Cosmic strings and scalar tensor gravity

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COSMIC STRINGS AND SCALAR TENSOR GRAVITY

by

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ABSTRACT

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This thesis is concerned with the study of cosmic strings. We studied the values for the Higgs mass and string coupling for which the gravitational effect of an infinite cosmic string in the context of the Einstein theory is not only locally but also globally weak. We conclude this happens for strings formed at scales less or equal to the Planck one with Higgs mass being less or equal to the boson vectorial mass.

Then we examined the metric of an isolated self-gravitating abelian-Higgs vortex in dilatonic gravity for arbitrary coupling of the vortex fields to the dilaton. We looked for solutions in both massless and massive dilaton gravity. We compared our results to existing metrics for strings in Einstein and Jordan-Brans-Dicke theories. We explored the generalisation of Bogomolnyi arguments for our vortices and commented on the effects on test particles. We then included the presence of an axion field and examined the metric of an isolated self-gravitating axionic-dilatonic string.

Finally we studied dilatonic strings through black hole solutions in string theory. We concluded that the horizon of non-extreme charged black holes supports the long-range fields of the Nielsen-Olesen string that can be considered as black hole hair and whose gravitational effect is in general the production of a conical deficit into the metric of the black hole background. We also concluded that the effect of the dilaton on the horizon of these black holes is to generate an additional charge.

Declaration

This thesis is the result of research carried out between March 1996 and March 1999. The work presented in this thesis has not been submitted in fulfilment of any other degree or professional qualification.

No claim of originality is made for chapter 1. Chapters 3 and 4 arose from collaboration with my supervisor, Dr. Ruth A. W. Gregory resulting in the publication *Cosmic strings in dilaton gravity* and *Vortices and black holes in dilatonic gravity* both in Physical Review D.

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Dedicated to

my supervisor, Dr. Ruth A. W. Gregory,

Gregory Michael Koningstein.

the Percival family

and

my family.

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Introduction

There has been an enduring interest in topological defects in general since their formation is predicted in a wide class of elementary particle models which assume phase transitions in the early universe, and a particular interest for strings that can lead to very interesting cosmological consequences [1].

When formed at grand unification (GUT) scales, strings may generate density fluctuations sufficient to explain the galaxy formation [2,3] and their gravitational effects in the context of the Einstein theory are not only locally but also globally weak even though they are infinite objects and therefore very massive. However other phase transitions may have happened in the early universe and strings may have formed at other scales and apart their non-significant cosmological consequences their properties are nonetheless important to be studied when regarded as solutions of the low energy action for the supersymmetric unifying model. Meanwhile it also seems likely that gravity is not given by the Einstein action, at least at sufficiently high energy scales, and the most promising alternative seems to be that offered by string theory, where gravity becomes scalar-tensor in nature [4]. Scalar-tensor gravity is not new, it was pioneered by Jordan, Brans and Dicke [5], who sought to incorporate Mach's principle into gravity. The implications of such actions on general Friedmann-Robertson-Walker cosmological models have been well explored [6–8], however, the implications for theories of structure formation have not been so well studied, in particular those for the perturbations of the microwave background [9] and for the radiation from a cosmic string networks [10]. Finally as infinitely strings are vortex solutions extended to spatial infinity, they also can provide hairs for black holes when threading them [11–13] being another example of physical systems where due to inclusion of matter fields in the horizon the non-hair theorem [14], stating that a stationary black hole is uniquely determined by its mass, electromagnetic charge and angular momentum, is not verified [15–18].

Therefore this thesis is organised as follows: in chapter 1 we present a brief introduction to topological defects, in particular to strings, explaining the conditions for their formation, and focusing on their relevance in the context of the hot big bang cosmology. We then review their gravitational effects in the context of Einstein gravity and introduce some notations and methods useful for the following chapters. In chapter 2 we study the values for the Higgs mass and string coupling for non-singular cosmic strings in Einstein gravity. In Chapter 3 we examine the gravi-dilaton field of a self-gravitating cosmic string in both massless and massive dilaton gravity for arbitrary coupling of the vortex fields to the dilaton. We then include the presence of an axion field and examine the metric of an isolated self-gravitating axionic-dilatonic string. In Chapter 4 we examine whether dilatonic strings can thread black holes in string theory and study their gravitational effects on the black holes backgrounds. Finally in Chapter 5 we summarise the most important results of our research.

Notations and conventions

In this thesis we use "vortex units", i.e. such that the string width is of order unity (i.e. $\sqrt{\lambda}\eta = 1$) with $\hbar = c = K_B = \sqrt{\lambda}\eta = 1$, where h is the Planck constant, c the velocity of light, K_B the Boltzman constant. The gravitational coupling is no more G, the Newton's constant, but instead $\epsilon = 8\pi G\eta^2$ which for example gives for cosmic strings formed at GUT scales ($\eta \sim 10^{16}\, GeV$) that $\epsilon \sim 10^{-6}$. Lengths and masses have inverse dimensions and for conversion to cgs units it is useful to remind the values at the Planck scale: $m_{pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\, Gev \sim 10\,\mu g$ and $l_{pl} = \sqrt{\frac{G\hbar}{c^3}} \sim 10^{-19}\, Gev^{-1} \sim 10^{-33}\, cm$.

We also take a mainly minus signature for the metrics, i.e., $(+, -, -, -)$.

Chapter 1

Introduction to cosmic strings

1.1 Introduction

This chapter presents a brief introduction to cosmic strings, which are topological defects that may have resulted from phase transitions in the early universe. They are very massive objects and therefore their gravitational effects may be significant. In this chapter we review some of these effects in the context of Einstein gravity, under some approximations for the string model, emphasising the most important results for leading chapters. Therefore this chapter can be roughly divided into two parts. The first part, including *Sections* 1.1 – 1.4, presents an introduction to cosmic strings explaining their formation in the context of the early universe by using topological arguments. In the second part (*Sections* 1.5 – 1.7) we concentrate on static straight strings and look for some of their gravitational effects in the context of the Einstein gravity. For a critical Higgs coupling (*Sections* 1.5 – 1.6) solutions for Einstein's equations are exact for a string model where the energy is uniformly distributed along a transversal plane of the string and two extreme regimes are discussed: the wire string and the supermassive string, whose gravitational effects are similar to those for the global strings which are reviewed in *Section*1.7.

1.2 The standard cosmological model

The standard cosmological model (SCM) assumes the cosmological principle (CP) which states that the universe is, at least on the large scale, homogeneous and isotropic, i.e., it appears much the same to any observer placed wherever in the universe. It also comes from the universality of the physical laws that they must be valid anywhere in the universe. For this reason this model uses Einstein's theory of general relativity [19] to describe gravity: the most prominent fundamental interaction on the large scales relevant to cosmology, as this is the most complete theory of gravity presently available.

Despite the several successes of the SCM we will only concentrate on the crucial results necessary to explain the cosmological phase transitions and therefore the existence of topological defects in the primitive universe. i.e.. that the universe cools while it expands.

The expansion of the universe. first observed by Hubble [20]. is predicted by the SCM which from the CP requires the spacetime to be described in terms of the Robertson-Walker metric. given by:

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right] \tag{1.1}$$

where t is the cosmological time. $a(t)$ is the scale factor. and k a constant which represents the spatial curvature of the universe. For $k = 1$ the universe is closed. while for $k = 0$ and $k = -1$ it is open. being flat for $k = 0$ [21].

The CP also requires by the Einstein's equations:

$$G_{ab} = \frac{1}{2} T_{ab} \tag{1.2}$$

(we set. for simplicity. $16\pi G = 1$ for the purposes of this discussion only) homogeneity and isotropy for the matter source, T_{ab} which therefore must take the form [22]

$$T_{ab} = -p g_{ab} + (p + \rho) u_a u_b \tag{1.3}$$

where u_a is the four-velocity of the source ($u_a u^a = 1$), ρ its energy density and p its pressure.

The Einstein's equations can then be written as [23]:

$$\Omega = 1 + \frac{k}{a^2 H^2} \tag{1.4a}$$

$$\dot{H} = -H^2 - \frac{1}{12} (\rho + 3p) \tag{1.4b}$$

where

$$\Omega = \frac{\rho}{6H^2} \tag{1.5}$$

and

$$H = \frac{\dot{a}}{a} \tag{1.6}$$

are respectively the cosmological density and the Hubble parameters.

For simplicity we take $k = 0$ and solve the equations of motion (1.4a)-(1.4b) for a universe either filled with dust, i.e., $p = 0$, to get

$$a \propto t^{\frac{2}{3}} \tag{1.7}$$

or filled with radiation, i.e., $p = \frac{1}{3}\rho$ [24] to get

$$a \propto t^{\frac{1}{2}} \ . \tag{1.8}$$

Considering now two comoving particles spatially located at l_1 and l_2 and separated by a distance l, with $l = \int_{l_1}^{l_2} a(t)dl$ where $dl^2 = r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$, it is clear that l grows in time with $a(t)$ and therefore in any of these models the universe expands. This also gives that the density of energy of an universe filled by radiation ($\rho_r \propto \frac{1}{a^4}$ [24]) decreases faster than that of an universe filled by matter ($\rho_m \propto \frac{1}{a^3}$) and therefore the early universe was much hotter as radiation dominates.

Equations (1.4a)- (1.4b) also imply a conservation law:

$$\frac{d}{dt}\left(a^3 \rho\right) + p\,\frac{d}{dt}\left(a^3\right) = 0 \tag{1.9}$$

which can be interpreted by using thermodynamics laws as the conservation of the entropy of a system in thermal equilibrium, i.e., the universe expands adiabatically. Therefore it is also possible to relate the energy density of universes filled with matter or radiation with their temperature. Treating radiation and matter as an ideal gas in thermal equilibrium of fermions or bosons respectively, that relation can be written as [10]

$$T \propto \rho^{\frac{1}{4}} \ . \tag{1.10}$$

Thus whether matter or radiation dominated the universe cools while it expands.

1.3 Phase transitions in the early universe

Most particle physicists believe that the standard model of electroweak and strong interactions is just the low energy limit of a grand-unified theory. This suggests that the universe underwent some process of symmetry breaking.

It was probably Kirzhnits, [25], who was the first to realize that each spontaneously broken symmetry in particle physics corresponds to a phase transition in the early universe. The number of phase transitions that breaks the full grand-unified symmetry down to $SU(3) \times U(1)_{em}$ depends on the unifying model. Although one can expect at least two: one at the energy scale about 10^{16} GeV where the strong interaction became distinct from the electroweak one, and another at 100 GeV where the electroweak symmetry was broken. Cosmological phase transitions can give rise to defects of various kinds as explained in the next section.

We now consider a toy model that produces cosmic strings after a cosmological phase transition. We take a complex scalar Higgs field, Φ, with a self-interacting "*Mexican hat*" potential given by

$$V(\Phi) = \frac{\lambda}{4}(\Phi^{\dagger}\Phi - \eta^2)^2 \qquad (1.11)$$

and represented in figure 1.1

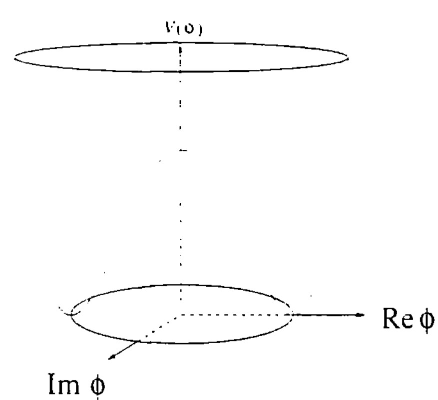

Figure 1.1: The Higgs potential for our toy model [26].

Finite temperature corrections add a $T^2\,\Phi^{\dagger}\Phi$ term to the effective potential [27] and so at high temperature $<\Phi>= 0$ while at lower temperatures, below the critical temperature, T_c, ($T_c^2 = \frac{\lambda}{2}\,\eta^2$) the Higgs field "*rolls down*" to the minimum of the potential where $<\Phi>= \eta\, e^{i\varphi}$. If the temperature falls rapidly below the critical one, as happened in the primitive universe, there is only enough time for equilibrium to become established in a restricted space limited by the coherence length of the field fluctuations. The dropping in temperature of the universe, i.e., the rate at which the phase transition happened, depends on the rate of the expansion of the universe and therefore the coherence length of the field fluctuations is limited to the Hubble length. Beyond this limit different regions of space are not causally connected and therefore the Higgs field can have different phases at different points in space and for whichever one we choose, the gauge symmetry is spontaneously broken, all the vacuum states of the circle in the bottom of the potential of figure 1.1 give the same physical results. Hence η is called the scale of symmetry breaking. At the boundaries of those regions, the Higgs field will arrange itself so as to minimise the energy resulting in the creation of a topological defect (a string for this potential) as was first recognised by Kibble [28]. In order to better visualise how this process happens we use the physical analogue of a phase transition giving rise to cosmic strings presented in the reference [29] where a forest of pencils connected by strings is taken. At the high temperatures, the rapidly oscillating pencils are at all angles and therefore their mean position, i.e., their vacuum expectation values, is standing straight up. Cooling the pencil forest the pencils fall down and it can happen that in some regions they will fall leaving a pencil standing erect which therefore was trapped in a false vacuum state being the analogue of the cosmic strings defect.

1.4 Topological defects

A topological defect is a discontinuity in the vacuum as already stated in the previous section, and in conventional field theory can be classified according to the topology of the vacuum manifold of the particular field theory being used to model the physical set up: disconnected vacuum manifolds give domain walls, non-simply connected manifolds, strings, and manifolds with non-trivial 2- and 3-spheres give monopoles and textures respectively. In this thesis, we are concerned with cosmic strings which are defects associated with non-simply connected vacuum manifolds, i.e., field configurations of the Higgs field which minimise the total energy and whose set of values for the Higgs field, which also minimise the potential, is not simply connected. We now consider the potential taken in (1.11) and show that it can give rise to strings.

For that we take a circle in space such as that represented in figure 1.2 and fix vacuum values for the Higgs in this circle. We take a winding number, N, of one as winding once around the circle in space in the figure 1.2 one winds once around the circle in the Higgs values in figure 1.1. If in all the disk bounded by the circle in figure 1.2 the Higgs states were vacuum, continuity in space for the Higgs field would give that this circle could be contractible into a point where the Higgs field would have any value. As this is impossible

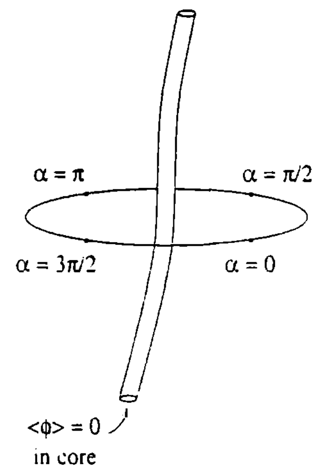

Figure 1.2: Explaining string formation using topological arguments [26].

(in the absence of non-trivial spatial topology) it means that somewhere inside that disk the Higgs field has to leave the vacuum state which from continuity implies that there is a point where $\Phi = 0$. In this way continuity in an orthogonal direction to the plane of the disk demands that $\Phi = 0$ at some point on any surface spanning the loop - this is the locus of the vortex as represented in figure 1.2. The width of the string, i.e., its core it is taken to be the size of the compton wavelength of the Higgs particle, i.e. $\sqrt{\lambda}\,\eta$. The vortex can have no end (otherwise the spatial circle would be contractible to a point) and therefore strings are one dimensional objects either closed in loops or infinite.

In this thesis we only consider straight infinite strings which are very massive objects and therefore one expects to have significant gravitational effects as examined in the next sections. In fact the width δ of a string formed at a grand-unification scale $\eta \sim 10^{16}$ GeV is of order $\delta \sim 10^{-30}$cm (take $\delta \sim \eta^{-1}$ for dimensional reasons) while its mass per unit length, μ, is of order $\mu \sim 10^{22}$g/cm (take $\mu \sim \eta^2$).

In conclusion: there is a string inside of every loop where the Higgs field is vacuum and its phase changes by 2π.

1.5 The Abelian-Higgs model

In order to establish notation and conventions we start to review the Nielsen-Olesen string, i.e., a U(1) vortex of the Abelian-Higgs model described by the lagrangian:

$$\mathcal{L}[\Phi, A_c] = D_a\Phi^\dagger D^a\Phi - \frac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab} - \frac{\lambda}{4}(\Phi^\dagger\Phi - \eta^2)^2 \tag{1.12}$$

where $D_c = \nabla_c + ieA_c$ is the usual gauge covariant derivative, and $\tilde{F}_{ab}$ the field strength of the gauge field A_c. At far distances from the core when the Higgs potential is minimum, i.e., $|\Phi| = \eta$, one requires that the covariant derivative of the Higgs field vanish, i.e.,

$$D_\mu\Phi = [\nabla_\mu + ieA_\mu]\,\Phi = 0 \tag{1.13}$$

which guarantees a finite energy per unit length for a local string. A gauge field A_μ satisfying (1.13) is given by

$$A_\mu = -\frac{\nabla_\mu \Phi}{i\,e\,|\Phi|} + f_{,\mu} \tag{1.14}$$

where from the gauge invariance we included the arbitrary gauge field $f_{,\mu}$ with f an arbitrary function. Therefore a natural way to rewrite the fields that makes manifest the physical degrees of our model is:

$$\Phi(x^\alpha) = \eta X(x^\alpha)e^{i\chi(x^\alpha)} \tag{1.15a}$$

$$A_c(x^\alpha) = \frac{1}{e}[P_c(x^\alpha) - \nabla_c\chi(x^\alpha)] \tag{1.15b}$$

where we set $f_{,\mu} = -\frac{\nabla_\mu\chi}{e}$, where X, χ and P_c are now real. In terms of these new variables, the lagrangian and equations of motion become

$$\mathcal{L} = \nabla_a X\nabla^a X - X^2 P_a P^a - \frac{\beta}{2}F_{ab}F^{ab} - \frac{1}{4}(X^2 - 1)^2 \tag{1.16a}$$

$$\Box X - P_a P^a X + \frac{1}{2}X(X^2 - 1) = 0 \tag{1.16b}$$

$$\nabla_a F^{ab} + \frac{X^2 P^b}{\beta} = 0 \tag{1.16c}$$

Thus P_b is the massive vector field in the broken symmetry phase, $F_{ab} = \nabla_a P_b - \nabla_b P_a$ its field strength, and X the residual real scalar field with which it interacts. χ is not in itself a physical quantity, however, it can contain physical information if it is non-single valued, in other words, if $\oint \nabla_a\chi dx^a = 2\pi N$ for some $N \in Z$ which is also related with the magnetic flux via the condition of the vanishing covariant derivative of the Higgs field giving $\oint A_a dl^a = \frac{2\pi N}{e}$ Thus the true physical content of this model is contained in the fields P_a and X plus boundary conditions on P_a and X representing vortices.

1.5.1 The Nielsen-Olesen solution

The simplest vortex solution is the Nielsen-Olesen (NO) vortex [30], an infinite, straight static $N = 1$ solution with cylindrical symmetry. In this case, we can choose a gauge in which

$$\Phi = \eta X_0(R) e^{i\varphi} \tag{1.17a}$$

$$A_c = \frac{1}{e}\left[P_0(R) - 1\right] \nabla_c \varphi \tag{1.17b}$$

where R and φ are cylindrical polar coordinates measured in "*vortex units*", i.e., in which the string width is of order unity ($\sqrt{\lambda}\,\eta \simeq 1$). In these units we note that the energy per unit length of the string, $\hat{\mu}$, is of order unity. Near the core where $X_0 = 0$ one can choose a gauge in which $A_\varphi = 0$, i.e., $P_0 = 1$. The energy and stresses of the vortex are given by:

$$\hat{T}^t_t = \mathcal{E} = X_0'^2 + \frac{X_0^2 P_0^2}{R^2} + \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \tag{1.18a}$$

$$\hat{T}^R_R = -\mathcal{P}_R = -X_0'^2 + \frac{X_0^2 P_0^2}{R^2} - \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \tag{1.18b}$$

$$\hat{T}^\theta_\theta = -\mathcal{P}_\theta = X_0'^2 - \frac{X_0^2 P_0^2}{R^2} - \beta \frac{P_0'^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \tag{1.18c}$$

$$\hat{T}^z_z = -\mathcal{P}_z = \hat{T}^t_t. \tag{1.18d}$$

and the magnetic field along the string is given by:

$$B^z = -\frac{2P_0'}{eR} \tag{1.19}$$

We look for vortex solutions, i.e. solutions of the vortex equations

$$-X_0'' - \frac{X_0'}{R} + \frac{P_0^2 X_0}{R^2} + \frac{1}{2} X_0 (X_0^2 - 1) = 0 \tag{1.20a}$$

$$-P_0'' + \frac{P_0'}{R} + \frac{1}{\beta} X_0^2 P_0 = 0 \tag{1.20b}$$

where $\beta = \frac{\lambda}{2e^2} = \frac{m_X^2}{m_P^2}$ is the Bogomolnyi parameter [31], such that the Higgs scalar field and the magnetic field are finite everywhere, which in particular requires $\frac{P_0'}{R}$ to be finite as $R \to 0$ and whose energy is finite. Therefore the appropriate boundary conditions for our problem are:

$$X_0(R \to 0) \to 0 \tag{1.21a}$$

$$P_0(R \to 0) \to 1 \tag{1.21b}$$

$$\vdots$$

and

$$X_0(R \to \infty) \to 1 \tag{1.22a}$$

$$P_0(R \to \infty) \to 0 \ . \tag{1.22b}$$

This string has winding number one: for winding number N, we replace χ by $N\chi$, and so P by NP. Those equations (1.20a-1.20b) are coupled differential equations of second order for $\beta \neq 1$ (or of first order for $\beta = 1$) which in any case do not have analytical solutions. One can get asymptotic solutions near the core ($R \to 0$) or at far distances from that ($R \to \infty$) as discussed in what follows.

