y be normed linear spaces, $\Omega \subset X$ closed and $f : \Omega \to Y$ continuous. Then f has a continuous extension $F : X \to Y$ such that* $F(X) \subset \operatorname{conv}(f(\Omega))$.

For a bounded set $\Omega \subset X$, we define the *set measure of noncompactness* $\alpha(\Omega)$ and the *ball measure of noncompactness* $\beta(\Omega)$ as

$\alpha(\Omega) = \inf\{\delta > 0 : \Omega$ admits a finite covering by sets of diameter $\leq \delta\}$;

$\beta(\Omega) = \inf\{\delta > 0 : \Omega$ can be covered by finitely many balls of diameter $\leq \delta\}$.

We note that $\alpha(\Omega) = \beta(\Omega) = 0$ iff $\overline{\Omega}$ is compact.

A continuous bounded map $T : X \to Y$ is called a *k-set contraction* if there is a constant $k \geq 0$ such that for all bounded sets $\Omega \subset \operatorname{dom} T$, $\alpha_Y(T(\Omega)) \leq k\alpha_X(\Omega)$ (the subscript indicates the space in which the measure is determined). If $\alpha_Y(T(\Omega)) < \alpha_X(\Omega)$ whenever $\alpha_X(\Omega) \neq 0$ then T is said to be *α-condensing.* Similar definitions exist for the ball measure of non-compactness β. That is, a continuous bounded map $T : X \to Y$ is called a *k-ball contraction* if there exists a number $k \geq 0$ such that $\beta_Y(T(\Omega)) \leq k\beta_X(\Omega)$ for all bounded sets $\Omega \subset \operatorname{dom} T$. T is called *ball-condensing* if $\beta_Y(T(\Omega)) < \beta_X(\Omega)$ unless $\overline{\Omega}$ is compact.

1.3 A-proper maps

The solution to infinite dimensional operator equations of the form $F(x) = y$ by limits of finite dimensional approximations $F_n(x_n) = y_n$ motivates the study of *Approximation-proper* (abbreviated *A-proper*) maps introduced by Petryshyn in [39]. It has been shown that many commonly encountered operators in applications are A-proper relative to an appropriate projection scheme. We precisely define such a scheme and the notion of A-proper maps.

Definition 1.3.1 *Let X and Y be separable Banach spaces, D a dense linear subspace of X, $\{X_n\} \subset D$ and $\{Y_n\} \subset Y$ sequences of oriented finite dimensional subspaces such that $\dim X_n = \dim Y_n$ for each n, $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$ and let $Q_n : Y \to Y_n$ be a sequences of continuous linear projections such that $Q_n y \to y$ in Y for every $y \in Y$. The projection scheme $\Gamma = \{X_n, Y_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y. (Also when the condition $Q_n y \to y$ holds the scheme is sometimes said to be projectionally complete.)*

Remark 1.3.2 *Here D is allowed to be the whole space X. We need to include D when we consider densely defined operators, as we shall do below.*

Remark 1.3.3 *If X and Y possess Schauder bases then there exist natural projection schemes. Recall that, a sequence $\{\varphi_i\}$ is called a Schauder basis for X if, for each $x \in X$, there exists a unique sequence of numbers $\{x_i\}$ such that $\sum_{i=1}^{n} x_i \varphi_i$ converges to x as $n \to \infty$. Suppose that $\{\varphi_i\}$ and $\{\psi_i\}$ are Schauder bases for X, Y respectively. Then we may take $X_n = [\varphi_1, ..., \varphi_n]$, $Y_n = [\psi_1, ..., \psi_n]$, where $[\cdots]$ denotes linear span, and $Q_n(y) = \sum_{i=1}^{n} y_i \psi_i$, By the Uniform Boundedness Theorem [52], there exists $0 < c < \infty$, such that $\|Q_n\| \leq c$ for all $n \in \mathbb{N}$.*

Definition 1.3.4 *Let S be a subset of D and $T : S \subset X \to Y$. T is called A-proper at y relative to the projection scheme Γ if*

(i) $Q_n T : S \cap X_n \to Y_n$ is continuous

(ii) for any bounded sequence $x_n \subset S \cap X_n$ such that $Q_n T x_n \to y$ for $y \in Y$, there exists a subsequence $x_{n_j} \to x \in S$ and which satisfies $Tx = y$. T is called A-proper if it is A-proper at all points $y \in Y$.

Remark 1.3.5 *The set of A-proper maps with a given projection scheme does not form a linear space, this is evident from the simple example that I and $-I$ are A-proper but their sum, the 0 operator, is not. However, as we shall prove in our next theorem, if T is A-proper and C is compact, then $T+C$ is A-proper. Also, it is clear that if T is A-proper and $\lambda \neq 0$ then λT is A -proper.*

Theorem 1.3.6 *(Petryshyn [39]) If $T : S \subset X \to Y$ is A-proper with respect to $\Gamma = \{X_n, Y_n, Q_n\}$ and $C : S \subset X \to Y$ is compact then $T + C$ is A-proper with respect to Γ.*

Proof. Let $\{x_n\} \in X_n$ be a bounded sequence such that $Q_n(T+C)x_n \to y \in Y$. Since C is compact, there exists a subsequence $\{x_{n_j}\}$ such that $Q_{n_j}Cx_{n_j} \to y_0$. Since T is A-proper, we may choose a subsequence, again denoted by $\{x_{n_j}\}$ converging to x such that $Tx = y - y_0$. By the continuity of C, $Cx_{n_j} \to Cx = y_0$ and therefore $(T+C)x = y$. Q.E.D.

Definition 1.3.7 *T is proper if $T^{-1}(K)$ is compact whenever K is compact .*

In [40], Petryshyn proved that continuous A-proper maps are indeed proper. We provide a proof of this assertion for completeness.

Theorem 1.3.8 *Let X and Y be Banach spaces, $\Omega \subset X$ open and $T : \overline{\Omega} \to Y$ continuous and A-proper with respect to a projection scheme $\Gamma = \{X_n, Y_n, Q_n\}$. Then the restriction of T to every closed bounded subset of Ω is proper.*

Proof. Let M be a closed bounded subset of Ω. Suppose that $\{x_n\}$ is a sequence in $M \cap T^{-1}(K)$ where $K \subset Y$ is compact. Then $\{T(x_n)\}$ is a sequence in K which we may and do assume converges to $y \in K$. For each $k \in \mathbb{N}$, choose $\epsilon_k > 0$ with $\epsilon_k \to 0$ as $k \to \infty$. By the continuity of T, there exists $\delta_k > 0$, $\delta_k \to 0$ as $k \to \infty$ such that if $\|z - x_k\| < \delta_k$ for $z \in \overline{\Omega}$, then $\|Tz - Tx_k\| < \epsilon_k$. By the properties of the admissible scheme, there exists $z_{n_k} \in X_{n_k} \cap \overline{\Omega}$, $n_k > k$, with $\|Tz_{n_k} - Tx_k\| < \epsilon_k$ and $\|z_{n_k} - x_k\| < \delta_k$. Thus

$$\begin{aligned} \|Q_{n_k}Tz_{n_k} - y\| &\leq \|Q_{n_k}Tz_{n_k} - Q_{n_k}Tx_k\| + \|Q_{n_k}Tx_k - Q_{n_k}y\| + \|Q_{n_k}y - y\| \\ &\leq c\,\|Tz_{n_k} - Tx_k\| + c\,\|Tx_k - y\| + \|Q_{n_k}y - y\| \end{aligned}$$

since the projections Q_n are uniformly bounded by a constant c. Then $Q_{n_k}Tz_{n_k} \to y$ as $k \to \infty$ and by the A-properness of T, there exists $x \in \overline{\Omega}$ such that (choosing a subsequence if necessary) $z_{n_k} \to x$ and $Tx = y$. Hence, $x_k \to x$ and since M is closed, $x \in M \cap T^{-1}(K)$ which is therefore compact. Q.E.D.

The class of *projectionally-compact* (abbreviated *P-compact*) maps introduced by Petryshyn in [38] were prototypical in the development of A-proper maps and will appear later in Chapter 5. They are defined as follows.

Definition 1.3.9 *$T : S \subset X \to Y$ is P_γ-compact if the map $\lambda I - T : S \to Y$ is A-proper with respect to Γ for each $\lambda \geq \gamma$ if $\gamma > 0$ or $\lambda > 0$ if $\gamma = 0$. (For $\gamma = 0$, T is simply said to be P-compact).*

Other examples of A-proper maps include strongly monotone and strongly accretive operators and their perturbations by compact or ball condensing maps, *cf.* Petryshyn [47].

Many of the proofs to our theorems involve homotopies that are A-proper; we define these maps as follows.

Definition 1.3.10 *A map $H : [0,1] \times S \subset X \to Y$ is called an A-proper homotopy at y relative to a projection scheme Γ if $Q_nH : [0,1] \times Q_nS \to Y_n$ is continuous and if $\{x_n\}$ is a bounded sequence in S and $\{t_n\} \subset [0,1]$ are such that $Q_nH(t_n, x_n) \to y$ for some $y \in Y$, then there exist subsequences $x_{n_j} \to x \in S$ and $t_{n_j} \to t \in [0,1]$ such that $H(t,x) = y$. H is said to be an A-proper homotopy if it is A-proper at all points y.*

Remark 1.3.11 *We shall be particularly interested in the case of A-proper at 0.*

1.4 Fredholm operators of index zero

The study of certain integral equations, initiated by I. Fredholm [17], when formulated as abstract operator equations, led to the general theory of Fredholm operators in Banach spaces. In this thesis we shall consider the subclass of Fredholm operators with index zero as the construction of our fixed point index depends on certain properties particular to them. We define these operators and mention some useful properties and their relation to A-proper maps.

Definition 1.4.1 *A closed, densely defined, linear operator* $T : \mathrm{dom}(T) \subset X \to Y$ *is said to be a Fredholm operator if* $\dim(\ker T) < \infty$ *and* $\mathrm{codim}\,(\mathrm{im}\,T) = \dim(Y/\,\mathrm{im}\,T) < \infty$*. We denote the class of all Fredholm operators from* $\mathrm{dom}(T) \subset X$ *to* Y *by* $\Phi\,(X,Y)$ *or* $\Phi\,(X)$ *if* $X = Y$*. The index of* $T \in \Phi\,(X,Y)$ *is defined as* $\dim(\ker T) - \dim(Y/\,\mathrm{im}\,T)$*. The subclass of Fredholm operators with index zero is denoted* $\Phi_0\,(X,Y)$*.*

Some examples of Fredholm operators of index zero are:

(i) $T : X \to Y$ where T is a bounded linear bijection [52]

(ii) $T = I - C$ where C is compact [52]

(iii) $T = L - C$ where $L \in \Phi_0\,(X,Y)$, C is compact and linear [25]

(iv) if $L : X \to X$ is a bounded linear operator and $|\lambda| > r_{ess}\,(L)$ where $r_{ess}\,(L) = \sup\{|\lambda| : \lambda \in \sigma_{ess}\,(L)$ the essential spectrum of $L\}$, then $\lambda I - L \in \Phi_0\,(X)$ [35]

(v) if $L \in \Phi_0\,(X,Y)$ and $T \in \Phi_0\,(Y,Z)$ then $TL \in \Phi_0\,(X,Z)$ [52]

(vi) $T : X \to Y$ where T is a bounded linear A-proper map with $\ker T = 0$ [47].

We provide a proof to example (vi) as it is of particular importance in our results and to illustrate the methods in general.

Theorem 1.4.2 *(Petryshyn [47]) If* $T : X \to Y$ *is a bounded linear A-proper map relative to* $\Gamma = \{X_n, Y_n\}$*with* $\ker T = 0$ *then* T *is Fredholm of index zero.*

Proof. We show that these conditions imply T is a homeomorphism whence the result readily follows. Since $\ker T = 0$, T is injective. To prove T is also onto we demonstrate first that there exists a constant $c > 0$ and $N_0 \in \mathbb{N}$ such that $\|Q_n T x_n\| \geq c \|x_n\|$ for every $x_n \in X_n$ with $n \geq N_0$. To obtain a contradiction, suppose the contrary. Then there is a sequence $\{x_n\}$ which, by linearity of $Q_n T$, we may choose with $\|x_n\| = 1$ for every $n \in \mathbb{N}$ such that $\|Q_n T x_n\| < \frac{1}{n} \|x_n\| = \frac{1}{n} \to 0$ as $n \to \infty$. By the A-properness of T, there exists a subsequence $\{x_{n_j}\}$ and $x \in X$ such that $x_{n_j} \to x$ with $\|x\| = 1$ and $Tx = 0$. This contradicts the injectivity of T and so we have proved the existence of c and N_0. Now for $n \geq N_0$, $Q_n T : X_n \to Y_n$ is injective and therefore onto since X_n and Y_n are of equal finite dimension n and $Q_n T$ is linear and continuous for such n. Thus, for each $y \in Y$ there exists a unique $x_n \in X_n$ such that $Q_n T x_n = Q_n y$ for every $n \geq N_0$. Now $c \|x_n\| \leq \|Q_n T x_n\| = \|Q_n y\| \leq k \|y\|$ since the sequence $\{Q_n\}$ is uniformly bounded, *cf.* Remark 1.3.3. So $\{x_n\}$ is a bounded sequence and $Q_n T x_n = Q_n y \to y$ as $n \to \infty$. By the A-properness of T, there exists $x \in X$ and a subsequence $x_{n_j} \to x$ with $Tx = y$. Hence T is onto and therefore a homeomorphism.

Now since T is a homeomorphism, $\operatorname{im} T = Y$ which is closed and $\dim (Y / \operatorname{im} T) = 0$ so that T is Fredholm of index zero. Q.E.D.

Remark 1.4.3 *That the image of $T \in \Phi(X, Y)$ is closed follows from* $\operatorname{codim}(\operatorname{im} T) < \infty$ *and Y is a Banach space [13].*

We now discuss some important properties of Fredholm operators of index zero which are essential to our results and will recur frequently throughout this work.

If $L \in \Phi_0(X, Y)$, then using known results for such operators, *cf.* Deimling [13], X and Y may be expressed as direct sums; $X = X_0 \oplus X_1$, $Y = Y_0 \oplus Y_1$ with continuous linear projections $P : X \to \ker L = X_0$ and $Q : Y \to Y_0$. The restriction of L to $\operatorname{dom} L \cap X_1$, denoted L_1, is a bijection onto $\operatorname{im} L = Y_1$ with continuous inverse $L_1^{-1} : Y_1 \to \operatorname{dom} L \cap X_1$ which is also bijective. Since X_0 and Y_0 have the same finite dimension, there exists a continuous bijection $J : Y_0 \to X_0$. If we let $H = L + J^{-1}P$ then $H : \operatorname{dom} L \subset X \to Y$ is a linear bijection and H^{-1} is bounded as we show in the following theorem.

Theorem 1.4.4 *The operator $H : \operatorname{dom} L \subset X \to Y$ where $H = L + J^{-1}P$ is a linear bijection and H^{-1} is bounded.*

Proof. H is clearly linear as it is the sum of two linear operators L and $J^{-1}P$. We prove first that H is injective, *i.e.,* one to one. Suppose $Hx = (L + J^{-1}P)\,x = 0$, then $Lx + J^{-1}Px = 0$ and we have $Lx = -J^{-1}Px$. Now, as $Lx \in \operatorname{im} L$ and $J^{-1}Px \in Y_0$ we must have $Lx = 0$ and $J^{-1}Px = 0$ because they are direct sums. So $x \in \ker L$ and $J^{-1}x = 0$ which gives $x = 0$ and hence H is injective.

To prove H is surjective, the preceding discussion showed that H is injective and therefore $\ker H = \{0\}$. Since H is Fredholm of index zero by (iii) above, $\dim \ker H = \dim Y \setminus \operatorname{im} H = 0$. Thus $\operatorname{im} H = Y$ and H is surjective.

Finally, we prove the boundedness of H^{-1}. We observe that H is closed since the graph of H is closed in $X \times Y$ and consequently, H^{-1} is closed as the graph of H^{-1} is closed in $Y \times X$. Now as $\operatorname{dom} H^{-1} = Y$, the Closed Graph Theorem [13] implies H^{-1} is continuous, *i.e.,* bounded. Q.E.D.

An admissible projection scheme Γ_L can now be constructed for $L \in \Phi_0(X, Y)$ such that L is A-proper with respect to Γ_L as first shown by Petryshyn in [42]. Let $Y_n \subset Y$ be a sequence of finite dimensional subspaces and $Q_n : Y \to Y_n$ a sequence of projections such that $Q_n y \to y$ as $n \to \infty$ for each $y \in Y$. If we let $X_n = H^{-1}(Y_n)$ then $\Gamma_L = \{X_n, Y_n, Q_n\}$ is an admissible scheme for maps $L : \operatorname{dom} L \subset X \to Y$ and L is A-proper relative to Γ_L. We prove these assertions in the following two theorems.

Theorem 1.4.5 *(Petryshyn [42]) For $L \in \Phi_0(X, Y)$, $\Gamma_L = \{X_n, Y_n, Q_n\}$ is an admissible scheme.*

Proof. We need only show that X_n and Y_n have the same finite dimension and that $\operatorname{dist}(x, X_n) \to 0$ for every $x \in X$. Since H is a linear homeomorphism, H^{-1} preserves subspace dimension so that $\dim X_n = \dim H^{-1}(Y_n) = \dim Y_n$ for each $n \in \mathbb{N}$. Now for each $x \in X$, there exists $y \in Y$ such that $H^{-1}y = x$ and

$$\begin{aligned} \operatorname{dist}(x, X_n) &= \operatorname{dist}(H^{-1}y, H^{-1}(Y_n)) \\ &= \inf_{H^{-1}y_n = x_n \in X_n} \left\| H^{-1}y - H^{-1}y_n \right\| \\ &\leq \inf_{y_n \in Y_n} \left\| H^{-1} \right\| \|y - y_n\| \\ &= \left\| H^{-1} \right\| \operatorname{dist}(y, Y_n) \to 0 \end{aligned}$$

as $n \to \infty$. Q.E.D.

Theorem 1.4.6 *(Petryshyn [42])If $L \in \Phi_0(X, Y)$, then L is A-proper with respect to Γ_L.*

Proof. Let $\{x_n\} \subset X_n$ be a bounded sequence such that $Q_n L x_n \to y \in Y$. Then $x_n = H^{-1}y_n$ for some $y_n \in Y_n$. L will be A-proper if we can find a subsequence $x_{n_j} \to x$ such that $Lx = y$. Let $H = L + C$ where $C = J^{-1}P$, then $Q_n H x_n = Q_n(L + C)x_n = Q_n L x_n + Q_n C x_n$. As C is compact, there exists a subsequence $\{x_{n_j}\}$ such that $Q_n C x_{n_j} \to z \in Y$ and $Q_n L x_{n_j} + Q_n C x_{n_j} \to y + z = h \in Y$. So $Q_{n_j} H x_{n_j} \to h$, that is $Q_{n_j} y_{n_j} = y_{n_j} \to h$, therefore, $x_{n_j} = H^{-1} y_{n_j} \to H^{-1}h = x$ for some $x \in X$. By the continuity of C, $Cx = z$ so that $Hx = Lx + Cx = y + z$ and therefore $Lx = y$. Q.E.D.

As many of the results of this thesis involve maps of the form $L - \lambda N$, the next theorem by Petryshyn [42] gives conditions on N so that $L - \lambda N$ is A-proper.

Theorem 1.4.7 *Let $L \in \Phi_0(X, Y)$, Ω be an open bounded set in X with $\Omega \cap \operatorname{dom} L \neq \emptyset$, Γ_L as constructed above an admissible projection scheme for L and let $N : \overline{\Omega} \to Y$ be a bounded continuous map. Then each of the following conditions implies $T_\lambda = L - \lambda N$:*

$\overline{\Omega} \cap \operatorname{dom} L \to Y$ *is A-proper relative to* Γ_L *for each* $\lambda \in (0,1]$.

(i) Either N *or* $L^{-1} : \operatorname{im} L \subset Y \to X$ *is compact.*

(ii) N *is k-ball contractive with* $k \in [0, l(L))$ *where*

$$l(L) \equiv \sup \{ r \in \mathbb{R}^+ : r\beta(\Omega) \leq \beta(L(\Omega)) \text{ for each bounded } \Omega \subset \operatorname{dom} L \}$$

and $\|Q_n\| = 1$.

(iii) $N(L+C)^{-1} : H(\overline{\Omega} \cap \operatorname{dom} L) \to Y$ *is ball condensing and* $\|Q_n\| = 1$.

(iv) $Y = X$, $\Omega = \operatorname{dom} L = X$, $X_n = Y_n$, $Q_n \subset Q_{n+1}$, $\|Q_n\| = 1$, $\Gamma_L = \Gamma$, $L = I$ *and* N *c-dissipative for* $c \in (0,1)$ *(i.e.* $(Nx - Ny, j(x-y)) \leq c\|x-y\|^2$ *for some* $c \in (0,1)$ *and any normalised duality mapping* $j : X \to 2^{X^*}$*).*

Proof. We provide a proof to (i) and (ii) as we shall refer to them later whilst proofs to the last two may be found in Petryshyn [42].

(i) If N is compact and as L is A-proper, Theorem 1.3.6 implies $L - \lambda N$ is A-proper with respect to $\Gamma_L = \{Y_n, Q_n\}$. Now suppose L_1^{-1} is compact and $\lambda \in (0,1]$ is fixed. Let $\{x_n\}$ be a bounded sequence in $X_n = H^{-1}(Y_n)$ such that $Q_n(L - \lambda N)x_n = y_n \to y \in Y$. Since $Q_n Hx = (L + J^{-1}P)x$ for every $x \in X_n$ we have $y_n = Hx_n - \lambda Q_n N x_n - Q_n J^{-1} P x_n \to y \in Y$. The compactness of $J^{-1}P$ and completeness of the projection scheme Γ_L imply $Lx_n - \lambda Q_n N x_n \equiv \tilde{y}_n \to y$. Then applying $(I - Q)$ to the equation gives

$$Lx_n - \lambda (I-Q) Q_n N x_n \equiv (I-Q)\tilde{y}_n \to (I-Q)y \in Y_1.$$

Hence

$$z_n = L_1^{-1}(I-Q)\tilde{y}_n = (I-P)x_n - \lambda L_1^{-1}(I-Q)Q_n N x_n \to L_1^{-1}(I-Q)y = z$$

and $z \in X_1 \cap \operatorname{dom} L$. Since $\lambda L_1^{-1}(I-Q)$ and P are compact and the sequences $\{x_n\}$ and $\{Q_n N x_n\}$ are bounded we may assume $Px_n \to x_0 \in \ker L$ and $\lambda L_1^{-1}(I-Q)Q_n N x_n \to x_1 \in X_1$. Thus

$$z_n = x_n - Px_n - \lambda L_1^{-1}(I-Q)Q_n N x_n$$

or

$$x_n = z_n + Px_n + \lambda L_1^{-1}(I-Q)Q_nNx_n \to z + x_0 + x_1 \equiv x \in \overline{\Omega} \subset X.$$

By the continuity of N, $Nx_n \to Nx$ and by property of Q_n, $Q_nNx \to Nx$ so $Q_nNx_n \to Nx$ and

$$Lx_n = \tilde{y}_n + \lambda Q_nNx_n \to y + \lambda Nx_n$$

in Y. Now since L is closed, $x \in \operatorname{dom} L$ so $x \in \overline{\Omega} \cap \operatorname{dom} L$ and $Lx = y + \lambda Nx$. Hence $Lx - \lambda Nx = y$ and $L - \lambda N$ is A-proper.

(ii) Assume N is k-ball contractive with $k < l(L)$ and let $\{x_n\}$ be a bounded sequence in $X_n = H^{-1}(Y_n)$ such that $Q_nLx_n - \lambda Q_nNx_n \equiv y_n \to y \in Y$. As in (i), since $Q_nHx = Hx$ for every $x \in X_n$ and $J^{-1}P$ is compact, we may assume that $Lx_n - \lambda Q_nNx_n \equiv \tilde{y}_n \to y \in Y$. From the ball measure of noncompactness, β, we obtain the inequalities $\beta(\{Q_nNx_n\}) \leq \beta(\{Nx_n\}) \leq k\beta(\{x_n\})$. Writing the preceding identity as $Q_nLx_n = y_n + \lambda Q_nNx_n$ and noting the sequential compactness of y_n, these inequalities imply $\beta(\{Q_nLx_n\}) \leq \lambda k\beta(\{x_n\}) \leq k\beta(\{x_n\})$. Since $k < l(L)$, we have $\beta(\{x_n\}) = 0$ which implies $\{x_n\}$ is relatively compact so we may assume $x_n \to x \in \overline{\Omega} \subset X$. Thus $Lx_n = \tilde{y}_n + \lambda Q_nNx_n \to y + \lambda Nx$ in Y and since L is closed, $x \in \operatorname{dom} L$ so $x \in \overline{\Omega} \cap \operatorname{dom} L$. Hence $Lx - \lambda Nx = y$ and $L - \lambda N$ is A-proper with respect to Γ_L. Q.E.D.

Remark 1.4.8 *Similarly, if we have $\mu_0 k < l(L)$ for some $\mu_0 > 1$ then $L - \lambda N$ is A-proper for $0 < \lambda \leq \mu_0$.*

1.5 Topological degrees

As the principal results of this thesis concern fixed point indices for A-proper maps which we define in terms of the topological degree, we shall mention those relevant concepts of the theory required in the sequel. We assume some knowledge of the classical Brouwer degree for continuous maps in finite dimensional spaces and the Leray-Schauder degree for identity minus compact maps in infinite dimensional spaces. We shall use a modification of the degree theory in Banach spaces so that the degree is determined over closed convex sets and cones in particular.

Comprehensive accounts of degree theory may be found in Cronin [10], Deimling [13] and Lloyd [32].

A generalised topological degree theory for A-proper maps was developed by Browder and Petryshyn [4]. This degree forms the basis of our definition of a fixed point index in Chapter 2 so we present the definition and pertinent properties here for future reference. The particular version of the degree we state is from Petryshyn [47] for densely defined A-proper maps.

Definition 1.5.1 *(Petryshyn [47]) Let X, Y be Banach spaces, $\Omega \subset X$ open bounded such that $\Omega \cap \operatorname{dom} T = G \neq \emptyset$ and $T : G \subset X \to Y$ A-proper at y with respect to $\Gamma = \{X_n, Y_n, Q_n\}$. Write $\overline{G} = \overline{\Omega} \cap \operatorname{dom} T$, $\partial G = \partial\Omega \cap \operatorname{dom} T$ and assume $y \notin T(\partial G)$ and $G_n = G \cap X_n \neq \emptyset$, then the A-proper degree $\operatorname{Deg}(T, G, y)$ is defined to be the set*

$$\left\{k \in \mathbb{Z} \cup \{-\infty, \infty\} : \deg\left(Q_{n_m} T\,|_{\Omega_{n_m}}, G_{n_m}, Q_{n_m} y\right) \to k \text{ for some } n_m \to \infty\right\}$$

where $\deg(\cdot,\cdot,\cdot)$ is the finite dimensional Brouwer degree. That is, $\operatorname{Deg}(T, G, y)$ is the set of all limit points of $\{\deg(T_n, G_n, Q_n y)\}$ (including $\pm\infty$).

Remark 1.5.2 *The A-properness of T and the assumption $y \notin T(\partial G)$ imply there exists n_0 such that for all $n \geq n_0$, $Q_n y \notin Q_n T(\partial G_n)$ so that $\deg(Q_n T\,|_{G_n}, G_n, Q_n y)$ is defined for all $n \geq n_0$. Since $Q_n T$ may not converge uniformly to T on G, the $\operatorname{Deg}(T, G, y)$ is in general multivalued; for example $\operatorname{Deg}(-I, B_1(0), 0) = \{-1, 1\}$. Consequently, the usual properties of classical degree theory must be modified in the multivalued context, but the utility of the theory remains.*

Remark 1.5.3 *It is well-known that the definition of degree can be extended to deal with unbounded sets Ω provided the set $T^{-1}(0)$ is bounded, for example [13]. For details of how this extension is carried out we refer to §2.4.*

The following properties of the A-proper degree are from Petryshyn [47].

Theorem 1.5.4 *Assuming the notation and conditions of the preceding definition, then:*
P1. If $\operatorname{Deg}(T, G, y) \neq \{0\}$, *then there exists* $x \in G$ *such that* $Tx = y$.
P2. If $G \subset G_1 \cup G_2$, $\overline{G} = \overline{G}_1 \cup \overline{G}_2$ *with* G_1 *and* G_2 *open bounded sets such that* $G_1 \cap G_2 = \emptyset$ *and* $y \notin T(\partial G_1) \cup T(\partial G_2)$ *then* $\operatorname{Deg}(T, G, y) \subseteq \operatorname{Deg}(T, G_1, y) + \operatorname{Deg}(T, G_2, y)$ *with equality if either of the terms on the right is a singleton (defining* $\infty + (-\infty) = \mathbb{Z} \cup \{\pm\infty\}$*).*
P3. If $H : [0,1] \times \overline{G} \to Y$ *is an A-proper homotopy such that* $H(t, x) \neq y$ *for* $t \in [0,1]$ *and* $x \in \partial G$, *then* $\operatorname{Deg}(H(0,x), G, y) = \operatorname{Deg}(H(1,x), G, y)$.
P4. If G *is symmetric about* 0, $0 \in G$ *and* $T : \overline{G} \to Y$ *is A-proper and odd and* $0 \notin T(\partial G)$ *then* $\operatorname{Deg}(T, G, 0)$ *is odd, i.e.,* $2m \notin \operatorname{Deg}(T, G, 0)$ *for every* $m \in \mathbb{Z}$.

For a proof of these properties see Petryshyn [47]. We point out that, in general, P2 is not an equality. To clarify this further, we provide a complete (new) proof of a rather more precise statement. We will show that equality holds if one of the degrees is a finite singleton or if one of the degrees is $+\infty$ (resp. $-\infty$) and the other does not contain $-\infty$ (resp. $+\infty$).

Proof of equality in P2. Suppose $\operatorname{Deg}(T, G_1, y)$ is a finite singleton, $\{m\}$ say. Then $d_n^1 := \deg(T_n, G_n^1, y_n) \to m$ so that there exists n_1 such that $d_n^1 = m$ for every $n \geq n_1$. Then

$$\begin{aligned} \operatorname{Deg}(T, G, y) &= \text{limit points of } \{d_n^1 + d_n^2\} \\ &= m + \text{ limit points of } \{d_n^2\} \\ &= m + \operatorname{Deg}(T, G_2, y) \end{aligned}$$

so equality holds in this case. Note that this includes the possibility that ∞ or $-\infty$ are in $\operatorname{Deg}(T, G_2, y)$.

Now suppose $\operatorname{Deg}(T, G_1, y) = \{\infty\}$. We claim equality holds provided $\{-\infty\} \notin \operatorname{Deg}(T, G_2, y)$. Observe that the only limit point of $\{d_n^1\}$ is ∞, that is, $d_n^1 \to \infty$ as $n \to \infty$. Now suppose that $\{-\infty\} \notin \operatorname{Deg}(T, G_2, y)$. Then $\{d_n^2\}$ is bounded below, that is, there exists M such that $d_n^2 \geq -M$ for every n. We claim that $d_n^1 + d_n^2 \to \infty$ so that $\operatorname{Deg}(T, G, y) = \{\infty\}$. Indeed, given $M_1 > 0$ there exists n_1 such that $d_n^1 > M_1 + M$ for every $n \geq n_1$ and therefore $d_n^1 + d_n^2 > M_1$ for every $n \geq n_1$. This proves that

$$\infty = \operatorname{Deg}(T, G, y) = \operatorname{Deg}(T, G_1, y) + \operatorname{Deg}(T, G_2, y)$$

(since $\infty + \gamma = \infty$ for $\gamma \neq -\infty$). A similar argument, also resulting in equality, applies if $\operatorname{Deg}(T, G_1, y) = \{-\infty\}$ and $\{\infty\} \notin \operatorname{Deg}(T, G_2, y)$.

If $\{-\infty\} \in \operatorname{Deg}(T, G_2, y)$ we cannot expect equality in general. Q.E.D.

We conclude this section with a theorem by Petryshyn [42].

Theorem 1.5.5 *Let $L \in \Phi_0(X, Y)$ and let $\Omega \subset X$ be an open bounded set with $G = \Omega \cap \operatorname{dom} L \neq \emptyset$, let $F : X \to Y$ be a bounded linear map such that $L - F : \operatorname{dom} L \to Y$ is A-proper with respect to Γ_L and $\ker(L - F) = \{0\}$, then $\operatorname{Deg}(L - F, G, 0) = \{0\}$ if $0 \notin \overline{G}$ and $\operatorname{Deg}(L - F, G, 0) \subseteq \{\pm 1\}$ if $0 \in G$.*

Proof. Since $\ker(L - F) = \{0\}$, $L - F$ is injective (one to one) so that $Lx - Fx \neq 0$ for all $x \neq 0$ and $\operatorname{Deg}(L - F, G, 0)$ is well defined for any open bounded set $G \subset X$ with $0 \notin \partial G$. Now if $0 \notin \overline{G}$, then P1 from above and the assumption $\operatorname{Deg}(L - F, G, 0) \neq \{0\}$ imply the existence of $x \in G$ such that $Lx - Fx = 0$ and $x \neq 0$ which contradicts $\ker(L - F) = \{0\}$.

