Instanton effects in supersymmetric SU(N) gauge theories

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Instanton Effects in Supersymmetric $SU(N)$ Gauge Theories

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by

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Abstract

We investigate nonperturbative effects due to instantons in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills models, with the aim of testing the exact results predicted for these models. In two separate semiclassical calculations we obtain the one-instanton contribution to the Higgs condensate $u_3 = \langle \mathrm{Tr} A^3 \rangle$ and to the prepotential $\mathcal{F}$. Comparing our results with the exact predictions, we find complete agreement except when the number of flavours of fundamental matter hypermultiplets, N_f, takes certain values. The source of the u_3 discrepancy is an ambiguity in the parameterization of the hyperelliptic curves from which the exact predictions are derived when $N_f \geq N$. This ambiguity can easily be fixed using the results of instanton calculations. The discrepancy associated with $\mathcal{F}$ appears in the finite $N_f = 2N$ models. For these models we are unable to modify the curves to agree with the instanton calculations when $N > 3$.

Our one-instanton calculation of the prepotential is facilitated by a multi-instanton calculus which we construct, starting from the general solution of Atiyah, Drinfeld, Hitchin and Manin. Our calculus comprises: (i) the super-multi-instanton background, (ii) the supersymmetric multi-instanton action and (iii) the supersymmetric semiclassical collective coordinate measure. Our calculus has application to supersymmetric Yang-Mills theory with gauge group $U(N)$ or $SU(N)$.

We employ our instanton calculus to derive results at arbitrary k-instanton levels. In $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory, we derive a closed form expression for the k-instanton contribution to the prepotential. This amounts to a solution, in quadratures, of the low-energy physics of the theory, obtained from first principles. In supersymmetric $SU(2)$ Yang-Mills theory, we use our calculus to investigate multi-instanton contributions to higher-derivative terms in the Wilsonian effective action. Using a scaling argument, based on general properties of the $SU(2)$ k-instanton action and measure, we show that in the finite, massless $\mathcal{N} = 2$ and $\mathcal{N} = 4$ models, all k-instanton contributions to the next-to-leading higher-derivative terms vanish. This confirms a nonperturbative nonrenormalization theorem due to Dine and Seiberg.

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Declaration

I declare that no material presented in this thesis has previously been submitted for a degree at this or any other university.

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Contents

Chapter 1

Introduction

In quantum field theory, instantons are important as configurations that dominate the path integral. They give rise to nonperturbative effects which can in principle be calculated by performing a semiclassical expansion of the path integral. Unfortunately, in a generic gauge theory, technical difficulties have restricted semiclassical calculations to the level of the simplest (topological charge unity) instanton configuration. Moreover, in a classically scale-invariant Yang-Mills model such as quantum chromodynamics (QCD), instanton calculations suffer from an infra-red problem; the contribution of large instanton configurations to the path integral is divergent. This problem cannot be resolved without knowledge of the theory in the strong-coupling regime. As a result of these difficulties, until recently there had been few direct quantitative predictions of instanton effects in Yang-Mills theory.

In 1994, spectacular advances in the study of quantum field theory were made by Seiberg and Witten. The focus of the Seiberg-Witten analysis was a particular class of $SU(2)$ Yang-Mills models blessed with the property of $\mathcal{N} = 2$ supersymmetry. These models are characterized by the presence of an adjoint Higgs field which can break the $SU(2)$ gauge symmetry to a $U(1)$ symmetry. Seiberg and Witten discovered that the low-energy physics of the models admits more than one effective field theoretic description. At weak-coupling the low-energy physics is naturally described in terms of 'electric' degrees of freedom, corresponding to the light $U(1)$ components of the microscopic fields. At strong-coupling one can transform these 'electric' degrees of freedom into a weakly-coupled set of 'magnetic' degrees of freedom that provide an equivalent, but more convenient, description of the low-energy physics. This phenomenon is known as duality. Using arguments based on the presence of duality, together with supersymmetry, Seiberg and Witten were able to predict *exact* results for the models, valid at both strong and weak values of the coupling.

The work of Seiberg and Witten has motivated new investigations of instanton effects in quantum field theory. Seiberg and Witten predicted a solution for a holomorphic function known as the prepotential, which describes the dynamics of particle interactions at low-energy. At weak-coupling, the prepotential has an expansion consisting of a one-loop perturbative contribution plus an infinite series of nonperturbative terms. The nonperturbative terms are directly associated with instanton effects. They should precisely match the results of semiclassical instanton calculations. Such calculations therefore provide a non-trivial test of the Seiberg-Witten analysis and, in particular, of the physical duality-based arguments that were used.

A number of instanton calculations in the $\mathcal{N} = 2$ supersymmetric $SU(2)$ models investigated by Seiberg and Witten have indeed been successfully performed. In most cases, the results of these calculations are in complete agreement with the predictions of Seiberg and Witten. However, in the models with $N_f = 3$ and $N_f = 4$ flavours of fundamental matter hypermultiplets, interesting discrepancies have been observed. The Seiberg-Witten exact results are expressed in terms of an elliptic curve construction which involves various parameters of the physical theory. It has been found that the source of the discrepancies in the $N_f = 3$ and $N_f = 4$ models can be attributed to an ambiguity in the original physical interpretation of the parameters in the corresponding curves. By reinterpreting the parameters, in accordance with the instanton results, the discrepancies can be resolved.

In addition to providing these important insights into Seiberg-Witten theory, the programme of instanton tests in $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory has stimulated dramatic progress in the development of the semiclassical instanton method itself. Employing the multi-instanton construction of Atiyah, Drinfeld, Hitchin and Manin (ADHM) and with the aid of supersymmetry, Dorey, Khoze and Mattis were able to perform the first complete semiclassical calculation in the background of a two-instanton configuration. In subsequent work, Dorey, Khoze and Mattis have elegantly combined the ADHM construction with supersymmetry to formulate a complete multi-instanton calculus for supersymmetric $SU(2)$ Yang-Mills theory.

Investigations of instanton effects in supersymmetric Yang-Mills theory have not been confined to models with gauge group $SU(2)$. The exact results of Seiberg and Witten have been generalized to $\mathcal{N}=2$ supersymmetric Yang-Mills models with arbitrary classical simple and product gauge groups. It is desirable to perform instanton tests of the exact predictions in these models just as for the $SU(2)$ models.

Thesis Outline

In this thesis we investigate instanton effects in supersymmetric Yang-Mills theory with general gauge group $SU(N)$. We perform explicit one-instanton calculations that provide tests of the exact results in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD. We also construct a multi-instanton calculus for $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $U(N)$ or $SU(N)$. This represents a generalization of the $SU(2)$ work of Dorey, Khoze and Mattis. Using our calculus we derive a closed form expression for the k-instanton contribution to the prepotential in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD. We also employ the calculus to verify a nonperturbative nonrenormalization theorem in finite supersymmetric $SU(2)$ Yang-Mills models.

The thesis is organized as follows. In Chapter 2, we review instantons in Yang-Mills theory. The chapter is divided into two parts. In the first part, we describe classical properties of instantons in pure Yang-Mills theory. We motivate the assignment of an integer index, the topological charge, to instanton solutions and we show that a defining feature of instantons is that they imply a self-dual field strength. We also present the explicit $SU(2)$ one-instanton solution obtained by Belavin, Polyakov, Schwartz and Tyupkin and use it to construct the general $SU(N)$ one-instanton solution. In the second part of the chapter, we discuss instantons in the context of quantum field theory. We outline the semiclassical procedure for calculating nonperturbative effects due to instantons. To illustrate the procedure, we describe the calculation of the one-instanton contribution to the vacuum-to-vacuum amplitude in pure $SU(N)$ gauge theory, drawing on the seminal work of 't Hooft. We then discuss the effect of fermion fields on an instanton calculation. Finally, we consider the case of a Yang-Mills model that includes a Higgs field. We show that when the Higgs acquires a symmetry-breaking vacuum expectation value, instanton solutions do not formally exist, but that instanton calculations can nonetheless be performed using the 'constrained instanton' approach of Affleck.

In Chapter 3, we review aspects of supersymmetric Yang-Mills theory. This chapter has two parts. In the first part, we present the supersymmetric models that are the subject of our instanton investigations. An elegant construction of these models is provided by the $\mathcal{N} = 1$ superfield formalism. In the second part of the chapter, to set the scene for our instanton investigations, we review the exact predictions in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills models. The low-energy (Coulomb branch) physics of these models is determined by a single holomorphic function, known as the prepotential. We describe the Seiberg-Witten arguments that predict an exact solution for this function in terms of an elliptic curve construction. We also discuss the generalization of the Seiberg-Witten exact results to $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD and, in particular, the hyperelliptic curves that were proposed for this purpose.

In Chapter 4, we describe instanton tests of the exact solutions in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory. We first review the calculations performed in the $SU(2)$ models. We describe, in some detail, the one-instanton calculation of Finnel and Pouliot, which provided the first instanton test of the exact results. We also summarize the $SU(2)$ two-instanton calculations and discuss the resolution of the discrepancies revealed by these calculations. After reviewing the $SU(2)$ results, we present a one-instanton calculation in the $SU(N)$ models with $N > 2$. For the $SU(3)$ models with $N_f < 6$ hypermultiplet flavours we obtain a complete result. We find that the proposed hyperelliptic curves do not correctly predict this result. We show explicitly how the discrepancies can be resolved.

In Chapter 5, we present a multi-instanton calculus for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with gauge group $U(N)$ or $SU(N)$, extending the $SU(2)$ work of Dorey, Khoze and Mattis. Using the multi-instanton construction of Atiyah, Drinfeld, Hitchin, and Manin we obtain the set of solutions comprising the supersymmetric k-instanton background. We show how the supersymmetry algebra can be realized directly on the overcomplete set of collective coordinates appearing in these solutions. We then construct the k-instanton action and collective coordinate measure. Supersymmetry plays an important role; since they are supersymmetry invariant quantities, the instanton action and measure must be constructed from supersymmetry invariant combinations of the collective coordinates.

In Chapter 6, we apply the multi-instanton calculus to further investigate instanton effects in supersymmetric $SU(N)$ Yang-Mills theory. For the $\mathcal{N}=2$ supersymmetric models, we derive a closed form expression for the contribution of an arbitrary k-instanton configuration to the prepotential, as a finite-dimensional integral over bosonic and fermionic collective coordinates. Using our expression, we proceed to completely evaluate the one-instanton contribution to the prepotential, for arbitrary N and N_f. We discuss how our result compares with the predictions of the curves. In the finite $N_f = 2N$ models, we find a discrepancy which we do not know how to interpret. As a separate application of the multi-instanton calculus, we also investigate higher-derivative terms in the low-energy Wilsonian effective action of $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory. We are able to derive a closed form expression for the contribution of an arbitrary k-instanton configuration to the single real function that determines the next-to-leading terms in the gradient expansion of the Wilsonian effective action. Using a simple scaling argument we show that, in the finite $SU(2)$ model with four flavours, all k-instanton contributions to the next-to-leading terms vanish. This verifies a nonperturbative nonrenormalization theorem of Dine and Seiberg. Using a slightly modified scaling argument we also confirm the Dine-Seiberg nonrenormalization theorem for the $\mathcal{N}=4$ supersymmetric $SU(2)$ model.

Finally, in Chapter 7, we present a summary of our results and our conclusions.

Chapter 2

Instantons in Yang-Mills Theory

2.1 Introduction

In four-dimensional Yang-Mills field theory, there exist certain nontrivial solutions to the Euler-Lagrange equations which locally minimize the (Euclidean space) action. These solutions have the property of being localized in the (imaginary) time dimension as well as in spatial dimensions. For this reason they have come to be known as 'instantons'. The first instanton solution was discovered in 1975 by Belavin, Polyakov, Schwartz and Tyupkin (BPST) [1]. A few years later Atiyah, Drinfeld, Hitchin and Manin (ADHM) showed how to construct the most general instanton solution [2] — the well-known ADHM multi-instanton.

In quantum field theory, instantons are important as configurations that dominate the path integral. They give rise to nonperturbative effects, which can in principle be calculated by performing a semiclassical expansion of the path integral. In his famous 1976 paper, 't Hooft succeeded in performing such a calculation in $SU(2)$ gauge theory with the BPST instanton as background [3]. Due to various technical difficulties, subsequent calculations of instanton effects in Yang-Mills theory have mostly been restricted to the one-instanton level.

In this chapter, we present a review of instantons in Yang-Mills theory. This review is divided into two parts. In Section 2.2 we discuss classical properties of instanton solutions in pure Yang-Mills theory. We show that, as a consequence of the nontrivial topology of the gauge group, the space of finite action field configurations can be naturally divided into sectors labelled by a single integer index k. A distinct instanton solution is associated with each of these sectors. In Subsection 2.2.2, we proceed to derive the self-dual Yang-Mills equation, which serves as the defining equation for instantons in pure Yang-Mills theory. Finally, in Subsection 2.2.3, we present the explicit one-instanton solution in pure $SU(2)$ Yang-Mills theory derived by Belavin, Polyakov, Schwartz and Tyupkin. We discuss the main features of this solution and describe its $SU(N)$ generalization.

In Sections 2.3 and 2.4, we consider the calculation of nonperturbative effects in gauge theory due to instantons. In the first of these sections we describe the basic approach. Essentially, this consists of performing a semiclassical expansion of the path integral about the instanton solution, which represents a saddle-point in configuration space. We shall see that an important subtlety arises in connection with the 'zero-modes' associated with a generic instanton solution. (These can be understood as directions in configuration space in which the action is invariant.) The subsequent section is designed to illustrate the semiclassical instanton method in the specific context of Yang-Mills theory. Our aim is to prime the reader for the instanton calculations performed in Chapters 4 and 6. We first outline the one-instanton calculation of the vacuum-to-vacuum amplitude in pure $SU(N)$ Yang-Mills theory. For this purpose we draw extensively on the seminal results obtained by 't Hooft [3] and generalized by Bernard [4]. We then discuss the effect of fermions on the instanton calculation. Finally, we consider applying the instanton method to a Yang-Mills model with a Higgs sector. We show that the presence of a nonzero Higgs expectation value spoils the instanton solutions but that instanton effects can nonetheless be calculated using an approach formalized by Affleck [5].

Throughout this chapter we work in Euclidean space, using the conventions given in Appendix A.

2.2 Classical Properties

In this section, we examine classical properties of instanton solutions in pure Yang-Mills theory. We first show that, as finite action configurations, instanton solutions are labelled by a single integer k. We present the simple formula for calculating this integer, which is proved in Appendix B. Using the requirement that instantons locally minimize the action, we then derive a first-order differential equation satisfied by instanton solutions. This is the self-dual Yang-Mills equation. Finally, in Subsection 2.2.3, we present the BPST instanton solution. We discuss its salient features and show how it can be used to construct the general one-instanton solution in pure $SU(N)$ Yang-Mills theory.

Throughout this section we make use of the reviews [6, 7].

2.2.1 The Topological Charge

The Euclidean action of pure Yang-Mills theory is given by

$$S = \frac{1}{2}\int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}). \tag{2.1}$$

A necessary condition for this action to be finite is

$$\lim_{|x|\to\infty} F_{\mu\nu} = 0. \tag{2.2}$$

A necessary and sufficient condition for the field strength to vanish at large distances is

$$\lim_{|x|\to\infty} A_\mu = \frac{i}{g}U\partial_\mu U^{-1}, \tag{2.3}$$

where $U(x)$ is an element of the gauge group, G. At first glance it appears that, for the action to be finite, the gauge field A_μ must approach pure gauge (a gauge transformation of the $A_\mu = 0$ vacuum) at large distances. However, this is not generally true because the matrix $U(x)$ cannot necessarily be identified with a standard gauge transformation matrix. We now elaborate on this crucial point.

A standard gauge transformation is described by a group matrix, say $V(x)$, which represents a continuous mapping from Euclidean space, E^4, to the gauge group G. Such a mapping is in fact continuously deformable to the trivial mapping, from E^4 to a single element in G. To see this, let us select a point in space, x_0, at which V has the value V_0. We can consider a nested series of hyperspheres, centred at x_0, with infinitesimally increasing radii. Given V as a function of the angles (ϕ_1, ϕ_2, ϕ_3) on any of these hyperspheres, we can continuously deform V on the neighbouring hyperspheres into exactly the same $V(\phi_1, \phi_2, \phi_3)$. Hence, starting with the infinitesimal hypersphere at the point x_0, one can continuously deform $V(x)$ into the single element V_0 throughout E^4.

In contrast, the matrix U represents a continuous mapping from the 3-sphere at infinity, S^3_∞, to the gauge group G, and generally can *not* be continuously deformed to the trivial mapping. The situation is more subtle, and it is convenient to first consider the specific case of gauge group $SU(2)$.

It is well-known that the group $SU(2)$ is topologically equivalent to a 3-sphere. So for this gauge group, the matrix U in Eq. (2.3) gives a mapping from one 3-sphere to another. This kind of mapping is like the continuous mapping from the circle S^1 to itself. In that situation it easy to see that the set of all continuous mappings can be divided into equivalence classes, such that:

- the elements of each class can be continuously deformed into each other, and
- each class can be labelled by an integer index, which gives the number of times that the circle is 'wound' onto itself.

More formally, these equivalence classes form a homotopy group, $\Pi_1(S^1) = \mathbb{Z}$.

The set of continuous mappings of S^3 to itself can be divided into equivalence classes in just the same way. The integer index associated with each equivalence class now represents the number of times the 3-sphere is 'wrapped' onto itself. We call this index the 'Pontryagin index' or the 'topological charge'. Only when the topological charge associated with $U(x)$ is zero can it be continuously deformed into the trivial mapping. So only in this case can we identify the finite action gauge configuration in Eq. (2.3) as pure gauge. When the topological charge is nonzero, it distinguishes different sectors in the space of finite action configurations. Instanton solutions are configurations that minimize the action in each of these sectors.

To complete the discussion, we must now consider the situation for a general gauge group G. The generalization is straightforward, thanks to a theorem due to Bott [8]. Bott's theorem states that if G is an arbitrary simple Lie group then any continuous mapping of S^3 into G can be continuously deformed into a mapping into an $SU(2)$ subgroup of G. So we can always divide the space of finite action configurations into sectors, according to the number of times $U(x)$ wraps the 3-sphere at infinity onto such an $SU(2)$ subgroup. Instanton solutions are then configurations that minimize the action in each of these topological sectors.

To round off this subsection, we present the formula which gives the topological charge, $k \in \mathbb{Z}$, of a finite action field configuration. This will be used in the derivation of the self-dual Yang-Mills equation presented in the next subsection. The formula reads

$$k = \frac{g^2}{16\pi^2} \int d^4x \, \mathrm{Tr} \left(F_{\mu\nu} \tilde{F}_{\mu\nu} \right) , \tag{2.4}$$

where the dual field strength is defined by

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda} . \tag{2.5}$$

For the proof, we refer the reader to Appendix B.

2.2.2 Self-Duality

In this subsection we show that the field strength associated with instantons is self-dual. This property follows directly from the requirement that instantons locally minimize the action. The self-duality condition is a non-linear first-order differential equation. It can be taken as the defining equation for instantons in pure Yang-Mills theory.

Let us first consider field configurations with positive definite topological charge k. The trick is to write the action (2.1) as follows:

$$S = \frac{1}{4} \int d^4x \, \mathrm{Tr} \left[\left(F_{\mu\nu} - \tilde{F}_{\mu\nu} \right)^2 \right] + \frac{1}{2} \int d^4x \, \mathrm{Tr} \left(F_{\mu\nu} \tilde{F}_{\mu\nu} \right) . \tag{2.6}$$

Using the expression for the topological charge (2.4), we can easily evaluate the second integral:

$$\frac{1}{2}\int d^4x \,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) = \frac{8\pi^2}{g^2}k. \tag{2.7}$$

Since the first integral cannot be less than zero, this gives a lower bound on the action,

$$S \geq \frac{8\pi^2}{g^2}k. \tag{2.8}$$

The lower bound is attained, and the action is locally minimized, when

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}. \tag{2.9}$$

This is the self-duality condition, also known as the self-dual Yang-Mills equation, satisfied by k-instantons of positive topological charge.

If we consider configurations with negative k, and again write the action in the form (2.6), then we arrive at a negative lower bound on the action. This bound cannot be attained because the Euclidean Yang-Mills action cannot be less than zero. We conclude that there are no instantons of negative topological charge that satisfy the self-dual Yang-Mills equation. It is more useful, in the case of negative k configurations, to write the action as

$$S = \frac{1}{4}\int d^4x \,\mathrm{Tr}\left[\left(F_{\mu\nu} + \tilde{F}_{\mu\nu}\right)^2\right] - \frac{1}{2}\int d^4x \,\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right). \tag{2.10}$$

This gives a positive lower bound on the action,

$$S \geq \frac{8\pi^2}{g^2}(-k), \tag{2.11}$$

which is attained when

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu}. \tag{2.12}$$

This is the anti-self-duality condition that defines instanton solutions in pure Yang-Mills theory with negative topological charge. Such solutions will be referred to as anti-instantons whenever we wish to distinguish them from instantons of positive topological charge.

Note that solutions to Eqs. (2.9) and (2.12) automatically satisfy the Euler-Lagrange equation

$$D_\mu F_{\mu\nu} = 0, \tag{2.13}$$

by virtue of the Bianchi identity

$$D_\mu \tilde{F}_{\mu\nu} = 0. \tag{2.14}$$

However, the converse is not true, i.e. solutions to the Euler-Lagrange equation are not necessarily solutions to (2.9) or (2.12). Whereas the Euler-Lagrange equation is a second order differential equation, the self-dual Yang-Mills equation is a first order equation.

In the next subsection we consider the simplest nontrivial solution to the self-dual Yang-Mills equation.

2.2.3 The One-Instanton Solution

In 1975, Belavin, Polyakov, Schwartz and Tyupkin derived an explicit instanton solution of topological charge unity for the case of gauge group $SU(2)$ [1]. Their solution (henceforth referred to as the BPST instanton) takes the form

$$A_\mu = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}u^\dagger \frac{\tau^a}{2}u, \tag{2.15}$$

where $u \in SU(2)$ and the tensor $\eta^a_{\mu\nu}$ is defined in Appendix A. We list below the main features of the BPST instanton solution:

1. The solution contains a total of eight free parameters: four space-time coordinates x_0, representing the centre of the instanton, one dilatation parameter ρ, representing the size of the instanton, and three parameters implicit in the matrix u, representing the $SU(2)$ iso-orientation of the instanton. These parameters are known as 'collective coordinates'. They are associated with the classical global symmetries of pure $SU(2)$ Yang-Mills theory that are broken by the instanton solution. Specifically, these symmetries comprise space-time translations, scale transformations and global gauge transformations.

2. The self-dual Yang-Mills equation is clearly invariant under local gauge transformations. The expression (2.15) therefore represents the BPST instanton in a specific local gauge. It is conventional to call this 'regular gauge'. We discuss below an alternative

gauge for the BPST instanton, which proves to be more convenient for semiclassical calculations.

3. The field strength of the BPST instanton reads

$$F_{\mu\nu} = -\frac{4}{g}\eta^a_{\mu\nu}\frac{\rho^2}{\left((x-x_0)^2+\rho^2\right)^2}u^\dagger\frac{\tau^a}{2}u. \tag{2.16}$$

This is manifestly self-dual, by virtue of the self-duality property of the tensor $\eta^a_{\mu\nu}$ (see Appendix A).

4. At large distances, the BPST instanton has the asymptotic form (2.3), with

$$U(x) = u^\dagger\frac{x_\mu e_\mu}{|x|}. \tag{2.17}$$

This matrix gives a one-to-one mapping from the 3-sphere at infinity to the gauge group $SU(2)$. So the topological charge of the BPST instanton is manifestly equal to unity.

5. The BPST instanton is the most general $SU(2)$ one-instanton solution. It was shown in [9] that the most general k-instanton solution in pure $SU(2)$ Yang-Mills theory contains $8k$ free parameters. This is intuitive, since one expects that within the k-instanton moduli space there is a region corresponding to k well-separated BPST instantons, each containing eight free parameters.

6. An anti-instanton solution, with topological charge -1, is easily obtained from the BPST instanton by replacing $\eta^a_{\mu\nu}$ with the anti-self-dual tensor $\bar{\eta}^a_{\mu\nu}$.

7. The BPST solution takes the form of a 'lump' localized in time as well as space. It is this fact that originally inspired 't Hooft to invent the term 'instanton'.[1]

The BPST instanton can be used to construct the general one-instanton solution in pure $SU(N)$ Yang-Mills theory. The first step is to write

$$A_\mu = \begin{pmatrix} A_\mu^{BPST} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.18}$$

[1]In this thesis we shall use the term more generically, to refer to any set of classical configurations that locally minimizes the Euclidean action and provides a suitable background for semiclassical calculations.

where A_μ^{BPST} is the BPST instanton (2.15). This is clearly a solution to the $SU(N)$ self-dual Yang-Mills equation, of topological charge unity, but it is not the most general solution. In pure $SU(N)$ Yang-Mills theory the general k-instanton solution contains $4Nk$ free parameters [10].[2] What we are missing is the set of collective coordinates associated with global gauge transformations that rotate the embedded BPST instanton into $SU(N)$ group space. An arbitrary global gauge transformation acts on the solution (2.18) as follows:

$$\begin{pmatrix} A_\mu^{BPST} & 0 \\ 0 & 0 \end{pmatrix} \to \Omega^\dagger \begin{pmatrix} A_\mu^{BPST} & 0 \\ 0 & 0 \end{pmatrix} \Omega, \qquad \Omega \in SU(N). \tag{2.19}$$

It can be seen that there are two independent subgroups of $SU(N)$ that do not effect rotations of the embedded BPST instanton:

1. A $U(1)$ subgroup.

$$\Omega = \mathrm{diag}\left(e^{i\theta}, e^{i\theta}, e^{-2i\theta/(N-2)}, e^{-2i\theta/(N-2)}, \ldots, e^{-2i\theta/(N-2)}\right), \qquad 0 \le \theta < 2\pi. \tag{2.20}$$

2. An $SU(N-2)$ subgroup.

$$\Omega = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & \Omega' \end{pmatrix}, \qquad \Omega' \in SU(N-2). \tag{2.21}$$

We conclude that the embedded BPST instanton is properly rotated into group space by elements of the coset $SU(N)/T_N$, where

$$T_N = SU(N-2) \times U(1) \tag{2.22}$$

is the 'stability group' of the instanton. Since this coset is $(2N-5)$-dimensional and the embedded BPST instanton contains 5 free parameters (ignoring the $SU(2)$ iso-orientation matrix which is absorbed into the $SU(N)/T_N$ orientation matrix) it follows that

$$A_\mu = \Omega^\dagger \begin{pmatrix} A_\mu^{BPST} & 0 \\ 0 & 0 \end{pmatrix} \Omega, \qquad \Omega \in \frac{SU(N)}{SU(N-2) \times U(1)}, \tag{2.23}$$

is the most general $SU(N)$ one-instanton configuration.

[2] In [10], the number of collective coordinates of the k-instanton is quoted as $4Nk - N^2 + 1$ for $k \ge N/2$ and $4k^2 + 1$ for $k < N/2$. However, these formulae exclude the collective coordinates associated with global gauge rotations of the k-instanton configuration. Our counting includes these collective coordinates.

In our discussion of the BPST instanton, it was pointed out that the solution (2.15) is unique only up to local gauge transformations. In semiclassical calculations of instanton contributions to gauge invariant physical quantities, the choice of local gauge for the background instanton configuration does not matter; the final answer must clearly be independent of this choice. However, as we shall see in Section 2.4, in practice it is convenient to work with a specific instanton gauge known as 'singular gauge'. To obtain the BPST instanton in singular gauge, we perform a gauge transformation of the regular gauge solution (2.15),

$$A_\mu \to V A_\mu V^\dagger + \frac{i}{g} V \partial_\mu V^\dagger, \tag{2.24}$$

using the singular transformation matrix

$$V(x) = u^\dagger \frac{(x-x_0)_\mu \bar{e}_\mu}{|x-x_0|} u. \tag{2.25}$$

The result is

$$A_\mu = \frac{2}{g} \bar{\eta}^a_{\mu\nu} \frac{\rho^2}{(x-x_0)^2} \frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2} u^\dagger \frac{\tau^a}{2} u. \tag{2.26}$$

Note that the gauge transformation (2.25) is strictly only valid if we exclude the point $x = x_0$ from E^4. The topological charge of the singular gauge instanton is concentrated on the infinitesimal 3-sphere surrounding this point, rather than on the 3-sphere at infinity.

2.3 The Semiclassical Instanton Method

In this section we outline the procedure for calculating nonperturbative effects in quantum field theory due to instantons. The method basically consists of semiclassically expanding the path integral about the classical instanton configuration, which represents a saddle-point of the action in configuration space. In the one-loop approximation, to which we restrict ourselves throughout this thesis, we only retain terms in the Taylor expanded action that are quadratic in the field fluctuations. The integration over the field fluctuations is therefore Gaussian, and easily accomplished. However, we shall see that in a generic instanton background, this integration (naïvely) leads to an ill-defined answer, due to the presence of so-called 'zero-modes'. We shall see how this difficulty is resolved by making a change of integration variables, from the parameters associated with field fluctuations in the direction of the zero-modes to the collective coordinates in the instanton solution.

We shall illustrate the semiclassical instanton method in the context of a general field theoretic model which involves a single real scalar field ϕ. We shall derive a general expression for an instanton contribution to the partition function Z, which is given by

$$Z = N \int [d\phi] e^{-S[\phi]} \tag{2.27}$$

in the path integral formalism. Here $S[\phi]$ is the Euclidean action of the model and N is an infinite normalization factor. An instanton solution $\phi^{\rm cl}$ is a classical field configuration that locally minimizes the action. Hence it must satisfy the Euler-Lagrange equation

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi^{\rm cl}} = 0. \tag{2.28}$$

To perform a semiclassical expansion of the path integral (2.27), we write

$$\phi = \phi^{\rm cl} + \phi^{\rm qu}, \tag{2.29}$$

and Taylor expand the action about the instanton background:

$$S[\phi] = S\left[\phi^{\rm cl}\right] + \frac{1}{2}\int d^4x\, \phi^{\rm qu}(x)\hat{M}(x)\phi^{\rm qu}(x) + \mathcal{O}((\phi^{\rm qu})^3), \tag{2.30}$$

where

$$\hat{M} = \left.\frac{\delta^2 S}{\delta \phi^2}\right|_{\phi=\phi^{\rm cl}}. \tag{2.31}$$

Note that there is no term linear in the quantum field fluctuation $\phi^{\rm qu}$ because $\phi^{\rm cl}$ satisfies the Euler-Lagrange equation (2.28). In what follows, we neglect the $\mathcal{O}((\phi^{\rm qu})^3)$ terms in the expanded action (2.30). This represents a one-loop approximation. As stated above, throughout this thesis we restrict ourselves to one-loop semiclassical instanton calculations.

The operator $\hat{M}$ generally possesses a complete, orthogonal set of orthogonal eigenfunctions $\{\phi_i\}$, with corresponding eigenvalues $\{\epsilon_i\}$. It is convenient to adopt the ϕ_i as a basis for the quantum field fluctuations,

$$\phi^{\rm qu} = \sum_i c_i \phi_i. \tag{2.32}$$

The quadratic term in the expanded action (2.30) then becomes

$$\frac{1}{2}\int d^4x\, \phi^{\rm qu}(x)\hat{M}(x)\phi^{\rm qu}(x) = \frac{1}{2}\sum_i c_i^2 ||\phi_i||^2 \epsilon_i. \tag{2.33}$$

In the semiclassical approach, we can define the functional integral measure to be

$$[d\phi] = \prod_i \frac{||\phi_i||}{\sqrt{2\pi}}\, dc_i, \tag{2.34}$$

where

$$||\phi_i||^2 = \int d^4x\, \phi_i(x)\cdot\phi_i(x). \tag{2.35}$$

After substituting Eqs. (2.30), (2.33) and (2.34) into the path integral (2.27), and performing the Gaussian integrations over the c_i, we obtain the following simple expression for the instanton contribution to the partition function:

$$Z^{(1)} = Ne^{-S[\phi^{\rm cl}]}(\det\hat{M})^{-\frac{1}{2}}, \tag{2.36}$$

where

$$\det\hat{M} = \prod_i \epsilon_i. \tag{2.37}$$

Unfortunately, this expression is too simple; we have not considered the possibility that some of the ϵ_i are vanishing. If this is the case then Eq. (2.36) is formally ill-defined.

In a generic instanton background, the small-fluctuations operator $\hat{M}$ does indeed possess zero-eigenvalues. These are directly related to the classical symmetries broken by the instanton solution. Let us suppose that the instanton solution $\phi^{\rm cl}$ breaks m classical symmetry generators and thus is parameterized by m collective coordinates, γ_i, $i = 1,\ldots,m$. Then it represents a family of classical field configurations which map out an m-dimensional region of configuration space throughout which the action is constant. In this region there are necessarily m independent directions in which the action is invariant. These directions are precisely the zero-eigenfunctions or 'zero-modes' of the operator $\hat{M}$.[3]

There is a standard procedure for dealing with zero-modes, known as the collective coordinate method [11]. Let us identify the zero-modes of the operator $\hat{M}$ with the first

[3]Typically, the zero-modes are just given by the m tangent vectors $\partial\phi^{\rm cl}/\partial\gamma_i$. However, this is not the case in a gauge theory, because the zero-modes are constrained by a gauge-fixing condition.

m eigenfunctions ϕ_i, $i = 1, \ldots, m$. It is clear that the quadratic term in the action (2.33) has no dependence on the corresponding parameters c_i, $i = 1, \ldots, m$. Consequently, the integration over these parameters is divergent. The collective coordinate method consists of changing the integration variables from the m parameters c_i to the m collective coordinates of the instanton solution, γ_i. In this way one avoids a divergent integration.

An efficient way to make the change of variables from the c_i to the collective coordinates is to insert the following factor of unity into the path integral:

$$1 = \int d\gamma_1 d\gamma_2 \cdots d\gamma_m \, (\det \Delta) \prod_{i=1}^{m} \delta[(\phi - \phi^{\rm cl}, \phi_i)], \tag{2.38}$$

where

$$\Delta_{ij} = \left(\frac{\partial \phi^{\rm cl}}{\partial \gamma_i}, \phi_j \right) + \mathcal{O}(\phi - \phi^{\rm cl}). \tag{2.39}$$

We have used the following definition of the inner product of two functions:

$$(f, g) = \int d^4x \, f(x) \cdot g(x). \tag{2.40}$$

In the one-loop approximation, the $\mathcal{O}(\phi - \phi^{\rm cl})$ terms in Eq. (2.39) can be neglected. We now obtain, after expanding $\phi^{\rm qu}$ as per Eq. (2.32) and integrating out the parameters c_i, $i > m$, associated with the nonzero modes, the following expression for the instanton contribution to the partition function:

$$Z^{(1)} = N \int \left(\prod_{i=1}^{m} d\gamma_i \right) \left(\prod_{i=1}^{m} dc_i \right) (\det \Delta) \left(\prod_{i=1}^{m} \frac{||\phi_i||}{\sqrt{2\pi}} \delta \left(c_i ||\phi_i||^2 \right) \right) (\det{}' \hat{M})^{-\frac{1}{2}} e^{-S[\phi^{\rm cl}]}, \tag{2.41}$$

where

$$\det{}' \hat{M} = \prod_{i>m} \epsilon_i. \tag{2.42}$$

(The prime on the determinant indicates that the zero eigenvalues are excluded.) The δ-functions in Eq. (2.41) saturate the integrations over the c_i, $i = 1, \ldots, m$. After integrating out these parameters we obtain

$$Z^{(1)} = N \int d\gamma_1 d\gamma_2 \ldots d\gamma_m \, (\det \Delta) \left(\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}} \frac{1}{||\phi_i||} \right) (\det{}' \hat{M})^{-\frac{1}{2}} e^{-S[\phi^{\rm cl}]}. \tag{2.43}$$

This is our desired general expression for the instanton contribution to the partition function (2.27). To normalize the contribution, we should divide by the one-loop perturbative expression for the partition function. This amounts to setting

$$N(\det{}'\hat{M})^{-\frac{1}{2}} \to \left(\frac{\det{}'\hat{M}}{\det\hat{M}^0}\right)^{-\frac{1}{2}} \tag{2.44}$$

in Eq. (2.43), where $\hat{M}^0$ is the operator $\hat{M}$ evaluated in the background $\phi^{\rm cl} = 0$.

In the above analysis, we have taken the field ϕ to be a real scalar. However, it is easy to see how the final expression is modified for other kinds of fields. For instance, if ϕ were a complex scalar then the determinant factor in appearing in Eq, (2.43) would become $(\det{}'\hat{M})^{-1}$. On the other hand, if we were dealing with a fermion field then we would get a factor $(\det{}'\hat{M})^{+1}$. In the case of fermion zero-modes, the collective coordinates are Grassmann parameters. We discuss the effects of fermion zero-modes on semiclassical computations in Yang-Mills theory in Section 2.4.2.

Our analysis has also focussed on a specific field theoretic object, namely the partition function. For a generic Green's function, the (one-loop) semiclassical instanton method yields a formula similar to (2.43), with the field insertions in the path integral saturated by the classical instanton background.

Finally, we note that in a generic renormalizable quantum field theory there will be one-loop ultra-violet divergences associated with the determinant factor in Eq. (2.43); the operator $\hat{M}$ has infinitely many large eigenvalues. The renormalization procedure must be applied to the instanton calculation just as it is applied in perturbation theory. In the following section, we shall use the example of a one-instanton calculation in pure $SU(N)$ Yang-Mills theory to demonstrate how the renormalization procedure can be implemented in an instanton calculation.

2.4 One-instanton Effects in Yang-Mills Theory

The purpose of this section is to illustrate the main features of the semiclassical instanton method as applied to Yang-Mills theory. To this end, we shall consider the calculation of one-instanton effects in particular $SU(N)$ Yang-Mills models.

To begin with, in Subsection 2.4.1, we focus on pure $SU(N)$ Yang-Mills theory. Following 't Hooft [3] and Bernard [4], we describe, in some detail, the derivation of the one-instanton contribution to the vacuum-to-vacuum amplitude in this theory. This will serve to illustrate a number of general features of instanton calculations in gauge theory. We shall see that there is a natural and convenient gauge for the quantum field fluctuations, which leads to a considerable simplification in the calculation of the small-fluctuations determinants [3]. We shall also see that the ultra-violet divergence of these determinants can be regularized using the Pauli-Villars method, and that, in the final answer, the regularization mass appears with exactly the right power to renormalize the instanton factor $\exp(-8\pi^2/g^2)$, in accordance with the one-loop perturbative β-function [3].

In Subsection 2.4.2, we discuss the effect of fermion fields on the semiclassical analysis. We consider an $SU(2)$ gauge theory that includes a single Dirac fermion. We see that, in the background of the BPST instanton, the Dirac operator possesses zero-modes. In the massless theory, these fermion zero-modes imply a vanishing result unless the path integral contains field insertions that saturate the integrations over the associated Grassmann collective coordinates.

Finally, in Subsection 2.4.3, we consider an $SU(2)$ Yang-Mills model that includes a Higgs field. We show that the presence of a non-vanishing Higgs expectation value spoils the classical scale invariance of the theory. Consequently, instanton solutions do not formally exist. One can nonetheless perform an instanton calculation, using a 'constrained' instanton background [5]. In this model, the integration over the instanton size is cut off by the (inverse) Higgs expectation value, so that the infra-red problem is avoided.

2.4.1 Pure $SU(N)$ Yang-Mills Theory

In this subsection we outline the derivation of the one-instanton contribution to the vacuum-to-vacuum amplitude[4] in pure $SU(N)$ Yang-Mills theory, following the analysis of 't Hooft [3]

[4]In physical terms, instanton contributions to the vacuum-to-vacuum amplitude represent tunnelling effects between topologically nontrivial vacua of the theory [12] (see also [13]). This was the original

and Bernard [4].

In the path integral formalism, the vacuum-to-vacuum amplitude in pure $SU(N)$ gauge theory has the following form:

$$W = N \int [dA_\mu] [d\bar{\eta}] [d\eta] \, e^{-S[A_\mu,\bar{\eta},\eta]} \tag{2.45}$$

where

$$S[A_\mu,\bar{\eta},\eta] = \int d^4x \left\{ \frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) + \frac{1}{2}C\left(A_\mu\right)^2 + \mathcal{L}_{\mathrm{gh}}(\bar{\eta},\eta) \right\}. \tag{2.46}$$

The terms $C(A_\mu)$ and $\mathcal{L}_{\mathrm{gh}}(\bar{\eta},\eta)$ are the usual terms associated with the Faddeev-Popov gauge-fixing procedure; $C(A_\mu)$ is the gauge-fixing term and $\mathcal{L}_{\mathrm{gh}}(\bar{\eta},\eta)$ is the corresponding ghost term ($\bar{\eta}$ and η are the ghost fields).

To perform a semiclassical expansion of the path integral, we write

$$A_\mu = A_\mu^{\mathrm{cl}} + A_\mu^{\mathrm{qu}}, \tag{2.47}$$

where A_μ^{cl} is the $SU(N)$ one-instanton configuration given by Eq. (2.23). Ultimately, it does not matter whether we take the embedded BPST instanton to be in regular gauge (2.15) or in singular gauge (2.26); as stated earlier this choice cannot affect the final, gauge-invariant result. However, as we see below, in singular gauge the instanton zero-modes die off sufficiently fast at large distances to make the calculation of the collective coordinate measure much easier [4]. Substituting (2.47) into the pure Yang-Mills action (2.1), one obtains

$$\begin{aligned}\frac{1}{2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) &= \frac{8\pi^2}{g^2} + \frac{1}{2}\int d^4x\,\mathrm{Tr}\Big(2\left(D_\mu^{\mathrm{cl}}A_\nu^{\mathrm{qu}}\right)^2 - 2\left(D_\mu^{\mathrm{cl}}A_\mu^{\mathrm{qu}}\right)^2 \\ &\quad -4igF_{\mu\nu}^{\mathrm{cl}}\left[A_\mu^{\mathrm{qu}},A_\nu^{\mathrm{qu}}\right] + \mathcal{O}\left(\left(A_\mu^{\mathrm{qu}}\right)^3\right)\Big),\end{aligned} \tag{2.48}$$

where D_μ^{cl} is the covariant derivative evaluated in the one-instanton background.

It was found by 't Hooft that the problem of evaluating the small-fluctuations determinants in the instanton background was simplified in the particular gauge (for the field fluctuations) defined by

$$D_\mu^{\mathrm{cl}}A_\mu^{\mathrm{qu}} = 0. \tag{2.49}$$

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motivatation for the introduction of the well-known θ-term to the Lagrangian of Yang-Mills theory; the θ-parameter labels the infinite set of vacua induced by instanton tunneling effects [12].

This gauge is known as (covariant) background gauge. The effect of background gauge is to restrict the quantum fluctuations $A_\mu^{\rm qu}$ to directions in configuration space that are orthogonal to the directions associated with infinitesimal gauge transformations of $A_\mu^{\rm cl}$. To see this, first observe that an infinitesimal gauge transformation of $A_\mu^{\rm cl}$ is given by

$$A_\mu^{\rm cl} \to A_\mu^{\rm cl} + D_\mu^{\rm cl}\Lambda. \tag{2.50}$$

The orthogonality property is then

$$\int d^4x \,{\rm Tr}\left(A_\mu^{\rm qu} D_\mu^{\rm cl}\Lambda\right) = 0. \tag{2.51}$$

By integrating by parts, and using the fact that Λ is arbitrary, it is easy to see that the covariant background gauge condition (2.49), is equivalent to this orthogonality condition.

To implement the background gauge condition (2.49), the gauge-fixing and ghost terms in the action (2.46) take the following form:

$$\int d^4x \,\frac{1}{2} C\left(A_\mu\right)^2 = \int d^4x \,{\rm Tr}\left(\left(D_\mu^{\rm cl} A_\mu^{\rm qu}\right)^2\right), \tag{2.52}$$

$$\int d^4x \,\mathcal{L}_{\rm gh}(\bar\eta,\eta) = -2\int d^4x \,{\rm Tr}\left(\bar\eta\left(D^{\rm cl}\right)^2\eta\right). \tag{2.53}$$

Note that the gauge-fixing term (2.52) cancels one of the terms in the expansion (2.48). To quadratic order in $A_\mu^{\rm qu}$, the expanded action now takes the form

$$S\left[A_\mu,\bar\eta,\eta\right] = \frac{8\pi^2}{g^2} + \frac{1}{2}\int d^4x \,A_\mu^{{\rm qu}\,a}\left(\hat M_A\right)_{\mu\nu}^{ab} A_\nu^{{\rm qu}\,b} + \int d^4x \,\bar\eta^a\left(\hat M_{\rm gh}\right)^{ab}\eta^b, \tag{2.54}$$

where

$$\hat M_A A_\mu^{{\rm qu}\,a} = -\left(D^{\rm cl}\right)^2 A_\mu^{{\rm qu}\,a} - 2g f_{abc} F_{\mu\nu}^{{\rm cl}\,b} A_\nu^{{\rm qu}\,c}, \tag{2.55}$$

$$\hat M_{\rm gh}\eta^a = -\left(D^{\rm cl}\right)^2\eta^a. \tag{2.56}$$

The operator $\hat M_{\rm A}$ possesses a total of $4N$ zero-modes $A_\mu^{(i)}$, $i = 1,\ldots,4N$, corresponding to the collective coordinates of the $SU(N)$ one-instanton solution (2.23). (In contrast, the operator $\hat M_{\rm gh}$ does not possess any zero-modes.) To deal with these zero-modes, one applies the collective coordinate method, as outlined in the previous section. This gives the following expression for the collective coordinate measure:

$$\int d^{4N}\gamma\,(\det\Delta)\prod_{i=1}^{4N}\frac{1}{\|A^{(i)}\|}\frac{1}{\sqrt{2\pi}}, \tag{2.57}$$

where

$$\Delta_{ij} = \sum_{\mu,\,a} \left(\frac{\partial A_\mu^{\mathrm{cl}\,a}}{\partial \gamma_i}, A_\mu^{(j)\,a} \right), \qquad \left\| A^{(i)} \right\| = \sum_{\mu,\,a} \left(A_\mu^{(i)\,a}, A_\mu^{(i)\,a} \right). \tag{2.58}$$

Here we have designated the collective coordinates x_0, ρ and those implicit in Ω, by the set $\{\gamma_i\}$. Now in background gauge, the zero-modes of $A_\mu^{(i)}$ take the following form:

$$A_\mu^{(i)} = \frac{\partial A_\mu^{\mathrm{cl}}}{\partial \gamma_i} + D_\mu^{\mathrm{cl}} \Lambda^{(i)}. \tag{2.59}$$

Here $\Lambda^{(i)}$ is an infinitesimal generator of a gauge transformation, determined by the background gauge requirement

$$D_\mu^{\mathrm{cl}} A_\mu^{(i)} = 0. \tag{2.60}$$

It was shown by Bernard that the calculation of the Jacobian factors in (2.57) is simplified if one works with the singular gauge instanton background. In this case, direct calculation shows that, at large distances,

$$\Lambda^{(i)\,a} A_\mu^{(j)\,a} < \mathcal{O}\left(\frac{1}{|x|^3} \right), \tag{2.61}$$

for all the zero-modes [4]. (Only the five zero-modes associated with translations and scale transformations have this long-distance behaviour in regular gauge.) It follows that, upon integrating by parts, we have

$$\sum_{\mu,\,a} \left(D_\mu^{\mathrm{cl}} \Lambda^{(i)\,a}, A_\mu^{(j)\,a} \right) = - \sum_{\mu,\,a} \left(\Lambda^{(i)\,a}, D_\mu^{\mathrm{cl}} A_\mu^{(j)\,a} \right). \tag{2.62}$$

The right-hand side of this equation vanishes by virtue of the background gauge condition Eq. (2.60). Hence $\Delta_{ij} = \delta_{ij} \left\| A^{(i)} \right\|$ and the expression for the collective coordinate measure (2.57) simplifies to

$$\int d^{4N}\gamma \prod_{i=1}^{4N} \frac{\left\| A^{(i)} \right\|}{\sqrt{2\pi}}. \tag{2.63}$$

With some straightforward algebra, and a somewhat careful treatment of the integration over group space collective coordinates, one now obtains the following final expression for the (bosonic) one-instanton measure in $SU(N)$ Yang-Mills theory [3, 4]:

$$\frac{\pi^{2N-2}}{(N-1)!(N-2)!} \int d\Omega \int d^4x_0 d\rho \, \frac{4}{\rho^5} \left(\frac{2\rho\sqrt{\pi}}{g} \right)^{4N}. \tag{2.64}$$

Here the integration over the group space orientation of the instanton is normalized so that

$$\int d\Omega = 1. \tag{2.65}$$

The 't Hooft-Bernard collective coordinate measure (2.64) (and its supersymmetric multi-instanton generalization) is an essential requirement for the instanton calculations performed in Chapters 4 and 6.

Let us now consider the determinant factors obtained by integrating out the quantum field fluctuations $A_\mu^{\rm qu}$ and η. When combined with the prefactor N that normalizes the path integral, the net result is a factor

$$\left(\frac{\det' \hat{M}_A}{\det \hat{M}_A^0}\right)^{-\frac{1}{2}} \frac{\det \hat{M}_{\rm gh}}{\det \hat{M}_{\rm gh}^0} \tag{2.66}$$

Formally, this expression is ill-defined because the determinants contain one-loop divergences. These divergences must be controlled by regularizing the theory in some way. Dimensional regularization is not applicable because the instanton calculation is explicitly performed in four dimensions. Instead, one can introduce Pauli-Villars fields into the Lagrangian, whose large mass, μ, acts as a regularization parameter. To subtract the ultra-violet divergences, the Pauli-Villars fields have opposite statistics to the physical fields. Their effect is to divide each determinant in (2.66) by a regulator determinant, $\det(\hat{M} + \mu^2)$ [3]. In particular, the primed determinant $\det' \hat{M}_A$ becomes

$$\det{}' \hat{M}_A \to \frac{\det' \hat{M}_A}{\det(\hat{M}_A + \mu^2)} = \mu^{-8N} \frac{\det' \hat{M}_A}{\det'(\hat{M}_A + \mu^2)} \tag{2.67}$$

We have extracted the $4N$ lowest eigenvalues from the regulator determinant, to leave a dimensionless ratio of (primed) determinants.