We start by defining the asymptotic vortex solutions near the core for $\beta \neq 1$ (but finite) using the regularity of the stress energy tensor. The vortex equations (1.20a)-(1.20b) near the core become

$$\frac{1}{R}\left[RX_0'\right]' \simeq \frac{X_0P_0^2}{R^2} \tag{1.23a}$$

$$R\left[\frac{P_0'}{R}\right]' \simeq \frac{X_0^2P_0}{\beta} \tag{1.23b}$$

as

$$\frac{X_0P_0^2}{R^2} \gg \frac{X_0}{2}(1 - X_0^2) \ . \tag{1.24}$$

Regularity of the stress energy tensor requires $\int dx^1\sqrt{-g}\,T_t^t \propto \int dR\,R\,T_t^t \to 0$ when $R \to 0$. Let us first take the "*worst possibility*" i.e., $T_t^t \sim \frac{1}{R}$. This gives from (1.18a) that $X_0^2P_0^2 \sim \mathcal{O}(R)$ and that $\beta\,P_0'^2 \sim \mathcal{O}(R)$ which integrated gives

$$P_0 \simeq 1 - \frac{K_1}{2\beta\sqrt{R}} \tag{1.25}$$

with K_1 a positive integrating constant. Therefore P_0 would diverge for $R \to 0$ which is absurd as the magnetic field would become infinite. The only possibility is therefore to take $K_1 = 0$, i.e., $P_0 = 1$. Now using the other condition, $X_0P_0 \sim \mathcal{O}(\sqrt{R})$, as well X_0 would diverge for $R \to 0$ as from (1.23a) it comes

$$X_0' = \frac{K_2}{R} - \frac{1}{2\sqrt{R}} \tag{1.26}$$

with K_2 an integrating constant. This means that the energy density does not behave like in the "*worst possibility*". Instead we take $T_t^t \sim b_1$ with b_1 a positive constant which now it is consistent with the required behaviour for the vortex fields. To argue that we note that now from (1.18a) $\frac{X_0 P_0}{R}$ and $\frac{P_0'}{R}$ have to be finite, i.e.,

$$X_0 P_0 \sim \mathcal{O}(R) \tag{1.27a}$$

$$P_0' \sim \mathcal{O}(R) \tag{1.27b}$$

We take $P_0' \simeq -\frac{b_2}{2} R$ and solve the vortex equations (1.23a)- (1.23b) to get $X_0' \simeq b_3$. i.e..

$$X_0 \simeq b_3 R \tag{1.28a}$$

$$P_0 \simeq 1 - b_2 \frac{R^2}{4} \ . \tag{1.28b}$$

with b_2 and b_3 positive integrating constants.

We now take $\beta = 1$ for which the vortex equations (1.20a)-(1.20b) can be reduced to the first order set of equations:

$$X_0' = \frac{X_0 P_0}{R} \tag{1.29a}$$

$$P_0' = \frac{R}{2} (X_0^2 - 1) \tag{1.29b}$$

which near the core become

$$X_0' \simeq \frac{X_0 P_0}{R} \tag{1.30a}$$

$$P_0' \simeq -\frac{R}{2} \tag{1.30b}$$

and whose solutions are still given by (1.28a)- (1.28b) with now $b_2 = 1$.

This gives that near the core the magnetic field is constant (up to this order) with magnitude

$$B^z = \frac{b_2}{e} \ . \tag{1.31}$$

We now study the asymptotic solutions at far distances from the core. For that we now use another method that proves to be useful for later chapters and that consists of an analysis of the equations written as an autonomous dynamical system (d.s.). In this case equations (1.20a)-(1.20b) give the d.s.:

$$X_0' = Y \tag{1.32a}$$

$$Y' = -SY + P_0^2 X_0 S^2 + \frac{X_0}{2}(X_0^2 - 1) \tag{1.32b}$$

$$P_0' = Q \tag{1.32c}$$

$$Q' = QS + \frac{X_0^2 P_0}{\beta} \tag{1.32d}$$

$$S' = -S^2 \tag{1.32e}$$

where $S = \frac{1}{R}$, Y and Q are the derivatives of X_0 and P_0, and which will be studied near the critical point (c.p.) $(1,0,0,0,0)$ with $(X_{0c}, Y_c, P_{0c}, Q_c, S_c)$ being generic critical coordinates. Hence we take $X_0 = X_{0c} + \delta X$, *etc*, with very small perturbations, i.e., $|\delta X| \ll 1$ and write this d.s up to lowest order in the perturbations. This gives $S \simeq 0$ and

$$(\delta X_0)' \simeq \delta Y \tag{1.33a}$$

$$(\delta Y)' \simeq \delta X_0 \tag{1.33b}$$

$$(\delta P_0)' \simeq \delta Q \tag{1.33c}$$

$$(\delta Q)' \simeq \frac{\delta P_0}{\beta} \tag{1.33d}$$

which gives immediately for the vortex fields:

$$X_0 \simeq 1 - K_3\, e^{-R} \tag{1.34a}$$

$$P_0 \simeq K_4\, e^{-\frac{R}{\sqrt{\beta}}} \tag{1.34b}$$

with K_3 and K_4 two integration constants, i.e., the string fields (and so the magnetic field) die off exponentially fast. From (1.34b) it follows that in these units the Compton wavelength for the vector boson, δ_v, is $\delta_v = \sqrt{\beta}$ while it is of order unity for the Higgs particle. In the presence of other fields such as the dilaton field or the gravitational one of a black hole it is then natural to compare the size of the dilaton particle or of the black hole with these scales as is done in chapters 3 and 4.

We now discuss qualitatively three limits where either $\beta \to 0$, $\beta = 1$ or $\beta \to \infty$.

(i) $\beta \to 0$

As the Higgs field has to be rearranged at the distance $\sim m_H^{-1}$, when β decreases this rearrangement has to take place at larger distances and the amount of energy of the vortex to be rearranged (proportional to m_H^2) decreases so that the spatial gradient of the Higgs field decreases vanishing in the limit $\beta \to 0$. Therefore X_0 is settled to its core value and a string never forms. Fixing η and m_v this limit can be obtained when $\lambda \to 0$ and therefore $V(\Phi) \to 0$ and then the only non-vanishing contribution for the lagrangian in (1.12) comes from the magnetic field, i.e., in this limit one recovers electromagnetism. In particular it includes the vacuum case.

(ii) $\beta = 1$

This is the critical Higgs coupling case for which the vortex is supersymmetrizable [10] as the vortex equations (1.20a)-(1.20b) can be reduced to the set of first order coupled differential equations given in (1.29a)- (1.29b) whose solutions for a winding number one are plotted in figure 1.3. In the presence of gravity this value for β is still a special value

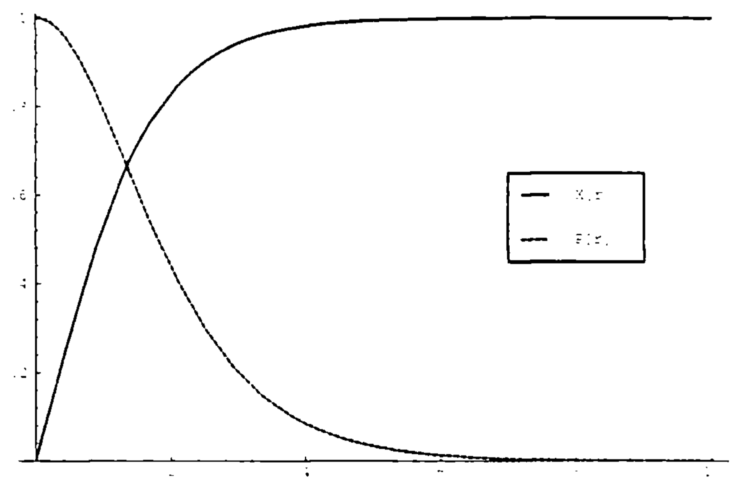

Figure 1.3: $X_0(R)$ and $P_0(R)$ for a $\beta = N = 1$ vortex [32].

as discussed in the next chapter.

(iii) $\beta \to \infty$

From (1.20b) one gets $P_0(R) = 1 - K_5 \frac{R^2}{2}$ with K_5 a positive integration constant and consistency with (1.34b) requires $K_4 = 1$ and $K_5 = 0$, i.e., $P_0(R) = 1$ and so $A_c(R) = 0$, i.e., the string turns into a global string as discussed later in *Section* 1.7. This limit will be taken for the axionic-dilatonic string considered in chapter 3 to get dyonic solutions.

1.6 The Abelian-Higgs model in Einstein gravity

It is also useful to review the self-gravitating Nielsen-Olesen vortex in Einstein gravity, as much of the formalism can be used directly in the next chapters. To include the self-gravity of the string (along the z axis), we require a metric which exhibits the symmetries of the source, namely, translational invariance along its length (i.e., $z \to z + h$, with h a constant) and rotational invariance around the core (i.e., $x \to -x$, $y \to -y$, $x \to y$) i.e. cylindrical symmetry.

The line element for a cylindrical system along the z axis is then given by

$$ds^2 = g_{00}dt^2 - g_{xx}(dx^2 + dy^2) - g_{zz}dz^2 \ . \tag{1.35}$$

In a surface where t and z are constants, $dl^2 = dx^2 + dy^2$ can also be written as $dl^2 = dR^2 + R^2\, d\varphi^2$ and therefore the general cylindrically symmetric metric is given by [33]

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dR^2) - e^{2\psi}dz^2 - \alpha^2 e^{-2\psi}d\varphi^2 \tag{1.36}$$

(where γ, ψ, α are independent of z, φ). The string couples to this metric via its energy momentum tensor T_{ab} from the Einstein's equations

$$G_{ab} = \epsilon\, T_{ab} \tag{1.37}$$

where

$$T_{ab} = 2\nabla_a X \nabla_b X + 2X^2 P_a P_b - 2\beta F_{ac} F_b^{\ c} - \mathcal{L} g_{ab} \tag{1.38}$$

where

$$\epsilon = 8\pi G\eta^2 \tag{1.39}$$

is the gravitational string coupling in these units.

Cylindrical symmetry of the source demands the vortex fields not to depend on z and φ. If in addition one requires a static string then $T_{zz} = -\mathcal{L}\, g_{zz}$ and as well $T_{tt} = -\mathcal{L}\, g_{tt}$, i.e., $T_t^t = T_z^z$ and the stress energy tensor can be seen to be boost invariant. This in turn implies through the Einstein's equations (1.37) that for a static metric

$$\gamma = 2\psi \tag{1.40}$$

which we will assume from now on and therefore:

$$ds^2 = e^{\gamma}\left[dt^2 - dR^2 - dz^2\right] - \alpha^2 e^{-\gamma} d\varphi^2 \ . \tag{1.41}$$

The gravity of cosmic strings within the context of the Einstein's theory has been well explored under various approaches that simplify the non-linear nature of the Einstein's equations [34]. In what follows we use the linearised gravity to find an approximation to the string metric while in the next section we review the Vilenkin [35] and Gott [36] approximations for the string model. The energy and stresses are given by

$$T_t^t = \mathcal{E} = e^{-\gamma} X'^2 + e^{\gamma} \frac{X^2 P^2}{\alpha^2} + \beta \frac{P'^2}{\alpha^2} + \frac{1}{4}(X^2 - 1)^2 \tag{1.42a}$$

$$T_R^R = -\mathcal{P}_R = -e^{-\gamma} X'^2 + e^{\gamma} \frac{X^2 P^2}{\alpha^2} - \beta \frac{P'^2}{\alpha^2} + \frac{1}{4}(X^2 - 1)^2 \tag{1.42b}$$

$$T_\varphi^\varphi = -\mathcal{P}_\varphi = e^{-\gamma} X'^2 - e^{\gamma} \frac{X^2 P^2}{\alpha^2} - \beta \frac{P'^2}{\alpha^2} + \frac{1}{4}(X^2 - 1)^2 \tag{1.42c}$$

$$T_z^z = -\mathcal{P}_z = T_t^t \tag{1.42d}$$

and the Einstein and vortex equations can then be read off respectively as [32]

$$\alpha'' = -\epsilon \alpha e^{\gamma} (\mathcal{E} - \mathcal{P}_R) \tag{1.43a}$$

$$(\alpha \gamma')' = \epsilon \alpha e^{\gamma} (\mathcal{P}_R + \mathcal{P}_\varphi) \tag{1.43b}$$

$$\alpha' \gamma' = \frac{1}{4} \alpha \gamma'^2 + \epsilon \alpha e^{\gamma} \mathcal{P}_R \tag{1.43c}$$

$$\frac{1}{\alpha} (\alpha X')' = \frac{X P^2}{\alpha^2} e^{2\gamma} + \frac{X}{2} (X^2 - 1) e^{\gamma} \tag{1.43d}$$

$$\alpha \left(\frac{P'}{\alpha} \right)' = -\gamma' P' + \frac{X^2 P}{\beta} e^{\gamma} \tag{1.43e}$$

Also for future reference, the Bianchi identity gives

$$\mathcal{P}_R' + (\mathcal{P}_R - \mathcal{P}_\varphi)\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{2}\right) + \gamma' \mathcal{P}_R + \gamma' \mathcal{E} = 0. \tag{1.44}$$

We now look for the gravitational effects of these strings assuming small string coupling, i.e., $\epsilon \ll 1$ which in particular is valid for strings formed at GUT scales ($\epsilon \sim 10^{-6}$). We then take a perturbative expansion for the fields to get for zeroth order (flat space)

$$\alpha = R \quad \psi = \gamma = 0 \quad X = X_0 \quad P = P_0, \tag{1.45}$$

with (1.44) giving

$$(R \mathcal{P}_{0R})' = \mathcal{P}_{0\varphi} \,. \tag{1.46}$$

These are the Nielsen-Olesen solutions discussed in the previous section which are valid near the core of the string where the symmetry of the Higgs potential remains unbroken and therefore gravity decouples from the gauge fields [26]. Thus near the core and up to terms of order $\mathcal{O}(R)$ the vortex fields X_0 and P_0 are given by (1.28a)- (1.28b) (where $b_2 = 1$ for $\beta = 1$).

To first order in ϵ the string metric is given by [37]

$$\alpha = \left[1 - \epsilon \int_0^R R(\mathcal{E}_0 - \mathcal{P}_{0R})dR\right] R + \epsilon \int_0^R R^2(\mathcal{E}_0 - \mathcal{P}_{0R})dR. \tag{1.47a}$$

$$\gamma = \epsilon \int_0^R R\mathcal{P}_{0R}dR \tag{1.47b}$$

where the subscript zero indicates evaluation in the flat space limit. Note that when the radial stresses do not vanish, there is a scaling between the time, z and radial coordinates for an observer at infinity and those for an observer sitting at the core of the string. The only case in which these stresses do vanish is when $\beta = 1$ which will be analysed in the next section.

We conclude this section by demonstrating the asymptotically conical nature of the corrected metric. Note that since the string functions X_0 and P_0 rapidly fall off to their vacuum values outside the core as shown in the previous section, the integrals in (1.47a) - (1.47b) rapidly converge to their asymptotic, constant, values. Let

$$A = \epsilon \int_0^R R(\mathcal{E}_0 - \mathcal{P}_{0R})dR \tag{1.48a}$$

$$B = \epsilon \int_0^R R^2(\mathcal{E}_0 - \mathcal{P}_{0R})dR \tag{1.48b}$$

$$D = \epsilon \int_0^R R\mathcal{P}_{0R} \tag{1.48c}$$

then the asymptotic form of the metric is

$$\begin{aligned} ds^2 &= e^D[dt^2 - dR^2 - dz^2] - R^2(1 - A + \frac{B}{R})^2 e^{-D} d\varphi^2 \\ &= d\hat{t}^2 - d\hat{R}^2 - d\hat{z}^2 - \hat{R}^2(1 - A)^2 e^{-2D} d\varphi^2 \end{aligned} \tag{1.49}$$

where

$$\hat{t} = e^{\frac{D}{2}} t, \quad \hat{z} = e^{\frac{D}{2}} z, \quad \hat{R} = e^{\frac{D}{2}}(R + \frac{B}{1 - A}) \tag{1.50}$$

This gives from (1.49) that in the spatial sections transverse to the string, i.e., where $\hat{R}$, z, and t are fixed, a circumference has a length

$$L = 2\pi\hat{R}(1 - A)e^{-D} \simeq 2\pi\hat{R}\,(1 - (A + D)) \tag{1.51}$$

which therefore represents the geometry of a conical space in which a wedge of angle

$$\Delta = 2\pi(A + D) = 2\pi\epsilon \int R\mathcal{E}_0 dR = 2\pi\epsilon\hat{\mu} \tag{1.52}$$

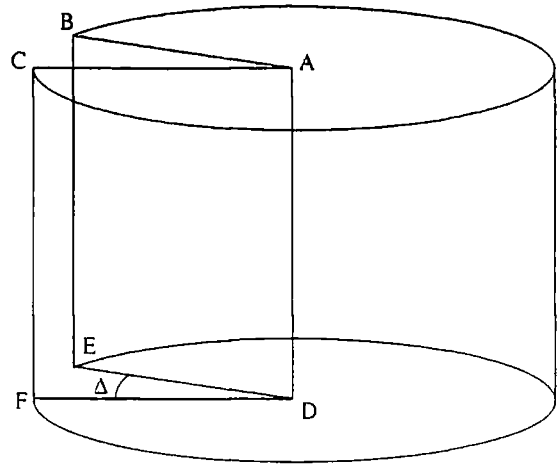

Figure 1.4: The conical nature of the string spacetime.

has been removed from locally flat space as represented in figure 1.4. The deficit angle is proportional to the energy per unit length of the string which in units natural to the vortex is given by $\hat{\mu}$ with

$$\hat{\mu} = \frac{A + D}{\epsilon} \tag{1.53}$$

and it is independent of the radial stresses. but that there is a red/blue-shift of time between infinity and the core of the string if they do not vanish. Finally this gives that the asymptotic string metrics in (1.49) can also be written as

$$ds^2 = d\tilde{t}^2 - d\hat{R}^2 - d\hat{z}^2 - \hat{R}^2(1 - \epsilon\,\hat{\mu})^2 d\varphi^2 \tag{1.54}$$

We now discuss the effects of the asymptotically conical string's spacetime in test particles which move along the geodesics of the spacetime that for a radial motion in a plane transverse to the string, $d\hat{z} = 0$, are given by:

$$\dot{\hat{R}}^2 + \frac{h^2}{\hat{R}^2} + k = E^2 \tag{1.55}$$

where the dot denotes a derivative with respect to the proper time along a timelike geodesic, or an affine parameter for photons. The parameter k is either one or zero. representing either a massive particle or photon respectively. E and h are constants of the motion representing energy and angular momentum respectively, and are given by:

$$E = g_{tt}\dot{\hat{t}} = \dot{\hat{t}} \tag{1.56a}$$

$$h = \frac{g_{\varphi\varphi}\,\dot{\varphi}}{1 - \epsilon\hat{\mu}} = -(1 - \epsilon\hat{\mu})\,\hat{R}^2\dot{\varphi} \tag{1.56b}$$

⋮

From (1.55) one sees that the radial motion of a geodesic is the same as the classical trajectory of a unit mass particle of energy $\frac{E^2}{2}$ with an effective potential given by:

$$V_{\text{eff}} = \frac{h^2}{2\hat{R}^2} + \frac{k}{2} \tag{1.57}$$

which is an identical effective potential to that of a particle moving in flat space. (The presence of the $\epsilon\hat{\mu}$ terms in the definition of h shows that the spacetime is not globally flat, but conical.) All non-static trajectories therefore escape to infinity, and satisfy

$$\hat{R} \geq \frac{h}{\sqrt{E^2 - k}} \, . \tag{1.58}$$

Now let us consider two particles approaching a string and freely moving in a normal plane to that as represented in figure 1.5 by the two parallel paths (dashed and dotted

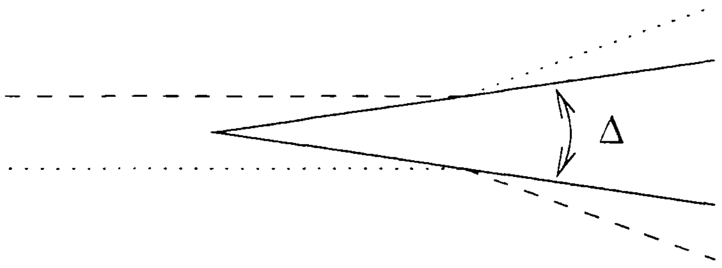

Figure 1.5: Effects of a conical string's spacetime on test particles [38].

lines). The string is located in the apex of the cone with angle Δ. According to the discussion for the geodesics presented above these particles will escape to infinity with h being constant, i.e., when they arrive at the edge of the wedge they continue on the other side at the same angle to the wedge as these sides are identified. By this way the string conical spacetime introduced an inward velocity in each particle. Thus the particles converge in the region behind the string and an over-dense wake forms generating density perturbations. This mechanism may seed galaxy formation as first pointed out by Silk and Vilenkin in reference [2].

If instead of particles one considers photons their trajectories will escape as well for infinity. This gives that two light rays coming from a point source placed behind a string create two images of the source, one on either side of the wedge. This is the gravitational lensing effect represented in figure 1.6 which for strings formed at GUT scales ($\Delta \sim 10^{-6}$) may be observable in images of distant galaxies and clusters [39].

We note that when the string coupling ϵ increases, i.e., the linear mass density increases,

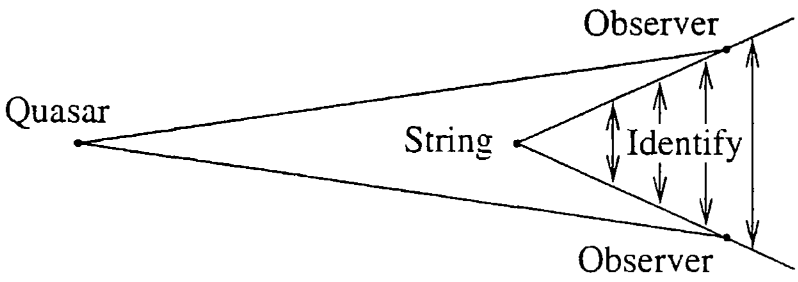

Figure 1.6: Effects of a conical string's spacetime on test photons [1].

the geometry of the spacetime around the string changes drastically beyond a certain value as examined in the next chapter.

1.6.1 The critical coupling

In this section we review the approximations taken by Vilenkin and Gott in references [35] and [36] respectively to determine the gravitational effects of a cosmic string.

In the Gott approximation [36] the energy of the string is distributed uniformly along a cylinder of finite radius, R_0, i.e.:

$$\mathcal{E}(R) = \rho_0 \qquad R \leq R_0 \tag{1.59a}$$

$$\mathcal{E}(R) = 0 \qquad R > R_0 \tag{1.59b}$$

with R_0 greater than or equal to the size of the core, i.e., $R_0 \geq 1$ in these units and ρ_0 a positive constant. The string energy per length in vortex units given in (1.53) is then

$$\hat{\mu} = \frac{R_0^2 \rho_0}{2} . \tag{1.60}$$

To get the gravitational effects of these strings we solve the geometry equations (1.43a)-(1.43c) but first we note that in this case the field equations (1.43d)-(1.43e) reduce to

$$X' = \frac{XP}{\alpha} \tag{1.61a}$$

$$P' = \frac{1}{2}\alpha(X^2 - 1) \tag{1.61b}$$

$$\alpha' = 1 - \epsilon[(X^2 - 1)P + 1] \tag{1.61c}$$

$$\gamma = 0 \tag{1.61d}$$

a first order set of coupled differential equations as one might expect from the fact that the model admits a supergravity extension.