Suppose now $0 \in G$, then $0 \in G$ for all n and by the injectivity of $Q_n(L - F) : X_n \to Y_n$, it follows that $\deg(Q_n(L - F), X_n \cap G, 0) = 1$ or -1 for all n. Hence $\operatorname{Deg}(L - F, G, 0) \subseteq \{\pm 1\}$. Q.E.D.

Chapter 2

A FIXED POINT INDEX DEFINED IN TERMS OF THE A-PROPER DEGREE

2.1 Introduction

In this chapter we define a fixed point index based upon the generalised topological degree of Browder and Petryshyn [4]. In our definition we use a retraction $\rho : X \to K$ with the properties that $\|x - \rho x\| \leq 2\,\mathrm{dist}\,(x, K)$ and $\rho\,(X_n) \subset X_n$. The following corollary of Theorem 18.1, *cf.* Krasnosel'skii and Zabreiko [26], shows that a retraction satisfying the inequality always exists.

Corollary 2.1.1 *If K is a closed convex subset of a Banach space X then there is a continuous retraction $\rho : X \to K$ with the property that $\|x - \rho x\| \leq 2\,\mathrm{dist}\,(x, K)$.*

However, the condition $\rho\,(X_n) \subset X_n$ is more restrictive and precludes the use of some retractions and projection schemes commonly employed in applications. An example of a retraction and projection scheme where this condition fails follows.

Let $X = C[0,1]$ and $X_n \subset X$ be the finite dimensional subspace of all $x \in X$ which are linear in the n equidistant subintervals partitioning $[0,1]$. If $\rho : X \to K$ is a retraction onto $K = \{x \in X : x(t) \geq 0\}$ defined by $\rho x = |x(t)|$ then $\rho(X_n) \nsubseteq X_n$. This is clear when one considers the action of ρ on a line segment that crosses the x-axis. The negative part is reflected above the x-axis creating another subinterval beginning at the reflection point. Thus, for such x, ρ maps to a higher dimensional subspace X_m where $m > n$.

We shall use this retraction and projection scheme in an application to differential equations in Chapter 6 but using results proved by the less restrictive index developed in Chapter 3. To prove that a retraction and projection scheme does exist that satisfy our requirements and has applications, we provide the following example similar to one from De Figueiredo [12].

Let S be an open cube in $\mathbb{R}^n$, $X = L^p(S)$ and $S^{(k)} = \{S_1, ..., S_{2^{nk}}\}$, $k \in \mathbb{N}$, be a family of disjoint n-cubes covering S obtained by successively dividing each length of S by one half. *E.g.*, for $S = \{x \in \mathbb{R}^n : a_i < x_i < b_i,\ 1 \leq i \leq n\}$ then $S^{(1)}$ is the family of half open subcubes of the form $\{x \in \mathbb{R}^n : a_i \leq x_i < (a_i + b_i)/2 \text{ or } (a_i + b_i)/2 \leq x_i < b_i,\ 1 \leq i \leq n\}$. Let $X_k \subset X$ be the finite dimensional subspace generated by the characteristic functions $\chi_1, \ldots, \chi_{2^{nk}}$ on the sets of $S^{(k)}$ and define the projection $P_k : X \to X_k$ by

$$P_k x = \sum_{i=1}^{2^{nk}} \frac{1}{\mu(S_i)} \left[\int_{S_i} x(\tau)\, d\tau \right] \chi_i.$$

We prove that the projection P_k has norm 1 as follows:

$$\begin{aligned} \|P_k x\|_{L^p}^p &= \int_S \sum_{i=1}^{2^{nk}} \frac{1}{\mu(S_i)^p} \left| \int_{S_i} x(\tau)\, d\tau \right|^p \chi_i d\tau \\ &= \sum_{i=1}^{2^{nk}} \mu(S_i)^{1-p} \left| \int_{S_i} x(\tau)\, d\tau \right|^p. \end{aligned}$$

By Hölder's inequality we obtain

$$\|P_k x\|_{L^p}^p \leq \sum_{i=1}^{2^{nk}} \mu(S_i)^{1-p} \left[\int_{S_i} |x(\tau)|^p\, d\tau \right] \mu(S_i)^{\frac{p}{q}}$$

$$= \sum_{i=1}^{2^{nk}} \int_{S_i} |x(\tau)|^p \, d\tau.$$

It follows that

$$\|P_k x\|_{L^p}^p \leq \int_S |x(\tau)|^p \, d\tau = \|x\|_{L^p}^p$$

and hence $\|P_k\| = 1$.

To prove $P_k x \to x$ in L^p as $k \to \infty$, we shall first use following definition and theorem from Rudin [48], to prove pointwise almost everywhere convergence.

Definition 2.1.2 *Suppose $t \in \mathbb{R}^n$, then a sequence $\{S_i\}$ of Borel sets in $\mathbb{R}^n$ is said to shrink to t nicely if there is a number $\alpha > 0$ with the following property: Each S_i lies in an open ball $B_{r_i}(t)$ with centre at t and radius at $r_i > 0$ such that*

$$\mu(S_i) \geq \alpha \mu(B_{r_i}(t)), \; i = 1, 2, \ldots$$

and $r_i \to 0$ as $i \to \infty$ where μ is Lebesgue measure on $\mathbb{R}^n$.

Theorem 2.1.3 *Suppose $x \in L^1(\mathbb{R}^n)$. Define the Lebesgue set L_x of x to be the set of all $t_0 \in \mathbb{R}^n$ such that*

$$\lim_{i \to \infty} \frac{1}{\mu(S_i)} \int_{S_i} |x(t) - x(t_0)| \, dt = 0$$

for every sequence $\{S_i\}$ that shrinks to t_0 nicely. Then almost all $t_0 \in \mathbb{R}^n$ belong to L_x.

We now show that $P_k x(t) \to x(t)$ pointwise *a.e.* for each $x \in L^p$. We have

$$|P_k x(t) - x(t)| = \left| \sum_{i=1}^{2^{nk}} \frac{1}{\mu(S_i)} \int_{S_i} x(\tau) \, d\tau \chi_i - x(t) \right| \tag{2.1}$$

and for a fixed $t_0 \in S$, (2.1) reduces to

$$\left| \frac{1}{\mu(S_i)} \int_{S_i} (x(t) - x(t_0)) \, dt \right| \leq \frac{1}{\mu(S_i)} \int_{S_i} |x(t) - x(t_0)| \, dt$$

where $t_0 \in S_i$. By Theorem 2.1.3,

$$\frac{1}{\mu(S_i)} \int_{S_i} |x(t) - x(t_0)| \, dt < \epsilon \text{ if } \mu(S_i) < \delta$$

noting that $\{S_j\}$ is a sequence that shrinks nicely to t_0.

Since $\mu(S_j) \to 0$ as $j \to \infty$, we have for every $k \geq k_0(\epsilon)$, $|P_k x(t_0) - x(t_0)| < \epsilon$, and this holds for almost all $t_0 \in S$, that is, we have pointwise almost everywhere convergence.

For a function x and $r > 0$ we define the *truncation* of x at level r by

$$x^{(r)}(t) = \begin{cases} -r & \text{if } x(t) < -r, \\ x(t) & \text{if } -r \leq x(t) \leq r, \\ r & \text{if } x(t) > r. \end{cases}$$

Then $x^{(r)}(t) \to x(t)$ pointwise and because $|x^{(r)}(t)| \leq |x(t)|$, by the dominated convergence theorem $\|x^{(r)} - x\|_{L^p} \to 0$ as $r \to \infty$.

By the above, for every fixed $r > 0$, $P_k x^{(r)}(t) \to x^{(r)}(t)$ for almost every t and since $x^{(r)}$ is bounded, by the dominated convergence theorem, $\|P_k x^{(r)} - x^{(r)}\|_{L^p} \to 0$ as $k \to \infty$. For $\epsilon > 0$ first choose $r = r(\epsilon)$ so that $\|x^{(r)} - x\|_{L^p} < \epsilon$ and then choose k_0 (depending on ϵ and $r(\epsilon)$ hence only on ϵ) so that $\|P_k x^{(r)} - x^{(r)}\|_{L^p} < \epsilon$ for all $k \geq k_0$. Then for $k \geq k_0$

$$\|P_k x - x\|_{L^p} \leq \|P_k x - P_k x^{(r)}\|_{L^p} + \|P_k x^{(r)} - x^{(r)}\|_{L^p} + \|x^{(r)} - x\|_{L^p} < 3\epsilon$$

since $\|P_k\| = 1$. This proves that $P_k x \to x$ in $L^p(S)$ as $k \to \infty$.

This projection scheme can be extended to $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is any bounded measurable set by choosing an n-cube S containing Ω and defining $\tilde{x}(t) = x(t)$ for $t \in \Omega$ and $\tilde{x}(t) = 0$ elsewhere.

Let $K = \{x \in X : x(t) \geq 0 \text{ a.e.}\}$ and define $\rho : X \to K$ by $\rho x = x^+(t) = \max\{x(t), 0\}$. We show that ρ satisfies the conditions $\|x - \rho x\| \leq \operatorname{dist}(x, K)$ and $\rho(X_k) \subset X_k$. Now from the inequality $|x(t) - \rho x(t)| \leq |x(t) - y(t)|$ for every $y \in K$, we obtain

$$\|x - \rho x\|_{L^p} = \left(\int_S |x(t) - \rho x(t)|^p \, dt\right)^{\frac{1}{p}} \leq \left(\int_S |x(t) - y(t)|^p \, dt\right)^{\frac{1}{p}}$$

for every $y \in K$. Hence

$$\|x - \rho x\|_{L^p} \leq \inf\left\{\left(\int_S |x(t) - y(t)|^p \, dt\right)^{\frac{1}{p}}\right\} = \operatorname{dist}(x, K).$$

Finally we show $\rho(X_k) \subset X_k$. Let

$$x_k(t) = \sum_{i=1}^{2^{nk}} \frac{1}{\mu(S_i)} \left[\int_{S_i} x(t)\, dt \right] \chi_i(t),$$

then it is clear that $\rho x_k \in X_k$ as those terms of the summation which had negative average value then have average value 0, so that dimension is preserved by ρ.

In our development of the fixed point index, we shall treat the simplest case first, *viz.* maps $T : Y \to Y$ with $I - T$ A-proper. We then demonstrate that certain A-proper maps of the form $L - N$ can be converted to the form $I - T$ and construct a fixed point index for such maps.

When L is an unbounded linear operator, $H = L + J^{-1}P$ is no longer a homeomorphism and $H\Omega$ may be unbounded in Y for open bounded sets $\Omega \subset X$. To remedy this inconvenience we conclude the chapter by modifying the index so that the A-proper degree is determined over open bounded sets V in $H\Omega$ such that $(I - T)^{-1}(0) \subset V$.

A fixed point index for P-compact maps was first defined by Wong [55], using his version of the generalised degree developed in that paper. As the difference, $\lambda I - T$ where T is P-compact is A-proper for each $\lambda > 0$, we shall obtain and extend this result by letting $\Omega = Y$, though without equality in the additivity over domains property of the index.

An early investigation involving generalised degree theory on possibly unbounded sets was Browder and Nussbaum's paper [5]. They considered continuous maps T such that $I - T$ is locally compact and obtained results for strictly contractive maps. Fitzpatrick and Petryshyn [16] defined an index for A-proper maps of the form $I - T$ where T maps a closed convex set into itself. This index was later extended to include unbounded sets by Lafferriere in [28]. The derivation of this index is analogous to that of the generalised degree for A-proper maps but uses limits of the finite dimensional Brouwer index instead of the Brouwer degree. In this chapter, we make use of the already established generalised degree for A-proper maps in defining a fixed point index but shall return to the idea of constructing an index in terms of limits of finite dimensional Brouwer degrees in Chapter 3.

2.2 Definition and properties of the index

Let $\Omega \subset K$ be open bounded and K a cone in a Banach space X. Unless otherwise stated, all topological notions for subsets of K refer to the relative topology of K, so for example, $\partial\Omega$ means $\partial_K\Omega$ the boundary of Ω relative to K.

Assume $I - T$ is A-proper at 0 relative to a projection scheme Γ, $T : K \to K$ and $Tx \neq x$ for $x \in \partial\Omega$. Let $\rho : X \to K$ be a retraction such that $\|x - \rho x\| \leq 2\,\mathrm{dist}\,(x, K)$ and $\rho\,(X_n) \subset X_n$.

Remark 2.2.1 *There is nothing special about the number 2, we could use any constant larger than 1 but 2 is a simple choice that seems to be always satisfied in the applications.*

Here and henceforward it will always be assumed, either explicitly or tacitly, that $Q_n\,(K) \subseteq K$ where Q_n is the projection used in the projection scheme Γ or Γ_L.

Definition 2.2.2 *We define* $\mathrm{ind}_K\,(T, \Omega) = \mathrm{Deg}\,(I - T\rho, \rho^{-1}\,(\Omega)\,, 0)$ *where the right hand side is the degree for A-proper at 0 maps from Definition 1.5.1 and Remark 1.5.3.*

We shall show that the A-properness of $I - T$ at 0 and the conditions $\|x - \rho x\| \leq 2\,\mathrm{dist}\,(x, K)\,,\ \rho\,(X_n) \subset X_n$ and $T : K \to K$ imply $I - T\rho$ is A-proper at 0. We then proceed to show that the index does not depend on the retraction chosen in the definition. Finally, we show that the index has the following properties of the classical Brouwer fixed point index which we prove in Theorem 2.2.7.

Proposition 2.2.3 *1. (Existence) If* $\mathrm{ind}_K\,(T, \Omega) \neq \{0\}$, *then T has a fixed point in Ω.*
2. (Normalisation) If $x_0 \in \Omega$, then $\mathrm{ind}_K\,(\hat{x}_0, \Omega) = \{1\}$ *where $\hat{x}_0\,(x) = x_0$ for all $x \in \Omega$.*

3. (Additivity) If $\Omega = \Omega_1 \cup \Omega_2$, Ω_1 *and* Ω_2 *open bounded with* $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $Tx \neq x$ *for* $x \in \partial\Omega_1 \cup \partial\Omega_2$, *then* $\text{ind}_K(T,\Omega) \subseteq \text{ind}_K(T,\Omega_1) + \text{ind}_K(T,\Omega_2)$.

4.(Homotopy) If $H(t,x) : [0,1] \times \overline{\Omega} \to K$ *is such that* $I - H(t,x)$ *is an A-proper homotopy at* 0 *and* $H(t,x) \neq x$ *for* $x \in \partial\Omega$, $t \in [0,1]$, *then* $\text{ind}_K(H(0,x),\Omega) = \text{ind}_K(H(1,x),\Omega)$.

We prove first that if $I - T$ is A-proper at a point $y \in K$ relative to a projection scheme Γ and the previous assumptions hold, then $I - T\rho$ is also A-proper at y relative to Γ.

Theorem 2.2.4 *Let* K *be a cone in* X, *let* $\rho : X \to K$ *be a retraction such that* $\|x - \rho x\| \leq 2\,\text{dist}(x,K)$ *and* $\rho(X_n) \subset X_n$. *Suppose* $T : K \to K$ *is such that* $I - T$ *is A-proper at a point* $y \in K$. *Then* $I - T\rho$ *is A-proper at* y.

Proof. Let $\{x_n \in X_n\}$ be a bounded sequence in X such that $(I - Q_nT\rho)x_n = y_n \to y \in K$. Then

$$\|x_n - \rho x_n\| = \|y_n + Q_nT\rho x_n - \rho x_n\| \leq 2\,\text{dist}(x_n,K).$$

Since $x_n - Q_nT\rho x_n \to y \in K$ and $Q_nT\rho x_n \in K$, we have

$$\text{dist}(x_n,K) \leq \|x_n - (Q_nT\rho x_n + y)\| = \|y_n - y\| = \epsilon_n.$$

Hence $\|x_n - \rho x_n\| \leq 2\epsilon_n \to 0$. Then

$$\rho x_n - Q_nT\rho x_n = y_n + (\rho x_n - x_n) \to y.$$

By the A-properness of $I - T$ applied to ρx_n, there exists $\rho x_{n_j} \to x_0 \in K$ such that $(I - T)x_0 = y$ and since $\|x_n - \rho x_n\| \to 0$, there exists $x_{n_j} \to x_0$. Therefore $I - T\rho$ is A-proper at y. Q.E.D.

The next theorem shows that the index is independent of the retraction chosen in the definition provided it satisfies $\|x - \rho x\| \leq 2\,\text{dist}(x,K)$ and $\rho(X_n) \subset X_n$. We point out that the proof requires convexity for the domain of T, consequently, in this chapter we define T on all of K. This limitation will be removed in defining the fixed point indices of Chapters 3 and 4.

Theorem 2.2.5 *Suppose $\Omega \subset K$ is open and bounded and ρ and τ are retractions of X onto K satisfying the conditions above. If $T : K \to K$ is a map such that $I - T$ is A-proper at 0, then* $\operatorname{Deg}(I - T\rho, \rho^{-1}(\Omega), 0) = \operatorname{Deg}(I - T\tau, \tau^{-1}(\Omega), 0)$, *i.e., the generalised degree and hence the index is independent of the retraction chosen.*

Proof. Let $\Omega \subset K$ be open, bounded and let ρ and τ be retractions of X onto K. Note that the fixed points of $T\rho x$ on $\rho^{-1}(\Omega)$ and $T\tau x$ on $\tau^{-1}(\Omega)$ are contained in $G = \rho^{-1}(\Omega) \cap \tau^{-1}(\Omega)$. By the additivity of the degree, it suffices to prove

$$\operatorname{Deg}(I - T\rho, G, 0) = \operatorname{Deg}(I - T\tau, G, 0).$$

Define $H(t,x) = x - T(t\rho x + (1-t)\tau x)$ for $x \in G$ and $t \in [0,1]$. Then $H(t,x)$ is A-proper relative to Γ, which we will prove in the following lemma. Assuming this for the moment, we continue the proof.

Now let $x \in \overline{G}$ and assume $H(t,x) = 0$ for some $t \in [0,1]$, then $T(t\rho x + (1-t)\tau x) = x$. Since ρx and τx are in K and $T : K \to K$, we have $x \in K$. This implies that $\rho x = x$ and $\tau x = x$ so that $T(t\rho x + (1-t)\tau x)$ reduces to $T(tx + (1-t)x) = Tx = x$. Since $Tx \neq x$ for $x \in \partial\Omega$ we have $x \in \Omega$ and noting that $\rho^{-1}x = x$ and $\tau^{-1}x = x$ we see that $x \in G$. By the invariance under homotopy property of the degree, we have

$$\operatorname{Deg}(I - T\rho, G, 0) = \operatorname{Deg}(I - T\tau, G, 0). \text{ Q.E.D.}$$

It remains to prove the A-properness of $H(t,x)$.

Lemma 2.2.6 *Assume the conditions of the theorem hold, then $H(t,x) : [0,1] \times K \to K$ where $H(t,x) = x - T(t\rho x + (1-t)\tau x)$ is A-proper at $y \in K$.*

Proof. Let $\{x_n\}$ be a bounded sequence in X, $t_n \subset [0,1]$, such that

$$x_n - Q_n T\left(t_n \rho x_n + (1-t_n)\,\tau x_n\right) = y_n \to y \in K.$$

Write $w_n = t_n \rho x_n + (1-t_n)\,\tau x_n$. Since ρ and τ map X to K and K is convex, we see that $w_n \in K$. As $T : K \to K$, and $Q_n(K) \subset K$, $Q_n T w_n + y \in K$. Now

$$\|x_n - (Q_n T w_n + y)\| = \|y_n - y\| = \epsilon_n \to 0,$$

so dist $(x_n, K) \leq \epsilon_n$ and we have

$$\begin{aligned} \|x_n - w_n\| &= \|x_n - [t_n \rho x_n + (1-t_n)\,\tau x_n]\| \\ &\leq t_n \|x_n - \rho x_n\| + (1-t_n)\,\|x_n - \tau x_n\| \\ &\leq 2\,\mathrm{dist}(x_n, K) \leq 2\epsilon_n. \end{aligned}$$

Then

$$w_n - Q_n T w_n = y_n + (w_n - x_n) \to y \in K.$$

By the A-properness of $I - T$ applied to w_n, there exists $w_{n_j} \to w \in K$ such that $(I-T)\,w = y$, and since $\|x_n - w_n\| \to 0$, there exists $x_{n_j} \to w$ which proves $H(t,x)$ is A-proper at y. Q.E.D.

We now formally present the properties of the fixed point index in the following theorem along with their proofs.

Theorem 2.2.7 *Let $\Omega \subset K$ be open bounded and $T : K \to K$ be a map such that $I - T$ is A-proper at 0 and assume $Tx \neq x$ on $\partial\Omega$. Then the fixed point index of Definition 2.2.2 has the following properties.*

P1. (Existence) If $\mathrm{ind}_K(T, \Omega) \neq \{0\}$*, then T has a fixed point in Ω.*

P2. (Normalisation) If $x_0 \in \Omega$, then $\mathrm{ind}_K(\hat{x}_0, \Omega) = \{1\}$ *where $\hat{x}_0(x) = x_0$ for $x \in \Omega$.*

P3. (Additivity) If $\Omega = \Omega_1 \cup \Omega_2$, Ω_1 and Ω_2 are open bounded, $\Omega_1 \cap \Omega_2 = \emptyset$ and $Tx \neq x$

for $x \in \partial\Omega_1 \cup \partial\Omega_2$, *then* $\operatorname{ind}_K(T,\Omega) \subseteq \operatorname{ind}_K(T,\Omega_1) + \operatorname{ind}_K(T,\Omega_2)$ *with equality if either of the indices on the right is a singleton.*

P4. (Homotopy) If $H(t,x) : [0,1] \times \Omega \to K$ *is such that* $I - H(t,x)$ *is an A-proper homotopy at 0 and* $H(t,x) \neq x$ *for* $x \in \partial\Omega$, $t \in [0,1]$, *then* $\operatorname{ind}_K(H(0,x),\Omega) = \operatorname{ind}_K(H(1,x),\Omega)$.

Proof. P1. By definition, $\operatorname{ind}_K(T,\Omega) \neq \{0\}$ implies $\operatorname{Deg}(I - T\rho, \rho^{-1}(\Omega), 0) \neq \{0\}$. Then if the degree is non-zero, there exists $x \in \rho^{-1}(\Omega)$ such that $T\rho x = x$ with $\rho x \in \Omega$. Since $\rho : X \to K$ and $T : K \to K$ we have $\rho x = x$. By hypothesis, $Tx \neq x$ on $\partial\Omega$, consequently, $Tx = x$ for $x \in \Omega$.

P2. Since $Q_n x_0 \in \Omega \cap X_n$ for every $n \geq n_0$, the Brouwer degree $\deg(I, \Omega \cap X_n, Q_n x_0) = 1$. The equation $Ix_n = Q_n x_0$ however is equivalent to $Ix_n - Q_n x_0 = 0$ or $(I - Q_n \hat{x}_0)(x) = 0$ where $\hat{x}_0$ is the constant mapping $\hat{x}_0(x) = x_0$. Hence

$$\deg(I, \Omega \cap X_n, Q_n x_0) = \deg(I - Q_n \hat{x}_0, \Omega \cap X_n, 0) = 1$$

for every $n \geq n_0$ which implies $\operatorname{Deg}(I - \hat{x}_0, \Omega, 0) = \{1\}$. Now as $x_0 \in \Omega \subset K$ and ρ is the identity on K, we have

$$\operatorname{Deg}(I - \hat{x}_0, \Omega, 0) = \operatorname{Deg}\left(I - \hat{x}_0\rho, \rho^{-1}(\Omega), 0\right) = \{1\}.$$

P3. It suffices to show

$$\operatorname{Deg}\left(I - T\rho, \rho^{-1}(\Omega), 0\right) \subseteq \operatorname{Deg}\left(I - T\rho, \rho^{-1}(\Omega_1), 0\right) + \operatorname{Deg}\left(I - T\rho, \rho^{-1}(\Omega_2), 0\right).$$

This follows immediately from the additivity over domains property of the A-proper degree. Also, equality is obtained if either of the degrees, and consequently the indices, on the right is a singleton.

P4. By definition of the index, it suffices to prove that

$$\operatorname{Deg}\left(I - H(0,\rho x), \rho^{-1}(\Omega), 0\right) = \operatorname{Deg}\left(I - H(1,\rho x), \rho^{-1}(\Omega), 0\right).$$

We show that if $H(t,\rho x) = x$ for some $x \in \overline{\rho^{-1}(\Omega)}$, $t \in [0,1]$, then $H(t,x) = x$ and $x \notin \partial\rho^{-1}(\Omega)$. So assume $H(t,\rho x) = x$ for some $x \in \overline{\rho^{-1}(\Omega)}$, $t \in [0,1]$. Since $\rho : X \to K$ we have $\rho x \in \overline{\Omega}$ and since $H : K \to K$, $x \in K$. Now ρ is the identity on K so $\rho x = x$ and $H(t,\rho x) = H(t,x) = x$. But by assumption, $H(t,x) \neq x$ for $x \in \partial\Omega$, thus $x \in \Omega$, so $\rho x = x \in \Omega$ and $x \in \rho^{-1}(\Omega)$. It follows from the homotopy property of the A-proper degree that the two degrees are equal. To prove that $I - H(t,\rho x)$ is A-proper at a point $y \in K$; let $\{x_n\}$ be a bounded sequence in X and $\{t_n\}$ a sequence in $[0,1]$ such that $x_n - Q_n H(t_n, \rho x_n) = y_n \to y \in K$. Then

$$\|x_n - (Q_n H(t_n, \rho x_n) + y)\| = \|y_n - y\| = \epsilon_n \to 0.$$

We know that $\|x_n - \rho x_n\| \leq 2\,\mathrm{dist}\,(x_n, K) \leq 2\epsilon_n$ and

$$\rho x_n - Q_n H(t_n, \rho x_n) = y_n + (\rho x_n - x_n) \to y \in K.$$

By the A-properness of $I - H(t,x)$, there exist subsequences $\rho x_{n_j} \to x \in K$ and $t_{n_j} \to t \in [0,1]$ with $x - H(t,x) = y$. Since $\|x_n - \rho x_n\| \leq 2\epsilon_n$ there exists a subsequence x_{n_j} of x_n with $x_{n_j} \to x$. Therefore, $I - H(t,\rho x)$ is A-proper at y. Q.E.D.

2.3 The index extended to maps of the form L-N

We now consider A-proper maps of the form $L - N$ where $L : \mathrm{dom}\, L \subset X \to Y$ is Fredholm of index zero and N is continuous and nonlinear. We are interested in solutions to the equation $Lx - Nx = w$ that lie in a cone K in X. Using the operators and subspaces of X and Y constructed previously for Fredholm operators of index zero, we rewrite the equation $Lx - Nx = w$ as follows.

$$\begin{aligned} Lx + J^{-1}Px - \left(N + J^{-1}P\right)x &= w \\ Hx - \left(N + J^{-1}P\right)x &= w \end{aligned}$$

$$\begin{aligned} y - \left(N + J^{-1}P\right) H^{-1}y &= w \\ (I - T)\, y &= w \end{aligned}$$

We will utilize the operator $T : Y \to Y$, $Ty = \left(N + J^{-1}P\right) H^{-1}y$ and the corresponding cone $K_1 = H\,(K \cap \operatorname{dom} L)$ in Y. Let $\rho_1 : Y \to K_1$ be a retraction such that $\|y - \rho_1 y\| \leq 2 \operatorname{dist}(y, K_1)$ and $\rho_1 (Y_n) \subset Y_n$.

We verify first that K_1 is indeed a cone in Y. To this end, let y_1 and y_2 be elements of K_1 and $\alpha, \beta \in \mathbb{R}^+$. We show that $\alpha y_1 + \beta y_2 \in K_1$ and that K_1 is closed in Y. Since $y_1 \in K_1$ there exists $x_1 \in K$ with $Hx_1 = y_1$ and similarly there exists $x_2 \in K$ with $Hx_2 = y_2$. Then $\alpha y_1 + \beta y_2 = \alpha H x_1 + \beta H x_2 = H\,(\alpha x_1 + \beta x_2)$. Now since K is a cone, $\alpha x_1 + \beta x_2 \in K$ and hence $\alpha y_1 + \beta y_2 \in K_1$. Finally, we prove K_1 is closed in the following proposition.

Proposition 2.3.1 $K_1 = H\,(K \cap \operatorname{dom} L)$ *is closed in* Y.

Proof. Suppose $y_n \to y \in Y$ where $\{y_n\} \subset H\,(K \cap \operatorname{dom} L) = K_1$. Now $y_n = Hx_n$ for $x_n \in K \cap \operatorname{dom} L$ so that $Hx_n \to y$ and since H maps onto Y, $y = Hx$ for some $x \in \operatorname{dom} L$. Then $H^{-1}(Hx_n) \to H^{-1}y$ and hence $x_n \to x \in \operatorname{dom} L$. Since $x_n \in K$ and K is closed, $x \in K$. Thus $x \in K \cap \operatorname{dom} L$ and we have $Hx = y \in H\,(K \cap \operatorname{dom} L) = K_1$ which proves K_1 is closed. Q.E.D.

Next we show that the A-properness of $L - N$ at a point $w \in Y$ relative to Γ_L implies $I - T$ is A-proper at w relative to Γ. It will then follow that $I - T\rho_1$ is A-proper at w provided certain operators map cones to cones.

Lemma 2.3.2 *Let L and N be as mentioned and assume $L - N$ is A-proper at $w \in Y$ relative to Γ_L. Then $I - T$ is A-proper at w relative to Γ.*

Proof. Let $\{y_n\} \in Y_n$ be a bounded sequence in Y such that

$$y_n - Q_n T y_n = w_n \to w \in Y$$

where $y_n = Hx_n$, $x_n \in H^{-1}(Y_n) = X_n$. Then

$$Hx_n - Q_n\left(N + J^{-1}P\right)x_n = w_n \to w$$

and $Q_nHx_n = Hx_n$ so

$$Q_n(Lx_n + J^{-1}Px_n) - Q_nNx_n - Q_nJ^{-1}Px_n \to w,$$

that is

$$Q_nLx_n - Q_nNx_n \to w.$$

By the A-properness of $L - N$, there exists a subsequence $x_{n_j} \to x$ with $Lx - Nx = w$. Now $x_{n_j} \to x$ implies $Nx_{n_j} \to Nx$ and $Q_nNx_{n_j} \to Nx$ because $\|Q_n\| \leq M$ and $Q_ny \to y$. Then

$$Hx_{n_j} \to w + Nx + J^{-1}Px$$

and hence $y_{n_j} \to y$ with $y - Ty = w$ which proves $I - T$ is A-proper at w. Q.E.D.

Before proving $I - T\rho_1$ is A-proper at w relative to Γ, we introduce a lemma and a proposition which we shall require.

Lemma 2.3.3 *$Nx + J^{-1}Px = H\tilde{x}$ where $\tilde{x} = (P + JQN)x + L_1^{-1}(I-Q)Nx = \tilde{x}_0 + \tilde{x}_1$ and $\tilde{x}$ is uniquely determined.*

Proof. Suppose

$$\left(P + JQN + L_1^{-1}(I-Q)N\right)(x) = \tilde{x},$$

for $x, \tilde{x} \in X$, $\tilde{x} = \tilde{x}_0 + \tilde{x}_1$ where $\tilde{x}_0 \in \ker L$ and $\tilde{x}_1 \in X_1$. Then

$$Px + JQNx = \tilde{x}_0 = P\tilde{x}$$

and

$$L_1^{-1}(I-Q)Nx = \tilde{x}_1 = (I-P)\tilde{x}.$$

Now

$$\begin{aligned} L\tilde{x} &= L(\tilde{x}_1 + \tilde{x}_0) = L_1\tilde{x}_1 \\ &= L_1 L_1^{-1}(I - Q) Nx = (I - Q) Nx \end{aligned}$$

and

$$J^{-1}P\tilde{x} = J^{-1}(Px + JQNx) = J^{-1}Px + QNx$$

so

$$\begin{aligned} L\tilde{x} + J^{-1}P\tilde{x} &= (I - Q) Nx + J^{-1}Px + QNx \\ &= Nx + J^{-1}Px \end{aligned}$$

or

$$H\tilde{x} = \left(N + J^{-1}P\right) x.$$

Now $\tilde{x}$ is uniquely determined since H is injective. Q.E.D.