In his investigation of a generic $SU(2)$ Yang-Mills model, 't Hooft discovered that, in the covariant background gauge, all of the normalized, regulated (and where necessary, primed) small-fluctuations determinants are essentially given by a single formula. For the case of a massless scalar field of isospin-t, the formula reads [3]

$$\frac{\det \hat{M}}{\det(\hat{M} + \mu^2)} \frac{\det(\hat{M}^0 + \mu^2)}{\det \hat{M}^0} = \exp\left(\frac{C(t)}{6} \ln(\mu\rho) + \alpha(t)\right) \tag{2.68}$$

(The functions $C(t)$ and $\alpha(t)$ are given in Appendix A). The $SU(2)$ gauge boson has isospin-1 and four Lorentz components. As a consequence, its small-fluctuations determinant is given by the right-hand side of Eq. (2.68), with $t = 1$, taken to the power four. The operator associated with the Faddeev-Popov ghosts is identical to the operator associated with the massless scalars, namely $(D^{\rm cl})^2$, so its determinant is given exactly by the right-hand side of Eq. (2.68), with $t = 1$. As for fermion fields, it turns out that the determinant associated with a massless two-component Weyl fermion is also given by precisely the right-hand side of Eq. (2.68). Dirac fermions are comprised of two Weyl fermions, so the associated determinant is just the square of this.

Because of the way the $SU(N)$ one-instanton solution is constructed by a simple embedding of the BPST instanton, it is straightforward to apply 't Hooft's $SU(2)$ results to the model at hand. The determinants of the operators $\hat{M}_A$ and $\hat{M}_{\rm gh}$ are invariant under global gauge rotations of the embedded BPST instanton, effected by the matrix Ω in Eq. (2.23). So without losing anything, we can take $A^{\rm cl}_\mu$ in the definitions of $\hat{M}_A$ and $\hat{M}_{\rm gh}$ to be the unrotated upper-left embedded BPST instanton (2.18). We now have to consider how the $N^2 - 1$ colour components of the fields $A^{\rm qu}_\mu$ and η transform under the action of the $SU(2)$ embedding subgroup. The $(N-2)^2$ colour components associated with the generators of the instanton stability group evidently transform as isospin singlets. The three components associated with the generators of the $SU(2)$ subgroup itself form an isospin triplet, and the remaining $4(N-2)$ components form isospin doublets. Using this information, together with 't Hooft's determinant formula (2.68), we deduce that the determinant factors (2.66) yield

$$\mu^{4N} \exp\left(-\frac{N}{3}\ln(\mu\rho) - \alpha(1) - 2(N-2)\alpha(\tfrac{1}{2})\right). \tag{2.69}$$

From the expression for the collective coordinate measure (2.64) and from the above expression for the regularized, normalized small-fluctuations determinant factors, we obtain the following expression for the one-instanton contribution to the vacuum-to-vacuum amplitude:

$$W_{1I} = \frac{\mu^{11N/3}e^{-8\pi^2/g^2}}{g^{4N}}\frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}e^{\left(-\alpha(1)-2(N-2)\alpha(\frac{1}{2})\right)}\int d^4x_0 d\rho\, \rho^{11N/3-5}. \tag{2.70}$$

(We have performed the trivial integration over group space.) With regard to this final expression we make two important comments:

1. The Pauli-Villars regulator mass μ appears with exactly the right power to renormalize the bare coupling in the instanton factor $\exp(-8\pi^2/g^2)$. This follows from the perturbative relation
$$\frac{8\pi^2}{g_{PV}^2} = \frac{8\pi^2}{g^2} - \frac{11N}{3}\ln(\mu), \tag{2.71}$$
obtained by integrating the one-loop perturbative β-function. (Here g and g_{PV} are the bare coupling and the Pauli-Villars renormalized coupling, respectively.) The factor of g^{-4N} appearing in (2.70) is expected to be renormalized by higher loop effects.

2. The integration over the instanton size ρ is divergent. This is the well-known infra-red problem associated with instanton effects in classically scale invariant Yang-Mills models. Strong-coupling effects presumably serve to cut off the integration over ρ at the characteristic length scale of the theory, Λ^{-1}, but this cannot be seen directly in the semiclassical approach. In Subsection 2.4.3, we shall see that in a theory with a symmetry-breaking Higgs sector, the scale provided by the Higgs expectation value serves to cut off the ρ integration, so that the infra-red problem is avoided.

2.4.2 Inclusion of Fermions

In this subsection, we examine the effect of fermion fields on instanton calculations in Yang-Mills theory. As a specific example, we consider the model whose (Euclideanized) action is
$$S = \frac{1}{2}\int d^4x\, \mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \int d^4x\, (\bar{\Psi}\hat{\gamma}_\mu D_\mu \Psi + m\bar{\Psi}\Psi). \tag{2.72}$$
Here Ψ is a Dirac spinor of mass m and the $\hat{\gamma}$-matrices are defined in Appendix A. We take the gauge group to be $SU(2)$ and Ψ to transform in the fundamental representation.

According to the approach of 't Hooft, the one-instanton background consists of the BPST instanton and the trivial fermion field configuration $\Psi = \bar{\Psi} = 0$. The operator associated with fermion field fluctuations is then given by
$$\hat{M}_\Psi = \hat{\gamma}_\mu D_\mu^{\mathrm{cl}} + m. \tag{2.73}$$

We first consider the situation for massless fermions. (The mass will later be introduced as a small perturbation.)

When $m = 0$, the operator (2.73) possesses zero-modes. In fact, in the background of the BPST instanton there are precisely two zero-modes, one for each Weyl component of the Dirac spinor. Let us write

$$\Psi = \begin{pmatrix} \lambda \\ \bar{\psi} \end{pmatrix}, \qquad \bar{\Psi} = (\psi \ \bar{\lambda}). \tag{2.74}$$

In terms of the Weyl components, the zero-modes are the solutions to

$$\bar{e}_\mu D^{\rm cl}_\mu \lambda = 0, \tag{2.75}$$

$$e_\mu D^{\rm cl}_\mu \bar{\lambda} = 0, \tag{2.76}$$

and similarly for ψ and $\bar{\psi}$. It is important to note that Eqs. (2.75) and (2.76) are not simply Hermitian conjugates of each other. (This reflects the fact that in Euclidean space, a spinor and its conjugate are treated as independent field variables.) The Atiyah-Singer index theorem predicts that in a k-instanton background, the number of solutions to (2.75) minus the number of solutions to (2.76) equals k [14]. Moreover, it is not difficult to show that in a self-dual background there are no solutions to the equation for $\bar{\lambda}$. The trick is to act on Eq. (2.76) with the operator $\bar{e}_\nu D^{\rm cl}_\nu$. Using Eq. (A.28) and the property $\bar{\eta}^a_{\mu\nu} F_{\mu\nu} = 0$ for a self-dual field strength, we obtain

$$\left(D^{\rm cl}\right)^2 \bar{\lambda} = 0. \tag{2.77}$$

The operator $\left(D^{\rm cl}\right)^2$ is positive definite and therefore cannot have a zero eigenvalue.

In the background of a singular gauge BPST instanton, the unique solution to Eq. (2.75) is [3]

$$\lambda^i_\alpha = \frac{\rho}{\pi} \frac{y_\mu (e_\mu)^i_\alpha}{\sqrt{y^2}(y^2+\rho^2)^{\frac{3}{2}}} \xi, \tag{2.78}$$

where $y_\mu = (x - x_0)_\mu$ and i is the isospin index. The parameter ξ is the Grassmann collective coordinate associated with the fermion zero-mode. We shall use the symbol ζ to represent the collective coordinate associated with the same zero-mode solution for the other Weyl spinor, ψ.

In a semiclassical instanton calculation, the fermionic zero-modes can be dealt with using the collective coordinate method, just as for bosonic zero-modes. Note that the solution (2.78) satisfies (after stripping away ξ)

$$\sum_{\alpha,i}\int d^4x\,\lambda^{*\,i}_{\alpha}\lambda^{i}_{\alpha}=1. \tag{2.79}$$

Hence it is normalized, so that the collective coordinate measure associated with the fermion zero-modes is simply $\int d\xi d\zeta$.

For the one-instanton contribution to a generic Green's function in the massless theory to be non-vanishing, the path integral must contain field insertions that saturate the integration over ξ and ζ. The instanton calculations of Chapters 4 and 6 will explicitly demonstrate this principle. Since the functional integral expression for the vacuum-to-vacuum amplitude contains no field insertions, the one-instanton contribution to this amplitude vanishes. More generally, one can argue that all topologically nontrivial instanton sectors give vanishing contributions to the vacuum-to-vacuum amplitude. Physically, the effect of massless fermions is to suppress the vacuum tunnelling associated with instantons.

Along with the collective coordinate measure associated with the zero-modes comes the t' Hooft determinant factor associated with the nonzero modes. From the master formula (2.68) we deduce that this determinant factor is

$$\frac{\det'\hat{M}_\Psi}{\det(\hat{M}_\Psi+\mu)}\,\frac{\det(\hat{M}^0_\Psi+\mu)}{\det\hat{M}^0_\Psi}=\mu^{-1}\exp\left(\tfrac{1}{3}\ln(\mu\rho)+2\alpha(\tfrac{1}{2})\right). \tag{2.80}$$

The factors of the Pauli-Villars mass μ appearing here combine with the factors arising from the bosonic 't Hooft determinants (see Eq. (2.69)) in exactly the right way to renormalize the instanton factor $\exp(-8\pi^2/g^2)$.

Let us now consider the effect of turning on the fermion mass m. When the mass is small compared to the inverse size of the instanton[5], $m\ll\rho^{-1}$, it is small compared to the nonzero eigenvalues of the massless operator $\hat{\mathcal{M}}_\Psi$. If we regard the mass term as perturbing

[5]Since ρ is an integration variable, we can really only take the mass to be small up to some critical (inverse) value of this parameter. In the next subsection we discuss a scenario in which the ρ-integration has a manifest cut-off. In this situation the mass is amenable to a perturbative treatment throughout the whole integration over ρ.

the eigenvalues of the massless operator then, in the lowest order approximation, only the zero-modes are affected. Their perturbed eigenvalue takes the value m. To account for this perturbation, we may write

$$m \int d^4x\, \bar{\Psi}\Psi \rightarrow m\xi\zeta \tag{2.81}$$

in the action (2.72), and continue to use the collective coordinate prescripton. In the calculation of the vacuum-to-vacuum amplitude, the mass term (2.81) then saturates the integration over the Grassmann collective coordinates ξ and ζ.

The replacement (2.81) is more obvious if we adopt the viewpoint that the zero-modes constitute part of the instanton background, rather than the trivial configuration $\bar{\Psi} = \Psi = 0$. The term $m\xi\zeta$ is then just part of the instanton action, obtained by evaluating the mass term at the zero-modes. This interpretation of the zero-modes is very natural in supersymmetric models, where the various fields comprising an instanton background form supersymmetry multiplets [15]. In the supersymmetric instanton calculus developed in Chapter 5 it is essential to regard fermion zero-modes in this way.

2.4.3 Inclusion of a Higgs Field and the Constrained Instanton

We now discuss the application of the semiclassical instanton method to models that possess a symmetry-breaking Higgs sector. For illustrative purposes we shall focus on one fairly simple example, which consists of pure $SU(2)$ Yang-Mills theory coupled to a complex Higgs doublet. It has the Euclidean action

$$S = \frac{1}{2}\int d^4x \,\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \int d^4x \left(D_\mu\phi^\dagger D_\mu\phi + \lambda\left(|\phi|^2 - |\langle\phi\rangle|^2\right)^2\right). \tag{2.82}$$

The associated Euler-Lagrange equations are:

$$D_\mu F^a_{\mu\nu} = ig\phi^\dagger T^a D_\mu\phi + c.c. \tag{2.83}$$

$$D^2\phi = 2\lambda(|\phi|^2 - |\langle\phi\rangle|^2)\phi \tag{2.84}$$

If the Higgs vacuum expectation value (VEV), $\langle\phi\rangle$, is zero, then a suitable background for a semiclassical calculation is given by the BPST instanton solution and $\phi = 0$. But when $\langle\phi\rangle^2 > 0$, there are no non-singular configurations that minimize the action. The underlying reason is that the Higgs potential breaks the classical scale invariance of the pure Yang-Mills theory. This can be seen using Derrick's theorem [16]. We perform the following scale transformation of the fields:

$$A_\mu(x) \to \frac{1}{a} A_\mu\left(\frac{x}{a}\right), \qquad \phi(x) \to \phi\left(\frac{x}{a}\right). \tag{2.85}$$

The effect on the action is

$$S \to \frac{1}{2}\int d^4x \,\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) + \int d^4x \left(a^2 D_\mu\phi^\dagger D_\mu\phi + a^4\lambda\left(|\phi|^2 - |\langle\phi\rangle|^2\right)^2\right). \tag{2.86}$$

If we choose $a < 1$, which amounts to shrinking the configurations, then the action is made smaller. Only in the singular limit, $a \to 0$, does the action reach a minimum.

How then, can there be any hope of performing instanton calculations in theories such as these? An answer to this question was first intuited by 't Hooft [3] and later refined by Affleck [5]. According to 't Hooft, one should be able to work with approximate instanton configurations that do not strictly minimize the action, provided that they dominate the path integral; since the action decreases with configuration size, the appropriate configurations are small. More precisely, a suitable instanton background is provided by solutions to the Euler-Lagrange equations in the region of configuration space where the scale parameter ρ satisfies

$$\rho \ll \frac{1}{M}, \tag{2.87}$$

where M is the mass scale set by the Higgs.

In the model at hand, the Higgs mass is $M = \sqrt{\lambda}\langle\phi\rangle$. In the small-$\rho$ region of configuration space, the Euler-Lagrange equations (2.83) and (2.84) approximate to:

$$D_\mu F_{\mu\nu} = 0, \tag{2.88}$$

$$D^2\phi = 0; \qquad \lim_{|x|\to\infty}\phi = \langle\phi\rangle. \tag{2.89}$$

Solutions to these equations are provided by the BPST instanton on the one hand and

$$\phi = \sqrt{\frac{x^2}{x^2+\rho^2}}\langle\phi\rangle \tag{2.90}$$

on the other [3].

Let us now consider calculating the contribution of this approximate one-instanton solution to some Green's function. Evaluating the action (2.82) in the instanton background, we find

$$S = \frac{8\pi^2}{g^2} + 2\pi^2\rho^2 |\langle\phi\rangle|^2 + \mathcal{O}\left(\lambda\rho^4 |\langle\phi\rangle|^4\right). \tag{2.91}$$

The second term in this instanton action dominates the collective coordinate integration over the scale ρ. In effect, this term cuts off the ρ-integration, at

$$\rho \sim \frac{1}{|\langle\phi\rangle|}. \tag{2.92}$$

Note that, since the semiclassical instanton analysis assumes g and λ to be small parameters, this cut-off belongs to the small-ρ region specified by (2.87), so that the ρ-integration can safely be performed. The essential point is that, in contrast to the pure Yang-Mills case discussed in Subsection 2.4.1, the ρ-integration in this model is convergent, and the infra-red problem is avoided.

In [5], it was shown that the approximate instanton background of 't Hooft can be motivated using a more rigorous approach. The idea is to impose a constraint on the Euler-Lagrange equations so that they do permit exact solutions. (Formally, the constraint can be introduced into the path integral using a δ-functional insertion.) The resulting 'constrained instanton' configurations can be solved for perturbatively at both long and short distances. At leading order, the short-distance constrained instanton is identical to 't Hooft's approximate instanton configuration. The long-distance expansion is also constraint-independent at leading order and is important in the calculation of low-energy Green's functions; it provides the appropriate field insertions. We now examine this constrained instanton formalism.

The Constrained Instanton Formalism

According to Derrick's theorem, the reason that the action (2.82) cannot be minimized is that it always decreases with the scale parameter ρ. However, in each region of configuration space corresponding to a fixed value of ρ, the action may, and generally will, have a minimum. The basic idea of the constrained instanton approach of Affleck is that if the path integral is broken up into an infinity of sectors, labelled by ρ, then in each sector one can perform a semiclassical expansion about a truly minimized action. The integration over ρ at the end of the calculation amounts to a summation of the contributions from all the sectors.

To implement this idea formally, Affleck proposed that the configuration size should be fixed by a constraint of the form [5]:

$$\int d^4x\, O = c\rho^{4-d}, \tag{2.93}$$

where O is some d-dimensional local operator of the fields. The constant c can be chosen to take any convenient value. To impose the constaint in the path integral, one can insert a factor of unity, written as

$$1 = \int d\rho\, \frac{dF}{d\rho}\delta[F], \qquad F = \int d^4x\, O - c\rho^{4-d}. \tag{2.94}$$

In the presence of the constraint (2.93), Affleck analysed the Euler-Lagrange equations (2.83) and (2.84) at long and short distances. Let us first consider the long-distance regime, defined by $|x| \gg \rho$. Provided the operator O is chosen to vanish quickly enough, the long-distance constrained instanton satisfies the following, linearized Euler-Lagrange equations [5]:

$$\left(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu + \frac{1}{2}g^2|\langle\phi\rangle|^2\delta_{\mu\nu}\right)A_\nu = 0, \tag{2.95}$$

$$\left(-\partial^2 + 4\lambda|\langle\phi\rangle|^2\right)\delta\phi = 0. \tag{2.96}$$

These equations reflect the Higgs mechanism, which gives a mass to the gauge bosons and to the Higgs field fluctuations, $\delta\phi = \phi - \langle\phi\rangle$. The solutions to (2.95) and (2.96) can be obtained perturbatively [5]:

$$A_\mu(x) = C_A\left(\frac{1}{g}\frac{x_\nu}{|x|^4} - \frac{g}{2}|\langle\phi\rangle|^2\frac{x_\nu}{|x|^2} + \dots\right)\bar{\eta}^a_{\mu\nu}\frac{\tau^a}{2}, \tag{2.97}$$

$$\phi(x) = \langle\phi\rangle + C_\phi\left(-\frac{1}{2}\frac{1}{|x|^2} - \lambda|\langle\phi\rangle|^2\ln\frac{k|x|}{\sqrt{\lambda}|\langle\phi\rangle|} + \dots\right)\langle\phi\rangle. \tag{2.98}$$

Here C_A, C_ϕ and k are constants which are determined using a 'patching' condition described below.

In the short-distance regime, given by $|x| \ll 1/M$, the constrained instanton configurations are obtained by perturbing the Euler-Lagrange equations, (2.83) and (2.84) in the small parameter ρM. One obtains the perturbative expansions [5]

$$A_\mu(x) = A_\mu^{(0)}(x) + A_\mu^{(1)}(x)\,\lambda|\langle\phi\rangle|^2\rho^2 + \ldots \tag{2.99}$$

$$\phi(x) = \phi^{(0)}(x) + \phi^{(1)}(x)\,\lambda|\langle\phi\rangle|^3\rho^3 \ln\left(\sqrt{\lambda}|\langle\phi\rangle|\rho\right) + \ldots \tag{2.100}$$

As one would hope, $A_\mu^{(0)}(x)$ and $\phi^{(0)}(x)$ can be identified with the approximate instanton configurations of 't Hooft, given by the BPST instanton, Eq. (2.26), and the configuration (2.90). The higher order terms depend on the choice of the operator O in the constraint equation (2.94).

The analysis of the constrained instanton solution is not yet complete. One must determine the constants in the long-distance expansions (2.97) and (2.98) and one must also supply boundary conditions for the higher order terms in the short-distance expansions (2.99) and (2.100). The way that both these requirements are met is through a 'patching' condition [5], which equates both expansions in the intermediate regime $\rho \ll |x| \ll 1/M$. In particular, by comparing the leading order short-distance terms $A_\mu^{(0)}(x)$ and $\phi^{(0)}(x)$ with the leading order long-distance terms we find that $C_A = 2\rho^2$ and $C_\phi = \rho^2$, independently of the specific choice of constraint.

Let us now consider semiclassical calculations in the background of the constrained instanton detailed above. In evaluating the instanton action, one finds that the short-distance 'core' of the instanton dominates, and the result is just 't Hooft's expression (2.91). An important point is that, due to the cut-off (2.92), the $\mathcal{O}\left(\lambda\rho^4\langle\phi\rangle^4\right)$ terms in the instanton action (2.91) are effectively $\mathcal{O}(\lambda)$. In the one-loop approximation it is therefore legitimate to discard these terms. We can also discard $\mathcal{O}(\rho M)$ corrections to small-fluctuations determinants. This means we can directly apply the formula (2.68) derived by 't Hooft. We can also apply the 't Hooft-Bernard collective coordinate measure (2.64), because at leading order in ρM (or equivalently, in the coupling), the effect of the insertion (2.94), is just to introduce ρ as a collective coordinate in the standard way.

It may be that we wish to evaluate a low-energy Green's function of the theory. In this case, the right expressions to use as field insertions in the semiclassical instanton calculation are provided by the leading order long-distance 'tail' of the constrained instanton configurations [5].

The above conclusions are of general validity. We apply the constrained instanton formalism to perform instanton calculations in supersymmetric Yang-Mills theory in Chapters 4 and 6.

Chapter 3

Supersymmetric Yang-Mills Theory

3.1 Introduction

In this chapter we review the supersymmetric Yang-Mills models that are the subject of our semiclassical instanton investigations. An elegant way to build supersymmetric models is provided by the superfield formalism. In Section 3.2 we present the $\mathcal{N} = 1$ superfield formalism and use it to construct the Lagrangians of the Yang-Mills models of interest. The main focus of our investigations is the $\mathcal{N} = 2$ class of models for which exact results have been predicted. To set the scene for our instanton calculations, we review, in Section 3.3, the Seiberg-Witten analysis in $\mathcal{N} = 2$ supersymmetric pure $SU(2)$ Yang-Mills theory [17, 18]. We also discuss the generalization of the Seiberg-Witten exact solutions to other $\mathcal{N} = 2$ supersymmetric $SU(N)$ models [19, 20, 21, 22].

3.2 Supersymmetry

In what follows we present the formalism used to describe supersymmetric Yang-Mills theory. To a large extent we follow the approach of Wess and Bagger [25]. (Our conventions are summarized in Appendix A.) For more details, we refer the reader to this and the other standard text [26].

To set up a supersymmetry algebra, one first defines the $\mathcal{N}$ supersymmetry generators

$Q_{A\,\alpha}$, $A = 1, 2, \ldots, \mathcal{N}$. The index α indicates that under the action of the Lorentz group, these generators transform like left-handed Weyl spinors (see Appendix A). One also defines $\mathcal{N}$ conjugate generators, $\bar{Q}_{A\,\dot{\alpha}}$, $A = 1, 2, \ldots, \mathcal{N}$, that transform like right-handed Weyl spinors. Taking these generators together with the generator of space-time translations, P_m, one now defines the following $\mathbb{Z}_2$ graded algebra:

$$\begin{aligned}
\left\{Q_{A\,\alpha}, \bar{Q}_{B\,\dot{\beta}}\right\} &= 2\sigma^m_{\alpha\dot{\beta}} P_m \delta_{AB}, \\
\left\{Q_{A\,\alpha}, Q_{B\,\beta}\right\} &= 2\sqrt{2}\epsilon_{\alpha\beta} Z_{AB}, \\
\left\{\bar{Q}_{A\,\dot{\alpha}}, \bar{Q}_{B\,\dot{\beta}}\right\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}} Z^*_{AB}, \\
\left[P_m, Q_{A\,\alpha}\right] &= 0, \\
\left[P_m, \bar{Q}_{A\,\dot{\alpha}}\right] &= 0, \\
\left[P_m, P_n\right] &= 0, \qquad (3.1)
\end{aligned}$$

where Z_{AB} and Z^*_{AB} are antisymmetric in A and B; they are the central charge matrices of the algebra.[1]

To construct a supersymmetric field theory, one must find multiplets of fields that form representations of the above supersymmetry algebra. An elegant way to do this is provided by the superfield formalism. We restrict our attention to the $\mathcal{N} = 1$ formalism.

3.2.1 $\mathcal{N} = 1$ Superspace and Superfields

Each point in $\mathcal{N} = 1$ superspace is labelled by eight coordinates. Four of these are the usual Minkowski space coordinates, x_m. The other four are given by Grassmann parameters, θ_α and their conjugates, $\bar{\theta}_{\dot{\alpha}}$. These objects transform like Weyl spinors, as their index structure suggests. In superspace, the generators of the supersymmetry algebra can be represented as linear differential operators,

$$P_m = i\partial_m, \qquad (3.2)$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \qquad (3.3)$$

[1]The supersymmetric models with which we are concerned realize the non-centrally extended supersymmetry algebra, so we set $Z = Z^* = 0$ henceforth.

$$\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\partial_m. \tag{3.4}$$

It is useful to define

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i\sigma^{m}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \tag{3.5}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^{\alpha}\sigma^{m}_{\alpha\dot{\alpha}}\partial_m. \tag{3.6}$$

The D operators satisfy the supersymmetry algebra (3.1), but with the sign of P_m reversed. They anticommute with the Q's:

$$\{D_{\alpha}, Q_{\beta}\} = \{D_{\alpha}, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_{\beta}\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0. \tag{3.7}$$

The next step is to generalize the notion of a field in space-time to superspace. We define a superfield F to be a function of the superspace coordinates, with the Taylor expansion

$$\begin{aligned} F\left(x,\theta,\bar{\theta}\right) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta^2 m(x) + \bar{\theta}^2 n(x) \\ &\quad + \theta\sigma^m\bar{\theta}v_m(x) + \theta^2\bar{\theta}\bar{\lambda}(x) + \bar{\theta}^2\theta\psi(x) + \theta^2\bar{\theta}^2 d(x). \end{aligned} \tag{3.8}$$

(Higher order terms in θ and $\bar{\theta}$ must vanish due to the anticommuting property of these parameters.) In the language of quantum field theory, the component fields of this expansion comprise Weyl spinors, scalar bosons and a vector boson v_m. Together they form a representation of the supersymmetry algebra. We can derive their transformation laws by acting on $F(x,\theta,\bar{\theta})$ with the infinitesimal supersymmetry generator

$$\delta_{\xi} = \xi Q + \bar{\xi}\bar{Q}, \tag{3.9}$$

where ξ_{α} and $\bar{\xi}_{\dot{\alpha}}$ are arbitrary Grassmann parameters and Q_{α} and $Q_{\dot{\alpha}}$ are the differential operators (3.3) and (3.4). The action of δ_{ξ} on $F(x,\theta,\bar{\theta})$ is to generate a new superfield with transformed components $\delta_{\xi}f(x)$, $\delta_{\xi}\phi(x)$, etc.

The representation provided by the component fields of $F(x,\theta,\bar{\theta})$ is too general for the purpose of constructing supersymmetric models. One can reduce the representation by imposing a supersymmetry covariant algebraic constraint on the superfield F. In what follows we shall specify the two constraints that define chiral and vector superfields. With these two superfields we shall be able to construct the supersymmetric Yang-Mills models of interest.

Chiral Superfields

A chiral (or scalar) superfield Φ satisfies the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{3.10}$$

The covariance of this constraint under supersymmetry transformations follows from the anticommutation property (3.7). It is easy to solve the constraint (3.10) for the component fields of Φ. Let us define shifted Minkowski space coordinates,

$$y^m = x^m + i\theta\sigma^m\bar{\theta}. \tag{3.11}$$

In terms of the superspace coordinates $(y, \theta, \bar{\theta})$ we can write

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}, \tag{3.12}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \tag{3.13}$$

Hence the general solution to (3.10) is

$$\begin{aligned} \Phi\left(y, \theta, \bar{\theta}\right) &= A\left(y\right) + \sqrt{2}\theta\psi\left(y\right) + \theta^2 F\left(y\right) \\ &\equiv A\left(x\right) + \sqrt{2}\theta\psi\left(x\right) + \theta^2 F\left(x\right) + i\theta\sigma^m\bar{\theta}\partial_m A\left(x\right) \\ &\quad + \frac{i}{\sqrt{2}}\theta^2\bar{\theta}\bar{\sigma}^m\partial_m\psi\left(x\right) + \frac{1}{4}\theta^2\bar{\theta}^2\partial_m\partial^m A\left(x\right). \end{aligned} \tag{3.14}$$

The Hermitian conjugate of Φ satisfies

$$D_\alpha\Phi^\dagger = 0. \tag{3.15}$$

Its component field expansion is given by conjugating each term in the above expansion of Φ.

Vector Superfields

A vector superfield V satisfies the constraint

$$V = V^\dagger. \tag{3.16}$$

In terms of component fields, the most general solution to this constraint is

$$\begin{aligned} V\left(x,\theta,\bar{\theta}\right) &= f\left(x\right)+\theta\phi\left(x\right)+\bar{\theta}\bar{\phi}\left(x\right)+\theta^2 m\left(x\right)+\bar{\theta}^2 m^\dagger\left(x\right) \\ &\quad +\theta\sigma^m\bar{\theta}v_m\left(x\right)+\theta^2\bar{\theta}\bar{\lambda}\left(x\right)+\bar{\theta}^2\theta\lambda\left(x\right)+\theta^2\bar{\theta}^2 D\left(x\right), \end{aligned} \tag{3.17}$$

where $f(x)$, $v_m(x)$, and $D(x)$ are Hermitian fields.

In the $\mathcal{N}=1$ superfield construction of supersymmetric Yang-Mills theory, the vector superfield may be regarded as the supersymmetric generalization of the vector potential. It transforms in the adjoint representation of the gauge group, so we write

$$V = V^a T^a. \tag{3.18}$$

Supersymmetric Yang-Mills theory is invariant under a generalized gauge transformation, which acts on the vector superfield as follows,

$$e^{-2gV} \rightarrow e^{2ig\Lambda^\dagger} e^{-2gV} e^{-2ig\Lambda}. \tag{3.19}$$

The parameter g is the gauge coupling and $\Lambda = \Lambda^a T^a$ is an adjoint chiral superfield. This generalized gauge transformation incorporates the usual gauge transformation of the vector potential,

$$v_m \rightarrow U v_m U^\dagger + \frac{i}{g} U \partial_m U^\dagger, \tag{3.20}$$

where $U(x)$ is an element of the gauge group.

In supersymmetric Yang-Mills theory, we can use the generalized gauge transformation (3.19) to eliminate the field components $f(x)$, $\phi(x)$, and $m(x)$ from V. This procedure fixes the theory to what is known as Wess-Zumino gauge. In this supersymmetric gauge, all that remains of the generalized gauge symmetry is the actual gauge symmetry. Throughout this thesis, our analysis of supersymmetric Yang-Mills theory will use the Wess-Zumino gauge description.

The supersymmetric generalization of the field strength tensor is defined as

$$W_\alpha = \frac{1}{8g}\bar{D}^2 e^{2gV} D_\alpha e^{-2gV}. \tag{3.21}$$

This is a chiral superfield. In terms of the superspace coordinates $(y, \theta, \bar{\theta})$ it has the expansion

$$W_\alpha = -i\lambda_\alpha(y) + \left[D(y)\,\delta_\alpha^\beta - \frac{i}{2}v_{mn}(y)\,(\sigma^m\bar{\sigma}^n)_\alpha^\beta\right]\theta_\beta + \left[\sigma^m D_m\bar{\lambda}(y)\right]_\alpha \theta^2, \tag{3.22}$$

where v_{mn} is the usual field strength tensor and

$$D_m\bar{\lambda}^{\dot{\alpha}} = \partial_m\bar{\lambda}^{\dot{\alpha}} - ig\left[v_m, \bar{\lambda}^{\dot{\alpha}}\right]. \tag{3.23}$$

(This indicates the form of the covariant derivative when acting on fields in the adjoint representation.) Under the transformation (3.19), the supersymmetric field strength and its conjugate obey

$$W_\alpha \to e^{2ig\Lambda}W_\alpha e^{-2ig\Lambda}, \qquad \bar{W}_{\dot{\alpha}} \to e^{2ig\Lambda^\dagger}\bar{W}_{\dot{\alpha}}e^{-2ig\Lambda^\dagger}. \tag{3.24}$$

The superfields V and W_α and chiral superfields in various representations of the gauge group are all that we require to construct the Lagrangians of renormalizable supersymmetric Yang-Mills theory.

3.2.2 Supersymmetric Yang-Mills Theory

In this subsection, we use the $\mathcal{N} = 1$ superfield formalism to construct the supersymmetric Yang-Mills models that are of interest in this work. We first consider the $\mathcal{N} = 1$ supersymmetric models and then proceed to the models with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ extended supersymmetry. The $\mathcal{N} = 2$ models are of primary concern in this thesis, since it is for these models that exact solutions have been predicted [17, 18, 19, 20, 21, 22]. (We review the exact results in the next section.)

The models we describe contain chiral superfields that transform in the adjoint, fundamental and conjugate-fundamental representations of the gauge group. An adjoint chiral superfield is a matrix,

$$\Phi = \Phi^a T^a. \tag{3.25}$$

We shall use the symbol Q to denote a chiral superfield in the fundamental representation. We shall denote its component fields (q,χ,G) to distinguish them from the components

(A,ψ,F) of the adjoint superfield Φ. We write $\tilde{Q}$ for a chiral superfield in the conjugate representation and $(\tilde{q},\tilde{\chi},\tilde{G})$ for its components. If the fundamental representation is N-dimensional then Q is understood to be an N-dimensional column vector. Likewise $\tilde{Q}$ is understood to be an N-dimensional row vector. With this in mind, we can suppress colour indices.

$\mathcal{N}=1$ Supersymmetric Pure Yang-Mills Theory

The pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory is given by the Lagrangian[2]

$$\mathcal{L}_{\mathcal{N}=1\ SYM} = \frac{1}{2}\int d^2\theta\, \mathrm{Tr}\,(W^\alpha W_\alpha) + c.c. \tag{3.26}$$

Both supersymmetry invariance and gauge invariance of this model are manifest in the $\mathcal{N}=1$ superfield formalism. Gauge invariance is a consequence of invariance of the Lagrangian under the generalized gauge transformation, which acts on W_α according to (3.24). To see supersymmetry invariance, we note two things. First, any product of chiral superfields is a chiral superfield. Second, the θ^2 component of a chiral superfield transforms into a total derivative under a supersymmetry transformation. Since the integral in (3.26) picks out just such a component, it follows that the Lagrangian transforms into a total derivative under a supersymmetry transformation and hence the action is invariant.

In component fields, the action of this model reads

$$S_{\mathcal{N}=1\ SYM} = \frac{1}{2}\int d^4x\, \mathrm{Tr}\left(-v_{mn}v^{mn} - 4i\bar{\lambda}\,\bar{\not{D}}\lambda + 2D^2\right). \tag{3.27}$$

It is clear that the field D plays no dynamical role in the model; it can immediately be integrated out of the action. As the other models we construct will demonstrate, this is a generic property of the field D. For this reason it is referred to as an 'auxiliary' field.

Without spoiling supersymmetry, we can make the action (3.27) slightly more general, by including a θ-term,

$$\frac{\theta}{16\pi^2}\int d^4x\, \mathrm{Tr}\left(-v_{mn}\tilde{v}^{mn}\right). \tag{3.28}$$

[2]For integration over the Grassmann parameters θ and $\bar{\theta}$ we use the conventions $\int d^2\theta\,\theta^2 = 1$ and $\int d^2\bar{\theta}\,\bar{\theta}^2 = 1$.

All of the supersymmetric Yang-Mills models considered in this thesis will be implicitly assumed to include this term.

$\mathcal{N} = 1$ Supersymmetric QCD

This theory is obtained by coupling the vector multiplet to chiral multiplets in a gauge invariant way. We write

$$\mathcal{L}_{\mathcal{N}=1\ SQCD} = \mathcal{L}_{\mathcal{N}=1\ SYM} + \mathcal{L}_{matter}, \tag{3.29}$$

where $\mathcal{L}_{\mathcal{N}=1\ SYM}$ is given by Eq. (3.26) and, in the simplest case,

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta}\, Q^\dagger e^{-2gV} Q. \tag{3.30}$$

The chiral superfield Q is in the fundamental representation of the gauge group. It obeys the generalized gauge transformation law

$$Q \to e^{2ig\Lambda} Q, \qquad Q^\dagger \to Q^\dagger e^{-2ig\Lambda^\dagger}, \tag{3.31}$$

so that, from (3.19), it is clear that $\mathcal{L}_{matter}$ is gauge invariant. Supersymmetry invariance follows because we are picking out the $\theta^2\bar{\theta}^2$ component of a vector superfield, which transforms to a total derivative.

In terms of the component fields, we have

$$S_{matter} = \int d^4x \left(-D_m q^\dagger D^m q - i\bar{\chi}\bar{\not{D}}\chi + G^\dagger G - \sqrt{2}ig\left(q^\dagger\lambda\chi - \bar{\chi}\bar{\lambda}q\right) - gq^\dagger Dq\right). \tag{3.32}$$

We observe that the field G exhibits the same property as the vector superfield component D, i.e. it is not dynamical and can be immediately integrated out of the theory. The field G (also $\tilde{G}$ and F) is therefore also regarded as an auxiliary field.

More generally, we can couple the vector multiplet to N_f flavours of fundamental chiral multiplets Q_f and N_f flavours of conjugate representation chiral multiplets $\tilde{Q}_f$, which in turn can be coupled via a mass term. In this case

$$\mathcal{L}_{matter} = \int d^2\theta d^2\bar{\theta} \sum_{f=1}^{N_f} \left(Q_f^\dagger e^{-2gV} Q_f + \tilde{Q}_f e^{2gV} \tilde{Q}_f^\dagger\right) + \left\{\int d^2\theta \sum_{f=1}^{N_f} m_f \tilde{Q}_f Q_f + c.c.\right\}. \tag{3.33}$$

Gauge invariance of these couplings is easily verified using (3.19) and (3.31) as well as the transformation law for a conjugate chiral superfield $\tilde{Q}$,

$$\tilde{Q} \rightarrow \tilde{Q} e^{-2ig\Lambda}, \qquad \tilde{Q}^{\dagger} \rightarrow e^{2ig\Lambda^{\dagger}} \tilde{Q}^{\dagger}. \tag{3.34}$$

In terms of component fields, the mass term in Eq. (3.33) has the expansion

$$\int d^2\theta \sum_{f=1}^{N_f} m_f \tilde{Q}_f Q_f + c.c. = \sum_{f=1}^{N_f} m_f \left(\tilde{q}_f G_f + \tilde{G}_f q_f - \tilde{\chi}_f \chi_f \right) + c.c. \tag{3.35}$$

$\mathcal{N} = 2$ Supersymmetric Pure Yang-Mills Theory

In the $\mathcal{N} = 1$ superfield formalism, the Lagrangian of this theory is

$$\mathcal{L}_{\mathcal{N}=2\ SYM} = \left\{ \frac{1}{2} \int d^2\theta \, \mathrm{Tr} \left(W^{\alpha} W_{\alpha} \right) + c.c. \right\} + 2 \int d^2\theta d^2\bar{\theta} \, \mathrm{Tr} \left(\Phi^{\dagger} e^{-2gV} \Phi e^{2gV} \right), \tag{3.36}$$

where Φ is in the adjoint representation. From the transformation law

$$\Phi \rightarrow e^{2ig\Lambda} \Phi e^{-2ig\Lambda}, \qquad \Phi^{\dagger} \rightarrow e^{2ig\Lambda^{\dagger}} \Phi^{\dagger} e^{-2ig\Lambda^{\dagger}}, \tag{3.37}$$

it is clear that the theory is gauge invariant. In component fields, we may write the action of the theory as

$$\begin{aligned} S_{\mathcal{N}=2\ SYM} &= 2 \int d^4x \, \mathrm{Tr} \left(-\frac{1}{4} v_{mn} v^{mn} - i\bar{\lambda} \bar{\mathcal{D}} \lambda + \frac{1}{2} D^2 - D_m A^{\dagger} D^m A - i\bar{\psi} \bar{\mathcal{D}} \psi + F^{\dagger} F \right. \\ & \left. + \sqrt{2} ig \left(\left[A^{\dagger}, \psi \right] \lambda + \bar{\lambda} \left[A, \bar{\psi} \right] \right) - gD \left[A, A^{\dagger} \right] \right) \end{aligned} \tag{3.38}$$

This model will come under close scrutiny in the next section, where we review the analysis of Seiberg and Witten.

$\mathcal{N} = 2$ Supersymmetric QCD

This theory is obtained by coupling the $\mathcal{N} = 2$ vector multiplet (V, Φ) to N_f flavours of matter hypermultiplets, $(Q_f, \tilde{Q}_f^{\dagger})$ and $(Q_f^{\dagger}, \tilde{Q}_f)$, transforming in the fundamental and conjugate representation respectively. The Lagrangian is

$$\mathcal{L}_{\mathcal{N}=2\ SQCD} = \mathcal{L}_{\mathcal{N}=2\ SYM} + \mathcal{L}_{matter} + \left\{ \sqrt{2} ig \int d^2\theta \sum_{f=1}^{N_f} \tilde{Q}_f \Phi Q_f + c.c. \right\} \tag{3.39}$$

where $\mathcal{L}_{\mathcal{N}=2\ SYM}$ and $\mathcal{L}_{matter}$ are given by Eqs. (3.36) and (3.33) respectively. The term in brackets may be regarded as the supersymmetric generalization of a Yukawa coupling. It has the component field expansion

$$\begin{aligned}\sqrt{2}ig\int d^2\theta\sum_{f=1}^{N_f}\tilde{Q}_f\Phi Q_f + c.c. &= \sqrt{2}ig\sum_{f=1}^{N_f}\Big(\tilde{q}_f A G_f + \tilde{G}_f A q_f + \tilde{q}_f F q_f \\ &\quad -\tilde{q}_f\psi\chi_f - \tilde{\chi}_f\psi q_f - \tilde{\chi}_f A\chi_f\Big) + c.c. \end{aligned} \tag{3.40}$$

We shall discuss the exact results that have been predicted for this theory in the next section.

$\mathcal{N}=4$ Supersymmetric Yang-Mills theory

This theory is obtained by coupling $\mathcal{N}=2$ supersymmetric pure Yang-Mills theory to an $\mathcal{N}=2$ hypermultiplet $(Q,\tilde{Q})$ transforming in the *adjoint* representation. The coupling of V to Q and $\tilde{Q}$ is accomplished in the usual gauge invariant way (see Eq. (3.36)) whilst Φ couples to these fields via the term

$$4\sqrt{2}ig\int d^2\theta\,\mathrm{Tr}\left(\tilde{Q}\Phi Q\right) + c.c. \tag{3.41}$$

The $\mathcal{N}=4$ model comes to our attention in Chapter 6, where we use instantons to verify a nonperturbative nonrenormalization theorem due to Dine and Seiberg [27].

3.3 Exact Results

In 1994, Seiberg and Witten analysed the low-energy physics of $\mathcal{N}=2$ supersymmetric Yang-Mills models with gauge group $SU(2)$ and were able to obtain *exact* results for this theory, valid at both weak and strong values of the coupling [17, 18]. Their work has been generalized to $\mathcal{N}=2$ theories with larger classical simple and product gauge groups and a variety of matter representations [19, 20, 23, 21, 22, 24]. In this section we review the exact results that have been predicted for $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theories. These results are the primary motivation for the instanton calculations performed in the chapters that follow. In Subsection 3.3.1, we describe the Seiberg-Witten analysis of $\mathcal{N} = 2$ supersymmetric pure $SU(2)$ Yang-Mills theory [17]. In particular, we show how Seiberg and Witten employed physical arguments based on a low-energy duality property of the theory to predict exact results for this model. In Subsection 3.3.2, we discuss the generalization of these exact results to $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD [18, 19, 20, 21, 22].

3.3.1 Seiberg-Witten Theory

In what follows we make use of the several excellent reviews [28]. The Lagrangian of the $SU(2)$ model investigated in [17] is given, in $\mathcal{N} = 1$ superfield notation, by Eq. (3.36). It is convenient to rescale all the fields by a factor of g, so that the only dependence on the coupling is through an overall g^{-2} prefactor.

Symmetry Breaking

Let us begin by examining the component field expansion of the action, given (prior to field rescalings) by Eq. (3.38). In the functional integral, we can immediately integrate out the auxiliary fields F and D since the action is quadratic in these fields. The $F^\dagger F$ term is thereby eliminated and the two D dependent terms are replaced with

$$V = g^{-2}\,\mathrm{Tr}\left(\left[A^\dagger, A\right]^2\right). \tag{3.42}$$

This term acts as a symmetry-breaking Higgs potential for the scalar field A. It is minimized when A takes the form of a diagonal matrix. (This is the most general solution up to gauge transformations.) Thus we have

$$\langle A\rangle = \frac{\mathrm{v}}{2}\tau^3, \tag{3.43}$$

where the vacuum expectation value (VEV) v is an arbitrary complex parameter.

The arbitrariness of v has an important consequence. It leads to the concept of a moduli space of vacua. Each point on this space corresponds to a physically distinct vacuum, associated with a particular value of v. Actually, we should identify vacua given by $\pm$v because the Weyl transformation v $\to -$v does not affect the physics. Hence the classical moduli space of vacua is $\mathbb{C}/\mathbb{Z}_2$. In the analysis of Seiberg and Witten a central concern is how this picture is modified by quantum effects.

To see how the VEV breaks the $SU(2)$ gauge symmetry, we write

$$A = \langle A \rangle + \delta A, \tag{3.44}$$

and expand the action about $\langle A \rangle$. From the Higgs kinetic term, we obtain a mass term for the gauge bosons:

$$2g^{-2}\mathrm{Tr}\left(\left[\langle A \rangle^\dagger, v_m\right]\left[v^m, \langle A \rangle\right]\right). \tag{3.45}$$

Through this term the first and second isospin components of v_m acquire a mass

$$M = \sqrt{2}|\mathrm{v}|. \tag{3.46}$$

However, the third component of v_m commutes with $\langle A \rangle$ and therefore remains massless. Examining the other terms in the action, one finds the same mass M generated for the first and second isospin components of all the fields in the $\mathcal{N} = 2$ vector multiplet. The third component of every field remains light. We deduce that for a generic nonzero v, the $SU(2)$ gauge symmetry is broken to a $U(1)$ symmetry of these light fields.

Wilsonian Effective Action

We shall be interested in the quantum physics of the model at low energies, where the gauge symmetry breaking is manifest. A convenient way to describe the low-energy particle dynamics is to use an effective action. Specifically, Seiberg and Witten considered a Wilsonian effective action. This effective action is discussed in detail in [29]. In principle, it is obtained by integrating out of the functional integral all the massive fields as well as all light field fluctuations whose energy is greater than some infra-red cut-off $\mu \sim M$. To leading order in a derivatives expansion[3], the Wilsonian effective action is strongly constrained by $\mathcal{N} = 2$

[3]The Wilsonian effective action is expected to contain an infinite number of terms involving any number of field derivatives. However, the contribution of an n-derivatives term to a physical process that is characterized by the momentum $p \ll M$ is suppressed by a factor p^n/M^n. The leading order approximation considered here consists of discarding all terms with $n > 2$. In Chapter 6 we shall investigate corrections due to terms with up to four derivatives.

supersymmetry. It must take the form [30, 31]

$$S_{eff} = \frac{1}{4\pi}\text{Im}\int d^4x \left[\int d^2\theta d^2\bar{\theta}\,\frac{d\mathcal{F}(\Phi)}{d\Phi}\,\Phi^\dagger \;+\; \frac{1}{2}\int d^2\theta\,\frac{d^2\mathcal{F}(\Phi)}{d\Phi^2}\,W^\alpha W_\alpha\right] \tag{3.47}$$

where

$$\Phi = (\text{v} + \delta A) + \dots \qquad \text{and} \qquad W_\alpha = -i\lambda_\alpha + \dots \tag{3.48}$$

represent the light $U(1)$ multiplets. The essential point is that S_{eff} is determined by a single holomorphic function $\mathcal{F}$, known as the prepotential [30, 31]. The discovery of the exact solution for this function is the remarkable acheivement of Seiberg and Witten.

The effective action (3.47) naturally induces a low-energy effective coupling. Expanding S_{eff} in terms of component fields, we pick out

$$\frac{1}{4\pi}\text{Im}\frac{d^2\mathcal{F}(\text{v})}{d^2\text{v}}\int d^4x\left(-\frac{1}{4}v_{mn}v^{mn} - \frac{1}{4}iv_{mn}\tilde{v}^{mn}\right). \tag{3.49}$$

It is convenient to define a complexified coupling that combines the usual θ-parameter with the gauge coupling,

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}. \tag{3.50}$$

We may now identify the low-energy effective complexified coupling as

$$\tau_{eff} = \frac{d^2\mathcal{F}(\text{v})}{d^2\text{v}}, \tag{3.51}$$

so that the terms (3.49) take the more familiar form

$$\int d^4x\left(-\frac{1}{4g^2_{eff}}v_{mn}v^{mn} - \frac{\theta_{eff}}{32\pi^2}v_{mn}\tilde{v}^{mn}\right). \tag{3.52}$$

Weak-coupling Expansion

In the region of quantum moduli space corresponding to large values of the VEV, the theory is weakly coupled. This is a consequence of asymptotic freedom. The one-loop β-function of the theory is given by

$$\beta(g) = \frac{dg}{d\ln\mu} = -\frac{b_0}{(4\pi)^2}g^3, \tag{3.53}$$

where $b_0 = 4$. (More generally, this coefficient is $b_0 = 2N - N_f$ for $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD with N_f fundamental matter hypermultiplets.) A remarkable fact is that, due to $\mathcal{N} = 2$ supersymmetry, there are no higher-loop corrections to this formula [32]. Integrating the β-function gives the running of the coupling,

$$g^2 = \frac{(4\pi)^2}{b_0 \ln \mu^2/\Lambda^2}. \tag{3.54}$$

Here Λ is the dynamically generated scale. It is clear from Eq. (3.54) that at scales $\mu \sim M \gg \Lambda$ the typical coupling is small.