This gives that the pressures $\mathcal{P}_{0R}$ and $\mathcal{P}_{0\varphi}$ in (1.42b)-(1.42c) vanish everywhere. In fact they only vanish everywhere for $\beta = 1$ [32], and the only gravity equation to solve is (1.43a) that becomes

$$\alpha'' = -\epsilon\,\alpha\,\mathcal{E} \tag{1.62}$$

whose regular solution at the core which is continuous at $R = R_0$ is:

$$\alpha(R) = \frac{\sin\left(\sqrt{\epsilon\rho_0}\,R\right)}{\sqrt{\epsilon\rho_0}} \qquad R \le R_0 \tag{1.63a}$$

$$\alpha(R) = \frac{\sin\left(\sqrt{\epsilon\rho_0}\,R_0\right)}{\sqrt{\epsilon\rho_0}} + \cos\left(\sqrt{\epsilon\rho_0}\,R_0\right)\,(R - R_0) \quad R > R_0\,. \tag{1.63b}$$

We now study two extreme limits for the magnitude of $\sqrt{\epsilon\rho_0}\,R_0$: either when it is very small or very large.

The wire model

We first take the limit where $\sqrt{\epsilon\rho_0}\,R_0 \ll 1$. Using equations (1.63a)-(1.63b) it is easy to see that one recovers the results of the previous section. i.e.. this limit corresponds to the weak field approximation where gravity is linearised. In particular taking the limit $R_0 \to 0$. and fixing the energy per unit length. we can see that T_t^t tends to the Dirac distribution. $\delta(R)$. in the two-surface where t and z are constants. i.e..

$$R\,\mathcal{E}(R) = \hat{\mu}\,\delta(R) \tag{1.64}$$

and the energy gets very well localised. In this way the stress energy tensor gets simplified in the core as:

$$T_i^i = \frac{\hat{\mu}}{R}\,[\delta(R),\ \ 0,\ \ 0,\ \ \delta(R)] \tag{1.65}$$

This approximation was first taken by Vilenkin [35]. It corresponds to the limit where $\lambda \to \infty$ (and $e \to \infty$ such that $\beta = \frac{\lambda}{2e^2} = 1$) and it is the so called "*wire approximation*" as such defects can be well approximated as lines with vanishing thickness. Although there are processes where the thickness of the string is relevant or the weak field approximation is not suitable and other methods have to be applied [34, 40, 36]. In particular one expects that in the presence of long range fields such as that of a massless dilaton or of an axion the thickness for these strings is non-negligible and therefore the "*wire approximation*" is no longer valid as it will be examined in chapter 3.

Supermassive strings

We now fix ρ_0 and R_0 to be finite and increase ϵ such that one gets

$$\sqrt{\epsilon\rho_0}\, R_0 = (2\,n + 1)\ \pi \tag{1.66}$$

for some large integer n, i.e., we are taking $\sqrt{\epsilon\rho_0}\, R_0 \gg 1$.

In the limit $\epsilon \to \infty$ it then follows from (1.63b) that $\alpha(R > R_0) < 0$. As near the core $\alpha(R) \simeq R$, by continuity we have that somewhere at a finite distance, R_*, from the core $\alpha(R_*) = 0$ (and therefore a surface where t, z, and R_* are constants), circles with finite radius R_* have vanishing length, i.e., the string's spacetime becomes singular. In fact that happens for $\epsilon > 1$ as explained in next chapter. In this limit strings formed at phase transitions at scales well above the grand-unification scale and which might have taken place in the early universe as discussed in the next chapter and therefore strings are called "*supermassive*" as they couple strongly to gravity. Their strong gravitational effects at large distances are then similar to those of the global strings, another type of cosmic strings presented in the next section.

1.7 Global strings

For the interpretation of some solutions presented in chapters 2 and 3 we introduce another type of string: the global string, which arises as a result of a global symmetry breaking and does not carry a gauge magnetic flux.

The lagrangian for an isolated $U(1)$ global string is [1]:

$$\mathcal{L}[\Phi, A_c] = (\nabla_a\Phi)^\dagger\, \nabla^a\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi - \eta^2)^2\ . \tag{1.67}$$

In order to make manifest the physical degrees we write for an infinite straight string the Higgs field as in (1.15a) with $\chi = \varphi$ where χ is the massless Nambu-Goldstone field and obtain

$$\mathcal{L} = (\nabla_a X)^2 + X^2\, (\nabla_a\chi)^2 - \frac{1}{4}\left(X^2 - 1\right)^2\ . \tag{1.68}$$

The energy and stresses of the vortex can be obtained immediately from those for the local strings, (1.18a)-(1.18d) setting $P_0 = 1$ (and $\beta = \infty$) giving:

$$\hat{T}^t_t = \mathcal{E} = X_0'^2 + \frac{X_0^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \tag{1.69a}$$

$$\hat{T}_R^R = -\mathcal{P}_R = -X_0'^2 + \frac{X_0^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \quad (1.69b)$$

$$\hat{T}_\varphi^\varphi = -\mathcal{P}_\varphi = X_0'^2 - \frac{X_0^2}{R^2} + \frac{1}{4}(X_0^2 - 1)^2 \quad (1.69c)$$

$$\hat{T}_z^z = -\mathcal{P}_z = \hat{T}_t^t \quad (1.69d)$$

which do not fall off to zero rapidly enough as for the gauge strings due to the presence of the massless Nambu-Goldstone field χ. In fact the Higgs asymptotic far field is still given by (1.34a) which now gives that

$$\hat{T}_i^i = \frac{1}{R^2}\,[1,\ 1,\ 1,\ -1] \quad (1.70)$$

This gives that the energy per unit length of the string at far distances from the core becomes

$$\mu \sim \ln(R) \quad (1.71)$$

i.e., much of the energy defect is distributed outside the core which is a very much different behaviour from the very localised energy of the Nielsen Olesen string. The energy per unit length of the string is then infinite.

We also note that the asymptotic stress energy momentum given in (1.70) is not Lorentz invariant ($\hat{T}_i^i \not\propto (1, -1, -1, -1)$). This gives that the spacetime is no more static [1].

In the presence of Einstein gravity and due to its coupling to the Nambu-Goldstone field such strings have long-range gravitational interactions and therefore the gravitational field associated with global strings is not asymptotically flat and due to the infinite vortex energy per unit length the spacetime outside the core is singular. Although there are non-singular string solutions when one requires the large amount of the vortex energy to result in a large effective kinetic mass, i.e., taking a non-static string spacetime as first proved by Gregory in reference [42].

Chapter 2

Non-singular cosmic strings in Einstein gravity

2.1 Introduction

The cosmic strings which produce relevant astrophysical effects are those formed by the GUT scale. Their gravitational fields are weak as their string coupling is small ($\epsilon \simeq 10^{-6}$) [10]. However other phase transitions may have happened in the early Universe and strings may have formed at other scales. Even though these strings are ruled out from observations, one cannot exclude the possibility of their formation and subsequent elimination by some other process such as inflation [43]. When formed at scales above the Planck mass, i.e., for $\epsilon > 1$ and for a critical Higgs coupling, their gravitational effects are strong and static strings become singular somewhere at a finite spatial distance from their cores. Although there is always the possibility of topological inflation [44]. Meanwhile strings formed at smaller scales, (i.e., when $\epsilon \leq 1$) are non-singular [32, 45, 46].

For a non-critical Higgs coupling, Laguna et. al. [47] showed that strings become singular for $\beta = 2$ with $\epsilon = 0.849$ and that their asymptotic spacetime is of a Kasner [48] type. They, [47], also present non-singular string solutions for $\beta = 2$ with $\epsilon = 0.817$ and $\epsilon = 0.502$ for which the asymptotic spacetime is Minkowski minus a wedge of $2\pi\epsilon\hat{\mu}$, as in the asymptotic string solution found first by Vilenkin [35] and deduced by using a weak field approximation. Fitting the values for β and ϵ for the singular string case one gets $\epsilon \simeq \beta^{-0.2}$.

In this chapter we extend the works in refs. [45-47] by studying for a non-critical

Higgs coupling the string's gravitational effects (with no approximations taken for the stress energy of the string.) The fully coupled Einstein-Abelian-Higgs non-linear system is studied by using a qualitative analysis of autonomous dynamical systems and it is argued that strings are non-singular for $\beta \leq 1$ and $\epsilon \leq 1$.

2.2 The dynamical system

We look for non-singular self gravitating string solutions of the Abelian-Higgs model in Einstein gravity already introduced in *Section* 1.6 of chapter 1. For that we write the equations of motion (1.43a)-(1.43e) as the autonomous d.s.:

$$X' = W_1 + g\,X\,a \tag{2.1a}$$

$$W_1' = -y\,W_1 - z\,g\,X\,a - g\,a\,W_1 - g\,X\,V_1 \tag{2.1b}$$

$$a' = V_1 + \frac{1}{2}\,(X^2 - 1) - y\,a \tag{2.1c}$$

$$V_1' = -z\,V_1 - \frac{1}{2}\,z\,(X^2 - 1) - X\,W_1 + (\frac{1}{\beta} - 1)\,X^2\,g\,a \tag{2.1d}$$

$$y' = -2\epsilon\,g^2\,X^2\,a^2 - \frac{1}{2}\,\epsilon\,g\,(X^2 - 1)^2 - y^2 \tag{2.1e}$$

$$g' = z\,g \tag{2.1f}$$

$$z' = -y\,z + 2\epsilon\,\beta\,g\,V_1^2 + 2\epsilon\,\beta\,g\,V_1\,(X^2 - 1) + \frac{1}{2}\,\epsilon\,g\,(\beta - 1)\,(X^2 - 1)^2 \tag{2.1g}$$

and the Bianchi identity (1.44) as the constraint:

$$yz = \frac{1}{4}z^2 + \epsilon W_1^2 + 2\epsilon W_1 g X a + \epsilon\beta g V_1^2 + \epsilon\beta g V_1(X^2 - 1) + \frac{1}{4}\epsilon g(\beta - 1)(X^2 - 1)^2 \tag{2.2}$$

where:

$$W_1 = X' - g\,X\,\frac{P}{\alpha} \tag{2.3a}$$

$$a = \frac{P}{\alpha} \tag{2.3b}$$

$$V_1 = \frac{1}{\alpha}\,P' - \frac{1}{2}\,(X^2 - 1) \tag{2.3c}$$

$$y = \frac{\alpha'}{\alpha} \tag{2.3d}$$

$$g = e^{\gamma} \tag{2.3e}$$

$$z = \gamma' \tag{2.3f}$$

and look for the values of ϵ and β for which string solutions are non-singular.

2.2.1 Vacuum solutions

Since we expect that non-singular strings will tend asymptotically to vacuum at far distances from the core, it is necessary to characterise the vacuum solutions. We take $\epsilon = \beta = 0$ and solve the d.s. (2.1a)-(2.1g) under the constraint (2.2) to get two branches of solutions given by

$$y_c = \frac{d}{dR + \alpha_0} \tag{2.4a}$$

$$z_c = 0 \tag{2.4b}$$

$$g_c = e^{\gamma_0} \tag{2.4c}$$

for branch I and

$$y_c = \frac{d}{dR + \alpha_0} \tag{2.5a}$$

$$z_c = \frac{4d}{dR + \alpha_0} \tag{2.5b}$$

$$g_c = e^{\gamma_0} \left[dR + \alpha_0\right]^4 \tag{2.5c}$$

for branch II with γ_0, d and α_0 being integrating constants.

The first class of solutions (2.4a) -(2.4c) is regular at $R = 0$ provided $d = 1$ and $\gamma_0 = \alpha_0 = 0$ and the spacetime is non-singular while the second one (2.5a) -(2.5c) gives for $\frac{\alpha_0}{d} < 0$ a singular spacetime at the distance $R = -\frac{\alpha_0}{d}$ from the core. For simplicity we take $\frac{\alpha_0}{d} < 0$ in further sections.

For branch I, consistency with the vortex equations requires g_c to be finite. As argued in what follows consistency of these solutions require one of two cases: either we take $d \neq 0$ with any value for α_0 or we take $d = 0$ with $\alpha_0 \neq 0$. The case $d = \alpha_0 = 0$ is excluded.

Let us first take $d \neq 0$ with any value for α_0. At asymptotic far distances from the core one obtains:

$$y_c = z_c = 0 \tag{2.6a}$$

$$g_c = e^{\gamma_0} \ . \tag{2.6b}$$

We note that in this case z converges asymptotically faster than y which is a useful result for *Section* 2.6.

We now take $d = 0$ with $\alpha_0 \neq 0$ which gives the same asymptotical values for the variables y, z and g as in (2.6a) and (2.6b). In this case z converges asymptotically at the same rate as y.

Otherwise when $d = \alpha_0 = 0$ it appears that apparently y is undetermined. However one can calculate α from the definition of y in (2.3d) to get $\alpha_c = 0$. By combining $\alpha_{core} > 0$ with $\alpha'_{core} > 0$ one can immediately conclude that $y_c < 0$. It is also shown in *Section* 2.6 that y is monotonically decreasing at finite distances from the core and therefore this case corresponds to $y_c = -\infty$ for which the vortex fields do not approach their vacuum expectation values and therefore this case is excluded.

We now consider branch II of vacuum solutions and proceed as above. Again consistency with the vortex equations require g_c and y_c to be finite which implies that $d \neq 0$ and $d = \alpha_0 = 0$ are excluded. We then take $d = 0$ with $\alpha_0 \neq 0$ and taking the limit $d \to 0$ first (and then $R \to \infty$) we get y_c and z_c as in (2.6a) with now

$$g_c = e^{\gamma_0}\, \alpha_0^{4} \quad . \tag{2.7}$$

We also note that the metrics for this branch can be written as:

$$ds^2 = e^{\frac{\gamma_0}{3}}\ (3d)^{\frac{4}{3}}\ \rho^{\frac{4}{3}}\ \left[dt^2 - dz^2\right] - d\rho^2 - e^{\frac{-2\gamma_0}{3}}\ (3d)^{\frac{-2}{3}}\ \rho^{\frac{-2}{3}}\ d\varphi^2 \tag{2.8}$$

with $d\rho = e^{\frac{\gamma}{2}}\, dR$ which is an analogy of a Kasner type metric [48] and which for $d = \alpha_0 = 0$ it is exactly the asymptotic singular solution presented in ref. [47].

In the following sections we look for the values for ϵ and β for which strings are non-singular.

2.3 The critical Higgs coupling case

The Equations of motion

We now take a critical Higgs coupling, i.e., $\beta = 1$, for which the equations of motion are given by (1.61a)-(1.61d). This results in

$$\alpha'' = -\epsilon\left[\frac{2X^2\,P^2}{\alpha} + (X^2 - 1)^2\,\frac{\alpha}{2}\right] \tag{2.9}$$

and therefore for $\alpha > 0$, $\alpha'' < 0$ which combined with (1.61c) gives $0 < \alpha' \leq \alpha'_{core} = 1$ when $\epsilon \leq 1$. Therefore the C-energy given by [33]:

$$E_c(R) = 4\pi \ln \left[\frac{g}{\alpha'}\right] \tag{2.10}$$

is well defined everywhere in the region at finite distances from the core and therefore regular strings at the core are non-singular for $\epsilon \leq 1$.

We now take $\epsilon > 1$. Using (1.61c), α' changes sign somewhere at a finite distance from the core becoming negative and therefore α vanishes somewhere at finite distances from the core. This means that in the spatial sections transverse to the string where the coordinates R, z and t are fixed, a circumference has vanishing length, i.e., the metric is singular.

Therefore we conclude that for the critical Higgs coupling, static strings are singular for $\epsilon > 1$ and non-singular for $\epsilon \leq 1$. We now show these results by using a qualitative analysis of the equations of motion written as an autonomous d.s..

Using the equations of motion (1.61a)-(1.61d), one obtains $z = W_1 = V_1 = 0$ and $g = 1$ with the constraint (2.2) being trivially verified. This means that the 7-dimensional d.s. in (2.1a) -(2.1g) is reduced to the 3-dimensional one given by

$$a' = \frac{h}{2} - a\,b + \epsilon\,a^2\,h + \epsilon\,a\,b \tag{2.11a}$$

$$b' = b^2\,(\epsilon - 1) + \epsilon\,b\,a\,h \tag{2.11b}$$

$$h' = 2a\,(h + 1) \tag{2.11c}$$

where

$$b = \frac{1}{\alpha} \tag{2.12a}$$

$$h = X^2 - 1\,. \tag{2.12b}$$

In these variables the coordinates for the vacuum asymptotic solutions are finite and therefore to find non-singular strings it is enough to study the d.s. near the critical points in the finite region of the phase space which in this case is

$$B_1 = (0, b_c, 0) \tag{2.13}$$

with (a_c, b_c, h_c) being generic coordinates. First we note that this critical point corresponds to asymptotic non-singular solutions of branch I (take $d = \gamma_0 = 0$ and $\alpha_0 \neq 0$) and of branch II (take $d = \gamma_0 = 0$ and $\alpha_0 = 1$).

We now proceed with the classification of B_1. We first take $\epsilon \neq 1$ for which $b_c = 0$ and write the d.s. (2.11a)-(2.11c) near that critical point by taking linear perturbations for the variables around that point. In this case the critical point is degenerate, that is the linear perturbations are not conclusive for its classification, higher order perturbations being necessary. A different method, specific to degenerate critical points, is then necessary. For example one can use the blow-up method [49]. In this method one substitutes the critical point in an n-dimensional space by an n-dimensional sphere. In this way it is possible to determine the directions for which the d.s. attracts or repels from the critical point and therefore remove its degenerate behaviour. Applying this method to the critical point B_1 one concludes that after the substitution of this point by a 3-sphere this critical point is transformed into several other degenerate critical points, each one requiring the blow-up method to be applied. As a result this method may not be efficient in this case, as it has to be applied several times with no guarantee of being successful. One could apply other methods to treat degenerate critical points [50], however in this case and for our aim it is enough to classify the critical point B_1 in the invariant surface $b = 0$ to which it belongs and where the d.s. can be written as

$$a' = h\left(\frac{1}{2} + \epsilon a^2\right) \tag{2.14a}$$

$$h' = 2a(h + 1). \tag{2.14b}$$

Linearising the d.s. around the projection of B_1 in this surface, i.e., $D_1 = (0, 0)$, where (a_c, h_c) are generic coordinates, one concludes that D_1 is a saddle critical point [49] and therefore so also is B_1. This means that for regular strings at the core there is at least one approaching direction to the critical point B_1 corresponding to non-singular string solutions and at least one repelling direction from that point corresponding to singular string solutions as proven in what follows. For that it is enough for a starting point to settle down the vortex field X in its vacuum value and analyse the evolution of the other variables. Therefore we analyse the d.s. at the surface $X = 1$ which is taken to be invariant.

When $X = 1$, $h = 0$ and the d.s. (2.11a)-(2.11c) becomes

$$a'_{X=1} = (\epsilon - 1)\, a\, b \tag{2.15a}$$

$$b'_{X=1} = (\epsilon - 1)\, b^2 \tag{2.15b}$$

$$h'_{X=1} = 2a. \tag{2.15c}$$

Taking $X = 1$ to be an invariant surface ($a = 0$) the d.s. (2.15a)-(2.15c) becomes

$$a'_{X=1} = 0 \tag{2.16a}$$

$$b'_{X=1} = (\epsilon - 1)\, b^2 \tag{2.16b}$$

$$h'_{X=1} = 0. \tag{2.16c}$$

Let us take $\epsilon < 1$. From regularity at the core one gets $b_{core} > 0$ with $b'_{core} < 0$ and therefore the variable b starts to decrease from positive values. Using (2.11b) for $\epsilon < 1$ the variable b decreases monotonically, i.e., b decreases monotonically from positive values until its vanishing critical value at asymptotically far distances from the core. Therefore the critical point B_1 is reached later than the invariant surface $X = 1$ as required. Therefore the evolution of the variable b still supports our initial assumption. However it is still necessary to check whether the evolution for the variable a still supports that assumption.

Using the regularity at the core one obtains $a_{core} > 0$ with $a'_{core} < 0$ and therefore the variable a starts to decrease from positive values. Using (2.11a) and $\epsilon < 1$, the variable a decreases monotonically, i.e., from positive values until it vanishes at far distances from the core. Again this supports our initial assumption.

Therefore there are suitable initial conditions for a regular string at the core so that the vortex fields are settled down into their vacuum values at far distances from the core, with the geometry evolving towards the vacuum one, meaning that the d.s. for these string solutions reaches the critical point B_1. Therefore for $\epsilon < 1$ and $\beta = 1$ strings are non-singular. The case $\epsilon = 1$ will be considered separately in the next section as the d.s. (2.11a)- (2.11c) is reducible. However, one can use the above results with the only difference being that from (2.16b), the d.s. reaches the critical point B_1 at the same time as it reaches the invariant surface $X = 1$.

Finally we take $\epsilon > 1$. We now argue that there are no regular conditions at the core for which the d.s. reaches the invariant surface $X = 1$ (where $a = 0$), i.e., the d.s. never stops in the surface $X = 1$ because that is non-invariant.

As already noted for a regular string in the core one obtains $b'_{core} < 0$ and from (2.16b) one gets $b'_{X=1} > 0$, i.e., b' vanishes somewhere at finite distances from the core where $0 < X < 1$. From the equation (2.11b) it results that b has two extrema, i.e., for $b_1 = 0$ and for $b_2 = -\frac{\epsilon a h}{2(\epsilon - 1)}$. As a starting point we take these extrema to be different so that one can take for the initial conditions the regular conditions at the core. Since between those extrema the variable b decreases, this means that we have to set $b_2 > b_1$ and therefore $ah < 0$. We now take the region where $b > 0$ (because the d.s. starts to evolve from there for regular strings at the core), and study whether it stops or still evolves having reached the surface $b = 0$.