Proposition 2.3.4 *The following three assertions are equivalent.*
(i) $P + JQN + L_1^{-1}(I - Q) N$ *maps* $K \cap \operatorname{dom} L$ *to* $K \cap \operatorname{dom} L$.
(ii) $N + J^{-1}P$ *maps* $K \cap \operatorname{dom} L$ *into* K_1.
(iii) T *maps* K_1 *to* K_1.

Proof. We have $T = (N + J^{-1}P) H^{-1}$ maps K_1 to K_1 iff $N + J^{-1}P$ maps $K \cap \operatorname{dom} L$ into K_1 (so (ii) $\Leftrightarrow$ (iii)). By the preceding lemma, $N + J^{-1}P$ maps $K \cap \operatorname{dom} L$ to K_1 iff $H\tilde{x} \in K_1$ for every $\tilde{x} \in K \cap \operatorname{dom} L$ where $\tilde{x} = \left(P + JQN + L_1^{-1}(I - Q) N\right) x$ so that $H\tilde{x} \in K_1$ iff $P + JQN + L_1^{-1}(I - Q) N$ maps $K \cap \operatorname{dom} L$ to $K \cap \operatorname{dom} L$ (hence (ii) $\Leftrightarrow$ (i)). Q.E.D.

Lemma 2.3.5 $I - T\rho_1$ *is A-proper at* $w \in K_1$ *relative to* Γ *if* $P + JQN + L_1^{-1}(I - Q) N$ *maps* K *to* K *and* $\rho_1(Y_n) \subset Y_n$.

Proof. This is an immediate consequence of Lemma 2.3.2 and Theorem 2.2.4. Q.E.D.

We can now extend the definition of the index to A-proper maps of the form $L - N$ acting on cones. We begin by assuming L to be bounded. The consequence of this is that H becomes a homeomorphism and a very simple correspondence between Ω in X and $U = H\Omega$ in Y results, most importantly, H maps open bounded sets in X to open bounded sets in Y. The case where L is unbounded requires some modification as the A-proper degree is not defined on unbounded sets. We will consider this case later in Section 2.4.

Definition 2.3.6 *Let K be a cone in a Banach space X and $\Omega \subset K$ an open (relative to K) bounded set such that $\Omega \neq \emptyset$. Let $L : \operatorname{dom} L = X \to Y$ be a bounded Fredholm operator of index zero and $N : \overline{\Omega} \to Y$ be continuous and nonlinear such that $L - N$ is A-proper at 0 relative to Γ_L. Assume $Lx \neq Nx$ for $x \in \partial_K \Omega$, $P + JQN + L_1^{-1}(I - Q)N$ maps K to K and write $U = H\Omega$. We define $\operatorname{ind}_K([L, N], \Omega) = \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(U), 0\right)$ where the degree is that for A-proper mappings defined by Browder and Petryshyn [4], $\rho_1 : Y \to K_1$ is a retraction satisfying $\|y - \rho_1 y\| \leq 2\operatorname{dist}(y, K_1)$ and $\rho_1(Y_n) \subset Y_n$.*

Before giving the properties of this index we show that the condition $Lx \neq Nx$ on $\partial_K\Omega$ implies $(I - T\rho_1)y \neq 0$ on $\partial\rho_1^{-1}(U)$ so that the index is well defined. With this objective, assume $Lx \neq Nx$ for $x \in \partial_K\Omega$ and let $y \in \partial\rho_1^{-1}(U)$. Then if $y = T\rho_1 y$ we have $y \in K_1$ since $\rho_1 : Y \to K_1$ and $T : K_1 \to K_1$ so that $\rho_1 y = y$ and $Ty = y$ on ∂U. By construction, this is equivalent to $Lx = Nx$ on $\partial_K\Omega$, a contradiction.

Theorem 2.3.7 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $\operatorname{ind}_K([L, N], \Omega) \neq \{0\}$, then $Lx = Nx$ has a solution in Ω.

P2. If $x_0 \in \Omega$, then $\operatorname{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \{1\}$ where $y_0 = Hx_0$ and $\hat{y}_0(x) = y_0$ for every $x \in \overline{\Omega}$.

P3. If $\Omega = \Omega_1 \cup \Omega_2$, Ω_1 and Ω_2 are open and bounded in K with $\Omega_1 \cap \Omega_2 = \emptyset$ and $Lx \neq Nx$ on $\partial\Omega_1 \cup \partial\Omega_2$ then $\operatorname{ind}_K([L,N],\Omega) \subseteq \operatorname{ind}_K([L,N],\Omega_1) + \operatorname{ind}_K([L,N],\Omega_2)$ with equality if either of the indices on the right is a singleton.

P4. If $L - N(\lambda, x)$ is A-proper for $\lambda \in [0,1]$, $(N(\lambda,x) + J^{-1}P)H^{-1} : K_1 \to K_1$ and $0 \notin (L - N(\lambda,x))(\operatorname{dom} L \cap \partial\Omega)$ then $\operatorname{ind}_K([L, N(\lambda,x)],\Omega)$ is independent of $\lambda \in [0,1]$.

Proof. P1. By definition, $\operatorname{ind}_K([L,N],\Omega) \neq \{0\}$ implies $\operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(U), 0\right) \neq \{0\}$. Then there exists $y \in \rho_1^{-1}(U)$ such that $T\rho_1 y = y$. Since $\rho_1 : Y \to K_1$ and $T : K_1 \to K_1$ we have $y \in K_1$ and so $\rho_1 y = y$ so $Ty = y$ for some $y \in U$ and there exists $x = H^{-1}y \in H^{-1}U = \Omega$ satisfying $Lx = Nx$.

P2. We have

$$\operatorname{ind}_K\left(\left[L, -J^{-1}P + \hat{y}_0\right],\Omega\right) = \operatorname{Deg}\left(I - \hat{y}_0\rho_1, \rho_1^{-1}(U), 0\right) = \{1\}$$

as in the proof to P2 of Theorem 2.2.7.

P3. It suffices to show

$$\operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(U), 0\right) \subseteq \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(U_1), 0\right) + \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(U_2), 0\right)$$

where $U_1 = H\Omega_1$ and $U_2 = H\Omega_2$. Since $H\Omega = H\Omega_1 \cup H\Omega_2$ and $H\Omega_1 \cap H\Omega_2 = \emptyset$, the additivity over domains property of the A-proper degree implies the desired result. The proof of equality is analogous to the proof of P3, Theorem 2.2.7.

P4. We first note that the A-properness of $L - N(\lambda,x)$ implies $I - T_\lambda$ is A-proper where $T_\lambda = (N(\lambda,x) + J^{-1}P)H^{-1}$. This follows from Lemmas 2.3.2 and 2.3.5. We show that if $T_\lambda\rho_1 y = y$ for some $y \in \rho_1^{-1}(U)$, $\lambda \in [0,1]$ then $T_\lambda y = y$ and $y \notin \partial U$. Consequently, by the invariance under homotopy property of the A-proper degree we'll have

$$\operatorname{Deg}\left(I - T_0\rho_1, \rho_1^{-1}(U), 0\right) = \operatorname{Deg}\left(I - T_1\rho_1, \rho_1^{-1}(U), 0\right).$$

So assume $T_\lambda\rho_1 y = y$ for some $y \in \rho_1^{-1}(U)$ and $\lambda \in [0,1]$. Since $\rho_1 : Y \to K_1$, $\rho_1 y \in U \subset K_1$ so $\rho_1 y = y$ and as $T_\lambda : K_1 \to K_1$, $y \in K_1$. Whence $T_\lambda\rho_1 y = T_\lambda y = y$ or

$Hx = N(\lambda, x) + J^{-1}Px$ which implies $Lx = N(\lambda, x)$. By hypothesis, $Lx \neq N(\lambda, x)$ on $\partial\Omega$ so that $T_\lambda y \neq y$ on ∂U. Hence $\text{ind}_K([L, N(\lambda, x)], \Omega)$ is invariant for $\lambda \in [0,1]$. Q.E.D.

2.4 The index defined on unbounded sets

We now consider the case where L is unbounded and N is bounded. Since H is no longer a homeomorphism, $H\Omega = U$ may be unbounded for open bounded sets Ω. To remedy this inconvenience, we modify the index so that the A-proper degree is determined over open bounded sets V in U such that $(I-T)^{-1}(0) \subset V$. That such V exist is a consequence of the boundedness of N as we demonstrate in the following proposition.

Proposition 2.4.1 *If $Lx = Nx$ for $x \in \Omega$, $\Omega \subset K$ open and bounded such that $\Omega \cap \text{dom}\, L \neq \emptyset$ and N bounded, then $(I-T)^{-1}(0)$ is bounded in Y.*

Proof. We observe that $y = Ty$, $y \in U$ gives $\|Hx\| = \|(N + J^{-1}P)x\| \leq \|Nx\| + \|J^{-1}Px\| \leq M$ where $x = H^{-1}y \in \Omega$. Q.E.D.

Definition 2.4.2 *Let K be a cone in a Banach space X and $\Omega \subset K$ be an open (relative to K) bounded set such that $\Omega \cap \text{dom}\, L \neq \emptyset$. Let $L : \text{dom}\, L \subset X \to Y$ be an unbounded Fredholm operator of index zero and $N : \overline{\Omega} \cap \text{dom}\, L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0 and $Lx \neq Nx$ on $\partial_K\Omega \cap \text{dom}\, L$. Assume $P + JQN + L_1^{-1}(I-Q)N$ maps K to K. Then we define* $\text{ind}_K([L, N], \Omega) = \text{ind}_{K_1}(T, V) = \text{Deg}\left(I - T\rho_1, \rho_1^{-1}(V), 0\right)$ *where $V \subset H(\text{dom}\, L \cap \Omega)$ is any bounded set open relative to $K_1 = H(\text{dom}\, L \cap K)$ with $(I-T)^{-1}(0) \subset V$ and $\rho_1 : Y \to K_1$ is a retraction satisfying* $\|y - \rho_1 y\| \leq 2\,\text{dist}(y, K_1)$ *and $\rho_1(Y_n) \subset Y_n$.*

In the following theorem we show that the index is well defined and that it is independent of the choice of V.

Theorem 2.4.3 *The index in the preceding definition is well defined and is independent of the choice of V, where V is any open bounded set such that $(I-T)^{-1}(0) \subset V$.*

Proof. Let $y \in \partial\rho_1^{-1}(V)$ where $(I-T)^{-1}(0) \subset V$ and assume $T\rho_1 y = y$. Then $y \in K_1$ since $\rho_1 : Y \to K_1$ and $T : K_1 \to K_1$. Now for $y \in K_1$, $\rho_1 y = y$ so $T\rho_1 y = Ty = y$ and since ρ_1 is the identity on K_1, $\rho_1^{-1} y = y$ so $y \in \partial V$. But this gives $Ty = y$ on ∂V which is a contradiction. Hence the degree and consequently the index is well defined.

Now we prove the index is independent of the choice of V. Suppose $(I-T)^{-1}(0) \subset V_1$ and $(I-T)^{-1}(0) \subset V_2$ where V_1 and V_2 are open bounded sets in $U = H\Omega$. Then $(I-T)^{-1}(0) \subset V_1 \cap V_2 = W$ and W is an open bounded set in U. By the additivity and excision properties of the A-proper degree,

$$\begin{aligned}
\operatorname{ind}_{K_1}(T, V_1) &= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_1), 0\right) \\
&= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(W), 0\right) + \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_1 \setminus W), 0\right) \\
&= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(W), 0\right).
\end{aligned}$$

Note that equality holds as $\operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_1 \setminus W), 0\right) = \{0\}$ is a singleton.

Similarly,

$$\begin{aligned}
\operatorname{ind}_{K_1}(T, V_2) &= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_2), 0\right) \\
&= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(W), 0\right) + \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_2 \setminus W), 0\right) \\
&= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(W), 0\right).
\end{aligned}$$

Thus $\operatorname{ind}_{K_1}(T, V_1) = \operatorname{ind}_{K_1}(T, V_2)$ which proves the index is independent of the choice of V. Q.E.D.

The usual properties of the classical fixed point index remain valid for this index defined on unbounded sets and are provided in the next theorem.

Theorem 2.4.4 *Assume the conditions and notation of Definition 2.4.2. Then the index thus defined has the following properties.*

P1. If $\operatorname{ind}_K([L,N],\Omega) = \operatorname{ind}_{K_1}(T,V) \neq \{0\}$, *then there exists* $x \in \Omega \supset H^{-1}(V)$ *such that* $Lx = Nx$.

P2. If $x_0 \in \Omega$, *then* $\operatorname{ind}_K([L,-J^{-1}P+\hat{y}_0],\Omega) = \operatorname{ind}_{K_1}(\hat{y}_0,V) = \{1\}$ *where* $y_0 = Hx_0$ *and* $\hat{y}_0(x) = y_0$ *for every* $x \in \Omega$.

P3. If $\Omega_1 \subset \Omega$ *and* $\Omega_2 \subset \Omega$ *with* $\Omega_1 \cap \Omega_2 = \emptyset$ *and* $Lx \neq Nx$ *for* $x \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$ *then* $\operatorname{ind}_K([L,N],\Omega) \subseteq \operatorname{ind}_K([L,N],\Omega_1) + \operatorname{ind}_K([L,N],\Omega_2)$ *with equality if either of the indices on the right is a singleton.*

P4. If $L-N(\lambda,x)$ *is an A-proper homotopy on* Ω *for* $\lambda \in [0,1]$ *and* $(N(\lambda,x)+J^{-1}P)H^{-1} : K_1 \to K_1$, $V \subset H\Omega$ *open bounded with* $(I-T_\lambda)^{-1}(0) \subset V$, *then* $\operatorname{ind}_K([L,N(\lambda,x)],\Omega) = \operatorname{ind}_{K_1}(T_\lambda,V)$ *is independent of* $\lambda \in [0,1]$.

Proof. P1. From the definition, $\operatorname{ind}_{K_1}(T,V) \neq \{0\}$ implies $\operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V), 0\right) \neq \{0\}$. Then if the degree is non-zero there exists $y \in \rho_1^{-1}(V)$ such that $T\rho_1 y = y$. Since $\rho_1 : Y \to K_1$ and $T : K_1 \to K_1$ we have $\rho_1 y = y$ so $Ty = y$ for some $y \in V$. By the construction of $I-T$, this is equivalent to $Lx = Nx$ for $x = H^{-1}y \in H^{-1}V \subset \Omega$.

P2. We have

$$\operatorname{ind}_{K_1}(\hat{y}_0,V) = \operatorname{Deg}\left(I - \hat{y}_0\rho_1, \rho_1^{-1}(V), 0\right) = \{1\}$$

as in the proof to P2 of Theorem 2.2.7.

P3. We observe that $Lx \neq Nx$ for $x \in \overline{\Omega}\setminus(\Omega_1 \cup \Omega_2)$ implies $Ty \neq y$ for $y \in \overline{U}\setminus(U_1 \cup U_2)$ where $U_1 = H\Omega_1$ and $U_2 = H\Omega_2$. By definition, $\operatorname{ind}_K([L,N],\Omega) = \operatorname{ind}_{K_1}(T,V)$ where $(I-T)^{-1}(0) \subset V \subset Y$. We consider the subsets $V_1 = V \cap U_1$ and $V_2 = V \cap U_2$, then V_1 and V_2 are open bounded and disjoint. So

$$\begin{aligned}
\operatorname{ind}_{K_1}(T,V) &= \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V), 0\right) \\
&\subseteq \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_1), 0\right) + \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V_2), 0\right) \\
&\quad + \operatorname{Deg}\left(I - T\rho_1, \rho_1^{-1}(V \setminus (V_1 \cup V_2)), 0\right)
\end{aligned}$$

where the last degree is 0 since $V \setminus (V_1 \cup V_2) \subset \overline{U} \setminus (U_1 \cup U_2)$ and $(I-T)^{-1}(0) \not\subseteq \overline{U} \setminus (U_1 \cup U_2)$. Hence

$$\mathrm{ind}_{K_1}(T,V) \subseteq \mathrm{ind}_{K_1}(T,V_1) + \mathrm{ind}_{K_1}(T,V_2).$$

The proof of equality is analogous to the proof of P3, Theorem 2.2.7.

P4. We note first that if $L - N(\lambda,x)$ is A-proper then the A-properness of $I - T_\lambda \rho_1$ follows from Lemmas 2.3.2 and 2.3.5 with $N(\lambda,x)$ replacing $N(x)$ and $T_\lambda = (N(\lambda,x) + J^{-1}P)\,H^{-1}$. We show that if $T_\lambda \rho_1 y = y$ for some $y \in \overline{\rho_1^{-1}(V)}$, $\lambda \in [0,1]$, then $T_\lambda y = y$ for $y \in V$ and $y \notin \partial \rho_1^{-1}(V)$. Then by the invariance under homotopy property of the A-proper degree we'll have

$$\mathrm{Deg}\left(I - T_0\rho_1, \rho_1^{-1}(V), 0\right) = \mathrm{Deg}\left(I - T_1\rho_1, \rho_1^{-1}(V), 0\right).$$

So assume $T_\lambda \rho_1 y = y$ for some $y \in \overline{\rho_1^{-1}(V)}$, $\lambda \in [0,1]$. By the continuity of ρ_1, $\rho_1 y \in \overline{V} \subset K_1$. Now $T_\lambda : K_1 \to K_1$ so $T\rho_1 y \in K_1$ and $\rho_1 y = y$ and $y \in \overline{V}$. Since $T_\lambda y \neq y$ on ∂V we have $y \in V$ and $y \notin \partial \rho_1^{-1}(V)$. Q.E.D.

Chapter 3

A FIXED POINT INDEX DEFINED IN TERMS OF LIMITS OF THE FINITE DIMENSIONAL BROUWER INDEX

3.1 Introduction

In the previous chapter, the definition of the index required the retraction $\rho_1 : Y \to K_1$ to map Y_n to Y_n and that T be defined on all of K_1. In this chapter we define an index without these restrictions obtaining a result of greater generality in applications but with some cost in simplicity of definition. However, the definition in this chapter fits into the A-proper methodology of obtaining results by means of finite dimensional approximations. We first define a finite dimensional index for continuous maps similar to that of Amann [2], and then, using A-proper theory, extend the definition to A-proper maps in infinite dimensional spaces. Once we establish the index for operators T acting on the same space X, we consider operators of the form $L - N$ from the space X to Y.

As we mentioned in the introduction to Chapter 2, Fitzpatrick and Petryshyn in [16] defined a fixed point index for A-proper maps based on limits of the corresponding finite dimensional Brouwer index. We shall use a similar method but with some differences in the Brouwer degree and index we use. We conclude this chapter by modifying the index to include unbounded sets in Y. This was done by Lafferriere [28] to Fitzpatrick and Petryshyn's index where $T : X \to X$.

3.2 Definition and properties of the finite dimensional index

Let $K \subseteq X$ be a closed convex set (for example, a cone) in a finite dimensional Banach space X and $\Omega \subset X$ be open and bounded with $\Omega \cap K = \Omega_K \neq \emptyset$. Let $T : \overline{\Omega_K} \to K$ be continuous such that $Tx \neq x$ on $\partial_K \Omega$, the boundary of Ω relative to K. Let $\rho : X \to K$ be an arbitrary retraction.

Definition 3.2.1 *We define*

$$i_K (T, \Omega) = \deg \left(I - T\rho, \rho^{-1} (\Omega), 0\right)$$

where the degree is the Brouwer degree for continuous maps.

Remark 3.2.2 *This is essentially a special case of results in Amann [2] but we sketch the simpler case here for completeness.*

The following lemma and proposition show that the index is well defined.

Lemma 3.2.3 $\partial \left(\rho^{-1}\Omega_K\right) \subseteq \rho^{-1} \left(\partial_K \Omega\right).$

Proof. $\rho^{-1}\Omega_K$ is open in X because ρ is continuous. Then

$$\begin{aligned}
\partial\left(\rho^{-1}\Omega_K\right) &= \overline{\rho^{-1}\left(\Omega_K\right)} \setminus \rho^{-1}\left(\Omega_K\right) \text{ in } X \\
&\subseteq \rho^{-1}\left(\overline{\Omega_K}\right) \setminus \rho^{-1}\left(\Omega_K\right) \text{ by continuity of } \rho \\
&= \rho^{-1}\left(\overline{\Omega_K} \setminus \Omega_K\right) \text{ by property of inverse image} \\
&= \rho^{-1}\left(\partial_K\Omega\right). \text{ Q.E.D.}
\end{aligned}$$

Proposition 3.2.4 *If $Tx \neq x$ for all $x \in \partial_K\Omega$ then $T\rho x \neq x$ for all $x \in \partial\left(\rho^{-1}\Omega_K\right)$.*

Proof. (By contrapositive argument) We have $T\rho x = x$ for $x \in \partial(\rho^{-1}\Omega_K) \subseteq \rho^{-1}\left(\partial_K\Omega\right)$ implies $\rho x \in \partial_K\Omega$ and $T\rho x \in K$ so that $x \in K$ and $\rho x = x$. Therefore $x \in \partial_K\Omega$ and $Tx = x$. Q.E.D.

We prove that the index defined above is independent of the retraction ρ.

Theorem 3.2.5 *The index of Definition 3.2.1 does not depend on the retraction chosen from X to K.*

Proof. Let ρ and τ be retractions of X to K and assume $T : \overline{\Omega_K} \to K$. We note that the fixed points of $T\rho$ and $T\tau$ are contained in $D = \rho^{-1}\Omega_K \cap \tau^{-1}\Omega_K$. We define the homotopy $H(\lambda, x) : \overline{\Omega_K} \to K$ by $H(\lambda, x) = (1-\lambda)T\rho x + \lambda T\tau x$ and show that $H(\lambda, x) \neq x$ on $\partial_K D$. If not, then $(1-\lambda)T\rho x + \lambda T\tau x = x$ for some $x \in \partial D$ and since $\rho x \in \overline{\Omega_K}$ and $\tau x \in \overline{\Omega_K}$ and $T : \overline{\Omega_K} \to K$, by convexity of K, $(1-\lambda)T\rho x + \lambda T\tau x \in K$ so $x \in K$ and hence $\rho(x) = x$ and $\tau(x) = x$. Thus $H(\lambda, x) = x$ on $\partial_K D$ reduces to $Tx = x$ on $\partial_K\Omega$ which is excluded by hypothesis. Then by the homotopy invariance property of the Brouwer degree we have

$$\deg\left(I - T\rho, D, 0\right) = \deg\left(I - T\tau, D, 0\right).$$

Hence the index is independent of the retraction chosen. Q.E.D.

We accordingly frame the properties of this index in the next theorem.

Theorem 3.2.6 *Let $T : \overline{\Omega_K} \to K$ be continuous and such that $Tx \neq x$ on $\partial_K \Omega$, then the index defined above has the following properties.*

P1. If $i_K(T, \Omega) \neq 0$, then T has a fixed point in Ω_K.

P2. If $x_0 \in \Omega_K$, then $i_K(\hat{x}_0, \Omega) = 1$ where $\hat{x}_0(x) = x_0$ for every $x \in \overline{\Omega_K}$.

P3. If Ω_1, Ω_2 are relatively open disjoint subsets of Ω_K such that $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$, then

$$i_K(T, \Omega) = i_K(T, \Omega_1) + i_K(T, \Omega_2).$$

P4. If $H(\lambda, x) : [0,1] \times \overline{\Omega_K} \to K$ is a continuous homotopy such that $H(\lambda, x) \neq x$ on $\partial_K \Omega$ for $\lambda \in [0,1]$, then $i_K(H(\lambda, x), \Omega)$ is independent of $\lambda \in [0,1]$.

Proof. P1. If $i_K(T, \Omega) \neq 0$, then by definition the Brouwer degree $\deg(I - T\rho, \rho^{-1}(\Omega_K), 0) \neq 0$ which implies the existence of $x \in \rho^{-1}(\Omega_K)$ such that $T\rho x = x$ with $\rho x \in \Omega_K$. Since $\rho : X \to K$ and $T : \overline{\Omega_K} \to K$ we have $\rho x = x$. By hypothesis, $Tx \neq x$ on $\partial_K \Omega$, consequently, $Tx = x$ for $x \in \Omega_K$.

P2. For $x_0 \in \Omega_K$ we have $\deg(I, \Omega_K, x_0) = 1$ by property of the Brouwer degree. The equation $x_0 = x$ is equivalent to $x_0 - x = 0$ or $(I - \hat{x}_0)x = 0$. Thus

$$\deg(I, \Omega_K, x_0) = \deg(I - \hat{x}_0, \Omega_K, 0).$$

Now as $x_0 \in \Omega_K$ and ρ is the identity on K, we obtain

$$\deg(I - \hat{x}_0, \Omega_K, 0) = \deg\left(I - \hat{x}_0\rho, \rho^{-1}(\Omega_K), 0\right) = 1.$$

P3. If $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ then $Tx \neq x$ on $\partial\Omega_1 \cup \partial\Omega_2$ and by Proposition 3.2.4, $T\rho x \neq x$ on $\partial\rho^{-1}(\Omega_1) \cup \partial\rho^{-1}(\Omega_2)$. We note that since $\Omega_1 \cap \Omega_2 = \emptyset$, $\rho^{-1}(\Omega_1) \cap \rho^{-1}(\Omega_2) = \emptyset$ so that the additivity property of the Brouwer degree gives

$$\deg\left(I - T\rho, \rho^{-1}(\Omega_K), 0\right) = \deg\left(I - T\rho, \rho^{-1}(\Omega_1), 0\right) + \deg\left(I - T\rho, \rho^{-1}(\Omega_2), 0\right).$$

The result then follows from the definition of the index.

P4. Suppose $H(\lambda, \rho x) = x$ for some $x \in \overline{\rho^{-1}(\Omega_K)}$ and $\lambda \in [0,1]$. Since $\rho : X \to K$ we have $\rho x \in \overline{\Omega_K}$ and since $H : \overline{\Omega_K} \to K$, $x \in K$. Now ρ is the identity on K so $\rho x = x$ and $H(\lambda, \rho x) = H(\lambda, x) = x$. By assumption, $H(\lambda, x) \neq x$ on $\partial_K \Omega$ so $x \in \Omega_K$. Hence, by the homotopy property of the Brouwer degree

$$\deg\left(I - H(0, \rho x), \rho^{-1}(\Omega_K), 0\right) = \deg\left(I - H(1, \rho x), \rho^{-1}(\Omega_K), 0\right)$$

from which the result follows. Q.E.D.

3.3 The index defined for infinite dimensional spaces

Let K be a cone in an infinite dimensional Banach space X with projection scheme Γ such that $Q_n(K) \subseteq K$ for every $n \in \mathbb{N}$. Let $\rho : X \to K$ be an arbitrary retraction and $\Omega \subset X$ an open bounded set such that $\Omega_K = \Omega \cap K \neq \emptyset$. Let $T : \overline{\Omega_K} \to K$ be such that $I - T$ is A-proper at 0. Write $K_n = K \cap X_n = Q_n K$ and $\Omega_{K_n} = \Omega_K \cap X_n$. Then $Q_n \rho : X_n \to K_n$ is a finite dimensional retraction.

Definition 3.3.1 *If $Tx \neq x$ on $\partial_K \Omega$ then we define*

$$\operatorname{ind}_K(T, \Omega) = \left\{k \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}\left(Q_{n_j} T, \Omega_{n_j}\right) \to k \text{ for some } n_j \to \infty\right\}$$

where the finite dimensional index is that defined in the previous section.

The following lemma indicates that the index is well defined.

Lemma 3.3.2 *Let $T : \overline{\Omega_K} \to K$ be such that $I - T$ is A-proper at 0 and that $Tx \neq x$ on $\partial_K \Omega$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $Q_n T x \neq x$ for $x \in \partial_{K_n} \Omega_n$ and $Q_n T(Q_n \rho) x \neq x$ on $\partial (Q_n \rho)^{-1}(\Omega_{K_n})$.*

Proof. If the assertion is false, then there exists a sequence $\{x_n\} \subset \partial_{K_n}\Omega_n$ such that $x_n - Q_nTx_n = 0$. Since $I - T$ is A-proper at 0, there is a subsequence $x_{n_j} \to x$ with $Tx = x$. Now for each n, $\partial_{K_n}\Omega_n \subset \partial_K\Omega$ which is closed so we have $x \in \partial_K\Omega$ and we obtain a contradiction to our hypothesis. Then, by Proposition 3.2.4, $Q_nTx \neq x$ on $\partial_{K_n}\Omega_n$ for every $n \geq n_0$ implies $Q_nT(Q_n\rho)x \neq x$ on $\partial(Q_n\rho)^{-1}(\Omega_{K_n})$. Q.E.D.

Remark 3.3.3 *That the index thus defined is independent of the retraction chosen follows from the finite dimensional result.*

The usual properties of this index are stated in the next theorem.

Theorem 3.3.4 *Let $T : \overline{\Omega_K} \to K$ be such that $I - T$ is A-proper at 0 and that $Tx \neq x$ for $x \in \partial_K\Omega$. Then the index defined above has the following properties.*

P1. If $\operatorname{ind}_K(T,\Omega) \neq \{0\}$, then T has a fixed point in Ω_K.

P2. If $x_0 \in \Omega_K$, then $\operatorname{ind}_K(\hat{x}_0,\Omega) = \{1\}$ where $\hat{x}_0(x) = x_0$ for every $x \in \Omega_K$.

P3. If Ω_1 and Ω_2 are disjoint relatively open subsets of Ω_K such that $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$, then

$$\operatorname{ind}_K(T,\Omega) \subseteq \operatorname{ind}_K(T,\Omega_1) + \operatorname{ind}_K(T,\Omega_2)$$

with equality if either of the two indices on the right is a singleton.

P4. If $H(\lambda,x) : [0,1] \times \overline{\Omega_K} \to K$ is such that $I - H(\lambda,x)$ is an A-proper at 0 homotopy and $H(\lambda,x) \neq x$ on $\partial_K\Omega$, $\lambda \in [0,1]$, then $\operatorname{ind}_K(H(\lambda,x),\Omega)$ is independent of $\lambda \in [0,1]$.

Proof. P1. If $\operatorname{ind}_K(T,\Omega) \neq \{0\}$ then there exists a subsequence $\{n_j\} \subset \mathbb{N}$ with $n_j \to \infty$ such that the sequence of finite dimensional indices $\{i_{K_n}(Q_nT,\Omega_n)\}$ has non-zero terms. Consequently, there exist $x_{n_j} \in (Q_{n_j}\rho)^{-1}\left(\Omega_{K_{n_j}}\right)$ such that $Q_{n_j}TQ_{n_j}\rho x_{n_j} = x_{n_j}$. Since $Q_{n_j}\rho : X \to K$ and $Q_{n_j}T : \overline{\Omega_K} \to K$ we have $Q_{n_j}\rho x_{n_j} = x_{n_j}$ so that $Q_{n_j}Tx_{n_j} = x_{n_j}$ and $x_{n_j} \in Q_{n_j}K$. By the A-properness of $I-T$, there exists a subsequence $x_{n_{j_k}} \to x \in \overline{\Omega_K}$ such that $Tx = x$.

P2. Since $Q_n x_0 \in \Omega_{K_n}$ for every $n \in \mathbb{N}$, $i_{K_n}(Q_n \hat{x}_0, \Omega_n) = 1$ for every $n \in \mathbb{N}$ and hence $\mathrm{ind}_K(\hat{x}_0, \Omega) = \{1\}$.

P3. If $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ then $Tx \neq x$ on $\partial_K \Omega_1 \cup \partial_K \Omega_2$. The A-properness of $I - T$ implies that there exists n_0 such that for every $n \geq n_0$, $Q_n T x_n \neq x_n$ on $\partial_{K_n} \Omega_{1_n} \cup \partial_{K_n} \Omega_{2_n}$. By Proposition 3.2.4, we have $Q_n T Q_n \rho x_n \neq x_n$ on $\partial (Q_n \rho)^{-1} (\Omega_{1_n}) \cup \partial (Q_n \rho)^{-1} (\Omega_{2_n})$ where $\Omega_{i_n} = \Omega_i \cap X_n$, $i = 1, 2$. We note also that for every $n \in \mathbb{N}$, $\Omega_{1_n} \cap \Omega_{2_n} = \emptyset$ so that $\rho^{-1}(\Omega_{1_n}) \cap \rho^{-1}(\Omega_{2_n}) = \emptyset$ and the additivity property of the finite dimensional index gives

$$i_{K_n}(Q_n T, \Omega_n) = i_{K_n}(Q_n T, \Omega_{1_n}) + i_{K_n}(Q_n T, \Omega_{2_n}).$$

Then passing to limits we obtain

$$\mathrm{ind}_K(T, \Omega) \subseteq \mathrm{ind}_K(T, \Omega_1) + \mathrm{ind}_K(T, \Omega_2).$$

The argument for equality is essentially the same as that in the proof of P3, Theorem 2.2.7, interchanging degree for index.