In the weak-coupling regime, the prepotential has the expansion

$$\mathcal{F}(\mathrm{v}) = \text{const.} + \frac{ib_0 \mathrm{v}^2}{8\pi} \ln \frac{\mathrm{v}^2}{\Lambda^2} - \frac{i}{\pi}\mathrm{v}^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathrm{v}}\right)^{b_0 k}. \tag{3.55}$$

This result was obtained by Seiberg [31] using the anomalous $U(1)_R$ symmetry of the theory. The numerical coefficients $\mathcal{F}_k$ are a priori unknown. The value of the constant term is unimportant since only derivatives of the prepotential appear in the Wilsonian effective action. The logarithmic term is the one-loop exact perturbative contribution. It may be derived simply by evaluating the running coupling (3.54) at the scale $\mu = M$ and equating this to the effective coupling g_{eff}. The remaining terms correspond to k-instanton effects. From Eq. (3.54) we see that the characteristic k-instanton factor $\exp(-8\pi^2 k/g^2)$ indeed gives a contribution proportional to $\Lambda^{b_0 k}$. Note that there cannot be any powers of g multiplying the characteristic k-instanton factor because these would appear as powers of $\ln \Lambda$ multiplying the $\Lambda^{b_0 k}$ in Eq. (3.55). (This amounts to a prediction that the semiclassical calculus in a k-instanton background is exact at one-loop.)

Duality Transformation

We expect the semiclassical expansion (3.55) to break down when the VEV approaches the strong-coupling scale Λ. How then, can we get a handle on the physics in the strongly coupled region of moduli space? Seiberg and Witten found that duality provides the key. The duality phenomenon was first investigated by Dirac [33], in the context of electromagnetic theory. Dirac observed that (in the presence of magnetic monopoles) Maxwell's equations are invariant under exchange of electric and magnetic variables. Moreover, to get a consistent quantum mechanics, the electric charge q and the magnetic charge g must satisfy the quantization condition $gq = 2\pi n$, where $n \in \mathbb{Z}$. Hence the duality transformation inverts the coupling, $q \to g = 2\pi n/q$. The low-energy duality found by Seiberg and Witten represents a version of Olive-Montonen duality [34], believed to be present in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We shall see that it incorporates analogues of the features observed by Dirac. The property of strong-weak coupling exchange is crucial for an analysis of the strong-coupling region of moduli space.

In order to demonstrate the duality property of the low-energy physics, we consider the partition function

$$Z = \int [d\Phi][dV] \exp(iS_{eff}). \tag{3.56}$$

We shall describe a duality transformation that maps the 'electric' degrees of freedom, represented by Φ and V, into dual 'magnetic' degrees of freedom, Φ_D and V_D, whilst leaving the partition function invariant. To begin with, following Seiberg and Witten, we write

$$\Phi_D = \mathcal{F}'(\Phi), \tag{3.57}$$

where the prime just indicates differentiation of $\mathcal{F}$ with respect to Φ. We can change integration variable from Φ to Φ_D in Eq. (3.56) and the associated Jacobian factor is in fact unity. If we define a dual prepotential $\mathcal{F}_D$ such that

$$\Phi = -\mathcal{F}'_D(\Phi_D), \tag{3.58}$$

then the first term in the Wilsonian effective action (3.47) can be written

$$\frac{1}{4\pi}\mathrm{Im} \int d^4x d^2\theta d^2\bar{\theta}\, \mathcal{F}'(\Phi)\, \Phi^\dagger = \frac{1}{4\pi}\mathrm{Im} \int d^4x d^2\theta d^2\bar{\theta}\, \mathcal{F}'_D(\Phi_D)\Phi^\dagger_D. \tag{3.59}$$

In contrast to Φ, the vector superfield V maps into its dual in a non-local manner. We introduce V_D as a Lagrange multiplier which implements the constraint

$$\mathrm{Im}(D_\alpha W^\alpha) = 0 \tag{3.60}$$

in the functional integral. This constraint is the supersymmetric generalization of the Bianchi identity, Eq. (2.14). If W_α is assumed to be an arbitrary chiral superfield then this condition is sufficient to fix W_α to be a supersymmetric field strength. So the functional integration over the vector superfield V is equivalent to an integration over a general chiral superfield W_α in the presence of the constraint (3.60). We can therefore write

$$\int [dV] \exp\left(\frac{i}{8\pi}\mathrm{Im}\int d^4x d^2\theta\, \mathcal{F}''(\Phi)\, W^\alpha W_\alpha\right) \cong$$

$$\int [dW_\alpha][dV_D] \exp\left(\frac{i}{8\pi}\,\mathrm{Im}\left[\int d^4x d^2\theta\, \mathcal{F}''(\Phi)\, W^\alpha W_\alpha + \int d^4x d^2\theta d^2\bar{\theta}\, V_D D_\alpha W^\alpha\right]\right) \quad (3.61)$$

We can rewrite the Lagrange multiplier term as follows:

$$\begin{aligned} \int d^4x d^2\theta d^2\bar{\theta}\, V_D D_\alpha W^\alpha &= -\int d^4x d^2\theta d^2\bar{\theta}\, (D_\alpha V_D) W^\alpha \\ &= \frac{1}{2}\int d^4x d^2\theta\, \bar{D}^2 (D_\alpha V_D) W^\alpha \\ &= -2\int d^4x d^2\theta\, (W_D)_\alpha W^\alpha. \end{aligned} \quad (3.62)$$

In the second line we have made use of the form of $\bar{D}_{\dot{\alpha}}$ given in Eq. (3.12) and in the final line we have used the Abelian version of Eq. (3.21) to define the dual supersymmetric field strength,

$$(W_D)_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V_D. \quad (3.63)$$

We can now perform the W_α integration in Eq. (3.61) to obtain

$$\int [dV_D] \exp\left(\frac{i}{8\pi}\mathrm{Im}\int d^4x d^2\theta \left[-\frac{1}{\mathcal{F}''(\Phi)} W_D^\alpha W_{D\,\alpha}\right]\right). \quad (3.64)$$

We can eliminate $\mathcal{F}''(\Phi)$ using the relation

$$\mathcal{F}''(\Phi) = -\frac{1}{\mathcal{F}_D''(\Phi_D)}, \quad (3.65)$$

which follows from Eqs. (3.57) and (3.58).

The partition function can now be expressed as

$$Z = \int [d\Phi_D][dV_D] \exp(iS_{D\,eff}), \quad (3.66)$$

where the dual low-energy effective action $S_{D\,eff}$ has the form of S_{eff}, given in Eq. (3.47), with all quantities replaced by their duals. In analogy with Eq. (3.51), the complexified coupling associated with the dual effective action is

$$\tau_{D\,eff} = \mathcal{F}_D''(\mathrm{v}_D), \quad (3.67)$$

where, from Eq. (3.57), the dual VEV is given by

$$\mathrm{v}_D = \mathcal{F}'(\mathrm{v}). \tag{3.68}$$

It follows from Eq. (3.65) that

$$\tau_D = -\frac{1}{\tau}, \tag{3.69}$$

which is the property of weak-strong coupling exchange. In what follows we show how Seiberg and Witten deduced useful information about the low-energy physics in the strong-coupling region of moduli space using the weakly-coupled dual description provided by $S_{D\,eff}$.

Quantum Moduli Space

A crucial step in the Seiberg-Witten analysis is the introduction of an object that serves as a convenient coordinate on quantum moduli space. This object is the gauge invariant condensate

$$u = \langle \mathrm{Tr}(A^2) \rangle. \tag{3.70}$$

The key to the solution of the theory is to regard v and v_D as functions of u and to study their behaviour on moduli space.

Let us begin by considering the behaviour of v and v_D in the weak-coupling regime. In the weak-coupling limit, we expect

$$\mathrm{v} \to \sqrt{2u}. \tag{3.71}$$

Hence $\mathrm{v}(u)$ possesses a branch point singularity at $u = \infty$. We can obtain an expression for $\mathrm{v}_D = \mathcal{F}'(\mathrm{v})$ using the semiclassical expansion of the prepotential given by Eq. (3.55). In the weak-coupling limit, we find

$$\mathrm{v}_D \to \frac{i\sqrt{2u}}{\pi} \ln \frac{2u}{\Lambda^2} + \frac{4i\sqrt{2u}}{\pi}. \tag{3.72}$$

where we have used Eq. (3.71) to eliminate v in favour of u. Hence $\mathrm{v}_D(u)$ also has a branch point singularity at the point $u = \infty$.

Let us now consider a closed contour in the compactified complex plane of u that encloses the point at infinity. As we perform a rotation around this contour, $u \to e^{2\pi i}u$, the VEV and its dual map into linear combinations of themselves. Specifically, from the asymptotic behaviours (3.71) and (3.72), we find

$$\begin{pmatrix} \mathrm{v}_D \\ \mathrm{v} \end{pmatrix} \to M_\infty \begin{pmatrix} \mathrm{v}_D \\ \mathrm{v} \end{pmatrix}, \tag{3.73}$$

where the monodromy matrix M_∞ is given by

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{3.74}$$

This matrix encodes information about the weak-coupling singularity that will later be used to solve for $\mathrm{v}(u)$ and $\mathrm{v}_D(u)$.

What other singularities are present on moduli space? There must be at least one more singularity because the branch cut extending from $u = \infty$ has to end somewhere. Since there is only one weak-coupling singularity we must look to the strong-coupling region of moduli space. Let us first suppose there is precisely one singularity in this region. Now it can be shown that, due to the anomalous $U(1)_R$ symmetry of the theory, there is a $\mathbb{Z}_2$ symmetry on moduli space under the change of sign $u \to -u$. So if there is one strong-coupling singularity then it must be at the point $u = 0$. In this case it is straightforward to analytically continue the weak-coupling expansion (3.55) into the full moduli space. We find

$$\mathcal{F}(\mathrm{v}) = \mathrm{const.} + \frac{i\mathrm{v}^2}{2\pi} \ln \frac{\mathrm{v}^2}{\Lambda^2} + f(\mathrm{v}^2), \tag{3.75}$$

where $f(\mathrm{v}^2)$ is an entire function. However, a solution of this form is inconsistent with the positivity requirement

$$\frac{1}{g_{eff}^2} > 0. \tag{3.76}$$

So there cannot be just one strong-coupling singularity.

Seiberg and Witten argued that there are precisely two strong-coupling singularities. One way to justify this is to consider the effect of adding to the original Lagrangian a mass term for Φ. This gives an $\mathcal{N} = 1$ theory whose vacua correspond to the strong-coupling singularities on the $\mathcal{N} = 2$ moduli space. An independent calculation of Witten's index for the $\mathcal{N} = 1$ theory predicts that there are indeed two vacua. Let us suppose that one of the strong-coupling singularities is located at the point $u = u_0$. By the $\mathbb{Z}_2$ symmetry the other singularity must be located at $u = -u_0$. The question we now ask is: what is the physical cause of these singularities?

The proposal of Seiberg and Witten was that the strong-coupling singularities correspond to points on quantum moduli space where certain particle states of the theory become massless. At such points the (dual) Wilsonian effective action description of the physics should break down because it does not account for the new light degrees of freedom. The particular states that are responsible derive from the classical, solitonic solutions that generically appear in theories whose gauge symmetry is broken to an Abelian subgroup. These solutions may carry magnetic charge (monopoles) or both magnetic and electric charge (dyons), and form 'short' multiplets of $\mathcal{N} = 2$ supersymmetry. After quantization, the corresponding set of physical states are also expected to form short multiplets. Consequently, they saturate a Bogomolnyi-Prasad-Sommerfield (BPS) lower bound on their mass [35]. The BPS mass formula reads

$$M_{BPS} = \sqrt{2}|n_m \mathrm{v}_D + n_e \mathrm{v}|, \tag{3.77}$$

where n_e and n_m are the electric and magnetic quantum numbers of the BPS state.

Let us suppose the massless BPS state associated with the singularity at $u = u_0$ is a magnetic monopole (this will be verified later). We can use this information to work out the behaviour of v and v_D close to the singularity. We employ the dual effective action $S_{D\,eff}$ which describes the physics in terms of weakly-coupled magnetic degrees of freedom. To account for the effect of the monopole on the low-energy physics in the vicinity of $u = u_0$ we should supplement $S_{D\,eff}$ with terms that describe the coupling of the light dual magnetic fields to the monopole. These terms are uniquely determined by $\mathcal{N} = 2$ supersymmetry. One obtains an effective supersymmetric QED-like theory whose one-loop β-function reads

$$\beta(g_D) = \frac{2}{(4\pi)^2} g_D^3. \tag{3.78}$$

After integrating this equation and setting the characteristic mass scale equal to v_D, one obtains

$$\tau_{D\,eff} \approx -\frac{i}{\pi} \ln \frac{\mathrm{v}_D}{\Lambda}. \tag{3.79}$$

Since $\mathrm{v} = -\mathcal{F}'_D(\mathrm{v}_D)$, we can integrate with respect to v_D to get

$$\mathrm{v} \approx \text{const.} + \frac{i}{\pi} \mathrm{v}_D \ln \frac{\mathrm{v}_D}{\Lambda}. \tag{3.80}$$

Now since v_D provides a good local coordinate on moduli space, and vanishes at $u = u_0$, we expect that close to the singularity,

$$\mathrm{v} = c_0(u - u_0), \tag{3.81}$$

where c_0 is a constant. Using Eqs. (3.80) and (3.81) we can determine the behaviour of v and v_D as we move on a closed contour around the point $u = u_0$. We find that in one complete rotation, $(u - u_0) \to e^{2\pi i}(u - u_0)$, there is a monodromy:

$$\begin{pmatrix} \mathrm{v}_D \\ \mathrm{v} \end{pmatrix} \to M_{u_0} \begin{pmatrix} \mathrm{v}_D \\ \mathrm{v} \end{pmatrix}, \qquad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{3.82}$$

A more general analysis shows that if one assumes a dyon of charge (n_m, n_e) to be responsible for the strong-coupling singularity then the associated monodromy matrix is

$$M_{(n_m, n_e)} = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix}. \tag{3.83}$$

Note that (n_e, n_m) is a left eigenvector of the monodromy matrix $M_{(n_m, n_e)}$. Consequently, the BPS mass formula for the dyon is invariant as one encircles the singularity, as we should expect.

In order to find out precisely which BPS states are responsible for the strong-coupling singularities, one can make use of a consistency requirement that follows from standard complex analysis. Let us consider a contour in the complex plane of u that encloses both the strong coupling singularities. The monodromy behaviour of v and v_D around this contour is clearly the product of the monodromy matrices at $u = u_0$ and $u = -u_0$. But the contour can also be viewed as enclosing the point at infinity. Hence the monodromy matrix at infinity must equal the product of the two strong-coupling monodromy matrices,

$$M_{u_0} M_{-u_0} = M_\infty. \tag{3.84}$$

Knowing M_∞, one can solve for M_{u_0} and M_{-u_0} using the general form (3.83). In this way one finds that the singularity at $u = u_0$ can indeed be attributed to a massless monopole, with the corresponding monodromy matrix (3.82). The singularity at the point $u = -u_0$ is caused by a massless dyon of charge $(n_m, n_e) = (1, -1)$. The associated monodromy matrix is

$$M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{3.85}$$

Solution for the Prepotential

The mathematical data represented by the monodromy matrices $M_{\pm u_0}$ and M_∞ together with the positivity constraint (3.76) is sufficient to uniquely determine the functions $\mathrm{v}_D(u)$ and $\mathrm{v}(u)$. It is possible to construct a second order linear differential equation whose two independent solutions are $\mathrm{v}(u)$ and $\mathrm{v}_D(u)$. However, we shall follow the original approach of Seiberg and Witten which uses an auxiliary elliptic curve to construct the solutions. (As we see in Section 3.3.2, in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD with $N>2$, the generalization of the elliptic curve is a genus $N-1$ hyperelliptic curve.)

Seiberg and Witten observed that any two of the monodromy matrices generate a particular subgroup $\Gamma(2)$ of the duality group $SL(2,\mathbb{Z})$. This subgroup is defined as

$$\Gamma(2)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}),\ b=0 \mod 2\right\}. \tag{3.86}$$

The quantum moduli space is therefore given by $H/\Gamma(2)$, where H is the upper half complex plane (we have 'modded out' the $\mathbb{Z}_2$ symmetry $u\to -u$). This is exactly the same as the moduli space of the elliptic curve defined by

$$y^2=(x-u_0)(x+u_0)(x-u). \tag{3.87}$$

At a generic value of u, this equation defines a two-dimensional surface that is topologically equivalent to the torus. However, at the three points $u=\pm u_0,\infty$, this torus collapses to a two-sphere. Thus the singularities of the curve on its moduli space correspond (by construction) to those of our physical theory on quantum moduli space.

The connection between the elliptic curve and the low-energy physics is realized more concretely through the period 'matrix' of the curve. This object is given by

$$\tau(u)=\frac{\oint_\beta \omega(u)}{\oint_\alpha \omega(u)}, \tag{3.88}$$

where α and β form a canonical basis of one-cycles on the torus and

$$\omega(u)=\frac{dx}{y(x;u)} \tag{3.89}$$

is the unique holomorphic one-form. The monodromy behaviour of τ derives purely from the behaviour of the one-cycles α and β around the moduli space singularities. We can specify α and β so that τ has exactly the same monodromy behaviour as τ_{eff} (as derived from that of v and v_D). Furthermore, it can be shown that τ satisfies the positivity condition, $\mathrm{Im}\tau > 0$. We therefore identify $\tau(u)$ as the exact solution for the effective coupling τ_{eff}.

Using the relation

$$\tau_{eff} = \frac{d\mathrm{v}_D}{du}\left(\frac{d\mathrm{v}}{du}\right)^{-1}, \tag{3.90}$$

we infer that

$$\frac{d\mathrm{v}_D}{du} = c\oint_\beta \omega(u), \tag{3.91}$$

$$\frac{d\mathrm{v}}{du} = c\oint_\alpha \omega(u). \tag{3.92}$$

The constant prefactor c can be found using the boundary condition imposed by the semi-classical limit. Upon integrating with respect to u, one finds

$$\mathrm{v}_D(u) = \sqrt{2}\pi\int_{-1}^{1} dx\,\frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}} = \frac{i}{2}(u-1)F(\tfrac{1}{2},\tfrac{1}{2},2;\tfrac{1-u}{2}), \tag{3.93}$$

$$\mathrm{v}(u) = \sqrt{2}\pi\int_{1}^{u} dx\,\frac{\sqrt{x-u}}{\sqrt{x^2-u_0^2}} = \sqrt{2(u+1)}F(-\tfrac{1}{2},\tfrac{1}{2},1;\tfrac{2}{1+u}), \tag{3.94}$$

where the hypergeometric function $F(\alpha,\beta,\gamma;z)$ is given by

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n\geq 0}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!}. \tag{3.95}$$

One can invert Eq. (3.94) to get u in terms of v and substitute $u(\mathrm{v})$ into Eq. (3.93). By integrating $\mathrm{v}_D(u) = \mathcal{F}'(\mathrm{v})$ with respect to v, one then obtains the promised exact solution for the prepotential.

As a final comment, we note that u_0 must be proportional to the strong-coupling scale of the theory, Λ^2. By choosing the proportionality factor to be one, we specify an implicit renormalization scheme; the Seiberg-Witten scheme. To compare first-principles instanton calculations with the Seiberg-Witten exact predictions one has to relate the scale defined in the (Pauli-Villars) scheme used for the instanton calculation to the Seiberg-Witten scale, defined by $u_0 = \Lambda^2$. This relation can be completely determined at one-loop in perturbation theory. We come back to this point in the next chapter.

3.3.2 Generalization of the Exact Results

In what follows we outline the generalization of the results described above to $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theories with matter. We begin with the supersymmetric $SU(2)$ QCD theory which was investigated by Seiberg and Witten in their second paper [18].

$\mathcal{N} = 2$ SUSY $SU(2)$ QCD

In [18], Seiberg and Witten applied their analysis to $\mathcal{N} = 2$ supersymmetric $SU(2)$ QCD with $N_f = 1, 2, 3, 4$ flavours of matter hypermultiplets. For $N_f \leq 3$ the first coefficient of the β-function is negative and the theory is asymptotically free. The $N_f = 4$ theory is finite because its β-function vanishes. It is treated as a special case.

We shall not give details here, but the basic line of reasoning is just as for the pure Yang-Mills theory. Through the potential term (3.42) the field A may acquire a nonzero expectation value[4] v and the theory is then broken to its Coulomb branch. The isospin components of the matter hypermultiplets correspondingly acquire a mass $|m_f \pm i\mathrm{v}/\sqrt{2}|$. We deduce that in the weak-coupling region of moduli space, there are singularities at $\mathrm{v} = \pm\sqrt{2}im_f$ as well as at $\mathrm{v} = \infty$. Except at these points, one can integrate out the matter hypermultiplets along with the massive components of the $\mathcal{N} = 2$ vector multiplet to get a Wilsonian effective action and prepotential, just as for the pure Yang-Mills theory.

For each value of N_f, Seiberg and Witten located the singularities on moduli space, identified the associated BPS states, and computed the monodromy matrices. This data was used to construct auxiliary elliptic curves, from which the solution for the prepotential could be obtained following the recipe outlined in the previous subsection. To specify the curves, Seiberg and Witten also made use of the fact that odd-instanton contributions vanish when any hypermultiplet bare mass m_f is set to zero. This is due to a $\mathbb{Z}_2$ parity symmetry

[4]Some of the conventions used in the second Seiberg-Witten paper [18] differ from those used in the first [17]. In particular, both the vacuum expectation value and the complexified coupling are rescaled by a factor of two. Throughout this work, we maintain the conventions of [17], so that v is defined by Eq. (3.43) and τ has the form (3.50).

that arises because the fundamental representation of $SU(2)$ is pseudoreal.

An important consistency check on the Seiberg-Witten curves is provided by sending one of the N_f bare masses, say m_{N_f}, to infinity. Then the associated matter hypermultiplet decouples and theory is effectively supersymmetric QCD with $N_f - 1$ hypermultiplets. So in the large m_{N_f} limit the curve describing N_f hypermultiplets should exactly reproduce the $N_f - 1$ curve, provided the dynamically generated scales Λ_{N_f} and Λ_{N_f-1} are properly matched according to the renormalization group. In the implicit renormalization scheme of Seiberg and Witten, the matching condition is[5]

$$m\Lambda_{N_f}^{2N-N_f} \to \Lambda_{(N_f-1)}^{2N-(N_f-1)}. \tag{3.96}$$

At the beginning of the next chapter, we discuss how the predictions of the Seiberg-Witten curves compare with first-principles instanton calculations. We shall see that both the $N_f = 3$ and the $N_f = 4$ curve do not completely agree with the instanton calculus, but that the discrepancies can be cured by reinterpreting the parameters in the curves [36, 37].

$\mathcal{N} = 2$ Supersymmetric Pure $SU(N)$ Yang-Mills

Before turning to the exact solutions in this theory, we consider the $SU(N)$ generalization of the basic formalism. On the Coulomb branch, the field A acquires the following matrix of expectation values

$$\langle A \rangle = \mathrm{diag}(\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_N). \tag{3.97}$$

This diagonal form ensures that $\langle A \rangle$ and $\langle A^\dagger \rangle$ commute, so that the potential (3.42) is minimized. The VEV's v_u are arbitrary complex numbers constrained by

$$\sum_{u=1}^{N} \mathrm{v}_u = 0, \tag{3.98}$$

which ensures that $\langle A \rangle$ belongs to the $SU(N)$ Lie algebra.

[5]When $N_f = 4$, the instanton factor $q = e^{2\pi i \tau}$ takes the place of Λ^{b_0}, where τ is the complexified coupling of the microscopic theory. Seiberg and Witten found that their $N_f = 4$ curve reproduced the $N_f = 3$ curve provided the matching relation (3.96) was modified to $64mq \to \Lambda_3$ [18]. The appearance of a factor of 64 seems quite mysterious. In the next chapter we show that it is naturally explained if the parameter τ appearing in the $N_f = 4$ curve is reinterpreted as a low-energy effective coupling [36].

There is a residual gauge symmetry which leaves the physics invariant but acts nontrivially on the matrix $\langle A\rangle$. This is the group of Weyl transformations, $S(N)$, which permutes the VEV's. Hence the classical moduli space is given by $\mathbb{C}^N/S(N)$. As Weyl invariant coordinates on this space we can use

$$u_n = \langle \mathrm{Tr}(A^n)\rangle. \tag{3.99}$$

These condensates also provide convenient coordinates on the full quantum moduli space of the $SU(N)$ theory.

Examining the microscopic action, one finds that the light components of the $\mathcal{N}=2$ vector multiplet are those components which commute with $\langle A\rangle$. They therefore correspond to the diagonal generators that comprise the $SU(N)$ Cartan subalgebra. So there are $N-1$ light field components, this being the rank of the group $SU(N)$. The remaining $N(N-1)$ field components acquire masses $M_{uv}=\sqrt{2}|v_u - v_v|$ for all $u\neq v$. Since the gauge group is broken to a direct product of $N-1$ Abelian $U(1)$ subgroups, the Wilsonian effective action must take the form

$$S_{eff} = \frac{1}{4\pi}\mathrm{Im}\int d^4x\left[\int d^2\theta d^2\bar\theta\,\frac{d\mathcal{F}(\Phi)}{d\Phi^i}\,\Phi^{i\,\dagger} + \frac{1}{2}\int d^2\theta\,\frac{d^2\mathcal{F}(\Phi)}{d\Phi^i d\Phi^j}\,W^{i\,\alpha}W^j_\alpha\right]. \tag{3.100}$$

The indices i and j label the light components of the $\mathcal{N}=1$ superfields and therefore run from 1 to $N-1$. The prepotential is a holomorphic function of all the light components Φ^i.

An exact solution for $\mathcal{F}$ was predicted independently in [19] and [20]. Both sets of authors began by assuming that the Seiberg-Witten elliptic curve generalizes to a genus $N-1$ hyperelliptic curve. Such a curve has an associated $(N-1)\times(N-1)$ period matrix which transforms under the group $Sp(2(N-1),\mathbb{Z})$. This coincides with the duality group of the low-energy theory. Moreover, the period matrix is guaranteed to have an imaginary part greater than zero. It is therefore natural to identify it with the matrix of effective couplings of the low-energy $SU(N)$ theory, $(\tau_{eff})_{ij} = \partial_i\partial_j\mathcal{F}(\mathrm{v})$.

After assuming this correspondence, the problem is to determine the precise parameterization of the $SU(N)$ hyperelliptic curve. This was achieved in [19, 20] using the physical constraints imposed by: (i) a $\mathbb{Z}_{2N}$ remnant of the anomalous $U(1)_R$ symmetry, (ii) the $\Lambda^{b_0 k}$ form of k-instanton effects, and (iii) the semiclassical limit $\Lambda\to 0$.

$\mathcal{N} = 2$ Supersymmetric $SU(N)$ QCD

The generalization of the hyperelliptic curve associated with supersymmetric pure $SU(N)$ Yang-Mills theory to $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD with $N_f \leq 2N$ flavours of matter hypermultiplets was first investigated in [21] and [22].[6] The first coefficient of the β-function in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD is given by $-b_0 = N_f - 2N$ (see Eq. (3.53)). Hence the models considered are asymptotically free, except for the $N_f = 2N$ theory, which is finite.

To find the right parameterization of the curves, the authors of both [21] and [22] utilized the general constraints imposed by R-symmetry, instanton effects and the semiclassical limit. An additional constraint was provided by the meromorphic one-form λ that gives the solutions $\mathrm{v}_{D\,u}$ and v_u (see Eqs. (4.91)–(4.97) of the following chapter). Its residues are restricted to have a particular dependence on the bare masses m_f. Apart from this general input, the approaches of [21] and [22] are somewhat different and the proposed curves are not the same for all N_f. In [21], a distinction is made between the cases $N_f < N$ and $N_f \geq N$. For $N_f < N$ the curve was found to be uniquely specified using the general constraints listed above. For $N_f \geq N$, these constraints proved insufficient to completely fix the curves and a certain amount of conjecture was required to get a definite parameterization.

In [22], the authors first considered an $SU(2N)$ theory with $2N$ flavours. When N of the VEV's are taken to infinity, and the bare masses m_f are tuned so that the hypermultiplets do not decouple, this theory flows into an $SU(N)$ theory with $2N$ flavours. It was argued that this leads to constraints on the form of the hyperelliptic curve for the finite $N_f = 2N$ model. Next, by sending a single VEV to infinity, and by tuning the bare masses in a different way, the $SU(N)$ theory with $2N$ flavours was made to flow into an $SU(N-1)$ theory with $N_f = 2(N-1)$ flavours. The argument then proceeded inductively, with the $N_f = 2N$ curve ultimately being determined by matching to the $N_f = 4$ elliptic curve of Seiberg and Witten. From the $N_f = 2N$ curve, all the $N_f \leq 2N$ curves are determined by decoupling

[6] These models were later investigated in [38], [39] and [40]. In [38] only the case $N = 3$ was considered and the analysis focussed on the finite $N_f = 6$ model. In [39], a very different approach was used, based on the connection between $\mathcal{N} = 2$ supersymmetric Yang-Mills and integrable systems. In [40], the hyperelliptic curves were *derived* using M-theory, but an explicit parameterization was not specified.

matter hypermultiplets and using the renormalization group matching condition (3.96).

It turns out that the parameterization of the curves proposed in [21] and [22] differs precisely when $N_f \geq N$. In the next chapter we shall perform a semiclassical instanton calculation to investigate these discrepancies for the case of gauge group $SU(3)$.

Chapter 4

Instanton Tests of the Exact Results

4.1 Introduction

The prepotential that describes the low-energy physics of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory has a weak-coupling expansion which takes a very particular form [31]. It consists of a one-loop perturbative term plus an infinite series of nonperturbative terms. The nonperturbative terms are associated with one-loop k-instanton effects. In principle, they can be derived from first principles using the semiclassical instanton method.

Following the prediction of Seiberg and Witten for the exact prepotential, Finnel and Pouliot performed a one-instanton calculation in in $\mathcal{N} = 2$ supersymmetric pure $SU(2)$ Yang-Mills theory [41]. From first principles they evaluated the leading nonperturbative term in the weak-coupling expansion of the prepotential. In a subsequent investigation of the same model, Dorey, Khoze and Mattis employed the multi-instanton construction of Atiyah, Drinfeld, Hitchin and Manin to evaluate the two-instanton contribution to the prepotential [42]. Their analysis was later extended to the $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills models with fundamental matter hypermultiplets [43, 44, 45, 37] (also to the model with one massive flavour in the *adjoint* representation [46]).

In Section 4.2, we review these $SU(2)$ instanton calculations and their comparison with the exact solutions. We present the one-instanton calculation of Finnel and Pouliot and summarize the results of the two-instanton calculations. In most cases, the instanton calculations completely agree with the predictions of the Seiberg-Witten curves. However, in the models with $N_f = 3$ and $N_f = 4$ flavours of matter hypermultiplets certain discrepancies have been found. We explain the origin of these discrepancies and indicate how they can be cured by reinterpreting the parameters of the curves [36].

However, our main concern is with instanton effects in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory with $N > 2$. In Section 4.3, we present an explicit one-instanton calculation that provides a direct test of the $SU(N)$ hyperelliptic curves [47]. Our approach is similar to that of Ito and Sasakura, who performed the first one-instanton calculations in the $SU(N)$ models with $N > 2$ [57, 58]. For gauge group $SU(3)$ we are able to perform a complete calculation and we find certain discrepancies in the models with $N_f \geq 3$ flavours. These discrepancies are similar in nature to the discrepancy found in the $SU(2)$ model with $N_f = 3$ flavours and we show that they can be resolved in the same way.

4.2 $SU(2)$ Instanton Tests

In this section we describe the instanton calculations that have been performed in $\mathcal{N} = 2$ supersymmetric $SU(2)$ QCD. For $N_f < 3$ flavours of matter hypermultiplets, the instanton calculations are in complete agreement with the exact solutions. For $N_f = 3$ flavours of matter hypermultiplets, the Seiberg-Witten curve does not give the correct two-instanton contribution to the condensate $u = \langle \mathrm{Tr} A^2 \rangle$. (Seiberg-Witten theory predicts an exact solution for this object as well as for the prepotential.) The source of the discrepancy can be traced to an ambiguity in the definition of the parameter appearing in the $N_f = 3$ curve. The instanton result can be used to fix this ambiguity [37, 36]. For the model with $N_f = 4$ flavours, there is quite substantial disagreement between the semiclassical analysis and the exact solutions. We describe the discrepancy and outline its resolution, according to the proposal of [36].

4.2.1 One-Instanton Test

In this subsection we describe the one-instanton calculation in $\mathcal{N}=2$ supersymmetric pure $SU(2)$ Yang-Mills theory performed by Finnel and Pouliot [41]. The focus of this calculation is the four-fermi correlator

$$\langle\bar{\lambda}(x_1)\bar{\lambda}(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\rangle, \tag{4.1}$$

where the fermions are light $U(1)$ fields of the low-energy effective theory.

Contact is made with the exact results via the low-energy Wilsonian effective action. Specifically, we extract a prediction for the amplitude (4.1) by expanding Eq. (3.47) in terms of component fields. The relevant term in the expanded action is

$$\frac{1}{2}\frac{1}{8\pi i}\frac{\mathcal{F}''''(\mathrm{v})}{2!}\int d^4x\,\lambda(x)\lambda(x)\psi(x)\psi(x). \tag{4.2}$$

A simple tree-level calculation gives

$$\langle\bar{\lambda}(x_1)\bar{\lambda}(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\rangle = \frac{\mathcal{F}''''(\mathrm{v})}{8\pi i}\int d^4x_0\, S_{\alpha\dot{\alpha}}(x_1,x_0)\bar{S}^{\dot{\alpha}\alpha}(x_2,x_0)S_{\beta\dot{\beta}}(x_3,x_0)\bar{S}^{\dot{\beta}\beta}(x_4,x_0), \tag{4.3}$$

where $S_{\alpha\dot{\alpha}}$ is the massless fermion propagator,

$$S_{\alpha\dot{\alpha}}(x,x_0) = \frac{1}{4\pi^2}\sigma^m_{\alpha\dot{\alpha}}\partial_m\frac{1}{(x-x_0)^2}. \tag{4.4}$$

Expression (4.3) should be exact in the low-energy limit. This is equivalent to the long-distance limit $|x_i - x_j| \to \infty$.

The weak-coupling expansion of the prepotential in $\mathcal{N}=2$ supersymmetric pure $SU(2)$ Yang-Mills theory is given by Eq. (3.55). Using this expansion, we extract the following prediction for the one-instanton contribution to the four-fermi correlator (4.1):

$$\langle\bar{\lambda}(x_1)\bar{\lambda}(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)\rangle_{1I} = \frac{15\mathcal{F}_1}{\pi^2}\frac{\Lambda^4}{\mathrm{v}^6}\int d^4x_0\, S_{\alpha\dot{\alpha}}(x_1,x_0)\bar{S}^{\dot{\alpha}\alpha}(x_2,x_0)S_{\beta\dot{\beta}}(x_3,x_0)\bar{S}^{\dot{\beta}\beta}(x_4,x_0). \tag{4.5}$$

From the Seiberg-Witten exact results, Eqs. (3.93) and (3.94), we obtain the prediction $\mathcal{F}_1 = 1/8$.

Note that the numerical coefficient $\mathcal{F}_1$ is renormalization scheme dependent. In order to compare the Seiberg-Witten prediction with the first-principles result, we have to relate the implicit renormalization scheme defined by the Seiberg-Witten curve with the renormalization scheme used in the instanton calculation, i.e. the Pauli-Villars scheme. Specifically, we have to relate the scale Λ defined in the Seiberg-Witten scheme with the scale Λ_{PV} defined in the Pauli-Villars renormalization scheme We show how this is done further on.

We now consider the instanton calculation. After continuation of the pure Yang-Mills action (3.38) to Euclidean space (see Appendix A), one obtains the following Euler-Lagrange equations,

$$\begin{aligned} D_\mu F_{\mu\nu} &= -ig\left([A, D_\nu A^\dagger] + [A^\dagger, D_\nu A]\right) \\ &\quad +ig\left(\lambda e_\nu \bar{\lambda} + \bar{\lambda}\bar{e}_\nu \lambda + \psi e_\nu \bar{\psi} + \bar{\psi}\bar{e}_\nu \psi\right), && (4.6) \\ \bar{\not{D}}\lambda &= \sqrt{2}ig[A, \bar{\psi}], && (4.7) \\ \not{D}\bar{\lambda} &= \sqrt{2}ig[A^\dagger, \psi], && (4.8) \\ \bar{\not{D}}\psi &= -\sqrt{2}ig[A, \bar{\lambda}], && (4.9) \\ \not{D}\bar{\psi} &= -\sqrt{2}ig[A^\dagger, \lambda], && (4.10) \\ D^2 A &= \sqrt{2}ig[\lambda, \psi] + g^2[[A, A^\dagger], A], && (4.11) \\ D^2 A^\dagger &= \sqrt{2}ig[\bar{\lambda}, \bar{\psi}] + g^2[[A^\dagger, A], A^\dagger]. && (4.12) \end{aligned}$$

(The auxiliary fields D and F have been eliminated in the usual manner.) We are interested in the physics on the Coulomb branch of the theory, where the Higgs field A has a nonzero expectation value v. In this case, Derrick's theorem predicts that there is no nontrivial solution to Eqs. (4.6)–(4.12). To proceed, we apply the constrained instanton formalism (see Section 2.4.3).

The constrained instanton formalism tells us that to obtain a suitable background configuration we should solve Eqs. (4.6)–(4.12) perturbatively, in the presence of some supplementary constraint. The relevant small parameter is ρM, where ρ represents the size of the configuration and $M = \sqrt{2}g\mathrm{v}$ is the W-boson mass.[1] At leading order in ρM, the Euler-Lagrange equations reduce to

$$D_\mu F_{\mu\nu} = 0, \qquad (4.13)$$

[1] Note that this differs by a factor of g from the formula (3.46) given in Section 3.3.1. This is because in the analysis of Seiberg and Witten all of the fields are rescaled by a factor of g. Here we choose to work with the original fields, and account for factors of g in the final comparison with the exact results.

$$\bar{\not{D}}\lambda = 0, \qquad \not{D}\bar{\lambda} = 0, \tag{4.14}$$
$$\bar{\not{D}}\psi = 0, \qquad \not{D}\bar{\psi} = 0, \tag{4.15}$$
$$D^2 A = \sqrt{2}ig[\lambda, \psi], \tag{4.16}$$
$$D^2 A^\dagger = 0. \tag{4.17}$$

In the approach of Finnel and Pouliot the Yukawa terms in the action are treated perturbatively.[2] This means that the source term on the right-hand side of Eq. (4.16) is neglected in the first approximation.

We now present the solutions to Eqs. (4.13)–(4.17) (without the source term) that represent the leading order short-distance constrained one-instanton background. The appropriate solution to Eq. (4.13) is the BPST instanton. It is convenient to define

$$u = u_\mu e_\mu, \qquad \bar{u} = u^\dagger = u_\mu \bar{e}_\mu. \tag{4.18}$$

If we impose the constraint $u_\mu u_\mu = 1$ then the matrix u belongs to the gauge group $SU(2)$. The singular gauge BPST instanton can now be written

$$A_\mu = \frac{2}{g}\bar{\eta}^a_{\mu\nu}\frac{\rho^2 y_\nu}{y^2(y^2+\rho^2)}u\frac{\tau^a}{2}\bar{u}, \tag{4.19}$$

where $y_\mu = (x - x_0)_\mu$ and the matrix u effects the iso-rotations.

In the background of the BPST instanton, the solutions to Eqs. (4.14)–(4.17) are well-known. For the Higgs field we have

$$A = \frac{y^2}{y^2+\rho^2}\frac{\mathrm{v}}{2}\tau^3. \tag{4.20}$$

Clearly this satisfies the boundary condition

$$\lim_{x\to\infty} A = \langle A \rangle = \frac{\mathrm{v}}{2}\tau^3. \tag{4.21}$$

[2]In the models with matter, this approach is inadequate because it fails to generate leading order terms in the instanton action that are quadrilinear in fermionic collective coordinates. In the $SU(N)$ semiclassical instanton calculation of Section 4.3 we obtain the full one-instanton solution to Eq. (4.16), in the presence of the Yukawa source, and use it to construct the instanton action. In Chapter 5 the $SU(N)$ ADHM formalism is used to construct the full k-instanton solution to Eq. (4.16).

As for the fermion fields, the index theorem predicts that there are four solutions of the Dirac equations (4.14) and (4.15) for each of λ and ψ. (The only solutions to the antifermion field equations are $\bar{\lambda} = 0$ and $\bar{\psi} = 0$.) Let us define

$$y = y_\mu e_\mu, \qquad \bar{y} = y_\mu \bar{e}_\mu. \tag{4.22}$$

The four zero-mode solutions[3] for λ can now be written

$$\lambda_{SS\,\alpha} = -\frac{\sqrt{2}\rho^2}{\pi}\frac{(y\bar{u}\tau^a u\bar{y}\xi_{SS})_\alpha}{y^2(y^2+\rho^2)^2}\frac{\tau^a}{2}, \tag{4.23}$$

$$\lambda_{SC\,\alpha} = -\frac{\rho}{\pi}\frac{(y\bar{u}\tau^a u\bar{\xi}_{SC})_\alpha}{(y^2+\rho^2)^2}\frac{\tau^a}{2}. \tag{4.24}$$

The zero-mode solutions for ψ take an identical form, but we replace the Grassmann collective coordinates $\xi_{SS\,\beta}$ and $\bar{\xi}^{\dot{\beta}}_{SC}$ with $\zeta_{SS\,\beta}$ and $\bar{\zeta}^{\dot{\beta}}_{SC}$.

In Eqs. (4.23) and (4.24) we have labelled the fermion zero-modes using the subscripts 'SS' and 'SC' which stand for 'supersymmetric' and 'superconformal' respectively [15]. The two supersymmetric zero-modes are related to the BPST instanton solution by an $\mathcal{N} = 1$ supersymmetry transformation (specifically, this is given by Eq. (C.2)). The superconformal zero-modes are associated with a superconformal transformation (see e.g. Appendix A of [42]).

Using the solutions listed above we can evaluate the leading order one-instanton action. We find

$$\begin{aligned} S^{\text{1-inst}}_{\mathcal{N}=2\ SYM} &= 2\int d^4x\,\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + D_\mu A^\dagger D_\mu A - \sqrt{2}igA^\dagger[\psi,\lambda]\right) \\ &= \frac{8\pi^2}{g^2} + 4\pi^2\rho^2|\mathrm{v}|^2 - \frac{ig\bar{\mathrm{v}}}{\sqrt{2}}(\bar{\zeta}_{SC\,1},\ \bar{\zeta}_{SC\,2})\bar{u}\tau^3 u\begin{pmatrix}\bar{\xi}_{SC\,1}\\ \bar{\xi}_{SC\,2}\end{pmatrix}. \end{aligned} \tag{4.25}$$

Note that the instanton action depends on the superconformal collective coordinates but not on the supersymmetric collective coordinates. The four supersymmetric zero-modes are *exact* zero-modes whereas the superconformal zero-modes are lifted by the Yukawa term in the action.

[3] As in Section 2.4.2, we refer to solutions of the Dirac equation as zero-modes, although we now regard these zero-modes as constituting part of the instanton background.

Our next consideration is the collective coordinate integration measure. When $N = 2$ the $SU(N)$ measure (2.64), associated with the bosonic zero-modes, simplifies to

$$\frac{2^{10}\pi^6}{g^8}\int d^4x_0 d\rho\,\rho^3. \tag{4.26}$$

We have omitted the group space integral because the integrand will turn out to have no dependence on the group space variables. (This simplifying feature is not present in the $N > 2$ models and the group integration turns out to be highly nontrivial.) Since the zero-modes (4.23) and (4.24) are already normalized, the measure associated with the fermion zero-modes is simply

$$\int d^2\xi_{SC} d^2\zeta_{SC} d^2\xi_{SS} d^2\zeta_{SS}. \tag{4.27}$$

Now we turn to the small-fluctuations determinants. In supersymmetric theories an important simplification of the instanton calculus occurs in connection with these determinants. Namely, the factors due to fermionic and bosonic field fluctuations exactly cancel each other in the background gauge.[4] In the present calculation, we can see the cancellation directly. In Eq. (2.68) we presented the 't Hooft determinant for a complex scalar field of isospin t. Let us refer to this determinant as D_t. In $\mathcal{N} = 2$ supersymmetric pure $SU(2)$ Yang-Mills theory, all fields have isospin equal to one. From the discussion following Eq. (2.68) we now ascertain that the gauge boson and its ghost contribute a factor D_1^{-1}, the two Weyl fermions contribute a factor D_1^2, and the complex scalar field contributes a factor D_1^{-1}. Hence the determinants manifestly cancel.

In Section 2.4, we described how the renormalization divergences of the 't Hooft determinants are regulated by powers of the Pauli-Villars mass μ. The number of powers of μ that appear depends on the number of bosonic and fermionic zero-modes. In the present calculation, we have the usual eight bosonic zero-modes associated with the BPST instanton as well as eight fermionic zero-modes. From the bosonic zero-modes we get a factor μ^8 and from the fermionic zero-modes a factor μ^{-4}. Together these factors have the right power, $b_0 = 2N$, to combine with the instanton factor $\exp(-8\pi^2/g^2)$ and generate the renormalization group

[4]More precisely, this cancellation occurs for any self-dual background configuration in four dimensions in the covariant background gauge [48].

invariant scale

$$\Lambda_{PV}^4 = \mu^4 e^{-8\pi^2/g^2}. \tag{4.28}$$

(The subscript 'PV' indicates that this scale has been defined in the Pauli-Villars renormalization scheme.)

It remains for us to consider the antifermion field insertions. Since we are calculating a low-energy Green's function, it is the long-distance 'tail' of the instanton that is relevant. By virtue of the patching relation described in Section 2.4.3, the long-distance instanton can be obtained from a Taylor expansion of the short-distance instanton. We therefore consider the next-to-leading order short-distance field equations for $\bar{\lambda}$ and $\bar{\psi}$,

$$\not{D}\bar{\lambda} = \sqrt{2}ig[A^\dagger, \psi], \qquad \not{D}\bar{\psi} = -\sqrt{2}ig[A^\dagger, \lambda]. \tag{4.29}$$

Since the four antifermion field insertions must saturate the integration over all four supersymmetric collective coordinates, we substitute $\lambda = \lambda_{SS}$ and $\psi = \psi_{SS}$ in these equations. The corresponding solutions are

$$\bar{\lambda}^a_{SS\,\dot{\alpha}} = -\frac{ig}{2\pi}\frac{\bar{v}\rho^2}{(y^2+\rho^2)^2}(\bar{u}\tau^3\tau^a u\bar{y}\zeta_{SS})_{\dot{\alpha}} \tag{4.30}$$

and similarly for $\bar{\psi}^a_{SS\,\dot{\alpha}}$, with $\zeta_{SS} \to \xi_{SS}$. In the long-distance limit, the light low-energy fields can simply be equated with the third isospin components of the microscopic fields. From the first term in a Taylor expansion of $\bar{\lambda}^3_{SS}$, we finally obtain the required long-distance field insertion,

$$\bar{\lambda}^{LD}_{\dot{\alpha}}(x) = ig\bar{v}\rho^2\pi\zeta^{\alpha}_{SS}S_{\alpha\dot{\alpha}}(x, x_0) \tag{4.31}$$

Importantly, this has the right long-distance behaviour to be associated with a massless fermion propagator.[5]

We have now considered all of the components of the semiclassical instanton method. Putting these components together we have

[5]If we were to calculate the one-instanton contribution to the Green's function $\langle\lambda(x_1)\lambda(x_2)\psi(x_3)\psi(x_4)\rangle$ then the field insertions would be obtained from the supersymmetric zero-modes, λ_{SS} and ψ_{SS}. But at large $|x|$, the solution (4.23) falls off more rapidly than a fermion propagator and therefore gives vanishing contribution after LSZ amputation. To get field insertions with the right long-distance behaviour to be associated with this Green's function we must turn to the *anti*-instanton sector.

$$\begin{aligned}\langle\bar\lambda(x_1)\bar\lambda(x_2)\bar\psi(x_3)\bar\psi(x_4)\rangle_{1I} &= \frac{2^{10}\pi^6\mu^4}{g^8}\int d^2\xi_{SC}d^2\zeta_{SC}\int d\rho\,\rho^3\exp\left(-S^{\text{1-inst}}_{\mathcal{N}=2\ SYM}\right)\\ &\times\int d^2\xi_{SS}d^2\zeta_{SS}\int d^4x_0\,\bar\lambda^{LD}(x_1)\bar\lambda^{LD}(x_2)\bar\psi^{LD}(x_3)\bar\psi^{LD}(x_4)\\ &= \frac{15}{2\pi^2}\frac{\Lambda^4_{PV}}{g^2\mathrm{v}^6}\int d^4x_0\,S_{\alpha\dot\alpha}(x_1,x_0)\bar S^{\dot\alpha\alpha}(x_2,x_0)S_{\beta\dot\beta}(x_3,x_0)\bar S^{\dot\beta\beta}(x_4,x_0).\end{aligned} \tag{4.32}$$

Note that all $\bar{\mathrm{v}}$ dependence cancels out after the collective coordinate integrations are performed so that we get the expected holomorphic dependence on v.

In order to compare the result of this semiclassical calculation with the Seiberg-Witten prediction we require the relation between Λ_{PV} and the scale defined implicitly by the Seiberg-Witten analysis. It is well-known that Λ's defined in different renormalization schemes are related by a factor that is completely determined at one-loop in perturbation theory. To find this factor, Finnel and Pouliot compared the one-loop expression for the effective coupling $\tau_{eff} = \mathcal{F}''(\mathrm{v})$ in the Pauli-Villars scheme with the one-loop expression derived from the Seiberg-Witten prepotential. These expressions are the same provided

$$\Lambda^{b_0} = 4\Lambda^{b_0}_{PV} \tag{4.33}$$

where $b_0 = 4$. This is the desired matching relation. After accounting for factors of g, the expression (4.32) perfectly agrees with the prediction (4.5), with $\mathcal{F}_1 = 1/8$.

As a final comment, we note that the Wilsonian effective action predicts solutions for other low-energy Green's functions, besides the four-fermi correlator (4.1). For instance, we can extract a four-fermi coupling of antifermion fields, which gives the low-energy amplitude $\langle\lambda(x_1)\lambda(x_2)\psi(x_3)\psi(x_4)\rangle$. The expression for this amplitude is just given by replacing v with $\bar{\mathrm{v}}$ on the right-hand side of Eq. (4.3). This is reflected in the semiclassical approach, since the relevant contributions to $\langle\lambda(x_1)\lambda(x_2)\psi(x_3)\psi(x_4)\rangle$ originate in the anti-instanton sector. Semiclassical calculations in this sector emulate those performed in the $(k > 0)$-instanton sector, and give the same results but with v and $\bar{\mathrm{v}}$ exchanged.