Let us first suppose $h < 0$. We now look for a region for the variable a for a regular string at the core so that $a \to 0$ and so that the d.s. stops evolving in the surface $b = 0$. We conclude that there is no suitable region for which $a \to 0$ when imposing initial conditions regular at the core or for any other initial conditions. To get this result we use the equation (2.11a). We first assume that there are two different extrema for the variable a, i.e., a_- and a_+, where $a_- a_+ = \frac{1}{2\epsilon} > 0$ and $a_- + a_+ = -\frac{b(\epsilon-1)}{h\epsilon} > 0$ as we are assuming a region where $b > 0$ with $h < 0$. This means $a_- > 0$ and $a_+ > 0$ and between them i.e., $a_- < a < a_+$, one gets $a' > 0$. This means that there is a region suitable with the initial conditions $a'_{core} < 0$ and $a_{core} > 0$ but for which the variable a does not stop at its vanishing value $a'_{a=0} < 0$. One concludes that under no other initial conditions can the d.s. stop at the surface $a = 0$. One gets the same conclusions if instead we take the extrema for the variable a to coincide, i.e., $a_- = a_+ = -\frac{(\epsilon-1)b}{2h\epsilon} > 0$ as the d.s. evolves so that $a \to a_+$, provided $a_{core} > a_+$ (which seems to be possible from regularity as $a_{core} = \infty$). One could think that this surface, i.e., $a = a_+$, could evolve towards the surface $a = 0$ while $b \to 0$ and $h \to 0$, but now we note that $a_- = a_+$ means that $(\epsilon - 1)^2 b^2 - 2\epsilon h^2 = 0$, i.e. the d.s. reached the surface where $b = -\frac{\sqrt{2\epsilon}h}{\epsilon-1}$ (as we are taking $b > 0$ for $h < 0$) and therefore $a_+ = a_- = \frac{1}{\sqrt{2}}$. Therefore the d.s. never evolves so that $a \to 0$. Therefore we conclude that there are no regular initial conditions at the core or other initial conditions so that the d.s. stops to evolve at the critical point B_1.

We suspected this impossibility would be related with the sign for h and then we supposed that $h > 0$ in the region where $b > 0$. This gives that both extrema for the variable a are negative, i.e., $a_- < 0$ and $a_+ < 0$ and the conclusion remains the same as there are no initial conditions regular at the core or other initial conditions so that the d.s. evolves towards the surface $a = 0$. Finally we suppose that the extrema for the variable a coincide, i.e., $a_- = a_+ = -\frac{(\epsilon-1)b}{2h\epsilon}$ which now gives $a \to a_+ = -\frac{1}{\sqrt{2}}$ and as well the d.s. never evolves so that $a \to 0$.

Therefore there are no initial conditions for regular strings at the core so that for $\epsilon > 1$ the d.s. evolves towards the critical point B_1.

In conclusion for $\epsilon > 1$ and $\beta = 1$ strings are singular.

2.3.1 The critical Higgs coupling case with $\epsilon = 1$

For $\epsilon = \beta = 1$ the variables a and h evolve separately from the variable b and therefore instead of studying the d.s. (2.11a)- (2.11c) one can study the 2-dimensional sub-dynamical system given by

$$a' = \frac{h}{2} + a^2 h \tag{2.17a}$$

$$h' = 2a\,(h + 1) \tag{2.17b}$$

whose phase space, i.e.. $a(h)$. is given in the figure 2.1 and was first presented in [51].

This figure proves that there are suitable initial conditions. i.e.. coordinate values at the core. so that the d.s. evolves into the critical point D_1. As b decreases monotonically this means that there are also suitable initial conditions for b so that the initial 3-dimensional d.s. in (2.11a)- (2.11c) evolves towards the 2-dimensional d.s. (2.17a) -(2.17b) and therefore strings are non-singular for $\epsilon = \beta = 1$.

2.4 The non-critical Higgs coupling case in a flat spacetime

In this section we take a non-critical Higgs coupling, i.e.. $\beta \neq 1$ in a flat spacetime. i.e., $\epsilon = 0$, where obviously string solutions are non-singular. We use this qualitative analysis of the equations of motion to develop several useful techniques for the analysis done in the next section.

2.4.1 The dynamical system

When $\epsilon = 0$ one obtains from the constraint (2.2) that $z = 0$ or $y = \frac{z}{4}$, the latter of which would not be regular at the core of the string, and is therefore excluded. From (2.1g)

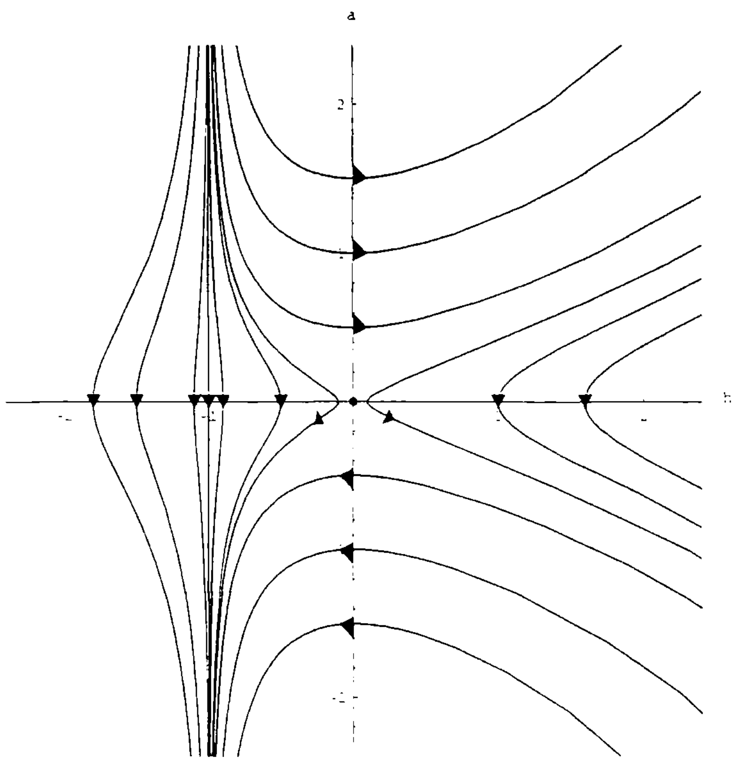

Figure 2.1: The phase space $a(h)$ for $\beta = \epsilon = 1$ [51]. The disk represents the point $D_1 = (0,0)$.

one also obtains that $z = 0$ is an invariant surface. As $z = 0$, i.e., $g = 1$ one gets the 7-dimensional d.s. (2.1a) -(2.1g) is reduced to the following 5-dimensional one:

$$X' = W_1 + Xa \tag{2.18a}$$

$$W_1' = -yW_1 - aW_1 - XV_1 \tag{2.18b}$$

$$a' = V_1 + \frac{1}{2}(X^2 - 1) - ya \tag{2.18c}$$

$$V_1' = -XW_1 + (\frac{1}{\beta} - 1)X^2 a \tag{2.18d}$$

$$y' = -y^2 \tag{2.18e}$$

whose critical points in the finite region of the phase space are:

$$A_1 = (0, 0, a_c, \frac{1}{2}, 0) \tag{2.19a}$$

$$A_2 = (\pm 1, 0, 0, 0, 0) \tag{2.19b}$$

with $A_i = (X_c, W_{1c}, a_c, V_{1c}, y_c)$ being a generic critical point. We now proceed with the analysis of each critical point.

The critical point A_1 corresponds to the non-evolving solution where the vortex fields remain at their core values, and is therefore not relevant.

For the critical point A_2 we note that as $X_c \neq 0$ and $a_c = 0$ one obtains that $\beta \neq 0$. For the + sign for X_c this critical point represents an asymptotic string solution and therefore we proceed with its classification. Linearising the d.s. around this critical point one gets that for $\beta < 1$ it is a saddle critical point while for $\beta > 1$ there are two 2-dimensional surfaces. one in which the trajectories approach the critical point as for as stable focus and another in which the trajectories repel the critical point as for an unstable focus as shown in *Sec.*2.7.

2.4.2 Necessary conditions for the convergence into the critical point and qualitative analysis of the d.s.

In order to prove that strings are non-singular we now show that there is always a physical direction for the d.s. to reach the critical point A_2 with $X_c = 1$ irrespective of the value of β. We first note that as $W_{1core} = W_{1c}$ and $V_{1core} = V_{1c}$ none of these variables can be monotonic. It follows from (2.18b) that they must have opposite signs at finite distances from the core and from (2.18d) that when $W_1 > 0$ we take $\beta < 1$ while for $W_1 < 0$ we take $\beta > 1$.

Let us first take $W_1 > 0$ and $V_1 < 0$ with $\beta < 1$ at finite distances from the core. This shows immediately from (2.18a) and (2.18c) that X and a are monotonic variables either increasing or decreasing respectively. Noting that y is monotonic shows immediately that there are suitable regular initial conditions at the core such that the d.s. converges into the surface S_2 with $X = 1$ and $a = y = 0$.

Meanwhile for $W_1 < 0$ and $V_1 > 0$ with $\beta > 1$ at finite distances from the core it is not immediate that the d.s. converges into S_2, because neither X nor a are monotonic. In order to prove that there are suitable regular initial conditions at the core so that the d.s. converges into the critical point A_2 with $X_c = 1$ we use the same method as in *section* 2.3. There we looked for the conditions for $X = 1$ with $a = 0$ to be an invariant surface which we now apply to the surface S_1 where $X = 1$ with $W_1 = V_1 = a = 0$. Taking the first order perturbations near S_1 the d.s. gets

$$(\delta X)' \simeq \delta W_1 + \delta a \tag{2.20a}$$

$$(\delta W_1)' \simeq -\delta V_1 - y\,\delta W_1 \tag{2.20b}$$

$$(\delta a)' \simeq \delta V_1 + \delta X - y\,\delta a \tag{2.20c}$$

$$(\delta V_1)' \simeq -\delta W_1 + \left(\frac{1}{\beta} - 1\right)\delta a \tag{2.20d}$$

with the constraint (2.2) being verified trivially and where $\delta X = X - X_c$, *etc* with $|\delta X| \ll 1$ and where y is finite, i.e., greater than the biggest perturbation.

As y evolves separately with $y \to 0$ after a certain distance to the core the terms $y\delta W_1$ and $y\delta a$ (respectively in equations (2.20b) and (2.20c)) become negligible because they are of second order in the perturbations. This gives that $(\delta W_1)' \to -\delta V_1$. Using (2.20c) one gets $(\delta a)'' \simeq \frac{\delta a}{\beta}$ and therefore convergence into S_1 is only guaranteed when $\delta a = 0$. This gives that $(\delta V_1)'' \simeq \delta V_1$ and $(\delta W_1)'' \simeq \delta W_1$ and therefore convergence into S_1 requires $\delta W_1 \to 0$ and $\delta V_1 \to 0$ at the same time. This means that up to this order it is not possible to distinguish which vortex field settles down first into its vacuum values Using (2.20c) it is possible to adjust the geometry, i.e., the value for y and therefore the distance to the core, so that one gets $y = y_1$ being finite where $y_1 = \frac{\delta V_1 - \delta X}{\delta a}$. This gives $(\delta a)' \simeq 0$ and therefore $\delta a \simeq$ *constant* which can be adjusted to the limit $\delta a = 0$. Similarly there is a distance from the core where $y = y_2$ with y being finite so that $y_2 = -\frac{\delta V_1}{\delta W_1}$ and therefore $\delta W_1 \simeq$ *constant* which as well can be adjusted to the limit $\delta W_1 = 0$. Now one requires that these distances coincide, i.e., that the d.s. tends to evolve in the direction

$$\frac{V_1 + X}{a} \to -\frac{V_1}{W_1} \tag{2.21}$$

so that $\delta W_1 \to 0$ and $\delta a \to 0$ happen simultaneously. Therefore for any value for β there is a physical direction so that the d.s. can evolve into the critical point A_2 with $X_c = 1$.

In conclusion we argued that there are suitable initial conditions at the core of regular strings so that the d.s. evolves towards the critical point A_2 with $X_c = 1$ irrespective of the values for β.

2.5 The non-critical Higgs coupling case in a non-flat spacetime

Finally we consider the non-critical Higgs coupling case in a non-flat spacetime, i.e., $\beta \neq 1$ with $\epsilon \neq 0$.

Now the d.s. to be studied is (2.1a)- (2.1g) with the constraint (2.2) and whose critical points in the finite region of the phase space, given generically by $q_c = (X_c, W_{1c}, a_c, V_{1c}, y_c, g_c, z_c)$, are given by

$$q_0 = (\pm 1, 0, 0, 0, 0, g_c, 0) \qquad (2.22a)$$

$$q_1 = \left(X_c, 0, a_c, -\frac{1}{2}(X_c^2 - 1), 0, 0, 0\right) . \qquad (2.22b)$$

Using the results of Sec.2.2.1, one gets that the critical point q_1 corresponds to a singular asymptotic solution, (as $g_c = 0$) either of branch I or of branch II, where circles of finite radius have infinite length. Therefore it is not relevant when searching for non-singular solutions.

Therefore the relevant critical point is q_0 with $g_c \neq 0$ (without loss of generality we take the positive root for q_0). This is a degenerate critical point. Applying the blow-up method once, one obtains several degenerate critical points in the 7-sphere that replace the critical point, applying this method again, it turns out that there are still degenerated critical points in the 7-spheres that replace the degenerate critical points. This indicates that the blow-up method is not efficient. Therefore in this case, and for our aim, it is enough to classify the critical point using the linearised method. Using this method one proves (see *Sec.*2.7) that there are always two 2-dimensional surfaces, one where trajectories approach the critical point as for a stable focus (for $\beta > 1$) or saddle critical point (for $\beta < 1$) and another one in which the trajectories repel from the critical point as for an unstable focus ($\beta > 1$) or saddle critical point ($\beta < 1$).

We now look for the values of β and ϵ so that the d.s. evolves towards the critical point q_0 with $g_c \neq 0$. For that we analyse the evolution of some of the dependent variables.

Study of the evolution of the variable y

We first note that if the surface $y = 0$ were an invariant surface as in the previous section, we could apply the same method and therefore the strings would be non-singular for any values for β. But in fact $y = 0$ is not an invariant surface for $\epsilon \neq 0$. Even so, when $\epsilon \ll 1$, this surface can be taken approximately invariant and we can use the argument of the last section to deduce that for $\epsilon \ll 1$ strings are non-singular irrespective of the values of β.

We now argue that it is necessary but not sufficient that $\epsilon \leq 1$ for regular strings at the core be non-singular. Writing $s = \frac{X'}{X}$, (2.1a)-(2.1g) gives

$$y' + \epsilon(hs)' = -y\,(y + \epsilon(hs)) + \epsilon\,(X^2 + 1)\,\left[s^2 - g^2\,a^2\right] \tag{2.23}$$

(where h is given by equation (2.12b)). Assuming a non-singular solution. we can implicitly integrate from the core to get

$$y = -\epsilon hs + \frac{1-\epsilon}{\alpha}\,Exp\left[\int_0^R \epsilon\frac{(X^2+1)}{y+\epsilon hs}\,(s^2 - g^2\,a^2)\right] \tag{2.24}$$

with $(1-\epsilon)$ given by examining (2.24) near the core. We now study the expression (2.24) near the critical point. For a non-singular solution this gives in particular that $W_1 \to 0$. and $h \to 0$ and therefore $s \to ga$ and $hs \to 0$. If one imposes that hs converges faster than $\frac{1}{\alpha}$ then one obtains:

$$y \to b(1-\epsilon) \tag{2.25}$$

(where b was already defined in (2.12a)) that is recognised as the same limit of y for the case $\beta = 1$. As the solution is non-singular one obtains $y \to 0$. Using the results of Sec.2.3 this gives that $\epsilon \leq 1$. This then shows that $\epsilon \leq 1$ is now a necessary (but not sufficient) condition for strings to be non-singular.

2.5.1 Necessary conditions for the convergence into the critical point q_0 with $g_c \neq 0$

We now look for directions so that the d.s. converges into the critical point q_0 with $g_c \neq 0$.

First we note that the surface S_3 where $X = 1$ with $W_1 = V_1 = a = z = 0$ and $g = g_c$ is invariant. Proceeding as in the previous section we write the d.s. near S_3 by taking the first order perturbations to get

$$(\delta X)' \simeq \delta W_1 + g_c\,\delta a \tag{2.26a}$$

$$(\delta W_1)' \simeq -g_c\,\delta V_1 - y\,\delta W_1 \tag{2.26b}$$

$$(\delta a)' \simeq \delta V_1 + \delta X - y\,\delta a \tag{2.26c}$$

$$(\delta V_1)' \simeq -\delta W_1 + \left(\frac{1}{\beta} - 1\right) g_c\,\delta a \tag{2.26d}$$

$$y' = -y^2 \tag{2.26e}$$

$$(\delta g)' \simeq g_c\,\delta z \tag{2.26f}$$

$$(\delta z)' \simeq -y\,\delta z \quad . \tag{2.26g}$$

The order of magnitude of g_c and y is bigger than the order of the biggest perturbation and is finite. One also obtains from the constraint (2.2) that

$$y\,\delta z \simeq 0 \tag{2.27}$$

i.e.. $\delta z \simeq 0$. Therefore z is the first geometric variable to reach its asymptotic vanishing value. as required in *section* 2.2.1 for branch I of vacuum solutions.

For generality we do not substitute the value for g_c.

Let us assume $W_1 \rightarrow 0$ with $W_1 < 0$ at finite distances from the core [1] (We are assuming $g_c = 1$. For $g_c \neq 1$ one shall take $W_1 > 0$). Then using (2.1a) and the fact that $X' > 0$ (from the spontaneous symmetry breaking) one gets $-gXa < W_1 < 0$. It is also assumed that the constraint (2.2) is verified in all of the region at finite distance from the core which from the analysis of the next section is guaranteed for $\beta < 1$. Now from Sec.2.2.1 one requires that $z \rightarrow 0$ faster than $y \rightarrow 0$. So let us take $W_1 \rightarrow 0$ with y finite. i.e.. $(\delta W_1)' \simeq 0$, at a distance $R = R_1$ from the core where $y = y_1$ with

$$y_1(R_1) = -\frac{\delta V_1}{\delta W_1}\,g_c \quad . \tag{2.28}$$

As for non-singular solutions $y > 0$ at finite distances from the core equation (2.28) then gives $\delta V_1 > 0$. i.e., $V_1 > 0$ near the critical point. The distance R_1 is then adjusted so that $\delta W_1 \simeq 0$.

It also proves consistent to assume $\delta a > 0$ for non-singular solutions as shown in Sec.2.3 for $\beta = 1$. We also take $\beta < 1$ as justified in the next section. Combining altogether in (2.26d) one obtains $(\delta V_1)' > 0$ and keeping $V_1(R) > 0$ for $R > R_1$ this would give that δV_1 would never converge into its vanishing asymptotic value. This means that one has to require $W_1 \rightarrow 0$ at the same time as $a \rightarrow 0$ so that $(\delta V_1)' \simeq 0$ and therefore, up to this

[1]This assumption comes from the study of the variable z.

order in the fluctuations, one can not distinguish which vortex field reaches its vacuum value first. We then require $(\delta a(R_1))' \simeq 0$ which from (2.26c) and (2.28) gives

$$y_1 = \frac{\delta V_1 + \delta X}{\delta a} = -\frac{\delta V_1}{\delta W_1}\, g_c \qquad (2.29)$$

and where now y_1 is adjusted so that $\delta a \simeq 0$. As $\delta W_1 \simeq 0$ with $\delta a \simeq 0$ this gives from (2.26a)- (2.26g) and (2.27) that $\delta z \simeq 0$, $\delta X \simeq 0$ and $\delta V_1 \simeq 0$ and therefore the surface S_3 is reached. After that the variable y still evolves into its vanishing critical value, which from a previous section requires $\epsilon \leq 1$.

Therefore we argue that when the d.s. evolves in the physical direction so that

$$\frac{V_1 + X}{a} \rightarrow -g\,\frac{V_1}{W_1} \qquad (2.30)$$

then for $\beta < 1$ and $\epsilon \leq 1$ the d.s. converges into the critical point q_0 with $g_c \neq 0$.

Analysis of the constraint

We now proceed into the analysis of the constraint (2.2) written as $\mathcal{P}(z) = 0$ with:

$$\mathcal{P}(z) = \frac{1}{4}z^2 - yz + \epsilon W_1^2 + 2\epsilon W_1 gXa + \epsilon\beta g V_1^2 + \epsilon\beta g V_1(X^2-1) + \frac{1}{4}\epsilon g(\beta-1)(X^2-1)^2 \quad (2.31)$$

to justify that the d.s. can evolve in the regions required in previous sections.

The constraint (2.2) is verified when the determinant of $\mathcal{P}(z)$ is positive, i.e., when $\mathcal{P}_1(W_1) = \Delta\,\mathcal{P}(z) > 0$ with

$$\mathcal{P}_1(W_1) = y^2 - \left(\epsilon W_1^2 + 2\epsilon W_1 gXa + \epsilon\beta g V_1^2 + \epsilon\beta g V_1(X^2-1) + \frac{1}{4}\epsilon g(\beta-1)(X^2-1)^2\right) \qquad (2.32)$$

where we exclude the case where it vanishes. To argue that we note that if $\mathcal{P}_1(W_1) = 0$ one gets that in particular the d.s. can evolve in the surface

$$y = \sqrt{\epsilon W_1^2 + 2\epsilon W_1 gXa + \epsilon\beta g V_1^2 + \epsilon\beta g V_1(X^2-1) + \frac{1}{4}\epsilon g(\beta-1)(X^2-1)^2}\ . \qquad (2.33)$$

The constraint gives $z = 2y$ which is not verified near the core of non-singular strings and therefore this case is excluded.

Now we note that for very large or very small values of W_1 and using (2.32) one gets $\mathcal{P}_1(W_1) < 0$ which is not what is wanted. In order to get $\mathcal{P}_1(W_1) > 0$ we therefore require 2 roots of $\mathcal{P}_1(W_1)$ and hence the determinant of $\mathcal{P}_1(W_1)$, $\mathcal{P}_1(V_1)$, to be positive, i.e.,

$$\mathcal{P}_1(V_1) = \epsilon\, g^2\, X^2\, a^2 + y^2 - \epsilon\,\beta\, g\, V_1^2 - \epsilon\,\beta\, g\, V_1\,(X^2-1) - \frac{1}{4}\,\epsilon\, g\,(\beta-1)\,(X^2-1)^2 > 0\ . \quad (2.34)$$

Again we note that for large positive or negative values of V_1, $\mathcal{P}_1(V_1) < 0$ and in order to get $\mathcal{P}_1(V_1) > 0$ we require the determinant of $\mathcal{P}_1(V_1)$ to be positive, i.e.,

$$\epsilon\,\beta\,g\,(X^2-1)^2 + 4\left(y^2 + \epsilon g^2 X^2 a^2 - \frac{1}{4}\,\epsilon\,g\,(\beta-1)\,(X^2-1)^2\right) > 0\ . \tag{2.35}$$

This shows immediately that $\beta < 1$ guarantees the existence of a region where $\mathcal{P}_1(V_1) > 0$, i.e., where $\mathcal{P}_1(W_1) > 0$ and consequently a region where the constraint (2.2) is verified.