P4. We have $H(\lambda, x) \neq x$ on $\partial_K \Omega$ and the A-properness $I - H(\lambda, x)$ imply there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $Q_n H(\lambda, x_n) \neq x_n$ on $\partial_{K_n} \Omega_n$, $\lambda \in [0, 1]$. Then by Proposition 3.2.4, $Q_n H(\lambda, Q_n \rho x_n) \neq x_n$ on $\partial (Q_n \rho)^{-1} (\Omega_{K_n})$ for $\lambda \in [0, 1]$ so that $i_{K_n}(Q_n H(\lambda, Q_n \rho x_n), \Omega_n)$ is well defined and independent of $\lambda \in [0, 1]$ for every $n \geq n_0$. Hence $\mathrm{ind}_K(H(\lambda, x), \Omega)$ is independent of $\lambda \in [0, 1]$. Q.E.D.

3.4 The index extended to maps of the form L-N

We now extend the index to the $L - N$ case and assume L to be bounded making $H = L + J^{-1}P$ a homeomorphism. We make the following assumptions in this section. Let K be a cone in the Banach space X, let $\Omega \subset X$ be open, bounded and such that $\Omega_K \neq \emptyset$, let $L : X \to Y$ be a bounded Fredholm operator of index zero, $N : \overline{\Omega_K} \to Y$ a bounded continuous nonlinear operator such that $L-N$ is A-proper at 0 and $N+J^{-1}P : K \to K$. Using the construction in Section 2.2, we write $Lx - Nx = w$ as $y - Ty = w$ in Y where $T = (N + J^{-1}P)\, H^{-1}$. By Lemma 2.3.2, $I - T$ is A-proper at 0 relative to Γ if $L - N$ is A-proper at 0 relative to Γ_L. By Proposition 2.3.4, T maps K_1 to K_1 where $K_1 = H(K)$.

Definition 3.4.1 *Under the above assumptions, let ρ_1 be a retraction from Y to K_1 and assume $Q_n K_1 \subset K_1$ and $Lx \neq Nx$ on $\partial_K \Omega$. We define the fixed point index of L − N over Ω_K as*

$$\text{ind}_K([L, N], \Omega) = \text{ind}_{K_1}(T, U)$$

where $U = H\Omega_K$ and the index on the right is that defined in the previous section.

The index is well defined since $Lx \neq Nx$ on $\partial_K \Omega$ implies $Ty \neq y$ on $\partial_{K_1} U$ and the A-properness of $I - T$ means there exists n_0 such that for every $n \geq n_0$, $Q_n T y_n \neq y_n$ on $\partial_{K_1} U_n$.

We list the properties of this index in the following theorem.

Theorem 3.4.2 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $\text{ind}_K([L, N], \Omega) = \text{ind}_{K_1}(T, U) \neq \{0\}$, then there exists $x \in \Omega_K$ such that $Lx = Nx$.

P2. If $x_0 \in \Omega_K$, then $\text{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \text{ind}_{K_1}(\hat{y}_0, U) = \{1\}$ where $y_0 = Hx_0$ and $\hat{y}_0(y) = y_0$ for every $y \in U$.

P3. If $Lx \neq Nx$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ where Ω_1 and Ω_2 are disjoint relatively open subsets of Ω_K then

$$\text{ind}_K([L, N], \Omega) \subseteq \text{ind}_K([L, N], \Omega_1) + \text{ind}_K([L, N], \Omega_2)$$

with equality if either of the indices on the right is a singleton.

P4. If $L - N(\lambda, x)$ is an A-proper homotopy on Ω_K for $\lambda \in [0, 1]$ and $(N(\lambda, x) + J^{-1}P)\, H^{-1}$

$: K_1 \to K_1$ *and* $0 \notin (L - N(\lambda, x))(\mathrm{dom}\, L \cap \partial_K \Omega)$ *for* $\lambda \in [0,1]$, *then* $\mathrm{ind}_K([L, N(\lambda, x)], \Omega)$ $= \mathrm{ind}_{K_1}(T_\lambda, U)$ *is independent of* $\lambda \in [0,1]$ *where* $T_\lambda = (N(\lambda, x) + J^{-1}P) H^{-1}$.

Proof. P1. If $\mathrm{ind}_{K_1}(T, U) \neq \{0\}$ then T has a fixed point in U by P1 of Theorem 3.3.4, *i.e.*, $Ty = y$ for $y \in U$ and $y = Hx$. By construction, this is equivalent to $Lx = Nx$, where $x = H^{-1}y \in \Omega_K$.

P2. Let $Hx_0 = y_0 \in H\Omega_K = U$, then $\mathrm{ind}_{K_1}(\hat{y}_0, U) = \{1\}$ by P2 of Theorem 3.3.4.

P3. We note first that $Lx \neq Nx$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ implies $Ty \neq y$ for $y \in \overline{U} \setminus (U_1 \cup U_2)$ where $U = H\Omega_K$, and $U_i = H\Omega_i$, $i = 1, 2$. Now noting that $U_1 \cap U_2 = \emptyset$, P3 of Theorem 3.3.4 gives

$$\mathrm{ind}_{K_1}(T, U) \subseteq \mathrm{ind}_{K_1}(T, U_1) + \mathrm{ind}_{K_1}(T, U_2).$$

The result then follows by definition of the index for $L - N$. The argument for equality is analogous to the proof of P3, Theorem 2.2.7.

P4. We observe that if $L - N(\lambda, x)$ is A-proper then the A-properness of $I - T_\lambda$ follows from Lemma 2.3.2 with $N(\lambda, x)$ replacing $N(x)$. By hypothesis, $Lx \neq N(\lambda, x)$ on $\partial_K \Omega$ so that $Hx \neq (N_\lambda + J^{-1}P)x$ on $\partial_K \Omega$ which implies $y \neq T_\lambda y$ on $\partial_{K_1} U$. Then by P4 of Theorem 3.3.4, $\mathrm{ind}_{K_1}(T_\lambda, U)$ is independent of $\lambda \in [0,1]$ and consequently $\mathrm{ind}_K([L, N(\lambda, x)], \Omega)$ is also. Q.E.D.

3.5 The index defined on unbounded sets

We conclude this chapter by extending the definition of the index to maps $L - N$ where L is unbounded and the set $U = H\Omega_K$ is unbounded in Y. Let $\Omega \subset X$ be open and bounded such that $\Omega_K \cap \mathrm{dom}\, L \neq \emptyset$, $L : \mathrm{dom}\, L \subset X \to Y$ an unbounded Fredholm operator of index zero, $N : \overline{\Omega_K} \cap \mathrm{dom}\, L \to Y$ a bounded continuous operator such that $L - N$ is A-proper relative to Γ_L at 0. Using the construction in Section 2.2, we write $L - N$ as $I - T$ in Y and assume $P + JQN + L_1^{-1}(I - Q)N$ maps K to K. Let $K_1 = H(K \cap \mathrm{dom}\, L)$

where $H = L + J^{-1}P$ and ρ_1 be a retraction from Y onto K_1. Assume $Q_n(K_1) \subset K_1$ and $Lx \neq Nx$ on $\partial_K \Omega$.

Definition 3.5.1 *We define*

$$\text{ind}_K([L,N],\Omega) = \text{ind}_{K_1}(T,V)$$

where $V \subset Y$ is an open bounded set relative to K_1 containing $(I-T)^{-1}(0)$.

Theorem 3.5.2 *The index of Definition 3.5.1 is well defined and independent of the choice of the open bounded set V containing $(I-T)^{-1}(0)$.*

Proof. The conditions $(I-T)^{-1}(0) \subset V$ and V open imply $Ty \neq y$ on ∂V so that $Ty \neq y$ on $\partial_{K_1} V \subseteq \partial V$. Thus, by Definition 3.3.1, $i_{K_1}(Q_nT, V_n)$ is defined for every $n \geq n_0$ and so $\text{ind}_{K_1}(T,V) = \text{ind}_K([L,N],\Omega)$ is well defined.

We need also show that the index is independent of the choice of V. Suppose $(I-T)^{-1}(0) \subset V_1$ and $(I-T)^{-1}(0) \subset V_2$ where V_1, V_2 are open bounded sets in Y. Then $(I-T)^{-1}(0) \subset V_1 \cap V_2 = W$ and W is an open bounded set in Y. Now $Ty \neq y$ on ∂W so $Ty \neq y$ on $\partial_{K_1} W$. Since $I - T$ is A-proper, there exists n_0 such that for every $n \geq n_0$, $(I - Q_nT)y_n \neq 0$ on $\partial_{K_1} W_n$. By Proposition 3.2.4, $(I - Q_nTQ_n\rho_1)y_n \neq 0$ on $\partial(Q_n\rho_1)^{-1}(W_{K_1})$ for every $n \geq n_0$. By the additivity and excision properties of the finite dimensional index of Definition 3.2.1, we have

$$\begin{aligned} i_{K_1}(Q_nT, V_{1_n}) &= i_{K_1}(Q_nT, W_n) + i_{K_1}(Q_nT, V_{1_n} \setminus W_n) \\ &= i_{K_1}(Q_nT, W_n). \end{aligned}$$

Similarly,

$$\begin{aligned} i_{K_1}(Q_nT, V_{2_n}) &= i_{K_1}(Q_nT, W_n) + i_{K_1}(Q_nT, V_{2_n} \setminus W_n) \\ &= i_{K_1}(Q_nT, W_n). \end{aligned}$$

Since this is true for all $n \geq n_0$, we have

$$\text{ind}_K([L,N],\Omega) = \text{ind}_{K_1}(T,V_1) = \text{ind}_{K_1}(T,V_2)\,. \text{ Q.E.D.}$$

We present the usual properties of this index in the following theorem.

Theorem 3.5.3 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $\text{ind}_K([L,N],\Omega) = \text{ind}_{K_1}(T,V) \neq \{0\}$, *then there exists* $x \in \Omega_K$ *such that* $Lx = Nx$.

P2. If $x_0 \in \Omega_K$, *then* $\text{ind}_K([L,-J^{-1}P+\hat{y}_0],\Omega) = \text{ind}_{K_1}(\hat{y}_0,V) = \{1\}$ *where* $y_0 = Hx_0$ *and* $\hat{y}_0(x) = y_0$ *for every* $x \in \Omega_K$.

P3. If Ω_1, Ω_2 *are disjoint relatively open subsets of* Ω_K *such that* $Lx \neq Nx$ *for* $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ *then*

$$\text{ind}_K([L,N],\Omega) \subseteq \text{ind}_K([L,N],\Omega_1) + \text{ind}_K([L,N],\Omega_2)$$

with equality if either of the indices on the right is a singleton.

P4. If $L - N(\lambda,x)$ *is an A-proper homotopy on* Ω_K *for* $\lambda \in [0,1]$ *and* $(N(\lambda,x)+J^{-1}P)H^{-1} : K_1 \to K_1$ *and* $(I-T_\lambda)^{-1}(0) \subset V$ *for every* $\lambda \in [0,1]$, $V \subset Y$ *open and bounded and* $0 \notin (L-N(\lambda,x))(\text{dom}\, L \cap \partial_K\Omega)$, *then* $\text{ind}_K([L,N(\lambda,x)],\Omega) = \text{ind}_{K_1}(T_\lambda,V)$ *is independent of* $\lambda \in [0,1]$.

Proof. P1. By properties of the earlier index defined for T in Theorem 3.3.4 we have $\text{ind}_{K_1}(T,V) \neq \{0\}$ implies T has a fixed point $y \in V_{K_1}$. Now $(I-T)^{-1}(0) \subset H(\Omega_K \cap \text{dom}\, L)$ so that $x = H^{-1}y \in \Omega_K$. Hence $Lx = Nx$ has a solution in Ω_K.

P2. This follows from P2 of Theorem 3.3.4, noting that $Hx_0 = y_0 \in H(\Omega_K \cap \text{dom}\, L)$.

P3. We note that $Lx \neq Nx$ for $x \in \overline{\Omega_K} \backslash (\Omega_1 \cup \Omega_2)$ implies $Ty \neq y$ for $y \in \overline{U} \backslash (U_1 \cup U_2)$ where $U = H\Omega_K$ and $U_i = H\Omega_i$, $i = 1,2$. Now by definition, $\text{ind}_K([L,N],\Omega) = \text{ind}_K(T,V)$ where $(I-T)^{-1}(0) \subset V \subset Y$ and V is open and bounded. We consider the subsets $V_1 = V \cap U_1$ and $V_2 = V \cap U_2$ of V, then V_1 and V_2 are open, bounded and disjoint. So, by the additivity and excision properties of the index in Theorem 3.3.4, we have

$$\operatorname{ind}_{K_1}(T, V) \subseteq \operatorname{ind}_{K_1}(T, V_1) + \operatorname{ind}_{K_1}(T, V_2) + \operatorname{ind}_{K_1}(T, V \setminus (V_1 \cup V_2))$$

where the last index is 0 since $(I - T)^{-1}(0) \not\subseteq \overline{V} \setminus (V_1 \cup V_2)$. The result then follows from the definition of the index for $L - N$. The argument for equality is analogous to the proof of P3, Theorem 2.2.7.

P4. We observe that if $L - N(\lambda, x)$ is A-proper, then the A-properness of $I - T_\lambda$ follows from Lemma 2.3.2 with $N(\lambda, x)$ replacing $N(x)$ and $T_\lambda = (N(\lambda, x) + J^{-1}P) H^{-1}$. The condition $0 \notin (L - N(\lambda, x))(\operatorname{dom} L \cap \partial_K \Omega)$ implies $(I - T_\lambda)^{-1}(0) \not\subseteq \overline{U} \setminus V$, $U = H\Omega_k$, so that by the invariance under homotopy property of Theorem 3.3.4 we have $\operatorname{ind}_{K_1}(T_\lambda, V)$ independent of $\lambda \in [0, 1]$. Hence it follows that $\operatorname{ind}_K([L, N(\lambda, x)], \Omega)$ is also independent of $\lambda \in [0, 1]$. Q.E.D.

Chapter 4

A FIXED POINT INDEX FOR WEAKLY INWARD A-PROPER MAPS OF THE FORM L-N

4.1 Introduction

The results of this chapter extend the fixed point index for weakly inward A-proper maps defined by Lan and Webb in [30] to maps of the form $L - N$. Weakly inward maps were apparently introduced by Halpern and Bergman [20] and have been extensively studied by Caristi [8], Deimling [13], Hu and Sun [24], Lan and Webb [30], Sun and Sun [51], and Webb [53]. Fixed point indices for weakly inward maps have been studied by Hu and Sun [24] for compact maps, Sun and Sun [51] for maps defined on compact convex sets, and by Lan and Webb [30] for A-proper maps.

Definition 4.1.1 *Let $K \subset X$ be a closed convex set. For each $x \in K$, the set $I_K(x) = \{x + c(z - x) : z \in K,\ c \geq 0\}$ is called the inward set of x with respect to K. A map $T : K \to X$ is called inward (respectively, weakly inward) if for all $x \in K$, $Tx \in I_K(x)$*

$(Tx \in \overline{I_K}(x))$.

Remark 4.1.2 *Geometrically, the inward set of x is the union of all rays originating at x and passing through some other point z of K. If x is an interior point of K then $I_K(x)$ forms the whole space X.*

Definition 4.1.3 *A map $T : \overline{\Omega_K} \to X$ is said to be inward (respectively, weakly inward) on $\overline{\Omega_K}$ relative to K if $Tx \in I_K(x)$ (respectively, $Tx \in \overline{I_K}(x)$) for $x \in \overline{\Omega_K}$ where $\Omega \subset X$ is open and bounded with $\Omega_K = \Omega \cap K \neq \emptyset$.*

The following two theorems give conditions for a map $T : K \subset X \to X$ to be weakly inward. The first involves a hypothesis of flow invariance from the theory of differential equations in Banach spaces.

Theorem 4.1.4 *(Caristi [8]) Let $K \subset X$ be a closed convex set in a Banach space X and $T : K \to X$. Then T is weakly inward iff*

$$\lim_{h \to 0^+} \frac{1}{h} \operatorname{dist}(x + h(Tx - x), K) = 0$$

for all $x \in K$.

Proof. (The argument follows that of Deimling [13]) Suppose

$$\lim_{h \to 0^+} \frac{1}{h} \operatorname{dist}(x + h(Tx - x), K) = 0$$

for all $x \in K$. Given $\epsilon > 0$, for each $\delta \in (0,1)$ there exists $y \in K$ such that

$$\|x + \delta(Tx - x) - y\| \leq \operatorname{dist}(x + \delta(Tx - x), K) + \delta\epsilon.$$

Then

$$\left\|Tx - \left[\left(1 - \delta^{-1}\right)x + \delta^{-1}y\right]\right\| \leq \delta^{-1} \operatorname{dist}(x + \delta(Tx - x), K) + \epsilon$$

and as ϵ was chosen arbitrarily, this and the assumed limit condition imply $Tx \in \overline{I_K}(x)$. Now suppose T is weakly inward, let $x \in K$, $\epsilon > 0$, and choose $y \in I_K(x)$ such that

$\|y - Tx\| \leq \epsilon$. Since $y \in I_K(x)$ and K is convex, there exists $\delta_0 > 0$ such that $x + \delta(y - x) \in K$ for $0 < \delta \leq \delta_0$. Then

$$\begin{aligned} \delta^{-1} \operatorname{dist}(x + \delta(Tx - x), K) &\leq \delta^{-1} \|(1 - \delta)x + \delta Tx - [x + \delta(y - x)]\| \\ &\leq \delta^{-1} \|\delta Tx - \delta y\| = \|Tx - y\| \leq \epsilon \end{aligned}$$

for $\delta \in (0, \delta_0]$. Hence, we have proved the converse. Q.E.D.

The second theorem is formulated in terms of tangent functionals and is often useful in applications and may be found in Deimling [13].

Theorem 4.1.5 *Let $K \subset X$ be a closed convex set in a Banach space X. Then $T : K \to X$ is weakly inward iff $x \in \partial K$, $x^* \in X^*$ and $x^*(x) = \sup_{y \in K} x^*(y)$ together imply $x^*(Tx - x) \leq 0$.*

Proof. Suppose T is weakly inward and $Tx = w = \lim_{n \to \infty} w_n$ where

$$w_n = x + c_n(z_n - x),$$

$z_n \in K$, $c_n \geq 0$ and $x \in \partial K$. Then for each $n \in \mathbb{N}$,

$$\begin{aligned} x^*(x + c_n(z_n - x)) &= x^*(x) + c_n x^*(z_n - x) \\ &= x^*(x) + c_n[x^*(z_n) - x^*(x)] \\ &\leq x^*(x), \end{aligned}$$

since $x^*(x) \geq x^*(z_n)$ for all n. Thus $x^*(Tx) \leq x^*(x)$ and $x^*(Tx - x) \leq 0$.

We shall prove the converse by contradiction. Suppose T is not weakly inward and the conditions of the theorem hold, then there exists an $x \in K$ such that $Tx \notin \overline{I_K}(x)$. Now by the separation theorem for convex sets, there exists $x^* \in X^*$ such that

$$\sup_{z \in \overline{I_K}(x)} x^*(z) < x^*(Tx),$$

noting that $\overline{I_K}(x)$ is convex and that there exists $\rho < \text{dist}\left(Tx, \overline{I_K}(x)\right)$ with $\overline{B_\rho}(Tx) \cap \overline{I_K}(x) = \emptyset$. Then

$$\begin{aligned} x^*(Tx - x) &= x^*(Tx) - x^*(x) \\ &> x^*(x + c(y - x)) - x^*(x) \\ &= cx^*(y - x) \end{aligned}$$

for all $c \geq 0$, and all $y \in K$. Hence, taking $c = 0$, $x^*(Tx - x) > 0$ and taking c arbitrarily large, $x^*(y - x) \leq 0$ for all $y \in K$ which contradicts our hypotheses. Q.E.D.

Before considering an index for the $L - N$ case, we discuss the derivation, definition and properties of the index established by Lan and Webb [30] for reference.

In this chapter, we adopt conventions corresponding to the notation of Lan and Webb for convenience. Denote $Q_n T$ as T_n, $\Omega \cap K$ as Ω_K and $\Omega_K \cap X_n$ as Ω_{K_n}.

We begin by recalling the Lan-Webb definition of an index for weakly inward maps in finite dimensional spaces and then, in a manner analogous to the index derived by Fitzpatrick and Petryshyn [16], extend the definition to the infinite dimensional case. The definition of the index requires a special retraction which we now mention.

Definition 4.1.6 *An ϵ-retraction of X onto K is a continuous map $r : X \to K$ which satisfies*

$$\|x - rx\| \leq (1 + \epsilon)\,\text{dist}(x, K) \text{ for every } x \in X.$$

Remark 4.1.7 *ϵ-retractions exist for every $\epsilon > 0$ by Dugundji's extension theorem, cf., [26]. For locally uniformly convex spaces, one may take $\epsilon = 0$ and a 0-retraction (equivalently, a metric projection) is possible.*

The following lemma can be used to help show that an index can be defined which is independent of the ϵ-retraction chosen provided ϵ is sufficiently small.

Lemma 4.1.8 *Let K be a closed convex set in a finite dimensional space X and let D be a closed bounded subset of K. Let $h : [0,1] \times D \to X$ be continuous and such that $h(t,\cdot)$ is weakly inward on D relative to K for each $t \in [0,1]$. Then, if $h(t,x) \neq x$ for all $x \in D$ and $t \in [0,1]$, there exists $\epsilon_0 > 0$ such that $r_\epsilon(h(t,x)) \neq x$ for all $x \in D$, $t \in [0,1]$, and $\epsilon \leq \epsilon_0$.*

Proof. If this were false, there would be sequences $\epsilon_n \to 0$, $x_n \in D$, $t_n \in [0,1]$ such that $r_\epsilon(h(t_n, x_n)) = x_n$. Thus

$$\begin{aligned} \|x_n - h(t_n, x_n)\| &= \|r_{\epsilon_n}(h(t_n, x_n) - h(t_n, x_n)\| \\ &\leq (1 + \epsilon_n) \operatorname{dist}(h(t_n, x_n), K). \end{aligned}$$

Passing then to subsequences, we may suppose that $x_n \to x \in D$, $t_n \to t$. Then $h(t_n, x_n) \to h(t,x)$ and we obtain $\|x - h(t,x)\| \leq \operatorname{dist}(h(t,x), K)$.

Now Lemma 2.2 of [30] states: if $f(x) \in \overline{I_K}(x)$ and $f(x) \notin K$ for $x \in K$, then $\operatorname{dist}(f(x), K) < \|f(x) - x\|$. The proof, for completeness, is as follows.

Let $f(x) = \lim w_n$ where $w_n = (1 - a_n)x + a_n y_n$ for some $y_n \in K$ and $a_n > 1$. Choose N sufficiently large so that $\|f(x) - w_N\| < \|f(x) - x\|$. Then

$$\begin{aligned} \operatorname{dist}(f(x), K) &\leq \|f(x) - y_N\| = \left\| f(x) - \left[\frac{1}{a_N} w_N + \left(1 - \frac{1}{a_N}\right) x \right] \right\| \\ &\leq \frac{1}{a_N} \|f(x) - w_N\| + \left(1 - \frac{1}{a_N}\right) \|f(x) - x\| \\ &< \|f(x) - x\|. \end{aligned}$$

Hence we obtain a contradiction unless $h(t,x) \in K$. But this implies $h(t,x) = x$ which contradicts our hypothesis and, therefore, proves the result. Q.E.D.

We now define the index for finite dimensional spaces which will later be extended to the infinite dimensional case.

Definition 4.1.9 *Let $\Omega \subset X$ be an open bounded set of the finite dimensional space X such that $\Omega_K \neq \emptyset$. Suppose that $f : \overline{\Omega_K} \to X$ is continuous and weakly inward on $\overline{\Omega_K}$ and suppose that $f(x) \neq x$ for all $x \in \partial_K \Omega$. Define the fixed point index by $i_K(f, \Omega) = i_K(r_\epsilon f, \Omega)$ for ϵ sufficiently small where $i_K(r_\epsilon f, \Omega)$ is the fixed point index defined in Section 3.2 above.*

Remark 4.1.10 *Lemma 4.1.8 implies the index is well defined and, by considering a homotopy argument, e.g., $h(t,x) = tr_{\epsilon_1} f(x) + (1-t) r_{\epsilon_2} f(x)$, it follows that the index is independent of the ϵ-retraction chosen for ϵ sufficiently small. If a continuous metric projection r_0 exists, then the index equals $i_K(r_0 f, \Omega_K)$ for Ω open and bounded and $\Omega \cap K \neq \emptyset$.*

In [30], this finite dimensional index is shown to have the standard fixed point index properties, *viz.*, existence, normalisation, additivity, and homotopy, as well as a fifth property which states criteria for the index to equal one.

Let K be a closed convex set in a Banach space X with projection scheme $\Gamma = \{X_n, Q_n\}$ and Ω an open bounded set in X such that $\Omega_K \neq \emptyset$. Let $T : \overline{\Omega_K} \to X$ be a weakly inward map where $I - T$ is A-proper at 0 and such that $Tx \neq x$ on $\partial_K \Omega$.

The fixed point index for weakly inward A-proper maps is then defined in terms of limits of these finite dimensional indices. An important step in the construction is the proof that if $T : \overline{\Omega_K} \to X$ is weakly inward relative to K, then $Q_n T : \overline{\Omega_K} \cap X_n \to X_n$ is weakly inward relative to $K_n = K \cap X_n$. This is evident if $Q_n I_K(x) \subseteq \overline{I_{K_n}}(x)$ for every $x \in K_n$ which Lan and Webb prove to be true iff $Q_n K \subseteq K$. As this assumption has been made throughout this thesis, it is not restrictive for our purposes and is usually easy to verify in applications.

Definition 4.1.11 *Write T_n for the map $Q_n T$ restricted to $\overline{\Omega_{K_n}}$. We define the fixed point index of T over Ω_K with respect to K as follows:*

$$i_K(T, \Omega_K) = \left\{ m \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}\left(T_{n_j}, \Omega_{n_j}\right) \to m \text{ for some } n_j \to \infty \right\}$$

where $i_{K_n}(T_n, \Omega_n)$ *is the fixed point index of Definition 4.1.9.*

Remark 4.1.12 *The A-properness of* $I-T$ *at 0 and the condition* $Tx \neq x$ *on* $\partial\Omega_K$ *imply* $T_n x_n \neq x_n$ *for all* $n \geq n_0$ *so that the indices* $i_{K_n}(T_n, \Omega_n)$ *are well defined for such* n. *This index generalises that of Fitzpatrick and Petryshyn [16] and is equivalent if* T *maps* $\overline{\Omega_K}$ *to* K *and* $I-T$ *is A-proper on* X. *Note that this index is, in general, multivalued as the derivation is similar to that of the degree for A-proper maps.*

The properties of this index are the content of the next theorem. We shall extend them to maps of the form $L-N$ in the next section.

Theorem 4.1.13 *(Lan and Webb [30]) Let* $T : \overline{\Omega_K} \to X$ *be a weakly inward map where* $I-T$ *is A-proper at 0 and such that* $Tx \neq x$ *for* $x \in \partial_K\Omega$. *Then the index defined above has the following properties.*

P1. (Existence) If $i_K(T, \Omega) \neq \{0\}$, *then* T *has a fixed point in* Ω_K.

P2. (Normalisation) If $x_0 \in \Omega_K$, *then* $i_K(\hat{x}_0, \Omega) = 1$, *where* $\hat{x}_0(x) = x_0$ *for every* $x \in \overline{\Omega_K}$.

P3. (Additivity) If Ω_1, Ω_2 *are disjoint relatively open subsets of* Ω_K *such that* $Tx \neq x$ *for* $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$, *then* $i_K(T, \Omega) \subseteq i_K(T, \Omega_1) + i_K(T, \Omega_2)$ *with equality holding if either* $i_K(T, \Omega_1)$ *or* $i_K(T, \Omega_2)$ *is a singleton.*

P4. (Homotopy property) Let $H : [0,1] \times \overline{\Omega_K} \to X$ *be such that* $H(t,.) : \overline{\Omega_K} \to X$ *is weakly inward for each* $t \in [0,1]$ *and* $Ix - H(t,x)$ *is an A-proper homotopy at 0. If* $H(t,x) \neq x$ *for* $x \in \partial_K\Omega$ *and* $t \in [0,1]$, *then* $i_K(H(0,.), \Omega) = i_K(H(1,.), \Omega)$.

P5. Suppose $T(K)$ *is bounded, then there exists* $r_0 > 0$ *such that* $i_K(T, B_r(0) \cap K) = \{1\}$ *for all* $r \geq r_0$, *and hence* T *has a fixed point in* K.

Proof. We prove properties P1 and P4 to illustrate the methods and refer to [30] for further details.

P1. If $i_K(T, \Omega) \neq \{0\}$ then there exists a subsequence $\{n_j\} \subset \mathbb{N}$ with $n_j \to \infty$ such that the sequence $\left\{i_{K_{n_j}}(T_{n_j}, \Omega_{n_j})\right\}$ has non-zero terms. Consequently, there exists $x_{n_j} \in$

$\Omega_{K_{n_j}}$ such that $T_{n_j}x_{n_j} = x_{n_j}$. Since $I - T$ is A-proper at 0, there exists a subsequence $x_{n_{j_k}} \to x \in \overline{\Omega_K}$ with $Tx = x$. As $Tx \neq x$ on $\partial_K \Omega$ we have $x \in \Omega_K$ which proves the result.

P4. From the definition of an A-proper homotopy, the condition $H(t, x) \neq x$ on $\partial_K \Omega$, $t \in [0, 1]$ implies $P_n H(t, x) \neq x$ for all $x \in \partial_{K_n} \Omega$, for sufficiently large n. Therefore, for such n, $P_n H(t, x)$ is defined on X_n and independent of $t \in [0, 1]$. Hence $i_K(H(t, x), \Omega)$ is independent of $t \in [0, 1]$. Q.E.D.

4.2 The index defined for maps of the form L-N

In this section we assume L to be bounded and develop an index and the corresponding properties accordingly. The case where L is unbounded will be the subject of a subsequent section. We begin by defining the concept of weakly inward for maps of the form $L - N$ on a closed convex set; in particular a cone K.

Let X, Y be Banach spaces with projection scheme $\Gamma_L = \{X_n, Y_n, Q_n\}$. Let $K \subset X$ be a cone and $\Omega \subset X$ be open bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Let $L : \operatorname{dom} L \to Y$ be a bounded Fredholm operator of index zero and $N : \overline{\Omega_K} \cap \operatorname{dom} L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0 relative to Γ_L.

Definition 4.2.1 *We say the pair $\{L, N\}$ is weakly inward on $\overline{\Omega_K}$ relative to K if $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_K}$.*

Remark 4.2.2 *When L is bounded, this is equivalent to*

$$Px + JQNx + L_1^{-1}(I - Q)Nx \in \overline{I_K}(x) \ \text{for } x \in \overline{\Omega_K}.$$

Proof of remark 4.2.2. Since H and H^{-1} are continuous, $H\left(\overline{I_K}(x)\right) = \overline{H(I_K(x))}$ so that if we write $(P + JQN)x + L_1^{-1}(I - Q)Nx = \tilde{x} \in \overline{I_K}(x)$ then $H\tilde{x} \in H\left(\overline{I_K}(x)\right) = \overline{I_{K_1}}(Hx)$. By Lemma 2.3.3, $H\tilde{x} = (N + J^{-1}P)x$.

Now assuming $(N + J^{-1}P)\,x \in \overline{I_{K_1}}\,(Hx)$ for every $x \in \overline{\Omega_K}$, we have $H\tilde{x} \in \overline{I_{K_1}}\,(Hx)$ by Lemma 2.3.3 since $H\tilde{x} = (N + J^{-1}P)\,x$. Also, $\overline{I_{K_1}}\,(Hx) = H\left(\overline{I_K}\,(x)\right)$ so $\tilde{x} \in \overline{I_K}\,(x)$. Then by Lemma 2.3.3, $\tilde{x} = (P + JQN)\,x + L_1^{-1}\,(I - Q)\,Nx \in \overline{I_K}\,(x)$, which proves equivalence. Q.E.D.

The justification for this definition is that we shall require the map T in Y to be weakly inward on $\overline{U} = \overline{H\Omega_K}$ relative to $K_1 = H(\operatorname{dom} L \cap K)$ in order to define our index. The following lemma shows that if the conditions of the preceding definition are satisfied, then T is indeed weakly inward on $\overline{U}$.