Other Green's functions that can be obtained from the Wilsonian effective action are $\langle v_{mn}(x_1)\bar\lambda(x_2)\bar\psi(x_3)\rangle$ and $\langle v_{mn}(x_1)v_{pq}(x_2)\rangle$. In [42] the one-instanton contributions to a general class of correlators including these were calculated. The results were all found to be consistent with the prediction $\mathcal{F}_1 = 1/8$.

4.2.2 Two-Instanton Tests

The analysis of Finnel and Pouliot was extended to the two-instanton level by Dorey, Khoze and Mattis [42]. Their calculation utilized the multi-instanton solution of Atiyah, Drinfeld, Hitchin, and Manin [2]. (We describe this solution in Chapter 5.) After completing the 32-fold integration over collective coordinates, Dorey, Khoze and Mattis determined the two-instanton coefficient $\mathcal{F}_2$ in Eq. (3.55) to be 5/256, in complete agreement with the exact results.

In subsequent work [43, 45], Dorey, Khoze and Mattis have considered the $SU(2)$ models with matter hypermultiplets (see also [44, 37]). As was mentioned in Section 3.3.2, odd-instanton effects in these models vanish when any hypermultiplet mass is set to zero. For nonzero hypermultiplet masses, a renormalization group matching condition ties the one-instanton contribution to the one-instanton effect in the pure Yang-Mills model. (This matching condition was described in Section 3.3.2; see Eq. (3.96) in particular.) Therefore a one-instanton calculation does not provide an independent test of the exact results in these models. The first tests are necessarily at the two-instanton level.

For the models with $N_f < 3$ flavours of matter hypermultiplets, the two-instanton calculations of [43, 44, 45, 37] were in complete agreement with the predictions of Seiberg-Witten theory. However, for $N_f = 3$ matter hypermultiplets the two-instanton contribution to the condensate $u = \langle \mathrm{Tr} A^2 \rangle$ was found not to match the prediction of Seiberg and Witten.[6] Let us denote by $\tilde{u}$ the solution obtained from the $N_f = 3$ Seiberg-Witten curve. Then the two-instanton discrepancy reads [37, 36]

$$\tilde{u} = u - u_0 \Lambda^2, \tag{4.34}$$

[6]The exact solution for u is obtained by inverting the formula for $\mathrm{v}(u)$. At weak-coupling it has an expansion consisting of a classical term $\mathrm{v}^2/2$ plus an infinite sum of nonperturbative terms. These nonperturbative terms have the form of instanton effects and they can be directly compared with the results of semiclassical calculations.

where $u_0 = -1/2^4 3^4$.

It was observed in [36] that the Seiberg-Witten analysis is insensitive to a constant shift in the curve parameter $\tilde{u}$. None of the global symmetries or monodromy properties that were built into the $N_f = 3$ curve are affected. Nor is the solution for the prepotential altered by a shift in $\tilde{u}$. The exact prepotential is obtained by inverting the function $\mathrm{v}(\tilde{u})$ and substituting into $\mathrm{v}_D(\tilde{u}) = \mathcal{F}'(\tilde{u})$. Clearly $\tilde{u}$ acts as a dummy variable in this procedure. Therefore the discrepancy can be resolved in a straightforward way, by simply reinterpreting the parameter $\tilde{u}$ appearing in the Seiberg-Witten curve as the shifted condensate (4.34).

Note that a k-instanton effect in the $N_f = 3$ model is proportional to Λ^k since the β-function coefficient, $b_0 = 2N - N_f$, is unity. From dimensional analysis it follows that a constant shift in u can only correspond to a two-instanton effect. Hence the curve should be completely fixed by the reparameterization described above, and there is no room for further discrepancies at higher order instanton levels.

For the $N_f = 4$ model, the disagreement between the instanton calculus and the exact results is more serious. Since the β-function vanishes when $N_f = 4$, there is no scale Λ in this model. The parameter that takes its place is the instanton factor $q = e^{2\pi i\tau}$, where τ is the complexified coupling of the microscopic $SU(2)$ theory. Now when all four hypermultiplet masses are zero there are no mass scales present. Seiberg and Witten assumed that in this case, the effective low-energy coupling is identical to the microscopic coupling,

$$\tau_{eff}^{(0)} = \tau. \tag{4.35}$$

(The superscript on τ_{eff} is to stress that this is the effective coupling of the massless theory.) The semiclassical calculations performed in [45, 36] show that in fact the effective coupling has the expansion

$$\tau_{eff}^{(0)} = \tau + \frac{i}{\pi}c_0 + \frac{i}{\pi}\sum_{k=2,4,\ldots}^{\infty} c_k q^k. \tag{4.36}$$

In the Pauli-Villars scheme,[7] a one-loop perturbative calculation gives $c_0 = 2\ln 2$ and a

[7] As was stressed in [36] it is important to be explicit about the renormalization scheme, even though the theory is finite. This is because the theory still generates divergent Feynman diagrams. It is only when diagrams are summed that the divergences cancel. The renormalization procedure fixes the usual ambiguity associated with this cancellation of infinities.

two-instanton calculation gives $c_2 = -7/2^7 3^5$.

To resolve this discrepancy it was proposed in [36] that the curve parameter q should be replaced by

$$q_{eff}^{(0)} = \exp(2\pi i \tau_{eff}^{(0)}). \tag{4.37}$$

For four flavours of massless matter hypermultiplets, the modified curve predicts the identity $\tau_{eff}^{(0)} = \tau_{eff}^{(0)}$ instead of Eq. (4.35). Hence the incorrect prediction (4.35) is avoided (although somewhat at the cost of the predictive power of the massless curve).

In Section 3.3.2, it was mentioned that the $N_f = 3$ curve can be obtained from the $N_f = 4$ curve if one uses a modified version of the renormalization group matching condition (3.96). The modified relation involves a factor of 64 that the Seiberg-Witten analysis fails to explain. On the other hand, if the curve parameter q is reinterpreted as $q_{eff}^{(0)}$ then the origin of this factor can be easily understood [36]. By substituting $q = q_{eff}^{(0)}$ in the modified version of the matching relation (3.96) (see associated footnote), and using the expansion (4.36), we obtain

$$4m_4 q_{PV} \to \Lambda_3. \tag{4.38}$$

(We have used the calculated value of c_0 in the Pauli-Villars scheme.) In the $N_f = 4$ model, the one-loop relation between scales, Eq. (4.33), becomes $q_{SW} = 4q_{PV}$. Substituting this into (4.38) gives

$$m_4 q_{SW} \to \Lambda_3, \tag{4.39}$$

which is the expected relation between parameters defined in the (implicit) Seiberg-Witten scheme. This result can be regarded as strong circumstantial evidence that the proposed fix of the $N_f = 4$ curve is correct. More stringent tests of the corrected curve would be provided by calculations at the three-instanton level and beyond.

Multi-instanton Test

Besides enabling the two-instanton tests summarized above, the $SU(2)$ multi-instanton calculus developed by Dorey, Khoze and Mattis [42, 45, 49] has been applied to verify a certain relation between u and $\mathcal{F}$ to *all* orders in the instanton expansion [50, 45]. This relation was originally derived by Matone [51] using the Seiberg-Witten curve for the pure $SU(2)$ Yang-Mills model. The Matone relation was shown to hold in $SU(2)$ models with matter in [52]. Physically, it can be understood as a Ward identity for superconformal invariance [53]. The Matone relation reads

$$\frac{\partial \mathcal{F}}{\partial \ln \Lambda} = \frac{b_0}{2\pi i} u. \tag{4.40}$$

The formula (4.40) was first checked by explicit one-instanton and two-instanton calculations in [54], and later shown to be true at all k-instanton levels in [50, 45].

Note that the relation (4.40) does not contradict the statement that in the $SU(2)$ model with $N_f = 3$ flavours there is a discrepancy associated with u but not with $\mathcal{F}$. Although the Matone relation does imply that a constant shift in u is tied to a constant shift in $\mathcal{F}$, a constant shift in $\mathcal{F}$ is not physically observable because only derivatives of $\mathcal{F}$ appear in the effective action. In contrast, the discrepancy associated with $\mathcal{F}$ in the finite model with $N_f = 4$ flavours *is* linked to a u discrepancy. The Matone relation for this model implies that the classical relation $u = \mathrm{v}^2/2$, expected to hold when all four hypermultiplet masses are zero [18], suffers the same quantum corrections as the effective coupling $\tau_{eff}^{(0)}$ (see Eq. (4.36) above). This u discrepancy can be fixed by reinterpreting the parameter u appearing in the $N_f = 4$ Seiberg-Witten curve in accordance with the Matone relation and the identification $\tau = \tau_{eff}^{(0)}$ [36].

The Matone relation has also been shown to follow from the hyperelliptic curves proposed for the $\mathcal{N} = 2$ supersymmetric $SU(N)$ models with $N > 2$ [55]. In Chapter 6, we shall confirm that the Matone relation holds in these models using the $SU(N)$ multi-instanton calculus constructed in Chapter 5.

4.3 $SU(N)$ Instanton Tests

Just as in the $SU(2)$ theory, the exact solutions in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory with $N > 2$ can be expanded in the semiclassical regime to give the one-loop perturbative contribution plus predictions for k-instanton corrections [31]. The weak-coupling expansion of the exact prepotential in the $SU(N)$ models has been performed in [56]. The k-instanton contributions take the form of rational functions of the vacuum expectation values v_u.

The first instanton tests of the $SU(N)$ exact results were performed at the one-instanton level by Ito and Sasakura in [57, 58]. These authors calculated the singular part[8] of the one-instanton contribution to the prepotential. When $N_f < 2N-2$ or $N_f = 2N-1$ there are no additional regular terms and the result is in full agreement with all of the hyperelliptic curves proposed in [19, 20, 21, 22, 38, 39].

In Section 3.3.2 it was mentioned that the the curve parameterizations that were suggested for the $SU(N)$ QCD models in [21] and [22] are not identical when $N_f \geq N$. Nor do they match either of the parameterizations that were later suggested in [38] and [39]. For the the case $N = 3$, Ito and Sasakura were able to calculate the regular terms that appear when $N_f \geq 4$ in the one-instanton contribution to u_2 [58]. Their results are in conflict with the predictions of all of the proposed curves and imply that none of the parameterizations in [21, 22, 38, 39] are correct.

In this section we perform a separate test of the proposed curves, by evaluating the one-instanton contribution to the quantum modulus u_3 in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD with $N > 2$ and $N_f < 2N$ flavours of matter hypermultiplets [47]. (The curves predict exact solutions for all the condensates $u_n = \langle \mathrm{Tr} A^n \rangle$, where $n = 2, 3, \ldots, N$.) Following the method of Ito and Sasakura, we determine the most singular part of the answer, which for $N_f < 2N-3$ or $N_f = 2N-2$ is the complete answer and agrees exactly with the prediction extracted from the curves. Our analysis also gives the coefficients of the regular terms which arise in the $SU(3)$ theory when $N_f \geq 3$. Here we find further disagreement with all the proposed curves.

To a large extent the semiclassical analysis we now describe parallels that of Section 4.2.1. Our first task is to write down the defining equations of the leading order short-distance constrained instanton. These are [42, 45, 37]:

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}, \tag{4.41}$$

[8]The one-instanton contribution can be decomposed into a 'regular' term and a 'singular' term, which diverges when any two VEV's coincide, corresponding to the restoration of a non-Abelian gauge symmetry.

$$\bar{\not{D}}\lambda = 0, \quad \bar{\not{D}}\psi = 0, \quad \bar{\not{D}}\chi = 0, \quad \bar{\not{D}}\tilde{\chi} = 0, \tag{4.42}$$

$$\not{D}\bar{\lambda} = 0, \quad \not{D}\bar{\psi} = 0, \quad \not{D}\bar{\chi} = 0, \quad \not{D}\bar{\tilde{\chi}} = 0, \tag{4.43}$$

$$D^2 A = \sqrt{2}ig[\lambda, \psi], \quad D^2 A^{\dagger a} = \sqrt{2}ig\tilde{\chi}T^a\chi, \tag{4.44}$$

$$D^2 q = \sqrt{2}ig\lambda\chi, \quad D^2\tilde{q} = -\sqrt{2}ig\tilde{\chi}\lambda, \quad D^2 q^\dagger = \sqrt{2}ig\tilde{\chi}\psi, \quad D^2\tilde{q}^\dagger = \sqrt{2}ig\psi\chi. \tag{4.45}$$

For notational clarity we have dropped the flavour indices on the squark and quark fields q and χ. On the Coulomb branch of the theory, the Higgs field A acquires the matrix of vacuum expectation values given by Eq. (3.97). This imposes a boundary condition on the solution for the Higgs field, since it must approach its matrix of VEV's at large distances.

The required self-dual solution to Eq. (4.41) of unit topological charge is given by the 'minimally embedded' BPST instanton (2.23). This configuration is subject to global gauge transformations which rotate it into $SU(N)$ group space. However, for the purposes of the instanton calculation we can choose to preserve the upper left embedding of the BPST instanton, and perform global gauge transformations of the matrix of VEV's (3.97) instead [59]. Therefore, in singular gauge, we have

$$A_\mu = \frac{2\rho^2}{g}\frac{y_\nu\bar{\eta}^a_{\mu\nu}}{y^2(y^2+\rho^2)}T^a, \tag{4.46}$$

and the boundary condition on the Higgs field becomes

$$\begin{aligned}\lim_{|y|\to\infty} A &= \Omega^\dagger\langle A\rangle\Omega; \qquad \Omega \in \frac{SU(N)}{SU(N-2)\times U(1)} \\ &= \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.\end{aligned} \tag{4.47}$$

The second equality indicates a convenient partitioning of the rotated VEV matrix; A_1 and A_4 are 2×2 and $(N-2)\times(N-2)$ matrix blocks respectively.

The leading order instanton action can be simplified by integrating by parts and using Eqs. (4.41)–(4.45). We obtain

$$\begin{aligned}S_0 &= \frac{8\pi^2}{g^2} + \int d^4x\,\partial_\mu\left\{2\mathrm{Tr}(A^\dagger D_\mu A) + (D_\mu q)^\dagger q + (D_\mu\tilde{q})^\dagger\tilde{q}\right\} + \sqrt{2}im\int d^4x\,\tilde{\chi}\chi \\ &+\sqrt{2}ig\int d^4x\,\left(\tilde{\chi}A\chi + q^\dagger\lambda\chi + \tilde{q}\psi\chi\right).\end{aligned} \tag{4.48}$$

The $\sqrt{2}$ prefactor of the quark mass term allies us with the usual curve convention.

We now present the remaining singular gauge solutions to the defining equations, which we shall use to evaluate the above action. The only solutions to the antifermion field equations (4.43) are trivial. The normalized zero-mode solutions for the gaugino λ are listed in [59, 60],

$$\lambda_{SC\alpha} = \frac{i\rho}{\pi} \frac{y_\nu \bar{\eta}^a_{\mu\nu} (e_\mu \bar{\xi}_{SC})_\alpha}{(y^2+\rho^2)^2} T^a, \tag{4.49}$$

$$\lambda_{SS\alpha} = \frac{\sqrt{2}\rho^2}{\pi} \frac{y_\nu y_\mu \bar{\eta}^a_{\lambda\nu} \eta^b_{\lambda\mu} (\tau^b \xi_{SS})_\alpha}{y^2(y^2+\rho^2)^2} T^a, \tag{4.50}$$

$$(\lambda_{M\alpha})_{uv} = -\frac{\rho}{\sqrt{2}\pi} \frac{y_\mu (e_\mu \epsilon)_{\alpha u}}{\sqrt{y^2}(y^2+\rho^2)^{3/2}} \xi_{Mv} \qquad (v = 3, 4, \ldots, N), \tag{4.51}$$

$$(\lambda_{N\alpha})_{uv} = \frac{\rho}{\sqrt{2}\pi} \frac{y_\mu (e_\mu)_{\alpha v}}{\sqrt{y^2}(y^2+\rho^2)^{3/2}} \xi_{Nu} \qquad (u = 3, 4, \ldots, N). \tag{4.52}$$

(Here ϵ is the antisymmetric tensor satisfying $\epsilon^{12} = 1$.) In addition to the two superconformal and two supersymmetric modes there are an additional $2(N-2)$ modes which we have chosen to partition such that the 'M' modes live in the upper right and the 'N' modes live in the lower left parts of the matrix representation of the $SU(N)$ Lie algebra. The analogous solutions for ψ are obtained by switching the Grassmann collective coordinates $\xi \to \zeta$.

The normalized solution for a quark flavour is [59, 60]

$$\chi_{\alpha u} = -\frac{\rho}{\pi} \frac{y_\mu (e_\mu \epsilon)_{\alpha u}}{\sqrt{y^2}(y^2+\rho^2)^{3/2}} \eta. \tag{4.53}$$

The conjugate quark solution is given by $\tilde{\chi}_{\alpha u} = \epsilon^{uv} \chi_{\alpha v}$ provided we exchange the collective coordinate η for $\tilde{\eta}$.

Turning to the scalar fields, we separate the solution for A into a part satisfying the homogeneous equation, A_h, and a particular solution A_p which arises in the presence of the Yukawa source term. The homogeneous solution was found in [57] to be

$$A_h = \begin{pmatrix} \frac{y^2}{y^2+\rho^2} A_{1(\text{tl})} + \frac{1}{2}\text{Tr}(A_1) I_2 & \sqrt{\frac{y^2}{y^2+\rho^2}} A_2 \\ \sqrt{\frac{y^2}{y^2+\rho^2}} A_3 & A_4 \end{pmatrix}, \tag{4.54}$$

where $A_{1(\text{tl})} = A_1 - \frac{1}{2}\text{Tr}(A_1) I_2$ and I_2 is the 2×2 identity matrix. This solution manifestly satisfies the boundary condition (4.47).

Linearity enables A_p to be decomposed further. If we define $A_{A/B}$ as the particular solution with fermionic modes λ_A and ψ_B inserted into the source term, then

$$A_p = \sum_{A,B=SC,SS,M,N} A_{A/B}. \tag{4.55}$$

We obtain the following list of independent solutions which enter the right-hand side of this equation:

$$A_{SC/SC} = \frac{ig}{4\sqrt{2}\pi^2}\frac{y^2(\bar{\xi}_{SC}\epsilon\tau^a\bar{\zeta}_{SC})}{(y^2+\rho^2)^2}T^a, \tag{4.56}$$

$$A_{SC/SS} = -\frac{g\rho}{4\pi^2}\frac{y_\mu\bar{\eta}^a_{\nu\mu}(\bar{\xi}_{SC}\epsilon\bar{e}_\nu\zeta_{SS})}{(y^2+\rho^2)^2}T^a, \tag{4.57}$$

$$A_{SS/SS} = -\frac{ig\rho^2}{2\sqrt{2}\pi^2}\frac{y_\nu y_\mu\bar{\eta}^a_{\lambda\nu}\eta^b_{\lambda\mu}(\xi_{SS}\epsilon\tau^b\zeta_{SS})}{y^2(y^2+\rho^2)^2}T^a, \tag{4.58}$$

$$(A_{SC/M})_{uv} = \frac{ig}{8\pi^2}\frac{\sqrt{y^2}}{(y^2+\rho^2)^{3/2}}\bar{\xi}^u_{SC}\zeta_{Mv}, \tag{4.59}$$

$$(A_{SC/N})_{uv} = \frac{ig}{8\pi^2}\frac{\sqrt{y^2}}{(y^2+\rho^2)^{3/2}}\zeta_{Nu}(\epsilon\bar{\xi}_{SC})_v, \tag{4.60}$$

$$(A_{SS/M})_{uv} = -\frac{ig\rho}{4\sqrt{2}\pi^2}\frac{y_\mu}{\sqrt{y^2}(y^2+\rho^2)^{3/2}}(\bar{e}_\mu\xi_{SS})^u\zeta_{Mv}, \tag{4.61}$$

$$(A_{SS/N})_{uv} = -\frac{ig\rho}{4\sqrt{2}\pi^2}\frac{y_\mu}{\sqrt{y^2}(y^2+\rho^2)^{3/2}}\zeta_{Nu}(\epsilon\bar{e}_\mu\xi_{SS})^v, \tag{4.62}$$

$$(A_{M/N})_{uv} = \frac{ig}{8\sqrt{2}\pi^2}\frac{1}{(y^2+\rho^2)}\left\{\delta_{uv}\delta_{u,v\leq 2}\sum_{w=3}^{N}\zeta_{Nw}\xi_{Mw} - 2\zeta_{Nu}\xi_{Mv}\delta_{u,v\geq 3}\right\}, \tag{4.63}$$

$$A_{M/M} = A_{N/N} = 0. \tag{4.64}$$

The solution for $A_{A/B}$ is deduced from the solution for $A_{B/A}$ by changing the sign and making the exchange $\xi \leftrightarrow \zeta$.

The conjugate Higgs also consists of a homogeneous and a particular solution. The homogeneous solution is simply the Hermitian conjugate of (4.54) whilst the particular solution is

$$(A_p^\dagger)_{uv} = \frac{ig}{4\sqrt{2}\pi^2}\frac{1}{(y^2+\rho^2)}\left\{\left(\frac{N-2}{2N}\right)\delta_{uv}\delta_{u,v\leq 2} - \frac{1}{N}\delta_{uv}\delta_{u,v\geq 3}\right\}\bar{\eta}\eta. \tag{4.65}$$

Finally, each squark solution is a sum of particular solutions [60],

$$q_{SCu} = \frac{ig}{4\sqrt{2}\pi^2}\frac{\sqrt{y^2}}{(y^2+\rho^2)^{3/2}}\bar{\xi}^u_{SC}\eta, \tag{4.66}$$

$$q_{SSu} = -\frac{ig\rho}{4\pi^2}\frac{y_\mu}{\sqrt{y^2}(y^2+\rho^2)^{3/2}}(\bar{e}_\mu\xi_{SS})^u\eta, \tag{4.67}$$

$$q_{Nu} = \frac{ig}{4\pi^2}\frac{1}{(y^2+\rho^2)}\xi_{Nu}\eta, \tag{4.68}$$

where q_A represents the solution with λ_A inserted in the source term. The solutions for $q^\dagger$ and the conjugate representation squarks may be obtained by straightforward manipulations of these configurations.

By plugging the above solutions into Eq. (4.48) we are immediately able to evaluate the leading order instanton action. Ignoring supersymmetric zero-modes which are not lifted and give no contribution, we find

$$2\int d^4x\,\partial_\mu \mathrm{Tr}(A^\dagger D_\mu A) = 8\pi^2\rho^2 F + g(\bar{\zeta}_{SC},\zeta_M,\zeta_N)M(\bar{\xi}_{SC},\xi_M,\xi_N)^t, \tag{4.69}$$

$$\int d^4x\,\partial_\mu((D_\mu q)^\dagger q) = 0, \tag{4.70}$$

$$\int d^4x\,\partial_\mu((D_\mu \tilde{q})^\dagger \tilde{q}) = 0, \tag{4.71}$$

$$\sqrt{2}im\int d^4x\,\tilde{\chi}\chi = -i\sqrt{2}m\tilde{\eta}\eta, \tag{4.72}$$

$$\sqrt{2}ig\int d^4x\,\tilde{\chi}A\chi = -\frac{ig}{\sqrt{2}}\mathrm{Tr}(A_1)\tilde{\eta}\eta - \frac{g^2}{24\pi^2\rho^2}\sum_{u=3}^{N}(\xi_{Mu}\zeta_{Nu}+\xi_{Nu}\zeta_{Mu})\tilde{\eta}\eta, \tag{4.73}$$

$$\sqrt{2}ig\int d^4x\,\left(q^\dagger\lambda\chi+\tilde{q}\psi\chi\right) = -\frac{g^2}{12\pi^2\rho^2}\sum_{u=3}^{N}(\xi_{Mu}\zeta_{Nu}+\xi_{Nu}\zeta_{Mu})\tilde{\eta}\eta. \tag{4.74}$$

In Eq. (4.69), F and M are the same as in [57], namely

$$F = \mathrm{Tr}(A^\dagger_{1(\mathrm{tl})}A_{1(\mathrm{tl})} + \frac{1}{2}(A_3A^\dagger_2 + A_2A^\dagger_3)), \tag{4.75}$$

and

$$M = i\begin{pmatrix} \sqrt{2}\epsilon A^\dagger_{1(\mathrm{tl})} & (A^\dagger_2)^t & \epsilon A^\dagger_3 \\ A^\dagger_2 & 0 & -\frac{\mathrm{Tr}A^\dagger_1}{\sqrt{2}}I_{N-2}+\sqrt{2}A^\dagger_4 \\ (\epsilon A^\dagger_3)^t & -\frac{\mathrm{Tr}A^\dagger_1}{\sqrt{2}}I_{N-2}+\sqrt{2}(A^\dagger_4)^t & 0 \end{pmatrix}, \tag{4.76}$$

where I_{N-2} is the $(N-2)\times(N-2)$ identity matrix.

Our next consideration is the collective coordinate integration measure. The bosonic measure is given by (2.64). Since the fermion zero-modes are all normalized, the complete one-instanton measure reads

$$\int d\mu = \frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}\frac{\mu^{2N-N_f}}{g^{4N}}\int d\Omega\int d^4x_0 d\rho\,\rho^{4N-5}\int d^{2N}\zeta d^{2N}\xi\int d^{N_f}\eta d^{N_f}\tilde{\eta}. \quad (4.77)$$

Here we have included all factors of the Pauli-Villars regularization mass μ that arise due to the bosonic and fermionic zero-modes. We can eliminate μ in favour of the renormalization group invariant scale

$$\Lambda_{PV}^{2N-N_f} = \mu^{2N-N_f}e^{-8\pi^2/g^2}. \quad (4.78)$$

To compare with the exact results, we need to switch from the Pauli-Villars scale to the scale Λ used in the hyperelliptic curves. In [58] it was shown using renormalization group matching arguments that these scales are related by

$$\Lambda^{2N-N_f} = 2^{2-N+N_f/2}i^{N_f}\Lambda_{PV}^{2N-N_f}. \quad (4.79)$$

Since there is complete cancellation between the small-fluctuations determinants associated with quadratic field fluctuations [48], we can now write down an expression for the one-instanton contribution to u_n. After assembling the relevant factors and performing the integration over the quark zero-modes we have

$$\begin{aligned} u_n^{1I} &= \frac{\Lambda^{2N-N_f}}{4}\sum_{p=0}^{N_f}2^{N-N_f+p}g^{N_f-p}\int d\tilde{\mu}\int d^2\zeta_{SS}d^2\xi_{SS}\int d^4x_0\,\mathrm{Tr}(A^n) \\ &\times t_p\left(\mathrm{Tr}(A_1)-\frac{ig}{4\sqrt{2}\pi^2\rho^2}\sum_{u=3}^{N}(\xi_{Mu}\zeta_{Nu}+\xi_{Nu}\zeta_{Mu})\right)^{N_f-p}\exp(-S_H), \end{aligned} \quad (4.80)$$

where the x_0 and SS mode integrations have been separated from $d\mu$, leaving

$$\int d\tilde{\mu} = \frac{2^{4N+2}\pi^{4N-2}}{(N-1)!(N-2)!}\frac{1}{g^{4N}}\int d\Omega\int d\rho\,\rho^{4N-5}\int d^{2N-2}\zeta d^{2N-2}\xi. \quad (4.81)$$

S_H is just the contribution of the Higgs kinetic term to the action as given by (4.69) and the t_p are symmetric polynomials in the hypermultiplet masses,

$$t_p = \sum_{i_1<i_2<\cdots<i_p}^{N_f} m_{i_1}m_{i_2}\ldots m_{i_p}. \quad (4.82)$$

In contrast to the low-energy Green's functions considered in Section 4.2.1, the condensate u_n is calculated using field insertions given by the instanton background at *short* distances. These insertions must saturate the integration over the collective coordinates corresponding to the exact supersymmetric zero-modes. It follows that only the part of $\mathrm{Tr}(A^n)$ which contains precisely four SS Grassmann variables can give a nonzero contribution. When $n = 2$ this is just $\mathrm{Tr}(A^2_{SS/SS})$ and using Eq. (4.58) we can perform the integration of the field operator over x_0 and the SS modes,

$$\int d^2\zeta_{SS} d^2\xi_{SS} \int d^4x_0 \,\mathrm{Tr}(A^2) = -\frac{g^2}{2^4\pi^2}. \tag{4.83}$$

In [57, 58], the authors considered the integral expression (4.80) when $n = 2$. Since the integration over group space was not generally tractable they studied the particular case of two VEV's being infinitesimally close. In this limit they found that the group integration linearized and could be carried out. The singularity structure of the answer is associated with the infra-red divergence caused by the restoration of a non-Abelian subgroup when any two VEV's coincide. Taking this to represent the only instance where the instanton integration diverges, and by considerations of dimensional analysis, gauge invariance and holomorphy, Ito and Sasakura deduced the full result

$$u_2^{1I} = \frac{\Lambda^{2N-N_f}}{2}\sum_{p=0}^{N_f} t_p \left(\sum_{u=1}^{N} \frac{\mathrm{v}_u^{N_f-p}}{\prod_{v\neq u}^{N}(\mathrm{v}_v - \mathrm{v}_u)^2} + \alpha_N \delta_{N_f-p,2N-2} + \beta_N \delta_{N_f-p,2N} \sum_{u=1}^{N} \mathrm{v}_u^2\right). \tag{4.84}$$

The analysis fails to determine the constant coefficients of the regular terms, α_N and β_N. However, in the specific case of $SU(3)$ Ito and Sasakura were able to directly evaluate the integral expression for u_2^{1I} and they found that $(\alpha_3, \beta_3) = (-3/8, -15/64)$. For a range of input values for the VEV's we have numerically verified these results.

We now employ the explicit solutions for A to evaluate u_3^{1I} along similar lines. For insertion into the integrand, we require the part of $\mathrm{Tr}(A^3)$ which has the necessary quadrilinear dependence on the SS Grassmann variables. This is

$$3\mathrm{Tr}\left\{A^2_{SS/SS}\left(A_h + \sum_{A,B\neq SS} A_{A/B}\right)\right\} + 3\mathrm{Tr}\left\{A_{SS/SS}\left(\sum_{A\neq SS} A_{A/SS} + \sum_{B\neq SS} A_{SS/B}\right)^2\right\}. \tag{4.85}$$

Since $A^2_{SS/SS}$ is proportional to the 2×2 identity matrix in the upper left block of the matrix representation, the first term reduces to two distinct nonzero components,

$$3\mathrm{Tr}(A^2_{SS/SS}A_h) = \frac{3}{2}\mathrm{Tr}(A^2_{SS/SS})\mathrm{Tr}(A_1), \tag{4.86}$$

$$3\mathrm{Tr}(A^2_{SS/SS}(A_{M/N} + A_{N/M})) = -\frac{3ig}{8\sqrt{2}\pi^2}\frac{\mathrm{Tr}(A^2_{SS/SS})}{y^2+\rho^2}\sum_{k=3}^{N}(\xi_{Mk}\zeta_{Nk} + \xi_{Nk}\zeta_{Mk}). \tag{4.87}$$

The second term simplifies because $A_{SS/SS}$ is composed of Pauli matrices living in the upper left corner of the matrix representation. Closer inspection shows that the only contributing component is

$$3\mathrm{Tr}(A_{SS/SS}(A_{M/SS}A_{SS/N} + A_{SS/M}A_{N/SS})) = -\frac{3ig}{8\sqrt{2}\pi^2}\frac{\mathrm{Tr}(A^2_{SS/SS})}{y^2+\rho^2}\sum_{u=3}^{N}(\xi_{Mu}\zeta_{Nu} + \xi_{Nu}\zeta_{Mu}). \tag{4.88}$$

Upon integrating over x_0 and the SS modes, we get

$$\int d^2\zeta_{SS}d^2\xi_{SS}\int d^4x_0\mathrm{Tr}(A^3) = \frac{3}{2}\left(-\frac{g^2}{2^4\pi^2}\right)\left(\mathrm{Tr}(A_1) - \frac{ig}{4\sqrt{2}\pi^2\rho^2}\sum_{u=3}^{N}(\xi_{Mu}\zeta_{Nu} + \xi_{Nu}\zeta_{Mu})\right). \tag{4.89}$$

The first factor in brackets is just the corresponding result (4.83) for the $\mathrm{Tr}(A^2)$ insertion whilst the second factor precisely matches the part of the instanton action which is pulled down by the integration over the quark collective coordinates.

This is a fortunate result since it allows us to immediately determine u_3^{1I} from knowledge of u_2^{1I}. Using Eq. (4.80) and Eq. (4.84) and after accounting for a rescaling of the Higgs field, we find that for $N_f < 2N$,

$$u_3^{1I} = \frac{3\Lambda^{2N-N_f}}{2}\sum_{p=0}^{N_f}t_p\left(\sum_{u=1}^{N}\frac{\mathrm{v}_u^{N_f-p+1}}{\prod_{v\neq u}^{N}(\mathrm{v}_v-\mathrm{v}_u)^2} + \tilde{\alpha}_N\delta_{N_f-p,2N-3} + \tilde{\beta}_N\delta_{N_f-p,2N-1}\sum_{u=1}^{N}\mathrm{v}_u^2\right), \tag{4.90}$$

where $(\tilde{\alpha}_N, \tilde{\beta}_N) \equiv (\alpha_N, \beta_N)$.

Let us now make the comparison with the exact results. For $N_f < 2N$ we can make use of the freedom to shift the x-variable to write all of the proposed curves [21, 22, 38, 39] in the following form,

$$y^2 = P(x)^2 - Q(x), \tag{4.91}$$

where

$$Q(x) = \Lambda^{2N-N_f} \sum_{p=0}^{N_f} t_p x^{N_f-p} \quad \text{and} \quad P(x) = \prod_{u=1}^{N} (x - e_u) + \Lambda^{2N-N_f} T(x). \tag{4.92}$$

The moduli space parameters e_u satisfy $\sum_{u=1}^{N} e_u = 0$ and are related to the moduli of the physical theory through

$$u_n = \sum_{u=1}^{N} e_u^n. \tag{4.93}$$

The function $T(x)$ satisfies

$$T(x) = \sum_{p=0}^{N_f} t_p T^{(N_f-p-N)}(x) \delta_{N_f-p \geq N}, \tag{4.94}$$

where the $T^{(N_f-p-N)}(x)$ are polynomials of degree $(N_f - p - N)$ in x, with possible dependence on the dynamical scale and also on the moduli space parameters. The precise form of the polynomials $T^{(N_f-p-N)}(x)$ distinguishes the various curves proposed in [21], [22], [38] and [39].[9]

The exact solutions are obtained from the curves through the periods

$$\mathrm{v}_{D\,u} = \frac{1}{2\pi i} \oint_{B_u} \lambda, \tag{4.95}$$

$$\mathrm{v}_u = \frac{1}{2\pi i} \oint_{A_u} \lambda, \tag{4.96}$$

where

$$\lambda = \frac{x(P' - \frac{PQ'}{2Q})}{y} dx, \tag{4.97}$$

and the A_u and B_u are a canonical basis of one-cycles enclosing branch cuts of the curves. These integrals can be expanded in powers of Λ^{2N-N_f} in the weak-coupling regime [56]. Here, we require only the expansion of Eq. (4.96),

$$\mathrm{v}_u = e_u + \sum_{m,n \geq 0; m+n \neq 0} \frac{(-1)^n (\Lambda^{2N-N_f})^{m+n}}{(m!)^2 n! 2^{2m}} \frac{\partial^{2m+n-1}}{\partial e_u} (S_u(e)^m R_u(e)^n), \tag{4.98}$$

[9] There is one requirement that is satisfied by all parameterizations. Namely, in the $N_f = 2N - 1$ curves, the x^{N-1} term in $T^{(N-1)}(x)$ has coefficient $\frac{1}{4}$. This ensures that the meromorphic one-form λ has no residue at infinity when the bare masses are zero.

where

$$S_u(e) = \frac{\sum_{p=0}^{N_f} t_p e_u^{N_f-p}}{\prod_{v\neq u}^N (e_u - e_v)^2} \quad \text{and} \quad R_u(e) = \frac{T(e_u)}{\prod_{v\neq u}^N (e_u - e_v)}. \tag{4.99}$$

At the one-instanton level it is a simple matter to invert this series and use the defining expression (4.93) to get the curve prediction for u_n^{1I}. The answer may be written in the form

$$u_n^{1I} = \frac{n(n-1)\Lambda^{2N-N_f}}{4} \sum_{p=0}^{N_f} t_p \left(\sum_{u=1}^{N} \frac{\mathrm{v}_u^{N_f-p+n-2}}{\prod_{v\neq u}^N (\mathrm{v}_v - \mathrm{v}_u)^2} + \frac{1}{n-1} r_n^{(N_f-p)} \right), \tag{4.100}$$

where $r_n^{(N_f-p)}$ is a regular function of the VEV's given by

$$\begin{aligned} r_n^{(N_f-p)} \;&=\; \sum_{u=1}^{N} \frac{\mathrm{v}_u^{N_f-p+n-2}}{\prod_{w\neq u}^N (\mathrm{v}_u - \mathrm{v}_w)^2} \left(2\mathrm{v}_u \sum_{v\neq u}^{N} \frac{1}{(\mathrm{v}_u - \mathrm{v}_v)} - (N_f - p + n - 1) \right) \\ &+4\delta_{N_f-p\geq N} \sum_{u=1}^{N} \frac{\mathrm{v}_u^{n-1} T^{(N_f-p-N)}(\mathrm{v}_u)|_{\Lambda=0}}{\prod_{w\neq u}^N (\mathrm{v}_u - \mathrm{v}_w)}. \end{aligned} \tag{4.101}$$

The non-singular nature of $r_n^{(N_f-p)}$ can be verified by expanding it in powers of the separation between two VEV's.

When $N_f - p < 2N - n$ or $N_f - p = 2N - n + 1$, the regular function $r_n^{(N_f-p)}$ vanishes and the full answer is unambiguously given by the singular term in Eq. (4.100). Setting $n = 3$ and comparing with Eq. (4.90), we conclude that when $N_f < 2N - 3$ or $N_f = 2N - 2$ all of the proposed curves predict the correct one-instanton contribution to u_3. By setting $n = 2$ and comparing with Eq. (4.84), we confirm the similar observation of Ito and Sasakura [58], i.e. the agreement of all the proposed curves with the one-instanton prediction for u_2 when $N_f < 2N - 2$ or $N_f = 2N - 1$.

When $N_f \geq 2N - 3$, the functions $r_2^{(2N-2)}$, $r_3^{(2N-3)}$ and $r_3^{(2N-1)}$ simplify to give regular terms of the expected form. (In fact, this form is fixed by dimensional considerations.) However, the associated numerical coefficients critically depend on the function $T(x)$. In Table 1 we summarize the curve predictions for the coefficients α_3, $\tilde{\alpha}_3$ and $\tilde{\beta}_3$ pertinent to the $SU(3)$ theory with $N_f < 6$ flavours, according to the various suggestions for $T(x)$ in [21, 22, 38, 39]. We see that none of the proposed curves give the numbers predicted by the instanton calculus.

For the $SU(3)$ models, the instanton results can be used in conjunction with Eq. (4.100) to fix the parameterization of the curves so that they are free from discrepancies at the one-instanton level. In Table 1, we give the polynomials $T^{(0)}(x)$, $T^{(1)}(x)$ and $T^{(2)}(x)$ that lead to the right regular term coefficients. Note that in fixing these polynomials we are essentially making shifts in the curve parameters e_u. Through Eq. (4.93), this equates to shifting the implicit curve parameters u_2 and u_3. So our fix of the $SU(3)$ curves is similar in nature to the fix of the Seiberg-Witten curve for the $SU(2)$ model with $N_f = 3$ flavours [37, 45]. Dimensional considerations imply that $T^{(0)}(x)$ and $T^{(1)}(x)$ are *uniquely* fixed so that in the $N_f = 3$ and $N_f = 4$ models there can be no discrepancies at higher order instanton levels. For the $N_f = 5$ model there is room for further corrections up to the three-instanton level.

Source of prediction	$T^{(0)}(x)$	$T^{(1)}(x)$	$T^{(2)}(x)$	$(\alpha_3, \tilde{\alpha}_3, \tilde{\beta}_3)$
Ref. [21]	$\frac{1}{4}$	$\frac{1}{4}x$	$\frac{1}{4}x^2$	(0,0,-1/4)
Refs. [38] and [22]	0	0	$\frac{1}{4}x^2 + \frac{\Lambda}{48}x + \frac{\Lambda^2}{1728} - \frac{1}{24}u_2$	(-1,-1/2,-1/3)
Ref. [39]	$\frac{1}{4}$	$\frac{1}{4}x$	$\frac{1}{4}x^2 + \frac{1}{8}u_2$	(0,0,0)
Instanton calculus	$\frac{1}{16}$	$\frac{5}{32}x$	$\frac{1}{4}x^2 + \frac{1}{128}u_2$	(-3/8,-3/8,-15/64)

Table 4.1: *Predictions for the coefficients of the regular terms appearing in the one-instanton contributions to the moduli u_2 and u_3 in $\mathcal{N} = 2$ supersymmetric $SU(3)$ QCD, according to suggested forms for $T(x)$ (defined by the polynomials $T^{(0)}(x)$, $T^{(1)}(x)$ and $T^{(2)}(x)$).*

In Section 4.2.2, we showed that the proposed fix of the $N_f = 3$ Seiberg-Witten curve did not affect the solution for the prepotential. In a similar way, the prepotential obtained from the $N_f < 2N$ hyperelliptic curves is insensitive to the function $T(x)$ [56]. In this respect, all of the $N_f < 2N$ curves are equivalent. However, we stress that the form of $T(x)$ critically affects the predictions for the condensates. For the physical correspondence to be complete, there must exist a definite form for $T(x)$ which determines the correct u_n. Our results show that the criteria used in [21, 22, 38, 39] to fix $T(x)$ cannot be valid. It would be interesting if some alternative a priori criterion could be found which is consistent with the instanton calculus.

The instanton calculation of this section has been limited because we were not able to perform the highly nontrivial integration over group space collective coordinates. In Chapter 6 we shall reformulate the problem using the ADHM instanton calculus developed in Chapter 5. This will enable us to completely evaluate the one-instanton contribution to the prepotential in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD.

4.4 Summary

In this chapter we have described instanton tests of the exact results in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD. The results of these tests can be summarized as follows:

- There is complete agreement with the Seiberg-Witten $SU(2)$ curves at both the one-instanton and two-instanton levels when there are $N_f = 0, 1, 2$ flavours of matter hypermultiplets [41, 42, 43, 44, 45, 37].

- When there are $N_f = 3$ flavours, the Seiberg-Witten prediction for the two-instanton contribution to the condensate $u = \langle \mathrm{Tr} A^2 \rangle$ does not match the result obtained from first principles [37]. This discrepancy can easily be resolved, by making a shift in the parameter u appearing in the Seiberg-Witten curve [36].

- In the special case of $N_f = 4$ flavours, there is significant disagreement between the first-principles analysis and the exact predictions. The reason for this disagreement is that when all hypermultiplet masses are zero Seiberg and Witten take the low-energy effective coupling $\tau_{eff}^{(0)}$ to be identical to the coupling of the microscopic theory, τ. In fact, there are corrections due to both perturbation theory and nonperturbative instanton effects. To resolve the discrepancy, it has been conjectured that the parameter τ appearing in the $N_f = 4$ curve should be replaced by the effective coupling $\tau_{eff}^{(0)}$ [36].

- For the $SU(N)$ models with $N > 2$, the one-instanton calculations that have been performed in [57, 58, 47] agree with the predictions of the proposed hyperelliptic curves when regular terms are absent.

- In the $SU(3)$ theory there are discrepancies at the one-instanton level in the predictions for the condensates u_2 and u_3 when there are $N_f = 3, 4, 5$ flavours [58, 47], similar in nature to the two-instanton discrepancy associated with the SU(2) model with $N_f = 3$ flavours [37]. The discrepancies are linked to an ambiguity in the parameterization of the hyperelliptic curves when $N_f \geq N$. The ambiguity can be fixed using the instanton results [47].

It is clearly very desirable to extend the analysis to higher order instanton levels. In the $SU(2)$ models with $N_f = 3$ and $N_f = 4$ flavours, this would provide an important check of the modified Seiberg-Witten curves. Much progress has been made along these lines by Dorey, Khoze and Mattis. Based on the ADHM formulation of multi-instanton solutions they have developed a complete $SU(2)$ supersymmetric multi-instanton calculus [42, 45, 49]. In the next chapter, we describe the generalization of this calculus to supersymmetric $SU(N)$ Yang-Mills theory. In Chapter 6 we apply the supersymmetric $SU(N)$ instanton calculus to dramatically improve upon the one-instanton calculations described in Section 4.3.

Chapter 5

Multi-Instanton Calculus

5.1 Introduction

An investigation of nonperturbative effects due to instantons of topological charge greater than one requires the multi-instanton construction of Atiyah, Drinfeld, Hitchin and Manin (ADHM) [2]. Unfortunately, there are various technical difficulties which generally prevent the use of ADHM multi-instantons in semiclassical calculations. One of the problems is that the collective coordinates appearing in the ADHM construction are not independent, but must satisfy certain nonlinear constraints. These constraints have only been solved explicitly for low values of the topological charge (specifically, for $k \leq 3$) [61, 62]. Moreover, when $k > 1$ the task of calculating the collective coordinate Jacobian factors and the small-fluctuations determinants proves to be highly nontrivial. In fact, the only success has been the calculation of the two-instanton Jacobian factors in $SU(2)$ Yang-Mills theory [63, 42].

Notwithstanding these difficulties, there have recently been significant advances in the study of ADHM multi-instanton effects, stimulated by the exact results in $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. In the pioneering work of Dorey, Khoze and Mattis [42, 45], the ADHM construction was employed not only to perform explicit two-instanton calculations, but also to derive results at the arbitrary k-instanton level.

Supersymmetry has played a central role in these developments. In particular, the two-instanton calculations have relied on the cancellation of the small-fluctuations determinants that occurs in supersymmetric Yang-Mills theory[48]. The presence of supersymmetry is also reflected in the multi-instanton background configuration [15]. Since the defining equations of the instanton are manifestly supersymmetric, it follows that the solutions to these equations are transformed into one another by supersymmetry transformations. So the solutions comprising the multi-instanton background form a supersymmetry multiplet.

Dorey, Khoze and Mattis observed that the supersymmetry algebra is realized in a very natural way on the overcomplete set of collective coordinates appearing in the ADHM construction [45]. Moreover, the ADHM constraints themselves are invariant under supersymmetry transformations. Since physically relevant quantities in the instanton calculus (such as the instanton action and the collective coordinate measure) are supersymmetry invariant [15], it follows that they depend upon combinations of the ADHM collective coordinates that are invariant under supersymmetry transformations. This fact was exploited by Dorey, Khoze and Mattis to obtain the leading order constrained k-instanton action for $\mathcal{N}=2$ supersymmetric $SU(2)$ QCD [45].

In subsequent work [49], Dorey, Khoze and Mattis have derived an expression for the k-instanton collective coordinate measure in supersymmetric $SU(2)$ Yang-Mills theory. Instead of directly calculating collective coordinate Jacobians, they introduced the measure as an ansatz, which takes the form of an integral over the full set of unconstrained ADHM collective coordinates, with the constraints imposed using δ-functions under the integral sign. Using the key requirement of supersymmetry invariance, together with other symmetries, they proved the uniqueness of this ansatz, up to a numerical prefactor. The prefactor was determined from the well-known one-instanton measure of 't Hooft using an inductive argument based on the property of cluster decomposition. Taken together, the results of [42, 45, 49] constitute a complete multi-instanton calculus for supersymmetric $SU(2)$ Yang-Mills theory.

In this chapter we construct a multi-instanton calculus for $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $U(N)$[1] or $SU(N)$ [64]. This represents a

[1]The ADHM formalism of [2, 61, 62, 63] is slightly more naturally suited to the gauge group $U(N)$ than to the gauge group $SU(N)$. We shall indicate what (minor) modifications of the formalism are required for the gauge group $SU(N)$ where necessary.

generalization of the work of Dorey, Khoze and Mattis [42, 45, 49]. In the next chapter we shall employ our calculus to investigate (multi-)instanton effects in supersymmetric $SU(N)$ Yang-Mills theory.

The chapter is organized as follows. In Section 5.2, we review the ADHM construction of the general k-instanton solution to the self-dual Yang-Mills equation [2, 61, 62, 63]. In particular, we follow the derivation given in [62]. The remaining sections are devoted to the development of the supersymmetric instanton calculus based on this construction. Throughout these sections we treat the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases in parallel. In Section 5.3, following [42], we construct the supersymmetric multiplets of instanton solutions. The adjoint fermion zero-modes were first derived in [62]. For the $\mathcal{N}=2$ case we also require the solution for the adjoint Higgs bosons. The construction of these solutions in the $SU(2)$ theory was one of the key results of [42, 45], and we show how to extend it to $U(N)$ (or $SU(N)$). In Section 5.4, we explain, following [45], how the $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry algebras may be realized directly on the space of unconstrained bosonic and fermionic ADHM collective coordinates, prior to the imposition of the nonlinear constraints that they are required to obey. In Section 5.5, generalizing [42, 45], we obtain the $\mathcal{N}=1$ and $\mathcal{N}=2$ multi-instanton actions for $U(N)$ (or $SU(N)$) gauge theory coupled to N_f flavours of matter hypermultiplets. Finally, in Section 5.6, following [49], we derive the $\mathcal{N}=1$ and $\mathcal{N}=2$ collective coordinate integration measures.

Throughout this chapter we work in Minkowski space. This has the advantage of keeping supersymmetry manifest. It is also convenient to set $g=1$.

5.2 The $U(N)$ ADHM Multi-Instanton

In this section we concern ourselves with pure $U(N)$ (or $SU(N)$) Yang-Mills theory, without fermions or scalars. We adhere to the ADHM tradition and work with an anti-Hermitian gauge field. This is arranged by writing $v_m \to iv_m$ and also $v_{mn} \to iv_{mn}$. Both v_m and v_{mn} are clearly $N\times N$ matrices and in the case of $SU(N)$, they are also traceless.