2.5.2 Qualitative analysis of the dynamical system

Combining the results from the previous sections we finally conclude that there are suitable initial conditions for the asymptotic solutions of regular strings at the core to converge into the critical point q_0 irrespective to the value for g_c ($g_c \neq 0$) when $\epsilon \leq 1$ with $\beta < 1$ provided that for $g_c = 1$ one verifies the conditions:

$$-gXa \ < \ W_1 \ < 0 \tag{2.36a}$$

$$y^2 - \left(\epsilon W_1^2 + 2\epsilon W_1 gXa + \epsilon\beta g V_1^2 + \epsilon\beta g V_1(X^2-1) + \frac{1}{4}\epsilon g(\beta-1)(X^2-1)^2\right) > 0 \tag{2.36b}$$

$$\epsilon\,(gXa)^2 + y^2 - \epsilon\left[\beta g V_1^2 + \beta g V_1(X^2-1) + \frac{g}{4}(\beta-1)(X^2-1)^2\right] > 0 \tag{2.36c}$$

with $W_1 \to 0$. Meanwhile for $g_c \neq 1$ and $g_c \neq 0$ strings are non-singular for $\epsilon \leq 1$ with $\beta < 1$ provided that one verifies the conditions (2.36b) and (2.36c) with $W_1 > 0$ and $W_1 \to 0$ [2].

2.6 Conclusions

2.6.1 Conclusions

In conclusion for $\epsilon \leq 1$ with $\beta = 1$, and for $\epsilon = 0$ or $\epsilon \ll 1$ with any value for β, strings are non-singular. For $\beta < 1$ and $\epsilon \leq 1$ we argued the existence of a direction so that the d.s. approaches the critical point q_0 with $g_c \neq 0$, i.e., that strings are non-singular. In particular we note that this result is consistent with the results in the ref. [47].

[2] This assumption comes from the study of the variable z.

2.7 Appendix: Classification of the critical point q_0 with $g_c \neq 0$.

In order to classify the critical point q_0 with $g_c \neq 0$ we write the d.s. (2.1a)- (2.1g) in a neighbourhood of q_0, i.e., we write $X = X_c + \delta X$, with $|\delta X| \ll 1$, *etc.* and take the corrections of first order to get the equations

$$(\delta X)' \simeq \delta W_1 + g_c\, \delta a \tag{2.37a}$$

$$(\delta W_1)' \simeq -g_c\, \delta V_1 \tag{2.37b}$$

$$(\delta a)' \simeq \delta V_1 + \delta X \tag{2.37c}$$

$$(\delta V_1)' \simeq -\delta W_1 + \left(\frac{1}{\beta} - 1\right) g_c\, \delta a \tag{2.37d}$$

$$(\delta y)' \simeq 0 \tag{2.37e}$$

$$(\delta g)' \simeq g_c\, \delta z \tag{2.37f}$$

$$(\delta z)' \simeq 0 \tag{2.37g}$$

with the constraint (2.2) being verified trivially. From (2.37e) and (2.37g) it follows that $\delta y \simeq 0$ and $\delta z \simeq 0$ respectively, which also gives from (2.37f) that $\delta g \simeq 0$, i.e., $y = y_c$, $z = z_c$ and $g = g_c$. This implies that the critical point is degenerate (up to this order the geometry is vacuum). This means that this method is not enough to classify the critical point. [49]. However for our aim it is enough to classify the projection of q_0 in the surface $y = z = 0$ with $g = 1$. For that we write (2.37a)- (2.37d) in a matrix form, i.e.,

$$\begin{bmatrix} (\delta X)' \\ (\delta W_1)' \\ (\delta a)' \\ (\delta V_1)' \end{bmatrix} = \mathcal{M} \begin{bmatrix} \delta X \\ \delta W_1 \\ \delta a \\ \delta V_1 \end{bmatrix} \tag{2.38}$$

with

$$\mathcal{M} = \begin{bmatrix} 0 & 1 & g_c & 0 \\ 0 & 0 & 0 & -g_c \\ 1 & 0 & 0 & 1 \\ 0 & -1 & g_c\left(\frac{1}{\beta} - 1\right) & 0 \end{bmatrix} \tag{2.39}$$

and determine the eigenvalues, λ_i, of $\mathcal{M}$ [49].

One gets that for $\beta < 1$:

$$\lambda_\pm^2 = \frac{g_c}{2}\left(\frac{1}{\beta} + 1\right)\left[1 \pm \sqrt{1 - \left(\frac{2}{\frac{1}{\beta} + 1}\right)^2}\right] \tag{2.40}$$

and therefore all the eigenvalues are real numbers with some being positive while others being negative, i.e., the projection of the critical point in this surface and therefore the critical point itself behaves like a saddle critical point [49].

Meanwhile for $\beta > 1$

$$\lambda_{\pm}^2 = \frac{g_c}{2}\left(\frac{1}{\beta}+1\right)\left[1 \pm i\sqrt{\left(\frac{2}{\frac{1}{\beta}+1}\right)^2 - 1}\right] \tag{2.41}$$

and therefore all the eigenvalues are complex numbers with $\lambda_{1/3} = a \pm ib$ and $\lambda_{2/4} = -a \mp ib$, with a and b real positive numbers. This means that from $\lambda_{1/3}$ there is a surface where the trajectories repel from the projection of the critical point like those for an unstable focus, while from the eigenvalues $\lambda_{2/4}$ there is a surface where the trajectories attract the projection of the critical point like those for a stable focus [49]. Therefore one concludes that as well for $\beta > 1$ there are directions for which trajectories attract the critical point and directions for which trajectories repel.

Chapter 3

Cosmic strings in axionic-dilatonic gravity

3.1 Introduction

This chapter is divided into two parts. In the first part, including subsections 3.1, we examine the metric of an isolated self-gravitating Abelian-Higgs vortex in dilatonic gravity for arbitrary coupling of the vortex fields to the dilaton. We look for solutions in both massless and massive dilaton gravity. We compare our results to existing metrics for strings in Einstein and Jordan-Brans-Dicke theory. We explore the generalisation of Bogomolnyi arguments for our vortices and comment on the effects on test particles. We then extend the previous analysis to the presence of an axion field and study in the second part (subsections 3.2) the metric of an axionic-dilatonic string. We get string asymptotic solutions by taking a supermassive Higgs limit in *Sec.*3.3.2. Finally we conclude emphasising the most important results for the next chapter.

Einstein's theory of general relativity is extremely successful at describing the dynamics of our solar system, and indeed the observable universe, nonetheless, it probably does not describe gravity accurately at all scales [52] as first postulated by Dirac [53]. Partially motivated by the Dirac's idea and the possible existence of extra dimensions of the space-time proposed by Kaluza and Klein [54], Jordan, Brans and Dicke [5] proposed a theory of gravity whose purpose was to incorporate Mach's Principle. In this way the variations on the inertial mass of a body caused by the surrounding Universe, assumed in that Principle, could be justified from the variations of the gravitational constant. Even so the Principle is not consistently justified, as it does not explain gravity in vacuum. Nonetheless JBD theory has led to various alternative theories for gravity, most notably the scalar tensor family [55] and is related to the gravitational lagrangian inspired by low energy string theory [4]. To see this we compare the actions, which for the bosonic string theory takes the form

$$S = \int d^4x\sqrt{-g}e^{-2\phi}\left(-R - 4(\nabla\phi)^2 + \frac{1}{12}H^2_{\mu\nu\lambda}\right) \tag{3.1}$$

where ϕ is the dilaton and $H_{\mu\nu\lambda}$ is the field strength of the two form $B_{\mu\nu}$ [56] and which for the JBD theory is given by

$$S_{JBD} = \int d^4x\sqrt{-g}\left[\Psi R - \omega\frac{(\nabla\Psi)^2}{\Psi} + \mathcal{L}\right] \tag{3.2}$$

where Ψ is the JBD scalar field and ω its coupling constant, and see that in the canonical representation for the kinetic term of the field Ψ their gravitational sectors are identical if $\omega = -1$.

The implications of such actions on general Friedman-Robertson-Walker cosmological models have been well explored [6–8] however, the implications for theories of structure formation have not been so well studied. Broadly speaking, there are two views on explaining structure formation – inflation or defects, the latter consisting of two subsets: cosmic string or texture [57] induced perturbations. While there is little to choose between these from the particle physics or large scale structure point of view, the implications of each of these theories for the perturbations of the microwave background are distinct. However, calculations on the microwave background multipole moments do assume Einstein gravity [9], therefore it is interesting to question whether these conclusions are still valid in the context of scalar-tensor gravity. Even if the dilaton acquires a mass at a fairly high energy scale (with respect to the recombination temperature of the universe), at the core of a defect symmetry is restored and the physics is determined by the GUT scale, at which the dilaton might have rather different properties, impacting back on the cosmic microwave background.

Calculations involving radiation from a cosmic string network generally make use of a "*worldsheet – approximation*" in which the string is treated as an infinitesimally thin source which moves according to, and has an energy momentum tensor appropriate for, a two-dimensional worldsheet governed by the Nambu action. That this action is appropriate for the local string has been convincingly argued in the absence of gravity [30, 58, 59] but as yet no proof exists in the presence of gravity. Nonetheless, the fact that the self-gravitating infinite local vortex has a relatively small effect on spacetime lends credence to the worldsheet approximation for the string.

In the presence of a dilaton. the worldsheet approximation may no longer be appropriate. If the dilaton is massless. there is no reason to expect that the string will not have a long range effect on the dilaton. and even if the dilaton is massive, it introduces an additional length scale which may still have significant impact. In the next section we take a modest step towards resolving this issue by examining the gravi-dilaton field of a self-gravitating cosmic string in dilaton gravity. In the final section we consider the impact of the axion.

Considering the fully coupled nonlinear field equations of a particular local string model with dilaton gravity we will also show that the Damour and Vilenkin [60] conclusion that a low mass superstring dilaton is incompatible with a local network of strings formed at a GUT phase transition. may not be valid as it is strongly dependent on the coupling of the defect to the dilaton.

3.2 The Abelian-Higgs vortex in dilaton gravity

We are interested in the behaviour of a static self gravitating string whose metric is given by (1.41) when the gravitational interactions take a form typical of low energy string theory [4]. We take an empirical approach to cosmic strings in this background theory. not concerning ourselves with the origin of the fields that form the vortex. but inputting "*by hand*" the abelian-Higgs lagrangian given in (1.12). To take account of the (unknown) coupling of the cosmic string to the dilaton we consider a reasonably general form for the interaction of the vortex with the dilaton assuming that the Abelian-Higgs lagrangian couples to the dilaton via an arbitrary coupling, $e^{2a\phi}\,\mathcal{L}$, in the string frame as in the action:

$$\hat{S} = \int d^4x\sqrt{-\hat{g}}\left[e^{-2\phi}\left(-\hat{R} - 4(\hat{\nabla}\phi)^2 - \hat{V}(\phi)\right) + e^{2a\phi}\hat{\mathcal{L}}\right]. \tag{3.3}$$

This action is written in terms of the string metric, i.e., the metric which appears in the string sigma model [61]. It proves useful to instead write the action in terms of the "*Einstein*" metric, which is defined via

$$g_{ab} = e^{-2\phi}\hat{g}_{ab} \quad . \tag{3.4}$$

in which the gravitational part of the action (written in "*vortex units*") appears in the more familiar Einstein form:

$$S = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\phi)^2 - V(\phi) + 2\epsilon e^{2(a+2)\phi}\mathcal{L}\{X, P, e^{2\phi}g\}\right] \tag{3.5}$$

where $V(\phi) = e^{2\phi}\hat{V}(\phi)$ and [23]

$$R = e^{2\phi}\left[\hat{R} + 6g^{ab}\nabla_a\nabla_b\phi - 6g^{ab}\nabla_a\phi\nabla_b\phi\right]. \tag{3.6}$$

Note however that this complicates the matter part of the lagrangian - a factor of $e^{-2\phi}$ being picked up each time $\hat{g}^{ab}$ is used:

$$T_{ab} = 2\frac{\delta\mathcal{L}[X, P, e^{2\phi}g]}{\delta g^{ab}} = 2e^{-2\phi}[\nabla_a X\nabla_b X + X^2 P_a P_b] - 2\beta e^{-4\phi}F_{ac}F_b^c - \mathcal{L}g_{ab}. \tag{3.7}$$

The "*Einstein*" equations are now

$$G_{ab} = \epsilon e^{2(a-2)\phi}T_{ab} + S_{ab} \tag{3.8}$$

where

$$S_{ab} = 2\nabla_a\phi\nabla_b\phi + \frac{1}{2}V(\phi)g_{ab} - (\nabla\phi)^2 g_{ab} \tag{3.9}$$

represents the energy-momentum of the dilaton, which has its equation of motion

$$\Box\phi = -\frac{1}{4}\frac{\partial V}{\partial\phi} + \epsilon(a+1)e^{2(a+2)\phi}\mathcal{L}[X, P, e^{2\phi}g] + \epsilon e^{2(a-2)\phi}\left[\beta F^2 e^{-4\phi} - \frac{1}{4}(X^2-1)^2\right]. \tag{3.10}$$

When the coupling of the vortex to the dilaton is bigger than to the geometry, i.e., $a > -1$ the vortex contribution to the dilaton field behaves like ordinary matter: if ϕ could be considered as a gravitational potential then for $a > -1$ there would be an attractive force as one can see comparing the dilaton equation with the Poisson one.

The stress energy then becomes:

$$\mathcal{E} = e^{2(a+2)\phi}\left[e^{-2\phi}\left(e^{-\gamma}X'^2 + e^{\gamma}\frac{X^2P^2}{\alpha^2}\right) + e^{-4\phi}\frac{\beta P'^2}{\alpha^2} + \frac{(X^2-1)^2}{4}\right] \tag{3.11a}$$

$$\mathcal{P}_R = e^{2(a+2)\phi}\left[e^{-2\phi}\left(e^{-\gamma}X'^2 - e^{\gamma}\frac{X^2P^2}{\alpha^2}\right) + e^{-4\phi}\beta\frac{P'^2}{\alpha^2} - \frac{(X^2-1)^2}{4}\right] \tag{3.11b}$$

$$\mathcal{P}_\varphi = e^{2(a+2)\phi}\left[e^{-2\phi}\left(-e^{-\gamma}X'^2 + e^{\gamma}\frac{X^2P^2}{\alpha^2}\right) + e^{-4\phi}\beta\frac{P'^2}{\alpha^2} - \frac{(X^2-1)^2}{4}\right] \tag{3.11c}$$

In terms of these variables, the full equations of motion for the gravitating vortex in dilaton gravity are

$$\alpha'' = -\alpha e^{\gamma}V(\phi) - \epsilon\alpha e^{\gamma}(\mathcal{E} - \mathcal{P}_R) \quad ; \tag{3.12a}$$

$$(\alpha\gamma')' = -\alpha e^{\gamma} V(\phi) + \epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_\varphi) \tag{3.12b}$$

$$\alpha'\gamma' = -\frac{1}{2}\alpha e^{\gamma} V(\phi) + \frac{\alpha\gamma'^2}{4} + \alpha\phi'^2 + \epsilon\alpha e^{\gamma}\mathcal{P}_R \tag{3.12c}$$

$$(\alpha\phi')' = \frac{\alpha e^{\gamma}}{4}\frac{\partial V}{\partial\phi} + \epsilon(a+1)\alpha e^{\gamma}\mathcal{E} - \frac{1}{2}\epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_\varphi) \tag{3.12d}$$

$$\frac{1}{\alpha}(\alpha X')' = -2(a+1)X'\phi' + \frac{XP^2}{\alpha^2}e^{2\gamma} + \frac{1}{2}X(X^2-1)e^{\gamma+2\phi} \tag{3.12e}$$

$$\alpha\left(\frac{P'}{\alpha}\right)' = -\gamma' P' - 2a\phi' P' + \beta^{-1}X^2 P e^{\gamma+2\phi} \tag{3.12f}$$

where ϵ is the gravitational strength of the string defined in (1.39), and $V(\phi)$ is the dilaton potential in units natural to the vortex. The Bianchi identity given in (1.44) in Einstein gravity now becomes

$$\epsilon(\alpha e^{\gamma}\mathcal{P}_R)' = \epsilon\alpha' e^{\gamma}\mathcal{P}_\varphi + \frac{1}{2}\epsilon\alpha\gamma' e^{\gamma}[\mathcal{P}_R - \mathcal{P}_\varphi - 2\mathcal{E}] - \alpha'\phi'^2 - (\alpha\phi'^2)' + \frac{1}{2}\alpha e^{\gamma}\phi'\frac{\partial V}{\partial\varphi}. \tag{3.13}$$

(i) $V(\phi) \equiv 0$: Massless dilatonic gravity

We start by examining the case $V(\phi) \equiv 0$, i.e. a massless dilaton, as this ought to be qualitatively the same as a cosmic string in Brans-Dicke gravity.

In the case that the dilaton is massless the equations (3.12a)- (3.12c) are rather reminiscent of the pure Einstein gravity vortex (1.43a) - (1.43c), however, there is one crucial difference - the constraint equation (3.12c) now contains an $\alpha\phi'^2$ term, and unless $a = -1$, $\alpha\phi'$ will definitely be nonzero. Integrating the dilaton equation (3.12d) one gets:

$$\phi = -\frac{D(R)}{2} + (a+1)\int_0^R \frac{A+D}{R} \sim -\frac{D(\infty)}{2} + (a+1)\epsilon\hat{\mu}\ln R \quad as \quad R \to \infty. \tag{3.14}$$

In order to explore this solution, let us first examine whether the thickness of the string can be neglected. For that we assume the "*wire approximation*" [35], namely we set

$$\alpha e^{\gamma}\mathcal{E}(R) = \hat{\mu}\delta(R) \quad ; \quad \mathcal{P}_R = \mathcal{P}_\varphi = 0 \tag{3.15}$$

and verify consistency of the Einstein's equations, (3.12a)- (3.12c) the first two of which can be readily integrated to give

$$\alpha(R) = (1 - \epsilon\hat{\mu})R \tag{3.16a}$$

$$\gamma(R) = 0 \tag{3.16b}$$

but now we find a contradiction – using the dilaton and geometry solutions presented in (3.14) and (3.16a)-(3.16b) respectively, it comes that the constraint (3.12c) is no longer satisfied unless $a = -1$. It is worth examining what has gone wrong here. The wire model is an approximate version of the stress-energy tensor which usually works well in Einstein gravity since the integral

$$\int_0^\infty \alpha e^\gamma (\mathcal{P}_R + \mathcal{P}_\varphi) = 0 \quad , \tag{3.17}$$

which is no longer necessarily true in the presence of the dilaton. A Bogomolnyi solution in flat space or Einstein gravity has the property that $\mathcal{P}_R = \mathcal{P}_\varphi \equiv 0$. therefore the fact that we cannot consistently use the wire approximation for these variables (unless $a = -1$) is an indication that a Bogomolnyi argument cannot exist unless $a = -1$.

Instead, let us examine consistent vacuum solutions to (3.12a) - (3.12c) which should represent asymptotic spacetimes for the string:

$$\alpha = dR + b \tag{3.18a}$$

$$\gamma = \gamma_0 + \frac{c}{d} \ln(dR + b) \tag{3.18b}$$

$$\phi = \phi_0 + \frac{f}{2d} \ln(dR + b) \tag{3.18c}$$

where $f = \pm\sqrt{4dc - c^2}$ from (3.12c). This gives a Levi-Civita [62] solution for the metric. (Note that if ϕ is constant, we have $c = 0$ or $4d$.) The constants b, c, d, f are given by integrating (3.12a)-(3.12d) and to order ϵ are

$$d = 1 - A, \quad b = B, \quad c = 0, \quad f = 2(a+1)(A+D) = 2(a+1)\epsilon\hat{\mu} \tag{3.19}$$

where A, B, D are the integrals given in (1.48a)-(1.48c). We can therefore see that c cannot remain zero, and to order ϵ^2, $c = (a+1)^2\epsilon^2\hat{\mu}^2$. So, unlike the Einstein self-gravitating vortex, the dilaton vortex for $a \neq -1$ has a strong gravitational effect far from the core, albeit an $O(\epsilon^2)$ one

$$ds^2 \approx R^{(a+1)^2\epsilon^2\hat{\mu}^2}(dt^2 - dR^2 - dz^2) - (1-\epsilon\hat{\mu})^2 R^{2-(a+1)^2\epsilon^2\hat{\mu}^2} d\varphi^2 \tag{3.20a}$$

$$\phi \approx (a+1)\epsilon\hat{\mu} \ln R \; . \tag{3.20b}$$

This metric agrees with Gundlach and Ortiz [63], who derived the metric for a Jordan-Brans-Dicke cosmic string. In the string frame,

$$d\hat{s}^2 = e^{2\phi} ds^2 = R^{2(a+1)\epsilon\hat{\mu} + (a+1)^2\epsilon^2\hat{\mu}^2} \left[dt^2 - dR^2 - dz^2 - (1-\epsilon\hat{\mu})^2 R^{2-2(a+1)^2\epsilon^2\hat{\mu}^2} d\varphi^2 \right] \tag{3.21}$$

which is almost, but not quite, a conformally rescaled cone. Note [63] that the radius at which non-conical effects become important is when $R \simeq e^{\frac{1}{(a+1)^2\epsilon^2\hat{\mu}^2}}$ or $r \simeq \sqrt{\lambda}\eta e^{\frac{1}{(a+1)^2\epsilon^2\hat{\mu}^2}}$, therefore, for a typical GUT string, $r = O(10^{10^{14}})$! i.e. well beyond any reasonable cosmological scale.

This is reminiscent of metric of the global string [64], already introduced in Sec. 1.7 of chapter 1, a system which has very strong asymptotic effects and was for some time thought to be singular [59]. The effect of the global string also becomes evident at very large radii ($e^{1/\epsilon}$), however, unlike the metric (3.20a) the global string metric is actually non-static and has an event horizon at finite distance from the core [37].

We now calculate the back-reaction of the solution (3.20a)-(3.20b) on the vortex fields which linearised in the asymptotic far distance, gives that the long-range fall-off of the X and P fields is changed to give:

$$1 - X \simeq \exp\left[-R^{1-\frac{1}{4}(a-1)\epsilon\hat{\mu}}\right] \tag{3.22a}$$

$$P \simeq \exp\left[-R^{\frac{1-\frac{1}{4}(a-1)\epsilon\hat{\mu}}{\sqrt{3}}}\right] \tag{3.22b}$$

which can be interpreted as a thickening of the core by a factor $(1 + \frac{1}{4}(a + 1)\epsilon\hat{\mu})$.