Lemma 4.2.3 *If $(N + J^{-1}P)\,x \in \overline{I_{K_1}}\,(Hx)$ for every $x \in \overline{\Omega_K}$, then $Ty \in \overline{I_{K_1}}\,(y)$ for every $y \in \overline{U} \subset K_1$.*

Proof. Using the construction and notation of Section 2.2 and letting $y = Hx$, we have $(N + J^{-1}P)\,x \in \overline{I_{K_1}}\,(Hx)$ for every $x \in \overline{\Omega_K}$ is equivalent to $(N + J^{-1}P)\,H^{-1}\,(y) \in \overline{I_{K_1}}\,(y)$ for every $y \in \overline{U}$ and hence $Ty \in \overline{I_{K_1}}\,(y)$ for every $y \in \overline{U}$, *i.e.*, T is weakly inward on $\overline{U}$ relative to K_1. Q.E.D.

We now define the index for weakly inward A-proper maps of the form $L - N$ as follows.

Definition 4.2.4 *Assume $\{L, N\}$ is weakly inward on $\overline{\Omega_K}$ relative to K, $Lx \neq Nx$ on $\partial_K\Omega$, $Q_n\,(K_1) \subseteq K_1$ for $n \in \mathbb{N}$ and write $U = H\Omega_K$. Under the above hypotheses, we define* $\operatorname{ind}_K\left([L, N]\,,\Omega\right) = i_{K_1}\,(T, U)$ *the fixed point index for weakly inward A-proper maps from Definition 4.1.11.*

Remark 4.2.5 *By Lemma 2.3.2, $I - T$ is A-proper with respect to Γ_L and Lemma 4.2.3 shows that T is weakly inward on the set $\overline{U}$ relative to K_1. As was shown in Section 2.2, the condition $Lx \neq Nx$ on $\partial_K\Omega$ implies $Ty \neq y$ on ∂U so that the index is well defined.*

In virtue of these definitions and remarks, we list the following properties of the index which extend those of Theorem 4.1.13.

Theorem 4.2.6 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $\operatorname{ind}_K([L,N],\Omega) \neq \{0\}$, *then there exists* $x \in \Omega_K$ *such that* $Lx = Nx$.

P2. If $x_0 \in \Omega_K$, *then* $\operatorname{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = \{1\}$ *where* $Hx_0 = y_0$ *and* $\hat{y}_0(x) = y_0$ *for every* $x \in \Omega_K$.

P3. If Ω_1, Ω_2 *are disjoint relatively open subsets of* Ω_K *such that* $Lx \neq Nx$ *for* $x \in \overline{\Omega_K} \backslash (\Omega_1 \cup \Omega_2)$, *then* $\operatorname{ind}_K([L,N],\Omega) \subseteq \operatorname{ind}_K([L,N],\Omega_1) + \operatorname{ind}_K([L,N],\Omega_2)$ *with equality if one of the indices on the right is a singleton.*

P4. If $L - N(\lambda, x)$ *is A-proper for* $\lambda \in [0,1]$ *and* $N(\lambda,x) + J^{-1}Px \in \overline{I_{K_1}}(Hx)$ *for every* $x \in K$, $\lambda \in [0,1]$ *and* $0 \notin (L - N(\lambda,x))(\partial_K \Omega)$, $\lambda \in [0,1]$, *then* $\operatorname{ind}_K([L, N(\lambda,x)],\Omega)$ *is independent of* $\lambda \in [0,1]$.

Proof. P1. If $\operatorname{ind}_K([L,N],\Omega) = i_{K_1}(T,U) \neq \{0\}$, then there exists $y \in U$ such that $Ty = y$ which is equivalent to $Lx = Nx$ for $x = H^{-1}y \in \Omega_K$.

P2. By definition,

$$\operatorname{ind}_K\left(\left[L, -J^{-1}P + \hat{y}_0\right], \Omega\right) = i_{K_1}(\hat{y}_0, U) = \{1\}$$

since $y_0 = Hx_0 \in U$.

P3. From $Lx \neq Nx$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ we have $Ty \neq y$ for $y \in \overline{U} \setminus (U_1 \cup U_2)$, where $U = H\Omega_K$ and $U_i = H\Omega_i$, $i = 1, 2$, noting that $H\Omega_1 \cap H\Omega_2 = \emptyset$ since H is a homeomorphism and that $H\Omega_1$ and $H\Omega_2$ are open bounded subsets of $H\Omega_K$. Then

$$\begin{aligned} \operatorname{ind}_K([L,N],\Omega) &= i_{K_1}(T,U) \\ &\subseteq i_{K_1}(T,U_1) + i_{K_1}(T,U_2) \\ &= \operatorname{ind}_K([L,N],\Omega_1) + \operatorname{ind}_K([L,N],\Omega_2). \end{aligned}$$

The argument for equality is analogous to the proof of P3, Theorem 2.2.7.

P4. The first condition implies $I - T_\lambda$ where $T_\lambda = \left(N(\lambda, x) + J^{-1}P\right)H^{-1}$ is A-proper at 0 by Lemma 2.3.2 with $N(\lambda, x)$ replacing N while the second condition implies T_λ is weakly inward by Lemma 4.2.3. From $0 \notin (L - N(\lambda, x))(\partial\Omega_K)$ we have $T_\lambda y \neq y$ on ∂U for $\lambda \in [0,1]$. Then by the invariance under homotopy property,

$$i_{K_1}(T_0, U) = i_{K_1}(T_1, U)$$

or equivalently,

$$\text{ind}_K([L, N(0,x)], \Omega) = \text{ind}_K([L, N(1,x)], \Omega).$$

Q.E.D.

We include a fifth property in the event that $\left(N + J^{-1}P\right)(K)$ is bounded where K is a closed convex set. We shall use this property in the proof to Theorem 5.4.13 to obtain multiple non-zero solutions to the equation $Lx = Nx$.

Theorem 4.2.7 *P5. If* $\left(N + J^{-1}P\right)(K)$ *is bounded and* $L - \lambda N$ *is A-proper for* $\lambda \in [0,1]$, *then there exists* $r_0 > 0$ *such that* $\text{ind}_K([L, N], B_r(0)) = \{1\}$ *for every* $r \geq r_0$.

Proof. Since $\left(N + J^{-1}P\right)(K)$ is bounded we have $T(HK) = T(K_1)$ bounded in Y so

$$\text{ind}_K([L, N], B_r(0)) = i_{K_1}(T, HB_r(0)) = \{1\}$$

by P5 of Theorem 4.1.13. Q.E.D.

4.3 The index extended to unbounded sets

Let $K \subset X$ be a closed convex set in a Banach space X with projection scheme $\Gamma = \{X_n, Q_n\}$. Let $\Omega \subset X$ be an open bounded set such that $\Omega_K \neq \emptyset$ and $T : \overline{\Omega_K} \to X$ be a weakly inward map on $\overline{\Omega_K}$ relative to K such that $I - T$ is A-proper at 0. Assume $(I - T)^{-1}(0)$ is bounded and $Tx \neq x$ on $\partial_K \Omega$.

Definition 4.3.1 *Define the fixed point index of T over Ω_K relative to K as $i_K(T,\Omega) = i_K(T,W)$ where $W = \Omega_K \cap V_K$ and $V \subset X$ is open, bounded such that $(I-T)^{-1}(0) \subset V$.*

Remark 4.3.2 *T is weakly inward on $\overline{W}$ relative to K since $W \subset \overline{\Omega_K}$ and $Tx \neq x$ on ∂W since $(I-T)^{-1}(0) \subset \Omega_K \cap V_K = W$ and W is the intersection of two relatively open sets. Thus the index is well defined.*

We show that the index is independent of the choice of the open bounded set V and consequently, W. Suppose $(I-T)^{-1}(0) \subset V_1$ and $(I-T)^{-1}(0) \subset V_2$ and let $W_1 = V_1 \cap \Omega_K$ and $W_2 = V_2 \cap \Omega_K$, then $(I-T)^{-1}(0) \subset W_1 \cap W_2 = W_0$. Now T is weakly inward on W_i, $i = 0,1,2$, relative to K and $Tx \neq x$ on ∂W_i, since W_i are open and $(I-T)^{-1}(0) \subset W_i$. By the additivity and excision properties of the Lan-Webb index we have

$$\begin{aligned} i_K(T,\Omega) &= i_K(T,W_1) = i_K\left(T, W_1 \setminus \overline{W_0}\right) + i_K(T,W_0) \\ &= i_K(T,W_0). \end{aligned}$$

Similarly,

$$\begin{aligned} i_K(T,\Omega) &= i_K(T,W_2) = i_K\left(T, W_2 \setminus \overline{W_0}\right) + i_K(T,W_0) \\ &= i_K(T,W_0). \end{aligned}$$

Hence the index is independent of the choice of V and W.

The properties of this index are given in the next theorem.

Theorem 4.3.3 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $i_K(T,\Omega) = i_K(T,W) \neq \{0\}$ then T has a fixed point in Ω_K.

P2. If $x_0 \in \Omega_K$ then $i_K(\hat{x}_0,\Omega) = i_K(\hat{x}_0,W) = \{1\}$, where $\hat{x}_0(x) = x_0$ for every $x \in \overline{\Omega_K}$.

P3. If Ω_1, Ω_2 are disjoint relatively open subsets of Ω_K such that $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ then

$$i_K(T, \Omega) \subseteq i_K(T, \Omega_1) + i_K(T, \Omega_2)$$

with equality if either of the indices on the right is a singleton.

P4. Let $H : [0,1] \times \overline{\Omega_K} \to X$ be such that $H(\lambda, \cdot) : \overline{\Omega_K} \to X$ is weakly inward relative to K for every $\lambda \in [0,1]$ and $Ix - H(\lambda, x)$ is an A-proper homotopy at 0. If $H(\lambda, x) \neq x$ for $x \in \partial_K \Omega$ and $\lambda \in [0,1]$ then $i_K(H(0, \cdot), \Omega) = i_K(H(1, \cdot), \Omega)$.

Proof. P1. If $i_K(T, \Omega) = i_K(T, W) \neq \{0\}$ then P1 of Theorem 4.1.13 implies T has a fixed point in W. Since $W \subset \Omega_K$ we have $Tx = x$ for some $x \in \Omega_K$.

P2. Since V is chosen such that $(I - T)^{-1}(0) = (I - \hat{x}_0)^{-1}(0) \subset V$ we have $x_0 \in V$ so that $x_0 \in V \cap \Omega_K = W$. Then P2 of Theorem 4.1.13 implies $i_K(\hat{x}_0, W) = \{1\}$.

P3. By definition $i_K(T, \Omega) = i_K(T, W)$ where $W = V \cap \Omega_K$ and $V \subset X$ is open bounded and $(I - T)^{-1}(0) \subset V$. We consider the subsets $W_1 = \Omega_1 \cap W$ and $W_2 = \Omega_2 \cap W$, then W_1 and W_2 are relatively open, bounded and the fixed points of T (if there are any) in Ω_1 are in W_1 and the fixed points of T (if there are any) in Ω_2 are in W_2. So that $i_K(T, \Omega_1) = i_K(T, W_1)$ and $i_K(T, \Omega_2) = i_K(T, W_2)$. We observe that $Tx \neq x$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ implies $Tx \neq x$ for $x \in \overline{W} \setminus (W_1 \cup W_2)$ since $\overline{W} \setminus (W_1 \cup W_2) \subset \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$. By the additivity and excision properties of the Lan-Webb index

$$i_K(T, \Omega) = i_K(T, W) \subseteq i_K(T, W_1) + i_K(T, W_2).$$

The argument for equality is analogous to the proof of P3, Theorem 2.2.7.

P4. We show that $i_K(H(\lambda, x), \Omega) = i_K(H(\lambda, x), W)$ is independent of $\lambda \in [0,1]$. If $H(\lambda, x) \neq x$ for $x \in \partial_K \Omega$, $\lambda \in [0,1]$ then $H(\lambda, x) \neq x$ for $x \in \partial_K W$ since $(I - H_\lambda)^{-1}(0) \subset \Omega_K \cap V = W$ where $V \subset X$ is open bounded such that $(I - H_\lambda)^{-1}(0) \subset V$. Now $H(\lambda, x)$ is weakly inward on $\overline{W}$ since $\overline{W} \subset \overline{\Omega_K}$. Then by P4 of Theorem 4.1.13

$$i_K(H(0, x), W) = i_K(H(1, x), W).$$

Which proves the result. Q.E.D.

We now consider the $L - N$ case, assuming L to be unbounded and develop an index in a manner similar to that of Sections 2.4 and 3.5.

Let $K \subset X$ be a cone and $\Omega \subset X$ be open bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Let $L : \operatorname{dom} L \subset X \to Y$ be an unbounded Fredholm operator of index zero and $N : \overline{\Omega_K} \cap \operatorname{dom} L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0 relative to Γ_L. Define $K_1 = H(\operatorname{dom} L \cap K)$.

Definition 4.3.4 *Assume $Lx \neq Nx$ on $\operatorname{dom} L \cap \partial_K \Omega$ and $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_K}$. We define $\operatorname{ind}_K([L, N], \Omega) = i_{K_1}(T, U)$ where $U = H\Omega_K$ and the index is that of Definition 4.3.1.*

Remark 4.3.5 *Lemma 2.3.2 implies that $I - T$ is A-proper at 0 while Lemma 4.2.3 implies T is weakly inward on $\overline{U} = \overline{H\Omega_K}$ relative to K_1. Also, $Ty \neq y$ on ∂U since $(I - T)^{-1}(0) \subset U$ and U is open. Hence the index is well defined.*

Theorem 4.3.6 *Assume the conditions and notation of the preceding definition. Then the index thus defined has the following properties.*

P1. If $\operatorname{ind}_K([L, N], \Omega) = i_{K_1}(T, U) \neq \{0\}$, then there exists $x \in \Omega_K$ such that $Lx = Nx$.

P2. If $x_0 \in \Omega_K$, then $\operatorname{ind}_K([L, -J^{-1}P + \hat{y}_0], \Omega) = i_{K_1}(\hat{y}_0, U) = \{1\}$ where $Hx_0 = y_0$ and $\hat{y}_0(x) = y_0$ for every $x \in \Omega_K$.

P3. If Ω_1, Ω_2 are disjoint relatively open subsets of Ω_K such that $Lx \neq Nx$ for $x \in \overline{\Omega_K} \backslash (\Omega_1 \cup \Omega_2)$, then $\operatorname{ind}_K([L, N], \Omega) \subseteq \operatorname{ind}_K([L, N], \Omega_1) + \operatorname{ind}_K([L, N], \Omega_2)$ with equality if one of the indices on the right is a singleton.

P4. If $L - N(\lambda, x)$ is an A-proper at 0 homotopy for $\lambda \in [0, 1]$ and $N(\lambda, x) + J^{-1}Px \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_K}, \lambda \in [0, 1]$ and $0 \notin (L - N(\lambda, \cdot))(\operatorname{dom} L \cap \partial_K \Omega)$ for $\lambda \in [0, 1]$, then $\operatorname{ind}_K([L, N(\lambda, x)], \Omega)$ is independent of $\lambda \in [0, 1]$.

Proof. P1. If $i_{K_1}(T,U) \neq \{0\}$, then T has a fixed point in $U = H\Omega_K$ by P1 of Theorem 4.3.3, *i.e.*, $Ty = y$ for some $y \in U$ and $y = Hx$. This is equivalent to $Lx = Nx$ for $x = H^{-1}y$ and as $y \in H\Omega_K$, $x \in \Omega_K$.

P2. Since $y_0 \in U$, the result follows from P2, Theorem 4.3.3.

P3. We note first that the condition $Lx \neq Nx$ for $x \in \overline{\Omega_K} \setminus (\Omega_1 \cup \Omega_2)$ implies $Ty \neq y$ for $y \in \overline{U} \setminus (U_1 \cup U_2)$ where $U_i = H\Omega_i$, $i = 1,2$. Now since $U_1 \cap U_2 = \emptyset$, P3 of Theorem 4.3.3 implies

$$i_{K_1}(T,U) \subseteq i_{K_1}(T,U_1) + i_{K_1}(T,U_2)$$

which proves the result. The argument for the equality case is analogous to the proof of P3, Theorem 2.2.7.

P4. The first condition implies $I - T_\lambda$ is an A-proper at 0 homotopy by Lemma 2.3.2 with $T_\lambda = (N(\lambda,x) + J^{-1}P)H^{-1}$. The second condition implies T_λ is weakly inward on $\overline{U} = \overline{H\Omega_K}$ by Lemma 4.2.3. From $0 \notin (L - N(\lambda,\cdot))(\mathrm{dom}\, L \cap \partial_K\Omega)$ we have $T_\lambda y \neq y$ on ∂U for $\lambda \in [0,1]$. Then by the invariance under homotopy property, P4 of Theorem 4.3.3, we have

$$i_{K_1}(T_0,U) = i_{K_1}(T_1,U)$$

that is

$$\mathrm{ind}_K([L, N(0,x)], \Omega) = \mathrm{ind}_K([L, N(1,x)], \Omega).$$

Q.E.D.

Chapter 5

EXISTENCE THEOREMS FOR SEMILINEAR EQUATIONS IN CONES

5.1 Introduction

In this chapter we use our fixed point indices to extend the existence theorems established by Webb [54] for semilinear equations in cones and, at the same time, obtain simpler proofs. We also extend a fixed point result of Petryshyn [41] and obtain a continuation theorem similar to that of Mawhin [34] for A-proper maps which we shall apply in Chapter 6 to the solution of an ordinary differential equation.

In Section 5.3, we extend some of the results of Lafferriere and Petryshyn [28] obtained for P_γ-compact maps to semilinear maps of the form $L - N$.

The contents of Section 5.4 concern weakly inward A-proper maps. We apply the results of Chapter 4 to obtain existence theorems related to those of Lan and Webb [30] but applicable to semilinear operator equations acting on different spaces.

5.2 Some existence and continuation theorems

We begin with a theorem and several simple lemmas which we shall need in the sequel. Throughout this chapter the operator notation is that which was introduced in Section 1.4 and the index used in the proofs may be that of Chapters 2, 3 or 4 but in practice. the index of Chapter 2 would depend on the retraction and projection scheme employed.

Theorem 5.2.1 *If $L : \operatorname{dom} L \subset X \to Y$ is Fredholm of index zero, $K \subset X$ is a cone and $\Omega \subset X$ is an open bounded set such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$, then*

$$\operatorname{ind}_K\left(\left[L, -J^{-1}P\right], \Omega\right) = \begin{cases} \{1\} & \text{if } 0 \in \Omega_K, \\ \{0\} & \text{if } 0 \notin \Omega_K. \end{cases}$$

Proof. From the results of Chapter 2, $L + J^{-1}P = H$ is a linear bijection that maps $\operatorname{dom} L \cap K \subset X \to K_1 \subset Y$ and $L + J^{-1}P$ is A-proper since it is the sum of a linear Fredholm operator and a compact map. We note that $\ker(L + J^{-1}P) = \{0\}$ so that $Lx + J^{-1}Px \neq 0$ for $x \in \partial_K\Omega$. Thus $\operatorname{ind}_K([L, -J^{-1}P], \Omega)$ is well defined.

Now for $0 \in \Omega_K$, we have $\operatorname{ind}_K([L, -J^{-1}P], \Omega) = \operatorname{ind}_{K_1}\left(\hat{0}, H\Omega\right) = \{1\}$ by P2 (the normalisation property) of the index with $x_0 = 0$ and $\hat{y}_0 = \hat{0}$, *i.e.*, the mapping $\hat{0}(x) = 0$ for every $x \in \bar{\Omega}_K$.

If $0 \notin \Omega_K$ and $\operatorname{ind}_K([L, -J^{-1}P], \Omega) \neq \{0\}$, then P1 (the existence property) of the index implies there exists $x \in \Omega_K$ such that $(L + J^{-1}P)x = 0$ and $x \neq 0$. This contradicts $\ker(L + J^{-1}P) = \{0\}$, therefore $\operatorname{ind}_K([L, -J^{-1}P], \Omega) = \{0\}$. Q.E.D.

Many of the proofs to our theorems require certain homotopies to map cones to cones. The subsequent lemma and propositions provide conditions that ensure this.

Lemma 5.2.2 *If $P + JQN$ maps K to K, then $J^{-1}P + QN$ maps K to K_1.*

Proof. Let $x \in K$ and assume $(P + JQN)x = \tilde{x} \in K$. Since P and JQN map to $\ker L$, $\tilde{x} \in \ker L$ and $H\tilde{x} = (L + J^{-1}P)\tilde{x} = J^{-1}P\tilde{x} = J^{-1}\tilde{x} \in K_1$ as H maps K to K_1. Now $J^{-1}(P + JQN)x = J^{-1}\tilde{x} \in K_1$. Q.E.D.

The following propositions are consequences of the preceding lemma, Proposition 2.3.4 and the definition and properties of a cone K.

Proposition 5.2.3 *If* $(1-\lambda)(P+JQN)$ *maps* K *to* K *for* $\lambda \in [0,1]$, *then* $(1-\lambda)(J^{-1}P+QN)$ *maps* K *to* K_1 *for* $\lambda \in [0,1]$.

Proof. By Lemma 5.2.2, if $(P+JQN)x \in K$ then $(J^{-1}P+QN)x \in K_1$ for every $x \in K$ so $(1-\lambda)(P+JQN)x \in K$ and $(1-\lambda)(J^{-1}P+QN)x \in K_1$ for $\lambda \in [0,1]$ since K and K_1 are cones and $(1-\lambda) \geq 0$. Q.E.D.

Proposition 5.2.4 *If* $\lambda\left[L_1^{-1}(I-Q)N+P+JQN\right]$ *maps* K *to* K *for* $\lambda \in [0,1]$, *then* $\lambda(N+J^{-1}P)$ *maps* K *to* K_1 *for* $\lambda \in [0,1]$.

Proof. By Proposition 2.3.4, if $\left[L_1^{-1}(I-Q)N+P+JQN\right]x \in K$ then $(N+J^{-1}P)x \in K_1$ for every $x \in K$. Since K and K_1 are cones, $\lambda\left[L_1^{-1}(I-Q)N+P+JQN\right]x \in K$ and $\lambda(N+J^{-1}P)x \in K_1$ for every $x \in K$, $\lambda \geq 0$. Q.E.D.

Proposition 5.2.5 *If* $P+JQN$ *and* $L_1^{-1}(I-Q)N+P+JQN$ *map* K *to* K, *then* $(1-\lambda)(P+JQN)+\lambda\left[L_1^{-1}(I-Q)N+P+JQN\right]$ *maps* K *to* K *for* $\lambda \in [0,1]$ *and* $(1-\lambda)(J^{-1}P+QN)+\lambda(N+J^{-1}P)$ *maps* K *to* K_1 *for* $\lambda \in [0,1]$.

Proof. If $(P+JQN)x \in K$ and $\left[L_1^{-1}(I-Q)N+P+JQN\right]x \in K$ for every $x \in K$, then $(1-\lambda)(P+JQN)x+\lambda\left[L_1^{-1}(I-Q)N+P+JQN\right]x \in K$ for every $x \in K$ and $\lambda \in [0,1]$ since cones are closed under addition and non-negative scalar multiplication. Then by Propositions 5.2.3 and 5.2.4 and the aforementioned properties of a cone, $(1-\lambda)(J^{-1}P+QN)x+\lambda(N+J^{-1}P)x \in K_1$. Q.E.D.

The theorems and corollaries which follow extend those found in Webb [54] where the map $L-\lambda NR$ is assumed to be A-proper and a generalised degree is employed. We shall use our fixed point index instead of the degree to weaken the hypothesis $L-\lambda NR$

A-proper and require only $L - \lambda N$ A-proper. Thus we avoid any explicit mention of a retraction in the formulation of our theorems.

We begin with a continuation theorem related to that of Mawhin [34] and Petryshyn [42] for semilinear equations where we extend their results to cones. The theorem presupposes the existence of an open bounded set Ω which is often difficult to establish in practice. Corollary 5.2.9 below provides a means of determining such a set.

Theorem 5.2.6 *Suppose that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ with $N : \overline{\Omega_K} \to Y$ bounded and $0 \in \Omega_K$ where $\Omega \subset X$ is open bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Assume that:*

(a) $Lx \neq \lambda Nx$ for $x \in \partial_K\Omega$, $\lambda \in [0,1]$

(b) $QNx \neq 0$ for $x \in \partial_K\Omega \cap \ker L$

(c) $\operatorname{ind}_K([L, QN], \Omega) \neq \{0\}$

(d) $P + JQN + L_1^{-1}(I - Q)N$ and $P + JQN$ map K to K.

Then there exists $x \in K \cap \operatorname{dom} L$ such that $Lx = Nx$.

Proof. We show that the A-proper homotopy $H_1(\lambda, x) = Lx - (1-\lambda)QNx - \lambda Nx \neq 0$ on $\partial_K\Omega$ for $\lambda \in [0,1]$. To obtain a contradiction, assume there exists $\lambda_0 \in [0,1]$ and $x_0 \in \partial_K\Omega$ such that $H_1(\lambda_0, x_0) = 0$. By (a), $\lambda_0 \neq 1$. If $\lambda_0 = 0$, then $H_1(0, x_0) = Lx_0 - QNx_0 = 0$ so $Lx_0 = QNx_0$ and $QNx_0 \in \operatorname{im} L \cap Y_2$ so that $QNx_0 = 0$, which contradicts (b). Now if $\lambda_0 \in (0,1)$, then $Lx_0 - \lambda_0 Nx_0 = (1-\lambda_0)QNx_0 \neq 0$ by (a), hence $QNx_0 \neq 0$. Applying Q to this relation, we obtain $-\lambda_0 QNx_0 = (1-\lambda_0)QNx_0$ which is impossible for $0 < \lambda_0 < 1$, $QNx_0 \neq 0$. By the homotopy property of the index, we have $\operatorname{ind}_K([L, N], \Omega) = \operatorname{ind}_K([L, QN], \Omega) \neq \{0\}$ by (c). Then by P1 (the existence property) of the index, there exists $x \in \Omega_K$ such that $Lx = Nx$. Q.E.D.

Condition (c) of the preceding theorem assumed $\operatorname{ind}_K([L, QN], \Omega) \neq \{0\}$, in the following corollary we shall replace this condition with one that implies it employing a certain bilinear form. We assume that there is a continuous bilinear form $[y.x]$ on $Y \times X$ such that $y \in \operatorname{im} L$ iff $[y, x] = 0$ for each $x \in \ker L$. This condition implies that if $\{x_1, x_2, \ldots, x_n\}$ is a basis in $\ker L$, then the linear map $J : Y_0 \to \ker L$, $Y = Y_0 \oplus \operatorname{im} L$, defined by $Jy = \beta \sum_{i=1}^{n} [y, x_i] x_i$, $\beta \in \mathbb{R}^+$, is an isomorphism and that if $y = \sum_{i=1}^{n} y_i x_i$ then $[J^{-1}y, x_i] = y_i/\beta$ for $1 \leq i \leq n$. This idea has been used by Cesari [9], Mawhin [34], and Petryshyn [42]; our results extend those of the last two.

Corollary 5.2.7 *Assume all conditions of Theorem 5.2.6 hold except (c) and assume* *(c_1)* $[QNx, x] \leq 0$ *for every* $x \in \ker L \cap \partial_K \Omega$.
Then the same conclusion holds.

Proof. We show that (c_1) implies (c), *i.e.*, $\operatorname{ind}_K([L, QN], \Omega) \neq \{0\}$. Let $H_2(\lambda, x) = Lx + (1 - \lambda) J^{-1}Px - \lambda QNx$ and suppose $H_2(\lambda, x) = 0$ for some $x \in \partial_K \Omega$, $\lambda \in [0, 1]$. If $\lambda = 0$, then $Lx + J^{-1}Px = 0$ which is a contradiction since $\ker(L + J^{-1}P) = 0$ and $x \neq 0$ ($x \in \partial_K \Omega$). If $\lambda = 1$, then $Lx = QNx$ and, as in the proof to Theorem 5.2.6, contradicts (b). For $\lambda \in (0, 1)$, $Lx = \lambda QNx - (1 - \lambda) J^{-1}Px$ which implies $Lx = 0$ and $(1 - \lambda) J^{-1}Px = \lambda QNx$. Applying the bilinear form to this relation, we obtain $(1 - \lambda)[J^{-1}Px, x] = \lambda [QNx, x]$ which contradicts our hypotheses since $(1 - \lambda)[J^{-1}Px, x] > 0$ by the definition of the bilinear form and $\lambda [QNx, x] \leq 0$ by (c_1). Thus $\operatorname{ind}_K([L, QN], \Omega) = \operatorname{ind}_K([L, -J^{-1}P], \Omega) \neq \{0\}$ by Theorem 5.2.1. The conclusion now follows as all conditions of Theorem 5.2.6 are satisfied. Q.E.D.

Remark 5.2.8 *Corollary 5.2.7 remains valid if the bilinear form condition is replaced with* $[QNx, x] \geq 0$ *for every* $x \in \ker L \cap \partial_K \Omega$ *as* J *may be replaced with* $-J$ *in the proof and the same conclusion obtained.*

A second corollary to Theorem 5.2.6 extends a result of Petryshyn [42] and Webb [54], and imposes a sublinear growth condition on N. It also provides conditions for establishing *a priori* bounds on the solution set and thus determining Ω.

Corollary 5.2.9 *Suppose for N bounded, $L-\lambda N : \Omega_K \to Y$ is A-proper at 0 relative to Γ_L for $\lambda \in [0,1]$ and $P + JQN + L_1^{-1}(I-Q)N$ maps K to K. Suppose also that*

(a) there exist $a \geq 0, b > 0$ such that $x \in \Omega_K$ and $QNx = 0$ imply

$$\left\|L_1^{-1}(I-Q)Nx\right\| \leq a\|x\| + b,$$

(b) there exist $\mu \geq 0$ and $r > 0$ such that $x \in \overline{\Omega_K}$ and $QNx = 0$ imply

$$\|Px\| < \mu\|(I-P)x\| + r,$$

(c) $a(1+\mu) < 1$,

(d) $\operatorname{ind}_K([L, QN], B_s(0)) \neq \{0\}$ for $s = \dfrac{(\mu+1)b + r}{1 - a(\mu+1)}$.

Then there exists $x \in \operatorname{dom} L \cap \Omega_K$ such that $Lx = Nx$.

Proof. We show that hypotheses (a) through (d) imply those of Theorem 5.2.6 from which the result follows. Suppose $Lx = \lambda Nx$ on $\partial_K B_s(0)$ for some $\lambda \in [0,1]$. Then this is equivalent to $x_1 = \lambda L_1^{-1}(I-Q)Nx$ and $QNx = 0$ where $Px = x_0$ and $(I-P)x = x_1$. Now $\|x_1\| = \lambda\left\|L_1^{-1}(I-Q)Nx\right\| \leq a\|x\| + b$ and

$$\begin{aligned}
\|x\| &\leq \|x_0\| + \|x_1\| \\
&< \mu\|(I-P)x\| + r + \|x_1\| \\
&= \mu\|x_1\| + r + \|x_1\| \\
&= (\mu+1)\|x_1\| + r \\
&\leq (\mu+1)(a\|x\| + b) + r.
\end{aligned}$$

Therefore

$$\|x\| < a\mu\|x\| + b\mu + a\|x\| + b + r.$$

Hence

$$\|x\| < \frac{b(\mu+1) + r}{1 - a\mu - a} = s$$

and $Lx \neq \lambda Nx$ on $\partial_K B_s(0)$ for $\lambda \in [0,1]$. So (a) of Theorem 5.2.6 is satisfied with $B_s(0) \cap K = \Omega_K$.

Now if $x \in \ker L$ and $QNx = 0$ then by (b)

$$\|x\| = \|x_0\| < \mu \|(I-P)x\| + r = r < s$$

so that $QNx \neq 0$ on $\partial B_s(0) \cap \ker L$ and (b) of Theorem 5.2.6 is satisfied.

Also, it follows that $Lx \neq QNx$ for $x \in \partial B_s(0)$. Otherwise, $QNx \in \operatorname{im} L \cap Y_0$ implies $QNx = 0 = Lx$ so $x \in \ker L \cap \partial B_s(0)$ which contradicts what we have just proved above. Thus $\operatorname{ind}_K([L, QN], B_s(0))$ is well defined and does not equal 0 by hypothesis, so (c) of Theorem 5.2.6 is satisfied.

Hence all conditions of Theorem 5.2.6 hold and the conclusion of the corollary follows. Q.E.D.

The next theorem gives conditions for the existence of a positive solution to a semilinear equation and extends a result of Webb [54].

Theorem 5.2.10 *Under the hypotheses of Corollary 5.2.9, if also*
(e) there exists $0 \neq e \in L(K \cap \operatorname{dom} L)$ *and* $r < s$ *such that* $Lx - Nx \neq \mu e$ *for every* $x \in K$ *with* $\|x\| = r$ *and all* $\mu \geq 0$*, then there exists* $x \in K$, $r \leq \|x\| \leq s$*, and* $Lx = Nx$.