For the particular case of $N=2$, the ADHM formalism reviewed here is slightly different to the $SU(2)$ formalism employed by Dorey, Khoze and Mattis. Their formalism is actually the one for the symplectic groups, and exploits the fact that $SU(2) \simeq Sp(1)$. A comparison of the ADHM construction for the different classical groups is given in [62, 61, 63].

5.2.1 Construction of the Solution

The ADHM multi-instanton is the general solution to the self-duality equation

$$v_{mn} = {}^*v_{mn}, \tag{5.1}$$

where the dual of v_{mn} is given by[2]

$${}^*v_{mn} = \frac{i}{2}\epsilon_{mnpq}v^{pq}. \tag{5.2}$$

The ADHM construction is discussed in [2, 62, 61, 63]. Here we follow, with minor modifications, the $U(N)$ formalism of [62].

The basic object in the ADHM construction is an $(N+2k)\times 2k$ complex-valued matrix $\Delta_{[N+2k]\times[2k]}$ which is taken to be linear in the space-time variable x_m :[3]

$$\Delta_{[N+2k]\times[2k]}(x) \equiv \Delta_{[N+2k]\times[k]\times[2]}(x) = a_{[N+2k]\times[k]\times[2]} + b_{[N+2k]\times[k]\times\underline{[2]}}\, x_{\underline{[2]}\times[2]} \,. \tag{5.3}$$

Here we have decomposed the column index of $\Delta_{[N+2k]\times[2k]}$ into the direct product of two indices, and have used a quaternionic representation of x_m,

$$x_{[2]\times[2]} = x_{\alpha\dot\alpha} = x_m\, \sigma^m_{\alpha\dot\alpha} \,. \tag{5.4}$$

The Hermitian conjugate of $\Delta_{[N+2k]\times[2k]}$ is given by

$$\bar\Delta_{[2k]\times[N+2k]}(x) \equiv \bar\Delta_{[2]\times[k]\times[N+2k]}(x) = \bar a_{[2]\times[k]\times[N+2k]} + \bar x_{[2]\times\underline{[2]}}\, \bar b_{\underline{[2]}\times[k]\times[N+2k]}, \tag{5.5}$$

---

[2]Clearly a Minkowski space self-dual solution v_m is not anti-Hermitian, due to the factor of i in the deinition of the Hodge dual (5.2). Throughout this chapter, we employ a conjugation operation in Minkowski space which is the continuation of the complex conjugation operation in Euclidean space. In terms of the σ-matrices (defined in Appendix A), which are central to the ADHM construction, the effect of this conjugation is simply $\sigma \to \bar\sigma$ and $\bar\sigma \to \sigma$. We can then regard the ADHM solution for v_m as anti-Hermitian under this *continued* operation of complex conjugation.

[3]For clarity we occasionally show matrix sizes explicitly; for example, the gauge field will be denoted $v^m_{[N]\times[N]}$. To represent matrix multiplication in this notation we underline contracted indices: $(AB)_{[a]\times[c]} = A_{[a]\times\underline{[b]}}\, B_{\underline{[b]}\times[c]}$. Also we adopt the shorthand $X_{[m}Y_{n]} = X_mY_n - X_nY_m$.

where

$$\bar{x}_{[2]\times[2]} = \bar{x}^{\dot{\alpha}\alpha} = x_m \, \bar{\sigma}^{m\,\dot{\alpha}\alpha} \,. \tag{5.6}$$

By counting the number of bosonic and fermionic zero-modes, we shall soon verify that k in Eqs. (5.3), (5.5) is indeed the topological charge, while N is the parameter in the gauge group $U(N)$ (or $SU(N)$). As we discuss in the next subsection, the complex-valued constant matrices a and b constitute a (highly overcomplete) set of k-instanton collective coordinates.

For generic x, the nullspace of $\bar{\Delta}_{[2k]\times[N+2k]}(x)$ is N-dimensional, as it has N fewer rows than columns. The basis vectors for this nullspace can be assembled into an $(N+2k)\times N$ complex-valued matrix $U(x)$,

$$\bar{\Delta}_{[2k]\times\underline{[N+2k]}}\, U_{\underline{[N+2k]}\times[N]} = 0 = \bar{U}_{[N]\times\underline{[N+2k]}}\, \Delta_{\underline{[N+2k]}\times[2k]} \;, \tag{5.7}$$

where U is orthonormalized according to

$$\bar{U}_{[N]\times\underline{[N+2k]}}\, U_{\underline{[N+2k]}\times[N]} = 1_{[N]\times[N]} \;. \tag{5.8}$$

The classical gauge field v_m is now constructed from U as follows,

$$v_{m[N]\times[N]} = \bar{U}_{[N]\times\underline{[N+2k]}}\, \partial_m \, U_{\underline{[N+2k]}\times[N]}. \tag{5.9}$$

Note that for the special case $k = 0$, the anti-Hermitian gauge configuration defined by this equation is pure gauge, and therefore represents the general solution to the self-duality equation (5.1) in the trivial vacuum sector. The ADHM ansatz is that Eq. (5.9) continues to give a solution to Eq. (5.1), even for nonzero k. As we shall see, this requires the additional condition

$$\bar{\Delta}_{[2]\times[k]\times\underline{[N+2k]}}\, \Delta_{\underline{[N+2k]}\times[k]\times[2]} = 1_{[2]\times[2]}\, f^{-1}_{[k]\times[k]}, \tag{5.10}$$

where f is an arbitrary (x-dependent) $k\times k$ Hermitian matrix.

To check the ADHM ansatz, we substitute Eq. (5.9) into the expression for the field strength,

$$\begin{aligned} v_{mn} &= \partial_{[m}v_{n]} + v_{[m}v_{n]} \\ &= \partial_{[m}(\bar{U}\partial_{n]}U) + (\bar{U}\partial_{[m}U)(\bar{U}\partial_{n]}U) \\ &= \partial_{[m}\bar{U}(1-U\bar{U})\partial_{n]}U. \end{aligned} \tag{5.11}$$

The last line follows from the orthonormality property (5.8). To proceed, we use Eq. (5.10) and the nullspace condition (5.7) to derive the completeness relation

$$\Delta_{[N+2k]\times\underline{[k]}\times\underline{[2]}}\, f_{\underline{[k]}\times\underline{[k]}}\, \bar{\Delta}_{\underline{[2]}\times\underline{[k]}\times[N+2k]} = 1_{[N+2k]\times[N+2k]} - U_{[N+2k]\times\underline{[N]}}\, \bar{U}_{\underline{[N]}\times[N+2k]}\,. \tag{5.12}$$

Substituting this into Eq. (5.11), we find

$$\begin{aligned} v_{mn} &= \partial_{[m}\bar{U}\Delta f\bar{\Delta}\partial_{n]}U \\ &= \bar{U}\partial_{[m}\Delta f\partial_{n]}\bar{\Delta}U \\ &= \bar{U}b\sigma_{[m}\bar{\sigma}_{n]}f\bar{b}U. \end{aligned} \tag{5.13}$$

To obtain the second line we differentiated the nullspace condition (5.7). Self-duality of the field strength now follows automatically from the well-known self-duality property of the tensor $\sigma_{mn} = \frac{1}{4}\sigma_{[m}\bar{\sigma}_{n]}$.[4]

Note that while the classical gauge configuration constructed above is not necessarily traceless, it can be made so by a $U(1)$ gauge transformation. The distinction between $U(N)$ and $SU(N)$ gauge groups is only really apparent when matter fields are included. In the sections that follow, we work with the $U(N)$ formalism, and do not explicitly impose the tracelessness condition on adjoint matter fields.

In the next subsection we count the number of independent degrees of freedom in the ADHM configuration and confirm that it has precisely the right number of collective coordinates needed to describe the most general k-instanton solution.

5.2.2 ADHM Constraints and Canonical Form

We have seen that the $U(N)$ ADHM construction requires the use of matrices of various sizes: $(N+2k)\times N$ matrices such as U, $(N+2k)\times 2k$ matrices such as Δ, a and b, $k\times k$ matrices such as f, and 2×2 matrices such as $\sigma^m_{\alpha\dot{\alpha}}$, $\bar{\sigma}^{m\,\dot{\alpha}\alpha}$, $x_{\alpha\dot{\alpha}}$, etc. Accordingly, we use a

[4] In Minkowski space the self-dual (SD) and anti-self-dual (ASD) components of an antisymmetric tensor X_{mn} are projected out using $X^{\rm SD}_{mn} = \frac{1}{4}(\eta_{mk}\eta_{nl} - \eta_{ml}\eta_{nk} + i\epsilon_{mnkl})X^{kl}$ and $X^{\rm ASD}_{mn} = (X^{\rm SD}_{mn})^*$, where $\epsilon_{0123} = -\epsilon^{0123} = -1$. Also, since $\sigma^{mn} = \frac{1}{4}\sigma^{[m}\bar{\sigma}^{n]}$ and $\bar{\sigma}^{mn} = \frac{1}{4}\bar{\sigma}^{[m}\sigma^{n]}$ are self-dual and anti-self-dual, respectively [25], it follows that $\sigma^{mn}{}_{\alpha}{}^{\beta}X_{mn} = \sigma^{mn}{}_{\alpha}{}^{\beta}X^{\rm SD}_{mn}$ and $\bar{\sigma}^{mn\dot{\alpha}}{}_{\dot{\beta}}X_{mn} = \bar{\sigma}^{mn\dot{\alpha}}{}_{\dot{\beta}}X^{\rm ASD}_{mn}$.

variety of index assignments:

$$\begin{aligned}\text{Topological charge indices } [k]: &\quad 1 \le i, j, \ldots \le k \\ \text{Colour indices } [N]: &\quad 1 \le u, v, \ldots \le N \\ \text{ADHM indices } [N+2k]: &\quad 1 \le \lambda, \mu, \ldots \le N+2k \\ \text{Quaternionic (Weyl) indices } [2]: &\quad \alpha, \beta, \dot\alpha, \dot\beta, \ldots = 1, 2 \\ \text{Lorentz indices } [4]: &\quad m, n, \ldots = 0, 1, 2, 3\end{aligned}$$

(No extra notation is required for the $2k$-dimensional index attached to Δ, a and b since this index can always be written as $[2k] = [k] \times [2]$.) With these index conventions, Eqs. (5.3) and (5.5) read

$$\Delta_{\lambda i\dot\alpha}(x) \;=\; a_{\lambda i\dot\alpha} \;+\; b^{\beta}_{\lambda i}\, x_{\beta\dot\alpha}\;, \qquad \bar\Delta^{\dot\alpha\lambda}_{i}(x) \;=\; \bar a^{\dot\alpha\lambda}_{i} \;+\; \bar x^{\dot\alpha\alpha}\, \bar b^{\lambda}_{\alpha i}, \tag{5.14}$$

while the factorization condition (5.10) becomes

$$\bar\Delta^{\dot\beta\lambda}_{i}\,\Delta_{\lambda j\dot\alpha} \;=\; \delta^{\dot\beta}_{\ \dot\alpha}\,(f^{-1})_{ij}\;. \tag{5.15}$$

On substituting Eqs. (5.14) into Eq. (5.15) we find that the ADHM factorization condition amounts to the following constraints on the matrices a and b:

$$\bar a^{\dot\alpha\lambda}_{i}\, a_{\lambda j\dot\beta} \;=\; (\tfrac{1}{2}\bar a a)_{ij}\,\delta^{\dot\alpha}_{\ \dot\beta} \;\propto\; \delta^{\dot\alpha}_{\ \dot\beta} \tag{5.16}$$

$$\bar a^{\dot\alpha\lambda}_{i}\, b^{\beta}_{\lambda j} \;=\; \bar b^{\beta\lambda}_{i}\, a^{\dot\alpha}_{\lambda j} \tag{5.17}$$

$$\bar b^{\lambda}_{\alpha i}\, b^{\beta}_{\lambda j} \;=\; (\tfrac{1}{2}\bar b b)_{ij}\,\delta_{\alpha}^{\ \beta} \;\propto\; \delta_{\alpha}^{\ \beta} \tag{5.18}$$

These three nonlinear constraints are known as the ADHM constraints [62, 61].

The elements of the matrices a and b correspond to the collective coordinates of the k-instanton gauge configuration. It was mentioned in Section 2.2.3 that there should be a total of $4Nk$ collective coordinates associated with the most general $SU(N)$ k-instanton solution. But even after accounting for the ADHM constraints (5.16)–(5.18), one finds that the number of independent elements in a and b appears to grow as k^2. It follows that there must be a certain amount of redundancy in the ADHM collective coordinates.

The source of this redundancy is a $U(N+2k)\times Gl(k,\mathbb{C})$ symmetry present in the ADHM construction. It is not hard to see that Eqs. (5.7)–(5.10) are invariant under the transformations

$$\begin{aligned}
\Delta_{[N+2k]\times[k]\times[2]} &\rightarrow \Lambda_{[N+2k]\times\underline{[N+2k]}}\,\Delta_{\underline{[N+2k]}\times\underline{[k]}\times[2]}\,B^{-1}_{\underline{[k]}\times[k]} \\
U_{[N+2k]\times[N]} &\rightarrow \Lambda_{[N+2k]\times\underline{[N+2k]}}\,U_{\underline{[N+2k]}\times[N]} \\
f_{[k]\times[k]} &\rightarrow B_{[k]\times\underline{[k]}}\,f_{\underline{[k]}\times\underline{[k]}}\,B^{-1}_{\underline{[k]}\times[k]}
\end{aligned} \tag{5.19}$$

where $\Lambda\in U(N+2k)$ and $B\in Gl(k,\mathbb{C})$. It is possible to use these transformations to eliminate all of the degrees of freedom in the matrix b [62]. In this way, one obtains the so-called 'canonical form':

$$b_{[N+2k]\times[2k]} = \begin{pmatrix} 0_{[N]\times[2k]} \\ 1_{[2k]\times[2k]} \end{pmatrix} \quad , \qquad a_{[N+2k]\times[2k]} = \begin{pmatrix} w_{[N]\times[2k]} \\ a'_{[2k]\times[2k]} \end{pmatrix} \tag{5.20}$$

We shall make the canonical form a little more explicit using a convenient decomposition of the $[N+2k]$ index. Schematically, we write each (N+2k)-valued ADHM index λ as[5]

$$\lambda = u + l\beta\ , \quad 1\le u\le N\ , \quad 1\le l\le k\ , \quad \beta=1,2\ . \tag{5.21}$$

This means that in Eq. (5.20) the upper, $[N]\times[2k]$ matrix blocks of a and b have rows indexed by u, and the lower, $[2k]\times[2k]$ matrix blocks have rows indexed by the pair $l\beta$. We now rewrite the canonical form (5.20) as

$$a_{\lambda i\dot\alpha} = a_{(u+l\beta)\,i\dot\alpha} = w_{ui\dot\alpha} + (a'_{\beta\dot\alpha})_{li} = \begin{pmatrix} w_{ui\dot\alpha} \\ (a'_{\beta\dot\alpha})_{li} \end{pmatrix} , \tag{5.22}$$

$$\bar a_i^{\dot\alpha\lambda} = \bar a_i^{\dot\alpha\,(u+l\beta)} = \bar w_{iu}^{\dot\alpha} + (\bar a'^{\dot\alpha\beta})_{il} = (\bar w_{iu}^{\dot\alpha}\ ,\ (\bar a'^{\dot\alpha\beta})_{il}) \quad , \tag{5.23}$$

$$b^{\alpha}_{\lambda i} = b^{\alpha}_{(u+l\beta)\,i} = \delta_\beta{}^\alpha\delta_{li} = \begin{pmatrix} 0 \\ \delta_\beta{}^\alpha\,\delta_{li} \end{pmatrix} \quad , \tag{5.24}$$

$$\bar b^{\lambda}_{\alpha i} = \bar b^{u+l\beta}_{\alpha i} = \delta_\alpha{}^\beta\,\delta_{il} = (0\ ,\ \delta_\alpha{}^\beta\,\delta_{il}) \quad . \tag{5.25}$$

[5]The Weyl index β in this decomposition is raised and lowered with the ϵ tensor in the usual manner [25], whereas for the $[N]$ and $[k]$ indices u and l there is no distinction between upper and lower indices.

With a and b in the form given above, the third ADHM constraint (5.18) is satisfied automatically, while the remaining constraints (5.16), (5.17) boil down to:

$$\begin{aligned} \mathrm{tr}_2\,(\tau^c \bar{a} a)_{ij} &= 0, && (5.26)\\ \bar{a}'^m_{ij} &= a'^m_{ij}\,. && (5.27) \end{aligned}$$

In Eq. (5.26) we have contracted $\bar{a}a$ with the Pauli matrix $(\tau^c)^{\dot\alpha}{}_{\dot\beta}$. In Eq. (5.27) the matrix a'^m gives the mth component of the matrix a' in a quaternionic expansion,

$$(a'_{\alpha\dot\alpha})_{ij} = (a'_m)_{ij}\,\sigma^m_{\alpha\dot\alpha}, \qquad (\bar{a}'^{\dot\alpha\alpha})_{ij} = (\bar{a}'_m)_{ij}\,\bar\sigma^{m\,\dot\alpha\alpha}. \tag{5.28}$$

Although the choice of canonical form eliminates much of the redundancy in a and b, there remains a $U(k)$ subgroup of the original $U(N+2k)\times Gl(k,\mathbb{C})$ symmetry which leaves the canonical form invariant. This residual symmetry acts as follows:

$$\Delta_{[N+2k]\times[2k]} \to \begin{pmatrix} 1_{[N]\times[N]} & 0_{[2k]\times[N]} \\ 0_{[N]\times[2k]} & \mathcal{R}^\dagger_{[2k]\times[2k]} \end{pmatrix} \Delta_{[N+2k]\times[2k]}\,\mathcal{R}_{[2k]\times[2k]}, \tag{5.29}$$

where $\mathcal{R}_{[2k]\times[2k]} = R_{ij}\,\delta^{\dot\beta}{}_{\dot\alpha}$ and $R_{ij}\in U(k)$. In terms of w and a', we have

$$w^{\dot\alpha}_{ui} \to w^{\dot\alpha}_{uj}\,R_{ji}\,, \quad (a'_{\beta\dot\alpha})_{ij} \to R^\dagger_{il}\,(a'_{\beta\dot\alpha})_{lp}\,R_{pj}. \tag{5.30}$$

In principle, we could use this transformation to eliminate some of the elements in the matrix a. However, we find it more convenient to work with the canonical form (5.22)–(5.25) and to leave the $U(k)$ symmetry manifest. To account for this, let us define M^k to be the moduli space of all solutions that satisfy the canonical ADHM constraints (5.26) and (5.27). Then the physical moduli space, $\mathrm{M}^k_{\mathrm{phys}}$, of gauge-inequivalent k-instanton solutions, is given by the quotient space

$$\mathrm{M}^k_{\mathrm{phys}} = \frac{\mathrm{M}^k}{U(k)}\,. \tag{5.31}$$

We are now in a position to count the number of truly independent collective coordinate degrees of freedom in the ADHM multi-instanton. The general complex matrix $a_{[N+2k]\times[2k]}$ has $4k(N+2k)$ real degrees of freedom. The two ADHM conditions (5.26) and (5.27) respectively impose $3k^2$ and $4k^2$ real constraints on this matrix, and 'modding out' the residual $U(k)$ symmetry removes another k^2 degrees of freedom. In total we therefore have

$$4k(N+2k) \ - \ 3k^2 \ - \ 4k^2 \ - k^2 \ = \ 4Nk \tag{5.32}$$

real degrees of freedom, precisely as required.

It is straightforward to extract the four degrees of freedom associated with the space-time location of the k-instanton solution. Let us linearly decompose a as

$$a_{\lambda i\dot\alpha} = -b^{\alpha}_{\lambda i}\,(x_0)_{\alpha\dot\alpha} + \ldots, \tag{5.33}$$

where $(x_0)_{\alpha\dot\alpha} = (x_0)_m\sigma^m_{\alpha\dot\alpha}$. From Eq. (5.14) we see that a shift in the space-time variables x is equivalent to a shift in the parameters x_0. So these parameters can be identified with the translational collective coordinates.

5.2.3 The Singular Gauge Solution

We now take a closer look at the ADHM solution (5.9). The matrix U can be eliminated in favour of the matrix Δ using the completeness relation (5.12). It is first convenient to make the decomposition

$$U_{[N+2k]\times[N]} \ = \ \begin{pmatrix} V_{[N]\times[N]} \\ \\ U'_{[2k]\times[N]} \end{pmatrix}, \qquad \Delta_{[N+2k]\times[2k]} \ = \ \begin{pmatrix} w_{[N]\times[2k]} \\ \\ \Delta'_{[2k]\times[2k]} \end{pmatrix} \tag{5.34}$$

From Eq. (5.12) we now obtain

$$V_{[N]\times\underline{[N]}}\ \bar V_{\underline{[N]}\times[N]} \ = \ 1_{[N]\times[N]} \ - \ w_{[N]\times\underline{[k]}\times\underline{[2]}}\ f_{\underline{[k]}\times\underline{[k]}}\ \bar w_{\underline{[2]}\times\underline{[k]}\times[N]}, \tag{5.35}$$

$$U'_{[2k]\times\underline{[N]}}\bar V_{\underline{[N]}\times[N]} \ = \ -\Delta'_{[2k]\times\underline{[k]}\times\underline{[2]}} f_{\underline{[k]}\times\underline{[k]}}\ \bar w_{\underline{[2]}\times\underline{[k]}\times[N]}. \tag{5.36}$$

Given a matrix V that satisfies Eq. (5.35), we can find another by right-multiplying it by an arbitrary $U(N)$ matrix. This precisely corresponds to performing a $U(N)$ gauge transformation of the instanton solution. By choosing a specific V we fix the (local) gauge of the instanton. The k-instanton generalization of singular gauge (see Section 2.2.3) is specified by choosing one of the 2^N matrix square roots:

$$V \ = \ (1 \ - \ wf\bar w)^{1/2}. \tag{5.37}$$

From Eq. (5.36) it follows that we also have

$$U' = -\Delta' f\bar{w}(1 - wf\bar{w})^{-1/2} \tag{5.38}$$

in singular gauge. Equations (5.37) and (5.38) determine U in (5.34), and hence the gauge field v_m via Eq. (5.9). For later use we list the leading large-$|x|$ asymptotic behaviour of several key ADHM quantities, assuming instanton singular gauge (5.37):

$$\Delta \to bx\ , \tag{5.39}$$

$$f_{kl} \to \frac{1}{|x|^2}\delta_{kl}\ , \tag{5.40}$$

$$U' \to -\frac{1}{|x|^2}x\,\bar{w}\ , \tag{5.41}$$

$$V \to 1_{[N]\times[N]}\ . \tag{5.42}$$

We can easily verify that the $SU(N)$ singular gauge one-instanton solution (2.23) follows from this construction. Unlike the $SU(2)$ constraints obtained using the $Sp(1)$ formalism, the $U(N)$ constraints (5.26) and (5.27) do not disappear in the one-instanton sector, even for $N = 2$. Instead, Eq. (5.27) implies that

$$a'_m \equiv -(x_0)_m, \tag{5.43}$$

where the $(x_0)_m$ are the four real degrees of freedom representing translational collective coordinates of the instanton (see Eq. (5.33)). The other constraint, Eq. (5.26), reduces to

$$\bar{w}_u^{\dot\alpha}\, w_{u\dot\beta} = \rho^2\, \delta^{\dot\alpha}_{\ \dot\beta}\ , \tag{5.44}$$

where ρ is an arbitrary parameter. (It will be identified with the scale parameter in the instanton solution.) The general solution to Eq. (5.44) can be written[6]

$$w_{[N]\times[2]} = \Omega_{[N]\times[N]}\begin{pmatrix} 1_{[2]\times[2]} \\ 0_{[N-2]\times[2]} \end{pmatrix}\rho\ , \qquad \Omega \in \frac{U(N)}{U(N-2)}\ . \tag{5.45}$$

[6]As a quick check, note that Eq. (5.44) imposes three real constraints on w, so the general solution should contain $4N - 3$ degrees of freedom. In Eq. (5.45), the coset element Ω has $N^2 - (N-2)^2 = 4N - 4$ real degrees of freedom, and the scale parameter ρ has one, so we get the required total. When we add in the four translational degrees of freedom x_0 and 'mod out' the residual $U(1)$ ADHM symmetry, we get the expected grand total of $4N$ independent collective coordinates.

It is convenient to initially set $\Omega = 1$. From the defining equations for Δ and f, Eqs. (5.3) and (5.10), we now obtain

$$\Delta_{[N+2]\times[2]} = \begin{pmatrix} \rho\cdot 1_{[2]\times[2]} \\ 0_{[N-2]\times[2]} \\ y_{[2]\times[2]} \end{pmatrix}, \qquad f = \frac{1}{y^2+\rho^2}, \tag{5.46}$$

where $y_{\alpha\dot\alpha} = (x - x_0)_{\alpha\dot\alpha}$. Using the singular gauge expressions for V and U', Eqs. (5.37) and (5.38), we find

$$\begin{aligned} V_{[N]\times[N]} &= \begin{pmatrix} \left(\frac{y^2}{y^2+\rho^2}\right)^{1/2} 1_{[2]\times[2]} & 0 \\ 0 & 1_{[N-2]\times[N-2]} \end{pmatrix}, \\ U'_{[2]\times[N]} &= \left(-\left(\frac{\rho^2}{y^2(y^2+\rho^2)}\right)^{1/2} y_{[2]\times[2]} \quad , \quad 0_{[2]\times[N-2]} \right). \end{aligned} \tag{5.47}$$

The gauge field now follows from Eq. (5.9),

$$v_m = \begin{pmatrix} v_m^{\rm BPST} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.48}$$

where $v_m^{\rm BPST}$ is the Minkowski space version of the singular gauge BPST instanton (2.26), with a fixed 'reference' iso-orientation:

$$v_m^{\rm BPST}(x) = \frac{1}{2}\frac{\rho^2}{y^2(y^2+\rho^2)} y^n \bar\sigma_{[n}\sigma_{m]}. \tag{5.49}$$

To obtain the more general one-instanton solution, given by $\Omega \neq 1$, let us now send $w \to \Omega w$. From Eqs. (5.37) and (5.38), it is easy to see that the effect of this transformation on the matrices V and U' is

$$\begin{aligned} V &\to \Omega V \Omega^\dagger, \\ U' &\to U'\Omega^\dagger. \end{aligned} \tag{5.50}$$

From Eq. (5.9), the effect on the gauge field is then

$$v_m \to \Omega\, v_m\, \Omega^\dagger. \tag{5.51}$$

The $U(N)$ ADHM formalism therefore yields the $k = 1$ solution

$$v_m = \Omega \begin{pmatrix} v_m^{\rm BPST} & 0 \\ 0 & 0 \end{pmatrix} \Omega^\dagger, \qquad \Omega \in \frac{U(N)}{U(1)\times U(N-2)}. \tag{5.52}$$

(The extra $U(1)$ factor in the stability group of the coset is due to the residual ADHM symmetry (5.30).) Equation (5.52) indeed gives the general $U(N)$ (or equivalently, $SU(N)$) singular gauge one-instanton solution (cf. Eq. (2.23)).

5.3 Construction of the Super-Multi-Instanton

In this section we construct the classical configurations that, together with the ADHM gauge configuration (5.9), constitute the (supersymmetric) multi-instanton background in supersymmetric $U(N)$ (or $SU(N)$) pure Yang-Mills theory. We consider both $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric models. In Section 5.5, we derive the associated multi-instanton actions. There we also consider the effect of including matter multiplets.

5.3.1 $\mathcal{N}=1$ Supersymmetric Yang-Mills Theory

In $\mathcal{N}=1$ supersymmetric Yang-Mills theory the gauge field v_m is accompanied by its fermionic superpartner, the gaugino λ. In the background of the k-instanton gauge configuration, the index theorem predicts that there are $2Nk$ zero-mode solutions to the massless Dirac equation $\bar{\not{\mathcal{D}}}\lambda=0$. As was shown in [15] in the one-instanton context, these zero-modes can be viewed as the $\mathcal{N}=1$ superpartners of the gauge instanton. Explicit expressions for the adjoint fermion zero-modes in the ADHM k-instanton background were first obtained in [62]. In our notation they read

$$(\lambda_\alpha)_{uv} \;=\; \bar{U}^\lambda_u \mathcal{M}_{\lambda i} f_{ij} \bar{b}^\rho_{\alpha j} U_{\rho v} \;-\; \bar{U}^\lambda_u b_{\lambda i\alpha}\, f_{ij} \bar{\mathcal{M}}^\rho_j U_{\rho v}\;. \tag{5.53}$$

Here $\mathcal{M}_{\lambda i}$ and $\bar{\mathcal{M}}^\rho_j$ are constant $(N+2k)\times k$ and $k\times(N+2k)$ matrices of Grassmann collective coordinates. They can be regarded either as two independent real Grassmann matrices or as two complex Grassmann matrices which are Hermitian conjugates of one another.

From Eq. (5.53) we calculate

$$\bar{\not{\mathcal{D}}}^{\dot\alpha\alpha}\lambda_\alpha \;=\; 2\bar{U}b^\alpha\, f(\bar\Delta^{\dot\alpha}\mathcal{M}+\bar{\mathcal{M}}\Delta^{\dot\alpha})f\,\bar{b}_\alpha\, U. \tag{5.54}$$

Hence the condition for a gaugino zero-mode is the following two sets of linear constraints on $\mathcal{M}$ and $\bar{\mathcal{M}}$ which ensure that the right-hand side of (5.54) vanishes (expanding $\Delta(x)$ as $a+bx$) [62]:

$$\bar{\mathcal{M}}^\lambda_i\, a_{\lambda j\dot\alpha} \;=\; -\bar{a}^\lambda_{i\dot\alpha}\,\mathcal{M}_{\lambda j}\;, \tag{5.55}$$

$$\bar{\mathcal{M}}^\lambda_i\, b^\alpha_{\lambda j} \;=\; \bar{b}^{\alpha\lambda}_i\,\mathcal{M}_{\lambda j}\;. \tag{5.56}$$

In a formal sense, discussed in Section 5.6 below, these fermionic constraints are the 'spin-1/2' superpartners of the original 'spin-1' ADHM constraints, (5.16) and (5.17). Note that Eq. (5.56) is easily solved when b is in the canonical form (5.22)–(5.25). With the ADHM index decomposition (5.21), we set

$$\mathcal{M}_{\lambda i} \equiv \mathcal{M}_{(u+l\beta)\, i} = \begin{pmatrix} \mu_{ui} \\ \\ (\mathcal{M}'_{\beta})_{li} \end{pmatrix} , \quad \bar{\mathcal{M}}_i^{\lambda} \equiv \bar{\mathcal{M}}_i^{u+l\beta} = \big(\bar{\mu}_{iu} \, , \, (\bar{\mathcal{M}}'^{\beta})_{il}\big) \, . \tag{5.57}$$

Equation (5.56) then collapses to

$$\bar{\mathcal{M}}'^{\alpha} = \mathcal{M}'^{\alpha} \tag{5.58}$$

which allows us to eliminate $\bar{\mathcal{M}}'$ in favour of $\mathcal{M}'$.

Counting the number of degrees of freedom, we find altogether $2k(N+2k)$ real Grassmann parameters in $\mathcal{M}$ and $\bar{\mathcal{M}}$; these are subject to $2k^2$ constraints from each of Eqs. (5.55) and (5.56), for a net of $2Nk$ gaugino zero-modes as required. Of these, four zero-modes can be distinguished, corresponding to

$$\mathcal{M}_{\lambda i} = 4 b^{\beta}_{\lambda i} \, \xi_{\beta} \, , \qquad \bar{\mathcal{M}}_i^{\lambda} = 4 \bar{b}^{\lambda}_{i\beta} \, \xi^{\beta} \tag{5.59}$$

and

$$\mathcal{M}_{\lambda i} = i \, a_{\lambda i \dot{\alpha}} \, \bar{\eta}^{\dot{\alpha}} \, , \qquad \bar{\mathcal{M}}_i^{\lambda} = -i \, \bar{a}_i^{\dot{\alpha}\lambda} \, \bar{\eta}_{\dot{\alpha}} \, , \tag{5.60}$$

where ξ_{β} and $\bar{\eta}^{\dot{\alpha}}$ are arbitrary spinor parameters. These are the generalization of the one-instanton supersymmetric and superconformal zero-modes (see Section 4.2.1) to sectors of arbitrary topological charge. They satisfy the fermionic constraints (5.55) and (5.56) by virtue of the ADHM constraints (5.16) and (5.17).

In the one-instanton sector, it is straightforward to check that the expression (5.53) does indeed yield the $SU(N)$ zero-mode solutions (4.49)–(4.52) listed in Chapter 4. It is simplest to initially take $\Omega = 1$ in the expression for the ADHM matrix w, Eq. (5.45). Using Eqs. (5.59) and (5.60) we obtain the supersymmetric and superconformal zero-modes, (4.50) and (4.49), respectively. The remaining $2N-4$ zero-modes, (4.51) and (4.52), are obtained by setting $\mathcal{M}' = 0$ and $\mu_{1,2} = \bar{\mu}_{1,2} = 0$, with arbitrary choices for μ_u (and $\bar{\mu}_u$) for $3 \leq u \leq N$. (This prescription can easily be seen to satisfy the constraints (5.55) and (5.56).) Turning on the orientation matrix Ω in Eq. (5.45) has the effect of rotating these choices of μ.

5.3.2 $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory

Next we turn to the $\mathcal{N} = 2$ case. The particle content of $\mathcal{N} = 2$ supersymmetric pure Yang-Mills theory comprises, in addition to the gauge field v_m and gaugino λ_α considered above, a Higgsino ψ_α and a complex Higgs boson A. All these fields transform in the adjoint representation of the $U(N)$ (or $SU(N)$) gauge group.

Adjoint Higgsino Zero-modes

The zero-mode solutions of the Higgsino ψ are constructed in an identical fashion to those of the gaugino, Eqs. (5.53)–(5.58):

$$(\psi_\alpha)_{uv} = \bar{U}^\lambda_u \mathcal{N}_{\lambda i} f_{ij} \bar{b}^\rho_{\alpha j} U_{\rho v} - \bar{U}^\lambda_u b_{\lambda i \alpha} f_{ij} \bar{\mathcal{N}}^\rho_j U_{\rho v} \; . \tag{5.61}$$

The Grassmann collective coordinate matrices $\mathcal{N}_{\lambda i}$ and $\bar{\mathcal{N}}^\rho_j$ are subject to the same linear constraints as $\mathcal{M}$ and $\bar{\mathcal{M}}$,

$$\bar{\mathcal{N}}^\lambda_i \, a_{\lambda j \dot{\alpha}} = -\bar{a}^\lambda_{i\dot{\alpha}} \mathcal{N}_{\lambda j} \; , \tag{5.62}$$

$$\bar{\mathcal{N}}^\lambda_i \, b^\alpha_{\lambda j} = \bar{b}^{\alpha\lambda}_i \mathcal{N}_{\lambda j} \; , \tag{5.63}$$

and are likewise decomposed as

$$\mathcal{N}_{\lambda i} \equiv \mathcal{N}_{(u+l\beta)\, i} = \begin{pmatrix} \nu_{ui} \\ (\mathcal{N}'_\beta)_{li} \end{pmatrix} , \quad \bar{\mathcal{N}}^\lambda_i \equiv \bar{\mathcal{N}}^{u+l\beta}_i = \left(\bar{\nu}_{iu} \, , \, (\bar{\mathcal{N}}'^\beta)_{il} \right) \; , \tag{5.64}$$

with

$$\bar{\mathcal{N}}'^\alpha = \mathcal{N}'^\alpha \tag{5.65}$$

when b is in canonical form.

The Adjoint Higgs Solution

The (leading order) Euler-Lagrange equation for the complex scalar field A reads[7]

$$D^2 A = \sqrt{2}\, i\, [\, \lambda \, , \psi \,] \tag{5.66}$$

[7]Following [42, 45], we take the only anti-Hermitian field to be the gauge field v_m; all other component fields are Hermitian.

where D^2 is the covariant Klein-Gordon operator in the background of the ADHM gauge configuration (5.9), and λ and ψ are given by (5.53) and (5.61), respectively. On the Coulomb branch of the theory, the solution must satisfy the boundary condition

$$\lim_{|x|\to\infty} A(x) = \mathrm{diag}(\mathrm{v}_1,\ldots,\mathrm{v}_N) \ , \tag{5.67}$$

where the v_u are the complex VEV's. Note that the $U(N)$ theory does not require the sum of the VEV's to equal zero, in contrast to the $SU(N)$ theory.

The construction of the solution to (5.66) is analogous to the '$SU(2)$ as $Sp(1)$' construction detailed in Sections 7.2–7.3 of [42], and goes as follows. The solution has the additive form

$$i\,A \;=\; \frac{1}{2\sqrt{2}}\bar{U}\,(\mathcal{N}f\bar{\mathcal{M}} - \mathcal{M}f\bar{\mathcal{N}})U \;+\; \bar{U}\,\mathcal{A}\,U\ . \tag{5.68}$$

Here $\mathcal{A}$ is a block-diagonal constant $(N+2k)\times(N+2k)$ matrix,

$$\mathcal{A}_\lambda^{\ \mu} \;\equiv\; \mathcal{A}_{u+l\alpha}^{\ \ v+m\beta} \;=\; \begin{pmatrix} \langle\mathcal{A}\rangle_{uv} & 0 \\ 0 & (\mathcal{A}_{\mathrm{tot}})_{lm}\,\delta_\alpha^{\ \beta} \end{pmatrix}\ , \tag{5.69}$$

where the $N\times N$ matrix $\langle\mathcal{A}\rangle$ is just i times the VEV matrix,

$$\langle\mathcal{A}\rangle \;=\; i\ \mathrm{diag}(\mathrm{v}_1,\ldots,\mathrm{v}_N)\ . \tag{5.70}$$

The $k\times k$ anti-Hermitian[8] matrix $\mathcal{A}_{\mathrm{tot}}$ is defined as the solution to the following inhomogeneous linear algebraic equation

$$\mathbf{L}\cdot\mathcal{A}_{\mathrm{tot}} \;=\; \Lambda + \Lambda_f\ , \tag{5.71}$$

where Λ and Λ_f are the $k\times k$ anti-Hermitian matrices

$$\Lambda_{ij} \;=\; \bar{w}_{iu}^{\dot\alpha}\ \langle\mathcal{A}\rangle_{uv}\ w_{vj\dot\alpha}\ , \tag{5.72}$$

[8]In the remainder of the chapter we distinguish between two different kinds of Hermitian conjugation. The first type, denoted by a dagger, does not turn fields into anti-fields, nor does it complex conjugate the VEV's. Thus: $\langle\mathcal{A}\rangle^\dagger_{uv} = -i\,\mathrm{diag}(\mathrm{v}_1,\ldots,\mathrm{v}_N)$. The second (standard) type of Hermitian conjugation, denoted by an overbar, does interchange fields and anti-fields and also complex conjugates the VEV's. Thus: $\langle\bar{\mathcal{A}}\rangle_{uv} = -i\,\mathrm{diag}(\bar{\mathrm{v}}_1,\ldots,\bar{\mathrm{v}}_N)$. For the remainder of this section, Hermitian conjugation is always of the first type.

and

$$(\Lambda_f)_{ij} = \frac{1}{2\sqrt{2}}\,(\bar{\mathcal{M}}\mathcal{N} - \bar{\mathcal{N}}\mathcal{M})_{ij}\,. \tag{5.73}$$

The linear operator $\mathbf{L}$ maps the space of $k \times k$ scalar-valued anti-Hermitian matrices onto itself. If Ω is such a matrix, then $\mathbf{L}$ acts as

$$\mathbf{L}\cdot\Omega = \tfrac{1}{2}\{\,\Omega\,,\,W\,\} - \tfrac{1}{2}\mathrm{tr}_2\left([\,\bar{a}'\,,\,\Omega\,]a' - \bar{a}'[\,a'\,,\,\Omega\,]\right), \tag{5.74}$$

where W is the Hermitian $k \times k$ matrix

$$W_{ij} = \bar{w}_{iu}^{\dot\alpha}\, w_{uj\dot\alpha}\,. \tag{5.75}$$

From Eqs. (5.71)–(5.75) we see that $\mathcal{A}_{\rm tot}$ transforms in the adjoint representation of the residual $U(k)$ ADHM symmetry (5.29) (i.e. like a', $\mathcal{M}'$ and $\mathcal{N}'$).

Defined in this way, the configuration (5.68) correctly satisfies the Euler-Lagrange equation (5.66). One can regard the four sets of constraints (5.16)–(5.17), (5.55)–(5.56), (5.62)–(5.63), and (5.71) as the 'spin-1', 'spin-1/2', 'spin-1/2', and 'spin-0' components of an $\mathcal{N}=2$ supermultiplet of constraints [45]. We shall exploit this observation in Section 5.6, when we construct the collective coordinate integration measure.

5.4 Realization of the Supersymmetry Algebra

Here we consider the supersymmetry transformation properties of the collective coordinates appearing in the multi-instanton configurations described above. The philosophy is as follows [15]. As the relevant component field configurations obey equations of motion which are manifestly supersymmetric, any non-vanishing action of the supersymmetry generators on a particular classical solution necessarily yields another solution. It follows that the 'active' supersymmetry transformations of the fields must be equivalent (up to a gauge transformation) to certain 'passive' transformations of the bosonic and fermionic collective coordinates that parameterize the superinstanton. As originally noted in [15] in the one-instanton context, physically relevant quantities such as the the superinstanton action must be constructed out of supersymmetry invariant combinations of the collective coordinates.

5.4.1 $\mathcal{N}=1$ Supersymmetric Yang-Mills Theory

The supersymmetry transformations that act on the component fields in supersymmetric Yang-Mills theory are listed in Appendix C. Here we require the $\mathcal{N}=1$ transformation laws for the gauge field v_m and the gaugino λ; these are given by Eqs. (C.1) and (C.2) respectively. As was demonstrated in [45], the supersymmetry algebra can be naturally realized in terms of passive transformations of the collective coordinate matrices a and $\mathcal{M}$ *before* implementing the respective algebraic constraints (5.16)–(5.18) and (5.55)–(5.56). For the ADHM gauge configuration (5.9), the passive supersymmetry transformation of a that implements (up to an infinitesimal gauge transformation) the active supersymmetry transformation (C.1) is[9]

$$\delta a_{\dot\alpha} = i\bar\xi_{\dot\alpha}\mathcal{M}\ , \qquad \delta\bar a^{\dot\alpha} = -i\bar{\mathcal{M}}\bar\xi^{\dot\alpha}. \tag{5.76}$$

To generate the supersymmetry transformation of the gaugino, (C.2), we require, in addition to the transformation of a given above, the following transformation of the gaugino collective coordinates:

$$\delta\mathcal{M} = -4b^\alpha\xi_\alpha\ , \qquad \delta\bar{\mathcal{M}} = -4\xi^\alpha\bar b_\alpha. \tag{5.77}$$

These results were derived using exactly the same algebraic manipulations employed in the '$SU(2)$ as $Sp(1)$' analysis of [45]. The reader interested in the calculational details is referred to Section 2 of that work.

5.4.2 $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

As in the $\mathcal{N}=1$ case, the $\mathcal{N}=2$ supersymmetry algebra may be realized directly on the unconstrained multi-instanton collective coordinates. The supersymmetry transformations of the component fields of the $\mathcal{N}=2$ superinstanton are given in Appendix C. They are generated by the following set of transformations of the collective coordinates:

$$\delta a_{\dot\alpha} = i\bar\xi_{1\dot\alpha}\mathcal{M} + i\bar\xi_{2\dot\alpha}\mathcal{N}\ , \qquad \delta\bar a^{\dot\alpha} = -i\bar{\mathcal{M}}\bar\xi_1^{\dot\alpha} - i\bar{\mathcal{N}}\bar\xi_2^{\dot\alpha}, \tag{5.78}$$

[9]Here and in the $\mathcal{N}=2$ case to follow, we find it convenient to redefine the infinitesimal supersymmetry parameters according to $\xi\to -i\xi$, $\bar\xi\to i\bar\xi$. Note also that we are dealing with an anti-Hermitian gauge field, which requires us to set $v_m\to iv_m$ and $v_{mn}\to iv_{mn}$ in the formulae of Appendix C.

$$\delta\mathcal{M} = -4b^\alpha\xi_{1\alpha} - i2\sqrt{2}\,\mathcal{C}_{\dot\alpha}\bar\xi_2^{\dot\alpha}\,, \qquad \delta\bar{\mathcal{M}} = -4\xi_1^\alpha\bar b_\alpha + i2\sqrt{2}\,\bar\xi_{2\dot\alpha}\mathcal{C}^{\dagger\dot\alpha}, \tag{5.79}$$

$$\delta\mathcal{N} = -4b^\alpha\xi_{2\alpha} + i2\sqrt{2}\,\mathcal{C}_{\dot\alpha}\bar\xi_1^{\dot\alpha}\,, \qquad \delta\bar{\mathcal{N}} = -4\xi_2^\alpha\bar b_\alpha - i2\sqrt{2}\,\bar\xi_{1\dot\alpha}\mathcal{C}^{\dagger\dot\alpha}. \tag{5.80}$$

Here $\mathcal{C}_{\dot\alpha}$ is the $(N+2k)\times k$ spinor-valued matrix

$$\mathcal{C}_{\lambda\, i\dot\alpha} \equiv \mathcal{C}_{(u+l\beta)\,i\dot\alpha} = \begin{pmatrix} \langle\mathcal{A}\rangle_{uv}w_{vi\dot\alpha} - w_{uj\dot\alpha}(\mathcal{A}_{\rm tot})_{ji} \\ \\ [\,\mathcal{A}_{\rm tot}\,,\,a'_{\beta\dot\alpha}\,]_{li} \end{pmatrix}, \tag{5.81}$$

$$\mathcal{C}^{\dagger\dot\alpha} = (\,\mathcal{A}_{\rm tot}\bar w^{\dot\alpha} - \bar w^{\dot\alpha}\langle\mathcal{A}\rangle\,,\,[\,\mathcal{A}_{\rm tot}\,,\bar a'^{\dot\alpha}\,]\,)\,. \tag{5.82}$$

Direct calculation (following Appendix A of [45]) shows that $\mathcal{A}_{\rm tot}$, as defined by Eq. (5.71) above, is a supersymmetry invariant:

$$\delta\mathcal{A}_{\rm tot} = 0\,. \tag{5.83}$$

5.5 Construction of the Multi-Instanton Action

5.5.1 $\mathcal{N}=1$ Supersymmetric Yang-Mills Theory

In the absence of matter multiplets, the k-instanton action of $\mathcal{N}=1$ supersymmetric $SU(N)$ pure Yang-Mills theory is simply $8\pi^2k/g^2$. An interesting result can only be obtained in the presence of a Higgs boson whose VEV breaks the classical scale invariance of the theory. We shall initially consider the simplest such theory, in which the gauge multiplet is minimally coupled to a single fundamental chiral multiplet $Q_u=(q_u,\chi_u)$, where the index u labels the N-dimensional fundamental representation. This model was constructed in Section 3.2; see Eqs. (3.29), (3.30) and (3.32). We refer to q_u as the Higgs field, and to χ_u as the Higgsino.

The fundamental fermion zero-modes were originally constructed in [62]. In our language, they read

$$\chi_u^\alpha = \bar U_{u\lambda}b_{\lambda i}{}^\alpha f_{ij}\mathcal{K}_j \tag{5.84}$$

where α is a Weyl spinor index, and $\mathcal{K}_j$ is a Grassmann number (as opposed to a Grassmann spinor). It is easily verified that the above expression satisfies the covariant Dirac equation in the ADHM background,

$$\bar{\not{\mathcal{D}}}\chi = 0\,. \tag{5.85}$$

The fundamental Higgs field q_u satisfies an inhomogeneous Euler-Lagrange equation,

$$D^2 q \;=\; -i\sqrt{2}\,\lambda\chi, \tag{5.86}$$

together with the VEV boundary condition

$$\lim_{|x|\to\infty} q_u = \langle q\rangle_u, \tag{5.87}$$

where $\langle q\rangle_u$ denotes the fundamental VEV. The right-hand side of Eq. (5.86) is the product of the classical configurations (5.53) and (5.84). A straightforward exercise in ADHM algebra yields the general solution to Eqs. (5.86)–(5.87). It reads

$$q_u \;=\; \bar{V}_{uv}\,\langle q\rangle_v \;+\; \frac{i}{2\sqrt{2}}\,\bar{U}_{u\lambda}\mathcal{M}_{\lambda i} f_{ij}\mathcal{K}_j\;, \tag{5.88}$$

generalizing the $SU(2)$ result given by Eq. (5.10) of [45]. Here V, defined by Eq. (5.34) above, is the upper $N\times N$ part of the ADHM matrix U.

We can now construct the (leading order) superinstanton action. The Maxwell term in the action yields $8\pi^2k/g^2$ as always. Following the method of [42, 45], we integrate by parts and use the Euler-Lagrange equation for the Higgs scalar, Eq. (5.86), to combine the remaining terms in the action into a single surface term:

$$\int d^4x\left(-D_m q^\dagger D^m q - \sqrt{2}iq^\dagger\lambda\chi\right) = -\int dS^3\, q^\dagger D_\perp q. \tag{5.89}$$

Here S^3 is the three-sphere at infinity and the normal covariant derivative $D_\perp$ is given by $(x^m/\sqrt{|x|^2})\,D_m$. The contribution of the surface term to the action is now extracted from the $1/|x|^3$ fall-off of $D_\perp q$. With the help of the asymptotic formulae (5.39)–(5.42), one calculates

$$D_\perp q_u \;\overset{|x|\to\infty}{\longrightarrow}\; \frac{1}{2|x|^3}\left(w_{\dot\alpha ui}\,\bar{w}_{iv}^{\dot\alpha}\,\langle q\rangle_v \;-\; \frac{i}{\sqrt{2}}\,\mu_{ui}\,\mathcal{K}_i\right)\;, \tag{5.90}$$

and hence

$$S^{k-\text{inst}}_{\mathcal{N}=1\ SQCD} \;=\; \frac{8k\pi^2}{g^2} \;+\; \pi^2\left(\langle q\rangle_u\,\langle\bar{q}\rangle_v\,w_{\dot\alpha vi}\,\bar{w}_{iu}^{\dot\alpha} \;-\; \frac{i}{\sqrt{2}}\,\langle\bar{q}\rangle_u\,\mu_{ui}\,\mathcal{K}_i\right). \tag{5.91}$$

This generalizes the $SU(2)$ expressions obtained in Appendix C of [45] and also in [65].