Now let us consider the special case $a = -1$. In this case we see that (setting $\gamma = \phi = 0$ at the core) $\gamma = -2\phi$ and the vortex field equations (3.12e) - (3.12f) reduce to their Einstein form. We therefore obtain the metric (1.49) and $e^{2\phi} = e^{-\gamma} \to e^{-D}$ which gives in the string frame

$$d\hat{s}^2 = dt^2 - dR^2 - dz^2 - \tilde{\alpha}_E^2 e^{-2\gamma_E} d\varphi^2 \tag{3.23a}$$

$$\sim_{R\to\infty} dt^2 - dR^2 - dz^2 - (1 - \epsilon\hat{\mu})^2 R^2 d\varphi^2 \tag{3.23b}$$

i.e. there is no red/blue-shift of time between the core and infinity in the string frame, no matter what the value of β.

Finally, let us consider $\beta = 1$. In this case, to linear order $\mathcal{P}_R = \mathcal{P}_\varphi = 0$, and $\gamma = \phi = 0$ to all orders, and indeed, the Bogmolonyi system (1.61a)-(1.61d) can be shown to provide the solution to the fully self-gravitating string in this case.

(ii) $V(\phi) = 2M^2\phi^2$ Massive dilatonic gravity

In the absence of a preferred potential to take for the dilaton, we will use $V(\phi) = 2M^2\phi^2$, where M is the dilaton mass in "*vortex units*". Of course, we do not expect that this will be the exact form of the dilaton potential, however, a quadratic approximation will be valid provided ϕ remains close to the minimum of the potential. For a GUT string we expect $10^{-11} \leq M \leq 1$, representing a range for the unknown dilaton mass of 10^3GeV - 10^{15}GeV. The dilaton equation (3.12d), then becomes

$$(\alpha\phi')' = \alpha e^{\gamma} M^2 \phi + \epsilon(a+1)\alpha e^{\gamma}\mathcal{E} - \frac{1}{2}\epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_{\varphi}). \tag{3.24}$$

Once again, we begin by considering the wire model for the string which again gives $\alpha(R)$ and $\gamma(R)$ as in (3.16a) - (3.16b). However, the presence of the mass term in (3.24) now alters the form of the dilaton. integrating (3.24) for the wire model gives

$$\phi_w = -(a+1)\epsilon\hat{\mu}K_0(MR) \tag{3.25}$$

where K_0 is the modified Bessel function. In this case. the constraint equation (3.12c) is satisfied for $R > M$. but for $R < M$ we once again require $O(\epsilon^2)$ corrections. this is not really surprising since this is within the Compton radius of the dilaton and we might expect a behaviour analogous to that of the massless dilaton. However. since these corrections are only significant for $R \simeq e^{1/\epsilon^2}$. we can in this case safely ignore them. At the string boundary. we have that $\phi \sim (a+1)\epsilon\hat{\mu}\ln M$ =O(ϵ). hence the quadratic approximation for the potential appears to be justified.

For an extended source. we may solve (3.24) implicitly using its Green's function:

$$\begin{aligned}
\phi &= -\epsilon K_0(MR)\int_0^R I_0(MR')R'\left[(a+1)\mathcal{E}(R') - \frac{1}{2}(\mathcal{P}_R(R') + \mathcal{P}_{\varphi}(R'))\right]dR' \\
&\quad -\epsilon I_0(MR)\int_R^{\infty} K_0(MR')R'\left[(a+1)\mathcal{E}(R') - \frac{1}{2}(\mathcal{P}_R(R') + \mathcal{P}_{\varphi}(R'))\right]dR' \\
&\simeq -(a+1)\epsilon\hat{\mu}K_0(MR) \text{ for } R > 1.\ M \ll 1
\end{aligned} \tag{3.26}$$

which, unlike the massless dilaton case, is now in agreement with the wire model estimate. A plot of $\phi(R)$ is illustrated in figure 3.1 for a $N = \beta = 1$ vortex with various values of M.

We may now write down the asymptotic solution for the cosmic string to order ϵ as:

$$ds^2 = e^{\gamma_E}\left[dt^2 - dR^2 - dz^2\right] - \alpha_E^2 e^{-\gamma_E} d\varphi^2 \tag{3.27a}$$

$$e^{\phi} = e^{-(a+1)\epsilon\hat{\mu}K_0(MR)}. \tag{3.27b}$$

Thus the spacetime is asymptotically conical in both string and Einstein frames.

Now consider $a = -1$. Now the dilaton is very strongly damped to zero outside the core therefore to a good approximation $\phi = 0$ outside the core, irrespective of M, and

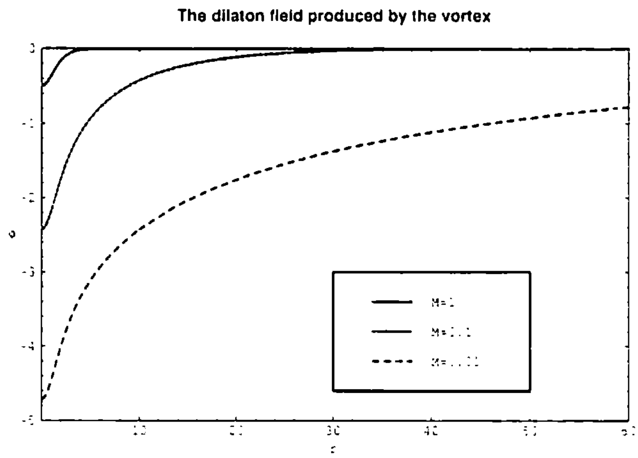

Figure 3.1: A plot of the dilaton field generated by $N = \beta = 1$ vortex for various values of the dilaton mass. The factor $(a+1)\epsilon\hat{\mu}$ has been scaled out of the dilaton. Note the reciprocal dependence of the dilaton fall-off on the mass, compared to the logarithmic dependence of the amplitude.

therefore in both the Einstein and the string frames there is a red or blue shift between the core and infinity.

Finally, if $\beta = 1$, we once again have $\gamma = \phi = 0$, and (1.61a)-(1.61d) gives the first order equations of motion which this system satisfies.

3.2.1 Bogomolnyi bounds for dilatonic cosmic strings

The results of the previous section suggest that $a = -1$ is a rather special point. Usually, for $\beta = 1$, the Bogomolnyi limit, the equations of motion for the cosmic string simplify - they become first order - and the vortex saturates an energy bound determined by the winding number of the vortex [65]. For the dilatonic vortex, this delicate balance appears to be destroyed, except in the special case $a = -1$. In this section we would like to formalise this by presenting an energetic argument that a topological bound can be saturated if and only if $\beta = -a = 1$.

Since the cosmic string is cylindrically symmetric, and we do not *a priori* wish to make any assumptions about the global behaviour of the spacetime, we use an energy tailored to the system at hand - the C-energy introduced by Thorne [33] and already presented in chapter 2:

$$E_c = 4\pi \left[\gamma - \ln \frac{\partial \alpha}{\partial R}\right] = 4\pi \left[\gamma - \ln \alpha'\right] \tag{3.28}$$

modified slightly to allow for the absence of the Newton constant, G. This energy can in turn be represented as the integral of the zeroth component of a covariantly conserved C-momentum vector:

$$E_c = \int \alpha e^{\gamma} \hat{P}^0 dR \tag{3.29}$$

where

$$\begin{aligned} \hat{P}^0 &= \frac{1}{2\pi\alpha e^{\gamma}} \frac{\partial E_c}{\partial R} = \frac{2}{\alpha e^{\gamma}} \left[\gamma' - \frac{\alpha''}{\alpha'}\right] \\ &= \frac{2}{\alpha'} \left[\epsilon\mathcal{E} + \frac{1}{4}\gamma'^2 e^{-\gamma} + e^{-\gamma}\phi'^2 + \frac{1}{2}V(\phi)\right]. \end{aligned} \tag{3.30}$$

Clearly every term in $\hat{P}^0$ is positive semi-definite, and all vanish only in flat space, the latter three vanishing if $\phi = \gamma = 0$. Now consider $\mathcal{E}$, we may rewrite this as

$$\begin{aligned} \mathcal{E} = e^{2(a+1)\phi} \Bigg\{ & e^{-\gamma} \left[X' - e^{\gamma} \frac{XP}{\alpha}\right]^2 + \left[\frac{P'}{\alpha} e^{-\phi} - \frac{1}{2} e^{\phi}(X^2 - 1)\right]^2 \\ & + (\beta - 1)\frac{P'^2}{\alpha^2} e^{-2\phi} + \frac{1}{\alpha}\left[(X^2 - 1)P\right]' \Bigg\} \end{aligned} \tag{3.31}$$

In order to get a "*topological*" value for the C-energy, we need $(\gamma - \ln \alpha')$ to be expressed in terms of X and P; alternatively, we require $\hat{P}^0$ to be a total derivative. For $a = -1$, $\beta = 1$, $\phi = \gamma = 0$ and the equations of motion given in (1.61a)-(1.61d) implies that all terms in $\hat{P}^0$ vanish except for the last expression in equation (3.31) for $\mathcal{E}$. We thus obtain

$$\begin{aligned} E_c &= \int \frac{2\epsilon}{\alpha'} \left[(X^2 - 1)P\right]' \\ &= -2 \int \left(\ln\left[1 - \epsilon[(X^2 - 1)P + 1]\right]\right)' \\ &= -2\ln(1 - \epsilon) = \epsilon + \ldots.. = \eta^2 + O(\eta^4) \end{aligned} \tag{3.32}$$

-the topological bound. For a string with winding number other than one, we replace P by NP and hence E_c becomes $-2\ln(1 - N\epsilon) = N\eta^2 + O(\eta^4)$.

For $\beta \neq 1$ it is immediately clear that this topological bound cannot be saturated due to the presence of the $(\beta - 1)P'^2$ term in the integral. Similarly, if $a \neq -1$, the equation of motion for ϕ shows that ϕ' must be nonzero due to the presence of the $(a+1)\mathcal{E}$ term on the right hand side of (3.12d), hence $\hat{P}^0$ is strictly greater than $\frac{2\epsilon}{\alpha'}\mathcal{E}$, and once again. the topological bound cannot be saturated.

Therefore, by considering a fully covariant relativistic definition of energy for cylindrically symmetric systems, we have shown that there exists a topological "*bound*" for the energy of the vortices. in a rather analogous fashion to the topological quantity originally derived by Bogomolnyi [31] for flat space vortices, and this bound is saturated only for $\beta = -a = 1$.

3.2.2 Geodesics

We finalise the section by discussing the motion of test particles following geodesics in the spacetimes presented in *Sec.* 3.1.2. According to experimental tests [66] any theory describing gravity has to verify the Weak Equivalence Principle (WEP). This principle states that any path through spacetime of a freely falling neutral test body is independent of its structure and composition. Therefore gravity has to couple in the same way to massive test particles and to photons. The obvious way is coupling directly to the metric in the string frame, which is what one usually does in scalar-tensor theories. In other words this principle states that one can transform away the gravitational field (which now includes the dilaton) locally by going to free-fall. Thus it is clear that when the gravitational effects are different from those in Einstein's theory. i.e., non-negligible contributions from the dilatonic sector, this principle is not verified. Therefore the WEP is not verified for $a \neq -1$. Clearly, since the string and Einstein frames are related by a conformal transformation, null geodesics will be the same in either frame, but the geodesics of massive particles will be different.

(i) $V(\phi) = 2M^2\phi^2$: **Massive dilatonic gravity**

We begin by commenting on the massive dilaton. Here the metric is given by the Einstein cousin (3.27a) outside the Compton radius of the dilaton, and is therefore conical. Geodesics are therefore the same as for the Einstein cosmic string, and indeed, since the corrections within the Compton radius of the dilaton are extremely small ($O(\epsilon^2)$), the geodesics throughout the whole spacetime in the Einstein frame are essentially the same as for the Einstein self-gravitating string already discussed in chapter 1.

(i) $V(\phi) = 0$: **Massless dilatonic gravity**

Now consider the massless dilaton. In the string frame the metric is given by eq. (3.21) and the radial motion of a test particle in a plane transverse to the string, $dz = 0$, is given by:

$$\dot{R}^2 + \frac{h^2}{R^{2(1-2\nu)}} + \frac{k}{R^{\nu(2+\nu)}} = \frac{E^2}{R^{2\nu(\nu+2)}} \tag{3.33}$$

where $\nu = (a+1)\epsilon\hat{\mu}$, and the dot denotes a derivative with respect to the proper time along a timelike geodesic, or an affine parameter for photons. The parameter k is either one or zero, representing either a massive particle or photon respectively. E and h are constants of the motion representing energy and angular momentum respectively, and are given by:

$$E = g_{tt}\dot{t} = R^{\nu(\nu+2)}\dot{t} \tag{3.34a}$$

$$h = \frac{g_{\varphi\varphi}\dot{\varphi}}{1-\epsilon\hat{\mu}} = -(1-\epsilon\hat{\mu})R^{2-\nu^2-2\nu}\dot{\varphi} \tag{3.34b}$$

For $a = -1$, $\nu = 0$, and irrespective of whether the dilaton is massive or massless, the geodesics are qualitatively the same as for the Einstein cosmic string already discussed in *Sec.*1.6 of chapter 1.

We then consider $a \neq -1$. For comparison with the effective potential given in (1.57), it is useful to redefine the radial coordinate R via

$$\rho = \frac{R^{(\nu+1)^2}}{(\nu+1)^2} \tag{3.35}$$

which gives the ρ-radial motion as that of a unit mass particle of energy $\frac{E^2}{2}$, with an effective potential given by:

$$U_{\text{eff}} = \frac{h^2}{2[(\nu+1)^2\rho]^{\frac{2(1-\nu^2)}{(\nu+1)^2}}} + \frac{k}{2}[(\nu+1)^2\rho]^{\frac{\nu(\nu+2)}{(\nu+1)^2}}. \tag{3.36}$$

Since $\nu = O(\epsilon)$, to leading order this is

$$U_{\text{eff}} \simeq \frac{h^2}{2(1+4\nu)\rho^{2(1-\nu)}} + \frac{k}{2}\rho^{2\nu}. \tag{3.37}$$

the terms involving ν from (3.36).

3.3 Cosmic strings in axionic-dilatonic gravity

In the final part of this chapter we study the metric of a self gravitating dilatonic string in the presence of an axion field. The gravi-axion-dilaton field is examined and its gravitational effects determined. Because there are no local axionic-dilatonic string solutions we take instead a supermassive Higgs limit to get asymptotic global string solutions known as dyonic universes.

3.3.1 Singular axionic-dilatonic strings

Assuming the gravitational interactions take a form typical of low energy string theory [4], isolated axionic-dilatonic cosmic strings can be described in the string frame by the action:

$$\hat{S} = \int d^4x\sqrt{-\hat{g}}\left[e^{-2\phi}\left(-\hat{R} - 4(\hat{\nabla}\phi)^2 - \hat{V}(\phi) + \frac{1}{12}\hat{H}^2_{\mu\nu\lambda}\right) + e^{2a\phi}\hat{\mathcal{L}}\right] \tag{3.38}$$

where apart the dilatonic string sector already described in the previous section, there is an axion field described by an antisymmetric tensor $B_{\nu\lambda}$ with antisymmetric field strength $\hat{H}_{\mu\nu\lambda}$ given by: $\hat{H}_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]}$ (where the brackets mean anti-symmetrisation).

Proceeding as for the dilatonic cosmic string, we write instead the action in terms of the "*Einstein*" metric, defined in (3.4) in which the gravitational part of the action appears in the more familiar Einstein form:

$$S = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\phi)^2 - V(\phi) + \frac{1}{12}e^{-4\phi}H^2_{\mu\nu\lambda} + 2\epsilon e^{2(a+2)\phi}\mathcal{L}\{X, P, e^{2\phi}g\}\right] \tag{3.39}$$

and write the "*Einstein*" equations to get

$$G_{ab} = \epsilon e^{2(a+2)\phi}T_{ab} + S_{ab} + N_{ab} \tag{3.40}$$

with T_{ab} and S_{ab} the energy-momentum tensors for the vortex and dilaton fields given in (3.7) and (3.9) respectively and N_{ab} the energy-momentum tensor for the axion given by:

$$N_{ab} = \frac{1}{12}e^{-4\phi}[3H_{a\lambda k}H_b{}^{\lambda k} - \frac{1}{2}g_{ab}H^2]. \tag{3.41}$$

Considering the axionic antisymmetric tensor, $B_{\nu\lambda}$, as an independent variable, the equation of motion for the axion is:

$$\nabla_\mu[e^{-4\phi}H^{\mu\nu\lambda}] = 0 \tag{3.42}$$

which can be simplified into a wave equation of motion for a massless scalar field, $h(t, r, z, \varphi)$, evolving coupled to the dilaton [7]

$$\Box h + 4\nabla^\mu\phi\nabla_\mu h = 0 \tag{3.43}$$

by writing

$$e^{-4\phi}H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda k}h_{,k} \tag{3.44}$$

with $\epsilon^{\mu\nu\lambda k}$ the antisymmetric Levi Civita tensor given by

$$\epsilon_{abcd}\epsilon^{abcf} = -3!\delta^f_d. \tag{3.45}$$

In terms of h, the energy-momentum tensor for the axion, (3.41), then becomes:

$$N_{ab} = \frac{1}{2}e^{4\phi}[h_{,a}h_{,b} - \frac{1}{2}g_{ab}h_{,i}h_{,j}g^{ij}]. \tag{3.46}$$

To include the self-gravity of the string we require a metric which exhibits the symmetries of the sources, namely the boost invariance for the matter and dilatonic sectors ($T^0_0 = T^z_z$, $S^0_0 = S^z_z$). Therefore we try first the static cylindrically symmetric metric given in (1.41). Consistency with the assumed symmetry then requires $N_{00} = -N_{zz}$, i.e., $h_{,t} = h_{,z} = 0$ and also $N_{R\varphi} = 0$, i.e., $h_{,R} = 0$ or $h_{,\varphi} = 0$. Therefore there are two possible *ansätze* for the axion: either $h(R)$ or $h(\varphi)$. Consistency of the constraint equation (given later by (3.48c)) also requires N^R_R to be a function of R and therefore for the ansatz $h(\varphi)$ we require $h_{,\varphi} = h$, i.e., $h(\varphi) = h\varphi$, with h a constant.

(i) $h(R)$

Let us first assume $h(R)$. Integrating the equation of motion for the axion (3.43) one gets:

$$\alpha h' = h_0 e^{-4\phi} \tag{3.47}$$

where h_0 is an integrating constant, with the energy-momentum tensor for the axion given by: $\hat{N}^0_0 = \hat{N}^z_z = \hat{N}^\varphi_\varphi = -\hat{N}^R_R = \frac{1}{4}e^{-\gamma+4\phi}h'^2$ where the prime means now the derivative with respect to R.

The equations of motion for the geometry and for the dilaton are respectively:

$$\alpha'' = -\alpha e^{\gamma} V(\phi) - \epsilon\alpha e^{\gamma}(\mathcal{E} - \mathcal{P}_R) \tag{3.48a}$$

$$(\alpha\gamma')' = -\alpha e^{\gamma} V(\phi) + \epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_{\varphi}) \tag{3.48b}$$

$$\alpha'\gamma' = -\frac{1}{2}\alpha e^{\gamma} V(\phi) + \frac{\alpha\gamma'^2}{4} + \alpha\phi'^2 + \epsilon\alpha e^{\gamma}\mathcal{P}_R + \frac{h_0^2}{4\alpha}e^{-4\phi} \tag{3.48c}$$

$$(\alpha\phi')' = \frac{\alpha e^{\gamma}}{4}\frac{\partial V}{\partial\phi} + \epsilon(a+1)\alpha e^{\gamma}\mathcal{E} - \frac{1}{2}\epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_{\varphi}) + \frac{h_0^2}{2\alpha}e^{-4\phi} \tag{3.48d}$$

while the equations of motion for the vortex sector are given by (1.43d)-(1.43e): the same as for the dilatonic string as the axion does not couple directly to matter. For further reference, the Bianchi identity now gives:

$$\begin{aligned}&\epsilon(\alpha e^{\gamma}\mathcal{P}_R)' - \epsilon\alpha' e^{\gamma}\mathcal{P}_{\varphi} - \frac{\epsilon}{2}\alpha\gamma' e^{\gamma}[\mathcal{P}_R - \mathcal{P}_{\varphi} - 2\mathcal{E}] \\ &+\alpha'\phi'^2 + (\alpha\phi'^2)' - \frac{\alpha}{2}e^{\gamma}\phi'\frac{\partial V}{\partial\phi} + \alpha e^{4\phi}h'[h'(\frac{\alpha'}{2\alpha} + \phi') + \frac{h''}{2}] = 0\end{aligned} \tag{3.49}$$

Let us start to look for the string's gravitational effects at very far distances from its core, where for non-singular local strings we are in vacuum.

Consider first the massless dilaton case, $V(\phi) \equiv 0$ for which the equations of motion for the geometry and the dilaton become:

$$\alpha'' = 0 \tag{3.50a}$$

$$(\alpha\gamma')' = 0 \tag{3.50b}$$

$$\alpha'\gamma' = \frac{\alpha\gamma'^2}{4} + \alpha\phi'^2 + \frac{h_0^2}{4\alpha}e^{-4\phi} \tag{3.50c}$$

$$(\alpha\phi')' = \frac{h_0^2}{2\alpha}e^{-4\phi}. \tag{3.50d}$$

These equations are only consistent for the trivial case where the axion is "*switched off*" and so results (3.20a) -(3.20b) for the dilatonic string still hold. Indeed one gets:

$$\alpha = dR + b \tag{3.51a}$$

$$\gamma = C_3 + \frac{C_2}{d}Ln\left[dR + b\right] \tag{3.51b}$$

$$\phi = Ln\left[e^{-8\phi_0}\frac{8C}{(Cosh(A))^2}\right]^{-\frac{1}{4}} \tag{3.51c}$$

with

$$A = \frac{\sqrt{8C}}{d}Ln\left[\frac{dR+b}{C_1}\right] \tag{3.52}$$

where C, C_1, C_2, C_3, d and b are integrating constants.

From the dilaton equation (3.50d) it comes that C is defined such that:

$$(\alpha\phi')^2 = -\frac{h_0^2}{4}e^{-4\phi} + 2C \geq 0 \tag{3.53}$$

and therefore $C > 0$. But now we find a contradiction:- integrating (3.48b) since the core and to lowest order in ϵ one gets $\alpha\gamma' = 0$, i.e., $C_2 = 0$. Using the constraint (3.50c), $dC_2 = \frac{1}{4}C_2^2 + 2C$, this gives $C = 0$ which is absurd. This means $h' = \phi' = 0$. Therefore the only vacuum solution is the trivial one where the axionic sector is "*switched off*" and the dilaton is constant. This result still holds in the string frame and is in agreement with those presented in reference [67] for massless dilatons in a spacetime of four dimensions.