Proof. We note first that conditions (a) through (d) imply $\operatorname{ind}_K([L, N], B_s(0)) \neq \{0\}$ so that there exists $x \in B_s(0)$ with $Lx = Nx$. From the proof to Corollary 5.2.9, we have $Lx \neq Nx$ on $\partial B_s(0)$ and by (e), $Lx \neq Nx$ on $\partial B_r(0)$ so by the additivity property of the index,

$$\operatorname{ind}_K([L, N], B_s(0) \setminus B_r(0)) + \operatorname{ind}_K([L, N], B_r(0)) \supseteq \operatorname{ind}_K([L, N], B_s(0)).$$

We prove that $\operatorname{ind}_K([L, N], B_r(0)) = \{0\}$, which will show that equality holds in the above. Let $H(\mu, x) = Lx - Nx - \mu e$. By (e), $H(\mu, x) \neq 0$ on $\partial B_r(0)$ so that

$$\operatorname{ind}_K([L, N], B_r(0)) = \operatorname{ind}_K([L, N + \mu e], B_r(0)).$$

Now if $\operatorname{ind}_K([L, N + \mu e], B_r(0)) \neq \{0\}$, then there would be an $x_\mu \in B_r(0)$ such that $Lx_\mu = Nx_\mu + \mu e$ or $L\left(\frac{x_\mu}{\mu}\right) = Nx_\mu/\mu + e$. Now letting $\mu \to \infty$ and noting the boundedness of N, we have $L\left(\frac{x_\mu}{\mu}\right) \to e$. This contradicts the fact that $\frac{x_\mu}{\mu} \to 0$ and L is a closed linear operator. Hence,

$$\operatorname{ind}_K([L, N], B_s(0) \setminus B_r(0)) \neq \{0\}$$

and there exists $x \in B_s(0) \setminus B_r(0) \cap K$ such that $Lx = Nx$. Q.E.D.

Our next theorem follows Webb [54] in using an idea of Cañada and Ortega [7] to weaken the *a priori* bound requirement (condition (a)) of Corollary 5.2.9. We introduce a mapping $\gamma : X \to Y^*$ that satisfies $(Lx, \gamma x) \geq 0$ for all $x \in \operatorname{dom} L$ and let $\Lambda^+ = \{x \in K : (Nx, \gamma x) \geq 0\}$.

Theorem 5.2.11 *Under the hypotheses of Corollary 5.2.9 or Theorem 5.2.10 without conditions (a) and (b), for the equation $Lx = Nx$ to have a solution in K, it is necessary that $\Lambda^+ = \{x \in K : (Nx, \gamma x) \geq 0\} \neq \emptyset$ and it is sufficient that*

(a_1) $\Lambda^+ \neq \emptyset$ and $\left\|L_1^{-1}(I - Q)Nx\right\| \leq a\|x\| + b$, $x \in K \cap \Lambda^+$

(b_1) $x \in K \cap \Lambda^+$ and $QNx = 0$ imply $\|Px\| \leq \mu\|(I - P)x\| + r$.

Proof. Necessity is a consequence of the following: if x is a solution to $Lx = Nx$, then $(Lx, \gamma x) = (Nx, \gamma x)$. For sufficiency the proof is similar to that of Corollary 5.2.9 in that if $H(\lambda, x) = L - (1 - \lambda)QN - \lambda N = L - N(\lambda, x) = 0$ for $\|x\| = s$ then $QNx = 0$ so that $Lx = \lambda Nx$ and $x \in \Lambda^+$. The rest of the proof is the same as that of Corollary 5.2.9, *q.v.* for particulars. Q.E.D.

The last result of this type from Webb [54] requires a certain homotopy to map K to K. This theorem is related to one proved by Mawhin [34] using coincidence degree where N is assumed to be compact. Santanilla [49] obtained a similar result on cones also using coincidence degree while Petryshyn proved a version for A-proper maps in [42] using the A-proper degree.

Theorem 5.2.12 *Suppose that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ where L is a linear Fredholm operator of index zero and N is nonlinear. Assume the following conditions hold:*

(a_1) there exist $a \geq 0$, $b > 0$ such that $\|Nx\| \leq a\|x\| + b$ for every $x \in K$ where $ac < 1$, $c = \left\|L_1^{-1}(I-Q)\right\|$

(b_1) there exists $r > 0$ such that for $M = \frac{arc+bc}{1-ac}$ we have $[QNx, Px] \leq 0$ for every $x \in K$ with $\|Px\| = r$, $\|(I-P)x\| \leq M$

(c_1) $L_1^{-1}(I-Q)N$ and $P + JQN$ map K to K.

Then there exists $x \in K$ such that $Lx = Nx$.

Proof. Let $\Omega = \{x \in K : \|Px\| < r,\ \|(I-P)x\| < M\}$ and define $H : [0,1] \times \overline{\Omega} \cap \operatorname{dom} L \to Y$ by

$$H(\lambda, x) = L + (1-\lambda) J^{-1}P - \lambda N = L - N(\lambda, x).$$

We prove $H(\lambda, x) \neq 0$ on $\partial\Omega \cap \operatorname{dom} L$ for $\lambda \in [0,1]$. If not, then $H(\lambda, x) = 0$ for some $x \in \partial\Omega \cap \operatorname{dom} L$, $\lambda \in [0,1]$, and

$$(I-P)x = x_1 = \lambda L_1^{-1}(I-Q)Nx$$

and

$$Px = x_0 = \lambda(P + JQN)x$$

so

$$x = x_0 + x_1 = \lambda(P + JQNx) + \lambda L_1^{-1}(I-Q)Nx.$$

Thus, by (c_1), we have $x \in K$.

Now applying the bilinear form to the relation

$$(1-\lambda)x_0 = \lambda JQNx$$

we have

$$(1-\lambda)\left[J^{-1}x_0, x_0\right] = \lambda[QNx, x_0]$$

obtaining by (b_1) a contradiction. Hence

$$\|Px\| = \|x_0\| < r$$

and $\|x_1\| = M$ since $x \in \partial\Omega$. By (a_1),

$$\begin{aligned} M &= \|x_1\| \leq \left\|L_1^{-1}(I-Q)Nx\right\| \\ M &\leq ac\|x\| + bc < ac(r+M) + bc \end{aligned}$$

which contradicts the definition of M. Thus $H(\lambda, x) = L - N(\lambda, x) \neq 0$ on $\partial\Omega \cap \operatorname{dom} L$ so that

$$\begin{aligned} \operatorname{ind}_K([L, N(1,x)], \Omega) &= \operatorname{ind}_K([L, N(0,x)], \Omega) \\ &= \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], \Omega\right) \neq \{0\} \end{aligned}$$

by Theorem 5.2.1. Since the index is non-zero, there exists $x \in K$ such that $Lx = Nx$. Q.E.D.

We conclude this section with a theorem that extends a result of Petryshyn [41] to semilinear maps.

Theorem 5.2.13 *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $N : X \to Y$ a bounded nonlinear map such that $L - \lambda N$ is A-proper for $0 \leq \lambda < 1$. Let $r_1, r_2 \in (0, \infty)$ and $L_1^{-1}(I-Q)N + P + JQN : K \to K$. Assume the following conditions hold:*

(a) there exists a compact map $C : \overline{B_{r_1}(0)} \cap K \to K_1$ such that $Lx \neq \mu Cx - J^{-1}Px$ for $\mu \in [0,1]$, $x \in \partial B_{r_1}(0) \cap K$ and $Lx \neq \lambda Nx + (1-\lambda)[Cx - J^{-1}Px]$ for $\lambda \in [0,1]$, $x \in \partial B_{r_1}(0) \cap K$

(b) there exists a bounded map $F : \overline{B_{r_2}(0)} \cap K \to K_1$, $\alpha > 0$ such that $\|Fx\| \geq \alpha > 0$ if $x \in \partial B_{r_2}(0) \cap K$ and $Lx - Nx - \lambda Fx$ is A-proper for every $\lambda \geq 0$ and $Lx \neq Nx + \mu Fx$ for $x \in \partial B_{r_2}(0) \cap K$, $\mu \geq 0$.

Then there exists $x \in K \cap \operatorname{dom} L$ *such that* $\min\{r_1, r_2\} \le \|x\| \le \max\{r_1. r_2\}$ *and* $Lx = Nx$.

Proof. From the second condition of (a), we have $\operatorname{ind}_K([L, N(\lambda, x)], B_{r_1}(0))$ is constant for $\lambda \in [0,1]$ where $N(\lambda, x) = \lambda Nx - (1-\lambda)[Cx - J^{-1}Px]$ so that

$$\operatorname{ind}_K([L,N], B_{r_1}(0)) = \operatorname{ind}_K\left(\left[L, C - J^{-1}P\right], B_{r_1}(0)\right).$$

Now by the first condition of (a), we have

$$\operatorname{ind}_K\left(\left[L, C - J^{-1}P\right], B_{r_1}(0)\right) = \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], B_{r_1}(0)\right) = \{1\}$$

by Theorem 5.2.1. We observe that the homotopy $H(\lambda, x) = L - \lambda N + (1-\lambda)[C - J^{-1}P]$ is A-proper for $\lambda \in [0,1]$ and show that $\operatorname{ind}_K([L,N], B_{r_2}(0)) = \{0\}$. We fix $y \in K_1$ with $\|y\| = 1$ and choose $\mu_0 \in (0, \infty)$ such that $Lx \neq Nx + \mu_0 Fx + my$ for $x \in \partial B_{r_2}(0) \cap K$, for every $m \in \mathbb{Z}^+$. It is possible to choose such a μ_0 since, if not, we would have $Lx = Nx + \mu Fx + my$ on $\partial B_{r_2}(0) \cap K$, $m \ge 0$, and there would exist sequences $\{\mu_n\} > 0$, $\mu_n \to \infty$, $\{m_n\} \ge 0$ and $\{x_n\} \subset \partial B_{r_2}(0) \cap K$ such that $Lx_n = Nx_n + \mu_n Fx_n + m_n y$. Dividing by μ_n, we have $L\left(\frac{x_n}{\mu_n}\right) = Nx_n/\mu_n + Fx_n + m_n y/\mu_n$ and as $\mu_n \to \infty$, $L\left(\frac{x_n}{\mu_n}\right) \to 0$, $Nx_n/\mu_n \to 0$, so that $Fx_n + m_n y/\mu_n \to 0$. Since $\{Fx_n\}$ is bounded, we may assume $m_n/\mu_n \to m_0 \in [0, \infty)$. By (b), $m_0 > 0$. Hence $Fx_n \to -m_0 y$. Since $y \in K_1$ implies $-m_0 y \in -K_1$ we have $-m_0 y \in \overline{F(\partial B_{r_2}(0) \cap K)} \cap (-K_1)$. As $F(\partial B_{r_2}(0) \cap K) \subset K_1$ and $K_1 \cap (-K_1) = \{0\}$, we obtain a contradiction.

We define the A-proper homotopy $H_{\mu_0}(\lambda, x) = Lx - Nx - \mu_0 Fx - \lambda my$ and note that $H_{\mu_0} \neq 0$ on $\partial B_{r_2}(0) \cap K$. By the homotopy property of the index, we have

$$\operatorname{ind}_K([L, N + \mu_0 F + \lambda my], B_{r_2}(0)) = \operatorname{ind}_K([L, N + \mu_0 F], B_{r_2}(0)).$$

Now if

$$\operatorname{ind}_K([L, N + \mu_0 F + \lambda my], B_{r_2}(0)) \neq \{0\},$$

then there exists $x \in B_{r_2}(0)$ such that $Lx = Nx + \mu_0 Fx + \lambda m y$ or $L\left(\frac{x}{m}\right) = Nx/m + \mu_0 Fx/m + \lambda y$. By the boundedness of N and F, we have $L\left(\frac{x}{m}\right) \to \lambda y \neq 0$ as $m \to \infty$. This contradicts the fact that $\frac{x}{m} \to 0$ and L is a closed linear operator. Thus

$$\mathrm{ind}_K\left(\left[L, N + \mu F\right], B_{r_2}(0)\right) = \{0\}.$$

By (b), we have $\mathrm{ind}_K\left(\left[L, N + \mu F\right], B_{r_2}(0)\right)$ is constant for every $\mu \geq 0$ and consequently,

$$\mathrm{ind}_K\left(\left[L, N + \mu F\right], B_{r_2}(0)\right) = \mathrm{ind}_K\left(\left[L, N\right], B_{r_2}(0)\right) = \{0\}.$$

By the additivity property of the index, we obtain

$$\mathrm{ind}_K\left(\left[L, N\right], B_{r_2}(0) \setminus \overline{B_{r_1}(0)}\right) = \mathrm{ind}_K\left(\left[L, N\right], B_{r_2}(0)\right) - \mathrm{ind}_K\left(\left[L, N\right], B_{r_1}(0)\right) \neq \{0\}.$$

Hence there exists $x \in K \cap \mathrm{dom}\, L$ with $\min\{r_1, r_2\} \leq \|x\| \leq \max\{r_1, r_2\}$ such that $Lx = Nx$. Q.E.D.

5.3 Existence theorems on quasinormal cones

The results of this section are established using the notion of a quasinormal cone introduced by Petryshyn [43]. In particular, this includes all normal cones. Many of the subsequent theorems and lemmas extend those of Lafferriere and Petryshyn [28] which pertained to P_γ compact maps, *q.v.* Chapter 1.

Definition 5.3.1 *A cone K is called quasinormal if there exist $y \in K \setminus \{0\}$ and a constant $\sigma > 0$ such that $\|x + y\| \geq \sigma \|x\|$ for all $x \in K$. Let*

$$\sigma(y) := \inf\left\{\|x + y\| / \|x\| : x \in K \setminus \{0\}\right\}.$$

Then we define the constant of quasinormality $\sigma(K) := \sup\{\sigma(y) : y \in K \setminus \{0\}\}$.

Remark 5.3.2 *Note that if K is normal then K is quasinormal with $\sigma = \gamma^{-1} \in (0.1]$ and any $y \in K \setminus \{0\}$. If X is a Hilbert space, then every cone $K \subset X$ is quasinormal with $\sigma = 1$ as shown in [29]. Also from this paper; if K is a cone in a Banach space X and $y \in K \setminus \{0\}$, then there exists $\sigma(y) \in (0,1]$ such that $\|x+y\| \geq \sigma(y)\|x\|$ for all $x \in K$, this implies every cone in a Banach space is quasinormal. It was proved in [11] that $\frac{1}{2} \leq \sigma(K) \leq 1$.*

Some examples of quasinormal cones which aren't normal are the set of non-negative functions in $C^k(\overline{Q})$, $k \geq 1$; the Hölder space $C^{\alpha}(\overline{Q})$ for $\alpha \in (0,1)$; and the Sobolev space $W^{k,p}(Q)$ for $p \in [1,\infty)$. As noted in [43], these cones have $\sigma = 1$ with $y(x) \equiv 1$, $x \in \overline{Q}$ and Q a bounded set in $\mathbb{R}^n$.

We begin with a definition and a theorem which we shall require later.

Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be a bounded nonlinear operator such that $L - N$ is A-proper relative to Γ_L.

Definition 5.3.3 *If $S = \{x \in K : Lx = Nx\}$ is bounded, we define* $\operatorname{ind}_K([L,N],K) = \operatorname{ind}_K([L,N], B_r(0) \cap K)$ *where $r > M$ and M is any bound for the set S.*

Remark 5.3.4 *Since all solutions to $Lx = Nx$ are contained in $\overline{B_M(0)}$, the* $\operatorname{ind}_K([L,N], B_r(0) \cap K)$ *is constant for all $r > M$ by the additivity property and hence the definition is independent of $r > M$. It is also clear that $Lx \neq Nx$ on $\partial B_r(0) \cap K$ so that the index is well defined.*

Theorem 5.3.5 *Suppose that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ where $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero and $N : X \to Y$ is bounded and nonlinear. If $L_1^{-1}(I-Q)N$ and $P + JQN$ map K to K and the set*

$$S = \left\{x \in K : Lx = \lambda Nx - (1-\lambda) J^{-1}Px, \lambda \in [0,1]\right\}$$

is bounded, then $\operatorname{ind}_K([L,N],K) = \{1\}$.

Proof. Let r be a bound for S. Consider the homotopy $L - \lambda N + (1-\lambda) J^{-1}P = L - N(\lambda, x)$ on $\overline{B_{r+\epsilon}}(0), \epsilon > 0$. Now $Lx \neq \lambda Nx - (1-\lambda) J^{-1}Px$ on $\partial B_{r+\epsilon}(0)$ so

$$\begin{aligned} \operatorname{ind}_K([L, N(1,x)], B_{r+\epsilon}(0)) &= \operatorname{ind}_K([L, N(0,x)], B_{r+\epsilon}(0)) \\ &= \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], B_{r+\epsilon}(0)\right) = \{1\}. \end{aligned}$$

Since ϵ is arbitrary, we have $\operatorname{ind}_K([L, N], K) = \{1\}$. Q.E.D.

The following lemma gives conditions for the index to be zero and will be employed in later arguments.

Lemma 5.3.6 *Let $0 \in \Omega \subset X$, Ω open and bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$ and let $L - \lambda N$ be A-proper for $1 \leq \lambda \leq \mu_0$ where $\mu_0 \sigma > 1$ and σ is the constant of quasinormality of K. Assume $L_1^{-1}(I-Q)N + P + JQN$ maps K to K and that the following conditions hold:*
(a) $\delta = \inf\left\{\left\|L_1^{-1}(I-Q)Nx + Px + JQNx\right\| : x \in \partial_K \Omega\right\} > d/\mu_0\sigma$ where
$d = \sup\{\|x\| : x \in \partial_K \Omega\}$
(b) $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ for $x \in \partial_K \Omega$ and $1 \leq \mu \leq \mu_0 \sigma$.
Then $\operatorname{ind}_K([L, N], \Omega) = \{0\}$.

Proof. Suppose $Lx = \mu Nx - (1-\mu) J^{-1}Px$ for some $x \in \partial_K \Omega$ and $1 \leq \mu \leq \mu_0$, then we obtain $x_0 = \mu(P + JQN)x$ and $x_1 = \mu L_1^{-1}(I-Q)Nx$ and

$$\begin{aligned} d &\geq \|x\| = \|x_0 + x_1\| \\ &= \mu\left\|L_1^{-1}(I-Q)Nx + (P + JQN)x\right\| \\ &\geq \mu\delta > \mu d/\mu_0\sigma \end{aligned}$$

so $\mu < \mu_0\sigma$. This and (b) imply $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ for $x \in \partial_K \Omega$ and $1 \leq \mu \leq \mu_0$. For $\mu_0 > 1/\sigma$, define the homotopy $L - N(\mu, x) = Lx - \mu Nx + (1-\mu) J^{-1}Px$ which is A-proper for $1 \leq \mu \leq \mu_0$ and $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ for $x \in \partial\Omega_K$ and $\mu \geq 1$.

Thus, by the homotopy property of the index, $\text{ind}_K([L, N(\mu, x)], \Omega)$ is well defined and independent of $\mu \geq 1$.

To prove $\text{ind}_K([L, N], \Omega) = \{0\}$, we choose $\epsilon \in (0, \sigma)$ such that $\delta > \frac{d}{\mu_0(\sigma-\epsilon)}$. Then by definition of σ, there exists $y \in K \setminus \{0\}$ such that $\sigma_y > \sigma - \epsilon$ and $\|x + y\| \geq \sigma_y \|x\| > (\sigma - \epsilon)\|x\|$ for every $x \in K \setminus \{0\}$. We note that for every $x \in K \setminus \{0\}$ and $p > 0$,

$$\begin{aligned} \|x + py\| &= \left\| p\left(\frac{x}{p} + y\right)\right\| \\ &= p\left\|\frac{x}{p} + y\right\| \\ &\geq p\sigma_y \left\|\frac{x}{p}\right\| = \sigma_y \|x\| > (\sigma - \epsilon)\|x\| \end{aligned}$$

so that $\|x + py\| > (\sigma - \epsilon)\|x\|$. We now consider the homotopy $H_{\mu_0}(\lambda, x) : [0, 1] \times \overline{\Omega_K} \to Y$ defined by $H_{\mu_0}(\lambda, x) = Lx - \mu_0 Nx + (1 - \mu_0) J^{-1}Px - \lambda m y_1$ where $y_1 = Ly$ (y as determined above) and any fixed $m \in \mathbb{N}$. We observe that $H_{\mu_0}(\lambda, x)$ is A-proper and $H_{\mu_0}(\lambda, x) \neq 0$ for $x \in \partial_K\Omega$, $\lambda \in [0, 1]$. If this were not true, then we would have $Lx = \mu_0 Nx - (1 - \mu_0) J^{-1}Px + \lambda m y_1$ for some $x \in \partial_K\Omega$ and $\lambda \in (0, 1]$ ($\lambda = 0$ is excluded by the previous argument).

Hence $x_0 = \mu_0(P + JQN)x$ and $x_1 = \mu_0 L_1^{-1}(I - Q)Nx + \lambda m L_1^{-1} y_1$ so that

$$\begin{aligned} d &\geq \|x\| = \|x_0 + x_1\| \\ &= \left\|\mu_0(P + JQN)x + \mu_0 L_1^{-1}(I - Q)Nx + \lambda m L_1^{-1} y_1\right\| \\ &\geq \mu_0(\sigma - \epsilon)\left\|(P + JQN)x + L_1^{-1}(I - Q)Nx\right\| \\ &\geq \mu_0(\sigma - \epsilon)\delta > \mu_0(\sigma - \epsilon)\left(\frac{d}{\mu_0(\sigma - \epsilon)}\right) = d, \end{aligned}$$

a contradiction. Hence $Lx \neq \mu_0 Nx - (1 - \mu_0) J^{-1}Px + \lambda m y_1$ on $\partial_K\Omega$ and $\lambda \in [0, 1]$. By the homotopy property of the index,

$$\text{ind}_K([L, N_{\mu_0}(0, x)], \Omega) = \text{ind}_K([L, N_{\mu_0}(1, x)], \Omega)$$

where $N_{\mu_0}(\lambda, x) = \mu_0 Nx - (1-\mu_0) J^{-1}Px + \lambda m y_1$. Now if $\operatorname{ind}_K([L, N_{\mu_0}(\lambda. x)].\Omega) \neq \{0\}$. then there exists $x_m \in \partial_K \Omega$ such that

$$Lx_m = \mu_0 N x_m - (1-\mu_0) J^{-1} P x_m + \lambda m y_1$$

or

$$L\left(\frac{x_m}{m}\right) = \frac{\mu_0}{m} N x_m - \frac{(1-\mu_0)}{m} J^{-1} P x_m + \lambda y_1.$$

Allowing $m \to \infty$ and using the boundedness of N and compactness of $J^{-1}P$, we have $L\left(\frac{x_m}{m}\right) \to \lambda y_1 \neq 0$. This contradicts the fact that $x_m/m \to 0$ and L is a closed linear operator. Thus we have shown that

$$\begin{aligned} \operatorname{ind}_K([L, N_{\mu_0}(\lambda, x)], \Omega) &= \operatorname{ind}_K([L, N(\mu, x)], \Omega) \\ &= \operatorname{ind}_K([L, N], \Omega) = \{0\}. \end{aligned}$$

Q.E.D.

We shall need the following lemma in the proof to Theorem 5.3.8.

Lemma 5.3.7 *Let $0 \in \Omega \subset X$, Ω open and bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$ and let $L - \lambda N$ be A-proper for $\lambda \in [0,1]$. Assume N is bounded and $L_1^{-1}(I-Q)N + P + JQN$ maps K to K. If $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ on $\partial_K \Omega$ for $\mu \in [0,1]$, then* $\operatorname{ind}_K([L, N], \Omega) = \{1\}$.

Proof. We consider the homotopy $L - N(\mu, x) = L - \mu N + (1-\mu) J^{-1}P$ on Ω_K for $\mu \in [0,1]$. Since $Lx \neq N(\mu, x)$ on $\partial_K \Omega$, $\mu \in [0,1]$, we have

$$\begin{aligned} \operatorname{ind}_K([L, N(1,x)], \Omega) &= \operatorname{ind}_K([L, N(0,x)], \Omega) \\ &= \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], \Omega\right) = \{1\} \end{aligned}$$

by Theorem 5.2.1. Q.E.D.

The next theorem gives conditions for the existence of a positive solution to a semi-linear equation.

Theorem 5.3.8 *Let $0 \in \overline{\Omega_1} \subset \Omega_2 \subset K$ where Ω_1 and Ω_2 are open and bounded. Let $L - \lambda N$ be A-proper for $0 \leq \lambda \leq \mu_0$ where $\mu_0 \sigma > 1$, N be bounded and $L_1^{-1}(I-Q)N + P + JQN$ map K to K. Assume the following conditions hold:*

(a) $\delta = \inf\left\{\left\|L_1^{-1}(I-Q)Nx + (P+JQN)x\right\| : x \in \partial\Omega_2\right\} > d/\mu_0\sigma$ where $d = \sup\{\|x\| : x \in \partial\Omega_2\}$

(b) $Lx \neq \mu Nx - (1-\mu)J^{-1}Px$ for $x \in \partial\Omega_2$ and $1 \leq \mu \leq \mu_0\sigma$

(c) $Lx \neq \mu Nx - (1-\mu)J^{-1}Px$ for $x \in \partial\Omega_1$ and $\mu \in [0,1]$.

Then there exists $x \in \overline{\Omega_2} \setminus \Omega_1$ such that $Lx = Nx$.

Proof. We assume $Lx \neq Nx$ on $\partial\Omega_1 \cup \partial\Omega_2$ otherwise the conclusion follows. From (c), we have $\text{ind}_K([L,N],\Omega_1) = \{1\}$. From Lemma 5.3.6, conditions (a) and (b) imply $\text{ind}_K([L,N],\Omega_2) = \{0\}$. By the additivity property of the index, we obtain

$$\begin{aligned} \text{ind}_K([L,N],\Omega_2 \setminus \Omega_1) &= \text{ind}_K([L,N],\Omega_2) - \text{ind}_K([L,N],\Omega_1) \\ &= \{0\} - \{1\} = \{-1\}. \end{aligned}$$

Since the index is non-zero, the existence property (P1) of the index implies there exists $x \in \overline{\Omega_2} \setminus \Omega_1$ such that $Lx = Nx$. Q.E.D.

Remark 5.3.9 *The conclusion of Theorem 5.3.8 is valid if conditions (a) and (b) hold on $\partial\Omega_1$ and (c) holds on $\partial\Omega_2$.*

We now establish several results of cone compression or expansion type which provide conditions for the existence of positive solutions. We shall require the following lemma in the proof to the succeeding theorem.

Lemma 5.3.10 *Let $0 \in \Omega \subset X$, Ω open and bounded such that $\Omega_K \cap \text{dom}\, L \neq \emptyset$ and let $L - \lambda N$ be A-proper for $1 \leq \lambda \leq \mu_0$ where $\mu_0\sigma > 1$ and N bounded. Suppose that $\left\|L_1^{-1}(I-Q)Nx + (P+JQN)x\right\| \geq \|x\|$ and $Lx \neq Nx$ on $\partial\Omega_K$. Assume $L_1^{-1}(I-Q)N + P + JQN$ maps K to K. Then $\text{ind}_K([L,N],\Omega) = \{0\}$.*

Proof. We note that if $Lx = \mu Nx - (1-\mu) J^{-1}Px$ for some $x \in \partial_K \Omega$, $\mu > 0$, then $x_0 = \mu (P + JQN) x$, $x_1 = \mu L_1^{-1} (I - Q) Nx$ and

$$\begin{aligned} \|x\| &= \|x_0 + x_1\| \\ &= \mu \left\| L_1^{-1} (I-Q) Nx + (P + JQN) x \right\| \\ &\geq \mu \|x\| \end{aligned}$$

by hypothesis. Therefore $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ for $x \in \partial_K \Omega$ and $\mu \geq 1$. Now as in the proof to Lemma 5.3.6, choose $\epsilon \in (0, \sigma)$ such that $\delta > \frac{d}{\mu_0(\sigma - \epsilon)}$ and $y \in K \setminus \{0\}$ such that $\sigma_y > \sigma - \epsilon$ and consider the homotopy $H_{\mu_0}(\lambda, x) : [0,1] \times \overline{\Omega}_K \to Y$ defined by $H_{\mu_0}(\lambda, x) = Lx - \mu_0 Nx + (1-\mu_0) J^{-1}Px - \lambda m y_1$ where $y_1 = Ly$ and $m \in \mathbb{N}$ is arbitrary. Then following verbatim the proof to Lemma 5.3.6, we obtain $\text{ind}_K ([L, N_{\mu_0}(\lambda, x)], \Omega) = \{0\}$. Finally, since $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ on $\partial \Omega_K$, $\mu \geq 1$, we have

$$\text{ind}_K ([L, N(\mu, x)], \Omega) = \text{ind}_K ([L, N], \Omega) = \{0\}$$

where $N(\mu, x) = \mu Nx - (1-\mu) J^{-1}Px$ and $L - N$ corresponds to $\mu = 1$. Q.E.D.

Theorem 5.3.11 *Let $0 \in \overline{\Omega_1} \subset \Omega_2 \subset K$, Ω_1 and Ω_2 open and bounded and let $L - \lambda N$ be A-proper for $0 \leq \lambda \leq \mu_0$ where $\mu_0 \sigma > 1$ and N bounded. Suppose that $L_1^{-1}(I-Q)N + P + JQN$ maps K to K and the following conditions hold:*
(a) $\left\| L_1^{-1}(I-Q)Nx + (P + JQN)x \right\| \geq \|x\|$ for $x \in \partial\Omega_1$;
(b) $Lx \neq \mu Nx - (1-\mu) J^{-1}Px$ for $x \in \partial\Omega_2$ and $\mu \in [0,1]$.
Then there exists $x \in \Omega_2 \setminus \Omega_1$ such that $Lx = Nx$.

Proof. We assume $Lx \neq Nx$ on $\partial\Omega_1$ otherwise we are done. From (b) and Lemma 5.3.7 we have $\text{ind}_K ([L, N], \Omega_2) = \{1\}$. From (a) and Lemma 5.3.10 we have $\text{ind}_K ([L, N], \Omega_1) = \{0\}$. By the additivity property of the index, we obtain

$$\begin{aligned} \text{ind}_K \left([L, N], \Omega_2 \setminus \overline{\Omega_1}\right) &= \text{ind}_K ([L, N], \Omega_2) - \text{ind}_K ([L, N], \Omega_1) \\ &= \{1\} - \{0\} = \{1\}. \end{aligned}$$

Since the index is non-zero, there exists $x \in \Omega_2 \setminus \overline{\Omega_1}$ such that $Lx = Nx$. Q.E.D.

Remark 5.3.12 *The conclusion of Theorem 5.3.11 is valid if (a) holds on $\partial\Omega_2$ and (b) holds on $\partial\Omega_1$.*

We end this section with a theorem related to norm type expansions of a cone.

Theorem 5.3.13 *Let $L - \lambda N$ be A-proper for $0 \leq \lambda \leq \mu_0$ where $\mu_0\sigma > 1$ and suppose that $L_1^{-1}(I-Q)N + P + JQN$ maps K to K. Assume that the following conditions hold:*
(a) $\left\|L_1^{-1}(I-Q)Nx + (P+JQN)x\right\| \geq \|x\|$ on $\partial_K B_r(0)$
(b) there exists $R > r$ such that $\left\|L_1^{-1}(I-Q)Nx + (P+JQN)x\right\| < \|x\|$ on $\partial_K B_R(0)$.
Then there exists $x \in (B_R(0) \setminus B_r(0)) \cap K$ such that $Lx = Nx$.

Proof. Applying Theorem 5.3.11, let $\Omega_1 = B_r(0) \cap K$ and $\Omega_2 = B_R(0) \cap K$, then (a) of Theorem 5.3.13 implies (a) of Theorem 5.3.11 on $\partial_K B_r(0)$ while (b) of Theorem 5.3.13 implies (b) of Theorem 5.3.11 on $\partial_K B_R(0)$. Otherwise, if $Lx = \mu Nx - (1-\mu)J^{-1}Px$ on $\partial_K B_R$ for $\mu \in [0,1]$, then $x_0 = \mu(P+JQN)x$, $x_1 = \mu L_1^{-1}(I-Q)Nx$ and

$$\begin{aligned}
\|x\| &= \|x_0 + x_1\| \\
&= \mu\left\|(P+JQN)x + L_1^{-1}(I-Q)Nx\right\| \\
&\leq \left\|(P+JQN)x + L_1^{-1}(I-Q)Nx\right\|
\end{aligned}$$

which contradicts condition (b) of the theorem. Q.E.D.

5.4 Existence theorems for weakly inward A-proper maps

The results of this section extend several theorems of Lan and Webb [30] to semilinear operators on cones. Since many of the subsequent proofs involve homotopies that are both A-proper and weakly inward, we shall make some preliminary observations in the form of propositions regarding weakly inward sets and maps. The proofs are immediate consequences of the definition and properties of a weakly inward set.