The k-instanton formula (5.91), although written in ADHM collective coordinates, is nonetheless easily compared with the one-instanton expression for the action found in [15]:

the first term in parentheses is equivalent to $\sum_i |q|^2 \rho_i^2$, summed over the k different instantons, where q is the fundamental VEV and ρ_i is the scale parameter of the ith instanton. Also, the second term in parentheses is the fermion bilinear necessary to promote this ρ_i^2 to $(\rho^2_{\rm inv})_i$ where $\rho_{\rm inv}$ is the supersymmetric invariant scale size constructed in [15]. Independently of one's choice of collective coordinates, the presence of the VEV's in the action (5.91) gives a natural cut-off to the integrations over instanton scale parameters, providing an infra-red safe application of the instanton calculus.

We can immediately generalize the expressions obtained above to phenomenologically more interesting models with N_f fundamental flavours of Dirac fermions. In this case case the gauge multiplet is minimally coupled to $2N_f$ chiral superfields Q_f and $\tilde{Q}_f$, $1 \le f \le N_f$, where Q_f transforms in the fundamental and $\tilde{Q}_f$ in the conjugate-fundamental representation of the gauge group (see Section 3.2 and in particular Eqs. (3.29) and (3.33)). When the bare masses of hypermultiplets vanish, the fundamental Higgs scalars q_f and $\tilde{q}_f$ can develop vacuum expectation values which spontaneously break the gauge group. (This is usually referred to as the Higgs branch of the theory.) The global symmetries of the theory can be used to put the VEV matrices $\langle q\rangle_{uf}$ and $\langle\tilde{q}\rangle_{fu}$ in the following form [66, 67]:

$$\langle q\rangle_{uf} \;=\; \begin{pmatrix} {\rm v}_1 & 0 & \dots & 0 & \dots & 0 \\ 0 & {\rm v}_2 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {\rm v}_N & \dots & 0 \end{pmatrix} , \quad \langle\tilde{q}\rangle_{fu} \;=\; \begin{pmatrix} \tilde{\rm v}_1 & 0 & \dots & 0 \\ 0 & \tilde{\rm v}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tilde{\rm v}_N \\ 0 & 0 & \dots & 0 \end{pmatrix} . \tag{5.92}$$

The VEV matrices in (5.92) correspond to the case $N_f \ge N$. The case $N_f < N$ is similar except that the VEV matrices have extra rows of zeroes rather than columns. The VEV's are not all independent; the D-flatness condition requires that for each value of u,

$$|{\rm v}_u|^2 \;=\; |\tilde{\rm v}_u|^2 + a^2 \ , \quad N_f \ge N \tag{5.93}$$

$$|{\rm v}_u|^2 \;=\; |\tilde{\rm v}_u|^2 \ , \qquad N_f < N \tag{5.94}$$

where a^2 is an arbitrary constant, independent of the colour index u.

For the more general model, Eqs. (5.84) and (5.88) become

$$\chi^\alpha_{uf} \;=\; \bar{U}_{u\lambda} b_{\lambda i}{}^\alpha f_{ij} \mathcal{K}_{jf} \ , \quad \tilde{\chi}_{f\alpha u} \;=\; \tilde{\mathcal{K}}_{fi} f_{ij} \bar{b}_{\alpha j\lambda} U_{\lambda u} \tag{5.95}$$

and

$$\begin{aligned} q_{uf} &= \bar{V}_{uv}\langle q\rangle_{vf} + \tfrac{i}{2\sqrt{2}}\,\bar{U}_{u\lambda}\mathcal{M}_{\lambda i}f_{ij}\mathcal{K}_{jf}\ , \\ \tilde{q}_{fu} &= \langle\tilde{q}\rangle_{fv}V_{vu} - \tfrac{i}{2\sqrt{2}}\,\tilde{\mathcal{K}}_{fi}f_{ij}\bar{\mathcal{M}}_{j\lambda}U_{\lambda u}\ , \end{aligned} \tag{5.96}$$

respectively, while the action generalizes to:

$$\begin{aligned} S^{k-\text{inst}}_{\mathcal{N}=1\ SQCD} &= \frac{8k\pi^2}{g^2} + \pi^2\Big(\langle q\rangle_{uf}\langle\bar{q}\rangle_{fv}w_{\dot{\alpha}vi}\,\bar{w}^{\dot{\alpha}}_{iu} - \frac{i}{\sqrt{2}}\,\langle\bar{q}\rangle_{fu}\,\mu_{ui}\,\mathcal{K}_{if} \\ &\quad + \langle\bar{\tilde{q}}\rangle_{uf}\langle\tilde{q}\rangle_{fv}w_{\dot{\alpha}vi}\,\bar{w}^{\dot{\alpha}}_{iu} + \frac{i}{\sqrt{2}}\,\langle\bar{\tilde{q}}\rangle_{uf}\,\tilde{\mathcal{K}}_{fi}\,\bar{\mu}_{iu}\Big). \end{aligned} \tag{5.97}$$

As mentioned in Section 5.4, on general principle this action must be a supersymmetry invariant [15, 45]. The $\mathcal{N}=1$ supersymmetry transformation properties of the collective coordinate matrices a and $\mathcal{M}$ (including the submatrices w and μ) were given above, in Eqs. (5.76) and (5.77). To check the invariance of the expression (5.97), we must also derive transformation properties for the Grassmann collective coordinates $\mathcal{K}$ and $\tilde{\mathcal{K}}$ associated with the fundamental fermions. Using straightforward algebraic manipulations we find that the $\mathcal{N}=1$ supersymmetry transformations of χ and $\tilde{\chi}$ are generated by

$$\delta\mathcal{K}_{if} = -2\sqrt{2}\ \bar{\xi}_{\dot{\alpha}}\ \bar{w}^{\dot{\alpha}}_{iu}\ \langle q\rangle_{uf}\ , \quad \delta\tilde{\mathcal{K}}_{fi} = -2\sqrt{2}\ \langle\tilde{q}\rangle_{fu}\ w_{\dot{\alpha}ui}\ \bar{\xi}^{\dot{\alpha}}\ . \tag{5.98}$$

It is now easily checked that the action (5.97) is invariant under the supersymmetry transformations (5.76), (5.77) and (5.98).

In the next subsection, we turn our attention to the multi-instanton action on the Coulomb branch of $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD. We will see that the $\mathcal{N}=1$ action (5.97) possesses two simplifying properties that the $\mathcal{N}=2$ action does not. First, Eq. (5.97) has the form of a disconnected sum of k single instantons; with our choice of ADHM coordinates there is no interaction between them. Second, the only gaugino zero-modes that are lifted (i.e. that appear in the action) are those associated with the top elements μ and $\bar{\mu}$ of the collective coordinate matrices $\mathcal{M}$ and $\bar{\mathcal{M}}$. This leaves $\mathcal{O}(k)$ unlifted gaugino modes after one implements the fermionic constraints (5.55) and (5.56). This counting contrasts sharply with the $\mathcal{N}=2$ theories in which the number of unlifted modes is independent of the topological charge k. Saturating each of these unlifted modes with an anti-gaugino field insertion (as per Affleck, Dine and Seiberg [68]) one sees that, unlike the $\mathcal{N}=2$ theory, here the sectors of different topological charge cannot interfere with one another, since the non-vanishing Green's functions are distinguished by different anti-fermion content.

5.5.2 $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

Now we derive the multi-instanton action in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD. We initially set $N_f=0$ and consider the pure Yang-Mills theory. As for the $\mathcal{N}=1$ models, the leading order supersymmetric multi-instanton action for this theory can be expressed as a surface term [42]:

$$\begin{aligned} S^{k\text{-inst}}_{\mathcal{N}=2\ SYM} &= \int d^4x\,\mathrm{Tr}\Big(\tfrac{1}{2}v_{mn}v^{mn} - 2\,D_mA^\dagger D^mA + 2\sqrt{2}\,i\,[A^\dagger,\psi]\,\lambda\Big) \\ &= \frac{8k\pi^2}{g^2} - 2\int d^3S\,\mathrm{Tr}(A^\dagger D_\perp A). \end{aligned} \tag{5.99}$$

To obtain the second line we integrated by parts and used the Euler-Lagrange equation for the Higgs field (5.66). Evaluating the asymptotic value of $A^\dagger D_\perp A$ with the help of Eqs. (5.39)–(5.42), we obtain the following expression for the k-instanton action:[10]

$$\begin{aligned} S^{k\text{-inst}}_{\mathcal{N}=2\ SYM} &= \frac{8k\pi^2}{g^2} + 8\pi^2\,\bar{w}^{\dot\alpha}_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\langle\mathcal{A}\rangle_{uu}w_{ui\dot\alpha} - 8\pi^2\,\bar\Lambda_{ij}(\mathcal{A}_{\rm tot})_{ji} \\ &\quad + 2\sqrt{2}\,\pi^2\left(\bar\mu_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\nu_{ui} - \bar\nu_{iu}\langle\bar{\mathcal{A}}\rangle_{uu}\mu_{ui}\right) \end{aligned} \tag{5.100}$$

This is the $SU(N)$ generalization of the $SU(2)$ action presented in Eq. (7.32) of [42].

Next we incorporate N_f flavours of fundamental matter hypermultiplets. We restrict our attention to the Coulomb branch of the theory, where the hypermultiplet squarks q_f do not acquire VEV's. Instead, the integrations over instanton scale parameters are regulated by the VEV's of the adjoint complex scalar A. The solutions for the quark and squark background fields χ_f, $\tilde\chi_f$, q_f and $\tilde q_f$ are just given by Eqs. (5.95)–(5.96), except that on the Coulomb branch, the first term on the right-hand side of Eqs. (5.96) is zero. The essential

[10] Note that $\langle\bar{\mathcal{A}}\rangle$ and $\bar\Lambda$ are Hermitian conjugations of the second type defined in Footnote 8, with complex conjugated VEV's. They are not to be confused with $\langle\mathcal{A}\rangle^\dagger$ and $\Lambda^\dagger$ defined in Section 5.3.

new feature in $\mathcal{N}=2$ supersymmetric QCD with $N_f>0$ is that the conjugate adjoint Higgs $A^\dagger$ acquires a fermion bilinear component due the inhomogeneous term in its Euler-Lagrange equation:

$$(\mathcal{D}^2 A^\dagger)_{uv} \;=\; \frac{1}{\sqrt{2}}\sum_{f=1}^{N_F} \chi_{uf}\tilde{\chi}_{fv}\;. \tag{5.101}$$

(The Euler-Lagrange equation for A, Eq. (5.66), is unchanged.)

The solution to Eq. (5.101) is similar to, but simpler than, the solution to Eq. (5.66). At the purely bosonic level, with all Grassmann parameters turned off, the solutions for A and $A^\dagger$ must coincide, except for $\mathrm{v}_u \leftrightarrow \bar{\mathrm{v}}_u$. In contrast, the fermion bilinear contributions to each of A and $A^\dagger$ are independent. The fermion bilinear contribution to $A^\dagger$ is straightforwardly obtained from (5.101) (the analysis parallels Section 5 of [45]). It takes the form

$$-i\bar{U}_{u'}{}^{u+l\alpha}\cdot\begin{pmatrix}0_{uv} & 0\\ 0 & (\mathcal{A}_{\rm hyp})_{lm}\,\delta_\alpha{}^\beta\end{pmatrix}\cdot U_{(v+m\beta)\,v'}\,, \tag{5.102}$$

where the $k\times k$ anti-Hermitian matrix $\mathcal{A}_{\rm hyp}$ is defined as the solution to the inhomogeneous linear algebraic equation

$$\mathbf{L}\cdot\mathcal{A}_{\rm hyp} \;=\; \Lambda_{\rm hyp}. \tag{5.103}$$

Here the $k\times k$ anti-Hermitian matrix $\Lambda_{\rm hyp}$ is given by (cf. Eq. (5.8) of [45]):

$$(\Lambda_{\rm hyp})_{ij} \;=\; \frac{i\sqrt{2}}{8}\sum_{f=1}^{N_f}\mathcal{K}_{if}\tilde{\mathcal{K}}_{fj}. \tag{5.104}$$

Note that $\Lambda_{\rm hyp}$ and $\mathcal{A}_{\rm hyp}$ are in fact anti-Hermitian matrices (in terms of the dagger operation described in Footnote 8) when it is understood that $\mathcal{K}^\dagger=\tilde{\mathcal{K}}$.

The derivation of the superinstanton action in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD follows the derivation of the $SU(2)$ action described in Section 5 of [45]; one finds

$$S^{k\text{-inst}}_{\mathcal{N}=2\ SQCD} \;=\; S^{k\text{-inst}}_{\mathcal{N}=2\ SYM} \;-\; 8\pi^2(\Lambda_{\rm hyp})_{ij}(\mathcal{A}_{\rm tot})_{ji} \;+\; \pi^2\sum_{f=1}^{N_F} m_f\tilde{\mathcal{K}}_{fi}\mathcal{K}_{if} \tag{5.105}$$

As with the $\mathcal{N}=1$ action (5.97), one can check that this expression is a supersymmetry invariant. On the Coulomb branch, the transformation laws for the quark collective coordinates (5.98) reduce to

$$0 \;=\delta\mathcal{K} \;=\; \delta\tilde{\mathcal{K}}. \tag{5.106}$$

This implies that $\Lambda_{\rm hyp}$ is a supersymmetry invariant quantity:

$$\delta\Lambda_{\rm hyp} = 0. \tag{5.107}$$

Verifying the supersymmetry invariance of the action (5.105) is now a straightforward exercise involving the various transformation laws, (5.78)–(5.80), (5.83), (5.106) and (5.107). It is possible to make the supersymmetry invariance of $S^{k\text{-inst}}_{\mathcal{N}=2\ SQCD}$ more manifest, by assembling the bosonic and fermionic collective coordinates into a space-time-constant $\mathcal{N} = 2$ 'superfield' and reexpressing the action as an $\mathcal{N} = 2$ 'F-term' constructed from this superfield. The reader is referred to [45] for the details of this construction in the '$SU(2)$ as $Sp(1)$' case.

5.6 The Collective Coordinate Integration Measure

5.6.1 Overall Strategy

In Sections 5.2–5.5 above, we constructed the k-instanton background configurations and the associated k-instanton actions for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory. As the small-fluctuations determinants in a self-dual background cancel between the bosonic and fermionic sectors in a supersymmetric theory [48], the remaining component of our instanton calculus is the collective coordinate integration measure. In principle, this measure could be obtained by evaluating the normalization matrices of the bosonic and fermionic zero-modes. In practice, this requires the solutions to the nonlinear ADHM constraints (5.26), and these have only been obtained for $k \leq 3$ [61, 62].

Following [49], we use an alternative approach to determine the measure. We write an ansatz for the measure in terms of the original overcomplete, unconstrained matrices of collective coordinates and introduce the requisite constraints, by hand, as δ-functions in the integrand. (An analogy would be the measure $dx\,dy\,\delta(x^2 + y^2 - 1)$ rather than $d\theta$ for integration on a circle.) The reason this construction can work is that the various bosonic and fermionic constraints together form a supermultiplet of constraints, as mentioned in Section 5.3. The requirement of supersymmetry invariance, together with other symmetries, is sufficient to prove that our ansatz is unique.

The first step in the construction of the measure is to formally undo the $U(k)$ quotient described in Eq. (5.31) and define an unidentified measure, $d\mu^{(k)}$, for integration over the larger moduli space M^k :

$$\int_{\mathrm{M}^k_{\mathrm{phys}}} d\mu^{(k)}_{\mathrm{phys}} = \frac{1}{\mathrm{Vol}(U(k))} \int_{\mathrm{M}^k} d\mu^{(k)}. \tag{5.108}$$

The correctly normalized volumes for the $U(k)$ groups,

$$\mathrm{Vol}(U(k)) = 2^{2k-1} \pi^{\frac{k^2+2k-1}{2}} \prod_{i=1}^{k-1} \frac{1}{\Gamma(i+\frac{1}{2})} \tag{5.109}$$

follow from

$$\frac{U(k)}{U(k-1) \times U(1)} = S^{2(k-1)} \tag{5.110}$$

together with the initial condition

$$\mathrm{Vol}(U(1)) = 2\pi \ . \tag{5.111}$$

Here $S^{2(k-1)}$ is the $2(k-1)$-sphere and

$$\mathrm{Vol}(S^{2(k-1)}) = \frac{2\pi^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \ . \tag{5.112}$$

In addition to being a supersymmetry invariant, the measure must transform as a singlet under this residual $U(k)$.

We now present explicit expressions for $d\mu^{(k)}$ in both the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases. (A similar construction works for the $\mathcal{N}=4$ case as well, while for the non-supersymmetric case complications arise due to the re-emergence of the small-fluctuations determinants [49].)

5.6.2 $\mathcal{N}=1$ Supersymmetric Yang-Mills Theory

Following the strategy outlined above, we present the following ansatz for the k-instanton collective coordinate integration measure in $\mathcal{N}=1$ supersymmetric $SU(N)$ pure Yang-Mills theory:

$$\begin{aligned}\int d\mu^{(k)}_{\rm phys} &\equiv \frac{1}{{\rm Vol}(U(k))}\int d\mu^{(k)}\\ &= \frac{C_1^k}{{\rm Vol}(U(k))}\int d^{2Nk}\bar{w}\; d^{2Nk}w\; d^{Nk}\bar{\mu}\; d^{Nk}\mu\; d^{4k^2}a'\; d^{2k^2}\mathcal{M}'\\ &\quad\times\left[\prod_{c=1}^{3}\delta^{(k^2)}({\rm tr}_2\,(\tfrac{1}{2}\tau^c\bar{a}a))\right]\;\delta^{(2k^2)}(\bar{\mathcal{M}}a+\bar{a}\mathcal{M})\end{aligned}\tag{5.113}$$

The differentials in Eq. (5.113) have the following explicit meanings:

$$\int d^{4k^2}a' = \int\prod_{m=0}^{3}\left[\prod_{i=1}^{k}da'^m_{ii}\right]\left[\prod_{1\le i<j\le k} d\,{\rm Re}(a'^m_{ij})\;d\,{\rm Im}(a'^m_{ij})\right]\tag{5.114}$$

$$\int d^{2Nk}\bar{w}\; d^{2Nk}w = \int\prod_{\dot\alpha=1,2}\prod_{u=1}^{N}\prod_{i=1}^{k} d\bar{w}^{\dot\alpha}_{iu}\; dw_{ui\dot\alpha}\tag{5.115}$$

$$\int d^{2k^2}\mathcal{M}' = \int\prod_{\alpha=1,2}\left[\prod_{i=1}^{k}d\mathcal{M}'_{\alpha ii}\right]\left[\prod_{1\le i<j\le k} d\,{\rm Re}(\mathcal{M}'_{\alpha ij})\;d\,{\rm Im}(\mathcal{M}'_{\alpha ij})\right]\tag{5.116}$$

$$\int d^{Nk}\bar{\mu}\; d^{Nk}\mu = \int\prod_{u=1}^{N}\prod_{i=1}^{k} d\bar{\mu}_{iu}\; d\mu_{ui}\tag{5.117}$$

Notice that these expressions presuppose the canonical form (5.22)–(5.25) for b, so that the collective coordinate matrices a' and $\mathcal{M}'$ are assumed from the outset to satisfy Eqs. (5.27) and (5.58), respectively. The remaining constraints, namely (5.26) and (5.55), are implemented in Eq. (5.113) via the δ-functions. These have the explicit meanings:

$$\begin{aligned}\prod_{c=1}^{3}\delta^{(k^2)}({\rm tr}_2\,(\tfrac{1}{2}\tau^c\bar{a}a)) &= \prod_{c=1}^{3}\left[\prod_{i=1}^{k}\delta({\rm tr}_2\,(\tfrac{1}{2}\tau^c\bar{a}a)_{ii})\right]\\ &\quad\times\left[\prod_{1\le i<j\le k}\delta({\rm tr}_2\,{\rm Re}(\tfrac{1}{2}\tau^c\bar{a}a)_{ij})\;\delta({\rm tr}_2\,{\rm Im}(\tfrac{1}{2}\tau^c\bar{a}a)_{ij})\right]\end{aligned}\tag{5.118}$$

$$\begin{aligned}\delta^{(2k^2)}(\bar{\mathcal{M}}a+\bar{a}\mathcal{M}) &= \prod_{\dot\alpha=1,2}\left[\prod_{i=1}^{k}\delta(\bar{\mathcal{M}}a_{\dot\alpha}+\bar{a}_{\dot\alpha}\mathcal{M})_{ii}\right]\\ &\quad\times\left[\prod_{1\le i<j\le k}\delta(\,{\rm Re}(\bar{\mathcal{M}}a_{\dot\alpha}+\bar{a}_{\dot\alpha}\mathcal{M}))_{ij}\;\delta(\,{\rm Im}(\bar{\mathcal{M}}a_{\dot\alpha}+\bar{a}_{\dot\alpha}\mathcal{M}))_{ij}\right]\end{aligned}\tag{5.119}$$

We now argue in support of the the proposed measure as follows:

1. In the one-instanton sector, Eq. (5.113) reduces to

$$\int d\mu_{\rm phys}^{(1)} = \frac{C_1}{2\pi}\int d^4a'\, d^2\mathcal{M}'\, d^{2N}\bar{w}\, d^{2N}w\, d^N\bar{\mu}\, d^N\mu \times \left[\prod_{c=1}^{3}\delta\big(\mathrm{tr}_2\,(\tfrac{1}{2}\sigma^c\bar{w}w)\big)\right]\, \delta^2(\bar{\mu}w+\bar{w}\mu). \qquad (5.120)$$

After resolving the δ-function constraints, this reduces to the one-instanton measure obtained by direct calculation of the collective coordinate Jacobians [3, 4]. In Section 5.2, upon resolving the one-instanton ADHM constraints (5.16) and (5.17), we were able to explicitly identify the collective coordinates corresponding to the position x_0, size ρ, and group orientation Ω of the instanton (see Eqs. (5.43) and (5.45)). By comparing the one-instanton adjoint fermion zero-modes obtained from the general solution (5.53) (see the discussion at the end of Section 5.3.1) with the normalized zero-modes (4.49)–(4.52) listed in Section 4.3, we can make the identifications $\xi_{SS\,1,2}=\sqrt{2}\pi\mathcal{M}'_{1,2}$, $\bar{\xi}_{SC\,1,2}=2\pi\mu_{1,2}$, $\xi_{M\,u}=\sqrt{2}\pi\bar{\mu}_u$ and $\xi_{N\,u}=\sqrt{2}\pi\mu_u$. In terms of the unconstrained collective coordinates, the one-instanton measure (5.120) becomes

$$\int d\mu_{\rm phys}^{(1)} = C_1\frac{2^{3N+1}\pi^{4N-2}}{(N-1)!(N-2)!}\int d\Omega\int d^4x_0 d\rho\rho^{4N-5}\int d^{2N}\xi. \qquad (5.121)$$

Since the fermion zero-modes (4.49)–(4.52) are normalized this bears immediate comparison with the 't Hooft-Bernard measure (2.64). From this comparison we deduce that $C_1=2^{N+1}$.

2. The mass dimension of the k-instanton measure should be $-b_0k=-3Nk$. Since $[a]=-1$, $[\mu]=-1/2$ and $[d\mu]=1/2$, the right-hand side of Eq. (5.113) does indeed have the right mass dimension.

3. The anomalous $U(1)_R$ symmetry present in $\mathcal{N}=1$ supersymmetric $SU(N)$ pure Yang-Mills theory requires a net of $2Nk$ exact fermion zero-modes. Since the multi-instanton action in this theory is exactly given by $8\pi^2k/g^2$, it follows that the δ-functions in the k-instanton measure should saturate all but $2Nk$ of the Grassmann integrations. It is easy to see that this counting is obeyed by the right-hand side of Eq. (5.113): $2k^2$ fermionic δ-functions saturate $2k^2$ out of the $2k^2+2Nk$ fermionic integrations over $\mathcal{M}'$, $\bar{\mu}$ and μ leaving $2Nk$ exact fermion zero-modes.

4. It is clear that the measure (5.113) is invariant under the action of the residual $U(k)$ ADHM symmetry (5.29).

5. In the dilute-gas limit of large space-time separation between instantons, the measure (5.113) correctly factors according to the property of cluster decomposition. The proof follows that given in [49] and is detailed in Appendix D.

6. Just as for the multi-instanton action, the k-instanton measure has to be a supersymmetry invariant. This important requirement can be checked directly using the supersymmetry transformations (5.76) and (5.77). The supersymmetry generator $-i\xi Q$ leaves the first δ-function in (5.113) trivially invariant and does not affect the argument of the second δ-function due to the constraint (5.17). As for the generator $i\bar{\xi}\bar{Q}$, the reasoning is as follows: the argument of the second δ-function in (5.113) is invariant, while that of the first δ-function transforms into itself plus an admixture of the second, so that the product of δ-functions is an invariant.

7. Finally, we construct a uniqueness argument, following [49]. Let us consider including an additional function of the collective coordinates, $f(a, \mathcal{M})$, in the integrand of the proposed measure, Eq. (5.113). To preserve supersymmetry, we require that f be a supersymmetry invariant. It is a fact that any non-constant function that is a supersymmetry invariant must contain fermion bilinear pieces (and possibly higher powers of fermions as well). By the rules of Grassmann integration, such bilinears would necessarily lift some of the adjoint fermion zero-modes contained in $\mathcal{M}$. Since Eq. (5.113) contains precisely the right number of unlifted fermion zero-modes, as dictated by the $U(1)_R$ anomaly, namely $2Nk$, this argument *rules out* the existence of a non-constant function f. Moreover, any constant f would be absorbed into the overall multiplicative factor, which is fixed inductively using the property of cluster decomposition.

The measure (5.113) is easily augmented to incorporate fundamental matter multiplets. Since the Higgsinos satisfy the normalization condition [62]

$$\int d^4x \; \tilde{\chi}_{f\alpha u} \chi^{\alpha}_{uf'} \;=\; \pi^2 \tilde{\mathcal{K}}_{fi} \mathcal{K}_{if'} \;, \tag{5.122}$$

the associated fermionic measure is simply

$$\int d\mu_{\rm hyp}^{(k)} = \frac{1}{\pi^{2kN_f}}\int \prod_{f=1}^{N_f} d\mathcal{K}_{1f}\cdots d\mathcal{K}_{kf}\, d\tilde{\mathcal{K}}_{f1}\cdots d\tilde{\mathcal{K}}_{fk}\,. \tag{5.123}$$

The total measure in $\mathcal{N}=1$ supersymmetric $SU(N)$ QCD is then simply the product of the measures (5.123) and (5.113).

5.6.3 $\mathcal{N}=2$ Supersymmetric Yang-Mills Theory

Our ansatz for the k-instanton measure in $\mathcal{N}=2$ supersymmetric $SU(N)$ pure Yang-Mills theory is an obvious extension of Eq. (5.113). The new features are induced by the second adjoint fermion ψ, described by the collective coordinate matrix $\mathcal{N}$, and the adjoint Higgs A, whose construction required a 'spin-0' constraint to be imposed on the collective coordinate matrix $\mathcal{A}_{\rm tot}$ (5.71). Accordingly, we postulate the measure:

$$\begin{aligned}\int d\mu_{\rm phys}^{(k)} &\equiv \frac{1}{\mathrm{Vol}(U(k))}\int d\mu^{(k)}\\ &= \frac{(C_1')^k}{\mathrm{Vol}(U(k))}\int d^{2Nk}\bar{w}\; d^{2Nk}w\; d^{Nk}\bar{\mu}\; d^{Nk}\mu\; d^{Nk}\bar{\nu}\; d^{Nk}\nu\\ &\quad\times d^{4k^2}a'\; d^{2k^2}\mathcal{M}'\; d^{2k^2}\mathcal{N}'\; d^{k^2}\mathcal{A}_{\rm tot}\\ &\quad\times\left[\prod_{c=1}^{3}\delta^{(k^2)}\big(\mathrm{tr}_2\,(\tfrac{1}{2}\tau^c\bar{a}a)\big)\right]\delta^{(2k^2)}(\bar{\mathcal{M}}a+\bar{a}\mathcal{M})\;\delta^{(2k^2)}(\bar{\mathcal{N}}a+\bar{a}\mathcal{N})\\ &\quad\times\delta^{(k^2)}(\mathbf{L}\cdot\mathcal{A}_{\rm tot}-\Lambda-\Lambda_f).\end{aligned} \tag{5.124}$$

All of the arguments of the previous subsection can be applied to justify this $\mathcal{N}=2$ measure, with two obvious modifications. First, there are twice as many adjoint fermionic zero-modes dictated by the $U(1)_R$ anomaly. Second, the $\mathcal{N}=2$ supersymmetry algebra incorporates the extra transformations (5.78)–(5.80) and (5.83). By matching to the 't Hooft-Bernard measure (2.64) we deduce that $C_1' = 4\pi^{-2N}$.

To obtain the measure for $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD we have only to include the factor (5.123) associated with the fundamental fermion zero-modes.

5.7 Summary

In this chapter we have constructed, using the $U(N)$ ADHM formalism [2, 61, 62, 63], a multi-instanton calculus for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric $U(N)$ or $SU(N)$ Yang-Mills theory. This represents a generalization of the $SU(2)$ multi-instanton calculus developed by Dorey, Khoze and Mattis in [42, 45, 49] using the $Sp(1)$ ADHM formalism.

Supersymmetry has played a key role in the construction of the instanton calculus. Since the Euler-Lagrange equations are manifestly supersymmetric, the field configurations that constitute the multi-instanton background form a supersymmetry multiplet. We have seen that the action of the supersymmetry generators on this multiplet can be effected by certain 'passive' transformations of the overcomplete set of collective coordinates that parameterize the superinstanton. These transformations were employed in the derivation of the instanton action and collective coordinate integration measure; both quantities are constrained to depend on supersymmetry invariant combinations of the collective coordinates [15].

In the next chapter, we apply the calculus to investigate various instanton effects in supersymmetric $SU(N)$ Yang-Mills theory. In particular, we use the formalism to completely evaluate the one-instanton contribution to the prepotential in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD with arbitrary numbers of matter hypermultiplets. This represents a significant improvement upon the one-instanton calculations described in Section 4.3 and will provide a concrete example of the usefulness of the measure constructed in Section 5.6.

Chapter 6

Application of the Multi-Instanton Calculus

In the previous chapter we presented a complete multi-instanton calculus for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric $U(N)$ or $SU(N)$ Yang-Mills theory, based upon the construction of multi-instanton solutions originally due to Atiyah, Drinfeld, Hitchin and Manin (ADHM) [2, 61, 62, 63]. A characteristic feature of these solutions is that they are parameterized by an overcomplete set of collective coordinates which must satisfy certain nonlinear constraints. Our multi-instanton calculus is phrased directly in terms of the overcomplete set of collective coordinates, without an explicit resolution of the constraints. In our approach the constraints are imposed *implicitly*, through the δ-functions appearing in the collective coordinate measure. In this chapter we apply our calculus in several ways to investigate instanton effects in supersymmetric $SU(N)$ Yang-Mills theory, and in doing so, we justify this approach.

In Section 6.1, we apply the multi-instanton calculus to investigate instanton effects in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD, with arbitrary N and N_f. Generalizing the $SU(2)$ analysis of Dorey, Khoze and Mattis [42, 50, 45, 49], we derive a closed form expression for the k-instanton contribution to the prepotential, as a finite dimensional integral over the bosonic and fermionic collective coordinates of the supersymmetric k-instanton configuration [64]. We then employ this expression to completely evaluate the one-instanton contribution to the prepotential for general N and an arbitrary number of flavours [64]. This represents a significant improvement upon the one-instanton calculations [57, 58, 47], that were described in Section 4.3. In particular, we completely determine the 'regular' terms that appear when $N_f = 2N - 2$ or $N_f = 2N$, for all N. (The method of Ito and Sasakura [57, 58] fails to determine such terms.) We compare our answer with the proposed hyperelliptic curves [19, 20, 21, 22, 38, 39] and find that for $N_f = 2N - 2$ and $N_f = 2N$, none of the curves predict the right regular terms. We end the section with a discussion of these discrepancies. In particular, we consider the implications for the finite $N_f = 2N$ models [64].

Our one-instanton calculation provides a concrete example of the usefulness of the measure constructed in Section 5.6. The calculation is accomplished by first exponentiating the δ-function constraints in the measure, through the introduction of a supermultiplet of Lagrange multipliers. The original collective coordinates in the problem can then be completely integrated out (the exponent is Gaussian in these variables). The final integrations over the Lagrange multipliers used to implement the constraints can be carried out by an application of Stokes' theorem. In this way, our approach overcomes the problem associated with integrating over group space collective coordinates that was encountered in Section 4.3.

In Section 6.2 we employ the instanton calculus to investigate higher-derivative terms in the low-energy Wilsonian effective action of $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory [69]. As we pointed out in Section 3.3.1, the prepotential governs the leading terms in the effective action, which involve up to two derivatives or four fermions. The next-to-leading terms, which involve up to four derivatives or eight fermions, are also determined by a single function, $\mathcal{H}$, which is constrained to be real rather than holomorphic [70, 71, 72]. We shall derive an expression for the k-instanton contribution to this function, by a straightforward extension of the analysis used in Subsection 6.1.1. Using this expression we show, using a simple scaling argument, that in the finite model with $N_f = 4$ massless hypermultiplets, all multi-instanton contributions to the next-to-leading higher-derivative terms vanish [69]. This verifies a nonperturbative nonrenormalization theorem due to Dine and Seiberg [27].[1] Using a slightly modified scaling argument we also confirm this theorem in the $\mathcal{N} = 4$ supersymmetric $SU(2)$ model [69].

[1] It is not known (to us) at present if the Dine-Seiberg theorem holds for finite $SU(N)$ models with $N > 2$.

6.1 Instanton Contributions to the Prepotential

6.1.1 Closed Form Expression for the Prepotential

Up to terms involving two derivatives or four fermions, the Wilsonian effective action of $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD is determined by the prepotential, $\mathcal{F}(A)$, as follows:

$$\mathcal{L}_{\text{eff}} \;=\; \frac{1}{4\pi}\,\text{Im}\Big[\int d^4\theta\,\frac{\partial\mathcal{F}(A)}{\partial A_i}\,\bar{A}_i \;+\; \frac{1}{2}\int d^2\theta\,\frac{\partial^2\mathcal{F}(A)}{\partial A_i\,\partial A_j}\,W_i\,W_j\Big]\,. \tag{6.1}$$

The index i labels the $N-1$ light superfields of the low-energy $U(1)^{N-1}$ effective theory. As was pointed out in Section 3.3.2, the light components of the microscopic superfields reside in the $SU(N)$ Cartan subalgebra. We choose the following basis of generators:

$$H^i = \frac{1}{2}\text{diag}(0,\ldots,0,+1,-1,0,\ldots,0); \qquad i=1,2,\ldots,N-1 \tag{6.2}$$

where $+1$ is the ith entry on the diagonal. Note that, in this basis, the Higgs expectation $\langle A\rangle$ has the natural expansion

$$\langle A\rangle = \sum_{i=1}^{N-1}\tilde{\mathrm{v}}_i H^i, \tag{6.3}$$

where the $\tilde{\mathrm{v}}_i$ form a set of $N-1$ vacuum expectation values. These parameters are independent, in contrast to the N parameters v_u, defined by Eqs. (3.97) and (3.98). In spite of this, we find it convenient to mostly work with the v_u rather than the $\tilde{\mathrm{v}}_i$. In much of our semiclassical analysis, it does not matter if we take the v_u to be independent (as would be the case for a $U(N)$ gauge group); where necessary we can regard the $SU(N)$ tracelessness constraint (3.98) as being imposed implicitly.

For general N and $N_f \leq 2N$, the weak-coupling expansion of the prepotential has the form [31]:

$$\mathcal{F} \;=\; \mathcal{F}_{1-loop} + \sum_{k=1}^{\infty}\mathcal{F}_k, \tag{6.4}$$

where the $\mathcal{F}_k$ are nonperturbative k-instanton contributions. In what follows, we use the $SU(N)$ instanton calculus of Chapter 5 to derive an explicit k-instanton expression for $\mathcal{F}_k$. Our analysis closely follows that of Dorey, Khoze and Mattis in [42, 50, 45, 49].

Our first task is to expand $\mathcal{L}_{\text{eff}}$ in terms of component fields and thereby to extract an expression for some low-energy Green's function in terms of the prepotential. In the one-instanton $SU(2)$ calculation of Finnel and Pouliot [41] described in Section 4.2.1, the analysis focussed on a particular four-antifermion Green's function. Here we shall be more general. Following [42] we extract the following three effective vertices from $\mathcal{L}_{\text{eff}}$:

$$\mathcal{L}_{\text{eff}} \supset \frac{1}{4\pi} \sum_{s=1,2,3} \mathcal{V}^s \circ \mathcal{F}(\text{v}) \tag{6.5}$$

$$\mathcal{V}^1 = \frac{i}{4} \sum_{i,j=1}^{N-1} (v^{\text{SD}}_{i\,mn} v^{\text{SD}}_{j\,mn}) \frac{\partial^2}{\partial \tilde{\text{v}}_i \partial \tilde{\text{v}}_j} , \tag{6.6}$$

$$\mathcal{V}^2 = \frac{i}{2\sqrt{2}} \sum_{i,j,k=1}^{N-1} \psi_i \sigma^{mn} \lambda_j v^{\text{SD}}_{k\,mn} \frac{\partial^3}{\partial \tilde{\text{v}}_i \partial \tilde{\text{v}}_j \partial \tilde{\text{v}}_k} , \tag{6.7}$$

$$\mathcal{V}^3 = -\frac{i}{8} \sum_{i,j,k,l=1}^{N-1} \psi_i \psi_j \lambda_k \lambda_l \frac{\partial^4}{\partial \tilde{\text{v}}_i \partial \tilde{\text{v}}_j \partial \tilde{\text{v}}_k \partial \tilde{\text{v}}_l} . \tag{6.8}$$

The superscript SD indicates the self-dual part of the gauge field strength $v_{i\,mn}$. From these vertices we obtain three *anti*-holomorphic low-energy Green's functions, $\langle \bar{\mathcal{O}}^s(x_1, \ldots, x_{s+1}) \rangle$, $s = 1, 2, 3$, with

$$\bar{\mathcal{O}}^1(x_1, x_2) = v^{\text{ASD}}_{i\,mn}(x_1)\, v^{\text{ASD}}_{j\,pq}(x_2) , \tag{6.9}$$

$$\bar{\mathcal{O}}^2(x_1, x_2, x_3) = \bar{\psi}_{i\,\dot{\alpha}}(x_1)\, v^{\text{ASD}}_{j\,mn}(x_2)\, \bar{\lambda}_{k\,\dot{\beta}}(x_3) , \tag{6.10}$$

$$\bar{\mathcal{O}}^3(x_1, x_2, x_3, x_4) = \bar{\psi}_{i\,\dot{\alpha}(x_1)}\, \bar{\psi}_{j\,\dot{\beta}(x_2)}\, \bar{\lambda}_{k\,\dot{\gamma}}(x_3)\, \bar{\lambda}_{l\,\dot{\delta}}(x_4) . \tag{6.11}$$

We first consider the k-instanton contribution to the familiar four-antifermion Green's function (6.11). In terms of the $SU(N)$ multi-instanton calculus, we have

$$\langle \bar{\psi}_{i\,\dot{\alpha}}(x_1)\, \bar{\psi}_{j\,\dot{\beta}}(x_2)\, \bar{\lambda}_{k\,\dot{\gamma}}(x_3)\, \bar{\lambda}_{l\,\dot{\delta}}(x_4) \rangle_{kI} \cong$$
$$\int d\mu^{(k)}_{\text{phys}}\, \bar{\psi}^{\text{LD}}_{i\,\dot{\alpha}}(x_1)\, \bar{\psi}^{\text{LD}}_{j\,\dot{\beta}}(x_2)\, \bar{\lambda}^{\text{LD}}_{k\,\dot{\gamma}}(x_3)\, \bar{\lambda}^{\text{LD}}_{l\,\dot{\delta}}(x_4)\, \exp(-S^{k\text{-inst}}_{SQCD}) , \tag{6.12}$$

where the k-instanton action $S^{k\text{-inst}}_{SQCD}$ is given by Eq. (5.105) and the measure $d\mu^{(k)}_{\text{phys}}$ by Eqs. (5.124) and (5.123).

In order to deduce $\mathcal{F}_k$ from Eq. (6.12), we need expressions for the field insertions λ_i and ψ_i [50, 45]. At leading order, these field insertions are approximated by the quantities

$\bar{\psi}_i^{\text{LD}}$ and $\bar{\lambda}_i^{\text{LD}}$ defined as follows [31, 42, 45]: first, one solves the Euler-Lagrange equations for $\bar{\psi}$ and $\bar{\lambda}$ in the classical background of the ADHM multi-instanton with all its fermionic zero-modes turned on; next, one projects the resulting $SU(N)$-valued configurations onto the generators of the $SU(N)$ Cartan subalgebra to get the required light components; and finally, one assumes that the insertion points x_s are far away from the instanton position x_0 and performs a long-distance (LD) expansion.

Now on the Coulomb branch of $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD there are precisely four exact fermionic zero-modes in the k-instanton background. (This is true for all k.) These are the supersymmetric zero-modes of λ and ψ that were described in Section 5.3. Indeed, it is straightforward to verify, using the definition (5.59), that the four Grassmann parameters ξ_α^1 and ξ_α^2 associated with these modes do not appear in either the instanton action (5.105) or in the constraints associated with the measure (5.124). As regards the k-instanton expression (6.12) this leads to a simplification in the first stage of the prescription described above for obtaining the field insertions. We only need to solve the antifermion field equations in the background of the supersymmetric zero-modes. The resulting partial solutions are linear in ξ_α^1 and ξ_α^2 and hence completely saturate the collective coordinate integration over these parameters.

The solutions to the antifermion field equations in the background of the supersymmetric zero-modes were obtained in [42, 45] (see especially Appendix B of [45]) using the $SU(2)$ multi-instanton formalism. These solutions are equally valid in the $SU(N)$ case. They read

$$\bar{\psi} = -i\sqrt{2}\xi_1 \bar{\mathcal{D}} A^\dagger, \qquad \bar{\lambda} = i\sqrt{2}\xi_2 \bar{\mathcal{D}} A^\dagger, \tag{6.13}$$

where $A^\dagger$ is the solution to the Euler-Lagrange equation (5.101), given by Eqs. (5.102)–(5.104). Note that $A^\dagger$ has a part that is bilinear in the fundamental fermion collective coordinates $\mathcal{K}$ and $\tilde{\mathcal{K}}$, so there are contributions to $\bar{\lambda}$ and $\bar{\psi}$ trilinear in Grassmann parameters.

The result of projecting the antifermion solutions onto the $SU(N)$ Cartan subalgebra and performing a long-distance expansion is another straightforward generalization of the

$SU(2)$ result of [42, 50, 45]. One finds

$$\bar{\psi}^{\text{LD}}_{i\,\dot{\alpha}}(x) \;=\; i\sqrt{2}\,\xi^{1\alpha}\,S_{\alpha\dot{\alpha}}(x,x_0)\,\frac{\partial S^{k\text{-inst}}_{SQCD}}{\partial\tilde{\text{v}}_i} \;+\;\cdots, \tag{6.14}$$

$$\bar{\lambda}^{\text{LD}}_{i\,\dot{\alpha}}(x) \;=\; -i\sqrt{2}\,\xi^{2\alpha}\,S_{\alpha\dot{\alpha}}(x,x_0)\,\frac{\partial S^{k\text{-inst}}_{SQCD}}{\partial\tilde{\text{v}}_i} \;+\;\cdots, \tag{6.15}$$

where $S_{\alpha\dot{\alpha}}$ is the usual Weyl spinor propagator,

$$S_{\alpha\dot{\alpha}}(x,x_0) \;=\; \sigma^m_{\alpha\dot{\alpha}}\partial_m\,G_0(x,x_0)\;,\quad G_0(x,x_0) \;=\; \frac{1}{4\pi^2(x-x_0)^2}\;, \tag{6.16}$$

and the derivatives $\partial/\partial\tilde{\text{v}}_i$ act on $S^{k\text{-inst}}_{SQCD}$ with the understanding that the complex conjugate parameters $\bar{\tilde{\text{v}}}_i$ are always to be treated as independent variables. The omitted terms in (6.14) and (6.15) represent terms that fall off faster than $(x_i - x_0)^{-3}$ (as well as terms that are independent of ξ_1 or ξ_2 and hence cannot saturate the integrations over these parameters).

The next step is to write the instanton measure (5.124) as follows [50]:

$$\int d\mu^{(k)}_{\text{phys}} \;=\; \int d^4x_0\,d^2\xi_1\,d^2\xi_2\int d\tilde{\mu}^{(k)}_{\text{phys}}. \tag{6.17}$$

Here $d\tilde{\mu}^{(k)}_{\text{phys}}$ is the 'reduced measure', obtained from the physical $\mathcal{N}=2$ measure by factoring out the integrations over the supersymmetric collective coordinates ξ_1 and ξ_2 and over the translational collective coordinates x_0. It is quite legitimate to do this because the δ-function constraints in (5.124) are independent of x_0, ξ_1, and ξ_2. (These collective coordinates can be interpreted as the global position of the multi-instanton in $\mathcal{N}=2$ superspace). From the defining equations (5.33) and (5.59) we can write x_0, ξ_1, ξ_2 as the linear combinations obtained by taking the trace of the $k\times k$ matrices a', $\mathcal{M}'$ and $\mathcal{N}'$ respectively [42]:

$$x_0 \;=\; \frac{1}{k}\text{Tr}_k\,a'\;,\quad \xi_1 \;=\; \frac{1}{4k}\text{Tr}_k\mathcal{M}'\;,\quad \xi_2 \;=\; \frac{1}{4k}\text{Tr}_k\mathcal{N}'\;. \tag{6.18}$$

After substituting the field insertions Eqs. (6.14) and (6.15) into Eq. (6.12) and performing the ξ_i integrals, extracted from the physical measure as per Eq. (6.17), we obtain

$$\int d^4x_0\;S^{\alpha}{}_{\dot{\alpha}}(x_1,x_0)S_{\alpha\dot{\beta}}(x_2,x_0)S^{\gamma}{}_{\dot{\gamma}}(x_3,x_0)S_{\gamma\dot{\delta}}(x_4,x_0)\,\frac{\partial}{\partial\tilde{\text{v}}_i\partial\tilde{\text{v}}_j\partial\tilde{\text{v}}_k\partial\tilde{\text{v}}_l}\int d\tilde{\mu}^{(k)}_{\text{phys}}\,\exp(-S^{k\text{-inst}}_{SQCD})\;. \tag{6.19}$$

There are two general properties of the multi-instanton calculus that have enabled us to pull the derivatives outside of the integral in this equation. First, the collective coordinate measure (5.124) is independent of the VEV's. At first glance, this might not seem the case, because the 'spin-0' constraint (5.71) does involve the VEV's, through the matrix Λ, defined in Eq. (5.72). However, note that we can straightforwardly integrate the collective coordinate matrix $\mathcal{A}_{\rm tot}$ out of the measure; the δ-function constraint is then replaced by a factor $(\det \mathbf{L})^{-1}$, which is manifestly VEV-independent. The second point is that $S^{k\text{-inst}}_{SQCD}$ is linear in the the VEV's, so that differentiating the exponentiated action four times does indeed generate the product of four '$\partial S/\partial\tilde{\rm v}$' factors.

In (6.19) we recognize the Feynman amplitude corresponding to the effective four-fermion vertex (6.8). We deduce that

$$\mathcal{F}_k({\rm v}) \;\equiv\; \mathcal{F}({\rm v})\Big|_{k\text{-inst}} \;=\; 8\pi i \int d\tilde{\mu}^{(k)}_{\rm phys}\,\exp(-S^{k\text{-inst}}_{SQCD}). \tag{6.20}$$

This collective coordinate integral expression for $\mathcal{F}_k$ constitutes a closed series solution, in quadratures, of the low-energy dynamics of the Coulomb branches of the $\mathcal{N}=2$ supersymmetric $SU(N)$ models.

Let us now consider the other two effective vertices, (6.6) and (6.7), that were extracted from $\mathcal{L}_{eff}$. In order to find the k-instanton contribution to these vertices we analyse the Green's functions (6.9) and (6.10), respectively. These Green's functions require field insertions associated with the anti-self-dual part of the field strength. Now in the leading order constrained instanton background, the field strength is purely self-dual, by definition. However, the prescription for obtaining long-distance field insertions requires us to first solve the next-to-leading Euler-Lagrange equations. In this way we do obtain an anti-self-dual contribution to the field strength (see Section 4.4 of [42]). In order to saturate the the integration over ξ_1 and ξ_2, we need the part of this contribution that is bilinear in these parameters; this was obtained in [42, 45]. Its long-distance expansion yields [42, 45]:

$$v^{\rm ASD,LD}_{i\,mn}(x) \;=\; \sqrt{2}\,\xi_1\sigma^{pq}\xi_2\,G_{mn,pq}(x,x_0)\,\frac{\partial S^{k\text{-inst}}_{SQCD}}{\partial\tilde{\rm v}_i} \;+\;\cdots \tag{6.21}$$

where $G_{mn,pq}$ is the gauge-invariant propagator of $U(1)$ field strengths:

$$G_{mn,pq}(x,x_0) \;=\; (\,\eta_{mp}\partial_n\partial_q - \eta_{mq}\partial_n\partial_p - \eta_{np}\partial_m\partial_q + \eta_{nq}\partial_m\partial_p\,)G_0(x,x_0)\;. \tag{6.22}$$

The omitted terms in (6.21) include terms that fall off faster than $(x_i - x_0)^{-4}$. An important property of $G_{mn,pq}$ is that it only connects $v^{\rm SD}_{imn}$ to $v^{\rm ASD}_{ipq}$ and vice versa (just as $S_{\alpha\dot\alpha}$ only connects λ to $\bar\lambda$, and ψ to $\bar\psi$). This property follows from the identity

$$\bar\sigma^{pq\dot\alpha}{}_{\dot\beta}\, G_{mn,pq}(x) \;=\; \frac{2}{\pi^2 x^6}\, \bar x^{\dot\alpha\alpha}\, \sigma_{mn\,\alpha}{}^{\beta}\, x_{\beta\dot\beta} \tag{6.23}$$

which implies

$$0 \;=\; \bar\sigma^{mn}_{\dot\gamma\dot\delta}\, \bar\sigma^{pq}_{\dot\alpha\dot\beta}\, G_{mn,pq}(x) \;=\; \sigma^{mn}_{\gamma\delta}\, \sigma^{pq}_{\alpha\beta}\, G_{mn,pq}(x)\ . \tag{6.24}$$

The Green's functions (6.9) and (6.10) may be calculated, just as before, by substituting the long-distance expressions (6.21), (6.14) and (6.15) into the collective coordinate integration, and performing the ξ_i integrals explicitly. Thanks to Eq. (6.24) one does indeed recover the effective vertices (6.6) and (6.7), and we again deduce that the k-instanton contribution to the prepotential is given by Eq. (6.20).