Consider now massive dilatons ($V(\phi) = 2M^2\phi^2$). Apart from an intermediate annular region (bounded by the Compton wavelength of the dilaton), the long-range structure of the fields is as for the Einstein case because for distances from the core greater than the Compton wavelength of the dilaton the dilaton field is essentially fixed. Hence the equations of motion for the geometry and the dilaton become:

$$\alpha'' = -\alpha e^{\gamma}2M^2\phi^2 \tag{3.54a}$$

$$(\alpha\gamma')' = -\alpha e^{\gamma}2M^2\phi^2 \tag{3.54b}$$

$$\alpha'\gamma' = -\alpha e^{\gamma}M^2\phi^2 + \frac{\alpha\gamma'^2}{4} + \frac{h_0^2}{4\alpha}e^{-4\phi} \tag{3.54c}$$

$$0 = \alpha e^{\gamma}M^2\phi + \frac{h_0^2}{2\alpha}e^{-4\phi}. \tag{3.54d}$$

whose solution is:

$$\alpha - \alpha_0 = C_4 R \tag{3.55a}$$

$$\gamma - \gamma_0 = -2\ln[\alpha_0 + C_4 R] \tag{3.55b}$$

$$h - h_1 = -2\phi_1\frac{C_5}{C_4}\ln[\alpha_0 + C_4 R] \tag{3.55c}$$

$$\phi = \phi_1 \tag{3.55d}$$

with

$$-\frac{h_0^2 e^{-4\phi_1}}{2\phi_1} = C_5 = \alpha^2 e^{\gamma} M^2 \tag{3.56}$$

which from (3.54c) gives

$$\phi_1 = -\frac{1}{4} - \frac{1}{4}\sqrt{1 + 48\frac{C_4^2}{C_5}} \tag{3.57}$$

where α_0, C_4, γ_0, h_1 are integrating constants and C_5 is a positive constant.

The asymptotic solution for the axionic-dilatonic cosmic string is then, to lowest order:

$$ds^2 = e^{\gamma_0}\left[\alpha_0 + C_4 R\right]^{-2}\left[dt^2 - dR^2 - dz^2\right] - e^{-\gamma_0}\left[\alpha_0 + C_4 R\right]^4 d\varphi^2 \tag{3.58}$$

with $\phi = \phi_1$ and the axion having very strong asymptotic effects as given in (3.55c).

(ii) $h(\varphi) = h\varphi$

Let us now consider the other ansatz for the axion, $h(\varphi) = h\varphi$. The axion is no more a dynamical field and so its equation of motion gives an identity. The elements of the energy-momentum tensor are: $\hat{N}^0_0 = \hat{N}^z_z = \hat{N}^R_R = -\hat{N}^\varphi_\varphi = \frac{1}{4\alpha^2}e^{4\phi}h^2e^{\gamma}$ and the the axionic sector in the initial action (3.39) is given by $\frac{1}{12}e^{-4\phi}H^2_{\mu\nu\lambda} = \frac{1}{2}e^{4\phi+\gamma}\frac{h^2}{\alpha^2}$.

The equations of motion for the geometry and the dilaton are now respectively:

$$\alpha'' = -\alpha e^{\gamma}V(\phi) - \epsilon\alpha e^{\gamma}(\mathcal{E} - \mathcal{P}_R) - \frac{h^2}{2\alpha}e^{2\gamma-4\phi} \tag{3.59a}$$

$$(\alpha\gamma')' = -\alpha e^{\gamma}V(\phi) + \epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_\varphi) \tag{3.59b}$$

$$\alpha'\gamma' = -\frac{1}{2}\alpha e^{\gamma}V(\phi) + \frac{\alpha\gamma'^2}{4} + \alpha\phi'^2 - \epsilon\alpha e^{\gamma}\mathcal{P}_R - \frac{h^2}{4\alpha}e^{2\gamma-4\phi} \tag{3.59c}$$

$$(\alpha\phi')' = \frac{\alpha e^{\gamma}}{4}\frac{\partial V}{\partial\phi} + \epsilon(a+1)\alpha e^{\gamma}\mathcal{E} - \frac{1}{2}\epsilon\alpha e^{\gamma}(\mathcal{P}_R + \mathcal{P}_\varphi) + \frac{h^2}{2\alpha}e^{2\gamma-4\phi} \tag{3.59d}$$

and again the equations of motion for the vortex fields are the same as for the dilatonic string. The Bianchi identity is now:

$$\begin{aligned}
&\epsilon(\alpha e^{\gamma}\mathcal{P}_R)' - \epsilon\alpha' e^{\gamma}\mathcal{P}_\varphi - \frac{\epsilon}{2}\alpha\gamma' e^{\gamma}[\mathcal{P}_R - \mathcal{P}_\varphi - 2\mathcal{E}] \\
&+\alpha'\phi'^2 + (\alpha\phi'^2)' - \frac{\alpha}{2}e^{\gamma}\phi'\frac{\partial V}{\partial\phi} - \frac{h^2}{\alpha}e^{4\phi-2\gamma}\phi' = 0.
\end{aligned} \tag{3.60}$$

Again, consider first the massless case, $V(\phi) \equiv 0$. Proceeding as for the previous ansatz we write the equations of motion for the geometry and the dilaton:

$$\alpha' = -\alpha\phi' + q \tag{3.61a}$$

$$\alpha\gamma' = p \tag{3.61b}$$

$$\alpha\frac{\alpha''}{2} + \alpha'(-2q - p) + (q^2 + \frac{1}{4}p^2) + \alpha'^2 = 0 \tag{3.61c}$$

$$(\alpha\phi')' = \frac{1}{2\alpha}h^2e^{2\gamma+4\phi} \tag{3.61d}$$

at far distances from the core where it is vacuum, with the Bianchi identity giving

$$\alpha'\phi'^2 + (\alpha\phi'^2)' - \frac{h^2}{\alpha}e^{4\phi+2\gamma}\phi' = 0 \tag{3.62}$$

:

where p and q are integrating constants. In order to get the asymptotic solutions to those equations we write the constraint (3.61c) as the autonomous d.s.:

$$u' = -3u^2 + 2(2q+p)ut - 2(q^2 + \frac{p^2}{4})t^2 \tag{3.63a}$$

$$t' = -ut \tag{3.63b}$$

where $t = \frac{1}{\alpha}$ and $u = \frac{\alpha'}{\alpha}$. Writing $x = \frac{u}{t} = \alpha'$ one gets:

$$dx = \left(3x + \frac{2}{x}(q^2 + \frac{p^2}{4}) - 2(2q+p)\right)\frac{dt}{t} \tag{3.64}$$

which integrated gives

$$t(x) = C\,(Q(x))^{\frac{1}{6}}\,Exp\left[-\frac{\sqrt{2}(p+2q)}{3\sqrt{p^2-8pq+4q^2}}\arctan[\frac{\sqrt{2}(p+2q-3x)}{\sqrt{p^2-8pq+4q^2}}]\right] \tag{3.65}$$

with C a positive integrating constant and $Q(x) = p^2 + 4q^2 - 4px - 8qx + 6x^2$, a non-invertible expression for $\alpha(R)$ whose asymptotic regimes are then analysed.

When $|x| \to \infty$ the result from expression (3.65) is that $t \to \infty$ where

$$t(x) = \frac{4}{C_3^4}|x|^{\frac{1}{3}} \tag{3.66}$$

Therefore the solution is:

$$\alpha = \alpha_0 + C_3 R^{\frac{1}{4}} \tag{3.67a}$$

$$\gamma = \gamma_0 + p\left(\frac{4\alpha_0^2}{C_3^3}R^{\frac{1}{4}} - \frac{2\alpha_0}{C_3^2}\sqrt{R} + \frac{4}{3C_3}R^{\frac{3}{4}} - \frac{4\alpha_0^3}{C_3^4}Ln\left[\alpha_0 + C_3R^{\frac{1}{4}}\right]\right) \tag{3.67b}$$

$$\phi = \phi_0 + q\frac{4\alpha_0^2}{C_3^3}R^{\frac{1}{4}} - q\frac{2\alpha_0}{C_3^2}\sqrt{R} + q\frac{4}{3C_3}R^{\frac{3}{4}} - (1 + q\frac{4\alpha_0^3}{C_3^4})Ln\left[\alpha_0 + C_3R^{\frac{1}{4}}\right] \tag{3.67c}$$

with C_3 a positive constant and α_0, γ_0 and ϕ_0 being integrating constants. But now an inconsistency comes- consistency of equation (3.61d) requires $C_3 = h = 0$, i.e., the axionic sector for these solutions is "*switched off*".

When $x \to x_\pm$, where $Q(x_\pm) = 0$, therefore from expression (3.65) $t \to 0$. As a result the solution is:

$$\alpha = \alpha_0 + x_\pm R \tag{3.68a}$$

$$\gamma = \gamma_0 + \frac{p}{x_\pm}Ln\left[\alpha_0 + x_\pm R\right] \tag{3.68b}$$

$$\phi = \phi_0 + (\frac{q}{x_\pm} - 1)Ln\left[\alpha_0 + x_\pm R\right] \tag{3.68c}$$

which from (3.61d), as in previous asymptotic regimes, gives $h = 0$, i.e., the axionic sector for these solutions it is as well "*switched off*".

Finally when $x \to 0$ then, from expression (3.65) $t \to t(0)$ and the solution is:

$$\alpha = \alpha_0 \tag{3.69a}$$

$$\gamma = \gamma_0 + \frac{p}{\alpha_0}R \tag{3.69b}$$

$$\phi = \phi_0 + \frac{q}{\alpha_0}R \tag{3.69c}$$

which, again, from (3.61d) gives $h = 0$. In conclusion the only vacuum solution is the trivial one where the axion is *"switched off"*. The same result is obtained in the string frame. Moreover no solution (in the Einstein frame) is regular at the origin where $\hat{N}_0^0 = \frac{1}{4\alpha^2}e^{4\phi+\gamma}h^2 \sim \frac{1}{R}$, i.e., $e^{4\phi} \sim R$ because from the dilaton equation (3.59d) one obtains $\phi \sim R$ whilst from the constraint (3.59c) one gets $\phi \sim \sqrt{R}$. Therefore solutions of the equations of motion are non-string type solutions.

It is worth examining the previous results. First we note that for the ansatz $h(\varphi) = h\varphi$ the elements of the energy-momentum tensor for the axion resemble those for a global string given in (1.70). This fact when combined with the absence of non-trivial vacuum solutions suggest that instead one shall look for global string type solutions whose form is given by [68]

$$ds^2 = \rho^4\left[dt^2 - dz^2 - d\rho^2\right] - \frac{D^2}{\rho^2}d\varphi^2. \tag{3.70}$$

On the other hand at large distances from the core the form for the global string metric is the asymptotic limit of the Melvin metric, an exact solution of the Einstein-Maxwell equations in the presence of a magnetic field given by [69]

$$ds^2 = (1 + \frac{1}{4}B^2R^2)^2\left[dt^2 - dR^2 - dz^2\right] - \frac{R^2}{(1+\frac{1}{4}B^2R^2)^2}d\varphi^2 \tag{3.71}$$

with B the strength of the magnetic field along the axis of symmetry. Therefore the asymptotic form at far distances from the core for the metric of global strings can be obtained by taking the limit case when the mass of the vector boson vanishes. The Higgs field then decouples and one is left with the Einstein-Maxwell equations in the presence of the dilaton and the axion whose exact metric is the dyonic Melvin universe. Therefore in the next section we present asymptotic global type solutions by taking the asymptotic form of dyonic universes.

3.3.2 Global string solutions: dyonic universes

Dilatonic Melvin Universe

The dilatonic Melvin magnetic universe is an exact solution of the Dilatonic-Einstein-Maxwell theory described by the action:

$$S = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right] \tag{3.72}$$

where R is the scalar curvature, ϕ is the dilaton, whose coupling to the matter is unknown and generically described by a, and F is the field strength of any $U(1)$ gauge field, A.

In the presence of a magnetic field B the equations of motion given by [70]:

$$\nabla_\mu\left[e^{-2a\phi}F^{\mu\nu}\right] = 0 \tag{3.73a}$$

$$\Box\phi + \frac{1}{2}ae^{-2a\phi}F^2 = 0 \tag{3.73b}$$

$$R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2a\phi}F_{\mu\rho}F^\rho_\nu - \frac{1}{2}g_{\mu\nu}e^{-2a\phi}F^2. \tag{3.73c}$$

admit an exact solution, the dilatonic Melvin magnetic universe, given by [70]

$$ds^2 = \left(1 + \frac{a^2+1}{4}B^2R^2\right)^{\frac{2}{1+a^2}}\left[dt^2 - dR^2 - dz^2\right] - \left(1 + \frac{a^2+1}{4}B^2R^2\right)^{-\frac{2}{1+a^2}}R^2d\varphi^2 \tag{3.74a}$$

$$e^{-2a\phi} = \left(1 + \frac{a^2+1}{4}B^2R^2\right)^{\frac{2a^2}{1+a^2}} \tag{3.74b}$$

$$A_\varphi = -\frac{2}{(1+a^2)B(1+\frac{a^2+1}{4}B^2R^2)} \tag{3.74c}$$

$$B^z = -\frac{B}{\left(1+\frac{a^2+1}{4}B^2R^2\right)^{2\frac{(2-a^2)}{1+a^2}}} \tag{3.74d}$$

The equations of motion are duality invariant as they admit an $SL(2, R)$ electric-magnetic duality defined by:

$$\phi' = -\phi \tag{3.75a}$$

$$F'_{\mu\nu} = \frac{1}{2}e^{-2a\phi}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \tag{3.75b}$$

which can be used to generate an electromagnetic Melvin universe and that will be generalised in the presence of an axion in the next section.

Dyonic Melvin Universe

In the presence of an axion, the Melvin magnetic universe can be generated after applying a Peccei-Quinn and a duality transformation to the dilatonic Melvin magnetic universe. Our starting point comes from the bosonic part of the four-dimensional effective action of low energy string theory that includes the terms [71] :

$$S = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\phi)^2 + \frac{1}{12}e^{-4\phi}H^2_{\mu\nu\lambda} - e^{-2\phi}F^2\right] \tag{3.76}$$

where $H_{\mu\nu\lambda}$ describes the axion, with now

$$H = \partial B + \frac{1}{4}A \wedge F. \tag{3.77}$$

If we write the axion as in (3.44) the equations of motion can equally be derived from the action [71] :

$$S = \int dx^4\sqrt{-g}\left[-R + 2(\nabla\phi)^2 + \frac{1}{2}e^{4\phi}(\nabla h)^2 - e^{-2\phi}F^2 - h\,F\,\tilde{F}\right] \tag{3.78}$$

with $\tilde{F}$ the dual of F.

It is also convenient to combine the axion, h, and the dilaton, ϕ, in a complex scalar field, $\lambda = \lambda_1 + i\lambda_2 = h + ie^{-2\phi}$ called the "*axidilaton*". In terms of λ the action simplifies to [71] :

$$S = \int d^4x\sqrt{-g}\left[-R + \frac{|\nabla\lambda|^2}{2\lambda_2^2} + \frac{i}{4}(\lambda F_+^2 - \bar{\lambda}F_-^2)\right] \tag{3.79}$$

where $F_\pm = F \pm i\tilde{F}$. The kinetic term for λ in (3.79) is invariant under the Peccei-Quinn shift of the axion, $h \to h + f$ and under the duality transformation $\lambda \to -\frac{1}{\lambda}$. The latter, for $h = 0$, reduces to the $\phi \to -\phi$ transformation of (3.75a).

The action as a whole is not invariant under this duality, because the Maxwell terms change sign, therefore the equations of motion given by [71]:

$$\frac{\nabla_\mu\partial^\mu\lambda}{\lambda_2^2} + i\frac{\partial_\mu\lambda\partial^\mu\lambda}{\lambda_2^3} - \frac{i}{2}F_-^2 = 0 \tag{3.80a}$$

$$\nabla_\alpha\left[\lambda F_+^{\alpha\beta} - \bar{\lambda}F_-^{\alpha\beta}\right] = 0 \tag{3.80b}$$

$$R_{\mu\nu} = \frac{\partial_\mu\bar{\lambda}\partial_\nu\lambda + \partial_\nu\bar{\lambda}\partial_\mu\lambda}{4\lambda_2^2} + 2\lambda_2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}\lambda_2\,g_{\mu\nu}F^2 \tag{3.80c}$$

are not invariant either. In fact the Einstein equation (3.80c) is not invariant in general under the duality transformation [71]

$$\phi' = -\phi \tag{3.81a}$$

$$F'_+ = -\lambda F_+ \tag{3.81b}$$

$$F'_- = -\bar{\lambda}F_- \tag{3.81c}$$

as the second and third terms are non-invariant [71] and transform into [71]:

$$2\lambda_2 F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}\lambda_2 g_{\mu\nu}F^2 - \frac{\lambda_1\lambda_2^2}{|\lambda|^2}\left(2F_{\mu\rho}\tilde{F}_\nu{}^\rho + 2F_{\nu\rho}\tilde{F}_\mu{}^\rho - g_{\mu\nu}F\tilde{F}\ .\right) \tag{3.82}$$

Only if the extra terms in this expression vanish will the duality transformation map solutions into solutions. A duality transformation of the offending terms above shows that they transform into themselves with

$$2F_{\mu\rho}\tilde{F}_\nu{}^\rho + 2F_{\nu\rho}\tilde{F}_\mu{}^\rho - g_{\mu\nu}F\tilde{F} \to \left(\lambda_1^2 - \lambda_2^2\right)\left(2F_{\mu\rho}\tilde{F}_\nu{}^\rho + 2F_{\nu\rho}\tilde{F}_\mu{}^\rho - g_{\mu\nu}F\tilde{F}\right) \tag{3.83}$$

and therefore the Einstein equation is invariant when [71]

$$2F_{\mu\rho}\tilde{F}_\nu{}^\rho + 2F_{\nu\rho}\tilde{F}_\mu{}^\rho - g_{\mu\nu}F\tilde{F} = 0 \tag{3.84}$$

which implies that if this expression vanishes for a particular solution. it will also vanish for its duality transformation of that solution. Thus. beginning with a solution one may build up a family of solutions. Indeed the Melvin magnetic universe verify the restriction (3.84) (the only non-vanishing field strength is $F_{R\varphi} = \frac{RB}{(1-\frac{B^2R^2}{4})^2}$) and therefore one can generate a axionic-dilatonic Melvin electromagnetic universe. To do this we start from the dilatonic Melvin magnetic solution (3.74a) - (3.74b) and obtain an axion from a shift f on a Peccei-Quinn transformation. $\lambda' = \lambda + f$. then dualise it. $\lambda' = -\frac{1}{\lambda}$. and finally rescale. $\lambda' = \lambda(f^2+1)$ [71]. One gets [71] :

$$\lambda' = -\frac{1}{\lambda + f}\left[f^2+1\right] \tag{3.85a}$$

$$h' = -\frac{f}{f^2 + e^{-4\phi}}\left[f^2+1\right] \tag{3.85b}$$

$$e^{-2\phi'} = \frac{e^{-2\phi}}{f^2 + e^{-4\phi}}\left[f^2+1\right] \tag{3.85c}$$

$$F' = \frac{1}{\sqrt{f^2+1}}\left[-fF + e^{-2\phi}\tilde{F}\right] \tag{3.85d}$$

$$A'_\varphi = -\frac{f}{\sqrt{f^2+1}}A_\varphi \tag{3.85e}$$

with A'_z, A'_t and A'_R unknown. For the dilatonic Melvin magnetic universe $F_{R\varphi} = \frac{RB}{k^2}$ and therefore:

$$F'_{tz} = \frac{e^{-2\phi}}{k^2R}B_1 \tag{3.86a}$$

$$F'_{R\varphi} = \frac{R}{k^2}B_2 \tag{3.86b}$$

with

$$B_1 = \frac{1}{\sqrt{f^2+1}} B \tag{3.87a}$$

$$B_2 = -\frac{f}{\sqrt{f^2+1}} B \tag{3.87b}$$

where B_2 is the magnetic field of the axionic sector and $B^2 = B_1^2 + B_2^2$ with $k = 1 + \frac{B^2 R^2}{2}$.

Using (3.74b) with $a = 1$ to rewrite the dilaton before the transformation one finally gets the exact dyonic Melvin electromagnetic universe, with the axion and dilaton fields given by:

$$h(B_1, B_2, R) = \frac{B_2 B_1}{B_1^2 + B_2^2} \frac{1}{\left(1 + \frac{1}{4} R^4 \left(B_1^2 + B_2^2\right) B_1^2 + R^2 B_1^2\right)} \tag{3.88a}$$

$$e^{-2\phi(B_1, B_2, R)} = \frac{B_1^2}{B_1^2 + B_2^2} \frac{1 + \frac{1}{2}\left(B_1^2 + B_2^2\right) R^2}{\left(1 + \frac{1}{4} R^4 \left(B_1^2 + B_2^2\right) B_1^2 + R^2 B_1^2\right)} \tag{3.88b}$$

with the metric given by:

$$ds^2 = k \left[dt^2 - dR^2 - dz^2\right] - \frac{1}{k} R^2 \, d\varphi^2. \tag{3.89}$$

As expected this solution is consistent with the dilatonic Melvin magnetic universe, (3.74a) - (3.74b), which corresponds to the case where there is no axion ($f = 0$) and so $B_1 = B$ ($B_2 = 0$) with $e^{-2\phi'} \equiv e^{2\phi} = \frac{1}{1 - \frac{1}{2} B^2 R^2}$.

Asymptotic global string solutions

The asymptotic global string solutions at very far distances from the core are then obtained from (3.88a)-(3.88b) by taking the limit $R \to \infty$. Therefore the axion and the dilaton are given by

$$h(B_1, B_2, R) \to \frac{4 B_2}{R^4 \, B^4 B_1} \tag{3.90a}$$

$$\phi(B_1, B_2, R) \to \ln \left[\frac{B\,R}{\sqrt{2}}\right] \tag{3.90b}$$

with the metric for the axionic-dilatonic global string given by

$$ds^2 = \left(\frac{BR}{\sqrt{2}}\right)^2 \left[dt^2 - dR^2 - dz^2\right] - \frac{2}{B^2} \, d\varphi^2. \tag{3.91}$$

⋮

3.4 Conclusions

In the first part of this chapter, we have derived the metric for U(1) local cosmic strings in dilaton gravity both with and without a potential for the dilaton. The (unknown) coupling of the Abelian-Higgs model to the dilaton is accounted for by coupling the Lagrangian to the gravitational sector by an arbitrary $e^{2a\phi}$ factor.