Proposition 5.4.1 *If* $t(N+J^{-1}P)x+(1-t)\hat{y}_0(x) \in \overline{I_{K_1}}(Hx)$ *for every* $x \in \overline{\Omega_K}$, $t \in [0,1]$, *where* $y_0 \in K_1$ *and* $\hat{y}_0(x) = y_0$ *for* $x \in K$ *then* $tTy + (1-t)\hat{y}_0(y) \in \overline{I_{K_1}}(y)$.

Proposition 5.4.2 *If* $(N+J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ *for every* $x \in \overline{\Omega_K}$ *and* $e \in K_1$, $t, \lambda \geq 0$, *then* $(N+J^{-1}P)x + t\lambda e \in \overline{I_{K_1}}(Hx)$ *and* $Ty + t\lambda e \in \overline{I_{K_1}}(y)$.

Proposition 5.4.3 *If* $(N+J^{-1}P)x + t\lambda_0 Fx \in \overline{I_{K_1}}(Hx)$ *for every* $x \in \overline{\Omega_K}$, $t, \lambda_0 \geq 0$ *and map* F, *then* $Ty + t\lambda_0 FH^{-1}y \in \overline{I_{K_1}}(y)$ *where* $H^{-1}y = x$.

Our first theorem employs a variation of the Leray-Schauder boundary condition to prove the existence of a solution to the equation $Lx = Nx$.

Theorem 5.4.4 *Let* $\Omega \subset X$ *be open bounded,* K *a cone in* X *such that* $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. *Assume* L, $N : \overline{\Omega_K} \to Y$ *are bounded such that* $L - \lambda N$ *is A-proper for* $\lambda \in [0,1]$ *and* $(N+J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ *for every* $x \in \overline{\Omega_K}$, $\lambda \in [0,1]$. *Suppose there exists* $y_0 \in H\Omega_K$ *such that* $Lx \neq \lambda Nx + (1-\lambda)[y_0 - J^{-1}Px]$ *for* $x \in \partial_K \Omega$, $\lambda \in [0,1]$. *Then* $Lx = Nx$ *for some* $x \in \Omega_K$.

Proof. Let $H(\lambda, x) = Lx - \lambda Nx - (1-\lambda)[y_0 - J^{-1}Px] = L - N(\lambda, x)$. Then $H(\lambda, x) \neq 0$ on $\partial_K \Omega$ and $H(\lambda, x)$ is weakly inward because $N(\lambda, x) + J^{-1}Px =$ $\lambda(Nx + J^{-1}Px) + (1-\lambda)y_0$ where $y_0 \in K_1$ and $\overline{I_{K_1}}(Hx)$ is a wedge containing K_1. By the homotopy property of the index, we have

$$\operatorname{ind}_K([L,N],\Omega) = \operatorname{ind}_K\left(\left[L, -J^{-1}P + y_0\right], \Omega\right) = \{1\},$$

by P2 of Theorem 4.2.6. Then P1 implies the existence of $x \in \Omega_K$ such that $Lx = Nx$. Q.E.D.

The next theorem gives conditions which imply that the index is 0.

Theorem 5.4.5 *Let $\Omega \subset X$ be open bounded and such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Assume $L, N : \overline{\Omega_K} \to Y$ are bounded such that $L - N$ is A-proper and $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_K}$. Suppose*

(E) there exists $e \in K_1 \setminus \{0\}$ such that $Lx \neq Nx + \lambda e$ for $x \in \partial_K \Omega$, $\lambda \geq 0$. Then $\operatorname{ind}_K([L, N], \Omega) = \{0\}$.

Proof. Let $L - N(\lambda, x) = Lx - (Nx + \lambda e)$ which is A-proper and weakly inward for each $\lambda \geq 0$ and $Lx - N(\lambda, x) \neq 0$ for $x \in \partial_K \Omega$, $\lambda \geq 0$. By the homotopy property of the index, we have

$$\operatorname{ind}_K([L, N], \Omega) = \operatorname{ind}_K([L, N + \lambda e], \Omega).$$

Now if $\operatorname{ind}_K([L, N + \lambda e], \Omega) \neq \{0\}$ then there exists $x_\lambda \in \Omega_K$ such that $Lx_\lambda = Nx_\lambda + \lambda e$ or $L\left(\frac{x_\lambda}{\lambda}\right) = Nx_\lambda/\lambda + e$. Allowing $\lambda \to \infty$ we have $L\left(\frac{x_\lambda}{\lambda}\right) \to e$ since N is bounded, but $x_\lambda/\lambda \to 0$ which contradicts the closed linearity of L. Hence $\operatorname{ind}_K([L, N], \Omega) = \{0\}$. Q.E.D.

The preceding result, in conjunction with a Leray-Schauder type boundary condition, is used in the following theorem to obtain a non-zero solution.

Theorem 5.4.6 *Let Ω_1 and Ω_2 be open and bounded in X with $0 \in \overline{\Omega_1} \subset \Omega_2$ and $\Omega_i \cap K \cap \operatorname{dom} L \neq \emptyset$ for $i = 1, 2$. Assume $L, N : \overline{\Omega_2} \cap K \to Y$ are bounded such that $L - \lambda N$ is A-proper for $\lambda \in [0, 1]$ and $\lambda(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_2} \cap K$, $\lambda \in [0, 1]$. Suppose that:*

(LS) $Lx \neq \lambda Nx + (\lambda - 1) J^{-1}Px$ for $x \in \partial\Omega_1 \cap K$, $\lambda \in [0, 1]$

(E) there exists $e \in K_1 \setminus \{0\}$ such that $Lx \neq Nx + \lambda e$ for $x \in \partial\Omega_2 \cap K$, $\lambda \geq 0$.

Then $Lx = Nx$ for some $x \in \left(\Omega_2 \setminus \overline{\Omega_1}\right) \cap K$.

Proof. Condition (LS) implies $\operatorname{ind}_K([L, N], \Omega_1) = \{1\}$ by Theorem 5.4.4 with $y_0 = 0$. From (E) we have $\operatorname{ind}_K([L, N], \Omega_2) = \{0\}$ by Theorem 5.4.5. The additivity property of the index gives

$$\operatorname{ind}_K\left([L,N],\Omega_2\setminus\overline{\Omega_1}\right)=\operatorname{ind}_K\left([L,N],\Omega_2\right)-\operatorname{ind}_K\left([L,N],\Omega_1\right)=\{-1\}.$$

As the index is non-zero, property P1 implies the existence of $x\in\left(\Omega_2\setminus\overline{\Omega_1}\right)\cap K$ such that $Lx=Nx$. Q.E.D.

Remark 5.4.7 *The conclusion to Theorem 5.4.6 is valid if (LS) holds on $\partial\Omega_2\cap K$ and (E) holds on $\partial\Omega_1\cap K$.*

Our next result, which extends Theorem 5.4.5 and a result of Lan and Webb [30], gives conditions that imply the index is zero and involves a weakly inward map F in addition to the maps L and N.

Theorem 5.4.8 *Let Ω_K be open bounded and such that $\Omega_K\cap\operatorname{dom}L$ is nonempty. Let $L,N:\overline{\Omega_K}\to Y$ be bounded maps such that $L-N$ is A-proper and assume $\left(N+J^{-1}P\right)x\in\overline{I_{K_1}}\,(Hx)$ for every $x\in\overline{\Omega_K}$. Suppose that there exists a bounded $F:\overline{\Omega_K}\to Y$ that satisfies $Fx\in\overline{I_{K_1}}\,(Hx)$ for every $x\in\overline{\Omega_K}$ and that the following conditions hold:*
(A1) $\overline{F\left(\partial_K\Omega\right)}\cap(-K)=\emptyset$
(A2) $\inf\{\|Fx\|:x\in\partial\Omega_K\}=\alpha>0$
(A3) $Lx\neq Nx+\lambda Fx$ for $x\in\partial\Omega_K$, $\lambda\geq 0$ and $L-N-\lambda F$ is A-proper for $\lambda\geq 0$.
Then $\operatorname{ind}_K\left([L,N],\Omega\right)=\{0\}$.

Proof. We prove first that for any $y\in K_1\setminus\{0\}$ with $\|y\|=1$ there exists $\lambda_0>0$ such that $Lx\neq Nx+\lambda_0Fx+\beta y$ for all $x\in\partial_K\Omega$ and $\beta\geq 0$. If not, there would exist sequences $\{\lambda_n\}>0$, $\lambda_n\to\infty$, $\{\beta_n\}\geq 0$, and $\{x_n\}\subset\partial_K\Omega$ such that

$$Lx_n=Nx_n+\lambda_nFx_n+\beta_ny.$$

Dividing by λ_n, we obtain

$$L\left(\frac{x_n}{\lambda_n}\right)=Nx_n/\lambda_n+Fx_n+\beta_ny/\lambda_n$$

and as $\lambda_n \to \infty$, $L\left(\frac{x_n}{\lambda_n}\right) \to 0$, $Nx_n/\lambda_n \to 0$ so that $Fx_n + \beta_n y/\lambda_n \to 0$. Since $\{Fx_n\}$ is bounded, we may assume $\beta_n/\lambda_n \to \beta_0 \in [0,\infty)$. By (A2), $\beta_0 > 0$. Thus $Fx_n \to -\beta_0 y$. Since $y \in K_1 \setminus \{0\}$, $-\beta_0 y \in -K_1$ so that $-\beta_0 y \in \overline{F(\partial_K \Omega)} \cap (-K_1)$ which contradicts (A1). By Theorem 5.4.5, $\operatorname{ind}_K([L, N+\lambda_0 F], \Omega) = \{0\}$. We define the A-proper and weakly inward homotopy

$$H_\lambda(t,x) = L - N - t\lambda_0 F = L - N(t,x)$$

and note that $H_\lambda(t,x) \neq 0$ on $\partial_K \Omega$ for $t \in [0,1]$ by (A3). The homotopy property of the index then gives

$$\operatorname{ind}_K([L,N],\Omega) = \operatorname{ind}_K([L, N+\lambda_0 F], \Omega) = \{0\}.$$

Q.E.D.

We now use Theorems 5.4.4 and 5.4.8 to obtain a result which ensures a non-zero solution to the equation $Lx = Nx$.

Theorem 5.4.9 *Let $0 \in \overline{\Omega_1} \subset \Omega_2$ be open bounded sets in X such that $\Omega_i \cap K \cap \operatorname{dom} L \neq \emptyset$ for $i = 1,2$. Suppose $L, N, F : \overline{\Omega_2} \cap K \to Y$ are bounded such that $L - \lambda N$ is A-proper for $0 \leq \lambda \leq 1$ and $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ and $Fx \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_2} \cap K$. Assume $L - N - \lambda F$ is A-proper for $\lambda \geq 0$ and*
(LS) $Lx \neq \lambda Nx + (\lambda - 1)J^{-1}Px$ for $x \in \partial_K \Omega_1$, $0 \leq \lambda < 1$ and (A1), (A2), (A3) of Theorem 5.4.8 hold on $\partial_K \Omega_2$.
Then $Lx = Nx$ for some $x \in (\Omega_2 \setminus \overline{\Omega_1}) \cap K$.

Proof. By condition (LS),

$$\operatorname{ind}_K([L,N],\Omega_1) = \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], \Omega_1\right) = \{1\}$$

by P2 with $\hat{y} = 0$. Then conditions (A1), (A2), (A3) imply $\operatorname{ind}_K([L,N],\Omega_2) = \{0\}$ by Theorem 5.4.8. From the additivity property of the index we obtain

$$\operatorname{ind}_K\left([L,N], \Omega_2 \setminus \overline{\Omega_1}\right) = \operatorname{ind}_K([L,N],\Omega_2) - \operatorname{ind}_K([L,N],\Omega_1) \neq \{0\}.$$

Thus, by P1, there exists $x \in (\Omega_2 \setminus \overline{\Omega_1}) \cap K$ such that $Lx = Nx$. Q.E.D.

Remark 5.4.10 *The same conclusion to Theorem 5.4.9 is valid if (LS) holds on $\partial_K \Omega_2$ and (A1), (A2), (A3) hold on $\partial_K \Omega_1$.*

A theorem similar to Theorem 5.4.8 for compact maps C is the content of our next result, a generalisation of a result by Lan and Webb [30].

Theorem 5.4.11 *Let $K \subset X$ be a cone and $H(\operatorname{dom} L \cap K) = K_1$ be the corresponding cone in Y and suppose that $\partial B_1(0) \cap K_1$ is not compact. Let $\Omega \subset X$ be open and bounded such that $\Omega_K \cap \operatorname{dom} L \neq \emptyset$. Suppose L, $N : \overline{\Omega_K} \to Y$ are bounded such that $L - N$ is A-proper and $C : \overline{\Omega_K} \to Y$ is weakly inward and compact. Suppose that (A2) and (A3) of Theorem 5.4.8 hold and $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_K}$. Then $\operatorname{ind}_K([L, N], \Omega) = \{0\}$.*

Proof. We first show that there exists $y \in K_1$ with $\|y\| = 1$ such that $\{ty : t \geq 0\} \cap -\overline{C(\partial_K \Omega)} = \emptyset$. If not, then for every $y \in K_1$ with $\|y\| = 1$ there exists t_y such that $t_y y \in -\overline{C(\partial_K \Omega)}$. Thus the set $Q = \{t_y y : \|y\| = 1\}$ is relatively compact and hence $\overline{co}(Q \cup \{0\})$ is compact. By (A2), $t_y \geq \alpha$ so that $\overline{co}(Q \cup \{0\}) \supseteq K_1 \cap \{\|y\| = \alpha\}$, a contradiction. As in the proof to Theorem 5.4.8, we find $\lambda_0 > 1$ such that $Lx \neq Nx + \lambda_0 Cx + \beta y$ for $x \in \partial_K \Omega$ and $\beta \geq 0$. By Theorem 5.4.5, we have $\operatorname{ind}_K([L, N + \lambda_0 C], \Omega) = \{0\}$. We define the A-proper and weakly inward homotopy (by Proposition 5.4.3) $L - N(t, x) = L - N - t\lambda_0 C$ and observe that $Lx - N(t, x) \neq 0$ on $\partial_K \Omega$, $t \in [0, 1]$ by (A3). The homotopy property of the index then gives

$$\operatorname{ind}_K([L, N], \Omega) = \operatorname{ind}_K([L, N + \lambda_0 C], \Omega) = \{0\}.$$

Q.E.D.

The preceding theorem, in conjunction with Theorem 5.4.9, is employed in the following result to obtain a non-zero solution to the equation $Lx = Nx$.

Theorem 5.4.12 *Let $K \subset X$ be a cone and $H(\operatorname{dom} L \cap K) = K_1$ be a cone in Y such that $\partial B_1(0) \cap K_1$ is not compact. Let Ω_1 and Ω_2 be open bounded sets in X such that $0 \in \Omega_1 \subset \overline{\Omega_2}$, $\Omega_i \cap K \cap \operatorname{dom} L \neq \emptyset$ for $i = 1, 2$. Assume $L, N : \overline{\Omega_2} \cap K \to Y$ are bounded such that $L - \lambda N$ is A-proper for $0 \leq \lambda \leq 1$ and $(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_2} \cap K$. Suppose $C : \overline{\Omega_2} \cap K \to Y$ is compact and $Cx \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_2} \cap K$ and (LS) holds on $\partial\Omega_1 \cap K$ and (A2), (A3) both hold on $\partial\Omega_2 \cap K$. Then $Lx = Nx$ for some $x \in \Omega_2 \setminus \overline{\Omega_1}$.*

Proof. As in the proof to Theorem 5.4.9, either $Lx = Nx$ on $\partial\Omega_2$ or (LS) implies

$$\operatorname{ind}_K([L, N], \Omega_1) = \operatorname{ind}_K\left(\left[L, -J^{-1}P\right], \Omega_1\right) = \{1\}$$

by Theorem 5.2.1. Then (A2) and (A3) imply $\operatorname{ind}_K([L, N], \Omega_2) = \{0\}$ by Theorem 5.4.11 so that

$$\operatorname{ind}_K\left([L, N], \Omega_2 \setminus \overline{\Omega_1}\right) = \operatorname{ind}_K([L, N], \Omega_2) - \operatorname{ind}_K([L, N], \Omega_1) \neq \{0\}.$$

Therefore, by P1 of the index, there exists $x \in (\Omega_2 \setminus \overline{\Omega_1}) \cap K$ such that $Lx = Nx$. Q.E.D.

We conclude this section with a theorem that ensures the existence of at least two non-zero solutions to the equation $Lx = Nx$.

Theorem 5.4.13 *Let $0 \in \Omega_1 \subset \overline{\Omega_2}$ be open bounded sets in X with $\Omega_i \cap K \cap \operatorname{dom} L \neq \emptyset$, $i = 1, 2$, where $K \subset X$ is a cone. Let $L, N : \overline{\Omega_2} \cap K \to Y$ be bounded such that $L - \lambda N$ is A-proper for $0 \leq \lambda \leq 1$ and $\lambda(N + J^{-1}P)x \in \overline{I_{K_1}}(Hx)$ for every $x \in \overline{\Omega_2} \cap K$, $\lambda \in [0, 1]$. Suppose $(N + J^{-1}P)(K)$ is bounded and:*
(LS) holds on $\partial_K\Omega_1$, i.e., $Lx \neq \lambda Nx + (1 - \lambda)J^{-1}Px$, $x \in \partial_K\Omega_1$, $\lambda \in [0, 1]$
(E) holds on $\partial_K\Omega_2$, i.e., there exists $e \in K_1 \setminus \{0\}$ such that $Lx \neq Nx + \lambda e$, $x \in \partial_K\Omega_2$, $\lambda \geq 0$.
Then $Lx = Nx$ has at least two solutions in $K \setminus \{0\}$.

Proof. Since $(N + J^{-1}P)(K)$ is bounded, property P5 of the index implies there exists an open bounded set Ω_3 with $\overline{\Omega_2} \subset \Omega_3$ such that $\text{ind}_K([L, N], \Omega_3) = \{1\}$. By (E). Theorem 5.4.5, and P3 we have

$$\text{ind}_K\left([L, N], \Omega_3 \setminus \overline{\Omega_2}\right) = \text{ind}_K([L, N], \Omega_3) - \text{ind}_K([L, N], \Omega_2) \neq \{0\}.$$

Hence there exists $x_1 \in \left(\Omega_3 \setminus \overline{\Omega_2}\right) \cap K$ such that $Lx_1 = Nx_1$. If $Lx = Nx$ on $\partial_K\Omega_1$ then the conclusion holds. If not, then $Lx \neq Nx$ on $\partial_K\Omega_1$ and by the proof to Theorem 5.4.4 and P3 we have

$$\text{ind}_K\left([L, N], \Omega_2 \setminus \overline{\Omega_1}\right) = \text{ind}_K([L, N], \Omega_2) - \text{ind}_K([L, N], \Omega_1) = \{-1\}$$

and therefore there exists $x_2 \in \left(\Omega_2 \setminus \overline{\Omega_1}\right) \cap K$ such that $Lx_2 = Nx_2$. Q.E.D.

Chapter 6

APPLICATIONS TO DIFFERENTIAL AND INTEGRAL EQUATIONS

6.1 Introduction

In this final chapter, we apply some of the theorems of Chapter 5 to prove the existence of non-negative solutions to several differential and integral equations. The equation we consider first, in Section 6.2, is the following boundary value problem.

$$-x''(t) = f(t, x(t), x'(t), x''(t)) \text{ where } x(0) = x(1) = 0. \tag{6.1}$$

This problem was studied by Lafferriere and Petryshyn [28] in which equation (6.1) was transformed to the fixed point operator equation $Ty = y$ in the space $Y = C[0,1]$. We shall formulate (6.1) in terms of a semilinear equation $L - N$ and obtain similar results.

Section 6.3 concerns the periodic boundary value problem:

$$-x''(t) = f(t, x(t), x'(t)) \text{ where } x(0) = x(1) \text{ and } x'(0) = x'(1). \tag{6.2}$$

We first transform (6.2) into a semilinear equation and then use a method similar to that of Mawhin [34] and Petryshyn and Yu [44] to obtain *a priori* bounds for the solution set.

Our last application involves the perturbed Volterra equation:

$$y'(t) = k(t, t, x(t)) + \int_0^t k_t(t, s, x(s))\, ds$$

obtained by differentiating the Volterra integral equation of the first kind,

$$\int_0^t k(t, s, x(s))\, ds = y(t). \tag{6.3}$$

This problem was studied by Deimling [13] in which he obtained a non-negative solution using a Leray-Schauder boundary condition argument. His results are proved for cones with nonempty interior and he remarks that it is unknown whether Theorem 20.4 of [13] remains true if the interior $\operatorname{int} K = \emptyset$. We shall use a weakly inward result in the framework of A-proper maps from the preceding chapter which applies to cones of both empty and nonempty interior and thus generalises the results in Deimling [13].

6.2 A positive solution to the Picard problem

We shall formulate the Picard boundary value problem, (6.1), as a semilinear equation in Banach spaces and place certain conditions on the nonlinearity $f(t, x, x', x'')$ so that we might apply Theorem 5.3.8 to obtain a positive solution to equation (6.1).

Let $X = C^2[0,1]$, $Y = C[0,1]$ and $K = \{x \in X : -x''(t) \geq 0, x(0) = x(1) = 0\}$ with norms $\|x\|_X = \max\{\|x\|_Y, \|x'\|_Y, \|x''\|_Y\}$ and $\|x\|_Y = \max_{t\in[0,1]}\{|x(t)|\}$. Define $L : \operatorname{dom} L \subset X \to Y$ by $Lx = -x''(t)$ where $\operatorname{dom} L = \{x \in X : x(0) = x(1) = 0\}$ and $Nx = f(t, x(t), x'(t), x''(t))$.

Theorem 6.2.1 *Under the above assumptions, suppose also that:*

(a) there exists $R > 0$, $k \in (0, l(L)/\mu_0)$, *such that* $f : [0,1] \times [0,R] \times [-R,R] \times \mathbb{R}^- \to \mathbb{R}^+$ *is continuous and* $|f(t,p,q,s_1) - f(t,p,q,s_2)| \leq k|s_1 - s_2|$ *for* $t \in [0,1]$, $p \in [0,R]$, $q \in [-R,R]$, $s_1, s_2 \in \mathbb{R}^-$,

(b) $f(t,p,q,s) \geq R$ *for every* $t \in [0,1]$, $p \in [0,R]$, $q \in [-R,R]$, $s \in \mathbb{R}^-$,

(c) there exists $r \in (0,R)$, $t_0 \in [0,1]$ *such that* $f(t_0,p,q,s) < r$ *for* $p \in [0,r]$, $q \in [-r,r]$, $s = -r$.

Then there exists a positive solution $x \in K$ *to equation (6.1) with* $r \leq \|x\|_X \leq R$.

Before verifying the hypotheses of Theorem 5.3.8, we show that problem (6.1), thus formulated, is a semilinear Fredholm operator equation. We state and prove this in the following subsidiary proposition.

Proposition 6.2.2 *Assume the conditions of the preceding theorem hold. Then*

(a) $L : \operatorname{dom} L \subset X \to Y$ *is Fredholm of index zero,*

(b) N *is k-ball contractive,*

(c) $L - \lambda N$ *is A-proper for* $0 \leq \lambda \leq \mu_0$ *relative to* Γ_L.

Proof of the proposition. To prove L is a Fredholm operator of index zero, we must show $\dim(\ker L) = \dim(Y \setminus \operatorname{im} L) < \infty$. To determine the $\ker L$; suppose $Lx = 0$, then $-x''(t) = 0$ and $-x'(t) = c_1$ by integration. Integrating a second time gives $-x(t) = c_1 t + c_2$ or $x(t) = -c_1 t - c_2$. By the boundary conditions we obtain $x(0) = -c_2 = x(1) = -c_1 - c_2 = 0$ so that $-c_2 = -c_1 - c_2$ which implies $c_1 = 0$ and $c_2 = 0$ by the last equality. Hence $\ker L = \{x \in \operatorname{dom} L : x(t) \equiv 0\}$ from which we deduce $\dim(\ker L) = 0$.

Now we prove $\operatorname{im} L = Y$. We shall show that for $y \in Y$, there exists $x \in \operatorname{dom} L$ such that $Lx = y$, *i.e.*, $-x''(t) = y(t)$ and $x(0) = x(1) = 0$. Let $y \in Y$ and

$$x(t) = \int_0^1 t(1-\tau)\, y(\tau)\, d\tau - \int_0^t (t-\tau)\, y(\tau)\, d\tau.$$

Then

$$x'(t) = \int_0^1 (1-\tau)\, y(\tau)\, d\tau - \int_0^t y(\tau)\, d\tau$$

and

$$x''(t) = -y(t) \text{ or } -x''(t) = y(t).$$

Also

$$x(0) = \int_0^1 0\,(1-\tau)\, y(\tau)\, d\tau - \int_0^0 (0-\tau)\, y(\tau)\, d\tau = 0$$

and

$$x(1) = \int_0^1 1\,(1-\tau)\, y(\tau)\, d\tau - \int_0^1 (1-\tau)\, y(\tau)\, d\tau = 0.$$

Therefore, $\operatorname{im} L = Y$ so that $\dim(Y \setminus \operatorname{im} L) = \dim(\ker L) = 0$ which proves L is Fredholm of index zero.

Now we show that condition (a) of the theorem implies (b) of the proposition. The proof is from Petryshyn [45]. We define a bivariate map $V : X \times X \to Y$ by $V(x,u) = f(t,x,x',u'')$ so that $N(x) = V(x,x)$. Let $Q \subset X$ be a bounded set and denote β_X and β_Y as the ball-measure of non-compactness in the spaces X and Y respectively. Let $r \equiv \beta_X(Q)$ and $\epsilon > 0$; we then cover Q with a finite number of balls in X with radii $r + \epsilon/k$ and centres u_i so that $Q \subset \bigcup_{j=1}^{p} B_{r+\epsilon/k}(u_j)$. Now Q is precompact in $C^1[0,1]$ since X is compactly embedded in $C^1[0,1]$, Q is bounded in X and the map $x \to V(x,u)$ is continuous from $C^1[0,1]$ to Y for each fixed u in X. This implies the set $V(Q,u)$ is precompact in Y for each $u \in X$ and so $\bigcup_{j=1}^{p} V(Q,u_j)$ is also precompact in Y. Now as $\bigcup_{j=1}^{p} V(Q,u_j)$ is precompact, we may choose $x_1, ..., x_q$ in X such that $\bigcup_{j=1}^{p} V(Q,u_j) \subset \bigcup_{n=1}^{q} B_\epsilon(x_n)$ for the given ϵ. Then for any $x \in Q$, we choose j such that $\|x - u_j\| \leq r + \epsilon/k$ and observe that

$$\begin{aligned} \|V(x,x) - V(x,u_j)\| &= \left\|f(t,x,x',x'') - f\left(t,x,x',u_j''\right)\right\| \\ &\leq k\left\|x'' - u_j''\right\| \\ &\leq k\|x - u_j\| \leq k(r + \epsilon/k). \end{aligned}$$

Now we choose $n \in \{1, ..., q\}$ such that $\|V(x, u_j) - x_n\| < \epsilon$. Thus

$$\begin{aligned}
\|N(x) - x_n\| &= \|V(x, x) - x_n\| \\
&\leq \|V(x, x) - V(x, u_j)\| + \|V(x, u_j) - x_n\| \\
&\leq k(r + \epsilon/k) + \epsilon = kr + 2\epsilon.
\end{aligned}$$

Hence $N(Q) \subset \bigcup_{n=1}^{q} B_{kr+2\epsilon}(x_n)$ so that $\beta(N(Q)) \leq kr + 2\epsilon$. As $\epsilon > 0$ was arbitrarily chosen, this implies $\beta(N(Q)) \leq k\beta(Q)$ so that N is k-ball-contractive. It follows from (ii) of Theorem 1.4.7 and Remark 1.4.8 that $L - \lambda N$ is A-proper for $0 \leq \lambda \leq \mu_0$. Q.E.D.

We mention that an example of an admissible scheme for maps from X into Y is the following. Let $Y_n \subset Y = C[0,1]$ be the subspace of all $y \in Y$ that are linear in every subinterval $[t_{n_i}, t_{n_{i+1}}]$ where $0 = t_{n_0} < t_{n_1} < ... < t_{n_n} = 1$ and $\max(t_{n_{i+1}} - t_{n_i}) \to 0$ as $n \to \infty$. Define $Q_n y(t) = y_n(t)$ where

$$y_n(t) = \begin{cases} y(t) & \text{for } t = t_{n_i} \;\; i = 0, ..., n, \\ y(t_{n_i}) + [y(t_{n_{i+1}}) - y(t_{n_i})] \dfrac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}} & \text{for } t \in (t_{n_i}, t_{n_{i+1}}), \;\; i = 0, ..., n. \end{cases}$$

We verify that this construction is an admissible scheme in the following proposition.

Proposition 6.2.3 *Let $Y_n = Q_n Y$ and $X_n = H^{-1}(Y_n)$, then $\Gamma_L = \{X_n, Y_{n,} Q_n\}$ is an admissible scheme for maps from X into Y.*

Proof of the proposition. Since $\{y_n(t)\} \subset Y_n$ are continuous for all $n \in \mathbb{N}$ and $t \in [0,1]$, it is clear that $Y_n \subset Y$ for all $n \in \mathbb{N}$.

We prove that Q_n is a continuous linear projection of Y onto Y_n with $\|Q_n\| = 1$. Let $y, z \in Y$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
Q_n(\alpha y + \beta z) &= \alpha y(t) + \beta z(t) \text{ for } t = t_{n_i}, i \in \{0, ..., n\} \\
&= Q_n(\alpha y(t)) + Q_n(\beta z(t)) \\
&= \alpha Q_n y(t) + \beta Q_n z(t).
\end{aligned}$$

For $t \in (t_{n_i}, t_{n_{i+1}})$ we have

$$\begin{aligned} Q_n(\alpha y(t) + \beta z(t)) &= \alpha y(t_{n_i}) + [\alpha y(t_{n_{i+1}}) - \alpha y(t_{n_i})] \frac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}} + \\ &\quad \beta z(t_{n_i}) + [\beta z(t_{n_{i+1}}) - \beta z(t_{n_i})] \frac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}} \\ &= \alpha Q_n y(t) + \beta Q_n z(t). \end{aligned}$$

Which proves the linearity of Q_n.

To prove Q_n is a projection, we observe that for $t = t_{n_i}$, $i \in \{0, ..., n\}$, $Q_n^2(y(t)) = Q_n(Q_n y(t)) = Q_n(y(t)) = y(t)$. For $t \in (t_{n_i}, t_{n_{i+1}})$, $i \in \{0, ..., n\}$, we have

$$\begin{aligned} Q_n^2(y(t)) &= Q_n(Q_n y(t)) \\ &= Q_n\left(y(t_{n_i}) + [y(t_{n_{i+1}}) - y(t_{n_i})] \frac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}}\right) \\ &= y(t_{n_i}) + [y(t_{n_{i+1}}) - y(t_{n_i})] \frac{t - t_{n_i}}{t_{n_{i+1}} - t_{n_i}}. \end{aligned}$$

So $Q_n^2(y(t)) = Q_n y(t)$.

We calculate the norm of Q_n as follows.

$$\begin{aligned} \|Q_n\| &= \sup\{\|Q_n y\| : \|y\|_Y = 1\} \text{ where } \|y\|_Y = \max_{t\in[0,1]} |y(t)| \\ &= \sup\left\{\max_{t\in[0,1]}(|Q_n y(t)|) : \|y\|_Y = 1\right\} \\ &\leq \sup\left\{\max_{t\in[0,1]}(|y(t)|) : \|y\|_Y = 1\right\} \\ &= 1. \end{aligned}$$

Now if we take the function $y(t) = 1$ for all $t \in [0,1]$, then $\|y\|_Y = \max_{t\in[0,1]} |y(t)| = 1$ and $\|Q_n y\|_Y = 1$ so that

$$\frac{\|Q_n y\|}{\|y\|} = 1 \leq \sup\{\|Q_n y\| : \|y\| = 1\} = \|Q_n\| \text{ for all } n \in \mathbb{N}.$$

Hence $\|Q_n\| = 1$ for each $n \in \mathbb{N}$.

The continuity of Q_n then follows from $\|Q_n\| = 1$.

Thus, for each $n \in \mathbb{N}$, Q_n is a continuous linear projection of Y onto Y_n with $\|Q_n\| = 1$.