We can easily extend the $SU(2)$ analysis of Dorey, Khoze and Mattis [50] to derive the $SU(N)$ Matone relation, Eq. (4.40). Consider the k-instanton expression for the condensate $u_2 = \langle {\rm Tr}(A^2)\rangle$. The relevant field insertions are given by the part of the background Higgs configuration (5.68) that is bilinear in ξ_1 and ξ_2. It was observed in [54] that this bilinear part has the form

$$A = \sqrt{2}\xi_2\sigma^{mn}\xi_1 v_{mn}. \tag{6.25}$$

After substituting this into the k-instanton expression for u_2 and integrating over both x_0 and ξ_i, one obtains

$$u_2\Big|_{k\text{-inst}} = -16\pi^2 k \int d\tilde\mu^{(k)}_{\rm phys}\, \exp(-S^{k\text{-inst}}_{SQCD}). \tag{6.26}$$

Comparing this with the expression (6.20) for $\mathcal{F}_k$, we deduce that

$$u_2({\rm v}_1,\ldots,{\rm v}_N)\Big|_{k\text{-inst}} \;=\; 2i\pi k\cdot \mathcal{F}_k({\rm v}_1,\ldots,{\rm v}_N). \tag{6.27}$$

Since a k-instanton effect is proportional to $\Lambda^{b_0 k}$, where $b_0 = 2N - N_f$, this result is equivalent to the Matone relation (4.40).

6.1.2 One-Instanton Contribution to the Prepotential

In Section 4.3, although we were able to derive integral expressions for one-instanton effects in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD, we were unable to evaluate these expressions in the general case. The source of the difficulty was the integration over group space collective coordinates. In what follows, we demonstrate that when the problem is reformulated using the ADHM instanton calculus, this difficulty is removed. Consequently, we are able to completely evaluate the one-instanton contribution to the prepotential, for general N and N_f.

Our starting point is the integral expression (6.20). From the formulae for the $\mathcal{N}=2$ action and reduced measure given respectively by Eq. (5.105) and by Eqs. (5.124) and (6.17), we write down:

$$\begin{aligned}\mathcal{F}_1 \;=\; & \frac{iC_1'\Lambda_{PV}^{b_0}}{2^6\pi^{2N_f}}\int d\mathcal{A}_{\rm tot}\cdot\prod_{u=1}^{N} d\bar\mu_u d\mu_u d\bar\nu_u d\nu_u d^2\bar w_u^{\dot\alpha} d^2 w_{u\dot\beta}\cdot\prod_{f=1}^{N_f} d\mathcal{K}_f d\tilde{\mathcal{K}}_f \\ & \times\delta(\mathbf{L}\cdot\mathcal{A}_{\rm tot}-\Lambda_{\rm tot})\prod_{c=1,2,3}\delta\big(\tfrac12(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar w_u^{\dot\beta}w_{u\dot\alpha}\big) \\ & \times\prod_{\dot\alpha=1,2}\delta(\bar\mu_u w_{u\dot\alpha}+\bar w_{u\dot\alpha}\mu_u)\delta(\bar\nu_u w_{u\dot\alpha}+\bar w_{u\dot\alpha}\nu_u) \\ & \times\ \exp\Big(-8\pi^2|{\rm v}_u|^2\bar w_u^{\dot\alpha}w_{u\dot\alpha}+2\sqrt2\,\pi^2 i(\bar\mu_u\bar{\rm v}_u\nu_u-\bar\nu_u\bar{\rm v}_u\mu_u) \\ & \qquad +8\pi^2(\bar\Lambda+\Lambda_{\rm hyp})\mathcal{A}_{\rm tot}-\pi^2\sum_{f=1}^{N_f} m_f\tilde{\mathcal{K}}_f\mathcal{K}_f\Big)\end{aligned}\tag{6.28}$$

In the $N_f=2N$ case, the scale $\Lambda_{PV}^{b_0}$ is replaced with the instanton factor $\exp(-8\pi^2/g^2)$.

To evaluate this integral, we exponentiate the various δ-functions by means of Lagrange multipliers, and then interchange the resulting order of integration. We integrate out the ADHM supermultiplet $\{a,\mathcal{M},\mathcal{N},\mathcal{A}_{\rm tot}\}$ first of all, followed by the hypermultiplet collective coordinates $\mathcal{K}_f$ and $\tilde{\mathcal{K}}_f$. Only then do we perform the integration over the Lagrange multipliers.

The spin-1 and spin-1/2 constraints in Eq. (6.28) are exponentiated in a straightforward manner, respectively as:

$$\prod_{c=1,2,3} \delta(\tfrac{1}{2}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar w_u^{\dot\beta} w_{u\dot\alpha}) \;=\; \frac{1}{\pi^3}\int d^3\mathbf{p}\, \exp(ip^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar w_u^{\dot\beta} w_{u\dot\alpha})\;, \tag{6.29}$$

and

$$\prod_{\dot\alpha=1,2} \delta(\bar\mu_u w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u) \;=\; 2\int d^2\xi\, \exp\big(\xi^{\dot\alpha}(\bar\mu_u w_{u\dot\alpha} + \bar w_{u\dot\alpha}\mu_u)\big), \tag{6.30}$$

$$\prod_{\dot\alpha=1,2} \delta(\bar\nu_u w_{u\dot\alpha} + \bar w_{u\dot\alpha}\nu_u) \;=\; 2\int d^2\eta\, \exp\big(\eta^{\dot\alpha}(\bar\nu_u w_{u\dot\alpha} + \bar w_{u\dot\alpha}\nu_u)\big). \tag{6.31}$$

Here we have introduced a triplet of bosonic Lagrange multipliers p^c, as well as two Grassmann spinor Lagrange multipliers $\xi^{\dot\alpha}$ and $\eta^{\dot\alpha}$. The exponentiation of the spin-0 constraint is best accomplished in a slightly trickier way involving a term in the action, as follows:

$$\begin{aligned}
&\int d\mathcal{A}_{\text{tot}}\, \delta(\mathbf{L}\cdot\mathcal{A}_{\text{tot}} - \Lambda_{\text{tot}})\exp\big(8\pi^2(\bar\Lambda + \Lambda_{\text{hyp}})\mathcal{A}_{\text{tot}}\big)\\
&= \frac{1}{\det\mathbf{L}}\, \exp\big(8\pi^2(\bar\Lambda + \Lambda_{\text{hyp}})\cdot\mathbf{L}^{-1}\cdot\Lambda_{\text{tot}}\big)\\
&= 8\pi\int d(\mathrm{Re}\,z)d(\mathrm{Im}\,z)\, \exp\big(-8\pi^2(\bar z\,\mathbf{L}\,z - (\bar\Lambda + \Lambda_{\text{hyp}})z - \bar z\Lambda_{\text{tot}})\big).
\end{aligned} \tag{6.32}$$

The second equality follows from the general Gaussian identity

$$\int\prod_i d(\mathrm{Re}\,z_i)d(\mathrm{Im}\,z_i)\, \exp\big(-\bar z_i K_{ij} z_j + \bar y_i z_i + \bar z_i y_i\big) \;=\; \frac{1}{\det(K/\pi)}\, \exp(\bar y_i K^{-1}_{ij} y_j), \tag{6.33}$$

which can be used to exponentiate the spin-0 constraint in an elegant way for arbitrary instanton number k. The advantage of the rewrite (6.32) is that $\mathbf{L}$ is easier to manipulate in the exponent than $\mathbf{L}^{-1}$ (which appears implicitly in the definition of $\mathcal{A}_{\text{tot}}$). In the present case, with $k=1$, the operator $\mathbf{L}$ collapses to a 1×1 c-number matrix:

$$\mathbf{L} \;=\; \det\mathbf{L} \;=\; \bar w_u^{\dot\alpha} w_{u\dot\alpha}\,. \tag{6.34}$$

Likewise $\bar\Lambda$ and Λ_{tot} collapse to

$$\bar\Lambda = -i\bar{\mathrm{v}}_u \bar w_u^{\dot\alpha} w_{u\dot\alpha}\;, \qquad \Lambda_{\text{tot}} = i\mathrm{v}_u \bar w_u^{\dot\alpha} w_{u\dot\alpha} - \frac{1}{2\sqrt{2}}(\bar\nu_u\mu_u - \bar\mu_u\nu_u)\;. \tag{6.35}$$

Now we consider the combined exponent formed from Eqs. (6.28)–(6.32). The linear shifts

$$\mu_u \to \mu_u + \frac{i\eta^{\dot\alpha} w_{u\dot\alpha}}{2\sqrt{2}\,\pi^2\bar\alpha_u}, \quad \bar\mu_u \to \bar\mu_u + \frac{i\eta^{\dot\alpha}\bar w_{u\dot\alpha}}{2\sqrt{2}\,\pi^2\bar\alpha_u},$$
$$\nu_u \to \nu_u - \frac{i\xi^{\dot\alpha} w_{u\dot\alpha}}{2\sqrt{2}\,\pi^2\bar\alpha_u}, \quad \bar\nu_u \to \bar\nu_u - \frac{i\xi^{\dot\alpha}\bar w_{u\dot\alpha}}{2\sqrt{2}\,\pi^2\bar\alpha_u} \tag{6.36}$$

eliminate the terms that are linear in these variables. By inspection, the Grassmann integrations over $\{\mu_u, \nu_u, \bar\mu_u, \bar\nu_u\}$ then simply bring down a factor of

$$\prod_{u=1}^{N} (2\sqrt{2}\,\pi^2 i\bar\alpha_u)^2 \tag{6.37}$$

In Eqs. (6.36)–(6.37), we have defined α_u and $\bar\alpha_u$ as the naturally appearing linear combinations

$$\alpha_u \;=\; \mathrm{v}_u + iz\;, \qquad \bar\alpha_u \;=\; \bar{\mathrm{v}}_u - i\bar z\;. \tag{6.38}$$

Next, the $\{w_u, \bar w_u, \mathcal{K}_f, \tilde{\mathcal{K}}_f\}$ integrations are accomplished, using the identities

$$\int d^2w_u d^2\bar w_u \,\exp\Big(- A^0 \bar w_u^{\dot\alpha} w_{u\dot\alpha} + i\sum_{c=1,2,3} A^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\bar w_u^{\dot\beta} w_{u\dot\alpha}\Big) \;=\; \frac{-4\pi^2}{(A^0)^2 + \sum (A^c)^2} \tag{6.39}$$

and

$$\int \prod_{f=1}^{N_f} d\mathcal{K}_f d\tilde{\mathcal{K}}_f \,\exp\Big(8\pi^2 \Lambda_{\rm hyp}\, z - \pi^2 \sum_{f=1}^{N_f} m_f \tilde{\mathcal{K}}_f \mathcal{K}_f \Big) \;=\; \pi^{2N_f} \prod_{f=1}^{N_f} (i\sqrt{2}\,z + m_f)\;. \tag{6.40}$$

In this way, all the original ADHM variables $\{a, \mathcal{M}, \mathcal{N}, \mathcal{A}_{\rm tot}, \mathcal{K}, \tilde{\mathcal{K}}\}$ are eliminated from the integral (6.28). One is left with an integral over Lagrange multipliers only:

$$\mathcal{F}_1 \;=\; \frac{iC_1'\Lambda_{PV}^{b_0}}{2\pi^2} \int d^3\mathbf{p}\, d^2\xi d^2\eta\, d(\mathrm{Re}\, z) d(\mathrm{Im}\, z)\, \mathcal{B} \prod_{f=1}^{N_f} (i\sqrt{2}\,z + m_f), \tag{6.41}$$

where

$$\mathcal{B} \;=\; \prod_{u=1}^{N} \frac{(2\sqrt{2}\,\pi^2 i\bar\alpha_u)^2(-4\pi^2)}{(8\pi^2|\alpha_u|^2)^2 + \sum_{c=1,2,3}(p^c + \Xi_u^c)^2} \tag{6.42}$$

and Ξ_u^c is the fermion bilinear

$$\Xi_u^c \;=\; \frac{1}{4\sqrt{2}\,\pi^2\bar\alpha_u}(\xi_{\dot\alpha}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\eta^{\dot\beta} - \eta_{\dot\alpha}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\,\xi^{\dot\beta})\;. \tag{6.43}$$

When $N_f = 0$ the product over flavours in Eq. (6.41) should simply be replaced by unity.

The $\{\xi, \eta\}$ Grassmann integrations in Eq. (6.41) must be saturated with two insertions of Ξ:

$$\int d^2\xi d^2\eta \, \Xi_u^b \, \Xi_v^c = \frac{\delta^{bc}}{16\pi^4 \bar{\alpha}_u \bar{\alpha}_v} \,. \tag{6.44}$$

Extracting these quadratic powers of Ξ from $\mathcal{B}$ can be done quite elegantly, thanks to the algebraic identity

$$\begin{aligned} \int d^2\xi d^2\eta \, \mathcal{B} &= \sum_{b,c=1}^{3} \sum_{u,v=1}^{N} \frac{\delta^{bc}}{16\pi^4 \bar{\alpha}_u \bar{\alpha}_v} \cdot \frac{1}{2} \frac{\partial^2}{\partial \Xi_u^b \, \partial \Xi_v^c} \mathcal{B} \Big|_{\Xi=0} \\ &= \frac{1}{32\pi^4 |\mathbf{p}|^2} \Big(\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \Big)^2 \mathcal{B} \Big|_{\Xi=0} \,. \end{aligned} \tag{6.45}$$

Pulling the VEV derivatives outside the integral, one therefore finds

$$\mathcal{F}_1 = \frac{iC_1' \Lambda_{PV}^{b_0}}{2\pi^2} \cdot \frac{1}{32\pi^4} \Big(\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \Big)^2 \int d(\mathrm{Re}\, z) d(\mathrm{Im}\, z) \, \Gamma \prod_{f=1}^{N_f} (i\sqrt{2}\, z + m_f) \,. \tag{6.46}$$

Here

$$\Gamma = \int d^3\mathbf{p} \, \frac{1}{|\mathbf{p}|^2} \prod_{u=1}^{N} \frac{(2\sqrt{2}\,\pi^2 i \bar{\alpha}_u)^2 (-4\pi^2)}{(8\pi^2 |\alpha_u|^2)^2 + |\mathbf{p}|^2} = 8\pi^6 \sum_{u=1}^{N} \frac{\bar{\alpha}_u}{\alpha_u} \prod_{v \neq u} \frac{\pi^2}{2} \frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4} \,, \tag{6.47}$$

the second equality following from a standard contour integration in the variable $|\mathbf{p}|$, extended to run from $-\infty$ to ∞.

In this fashion, the original expression (6.28) has collapsed to a two-dimensional integral over the xy plane (with $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$ henceforth). We now observe that the dependence on $\bar{\mathrm{v}}_u$ in this integral is entirely through the variables $\bar{\alpha}_u = \bar{\mathrm{v}}_u - i\bar{z}$. Therefore, it is tempting—but incorrect—to pull the $\bar{\mathrm{v}}_u$ derivatives back inside the integrand, and to make the naive replacement

$$\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \to i \frac{\partial}{\partial \bar{z}} \,, \qquad \Big(\sum_{u=1}^{N} \frac{\partial}{\partial \bar{\mathrm{v}}_u} \Big)^2 \to -\Big(\frac{\partial}{\partial \bar{z}} \Big)^2 \,. \tag{6.48}$$

The error here is due to the fact that the two sides of Eq. (6.48) can differ by δ-function contributions which arise at the locations of poles in the z variable. As a simple example, whereas obviously $(\sum \partial/\partial\bar{\mathrm{v}})\, z^{-1} = 0$, one also has, in contrast,[2]

$$\frac{\partial}{\partial\bar{z}}\frac{1}{z} \;=\; \pi\,\delta(x)\delta(y)\;, \tag{6.49}$$

$$\Big(\frac{\partial}{\partial\bar{z}}\Big)^2\frac{1}{z} \;=\; \pi\,\frac{\partial}{\partial\bar{z}}\,\delta(x)\delta(y) \;=\; \frac{\pi}{2}\,(\delta'(x)\delta(y) + i\delta(x)\delta'(y))\;. \tag{6.50}$$

The lesson is that one can legitimately trade $\bar{\mathrm{v}}_u$ differentiation for $\bar{z}$ differentiation as per Eq. (6.48)—but only if one explicitly subtracts off the extraneous δ-function pieces that are generated at the locations of the poles in z. Accordingly, we can split up $\mathcal{F}_1$ into two parts,

$$\mathcal{F}_1 \;=\; \mathcal{F}_\delta \;+\; \mathcal{F}_\partial\;, \tag{6.51}$$

where $\mathcal{F}_\delta$ is the contribution of these δ-function corrections, while $\mathcal{F}_\partial$ is a boundary term arising from judicious use of Stokes' theorem applied to $\partial^2/\partial\bar{z}^2$. We now evaluate each of these parts, in turn.

Calculation of $\mathcal{F}_\delta$

As stated, to calculate $\mathcal{F}_\delta$, one converts $(\sum\partial/\partial\bar{\mathrm{v}}_u)^2$ into $-\partial^2/\partial\bar{z}^2$ as per Eq. (6.48), then subtracts off the spurious δ-function contributions that correspond to the poles in z of the expression Γ given in Eq. (6.47). The relevant poles lie at the N distinct values

$$0 \;=\; \alpha_u \;= \mathrm{v}_u + iz \;=\; (\mathrm{Re}\,\mathrm{v}_u - y) + i(\mathrm{Im}\,\mathrm{v}_u + x)\;. \tag{6.52}$$

There also appear to be poles in Γ when $|\alpha_v|^2 = \pm|\alpha_u|^2$ but these are irrelevant: the poles at $|\alpha_v|^2 = -|\alpha_u|^2$ lie away from the real domain of integration $(x,y) \in \mathbb{R}^2$, whereas the poles at $|\alpha_v|^2 = +|\alpha_u|^2$ have residues that cancel pairwise among the terms in Eq. (6.47) (these pairs correspond to interchanging the indices u and v). Restricting our attention to the singularities (6.52), we therefore find:

$$\mathcal{F}_\delta \;=\; \frac{iC_1'\Lambda_{PV}^{b_0}}{2\pi^2}\cdot\frac{1}{32\pi^4}\cdot 8\pi^6\int dx\,dy\sum_{u=1}^{N}\Big[\Big(\frac{\partial^2}{\partial\bar{z}^2}\frac{1}{\alpha_u}\Big) \;+\; 2\Big(\frac{\partial}{\partial\bar{z}}\frac{1}{\alpha_u}\Big)\frac{\partial}{\partial\bar{z}}\Big]$$
$$\times\Big[\bar{\alpha}_u\prod_{v\neq u}\frac{\pi^2}{2}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4 - |\alpha_u|^4}\Big]\prod_{f=1}^{N_f}(i\sqrt{2}\,z + m_f)\;. \tag{6.53}$$

---

[2]The normalization factor on the right-hand side of Eq. (6.49) is easily fixed by integrating both sides against $\exp(-\lambda z\bar{z})$.

Integrating the first term on the right-hand side (the $\partial^2/\partial\bar{z}^2$ term) once by parts cancels half the second term, whereupon the identity

$$\frac{\partial}{\partial\bar{z}}\frac{1}{\alpha_u} = -i\pi\,\delta(\mathrm{Im}\,\mathrm{v}_u + x)\delta(\mathrm{Re}\,\mathrm{v}_u - y) \tag{6.54}$$

[cf. Eqs. (6.49) and (6.52)] quickly leads to

$$\mathcal{F}_\delta = -\frac{iC_1'\Lambda_{PV}^{b_0}\pi^{2N-1}}{2^{N+2}}\sum_{u=1}^{N}\prod_{v\neq u}\frac{1}{(\mathrm{v}_v-\mathrm{v}_u)^2}\prod_{f=1}^{N_f}(-\sqrt{2}\,\mathrm{v}_u + m_f)\,. \tag{6.55}$$

Calculation of $\mathcal{F}_\partial$

Next we evaluate the boundary term $\mathcal{F}_\partial$ implied by the naive replacement (6.48). It is useful to switch to polar coordinates, $(x,y)\to(r,\theta)$, in terms of which

$$\frac{\partial^2}{\partial\bar{z}^2} = \frac{1}{r}\frac{\partial}{\partial r}\circ\mathcal{D}_r + \frac{\partial}{\partial\theta}\circ\mathcal{D}_\theta \tag{6.56}$$

where

$$\mathcal{D}_r = \tfrac{1}{4}e^{2i\theta}\big(2 + r\frac{\partial}{\partial r}\big)\,,\qquad \mathcal{D}_\theta = \frac{i}{4r^2}e^{2i\theta}\big(1 + 2r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\big)\,. \tag{6.57}$$

Since the integrand in Eq. (6.46) is a single-valued function of θ, the $(\partial/\partial\theta)\,\mathcal{D}_\theta$ term can be neglected. Stoke's theorem then equates the two-dimensional integral (6.46) to the angularly integrated action of $\mathcal{D}_r$ evaluated on the circle of infinitely large radius:

$$\begin{aligned}\mathcal{F}_\partial =& -\frac{iC_1'\Lambda_{PV}^{b_0}}{2\pi^2}\cdot\frac{1}{32\pi^4}\cdot 8\pi^6\lim_{r\to\infty}\tfrac{1}{4}\big(2 + r\frac{\partial}{\partial r}\big)\\ &\times\int_0^\infty d\theta\,e^{2i\theta}\Big[\sum_{u=1}^{N}\frac{\bar{\alpha}_u}{\alpha_u}\prod_{v\neq u}\frac{\pi^2}{2}\frac{\bar{\alpha}_v^2}{|\alpha_v|^4-|\alpha_u|^4}\Big]\prod_{f=1}^{N_f}(i\sqrt{2}\,re^{i\theta}+m_f)\,,\end{aligned} \tag{6.58}$$

where $\alpha_u = \mathrm{v}_u + ire^{i\theta}$ and $\bar{\alpha}_u = \bar{\mathrm{v}}_u - ire^{-i\theta}$. The remaining θ integral is best evaluated by changing variables to $\xi = e^{i\theta}$, and summing the poles in ξ which sit within the unit circle. These lie at the points where $|\alpha_v|^2 = \pm|\alpha_u|^2$ or $\alpha_u = 0$. As before, the poles with $|\alpha_v|^2 = +|\alpha_u|^2$ may be omitted as they have pairwise canceling residues between terms with indices u and v interchanged. The poles with $|\alpha_v|^2 = -|\alpha_u|^2$ correspond to

$$\xi = \frac{-(|\mathrm{v}_u|^2+|\mathrm{v}_v|^2+2r^2)+\sqrt{4(r^2-\mathrm{Re}\,\mathrm{v}_u\bar{\mathrm{v}}_v)^2+|\mathrm{v}_u^2-\mathrm{v}_v^2|^2}}{2ir(\bar{\mathrm{v}}_u+\bar{\mathrm{v}}_v)} = \frac{i}{2r}(\mathrm{v}_u+\mathrm{v}_v)+\mathcal{O}(r^{-3})\,. \tag{6.59}$$

These contribute

$$\pi^3 \sum_{u=1}^{N} \sum_{v\neq u} \frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2} \prod_{w\neq u,v} \frac{\pi^2}{2} \frac{1}{(\mathrm{v}_w - \mathrm{v}_u)(\mathrm{v}_w - \mathrm{v}_v)} \prod_{f=1}^{N_f} \Big(-\frac{1}{\sqrt{2}}(\mathrm{v}_u + \mathrm{v}_v) + m_f \Big) \; + \; \mathcal{O}(r^{-2}) \tag{6.60}$$

to the θ integral in Eq. (6.58). Likewise, the poles at $\alpha_u = 0$, corresponding to $\xi = i\mathrm{v}_u/r$, contribute

$$-2\pi \sum_{u=1}^{N} \prod_{v\neq u} \frac{\pi^2}{2} \frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2} \prod_{f=1}^{N_f} (-\sqrt{2}\,\mathrm{v}_u + m_f) \; + \; \mathcal{O}(r^{-2}) \tag{6.61}$$

to the θ integral. Adding these two contributions gives, finally:

$$\begin{aligned} \mathcal{F}_\partial \;=\; & -\frac{iC_1' \Lambda_{PV}^{b_0} \pi^{2N-1}}{2^{N+2}} \sum_{u=1}^{N} \Big\{ \sum_{v\neq u} \frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2} \prod_{w\neq u,v} \frac{1}{(\mathrm{v}_w - \mathrm{v}_u)(\mathrm{v}_w - \mathrm{v}_v)} \\ & \times \prod_{f=1}^{N_f} \Big(-\frac{1}{\sqrt{2}}(\mathrm{v}_u + \mathrm{v}_v) + m_f \Big) \; - \; \prod_{v\neq u} \frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2} \prod_{f=1}^{N_f} (-\sqrt{2}\,\mathrm{v}_u + m_f) \Big\} \end{aligned} \tag{6.62}$$

For $N_f < 2N - 2$ this expression is in fact identically zero. Direct calculation shows that the residues of all the simple and double poles cancel amongst the various terms, so that the rational function $\mathcal{F}_\partial$ must be a polynomial in the parameters $\{\mathrm{v}_u, m_f\}$. By dimensional analysis, this polynomial must have degree $N_f - 2N + 2$. It follows that $\mathcal{F}_\partial$ must vanish for $N_f < 2N - 2$, as stated.

Final Expression for $\mathcal{F}_1$

Upon adding together $\mathcal{F}_\delta$ and $\mathcal{F}_\partial$ we notice that the final term in Eq. (6.62) precisely cancels the expression for $\mathcal{F}_\delta$, as given by Eq. (6.55). Our final one-instanton expression for the prepotential is therefore:

$$\begin{aligned} \mathcal{F}_1 \;\equiv\; \mathcal{F}_\delta + \mathcal{F}_\partial \;=\; & -\frac{iC_1' \Lambda_{PV}^{b_0} \pi^{2N-1}}{2^{N+2}} \sum_{u=1}^{N} \sum_{v\neq u} \frac{1}{(\mathrm{v}_v - \mathrm{v}_u)^2} \\ & \times \prod_{w\neq u,v} \frac{1}{(\mathrm{v}_w - \mathrm{v}_u)(\mathrm{v}_w - \mathrm{v}_v)} \cdot \prod_{f=1}^{N_f} \Big(-\frac{1}{\sqrt{2}}(\mathrm{v}_u + \mathrm{v}_v) + m_f \Big) \,. \end{aligned} \tag{6.63}$$

We reiterate that the product over N_f flavours is to be replaced by unity when $N_f = 0$; similarly the product over $w \neq u, v$ is to be replaced by unity when $N = 2$.

As a simple check on this answer, we observe that when $N = 2$ and $N_f > 0$ and all the masses are set to zero, this expression vanishes identically, since $\mathrm{v}_1 + \mathrm{v}_2 = 0$ by the tracelessness condition (3.98). This exactly what we expect since, as mentioned in Section 3.3.2, the $SU(2)$ models with massless hypermultiplets possess a $\mathbb{Z}_2$ symmetry which rules out contributions from odd-instantons [18, 45].

6.1.3 Discussion of the One-Instanton Result

The above expression for the one-instanton contribution to the prepotential, Eq. (6.63), is consistent with the one-instanton calculation of Ito and Sasakura [58]. As we saw in Section 4.3, the method of Ito and Sasakura is limited in two significant ways. First, it assumes that the final answer depends only on the VEV's $\{\mathrm{v}_1, \cdots, \mathrm{v}_N\}$ and not on the complex conjugate parameters $\{\bar{\mathrm{v}}_1, \cdots, \bar{\mathrm{v}}_N\}$. (This is the property of holomorphy.) Second, the method only extracts part of the full answer, namely the part that becomes maximally singular in the limit that two of the VEV's approach one another.

In the calculation described above, thanks to the collective coordinate measure (5.124), we have been able to overcome both these limitations. The reason is the intrinsic simplicity of the (super-)ADHM collective coordinate parameterization: the integration variables are all Cartesian, endowed with a flat measure save for the δ-function insertions. Consequently, we have been able to derive, rather than assume, holomorphy. For the special case of $SU(2)$, this holomorphy property is built into the instanton calculus from the outset. It emerges from a simple rescaling of the bosonic and fermionic collective coordinates in the k-instanton action [69].[3] But for $SU(N)$ with $N > 2$ no such rescaling removes the $\bar{\mathrm{v}}_u$ from the problem, and the ultimate emergence of the purely holomorphic answer (6.63) seems miraculous from the instanton approach.

For general N, the calculation of [58] fails to determine the 'regular' terms that appear in the one-instanton contribution to the prepotential when $N_f = 2N - 2$ and $N_f = 2N$. To be more precise, the method of Ito and Sasakura is insensitive to shifts in the prepotential

[3] We see explicitly how this rescaling works in Section 6.2.

of the following form:

$$N_f = 2N - 2 \quad : \quad \mathcal{F}_1 \to \mathcal{F}_1 + C_{2N-2}\,\Lambda^2 \,, \tag{6.64}$$

$$N_f = 2N \quad : \quad \mathcal{F}_1 \to \mathcal{F}_1 + C_{2N}\, e^{-8\pi^2/g^2} \sum_{u=1}^{N} \mathrm{v}_u^2 \tag{6.65}$$

where C_{2N-2} and C_{2N} are numerical constants. The form of the regular terms is clear from dimensional considerations. Our calculation demonstrates that, when $\mathcal{F}_1$ has the specified form (6.63), $C_{2N-2} = C_{2N} = 0$. This is consistent with the $N = 3$ computation of Ito and Sasakura [58].

Comparison with the Proposed Curves

We now compare Eq. (6.63) with the predictions of the hyperelliptic curves [19, 20, 21, 22, 38, 39]. The curve predictions for the one-instanton contribution to the prepotential were derived in [56, 58]. For $N_f < 2N - 2$ or $N_f = 2N - 1$, Eq. (6.63) is in perfect accord with the curves.

For $N_f = 2N - 2$, the curves proposed in [21, 22, 38, 39] give values of C_{2N-2} which differ from one another, and from the value $C_{2N-2} = 0$. The addition of a constant to the prepotential does not affect the low-energy Lagrangian (6.1) which depends only on derivatives of $\mathcal{F}$. But a constant shift does affect the quantum modulus u_2 whose k-instanton component is proportional to $\mathcal{F}_k$ via the Matone relation, Eq. (6.27). We conclude that the u_2 discrepancy discovered in the $SU(3)$ model with $N_f = 4$ flavours [57] is present in all of the $SU(N)$ models with $N_f = 2N - 2$ flavours. We can make corrections to the $SU(N)$ curves with $N > 3$ in the same way that we fixed the $SU(3)$ curves in Section 4.3; essentially, one simply has to shift the (implicit) curve parameter u_2 by an amount proportional to $C_{2N-2}\,\Lambda^2$. Only after it has been shifted in this way can the curve parameter u_2 properly be identified with the physical condensate $\langle \mathrm{Tr}\, A^2 \rangle$.

From the relation (4.90), discovered in [47], which ties the one-instanton contribution to u_3 in a model with N_f hypermultiplet flavours to the one-instanton contribution to u_2 in a model with $N_f + 1$ hypermultiplet flavours, it is apparent that the u_3 discrepancies found in the $SU(3)$ models with $N_f = 3$ and $N_f = 5$ flavours [47] are also generic. Moreover, since the curve parameterizations of [21, 22, 38, 39] are not uniquely fixed when $N_f \geq N$, we anticipate discrepancies in the predictions for all of the condensates u_n for this class of models. The discrepancies will always involve the addition of regular terms. This implies the following arithmetic. On the one hand, using dimensional analysis, such quantum shifts in u_n can only be proportional to $u_m \Lambda^{n-m}$ where $0 \leq m < n$; for an $SU(N)$ rather than a $U(N)$ theory we further require $m \neq 1$ since $u_1 \equiv 0$. On the other hand, a k-instanton effect is proportional to $\Lambda^{(2N-N_f)k}$. Consequently, equating powers of Λ, we generically expect a k-instanton additive shift to u_n when k, N and N_f satisfy

$$n - m \;=\; (2N - N_f)k \;, \quad 0 \leq m < n \;, \; m \neq 1 \;. \tag{6.66}$$

When $N_f = 2N$ the situation is more complicated: *all* instanton orders k can in principle contribute regular term shifts, as we now discuss.

The models with $N_f = 2N$ are finite theories; the β-function vanishes, and no dynamical scale is generated. Instead, the curves are functions of a dimensionless complexified coupling τ. Thus the dimensional analysis of the previous paragraph no longer applies; in order to agree with conventional definitions of condensates u_n and effective couplings $\tau_{\rm eff}$, parameters in the curves can in principle be shifted at all instanton orders, i.e, by a Taylor series in the dimensionless one-instanton parameter $q = \exp(i\pi\tau_{\rm micro})$, where $\tau_{\rm micro}$ is the renormalized coupling of the microscopic $SU(N)$ theory.

We discussed this feature in the context of the $SU(2)$ model with $N_f = 4$ flavours in Section 4.2.2. There we saw that for the exact results to make sense the parameter τ appearing in the Seiberg-Witten curve [18], should be interpreted as the effective $U(1)$ coupling evaluated in the region of moduli space where the four bare hypermultiplet masses vanish [36]. This effective coupling was denoted $\tau_{\rm eff}^{(0)}$, the superscript reminding us of this masslessness condition. The relation between the various complexified couplings reads [36]:

$$\tau_{\rm SW} \;=\; \tau_{\rm eff}^{(0)} \;=\; \tau_{\rm micro} \;+\; \frac{i}{\pi} \sum_{k=0,2,4\cdots} c_k \, q^k \;, \qquad q = \exp(i\pi\tau_{\rm micro}) \;, \tag{6.67}$$

where c_0 and c_2 have been calculated (and are nonzero) [45, 36]. A similar relation exists between $\tilde{u}$ (the parameter in the Seiberg-Witten curve) and $u_2 = \langle {\rm Tr}\, A^2 \rangle$ [36]. Importantly, the series (6.67) in no way contradicts the conformal invariance of the model, since the right-hand side is a purely numerical, scale-independent renormalization of the effective coupling [45, 36].

Now let us consider the curves for $SU(N)$ gauge theory with $N > 2$ and $N_f = 2N$. The three curve parameterizations proposed in [21, 22, 38] are ostensibly different.[4] For none of the three curves can the τ parameters be equated with τ_{micro}. This is seen explicitly at the one-instanton level: all three curves give values of C_{2N} different from one another, and different from the value $C_{2N} = 0$ [58].

What, then, is the physical meaning of the τ parameters in these curves? By analogy with the $SU(2)$ case (6.67), it is natural to guess that these τ's should be equated to some effective coupling, τ_{eff}, rather than to τ_{micro}. The trouble with such an identification is that, for $SU(N)$ with $N > 2$, the effective coupling is an $(N-1) \times (N-1)$ dimensional *matrix* rather than a scalar; furthermore, it is VEV-dependent (equivalently, u_n dependent):

$$(\tau_{\text{eff}})_{ij} \;=\; \frac{\partial^2}{\partial \tilde{\mathrm{v}}_i \, \partial \tilde{\mathrm{v}}_j} \, \mathcal{F}(\mathrm{v}) \; . \tag{6.68}$$

How then should the results of (multi-)instanton calculations enter into these curve parameters?

A potential answer to this question can be given for the special case of $SU(3)$. Here there is a distinguished line in moduli space where the 2×2 matrix τ_{eff} is effectively one-dimensional [74]. This line corresponds to setting the six bare hypermultiplet masses and the modulus u_2 equal to zero. On the distinguished line, the matrix of effective couplings is proportional to the classical form [74][5]

$$\tau_{\text{eff}} \;\propto\; \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{6.69}$$

Semiclassical calculations predict a relation between this τ_{eff} and τ_{micro} similar to the relation (6.67) that was observed in the finite $SU(2)$ model. By analogy with the $SU(2)$

[4]According to [74, 73], the $SU(3)$ curves in [22] and [38] can be transformed into one another by a modular redefinition of their respective τ parameters, but no such transformation has yet been found that equates these curves, in turn, with that of [21].

[5]In terms of the three constrained VEV's v_u, defined by Eqs. (3.97) and (3.98), the (properly normalized) classical prepotential reads $\mathcal{F} = \tau(\mathrm{v}_1^2 + \mathrm{v}_2^2 + \mathrm{v}_3^2)$. In terms of the two independent VEV's $\tilde{\mathrm{v}}_i$, defined by Eq. (6.3), this becomes $\mathcal{F} = \frac{1}{2}\tau(\tilde{\mathrm{v}}_1^2 + \tilde{\mathrm{v}}_2^2 - \tilde{\mathrm{v}}_1\tilde{\mathrm{v}}_2)$. Differentiating with respect to the $\tilde{v}_i$ twice then gives the classical form for τ_{eff} shown.

case [36], we envisage that this relation determines the meaning of the τ parameters in the proposed curves.

In contrast, for $N > 3$ with $N_f = 2N$, it can be proved that there are no points on moduli space where τ_{eff} is proportional to the classical form [74, 75]. The authors of [74] argue that the corresponding curves are underdetermined (the global symmetry requirements do not single out a unique set of modular forms). From the instanton perspective, it is not then clear how to reconcile the τ parameters in the curves of [21, 22] with the results of multi-instanton calculations, such as the explicit one-instanton expression (6.63) derived above. Without a definite interpretation of the τ parameters used in [21, 22] the meaning of the $\tau \to -1/\tau$ duality built into these curves is also unclear.

Nevertheless, we can make the following interesting observation, which might provide a clue to the eventual resolution of these issues. Consider the case of $SU(4)$ with $N_f = 8$. Let us examine the line in moduli space on which all eight hypermultiplet masses vanish and $u_2 = u_3 = 0$. At the classical level, the matrix τ_{eff} is proportional to

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{6.70}$$

by definition. It was shown in [74], that at the one-loop perturbative level, on the distinguished line in moduli space, τ_{eff} is corrected by an amount proportional to the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.71}$$

From our first-principles result (6.63) we find that on the distinguished line, the one-instanton contribution to τ_{eff} is proportional to this one-loop form. If this coincidence persist to arbitrary multi-instanton levels, then an all-instanton-orders renormalization of the coupling of the type (6.67) may be possible after all.

6.2 Instantons and Higher-Derivative Terms

In this section we specialize to $\mathcal{N} = 2$ (including $\mathcal{N} = 4$) supersymmetric $SU(2)$ Yang-Mills theory. Although the $SU(2) \simeq Sp(1)$ multi-instanton formalism [42, 45, 49] is better adapted to this case, we choose to work with the $SU(N)$ formalism that we have already described. The interested reader will find the same results obtained using the $SU(2) \simeq Sp(1)$ formalism in [69].

Following the work of Seiberg and Witten [17, 18], the SU(2) models have been the subject of extensive study. In particular, properties of the Wilsonian effective action, which describes the low-energy physics on the Coulomb branch of the theory, have been investigated. This effective action admits a (supersymmetrized) gradient expansion, the form of which is constrained both by gauge invariance and by $\mathcal{N} = 2$ supersymmetry [70]. In Section 3.3.1, we pointed out that the holomorphic prepotential $\mathcal{F}$, whose exact solution was obtained by Seiberg and Witten, determines the leading terms in this expansion, which involve up to two derivatives or four fermions. It turns out that the next-to-leading terms, involving up to four derivatives or eight fermions, are again determined by a single function, $\mathcal{H}$, which is real rather than holomorphic [70, 71].

In contrast to the prepotential, comparatively little is known in general about the function $\mathcal{H}$ [70, 71, 72, 76, 77]. However, it turns out that for the finite models exact statements can be made. In particular, Dine and Seiberg have argued that in such models, $\mathcal{H}$ is one-loop exact: the one-loop result receives corrections neither from higher orders in perturbation theory nor from nonperturbative effects such as instantons [27].

Our first task in this section will be to derive a formula for the k-instanton contribution to the function $\mathcal{H}$. We accomplish this in much the same way that we obtained the k-instanton contribution to the prepotential in Section 6.1.1. We shall illustrate our formula by calculating the one-instanton contribution to $\mathcal{H}$ in the $\mathcal{N} = 2$ pure Yang-Mills model. Then we shall use our formula to verify the Dine-Seiberg nonrenormalization theorems for $\mathcal{N} = 2$ supersymmetric $SU(2)$ QCD with $N_f = 4$ matter hypermultiplets and for the $\mathcal{N} = 4$ theory. Specifically, we show, by means of a simple rescaling argument, that all

k-instanton contributions to the next-to-leading higher-derivative terms identically vanish in these models.

6.2.1 Multi-Instanton Contribution to $\mathcal{H}$

In $\mathcal{N}=2$ superspace formalism the function $\mathcal{H}$ contributes to the Wilsonian effective action as follows [70, 71]:

$$\mathcal{L}_{\text{4-deriv}} = \int d^4\theta d^4\bar{\theta}\,\mathcal{H}(\Psi,\bar{\Psi}). \tag{6.72}$$

Here Ψ is an $\mathcal{N}=2$ chiral superfield [78], which contains the familiar $\mathcal{N}=1$ multiplets $V=(\lambda, v_m)$ and $\Phi=(A,\psi)$. A component field expansion of $\mathcal{L}_{\text{4-deriv}}$ yields the nine effective vertices[6]

$$\mathcal{L}_{\text{4-deriv}} \supset 4 \sum_{s,s'=1,2,3} \mathcal{V}^s \circ \bar{\mathcal{V}}^{s'} \circ \mathcal{H}(\mathrm{v},\bar{\mathrm{v}}). \tag{6.73}$$

where the $\mathcal{V}^s$ are the holomorphic vertices (6.6), (6.7) and (6.8) and the $\bar{\mathcal{V}}^s$ are their Hermitian conjugates (e.g. $\bar{\mathcal{V}}^2 = -\frac{i}{2\sqrt{2}}\,\bar{\psi}\bar{\sigma}^{mn}\bar{\lambda}\,v_{mn}^{\text{ASD}}\,\partial^3/\partial\bar{\mathrm{v}}^3$).[7] Associated with these vertices are the nine antiholomorphic $\times$ holomorphic Green's functions

$$\mathbf{G}^{s,s'}(x_1,\ldots,x_{s+1},y_1,\ldots,y_{s'+1}) = \langle \bar{\mathcal{O}}^s(x_1,\ldots,x_{s+1})\,\mathcal{O}^{s'}(y_1,\ldots,y_{s'+1})\rangle, \quad (s,s'=1,2,3), \tag{6.74}$$

where the $\bar{\mathcal{O}}^s$ are given by Eqs. (6.9), (6.10) and (6.11) and the $\mathcal{O}^{s'}$ are their Hermitian conjugates.

We require, in addition to the long-distance expansions of $\bar{\psi}$, $\bar{\lambda}$ and v_{mn}^{ASD} (given by Eqs. (6.14), (6.15) and (6.21) respectively), the long-distance expansions of the fields ψ, λ and v_{mn}^{SD}. These are easily derived from the k-instanton solutions for these fields, Eqs. (5.53), (5.61) and (5.13). Projecting onto the unbroken $U(1)$ direction (i.e. the τ^3 direction) and utilizing the asymptotic formulae (5.39)–(5.42), we obtain

[6] In terms of the $\mathcal{N}=1$ superspace formalism these nine vertices are all contained in the last term in Eq. (4.7) in [70], which reads $\frac{1}{4}\int d^2\theta d^2\bar{\theta}\,W^2\bar{W}^2\partial^4\mathcal{H}(\Phi,\bar{\Phi})/\partial^2\Phi\partial^2\bar{\Phi}$.

[7] Since we are specializing to models with gauge group $SU(2)$, we can clearly omit the $(N-1)$-valued index i that labels the low-energy fields. Also, in the $SU(2)$ models we can identify the VEV '$\tilde{\mathrm{v}}_1$', defined by Eq. (6.3), with the more familiar 'v', defined by Eq. (3.43).

$$\begin{aligned} v^{\rm SD,LD}_{mn}(y) &= \frac{4i}{(y-x_0)^6}\,\bar w^{\dot\alpha}_{iu}\tau^3_{uv}w_{vi\dot\beta}\,\left[(\bar y-\bar x_0)\,\sigma_{mn}\,(y-x_0)\right]^{\dot\beta}_{\ \dot\alpha}+\cdots \\ &= 2i\pi^2\,G_{mn,pq}(y,x_0)\,\bar w^{\dot\alpha}_{iu}\tau^3_{uv}w_{vi\dot\beta}\,(\bar\sigma^{pq})^{\dot\beta}_{\ \dot\alpha}+\cdots\,, \qquad (6.75)\\ \lambda^{\rm LD}_\alpha(y) &= 2i\pi^2\,S_{\alpha\dot\alpha}(y,x_0)\,(\bar w^{\dot\alpha}_{iu}\tau^3_{uv}\mu_{vi}+\bar\mu_{iu}\tau^3_{uv}w^{\dot\alpha}_{iv})+\cdots\,, \qquad (6.76)\\ \psi^{\rm LD}_\alpha(y) &= 2i\pi^2\,S_{\alpha\dot\alpha}(y,x_0)\,(\bar w^{\dot\alpha}_{iu}\tau^3_{uv}\nu_{vi}+\bar\nu_{iu}\tau^3_{uv}w^{\dot\alpha}_{iv})+\cdots\,, \qquad (6.77)\end{aligned}$$

omitting terms with a faster fall-off. The second equality in Eq. (6.75) follows from the identity (6.23).

We can now calculate, for example, the k-instanton contribution to the effective 8-fermi vertex

$$\frac{1}{16}\,\psi^2\,\lambda^2\,\bar\psi^2\,\bar\lambda^2\,\frac{\partial^4}{\partial {\rm v}^4}\,\frac{\partial^4}{\partial \bar{\rm v}^4}\,\mathcal{H}({\rm v},\bar{\rm v})\,. \qquad (6.78)$$

Inserting

$$\bar\psi^{\rm LD}_{\dot\alpha}(x_1)\,\bar\psi^{\rm LD}_{\dot\beta}(x_2)\,\bar\lambda^{\rm LD}_{\dot\gamma}(x_3)\,\bar\lambda^{\rm LD}_{\dot\delta}(x_4)\,\psi^{\rm LD}_{\alpha}(y_1)\,\psi^{\rm LD}_{\beta}(y_2)\,\lambda^{\rm LD}_{\gamma}(y_3)\,\lambda^{\rm LD}_{\delta}(y_4) \qquad (6.79)$$

into the collective coordinate integration and performing the ξ_i integrals leaves

$$\begin{aligned} &\int d^4x_0\,\epsilon^{\kappa\lambda}S_{\kappa\dot\alpha}(x_1,x_0)S_{\lambda\dot\beta}(x_2,x_0)\,\epsilon^{\rho\sigma}S_{\rho\dot\gamma}(x_3,x_0)S_{\sigma\dot\delta}(x_4,x_0)\\ &\quad\times\,\epsilon^{\dot\kappa\dot\lambda}S_{\alpha\dot\kappa}(y_1,x_0)S_{\beta\dot\lambda}(y_2,x_0)\,\epsilon^{\dot\rho\dot\sigma}S_{\gamma\dot\rho}(y_3,x_0)S_{\delta\dot\sigma}(y_4,x_0)\\ &\quad\times\,\frac{(2i\pi^2)^4}{4}\,\frac{\partial^4}{\partial{\rm v}^4}\int d\tilde\mu^{(k)}_{\rm phys}\,(\bar\Upsilon_{1\dot\alpha}\bar\Upsilon^{\dot\alpha}_1)(\bar\Upsilon_{2\dot\beta}\bar\Upsilon^{\dot\beta}_2)\,\exp(-S^{k\text{-inst}}_{SQCD}), \qquad (6.80)\end{aligned}$$

where

$$\begin{aligned} \bar\Upsilon^{\dot\alpha}_1 &= (\bar w^{\dot\alpha}_{iu}\tau^3_{uv}\mu_{vi}+\bar\mu_{iu}\tau^3_{uv}w^{\dot\alpha}_{iv}), \qquad (6.81)\\ \bar\Upsilon^{\dot\alpha}_2 &= (\bar w^{\dot\alpha}_{iu}\tau^3_{uv}\nu_{vi}+\bar\nu_{iu}\tau^3_{uv}w^{\dot\alpha}_{iv}). \qquad (6.82)\end{aligned}$$

(These are just the Grassmann-spinor-valued combinations of collective coordinates that appear in the long-distance expressions (6.76) and (6.77).)

In Eq. (6.80) we recognize the Feynman amplitude associated with the vertex $\psi^2\lambda^2\bar\psi^2\bar\lambda^2$, with an effective coupling given by the last line in (6.80). By comparison with (6.78) we deduce the following expression for the k-instanton contribution to $\mathcal{H}$, valid to leading order in the semiclassical expansion:

$$\frac{\partial^4}{\partial\bar{\rm v}^4}\,\mathcal{H}({\rm v},\bar{\rm v})\Big|_{k\text{-inst}} = 4\pi^8\int d\tilde\mu^{(k)}_{\rm phys}\,(\bar\Upsilon_1)^2(\bar\Upsilon_2)^2\,\exp(-S^{k\text{-inst}}_{SQCD}), \qquad (6.83)$$

This is the analog of Eq. (6.20) for the prepotential. Somewhat different (although necessarily consistent) expressions for $\partial^2\mathcal{H}/\partial\bar{\mathrm{v}}^2$ and $\partial^3\mathcal{H}/\partial\bar{\mathrm{v}}^3$ can be derived in the same way, by examining the Green's functions $\mathbf{G}^{k,1}$ and $\mathbf{G}^{k,2}$, respectively. By exchanging v and $\bar{\mathrm{v}}$ in Eq. (6.83) we obtain the contributions from anti-instantons of topological charge $-k$. There may also in general be mixed (k_+)-instanton, (k_-)-anti-instanton contributions to $\mathcal{H}$ (holomorphy prohibits such contributions to the prepotential), but these lie beyond the scope of our analysis.