We now determine the dimension of Y_n and show that X_n and Y_n are of the same finite dimension. Let $\{e_1, ..., e_{n+1}\}$ be the usual orthonormal basis in $\mathbb{R}^{n+1}$. We have for each $y_n \in Y_n$, $(y(t_1), ..., y(t_{n+1})) \in \mathbb{R}^{n+1}$ uniquely represents y_n in $\mathbb{R}^{n+1}$. Letting $(y_n(t_1), ..., y_n(t_{n+1})) = (\alpha_1, ..., \alpha_{n+1})$ then y_n has the unique representation $\sum_{j=1}^{n+1} \alpha_j e_j$ in $\mathbb{R}^{n+1}$ from which it is clear $\dim Y_n = n + 1$. Defining $X_n = H^{-1}(Y_n)$, where in this case $H^{-1} = L^{-1}$ (L being invertible), we have $\dim X_n = n + 1$ since the homeomorphism H^{-1} preserves dimension.

Finally, we show that $\operatorname{dist}(y, Y_n) \to 0$ as $n \to \infty$ by proving the norm convergence of $Q_n y \to y$ as $n \to \infty$ for each $y \in Y$. From the definition of the norm we have

$$\begin{aligned}
\|Q_n y - y\| &= \max_{t\in[0,1]} \{|Q_n y(t) - y(t)|\} \\
&= \max_{t\in(t_i,t_{i+1}),\, i=0,...,n-1} \left\{ \left| y(t_i) + (y(t_{i+1}) - y(t_i)) \frac{t - t_i}{t_{i+1} - t_i} - y(t) \right| \right\} \\
&\leq \max_{t\in(t_i,t_{i+1}),\, i=0,...,n-1} \{|y(t_i) - y(t)|\} \\
&\quad + \max_{t\in(t_i,t_{i+1}),\, i=0,...,n-1} \left\{ |y(t_{i+1}) - y(t_i)| \left| \frac{t - t_i}{t_{i+1} - t_i} \right| \right\}.
\end{aligned}$$

Now as $n \to \infty$, $|y(t_i) - y(t)| \to 0$ and $|y(t_{i+1}) - y(t_i)| \to 0$ and since $\left|\frac{t-t_i}{t_{i+1}-t_i}\right| < 1$, both terms on the right hand side of the inequality converge to zero.

Theorem 1.4.5 then implies that Γ_L, thus constructed, is an admissible scheme for maps from X into Y. Q.E.D.

Proof of Theorem 6.2.1. Let Γ_L be as above. We verify the hypotheses of Theorem 5.3.8. Let $x \in K \cap \operatorname{dom} L$ with $\|x\|_X = R$. Then $\|Lx\|_Y = \|-x''\|_Y = R$ and there exists $t_1 \in [0,1]$ such that $-x''(t_1) = R$ or $x''(t_1) = -R$. So we have for $t \in [0,1]$, $x(t) \in [0, R]$, $x'(t_1) \in [-R, R]$ and $x''(t_1) = -R$.

By (b) we have

$$\begin{aligned}\left\|L^{-1}Nx\right\|_X &= \|Nx\|_Y \geq |Nx(t_1)| \\ &= Nx(t_1) = f(t_1, x(t_1), x'(t_1), x''(t_1)) \\ &\geq R > R/\mu_0 \text{ (since } \mu_0 > 1).\end{aligned}$$

Hence

$$\delta = \inf\left\{\left\|L^{-1}Nx\right\| : x \in \partial B_R(0) \cap K\right\} > R/\mu_0$$

which satisfies (a) of Theorem 5.3.8.

Let r be as in (c) and $x \in \operatorname{dom} L \cap K$ such that $\|x\|_X = r$. Then there exists $t_0 \in [0,1]$ such that $|-x''(t_0)| = \|Lx\|_Y = r = -x''(t_0)$ or $x''(t_0) = -r$ and we have $x(t) \in [0,r]$, $x'(t) \in [-r,r]$ for every $t \in [0,1]$ and $x''(t_0) = -r$. From (c) we obtain

$$f(t_0, x(t_0), x'(t_0), x''(t_0)) < r.$$

Then if $Lx = \mu Nx$ for some $\mu < 1$ and $x \in K$ with $\|x\|_X = r$ we would have

$$Lx = -x''(t) = \mu f(t, x(t), x'(t), x''(t))$$

for every $t \in [0,1]$ including t_0. This would give

$$-x''(t_0) = r = \mu f(t_0, x(t_0), x'(t_0), x''(t_0)) < \mu r,$$

a contradiction. This satisfies condition (c) of Theorem 5.3.8 while condition (b) is immediate. Thus there exists $x \in K$ with $r \leq \|x\|_X \leq R$ such that $Lx = Nx$. Q.E.D.

6.3 Non-negative solutions to the periodic boundary value problem

As with the previous problem, we first convert equation (6.2) into a semilinear operator equation in Banach spaces. We then place certain restrictions on $f(t, x(t), x'(t))$ in Theorem 6.3.1 so that the conditions of Corollary 5.2.7 are satisfied.

Let $X = \{x \in C^2[0,1] : x(0) = x(1),\ x'(0) = x'(1)\}$ with $\|x\|_X = \max\{\|x\|_Y, \|x'\|_Y\}$ and $Y = C[0,1]$ with $\|y\|_Y = \max_{t\in[0,1]} |y(t)|$ and $K = \{x \in X : x(t) \geq 0,\ t \in [0,1]\}$. Define $L : X \to Y$ by $Lx = -x''(t)$ and $N : X \to Y$ by $Nx(t) = f(t, x(t), x'(t))$. We assume f to be continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$. Before stating our existence theorem, we prove that L, as so defined, is indeed Fredholm of index zero and that $L - \lambda N$ is A-proper.

To determine the $\ker L$; suppose $Lx = 0$, then $-x''(t) = 0$ and so $x'(t) = c_1$ by integration. Integrating a second time gives $x(t) = c_1 t + c_2$. By the boundary conditions we obtain $x(0) = c_2 = x(1) = c_1 + c_2$ which implies $c_1 = 0$. Also, $x'(0) = c_1 = x'(1) = c_1$ so that $\ker L$ is the set of constants $\{c \in \mathbb{R}\}$. Hence $\dim(\ker L) = 1$.

Now $\operatorname{im} L = \{y \in Y : -x''(t) = y(t) \text{ and } x(0) = x(1),\ x'(0) = x'(1)\}$ and the general solution to $-x''(t) = y(t)$ is

$$x(t) = c_1 + c_2 t - \int_0^t (t - \tau)\, y(\tau)\, d\tau.$$

The periodic boundary conditions require

$$x(0) = c_1 = x(1) = c_1 + c_2 - \int_0^1 (1 - \tau)\, y(\tau)\, d\tau$$

so that

$$c_2 = \int_0^1 (1 - \tau)\, y(\tau)\, d\tau.$$

As

$$x'(t) = c_2 - \int_0^t y(\tau)\, d\tau,$$

the boundary conditions require

$$x'(0) = c_2 = x'(1) = c_2 - \int_0^1 y(\tau)\, d\tau$$

so that

$$\int_0^1 y(\tau)\, d\tau = 0.$$

Hence the specific solution is

$$x(t) = c_1 + \int_0^1 t(1-\tau)\, y(\tau)\, d\tau - \int_0^t (t-\tau)\, y(\tau)\, d\tau$$

and $\operatorname{im} L = \left\{ y \in Y : \int_0^1 y(\tau)\, d\tau = 0 \right\}$.

Let $Y_0 = \{constants\}$. Then $Y = Y_0 \oplus Y_1$ and we may take the projection $Q : Y \to Y_0$ to be $Qy = \int_0^1 y(t)\, dt$. Therefore, $\dim(\ker L) = 1 = \dim(Y_0)$ and L is Fredholm of index zero.

Since $Nx = f(t, x, x')$ is continuous and X is compactly embedded in Y by Sobolev's embedding theorem [57], N is therefore compact. Hence $L - \lambda N$ is A-proper by (i) of Theorem 1.4.7.

We mention that the projection scheme constructed in Section 6.2 is valid also for this periodic problem with the modification that $H = L + J^{-1}P$ where $P : X \to \ker L$ and $J : Y_0 \to \ker L$. For this particular problem, we shall define $Px = \int_0^1 x(t)\, dt$ which is a constant and therefore in the $\ker L$ and $Jy = \beta I$ where $\beta \in \mathbb{R}^+$ will be specified later.

We are now prepared to state our existence theorem for equation (6.2).

Theorem 6.3.1 *Suppose*

(a) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exist* $a, b, c \in \mathbb{R}^+$ *such that* $|f(t, x, p)| \leq a + b|x| + c|p|$ *for every* p, $x \geq 0$ *and* $t \in [0,1]$ *where* $b + \pi c < 2\pi^2$

(b) there exists $M_1 > 0$ *such that* $\int_0^1 f(t, x, x')\, dt \neq 0$ *for* $x \in X$ *with* $x(t) \geq M_1$, $t \in [0,1]$

(c) there exists $M_2 \geq M_1$ *such that for* $x \in \ker L = \mathbb{R}$ *with* $x \geq M_2$ *implies* $xf(t, x, 0) \geq 0$ *for every* $t \in [0,1]$ *(or* $xf(t, x, 0) \leq 0$*)*

(d) there exists $\alpha \in (0, 8]$ *such that* $f(t, x, p) \geq -\alpha x$ *for every* $x \geq 0$, $t \in [0,1]$.

Then there exists $x \in \operatorname{dom} L, x(t) \geq 0$ *such that* $Lx = Nx$.

Proof. To verify the conditions of Corollary 5.2.7 we first obtain a set

$$\Omega = \{x \in K : \|x\|_2 < r\}$$

such that $Lx \neq \lambda Nx$ on $\partial\Omega$ for $\lambda \in [0,1]$. We then show that if $x \in K$ is a solution of $-x'' = \lambda f(t,x,x')$ for some $\lambda \in [0,1]$ then $x(t) \leq M$ for some $M > 0$ independent of $x(t)$ and λ.

Let x be a solution of $-x'' = \lambda f(t,x,x')$ for some $\lambda \in [0,1]$, then

$$\begin{aligned} -\int_0^1 x''(t)\,dt &= \lambda \int_0^1 f(t,x,x')\,dt \\ -[x'(1) - x'(0)] &= 0 = \lambda \int_0^1 f(t,x,x')\,dt. \end{aligned}$$

From (b) there exists $t_0 \in [0,1]$ such that $x(t_0) < M_1$. We write $x(t) = a_0 + u(t)$ with

$$a_0 = \int_0^1 x(t)\,dt$$

then

$$\begin{aligned} \int_0^1 u(t)\,dt &= 0 \\ x'(t) &= u'(t) \\ x(t) &= x(t_0) + \int_{t_0}^t x'(s)\,ds \end{aligned}$$

and

$$x(t) \leq M_1 + \|x'\|_2 = M_1 + \|u'\|_2 \text{ where } \|\|_2 \text{ is the } \mathrm{L}^2 \text{ norm.}$$

Next we prove

$$\|x\|_2 \leq M_1 + \frac{1}{\pi}\|x'\|_2 = M_1 + \frac{1}{\pi}\|u'\|_2 .$$

Let

$$w(t) = \begin{cases} x(t+t_0-1) - x(t_0) & \text{for } 1 - t_0 \leq t \leq 1, \\ x(t+t_0) - x(t_0) & \text{for } 0 \leq t < 1 - t_0. \end{cases}$$

Since $w(0) = w(1) = 0$ and $w \in C^1[0,1]$, Theorem 257 of [21] implies

$$\|w\|_2 \leq \frac{1}{\pi}\|w'\|_2$$

and

$$\begin{aligned} \|w(t) + x(t_0)\|_2^2 &= \int_0^{1-t_0} |x(t+t_0)|^2\,dt + \int_{1-t_0}^1 |x(t+t_0-1)|^2\,dt \\ &= \int_0^1 |x(t)|^2\,dt. \end{aligned}$$

Similarly we obtain

$$\|w'\|_2 = \|x'\|_2 = \|u'\|_2 .$$

These inequalities then give

$$\|x\|_2 \leq M_1 + \|w\|_2 \leq M_1 + \frac{1}{\pi} \|w'\|_2 = M_1 + \frac{1}{\pi} \|u'\|_2 .$$

Integrating the left side of $-x'' \cdot x = \lambda f(t, x, x') \cdot x$ by parts, we obtain

$$\begin{aligned} -\int_0^1 x''(t) \cdot x(t)\, dt &= -x'(t) \cdot x(t) \mid_0^1 + \int_0^1 x'(t) \cdot x'(t)\, dt \\ &= -\left[x'(1) \cdot x(1) - x'(0) \cdot x(0)\right] + \int_0^1 \left[x'(t)\right]^2 dt. \end{aligned}$$

So we have

$$\int_0^1 \left[x'(t)\right]^2 dt = \lambda \int_0^1 f(t, x, x') \cdot x(t)\, dt.$$

By (a) and Hölder's inequality we have

$$\begin{aligned} \|x'\|_2^2 &= \|u'\|_2^2 \leq \int_0^1 |f(t, x, x')|\, |u(t)|\, dt \\ &\leq \int_0^1 (a + b\,|x(t)| + c\,|x'(t)|)\, |u(t)|\, dt \\ &\leq \|a + b\,|x| + c\,|x'|\|_2 \|u\|_2 \\ &\leq (a + b\,\|x\|_2 + c\,\|u'\|_2)\, \|u\|_2 . \end{aligned}$$

Extending $u(t)$ periodically to all of $\mathbb{R}$ with period one and using Wirtinger's inequality [21], $\|u\|_2 \leq (1/2\pi)\|u'\|_2$, we obtain

$$\|x'\|_2^2 = \|u'\|_2^2 \leq (a + bM_1)\frac{1}{2\pi}\|u'\|_2 + (b + \pi c)\frac{1}{2\pi^2}\|u'\|_2^2 .$$

Since by (a), $b + \pi c < 2\pi^2$ and $\|x'\|_2 = \|u'\|_2 \leq A_1$ where

$$A_1 = \frac{\pi(a + bM_1)}{2\pi^2 - (b + \pi c)}.$$

So $x(t) \leq M_1 + A_1$ for $t \in [0, 1]$ and $\|x\|_2 \leq M_1 + A_1/\pi$.

Thus, if we choose $r > M_1 + A_1/\pi$ we shall have $-x'' \neq \lambda f(t,x,x')$ on $\partial\Omega$ where $\Omega = \{x \in K : \|x\|_2 < r\}$ and (a_1) of Corollary 5.2.7 is satisfied.

For $Q : Y \to Y_2$, $Qy = \int_0^1 y(t)\,dt$ and $x \in \ker L \cap \partial\Omega$, then $x(t) \equiv c$, a constant, and $x(t) = c = r > M_1$. Condition (b) gives

$$QNx = \int_0^1 f(t,x,0)\,dt \neq 0$$

and (b_1) of Corollary 5.2.7 is verified.

To prove (c_1) of Corollary 5.2.7, we define the bilinear form $[\cdot,\cdot] : Y \times Y \to \mathbb{R}$ as

$$[y,x] = \int_0^1 y(t)\,x(t)\,dt.$$

It is clear that $[\cdot,\cdot]$ is continuous and satisfies $[y,x] = 0$ for every $x \in \ker L$, $y \in \operatorname{im} L$. Now we show that condition (c) implies (c_1) of Corollary 5.2.7. Let $x \in \ker L \cap \partial\Omega$, then $x(t) = c = r > M_1$ so choose $M_2 = r$ and

$$[QNx,x] = \int_0^1 \int_0^1 f(t,c,0)\,dt \cdot c\,ds \leq 0$$

since $f(t,x,0) \cdot x \leq 0$ for $x \geq M_1$.

To verify (d), let $x \in K$ with $Px = \int_0^1 x(t)\,dt$, $Qy = \int_0^1 y(t)\,dt$ and

$$L_1^{-1}(I-Q)N = \int_0^1 G(s,t)\left[f(s,x(s),x'(s)) - \int_0^1 f(t,x(t),x'(t))\,dt\right] ds$$

where

$$G(s,t)^1 = \begin{cases} (s/2)(1-2t+s) & \text{for } 0 \leq s < t \\ (1/2)(1-s)(2t-s) & \text{for } t \leq s \leq 1 \end{cases}.$$

Then

$$\begin{aligned} Px + JQNx \;&+\; L_1^{-1}(I-Q)Nx \\ &= \int_0^1 x(s)\,ds + \beta \int_0^1 f(s,x(s),x'(s))\,ds \\ &+ \int_0^1 G(s,t)\left[f(s,x(s),x'(s)) - \int_0^1 f(t,x(t),x'(t))\,dt\right] ds \\ &= \int_0^1 x(s)\,ds + \int_0^1 H(s,t)\,f(s,x(s),x'(s))\,ds \end{aligned}$$

[1]The derivation of $G(s,t)$ is given in the appendix to this section.

where

$$H(s,t) = \beta + G(s,t) - \int_0^1 G(s,t)\, ds$$

or

$$H(s,t) = \begin{cases} \beta + (s/2)(1 - 2t + s) - \int_0^1 G(s,t)\, ds, & 0 \leq s < t \\ \beta + (1/2)(1 - s)(2t - s) - \int_0^1 G(s,t)\, ds, & t \leq s \leq 1 \end{cases}.$$

We show in the appendix to this section that $0 \leq H(s,t) \leq 1/8$ if β is chosen to be $1/24$. Then

$$\begin{aligned} \int_0^1 x(s)\, ds + \int_0^1 H(s,t) f(s, x(s), x'(s))\, ds &\geq \int_0^1 x(s)\, ds - \alpha \int_0^1 H(s,t)\, x(s)\, ds \\ &\geq \int_0^1 x(s)\, ds - \frac{\alpha}{8} \int_0^1 x(s)\, ds \\ &= \left(1 - \frac{\alpha}{8}\right) \int_0^1 x(s)\, ds \geq 0. \end{aligned}$$

Thus all conditions of Corollary 5.2.7 are satisfied and there exists $x \in \operatorname{dom} L \cap K$ such that $Lx = Nx$. Q.E.D.

6.4 A non-negative solution to a perturbed Volterra equation

We consider the system of Volterra integral equations of the first kind

$$y(t) = \int_0^t k(t, s, x(s))\, ds,\ t \in J = [0,1] \tag{6.4}$$

where k and y are $\mathbb{R}^n$-valued and all functions are known except $x(s)$. Differentiation with respect to t yields

$$y'(t) = k(t, t, x(t)) + \int_0^t k_t(t, s, x(s))\, ds \tag{6.5}$$

where $k_t(t, s, x(s)) = \partial k / \partial t$. We convert this to an operator equation of the form $x = T_1 x + T_2 x$ where

$$T_1 x = x(t) - k(t, t, x(t))$$

and

$$T_2 x = y'(t) - \int_0^t k_t(t, s, x(s))\, ds.$$

Or, more concisely, $x = Nx$ (in the form $Lx = Nx$ with $L = I$ and $N = T_1 + T_2$).

Let $X = L^2[0,1]$ and $K \subset X$ where $K = \{x \in X : x(t) \geq 0 \; a.e.\}$. We shall prove that (6.4) has a solution $x \in K$ if the following conditions are satisfied.

(i) $k : \{(t,s) \in J \times J : s \leq t\} \times \mathbb{R}^{n^+} \to \mathbb{R}^n$ and $k_t(t, s, x(s))$ satisfy Carathéodory conditions on J and there exists $M > 0$ such that $|k(t, s, x)|, |k_t(t, s, x)| \leq M(1 + |x|)$.

(ii) $(k(t,t,x) - k(t,t,y), x - y) \geq \alpha |x - y|^2$ on $J \times \mathbb{R}^{n^+} \times \mathbb{R}^{n^+}$ for some $\alpha \in (0,1)$.

(iii) $k_t(t, s, x(s)) \leq 0$.

(iv) $k_i(t, t, x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \leq 0$ for $i = 1, ..., n$.

(v) $y(t) \in W^{1,\infty}$, $y(0) = 0$ and $y'(t) \geq 0$ for $a.e.$ $t \in J$.

Theorem 6.4.1 *Assume that conditions (i) through (v) hold. Then there exists* $x \in L^2[0,1]$, $x(t) \geq 0$ *a.e., that is (6.4) has a non-negative solution.*

Proof. To apply Theorem 5.4.4, we must show N is A-proper, weakly inward and the solutions of $x = \lambda Nx$ are bounded for $\lambda \in [0,1]$. We note that $L = I$ is clearly Fredholm of index zero and that the projection scheme mentioned in the introduction to Chapter 2 suffices for $X = L^2[0,1]$.

Now (ii) implies T_1 is c-dissipative since

$$\begin{aligned}
(T_1 x - T_1 y, x - y) &= (x - k(t,t,x) - y + k(t,t,y), x - y) \\
&= (x - y + k(t,t,y) - k(t,t,x), x - y) \\
&= (x - y - (k(t,t,x) - k(t,t,y)), x - y) \\
&= (x - y, x - y) - (k(t,t,x) - k(t,t,y), x - y) \\
&= |x - y|^2 - (k(t,t,x) - k(t,t,y), x - y) \\
&\leq |x - y|^2 - \alpha |x - y|^2 = (1 - \alpha) |x - y|^2 .
\end{aligned}$$

So $(T_1x - T_1y, x-y) \le c|x-y|^2$ where $c \in (0,1)$. And since T_2 is compact, $N = T_1 + T_2$ is A-proper by (iv) of Theorem 1.4.7.

To prove N is weakly inward we observe first that conditions (iii) and (v) imply T_2 maps K to K so we need only show T_1 is weakly inward. Let $x \in \partial K$ such that $x^*(x) = 0$ for some $x^* \in K^*$, then we identify x^* with an L^2 function, $x^*(t) \ge 0$ *a.e.* and $\int_0^1 x^*(t)\, x(t)\, dt = 0$. Thus $x^*(t) = 0$ *a.e.* on the set $\{t : x(t) \ne 0\}$. Applying x^* to T_1x we obtain

$$\begin{aligned} x^*(T_1x) &= \sum_{i=1}^{n}\int_0^1 x_i^*(t)\, x_i(t)\, dt - \sum_{i=1}^{n}\int_0^1 x_i^*(t)\, k_i(t,t,x_1(t),\dots,x_n(t))\, dt \\ &= 0 - \sum_{i=1}^{n}\int_{\{t:x_i(t)=0\}} x_i^*(t)\, k_i(t,t,x_1(t),\dots,x_{i-1}(t),0,x_{i+1}(t),\dots,x_n(t))\, dt \\ &\ge 0 \end{aligned}$$

by (iv). Hence N is weakly inward on K.

Before showing the solutions of $x = \lambda Nx$, $\lambda \in [0,1]$, are bounded, we state and prove a Gronwall type inequality for non-negative a.e. functions of L^1.

Lemma 6.4.2 *Let $x \in L^1[0,a]$ such that $x(t) \ge 0$ a.e. and suppose that*

$$x(t) \le C + M\int_0^t x(s)\, ds \ a.e. \tag{6.6}$$

where C and M are non-negative constants. Then $x(t) \le Ce^{Ma}$ a.e..

Proof. From (6.6) we have

$$\frac{Mx(t)}{C + M\int_0^t x(s)\, ds} \le M.$$

Then integrating both sides from 0 to t gives

$$\int_0^t \frac{Mx(t)}{C + M\int_0^t x(s)\, ds} dt \le \int_0^t M dt = Mt.$$

To integrate the left side we use the change of variable formula of Lebesgue integration, a special case of which we state summarily; *cf.* [22] for a proof.

Theorem 6.4.3 *If $\varphi : [a,b] \to [\alpha,\beta] \subset \mathbb{R}$ is a monotone, absolutely continuous function and $f \in L^1[\alpha,\beta]$ then $(f \circ \varphi)|\varphi'| \in L^1[a,b]$ and*

$$\int_{\alpha=\varphi(a)}^{\beta=\varphi(b)} f(u)\,du = \int_a^b (f\circ\varphi)\,|\varphi'|\,dt.$$

We let $f(u) = \frac{1}{u}$ on $[C,D]\,; C,D > 0$, and $\varphi(t) = C + M\int_0^t x(s)\,ds$ on $[0,t]$ and note that $f \in L^1[C,D]$ and φ is absolutely continuous since it is defined as an integral and monotonic since $x(s) \geq 0$ *a.e.*. Thus

$$\begin{aligned}\int_0^t \frac{Mx(t)}{C + M\int_0^t x(s)\,ds}dt &= \int_C^{C+M\int_0^t x(s)ds} \frac{1}{u}du \\ &= \ln\left(C + M\int_0^t x(s)\,ds\right) - \ln C \\ &= \ln\left(\frac{C + M\int_0^t x(s)\,ds}{C}\right).\end{aligned}$$

So the inequality above yields

$$\frac{C + M\int_0^t x(s)\,ds}{C} \leq e^{Mt}$$

that is

$$C + M\int_0^t x(s)\,ds \leq Ce^{Mt}.$$

Then by (6.6)

$$x(t) \leq C + M\int_0^t x(s)\,ds \leq Ce^{Mt} \leq Ce^{Ma} \text{ for } a.e.,\ t \leq a \in \mathbb{R}^+. \text{ Q.E.D.}$$

Now from $x = \lambda N x$ we have

$$x = \lambda\left(x(t) - k(t,t,x(t)) + y'(t) - \int_0^t k_t(t,s,x(s))\,ds\right)$$

and by (ii) with $y = 0$,

$$|k(t,t,x) - k(t,t,0)|\,|x| \geq \lambda\,|x|^2$$

or

$$|k(t,t,x) - k(t,t,0)| \geq \lambda\,|x|\,.$$

Substituting $k(t,t,x) = y'(t) - \int_0^t k_t(t,s,x(s))\,ds$ we obtain

$$\begin{aligned} \lambda|x| &\leq \left| y'(t) - \int_0^t k_t(t,s,x(s))\,ds - k(t,t,0) \right| \\ &\leq |y'(t)| + |k(t,t,0)| + \int_0^t |k_t(t,s,x(s))|\,ds \\ &\leq |y'(t)| + |k(t,t,0)| + \int_0^t M(1+|x|)\,ds \\ &= |y'(t)| + |k(t,t,0)| + Mt + M\int_0^t |x(s)|\,ds. \end{aligned}$$

Lemma 6.4.2 then gives $|x| \leq Ce^{Ma}$. Theorem 5.4.4 can now be applied taking $\Omega_K = B_r(0) \cap K$ with $r > Ce^{Ma}$ to obtain the required result. Q.E.D.

Appendix A

A.1 The derivation of $G(s,t)$

As demonstrated in Section 6.3, the solution to $-x''(t) = y(t)$ with periodic boundary conditions $x(0) = x(1)$ and $x'(0) = x'(1)$, is

$$
\begin{aligned}
x(t) &= x(0) + \int_0^1 t(1-s)\,y(s)\,ds - \int_0^t (t-s)\,y(s)\,ds \\
&= x(0) + \left(\int_0^t t(1-s)\,y(s)\,ds + \int_t^1 t(1-s)\,y(s)\,ds\right) - \int_0^t (t-s)\,y(s)\,ds \\
&= x(0) + \int_0^t [t(1-s) - (t-s)]\,y(s)\,ds + \int_t^1 t(1-s)\,y(s)\,ds \\
&= x(0) + \int_0^t s(1-t)\,y(s)\,ds + \int_t^1 t(1-s)\,y(s)\,ds \\
&= x(0) + \int_0^1 g(s,t)\,y(s)\,ds
\end{aligned}
$$

where

$$
g(s,t) = \begin{cases} s(1-t) & \text{for } 0 \le s < t \\ t(1-s) & \text{for } t \le s \le 1 \end{cases}.
$$

The projection $Px = \int_0^1 x(t)\,dt$ applied to the solution $x(t)$ above gives

$$
\begin{aligned}
Px &= P\left(x(0) + \int_0^1 g(s,t)\,y(s)\,ds\right) \\
&= \int_0^1 x(0)\,dt + \int_0^1 \int_0^1 g(s,t)\,y(s)\,dsdt
\end{aligned}
$$

$$= x(0) + \int_0^1 \int_0^1 g(s,t)\, y(s)\, ds dt.$$

Now $L_1^{-1} y = x_1(t) = x(t) - Px(t)$ so that

$$\begin{aligned}
x_1(t) &= \int_0^1 g(s,t)\, y(s)\, ds - \int_0^1 \int_0^1 g(s,t)\, y(s)\, ds dt \\
&= \int_0^1 g(s,t)\, y(s)\, ds - \int_0^1 \left(\int_0^1 g(s,t)\, dt \right) y(s)\, ds \\
&= \int_0^1 \left[g(s,t) - \int_0^1 g(s,t)\, dt \right] y(s)\, ds. \qquad \text{(A.1)}
\end{aligned}$$

Evaluation of the integral $\int_0^1 g(s,t)\, dt$ gives

$$\begin{aligned}
\int_0^1 g(s,t)\, dt &= \int_0^s g(s,t)\, dt + \int_s^1 g(s,t)\, dt \\
&= \int_0^s t(1-s)\, dt + \int_s^1 s(1-t)\, dt \\
&= \frac{1}{2}\left[t^2 (1-s) \right]_0^s + \frac{1}{2}\left[-s(1-t)^2 \right]_s^1 \\
&= (s^2/2)(1-s) + (s/2)(1-s)^2 \\
&= (s/2)(1-s).
\end{aligned}$$

Equation (A.1) can then be expressed as $\int_0^1 G(s,t)\, y(s)\, ds$ where

$$G(s,t) = g(s,t) - (s/2)(1-s) = \begin{cases} (s/2)(1-2t+s) & \text{for } 0 \le s < t \\ (1/2)(1-s)(2t-s) & \text{for } t \le s \le 1 \end{cases}.$$

A.2 Bounds for $H(s,t)$

We show that if β is chosen to be $1/24$ then $0 \le H(s,t) \le 1/8$ where

$$H(s,t) = \begin{cases} \beta + (s/2)(1-2t+s) - \int_0^1 G(s,t)\, ds & \text{for } 0 \le s < t \\ \beta + (1/2)(2t - 2st - s + s^2) - \int_0^1 G(s,t)\, ds & \text{for } t \le s \le 1 \end{cases}.$$

Evaluation of the integral $\int_0^1 G(s,t)\, ds$ gives

$$\int_0^1 G(s,t)\, ds = \frac{1}{2} \int_0^t s - 2st + s^2 ds + \frac{1}{2} \int_t^1 2t - 2st - s + s^2 ds$$

$$= \frac{1}{2}\left[s^2/2 - ts^2 + s^3/3\right]_0^t + \frac{1}{2}\left[2ts - ts^2 - s^2/2 + s^3/3\right]_t^1$$
$$= -1/12 \text{ (after some simplification).}$$

So

$$H(s,t) = \begin{cases} \beta + (s/2)(1-2t+s) + 1/12 \text{ for } 0 \le s < t \\ \beta + (1/2)(2t - 2st - s + s^2) + 1/12 \text{ for } t \le s \le 1 \end{cases}.$$

To determine the bounds for $H(s,t)$ on the region $R = [0,1] \times [0,1]$ in $\mathbb{R}^2$ we use the theory of maximization and minimization of a function of two variables. Now for $0 \le s < t$, $\partial H/\partial s = 1/2 - t + s$ and $\partial H/\partial t = -s$ so that the solution to the system $\partial H/\partial s = 0$, $\partial H/\partial t = 0$ is $s = 0$, $t = 1/2$ which is on the boundary of R. For $t \le s \le 1$, $\partial H/\partial s = -t - 1/2 + s$ and $\partial H/\partial t = 1 - s$ so that the solution to the system $\partial H/\partial s = 0$, $\partial H/\partial t = 0$ is $s = 1$, $t = 1/2$ which is also on ∂R. Thus we need only look for the extreme points of $H(s,t)$ on ∂R. We divide the boundary of R into four intervals: S_1 whose points have coordinates $(0,t)$, S_2 whose points have coordinates $(s,1)$, S_3 whose points have coordinates $(s,0)$ and S_4 whose points have coordinates $(1,t)$.

For the interval S_1, $H(0,t) = \beta + 1/12$ and hence constant.

For the interval S_2, $H(s,1) = \beta - s/2 + s^2/2 + 1/12$ and $\partial H/\partial s = -1/2 + s$. Then $H(s,1)$ has a minimum for $s = 1/2$, $t = 1$ (by second derivative test or by parabolic, concave up nature of $H(s,1)$) and $H(1/2,1) = \beta - 1/24$. The maximum values of $H(s,1)$ occur at the end points of S_2 and we have $H(0,1) = H(1,1) = \beta + 1/12$.

For the interval S_3, $H(s,0) = \beta - s/2 + s^2/2 + 1/12$ and $\partial H/\partial s = -1/2 + s$. Then $H(s,0)$ has a minimum for $s = 1/2$, $t = 0$ and $H(1/2,0) = \beta - 1/24$. As before, the maximum values occur at the end points of S_3 and we have $H(0,0) = H(1,0) = \beta + 1/12$.

Finally, on S_4, $H(1,t) = \beta + 1/12$ and hence constant. Thus the minimum value of $H(s,t)$ on R is $\beta - 1/24$ and the maximum value is $\beta + 1/12$. Therefore, if β is chosen to be $1/24$ we have $0 \le H(s,t) \le 1/8$.

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