As a simple illustration, we now calculate the one-instanton contribution to $\mathcal{H}$ in the case of the $\mathcal{N}=2$ supersymmetric pure Yang-Mills theory. The instanton action (5.100) reduces to:

$$S^{k\text{-inst}}_{SYM} = \frac{8\pi^2}{g^2} + 2\pi^2|\mathrm{v}|^2\bar{w}_u^{\dot\alpha} w_{u\dot\alpha} + \sqrt{2}\,\pi^2\bar{\mathrm{v}}\big(\bar{\mu}_u\tau^3_{uv}\nu_v - \bar{\nu}_u\tau^3_{uv}\mu_v\big), \tag{6.84}$$

where $|\mathrm{v}| = \sqrt{\mathrm{v}\bar{\mathrm{v}}}$. In fact, we can ignore the fermionic terms in this action because the eight fermionic field insertions completely saturate the Grassmann integrations associated with the four superconformal zero-modes as well as those associated with the four supersymmetric zero-modes. Moreover, it is not hard to show, after making the identification $w_{u\dot\alpha} = \rho u$, with $u \in SU(2)$ (see Section 5.2.3), and with the aid of the fermionic constraints (5.55) and (5.62), that $(\bar{\Upsilon}_1)^2 = 4\rho^2\bar{\mu}_u\mu_u$ and similarly $(\bar{\Upsilon}_2)^2 = 4\rho^2\bar{\nu}_u\nu_u$. The collective coordinate integration is easily performed using the 't Hooft-Bernard measure (2.64) and using the fact that the Grassmann parameters $\bar{\xi}_{SC\,1,2} = 2\pi\mu_{1,2}$ and $\bar{\zeta}_{SC\,1,2} = 2\pi\mu_{1,2}$ correspond to normalized (superconformal) zero-modes. The result is[8]

$$\mathcal{H}(\mathrm{v},\bar{\mathrm{v}})\Big|_{\text{1-inst}} = -\frac{1}{8\pi^2}\frac{\Lambda^4}{\mathrm{v}^4}\log\bar{\mathrm{v}}\ . \tag{6.85}$$

This result agrees with an instanton calculation by Yung [77], which used a very different approach. In their early investigations of instanton effects in QCD, Callan, Dash and Gross found that it was possible to describe instanton effects by adding an effective vertex to the tree level Lagrangian [79]. In the model at hand, Yung employed the constraints of $\mathcal{N}=2$ supersymmetry to construct a (one-instanton) effective vertex of this type. It was then possible to directly extract one-instanton contributions to both leading and higher-derivative terms in the Wilsonian effective action.

[8] In the component field expansion of $\mathcal{L}_{4\text{-deriv}}$, only mixed derivatives of $\mathcal{H}$ with respect to v and $\bar{\mathrm{v}}$ appear. As a consequence, the function $\mathcal{H}$ can be written in a variety of equivalent ways.

Note that for $N_f > 0$, the first non-vanishing contribution to $\mathcal{H}$ is at the two-instanton level, due to the same $\mathbb{Z}_2$ symmetry that forbids all odd-instanton contributions to the prepotential [17, 18, 45].

6.2.2 Nonperturbative Nonrenormalization Theorems

The Finite $\mathcal{N} = 2$ $SU(2)$ Model

We now prove that all k-instanton contributions to $\mathcal{H}$ vanish in $\mathcal{N} = 2$ supersymmetric $SU(2)$ QCD with $N_f = 4$ massless matter hypermultiplets. As was originally observed in [42, 45], when the hypermultiplets have zero mass, the k-instanton action associated with the $SU(2)$ models has the following property: all dependence on v and $\bar{\mathrm{v}}$ can be eliminated by performing the collective coordinate rescaling

$$\begin{aligned} a \quad &\to \quad a/|\mathrm{v}| \,, \\ \mathcal{M} \to \mathcal{M}/\sqrt{\bar{\mathrm{v}}} \,, \qquad & \mathcal{N} \to \mathcal{N}/\sqrt{\bar{\mathrm{v}}} \,, \\ \mathcal{K} \to \mathcal{K}/\sqrt{\mathrm{v}} \,, \qquad & \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}/\sqrt{\mathrm{v}} \,. \end{aligned} \tag{6.86}$$

From Eqs. (5.124) and (5.123) we find that the effect of this rescaling on the reduced collective coordinate integration measure is:

$$d\tilde{\mu}^{(k)}_{\rm phys} \;\to\; |\mathrm{v}|^{4-8k}(\sqrt{\bar{\mathrm{v}}}\,)^{8k-4}(\sqrt{\mathrm{v}}\,)^{2kN_f}\cdot d\tilde{\mu}^{(k)}_{\rm phys} \;=\; \mathrm{v}^{2-(4-N_f)k}\cdot d\tilde{\mu}^{(k)}_{\rm phys}. \tag{6.87}$$

From the k-instanton formulae (6.83) and (6.20) we deduce that

$$\mathcal{H}(\mathrm{v},\bar{\mathrm{v}})\Big|_{k\text{-inst}} \sim \frac{\log\bar{\mathrm{v}}}{\mathrm{v}^{(4-N_f)k}}\,, \qquad \mathcal{F}(\mathrm{v})\Big|_{k\text{-inst}} \sim \frac{1}{\mathrm{v}^{(4-N_f)k-2}} \tag{6.88}$$

Setting $N_f = 4$, we find that $\mathcal{H}\,|_{k\text{-inst}} \sim \log\bar{\mathrm{v}}$. Consequently, the effective component vertices contained in $\mathcal{L}_{4\text{-deriv}}$ (all of which involve differentiating $\mathcal{H}$ with respect to both v and $\bar{\mathrm{v}}$) automatically vanish. We likewise conclude that anti-instanton contributions have no physical effect on $\mathcal{H}$. (Anti-instanton contributions are obtained simply by exchanging v and $\bar{\mathrm{v}}$.) This confirms the nonperturbative nonrenormalization theorem of Dine and Seiberg in this model.

Note that the scaling argument correctly predicts that in the $N_f = 4$ model one has $\mathcal{F}\,|_{k\text{-inst}} \sim \mathrm{v}^2$. This does not rule out instanton effects and, as we have seen, the prepotential *does* receive contributions from all (even)-instantons. Note further that giving any of the hypermultiplets a mass spoils the scaling argument, since m_f rescales to m_f/v, and this rescaled mass can be pulled down from the exponent.

The $\mathcal{N} = 4$ $SU(2)$ Model

Next we consider the $\mathcal{N} = 4$ theory, described in Section 3.2.2. The k-instanton action and measure for this model have been constructed, using the '$SU(2) \simeq Sp(1)$' formalism, in [46, 49]. It suffices to know only a few general properties of the action and measure, and it is straightforward to phrase these properties in terms of the $SU(N)$ formalism of Chapter 5.

The $\mathcal{N} = 4$ model is constructed by coupling $\mathcal{N} = 2$ supersymmetric pure Yang-Mills theory to a pair of chiral superfields transforming in the *adjoint* representation (see Section 3.2.2). Associated with these superfields are two adjoint Weyl fermions, χ and $\tilde{\chi}$. Accordingly, the the $\mathcal{N} = 2$ superinstanton background (described in Section 5.3) is supplemented by $2Nk + 2Nk$ new fermion zero-modes; these have the same form as the solutions for the gaugino and Higgsino zero-modes (5.53) and (5.61), but in place of $\mathcal{M}$ and $\mathcal{N}$ one has new collective coordinate matrices $\mathcal{R}$ and $\tilde{\mathcal{R}}$.

After spontaneous symmetry breakdown of the model, the low-energy dynamics involves a set of massless fields corresponding to a single $\mathcal{N} = 4$ $U(1)$ multiplet. Concomitantly, the instanton action is independent of four additional Grassmann collective coordinates [46]: the 'trace' components ξ_3 and ξ_4 of the matrices $\mathcal{R}'$ and $\tilde{\mathcal{R}}'$, defined as per Eq. (6.18). These components correspond to four new supersymmetric zero-modes.

An appropriate Green's function to consider, whose field insertions saturate all eight ξ_i integrals, is given by

$$\mathbf{G}^8(x_1,\ldots,x_8) = \langle \bar{\psi}_{\dot\alpha}(x_1)\,\bar{\psi}_{\dot\beta}(x_2)\,\bar{\lambda}_{\dot\gamma}(x_3)\,\bar{\lambda}_{\dot\delta}(x_4)\,\bar{\chi}_{\dot\kappa}(x_5)\,\bar{\chi}_{\dot\lambda}(x_6)\,\bar{\tilde{\chi}}_{\dot\rho}(x_7)\,\bar{\tilde{\chi}}_{\dot\sigma}(x_8)\rangle\ . \tag{6.89}$$

Using the methods described above, we deduce that the associated effective coupling can be written

$$\int d^4x_0\, d^2\xi_1 d^2\xi_2 d^2\xi_3 d^2\xi_4\, \mathcal{B}_k(\mathrm{v},\bar{\mathrm{v}}) \ , \tag{6.90}$$

where $\mathcal{B}_k$ is the k-instanton contribution to what one might call the 'anteprepotential' in analogy to Eq. (6.20):

$$\mathcal{B}_k(\mathrm{v},\bar{\mathrm{v}}) \ = \ \int d\tilde{\mu}^{(k)}_{\text{phys}} \exp(-S^{k\text{-inst}}_{\mathcal{N}=4}) \ . \tag{6.91}$$

Here $d\tilde{\mu}^{(k)}_{\text{phys}}$ is the reduced $\mathcal{N}=4$ integration measure, which excludes the $\mathcal{N}=4$ superspace position variables $(x_0,\xi_1,\xi_2,\xi_3,\xi_4)$.

Just as for the massless $\mathcal{N}=2$ models, we can remove all VEV dependence from the $SU(2)$ instanton action by making the rescalings (6.86) and also [46]

$$\mathcal{R}\to\mathcal{R}/\sqrt{\mathrm{v}}\ , \qquad \tilde{\mathcal{R}}\to\tilde{\mathcal{R}}/\sqrt{\bar{\mathrm{v}}}\ . \tag{6.92}$$

The effect on the measure is as follows:

$$d\tilde{\mu}^{(k)}_{\text{phys}} \ \to \ |\mathrm{v}|^{4-8k}(\sqrt{\bar{\mathrm{v}}}\,)^{8k-4}(\sqrt{\mathrm{v}}\,)^{8k-4}\cdot d\tilde{\mu}^{(k)}_{\text{phys}} \ = \ d\tilde{\mu}^{(k)}_{\text{phys}} \ . \tag{6.93}$$

Hence $\mathcal{B}_k(\mathrm{v},\bar{\mathrm{v}})$ is a constant, independent of v and $\bar{\mathrm{v}}$.

The action $S^{k\text{-inst}}_{\mathcal{N}=4}$ also possesses the discrete symmetry $\{\mathcal{M},\mathcal{N},\mathrm{v}\}\leftrightarrow\{\mathcal{R},\tilde{\mathcal{R}},\bar{\mathrm{v}}\}$ [46]. This symmetry, together with the long-distance expressions (6.14) and (6.15) implies that

$$\bar{\chi}^{\text{LD}}_{\dot\alpha}(x_i) \ = \ i\sqrt{2}\,\xi^{3\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial\bar{\mathrm{v}}} \ + \ \cdots \tag{6.94}$$

$$\bar{\chi}^{\text{LD}}_{\dot\alpha}(x_i) \ = \ -i\sqrt{2}\,\xi^{4\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial\bar{\mathrm{v}}} \ + \ \cdots \tag{6.95}$$

Using the long-distance expressions (6.14), (6.15), (6.94) and (6.95) to obtain a k-instanton expression for $\mathbf{G}^8$ and comparing with the expression (6.91) for $\mathcal{B}_k$, we deduce that

$$\mathbf{G}^8\big|_{k\text{-inst}} \propto \frac{\partial^8\mathcal{B}_k}{\partial\mathrm{v}^4\partial\bar{\mathrm{v}}^4}. \tag{6.96}$$

Thus all (multi-)instanton contributions to $\mathbf{G}^8$ vanish, which confirms the prediction of Dine and Seiberg for the $\mathcal{N}=4$ theory.

6.3 Summary

In this chapter we have employed the instanton calculus developed in Chapter 5 in several ways to investigate instanton effects in supersymmetric $SU(N)$ Yang-Mills theory.

Following the $SU(2)$ analysis of [50], we have derived a closed form expression for the k-instanton contribution to the prepotential in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD, as a definite integral over the bosonic and fermionic collective coordinates of the instanton configuration. This is a solution, in quadratures, of the low-energy dynamics of the Coulomb branches of the $\mathcal{N}=2$ supersymmetric $SU(N)$ models. It was extracted directly from the k-instanton expressions for three specific Green's functions of the low-energy theory, using the solutions for the long-distance field insertions obtained in [42, 45].

We have also seen that the $SU(2)$ multi-instanton proof of the Matone relation [51, 52, 55] presented in [50] (see also [54]) straightforwardly generalizes to $SU(N)$.

We have evaluated the one-instanton contribution to the prepotential in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD with an arbitrary number of matter hypermultiplets. This calculation was accomplished using the collective coordinate measure of Section 5.6. We found that, after exponentiating the δ-function constraints in the measure, the collective coordinate integrations are Gaussian and can be straightforwardly carried out. The difficulty associated with groups space collective coordinate integration that was encountered in Section 5.6 was completely avoided.

Comparing our complete one-instanton result with the predictions of the proposed hyperelliptic curves [21, 22, 38], we have found discrepancies for $N_f = 2N-2$ and $N_f = 2N$ flavours of matter hypermultiplets. For the case of $N_f = 2N-2$ flavours, the curve predictions for the one-instanton contribution to the condensate $u_2 = \langle \mathrm{Tr}(A^2) \rangle$ differ from our result by a constant regular term. So the one-instanton discrepancy discovered by Ito and Sasakura [58] in the $SU(3)$ model with $N_f = 4$ flavours is generic. More generally, we expect that when $N_f \geq N$ the curve predictions for the instanton contributions to all of the condensates $u_n = \langle \mathrm{Tr}(A^n) \rangle$, $n = 1, 2, \ldots, N-1$, are correct only up to regular term shifts; this reflects an ambiguity in the curve parameterization for this class of models. For $N_f < 2N$

the curves can be fixed by making shifts in the quantum moduli u_n that parameterize the curves.

For the finite $N \geq 3$ models with $N_f = 2N$ flavours, the disagreement between the curve predictions and the instanton calculus is more difficult to interpret. In analogy with the $SU(2)$ case, one expects that to resolve the discrepancy, the curve parameter τ should be identified with the effective coupling of the low-energy theory on a particular complex line in moduli space. However, we cannot directly make this identification, since the parameter τ is a scalar, whereas the effective coupling is an $(N-1) \times (N-1)$ matrix. For the special case $N = 3$, it turns out that the effective coupling matrix on the complex line is proportional to its classical form [74]. The proportionality factor gives an effective scalar coupling which can naturally be identified with the curve parameter. But for $N > 3$, the effective coupling on the conformally invariant line is not proportional to its classical form [74], so we do not have a natural interpretation of the parameter τ appearing in the curves.

In the same way that we derived a closed form expression for the k-instanton contribution to the prepotential, we have derived a closed form expression for the k-instanton contribution to the real function $\mathcal{H}$, which determines the next-to-leading terms in the Wilsonian effective action in $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory. We considered Green's functions which required, in addition to the field insertions associated with the long-distance antifermion fields and the anti-self-dual field strength, insertions associated with the long-distance fermion fields and self-dual field strength; these were easily extracted from the solutions of Chapter 5. For the $\mathcal{N} = 2$ supersymmetric QCD model with $N_f = 4$ flavours of massless hypermultiplets, we have used a general scaling property of the instanton action and measure to show that all multi-instanton contributions to the higher-derivatives terms vanish, thus confirming the nonrenormalization theorem of Dine and Seiberg [27]. Using the same scaling argument, we have also verified the Dine-Seiberg renormalization theorem in the $\mathcal{N} = 4$ supersymmetric $SU(2)$ model.

Chapter 7

Conclusions

In this thesis we have investigated instanton effects in supersymmetric $SU(N)$ Yang-Mills theory. In particular, we have studied instanton effects in the $\mathcal{N}=2$ supersymmetric models with $N>2$. We have presented two explicit one-instanton calculations that have provided tests of the exact predictions in these models.

In Chapter 4, we presented a one-instanton calculation of the condensate $u_3 = \langle \mathrm{Tr} A^3 \rangle$. In this calculation we faced a $(4N-5)$-dimensional integral over group space collective coordinates which we were not able to solve in the general case. We were nonetheless able to extract a 'maximally singular' part of the complete one-instanton contribution to u_3 by following the analysis of Ito and Sasakura. For the models with $N_f < 2N-2$ or $N_f = 2N-3$ flavours of fundamental matter hypermultiplets, dimensional considerations show that there are no additional 'regular' terms so that the singular contribution represents the complete contribution. In these models, we found that the predictions of the hyperelliptic curves completely agreed with the instanton calculation.

For the particular case of $SU(3)$ we were able to fully determine the one-instanton contribution to u_3, for all $N_f < 6$. We found that none of the proposed hyperelliptic curves predict the correct values for the regular terms which appear when $N_f = 3$ or $N_f = 5$. These discrepancies are similar in nature to the discrepancy found in the $SU(2)$ model with $N_f = 3$ flavours. They reflect an ambiguity in the parameterization of the hyperelliptic curves when $N_f \geq N$; we expect that the predictions of the $N_f \geq N$ curves for all the condensates $u_n = \langle \mathrm{Tr}(A^n) \rangle$, $n = 1, 2, \ldots, N-1$, are correct only up to regular terms. The ambiguity can easily be fixed by reinterpreting the parameters appearing in the curves in accordance with the instanton predictions. (Essentially one has to make regular term shifts in the moduli u_n.) We demonstrated this explicitly in the $SU(3)$ case.

In Chapter 6 we presented a complete calculation of the one-instanton contribution to the prepotential in $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD with an arbitrary number of matter hypermultiplets. This calculation was accomplished using the formalism of the multi-instanton calculus presented in Chapter 5. In particular, we made use of the the collective coordinate measure of Section 5.6. We found that, after exponentiating the δ-function constraints in the measure, the collective coordinate integrations are Gaussian and could be straightforwardly carried out. The difficulty associated with group space collective coordinate integration that one encounters using the standard 't Hooft-Bernard measure was completely avoided.

Comparing our complete one-instanton result with the predictions of the proposed hyperelliptic curves, we observed discrepancies for $N_f = 2N - 2$ and $N_f = 2N$ flavours of matter hypermultiplets. For the case of $N_f = 2N - 2$ flavours, the curve predictions for the one-instanton contribution to the condensate $u_2 = \langle \mathrm{Tr}(A^2) \rangle$ differ from our result by a constant regular term. This result is not unexpected; it reflects the general parameterization ambiguity associated with the moduli u_n when $N_f \geq N$. For the finite models with $N_f = 2N$ flavours, the predictions of the curves for the one-instanton contribution to the prepotential differ from our calculated expression by a regular term that is quadratic in the VEV's. It is unclear how to interpret this discrepancy. The similar discrepancy found in the $SU(2)$ model with four fundamental fermions was resolved by identifying the parameter τ appearing in the Seiberg-Witten curve with the effective coupling of the massless low-energy theory. However, for general $N > 2$, the effective coupling of the low-energy theory is an $(N-1) \times (N-1)$ matrix, in contrast to the parameter appearing in the hyperelliptic curves, which is a scalar. For the special case $N = 3$, it turns out that there is a complex line in moduli space on which the low-energy effective coupling matrix is proportional to its classical form. Consequently, it is possible to identify the curve parameter τ with the associated proportionality factor. However, for $N > 3$ there is no region of moduli space where the effective coupling is proportional to its classical form, so we do not have a natural interpretation of the parameter τ.

In conclusion, our one-instanton calculations have provided important tests of the exact solutions predicted for $\mathcal{N}=2$ supersymmetric $SU(N)$ Yang-Mills theory. In most cases, our calculations completely agree with the exact results. For $N_f \geq N$ we have detected minor discrepancies associated with the condensates u_n; these can easily be resolved by reinterpreting the corresponding parameters in the hyperelliptic curves. For the finite $N_f = 2N$ models there is a more serious discrepancy associated with the prepotential and in the general case it is unclear how this discrepancy can be resolved.

An important part of this thesis was devoted to the construction of a multi-instanton calculus for supersymmetric $U(N)$ or $SU(N)$ Yang-Mills theory. The calculus is based on the multi-instanton solution of Atiyah, Drinfeld, Hitchin and Manin and naturally incorporates supersymmetry. Following the $SU(2)$ analysis of Dorey, Khoze and Mattis, we obtained the solutions comprising the super-multi-instanton background and derived the associated action and collective coordinate measure for $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetric models.

Our calculus has enabled us not only to perform a complete one-instanton calculation in $\mathcal{N}=2$ supersymmetric $SU(N)$ Yang-Mills theory, but also to obtain results at arbitrary k-instanton levels. In Chapter 6 we presented a closed form expression for the k-instanton contribution to the prepotential in $\mathcal{N}=2$ supersymmetric $SU(N)$ QCD, as a definite integral over the bosonic and fermionic collective coordinates of the instanton configuration. This expression represents a solution, in quadratures, of the low-energy physics on the Coulomb branches of the $\mathcal{N}=2$ supersymmetric $SU(N)$ models. We were also able to verify the $SU(N)$ version of the Matone relation, at all k-instanton levels, by a simple generalization of the analysis of Dorey, Khoze and Mattis.

In a separate investigation in Chapter 6, we employed our calculus to investigate higher-derivative terms in the Wilsonian effective actions of supersymmetric $SU(2)$ models. We derived a closed form expression for the k-instanton contribution to the real function $\mathcal{H}$, which determines the next-to-leading terms in the Wilsonian effective action in $\mathcal{N}=2$ supersymmetric $SU(2)$ Yang-Mills theory. Using a scaling property of the $SU(2)$ multi-instanton action and measure we showed that, in the finite $SU(2)$ models, all multi-instanton contributions to the higher-derivatives terms vanish. This result confirms the nonperturbative nonrenormalization theorem of Dine and Seiberg.

There is much scope for further research into instanton effects in supersymmetric $SU(N)$ Yang-Mills theory. It would desirable to extend the instanton tests of the exact results to higher multi-instanton levels. We would particularly like to perform further calculations in those models in which discrepancies have been observed. More generally, we envisage employing our multi-instanton calculus, whose development was stimulated by the exact results in $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory, to investigate models for which exact solutions have not been predicted.

Appendix A

Conventions

Throughout Chapters 3, 5 and 6 we work in Minkowski space and utilize the conventions of Wess and Bagger [25]. In particular, this means that the metric is

$$\eta_{mn} = \mathrm{diag}(-1, 1, 1, 1). \tag{A.1}$$

$SU(N)$ Yang-Mills Theory

We choose a Hermitian basis of generators T^a, that satisfy

$$\begin{aligned} \left[T^a, T^b\right] &= i f_{abc} T^c, && \text{(A.2)} \\ \mathrm{Tr}\left(T^a T^b\right) &= \frac{1}{2}\delta^{ab}, && \text{(A.3)} \\ & && \text{(A.4)} \end{aligned}$$

where $a = 1, \ldots, N^2 - 1$. For the first three values of a, it is convenient to specify

$$T^a = \begin{pmatrix} \frac{1}{2}\tau^a & 0 \\ 0 & 0 \end{pmatrix} \qquad (a = 1, 2, 3), \tag{A.5}$$

where the τ^a are the Pauli matrices.

The gauge field is denoted by

$$v_m = v_m^a T^a. \tag{A.6}$$

In the fundamental representation, the covariant derivative reads

$$D_m = \partial_m - igv_m. \tag{A.7}$$

This entails a field strength

$$v_{mn} = \frac{i}{g}[D_m, D_n] = \partial_m v_n - \partial_n v_m - ig[v_m, v_n]. \tag{A.8}$$

Spinor Conventions

A left-handed Weyl spinor, transforming under the $(\frac{1}{2}, 0)$ representation of the Lorentz group, is given by ψ_α. Its conjugate $\bar{\psi}^{\dot{\alpha}}$ is a right-handed Weyl spinor, transforming under the $(0, \frac{1}{2})$ representation. The indices on these objects may be raised or lowered using the antisymmetric tensors

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.9}$$

We define the contraction of two Weyl spinors according to the rules [25]

$$\chi\psi = \chi^\alpha \psi_\alpha, \qquad \bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}. \tag{A.10}$$

The massless Dirac equations for Weyl spinors are

$$\bar{\not{D}}\psi = \bar{\sigma}^m D_m \psi = 0, \qquad \not{D}\bar{\psi} = \sigma^m D_m \bar{\psi} = 0, \tag{A.11}$$

where [25]

$$\sigma^m_{\alpha\dot{\alpha}} = \left(-\mathbf{1}, \tau^i\right), \qquad \bar{\sigma}^{m\,\dot{\alpha}\alpha} = \left(-\mathbf{1}, -\tau^i\right). \tag{A.12}$$

The sigma matrices are related by

$$\sigma^m_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}^{m\,\dot{\beta}\beta}. \tag{A.13}$$

We can form a Dirac spinor, Ψ, from two Weyl spinors, ψ_α and $\bar{\chi}^{\dot{\alpha}}$, by writing

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.14}$$

The γ-matrices are then represented as

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix}. \tag{A.15}$$

We refer the reader to Appendix A of [25] for many useful identities involving the spinor objects defined above.

Continuation to Euclidean Space

In Chapters 2 and 4 we work in Euclidean space. Here we present our conventions for the continuation from Minkowski space to Euclidean space. Note that to label tensor indices in Euclidean space we use the Greek symbols μ, ν, etc., in place of the letters m, n, etc., that are used in Minkowski space. The Euclidean space indices run from one to four instead of from zero to three. Also, in Euclidean space we denote the gauge potential by the standard 'A_μ', in place of the Minkowski space 'v_m' of Wess and Bagger.

First, we convert to the Minkowski metric $\eta_{mn} = \text{diag}(1,-1,-1,-1)$ by multiplying each tensor object by $(-1)^p$ where p is the number of raised Lorentz indices. Our continuation from Minkowski space to Euclidean space now closely follows the procedure given in [7]. We rotate the time coordinate and its derivative according to

$$x^0 \to -ix^4, \qquad \partial_0 \to i\partial_4. \tag{A.16}$$

The space coordinates x^i $(i = 1,2,3)$ are identical in Minkowski and Euclidean space. To continue the gauge potential v_m, we write

$$v_0 \to iA_4, \qquad v_i \to A_i. \tag{A.17}$$

We continue the covariant derivative and the field strength according to

$$D_0 \to iD_4, \qquad D_i \to D_i, \tag{A.18}$$

$$v_{0i} \to iF_{4i}, \qquad v_{ij} \to F_{ij}. \tag{A.19}$$

This ensures that their usual forms are preserved,

$$D_\mu = \partial_\mu - igA_\mu, \quad (A.20)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \quad (A.21)$$

To continue fermionic terms in the action, we define Euclidean space γ-matrices as follows,

$$\hat{\gamma}_4 = \gamma_0, \quad (A.22)$$

$$\hat{\gamma}_i = i\gamma_i. \quad (A.23)$$

The $\hat{\gamma}_\mu$ satisfy a Euclidean space Clifford algebra,

$$\{\hat{\gamma}_\mu, \hat{\gamma}_\nu\} = 2\delta_{\mu\nu}. \quad (A.24)$$

From the Weyl representation of the the γ-matrices, Eq. (A.15), we deduce that

$$\sigma_{m\,\alpha\dot{\alpha}} \rightarrow e_{\mu\,\alpha\dot{\alpha}} = \left(i\tau^i, \mathbf{1}\right), \quad (A.25)$$

$$\bar{\sigma}_m^{\dot{\alpha}\alpha} \rightarrow \bar{e}_\mu^{\dot{\alpha}\alpha} = \left(-i\tau^i, \mathbf{1}\right). \quad (A.26)$$

Notation of 't Hooft [3]

The matrices e_μ and $\bar{e}_\mu$ given in Eqs. (A.25) and (A.26) can be used to define 't Hooft's η-symbols,

$$e_\mu \bar{e}_\nu = \delta_{\mu\nu} + i\eta^a_{\mu\nu}\tau^a, \quad (A.27)$$

$$\bar{e}_\mu e_\nu = \delta_{\mu\nu} + i\bar{\eta}^a_{\mu\nu}\tau^a. \quad (A.28)$$

These objects are respectively self-dual and anti-self-dual,

$$\eta^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\eta^a_{\rho\lambda}, \quad (A.29)$$

$$\bar{\eta}^a_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}\bar{\eta}^a_{\rho\lambda}, \quad (A.30)$$

where $\epsilon_{1234} = 1$.

The one-instanton small-fluctuations determinants derived by 't Hooft contain the following functions of isospin:

$$C(t) = \frac{2}{3}t(t+1)(2t+1), \tag{A.31}$$

$$\alpha(t) = C(t)\left(2R - \frac{1}{6}\ln 2 - \frac{1}{9} - \frac{1}{6}t(t+1) + \frac{1}{2}\sum_{s=1}^{2t+1} s(2t+1-s)\left(s-t-\frac{1}{2}\right)\ln s\right), \tag{A.32}$$

where

$$R = \frac{\ln 2\pi + \gamma_E}{12} + \frac{1}{2\pi^2}\sum_{s=1}^{\infty}\frac{\ln s}{s^2} \approx 0.249. \tag{A.33}$$

Appendix B

Proof of the Topological Charge Formula

In this appendix we prove the topological charge formula,

$$k = \frac{g^2}{16\pi^2} \int d^4x \mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right). \tag{B.1}$$

We take the gauge group to be $SU(2)$, although the formula (B.1) holds in the general case, provided the group generators are normalized appropriately. (Throughout this thesis our conventions ensure this for the group $SU(N)$.)

The first step consists of rewriting the integral over space-time as an integral over the 3-sphere at infinity. We define a current

$$K_\mu = \frac{g^2}{8\pi^2}\epsilon_{\mu\nu\rho\lambda}\mathrm{Tr}\left(A_\nu\partial_\rho A_\lambda - \frac{2ig}{3}A_\nu A_\rho A_\lambda\right), \tag{B.2}$$

whose divergence identically satisfies

$$\partial_\mu K_\mu = \frac{g^2}{16\pi^2}\mathrm{Tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right). \tag{B.3}$$

It follows from Stokes' theorem that

$$k = \oint_{S^3_\infty} dS_\mu K_\mu. \tag{B.4}$$

Next we input information about the large-distance behaviour of finite action field configurations. In the large-distance limit, the field strength vanishes (see Eq. (2.2)), so that

$$\epsilon_{\mu\nu\rho\lambda}\partial_\rho A_\lambda = ig\epsilon_{\mu\nu\rho\lambda}A_\rho A_\lambda. \tag{B.5}$$

Consequently, we have

$$\lim_{|x|\to\infty} K_\mu = \frac{ig^3}{24\pi^2}\epsilon\mu\nu\rho\lambda\mathrm{Tr}\left(A_\nu A_\rho A_\lambda\right). \tag{B.6}$$

Using the asymptotic form of the gauge potential, Eq. (2.3), we can now write (B.4) as

$$k = \frac{1}{24\pi^2}\oint_{S^3_\infty} dS_\mu \epsilon_{\mu\nu\rho\lambda}\mathrm{Tr}\left(U\partial_\nu U^{-1}U\partial_\rho U^{-1}U\partial_\lambda U^{-1}\right). \tag{B.7}$$

It is convenient to deform the integration surface into a large hypercube, with faces at $x_i = \pm\infty$, so that (B.7) can be expressed in terms of Cartesian coordinates. We obtain

$$k = \frac{1}{24\pi^2}\int_{x_4=+\infty} dx_1 dx_2 dx_3 \epsilon_{lmn}\mathrm{Tr}\left(U\partial_l U^{-1}U\partial_m U^{-1}U\partial_n U^{-1}\right) + \ldots, \tag{B.8}$$

where the dots represent similar contributions from the other seven faces of the hypercube. Now let us define parameters (ξ_1, ξ_2, ξ_3) which act as coordinates for the matrix U in group space. The Cartesian coordinates on the surface of the hypercube implicitly depend on these parameters. Upon changing integration variables we get

$$k = \frac{k}{24\pi^2}\int_{SU(2)} d\xi_1 d\xi_2 d\xi_3 \epsilon_{lmn}\mathrm{Tr}\left(U\frac{\partial}{\partial\xi_l}U^{-1}U\frac{\partial}{\partial\xi_m}U^{-1}U\frac{\partial}{\partial\xi_n}U^{-1}\right). \tag{B.9}$$

The form of the integral is invariant under the change of variables because the Jacobian determinant cancels with the determinant that appears when the Cartesian derivatives ∂_i are written in terms of the $\partial/\partial\xi_i$. However, there is a significant difference between the integrals (B.8) and (B.9). In (B.8), if we integrate over the hypersurface once then the $SU(2)$ group space is covered precisely k times. On the other hand, in (B.9), the integration variables are in one to one correspondence with the group matrices U. The appearance of the factor k in front of the integral (B.9) is precisely what is needed to account for this difference when we change from the x_i to the ξ_i variables.

The integral over the ξ_i in Eq. (B.9) is in fact an invariant measure taken over group space. It gives a constant factor representing the volume of the group $SU(2)$. To complete the proof of Eq. (B.1) we need to show that this volume factor is exactly $24\pi^2$. We shall do this indirectly, by evaluating the right hand side of Eq. (B.7) for a specific function U that manifestly has topological charge equal to unity. This function is

$$U = \frac{x_\mu e_\mu}{|x|}. \tag{B.10}$$

After substituting into Eq. (B.7) we get, after a little algebra,

$$\begin{aligned} k &= \frac{1}{24\pi^2} \cdot 12 \int dS_\mu \frac{1}{|x|^4} x_\mu \\ &= 1 \end{aligned}$$

Q.E.D.

Appendix C

Supersymmetry Transformations

In this appendix we list supersymmetry transformation laws for the $\mathcal{N}=1$ vector multiplet $V=(v_m,\lambda,D)$ and the $\mathcal{N}=1$ adjoint chiral multiplet $\Phi=(A,\psi,F)$.

$\mathcal{N}=1$ Transformation Laws

We first consider the action of the supersymmetry operator $\delta_1=\xi_1 Q_1+\bar{\xi}_1\bar{Q}_1$ in Wess-Zumino gauge. For the $\mathcal{N}=1$ vector multiplet we have

$$\delta_1 v^m = i\xi_1\sigma^m\bar{\lambda}+i\bar{\xi}_1\bar{\sigma}^m\lambda, \tag{C.1}$$

$$\delta_1\lambda = -\xi_1\sigma^{mn}v_{mn}+i\xi_1 D, \tag{C.2}$$

$$\delta_1\bar{\lambda} = -\bar{\xi}_1\bar{\sigma}^{mn}v_{mn}-i\bar{\xi}_1 D, \tag{C.3}$$

$$\delta_1 D = -\xi_1\not{\mathcal{D}}\bar{\lambda}+\bar{\xi}_1\bar{\not{\mathcal{D}}}\lambda. \tag{C.4}$$

The transformation rules for the $\mathcal{N}=1$ adjoint chiral multiplet (and its Hermitian conjugate) are

$$\delta_1 A = \sqrt{2}\xi_1\psi, \tag{C.5}$$

$$\delta_1\psi = -\sqrt{2}i\bar{\xi}_1\bar{\not{\mathcal{D}}}A+\sqrt{2}\xi_1 F, \tag{C.6}$$

$$\delta_1 F = \sqrt{2}i\bar{\xi}_1\bar{\not{\mathcal{D}}}\psi-2ig\bar{\xi}_1[\bar{\lambda},A], \tag{C.7}$$

$$\delta_1 A^\dagger = \sqrt{2}\bar{\xi}_1\bar{\psi}, \tag{C.8}$$

$$\delta_1\bar{\psi} = -\sqrt{2}i\xi_1 \not{\mathcal{D}} A^\dagger + \sqrt{2}\bar{\xi}_1 F^\dagger, \tag{C.9}$$

$$\delta_1 F^\dagger = \sqrt{2}i\xi_1 \not{\mathcal{D}}\bar{\psi} + 2ig\xi_1[A^\dagger, \lambda]. \tag{C.10}$$

$\mathcal{N} = 2$ Transformation Laws

Next we consider the action of the supersymmetry operator $\delta_2 = \xi_2 Q_2 + \bar{\xi}_2\bar{Q}_2$ in Wess-Zumino gauge. Together $V = (v_m, \lambda, D)$ and $\Phi = (A, \psi, F)$ form an $\mathcal{N} = 2$ vector multiplet which transforms as follows,

$$\delta_2 v^m = i\xi_2\sigma^m\bar{\psi} + i\bar{\xi}_2\bar{\sigma}^m\psi, \tag{C.11}$$

$$\delta_2\lambda = \sqrt{2}i\bar{\xi}_2 \not{\mathcal{D}} A - \sqrt{2}\xi_2 F, \tag{C.12}$$

$$\delta_2\bar{\lambda} = \sqrt{2}i\xi_2 \not{\mathcal{D}} A^\dagger - \sqrt{2}\bar{\xi}_2 F^\dagger, \tag{C.13}$$

$$\delta_2 D = -\xi_2 \not{\mathcal{D}}\bar{\psi} + \bar{\xi}_2 \bar{\not{\mathcal{D}}}\psi, \tag{C.14}$$

$$\delta_2 A = -\sqrt{2}\xi_2\lambda, \tag{C.15}$$

$$\delta_2\psi = -\xi_2\sigma^{mn}v_{mn} + i\xi_2 D, \tag{C.16}$$

$$\delta_2 F = -\sqrt{2}i\bar{\xi}_2 \not{\mathcal{D}}\lambda + 2ig\bar{\xi}_2[A, \bar{\psi}], \tag{C.17}$$

$$\delta_2 A^\dagger = -\sqrt{2}\bar{\xi}_2\bar{\lambda}, \tag{C.18}$$

$$\delta_2\bar{\psi} = -\bar{\xi}_2\bar{\sigma}^{mn}v_{mn} - i\bar{\xi}_2 D, \tag{C.19}$$

$$\delta_2 F^\dagger = -\sqrt{2}i\xi_2 \not{\mathcal{D}}\bar{\lambda} + 2ig\xi_2[A^\dagger, \psi]. \tag{C.20}$$

Appendix D

Cluster Decomposition

In this appendix we demonstrate the clustering property of the $SU(N)$ k-instanton measures constructed in Section 5.6. We proceed along the lines of [49]. The matrix a' is understood to be a $k \times k$ matrix with 2×2 quaternion-like entries, $a'_{ij} = (a'_m)_{ij}\, \sigma^m$. In the limit of large separation, the space-time positions of the k individual instantons making up the k-instanton configuration may be identified with the k diagonal elements a'_{ii} [61]. In accordance with the property of cluster decomposition, when we take a single diagonal element, say a'_{kk}, to be large, the measure should factor into the product of a one-instanton and a $(k-1)$-instanton measure.

Where the unidentified k-instanton measure $d\mu^{(k)}$ is concerned, it is important to understand cluster decomposition as a $U(k)$ invariant effect. We therefore take the a'_{kk}-dependent submatrix of a',

$$h \; = \; a'_{kk} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (0, \cdots, 0, 1) \;, \tag{D.1}$$

and act on it with the residual $U(k)$ ADHM symmetry (5.30)

$$h \; \to \; g^\dagger \, h \, g \; . \tag{D.2}$$

There is a $U(k-1) \times U(1)$ subgroup of $U(k)$ that leaves h invariant, so that in fact g is restricted to the coset $U(k)/(U(k-1)\times U(1))$. Choosing the parameterization

$$g = \exp \begin{pmatrix} 0 & \cdots & 0 & -\alpha_{1k} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha_{k-1,k} \\ \bar{\alpha}_{1k} & \cdots & \bar{\alpha}_{k-1,k} & 0 \end{pmatrix} , \tag{D.3}$$

where the α_{ik} are complex numbers, the action of this coset on h is given by

$$\begin{aligned} a'_{kk} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (0,\cdots,0,1) &\longrightarrow a'_{kk}\, g^{\dagger} \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (0,\cdots,0,1) \cdot g \\ &= a'_{kk} \cdot \begin{pmatrix} 0 & \cdots & 0 & \alpha_{1k} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_{k-1,k} \\ \bar{\alpha}_{1k} & \cdots & \bar{\alpha}_{k-1,k} & 1 \end{pmatrix} + \mathcal{O}(|\alpha|^2) \end{aligned} \tag{D.4}$$

The second line gives the infinitesimal action of the coset $U(k)/(U(k-1)\times U(1))$ on the matrix h.

With the transformation (D.4) in mind, we can now consider the large $|a'_{kk}|$ limit of the unidentified measure in a meaningful way. The clustering condition should take the form

$$d\mu^{(k)} \overset{|a'_{kk}|\to\infty}{\longrightarrow} d\mu^{(k-1)} \times d\mu^{(1)} \times dS^{2(k-1)} , \tag{D.5}$$

where $dS^{2(k-1)}$ is just the Haar measure for the coset $U(k)/(U(k-1)\times U(1))$. We note here the result [80] that for infinitesimal α_{ik},

$$dS^{2(k-1)} = \prod_{i=1}^{k-1} d^2\alpha_{ik}. \tag{D.6}$$

To proceed, it is first convenient to make the following change of variables:

$$a'_{ik} \;=\; a'_{kk}\,\hat{a}_{ik}; \qquad 1 \le i \le k-1. \tag{D.7}$$

It is also useful to split $\hat{a}_{ik}$ into a scalar (S) part and a non-scalar (NS) part:

$$\hat{a}_{ik} \;=\; \hat{a}^{\mathrm{S}}_{ik} + \hat{a}^{\mathrm{NS}}_{ik} , \quad \hat{a}^{\mathrm{S}}_{ik} \;=\; (\hat{a}_0)_{ik}\,\sigma^0 , \quad \hat{a}^{\mathrm{NS}}_{ik} \;=\; \sum_{m=1}^{3} (\hat{a}_m)_{ik}\,\sigma^m . \tag{D.8}$$

As far as the measure is concerned, the change of variables has the effect

$$\int \prod_{i=1}^{k-1} d^8 a'_{ik} \;=\; |a'_{kk}|^{8(k-1)} \int \prod_{i=1}^{k-1} d^6 \hat{a}^{\rm NS}_{ik}\, d^2 \hat{a}^{\rm S}_{ik}\;. \tag{D.9}$$

The $\hat{a}^{\rm S}$ variables can now be identified with the infinitesimal group transformation parameters α. Then, using Eq. (D.6) above, we straightforwardly extract the expected group integration factor, $dS^{2(k-1)}$, from the measure.

We now examine the δ-function constraints in the clustering limit. We first examine the $\mathcal{N}=1$ measure, given by Eq. (5.113). Ignoring the infinitesimals $\hat{a}^{\rm S}$, the δ-function constraint on purely bosonic collective coordinates, Eq. (5.118), can be written as

$$\begin{aligned}
&\prod_{c=1}^{3} \delta^{(k^2)}\left(\tfrac{1}{2}{\rm tr}_2\,\tau^c(\bar{a}a)\right) \\
&= \prod_{c=1}^{3}\prod_{i=1}^{k-1} \delta^{(2)}\left(\tfrac{1}{2}{\rm tr}_2\,\tau^c\left((\bar{w}w)_{ik} + \sum_{j=1}^{k-1} \bar{a}'_{ij}\, a'_{kk}\, \hat{a}^{\rm NS}_{jk} - |a'_{kk}|^2\, \hat{a}^{\rm NS}_{ik}\right)\right) \\
&\times \prod_{c=1}^{3}\left[\prod_{i=1}^{k-1} \delta\left(\tfrac{1}{2}{\rm tr}_2\,\tau^c\left((\bar{\tilde{a}}\tilde{a})_{ii} - |a'_{kk}|^2\, \hat{a}^{\rm NS}_{ik}\, \hat{a}^{\rm NS}_{ki}\right)\right)\right]\left[\prod_{i<j}^{k-1} \delta^{(2)}\left(\tfrac{1}{2}{\rm tr}_2\,\tau^c\left((\bar{\tilde{a}}\tilde{a})_{ij} - |a'_{kk}|^2\, \hat{a}^{\rm NS}_{ik}\, \hat{a}^{\rm NS}_{kj}\right)\right)\right] \\
&\times \prod_{c=1}^{3} \delta\left(\tfrac{1}{2}{\rm tr}_2\,\tau^c\left((\bar{w}w)_{kk} - |a'_{kk}|^2 \sum_{j=1}^{k-1} \hat{a}^{\rm NS}_{kj}\, \hat{a}^{\rm NS}_{jk}\right)\right)\;.
\end{aligned} \tag{D.10}$$

Here $\tilde{a}$ is the matrix left behind when a has its last row and column removed. The δ-functions comprising the first line on the right-hand side of this equation saturate the integration over the $\hat{a}^{\rm NS}$ variables in Eq. (D.9). The effect of this integration is two-fold. First, it introduces a factor $|a'_{kk}|^{-12(k-1)}$ into the measure. Second, it requires the replacement of $\hat{a}^{\rm NS}_{ik}$ in the other δ-functions with an $\mathcal{O}(1/|a'_{kk}|^2)$ quantity. Consequently, in the limit $|a'_{kk}| \to \infty$, the δ-functions on the second and third lines become just the constraints that appear in the $(k-1)$-instanton and the one-instanton measure respectively.

Turning to the second, fermionic, δ-function constraint in the $\mathcal{N}=1$ measure, we see that it similarly factorizes into three pieces:

$$\begin{aligned}
&\delta^{(2k^2)}\left(\bar{\mathcal{M}}a + \bar{a}\mathcal{M}\right) \\
&= \prod_{i=1}^{k-1} \delta^{(4)}\left((\bar{\mu}w + \bar{w}\mu)_{ik} + \sum_{j=1}^{k-1}\left(\bar{a}'_{ij}\,\mathcal{M}'_{jk} + \bar{\mathcal{M}}'_{ij}\,a'_{kk}\,\hat{a}^{\rm NS}_{jk}\right) - \hat{a}^{\rm NS}_{ik}\,a'_{kk}\,\mathcal{M}'_{kk} + \bar{\mathcal{M}}'_{ik}\,a'_{kk}\right) \\
&\times \left[\prod_{i=1}^{k-1}\delta^{(2)}\left(\left(\bar{\tilde{\mathcal{M}}}\tilde{a} + \bar{\tilde{a}}\tilde{\mathcal{M}}\right)_{ii} + \ldots\right)\right]\left[\prod_{i<j}\delta^{(4)}\left(\left(\bar{\tilde{\mathcal{M}}}\tilde{a} + \bar{\tilde{a}}\tilde{\mathcal{M}}\right)_{ij} + \ldots\right)\right] \\
&\times\ \delta^{(2)}\left((\bar{\mu}w + \bar{w}\mu)_{kk} + \ldots\right).
\end{aligned} \tag{D.11}$$

Here $\tilde{\mathcal{M}}$ is the matrix left behind when $\mathcal{M}$ has its last row and column removed. The first δ-function factor above saturates the integration over the Grassmann collective coordinates $\mathcal{M}'_{ik}$ $(i = 1,\ldots,k-1)$. In performing this integration, a factor $|a'_{kk}|^{4(k-1)}$ is introduced into the measure. This exactly cancels the factors that appeared earlier. Further, in the large $|a'_{kk}|$ limit, the omitted terms in the arguments of the second and third δ-function factors in Eq. (D.11) vanish, and we are left with precisely the fermionic constraints that appear in $d\mu^{(k-1)}$ and $d\mu^{(1)}$ respectively. Since the numerical prefactor C_1^k also factorizes correctly, this completes the proof of the clustering property, Eq. (D.5), for the $\mathcal{N} = 1$ k-instanton measure (5.113).

In the case of the $\mathcal{N} = 2$ measure, Eq. (5.124), there are two further δ-function constraints to consider. The δ-function associated with the Higgsino collective coordinates can be factorized in exactly the same way as the gaugino δ-function in Eq. (D.11). The integration over the $\mathcal{N}'_{ik}$ $(i = 1,\ldots,k-1)$ yields a Jacobian factor $|a'_{kk}|^{4(k-1)}$ and leaves, in the large $|a'_{kk}|$ limit, the required one-instanton and $(k-1)$-instanton constraints. As for the δ-function associated with the Higgs collective coordinate matrix $\mathcal{A}_{\rm tot}$, we can write:

$$\begin{aligned}
\delta^{(k^2)}(\mathbf{L}\cdot\mathcal{A}_{\rm tot} - \Lambda - \Lambda_f) \;=\;& \delta^{(2(k-1))}(|a'_{kk}|^2\mathcal{A}_{\rm tot} + \ldots) \\
&\times\delta^{((k-1)^2)}(\tilde{\mathbf{L}}\cdot\tilde{\mathcal{A}}_{\rm tot} - \tilde{\Lambda} - \tilde{\Lambda}_f + \ldots) \\
&\times\delta^{(1)}({\rm tr}_2\,(\bar{w}w)_{kk}(\mathcal{A}_{\rm tot})_{kk} - \Lambda_{kk} - (\Lambda_f)_{kk} + \ldots)
\end{aligned} \tag{D.12}$$

where $\tilde{\mathbf{L}}$, $\tilde{\mathcal{A}}_{\rm tot}$, $\tilde{\Lambda}$ and $\tilde{\Lambda}_f$ are constructed using the truncated collective coordinate matrices $\tilde{a}$, $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$. The omitted terms are subleading in $|a'_{kk}|$. After integrating over $(\mathcal{A}_{\rm tot})_{ik}$ for $i = 1,\ldots,k-1$, we get a Jacobian factor $|a'_{kk}|^{-4(k-1)}$, which cancels the previous factor, and the one-instanton and $(k-1)$-instanton constraints remain. This confirms the clustering property (D.5) for the $\mathcal{N} = 2$ measure (5.124).

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