For a massless dilaton, the results are qualitatively the same as those of Gundlach and Ortiz [63], who considered cosmic strings in JBD theory. Essentially, the metric is the same as the usual cosmic string, i.e. conical, in the Einstein frame, and conformally conical in the string frame on scales of cosmological interest. The dilaton field has an effect that to $\mathcal{O}(\epsilon^2)$, the geometry acquires long range corrections, and on the very large scale, $(r \sim \sqrt{\lambda}\eta e^{\frac{1}{(a+1)^2\epsilon^2\beta^2}})$, there is additional curvature, and the spacetime is not asymptotically locally flat in either frame. The exception is the special case $a = -1$, in which the massless dilatonic cosmic string has no long range effects (other than the deficit angle) and merely shifts the value of the dilaton between the core and infinity by a constant of order ϵ. For the special case $\beta = 1$ there is no effect at all on the dilaton field, and the dilatonic string is the same as the Einstein one.

For a massive dilaton, we find that, apart from an intermediate annular region, the long-range structure of the string is as for Einstein gravity, as might be expected and so the metric asymptotes a conical metric, in both string and Einstein frames. The string

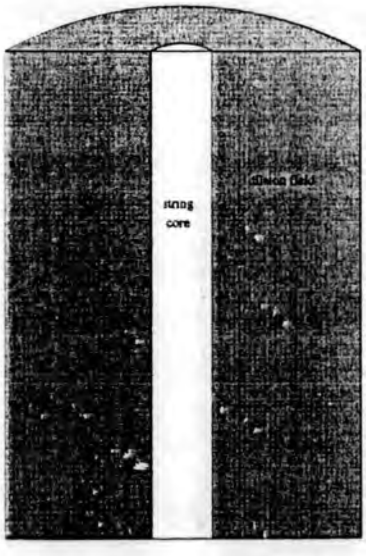

Figure 3.3: A representation of the dilaton field surrounding the cosmic string for $a \neq -1$.

generates a dilaton "*cloud*", approximately of width $\frac{m_H}{m_\phi}$, which is schematically depicted in figure 3.3, for $a \neq -1$. The main exception to this qualitative and expected picture is that for a special value ($a = -1$) of the coupling of the dilaton to the fields which constitute the vortex the dilaton effectively decouples from the string, showing little or no reaction to its presence. This occurs independent of whether the dilaton is massive, and independent of the specifics of the U(1) model, i.e. independent of the β values.

The Bogomolnyi bounds for the dilatonic string were also considered and it was shown these can only be saturated in the special case $a = -1$. In this case, the dilaton effectively decouples from the string.

The effects of these strings on test particles were also explored to conclude that in the string frame and for a massive dilaton geodesics are essentially the same as for the Einstein self gravitating string, i.e., all non-static trajectories escape to infinity. The same conclusions hold for $a = -1$. For a massless dilaton and $a \neq -1$ photons escape to infinity and are infinitely redshifted ($a > -1$) or blueshifted ($a < -1$) while for massive particles trajectories are bound for $a > -1$ or escape to infinity for $a < -1$. In the Einstein frame photon trajectories are identical to that of the string frame while all massive particle trajectories are bound.

Although it is beyond the scope of this chapter to derive the effective action of the cosmic string, the results do support a Nambu approximation for the string, since they show that the metric is little affected on cosmological length scales, and remains approximately flat locally (unlike the global string [64]). Damour and Vilenkin [60] have recently explored the impact of a massive dilaton on string networks using a model for the interactions which modifies the Nambu approximation by making the mass per unit length interact with the (massive) dilaton. In other words, the worldsheets act as sources for the dilaton which has a mass m_ϕ. They concluded that a TeV mass dilaton was incompatible with a GUT string network. Our results largely back up this calculation, but with one important caveat: The model used by Damour and Vilenkin makes no reference to the details of the dilaton coupling to the particle physics model producing the strings, the abelian-Higgs lagrangian, *i.e. their coupling is independent of our variable a.* Therefore, one should renormalise their calculations by factors of $(a + 1)$. This means that the conclusion that a TeV mass dilaton is incompatible with string theories of structure formation is only valid if a is not close to -1. For $a = -1$, there will be little dilatonic radiation from the cosmic string network, and hence a much weaker constraint.

To sum up: the gravitational field of a cosmic string in dilaton gravity is surprisingly close to that of an Einstein cosmic string on cosmological distance scales. However, it is the microwave background rather than cosmological observations, that provides the tightest constraint on the cosmic string theory of structure formation. If the strings couple to the dilaton directly ($a = -1$), then such constraints are identical to those derived in Einstein gravity. However, if the string couples with a different from -1, then the constraints of Damour and Vilenkin [60] apply, and a "*low*" (i.e. close to electroweak) mass for the dilaton rules out the cosmic string scenario of galaxy formation.

In the presence of an axion and for massless dilatons there are no local string type solutions and the asymptotic solutions for the gravi-axion-dilaton are given by the asymptotic limit of the dilatonic Melvin magnetic universe in the presence of an axion. The axion is strongly damped to zero while the dilaton has very strong asymptotic effects similar to those for the dilatonic string for massless dilaton.

Chapter 4

Vortices and black holes in dilatonic gravity

4.1 Introduction

In this chapter we study analytically black holes pierced by a thin vortex in dilatonic gravity for an arbitrary coupling of the vortex to the dilaton. We show that the horizon of the charged black hole supports the long-range fields of the Nielsen-Olesen vortex that can be considered as black hole hair for both massive and massless dilatons. We discuss the gravitational back-reaction of the thin vortex on the spacetime geometry and dilaton. The effect of the vortex on the massless dilaton is to generate an additional dilaton flux across the horizon.

The extrapolation of the black hole 'no-hair' conjecture, initially proposed by Ruffini and Wheeler [14] and stating that a stationary black hole is uniquely determined by its mass, electromagnetic charge and angular momentum, to the stronger statement of 'no dressing' of the horizon, has been proven to be false [11, 16–18]. A common feature of such 'counterexamples' is that they involve nontrivial topology of the matter fields. In particular, in reference [11], it was shown that for the Abelian-Higgs model in Einstein gravity, (see [15, 72] for the relevant no hair theorems), a Schwarzschild black hole could indeed support long hair, namely, a $U(1)$ vortex, which could either pierce, or end on the black hole horizon. This latter case is particularly interesting as it provides a decay channel for the disintegration of otherwise stable topological vortices [73–75].

It was also established in reference [11] that the gravitational effect of a vortex which is thin relative to the Schwarzschild radius of the black hole is to change its metric to a smooth version of the Aryal, Ford and Vilenkin solution [76]:

$$ds^2 = \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2(1 - \epsilon\hat{\mu})^2 \sin^2\theta d\varphi^2 \qquad (4.1)$$

in which spacetime is asymptotically locally flat, but has a conical deficit angle $2\pi\epsilon\hat{\mu}$ for a string with energy density per unit length of $\hat{\mu}$.

This work was then extended to other black hole solutions, namely to the Reissner-Nordstrom black hole in Einstein-Maxwell theory [13,77,78] and to non-extreme electrically charged black holes with a massless dilaton [12] in low energy string theory, and the main conclusion remains the same, i.e., in the thin vortex limit the Abelian-Higgs vortex also provides hair for these black holes (although reference [12] has an incorrect back reaction analysis).

In this chapter we extend the work of reference [11] to consider the Abelian-Higgs model coupled to dilatonic gravity, where the dilaton may be massless or massive. Using the same method as in [11] we show that a Schwarzschild black hole can indeed support long hair, namely, a $U(1)$ vortex. To lowest order the vortex (with an arbitrary dilaton coupling "a") introduces the same corrections on the geometry of the Schwarzschild black hole background as in [11], and when the coupling of the dilaton to the vortex is non-canonical in the string frame ($a \neq -1$), the vortex switches on non-constant values of the dilaton along the horizon. We then extend these results to charged black holes, and using similar arguments we show that for a massless dilaton, magnetically charged black holes can support the Abelian-Higgs vortex for reasonable dilaton couplings to the vortex ($|a| \ll$O(E^2)). For weak electrically charged black holes we prove analytically that they can support the Abelian-Higgs vortex, again for reasonable values of the dilaton couplings to the vortex. To leading order, the gravitational effect of the vortex on those black holes is to change their background geometries in an analogous fashion to the AFV metric, namely that a conical slice is removed from the geometry. However, for $a \neq -1$ the deficit angle is no longer constant, and acquires a dependence on the background dilaton; in addition there are strong long range gravitational effects to O(ϵ^2). The dilaton becomes modified by a correction which has the same sign for both magnetic and electric black holes, so that if its magnitude is decreased for the magnetic, it is increased for the electric, and vice versa.

We also consider black holes with a massive dilaton which are qualitatively different [80, 81] from their massless cousins [82]. As opposed to the single horizon plus space-like singularity causal structure of the massless dilatonic black holes, massive dilaton black holes can have two or three horizons and extremal solutions with a double or triple degenerated horizon.

4.2 Black holes in string theory

Charged black holes in string theory are solutions of the equation of motion of the low energy effective action in the Einstein frame [80–82]:

$$S_{dil-Max} = \int d^4x\sqrt{-g}\left[-R + 2(\nabla\phi)^2 - V(\phi) - e^{-2\phi}F^2\right] \tag{4.2}$$

which are given by

$$\nabla_\mu\left(e^{-2\phi}F^{\mu\nu}\right) = 0 \tag{4.3a}$$

$$\Box\phi + \frac{1}{4}\frac{\partial V}{\partial\phi} + \frac{1}{2}e^{-2\phi}F^2 = 0 \tag{4.3b}$$

$$R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}V(\phi) + 2e^{-2\phi}F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2 \tag{4.3c}$$

where F is the electromagnetic field strength of the Maxwell field which does not interact directly with the Higgs field, and where we will take $V(\phi) = 2M^2\phi^2$ as the dilaton potential.

A general spherically symmetric black hole solution has a metric of the form

$$ds^2 = \lambda(r)\,dt^2 - \frac{1}{\lambda(r)}\,dr^2 - C^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{4.4}$$

in which the electromagnetic equation of motion has the general magnetic solution

$$F = Q\sin\theta d\theta\wedge d\varphi \tag{4.5}$$

and the equations of motion in the Einstein frame are [80, 81]

$$\left[C^2\lambda\phi'\right]' = M^2C^2\phi - \frac{Q^2}{C^2}e^{-2\phi} \tag{4.6a}$$

$$\left[C^2\lambda'\right]' = -2M^2C^2\phi^2 + \frac{2Q^2}{C^2}e^{-2\phi} \tag{4.6b}$$

$$\left[\lambda\left(C^2\right)'\right]' = 2 - 2M^2C^2\phi^2 - \frac{2Q^2}{C^2}e^{-2\phi} \tag{4.6c}$$

$$0 = C''(r) + C\phi'^2; \tag{4.6d}$$

The electric solution is obtained by applying an electromagnetic duality transformation to the equations of motion that preserves the metric but changes the sign of the dilaton and is explicitly given by (3.75a)-(3.75b) in chapter 3.

In general (i.e. for nonvanishing dilaton potential) the solutions to these cannot be expressed in closed analytic form. Although when the dilaton is massless ($M = 0$), the black hole solution of the equations (4.6) with a pure magnetic charge Q is [82]

$$ds^2 = \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r\left(r - \frac{Q^2}{E}\right)(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad (4.7a)$$

$$e^{-2\phi} = 1 - \frac{Q^2}{Er} \quad (4.7b)$$

the mass, E, and the charge, Q, are written in "vortex units", and are related by $Q^2 \leq 2E^2$, while for the pure electrically charged black hole with a massless dilaton the metric is the same as for the magnetic black hole, but the dilaton is now given by

$$e^{2\phi} = 1 - \frac{Q^2}{Er} . \quad (4.8)$$

4.3 Strings through black holes

We now consider an isolated system of a dilatonic string threading a black hole and argue the existence of a vortex solution in the absence of gravitational back reaction. We begin by reviewing the argument of ref. [11] for the existence of a vortex solution in the Schwarzschild black hole background:

$$ds^2 = \left(1 - \frac{2E}{r}\right) dt^2 - \left(1 - \frac{2E}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \quad (4.9)$$

(where E is the mass of the black hole measured in "vortex units"), since this is also a solution of an uncharged dilatonic black hole.

We can choose a gauge in which the Higgs field Φ and the gauge field A_μ have the form

$$\Phi = \eta X(r,\theta) e^{i\varphi} \quad (4.10a)$$

$$A_\mu = \frac{1}{e}(P(r,\theta) - 1)\delta^\varphi_\mu \quad (4.10b)$$

i.e. we are considering a winding number 1 vortex. Substituting these forms into the vortex equations of motion we obtain

$$-\frac{1}{r^2}\left[r\,(r-2E)\,X_{,r}\right]_{,r}-\frac{1}{r^2\sin\theta}\left[\sin\theta X_{,\theta}\right]_{,\theta}+\frac{X}{2}(X^2-1)+\frac{XP^2}{r^2\sin^2\theta}=0 \quad (4.11a)$$

$$\left[\left(1-\frac{2E}{r}\right)P_{,r}\right]_{,r}-\frac{X^2P}{\beta}+\frac{\sin\theta}{r^2}\left[\frac{P_{,\theta}}{\sin\theta}\right]_{,\theta}=0. \quad (4.11b)$$

To argue the existence of a vortex solution analytically. we assume that the black hole is large compared to the string width. i.e. $E \gg 1$. We then take $X = X(R)$, $P = P(R)$ with $R = r\sin\theta$, and substituting in the vortex equations above. denoting the derivative with respect to R by a prime, we get

$$\left[-1+\frac{2E}{r}\sin^2\theta\right]\left[X''+\frac{X'}{R}\right]+\frac{X}{2}\left(X^2-1\right)+\frac{XP^2}{R^2} = 0 \quad (4.12a)$$

$$\left[1-\frac{2E}{r}\sin^2\theta\right]\left[P''-\frac{P'}{R}\right]-\frac{X^2P}{\beta} = 0. \quad (4.12b)$$

These can be seen to be the Nielsen-Olesen equations. given in (1.20a)-(1.20b) in chapter 1. up to terms of the form $\frac{2E}{r}\sin^2\theta$ times derivatives of X and P. In and near the core. where $R = r\sin\theta \leq 1$, $\sin\theta = \mathcal{O}(\frac{1}{r}) \leq \mathcal{O}(\frac{1}{E})$; so in this thin vortex limit. these corrections are negligible. and therefore to a good approximation the vortex equations are identical to the Nielsen-Olesen ones [30]. and the Nielsen-Olesen solution is still a good solution in and near the core of a thin vortex even at the event horizon (as proven in [11] using Kruskal coordinates) and the string simply continues regardless of the black hole as confirmed numerically in [11].

We now generalise these results to charged black holes in the presence of a dilaton. Proceeding as in [11] and [78] we look for a vortex solution by taking $X = X(\sigma)$, $P = P(\sigma)$, with $\sigma = Ce^{\Phi}\sin\theta$ which gives the vortex equations

$$\frac{\dot X}{\sigma}\left[-1+\sin^2\theta\left(2-\lambda C^2\frac{(Ce^{\Phi})''}{Ce^{\Phi}}-C^2\frac{(Ce^{\Phi})'}{Ce^{\Phi}}\left[2\lambda\frac{(Ce^{\Phi})'}{Ce^{\Phi}}+\lambda'+2\,a\,\lambda\,\phi'\right]\right)\right]$$
$$+\ddot X\left[-1+\sin^2\theta\left(1-\lambda C^2\left[\frac{(Ce^{\Phi})'}{Ce^{\Phi}}\right]^2\right)\right]+\frac{XP^2}{\sigma^2}+\frac{X}{2}(X^2-1)=0 \quad (4.13a)$$

$$\frac{\dot P}{\sigma}\left[-1+\sin^2\theta\left(\lambda' C^2\frac{(Ce^{\Phi})'}{Ce^{\Phi}}+\lambda C^2\frac{(Ce^{\Phi})''}{Ce^{\Phi}}+2\,a\,\lambda\,\phi' C^2\frac{(Ce^{\Phi})'}{Ce^{\Phi}}\right)\right]$$
$$+\ddot P\left[1-\sin^2\theta\left(1-\lambda C^2\left[\frac{(Ce^{\Phi})'}{Ce^{\Phi}}\right]^2\right)\right]-\frac{X^2P}{\beta}=0 \quad (4.13b)$$

where a dot means the derivative with respect to σ. These equations (4.13a)-(4.13b) are the Nielsen-Olesen ones up to terms which may be written as

$$\mathcal{T}_1 = \frac{\sigma^2}{C^2 e^{2\phi}} \left(1 - \lambda\, C^2 \left[\frac{(Ce^{\phi})'}{Ce^{\phi}} \right]^2 \right) \quad (4.14a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{C^2 e^{2\phi}} \left(\lambda'\, C^2 \, \frac{(Ce^{\phi})'}{Ce^{\phi}} + \lambda\, C^2 \, \frac{(Ce^{\phi})''}{Ce^{\phi}} + 2\, a\, \lambda\, \phi'\, C^2 \, \frac{(Ce^{\phi})'}{Ce^{\phi}} \right) \quad (4.14b)$$

multiplied by derivatives of the vortex fields. Provided these correction terms are negligible in and near the core of a thin vortex, the Nielsen-Olesen solutions will be a good approximation to the string threading the black hole.

Having derived the general equations. we now look at electrically and magnetically charged black holes with a massless and massive dilaton in turn.

4.3.1 Charged black holes with massless dilaton.

When the dilaton is massless ($M = 0$). the black hole solution with a pure magnetic charge Q is given by (4.7a)-(4.7b) and we may therefore read off

$$\mathcal{T}_1 = \frac{\sigma^2}{r^2} \frac{2E}{r} \left[1 + \frac{Q^2}{2E^2} - \frac{Q^2}{Er} \right] \quad (4.15a)$$

$$\mathcal{T}_2 = \frac{\sigma^2}{r^2} \left[\frac{2E}{r} \left(1 - \frac{Q^2}{Er} \right) - \frac{aQ^2}{Er} \left(1 - \frac{2E}{r} \right) \right] \quad (4.15b)$$

In and near the core of a thin vortex the charged correcting terms like (4.15a) are always negligible when compared with the Nielsen-Olesen ones. as they are of order $\mathcal{O}(\frac{1}{E^2})$. while the dilatonic coupling ones like the second part of (4.15b) are of order $\mathcal{O}(\frac{aQ^2}{E^4})$ and therefore could be relevant for extremely large couplings of the dilaton to the vortex $|a| \geq \mathcal{O}(\frac{E^4}{Q^2}) \geq \mathcal{O}(E^2)$, however, these are not particularly realistic values (e.g. $|a| = 0, 1, \sqrt{3}$ is usual). Therefore to a good approximation the vortex solution is given by the Nielsen-Olesen solution, and since $\sigma = r \sin\theta$ for the magnetic black hole, the solution is in fact identical to the Schwarzschild vortex.

We also note that these conclusions do not change with $\frac{Q}{E}$ and so still apply in the particular case where the black hole is extremal $Q^2 = 2E^2$. In this case the horizon is singular in the Einstein frame with a vanishing area [82], however in the string frame

$$ds^2 = dt^2 - \left(1 - \frac{2E}{r} \right)^{-2} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2), \quad (4.16)$$

and the previously singular horizon $r = 2E$ has been pushed off to infinite proper distance. Whether one could say that the string was or was not piercing the horizon is a moot point.

Let us now consider a pure electrically charged black hole with a massless dilaton, given by (4.7a)-(4.8).

Now we obtain

$$\mathcal{T}_1 = \frac{\sigma^2}{(r - Q^2/E)^2} \frac{(2E - Q^2/E)}{(r - Q^2/E)} \tag{4.17a}$$

$$\mathcal{T}_2 = \frac{\sigma^2}{(r - Q^2/E)^2} \left[\frac{2E}{r} + \frac{Q^2 a}{Er} \frac{(r - 2E)}{(r - Q^2/E)} \right] \tag{4.17b}$$

Clearly, when $Q^2 < E^2$ these terms are negligible for similar reasons as before. However, consider now the extremal (or near extremal) case $Q^2 = 2E^2 - qE$. In this case, we see that

$$\mathcal{T}_1 = \frac{\sigma^2 q}{(r - 2E + q)^3} < O(\sigma^2/q^2) \tag{4.18a}$$

$$\mathcal{T}_2 = \frac{\sigma^2}{(r - 2E + q)^3} \frac{2E}{r} \left[q + (r - 2E)(1 + a - \frac{aq}{2E}) \right] < O(\sigma^2/q^2) \tag{4.18b}$$

We now see that close to the extremal limit, the Nielsen-Olesen approximation breaks down in the vicinity of the horizon. What this means is that the thin vortex limit has broken down, and our analytic approximation is no longer valid. However, if we examine the area of the horizon, $4\pi C^2 = 8\pi Eq$, we see that we might only reasonably expect a thin vortex approximation to work for $Eq \gg 1$, (or $q \gg 1$ if we look at the string frame), therefore, the breakdown of this method is due to the breakdown of the coordinate system at the horizon, which becomes singular in the extremal limit.

At extremality, $\mathcal{T}_1 = 0$, and $\mathcal{T}_2 = \frac{2E\sigma^2(1+a)}{r(r-2E)^2}$. For $a \neq -1$ these terms eventually become important in the core when $(r - 2E)^2 r \leq 2E$ i.e. close to the horizon (which is also singular). As the size of the black hole is now zero this means that in fact the string, instead of penetrating the black hole, swallows it. Again this result does not depend on the frame. Note however, that for $a = -1$, our analytic approximation is exact and the Nielsen-Olesen solution gives the form of the string. Since $\sigma = 0$ on the horizon, one could say that the flux of the string was expelled.

4.3.2 Charged black holes with massive dilatons.

When the dilaton is massive the character of the black hole background is in general different from the massless one as (4.7a), (4.7b) and (4.8) are no longer solutions of the geometry equations (4.3c). Qualitatively speaking there are three distinct types of black hole [80,81], depending on the relative sizes of the black hole, E, and the Compton wavelength of the dilaton, M^{-1}. Black holes which are small compared to the Compton wavelength of the dilaton ($EM \ll 1$) resemble the massless dilaton solutions already discussed, which have the causal structure of a Schwarzschild black hole - a single horizon and spacelike singularity. Those black holes which are large compared to the Compton wavelength of the dilaton ($EM \gg 1$) resemble the Reissner-Nordstrøm solution in the region exterior to the horizon, although it is possible that their overall causal structure is quite different in that there can be one, two or even three horizons [80,81]. The intermediate case $EM = O(1)$, is the borderline between these two behaviours, where additional horizons are possible and even a special extremal solution with a triply degenerate horizon occurs. These black holes have no approximate analytic description.

When the Schwarzschild radius E is less than the Compton wavelength of the dilaton, i.e. $E \ll M^{-1}$, the black hole does not see the mass of the dilaton and behaves like the massless case, and therefore (4.7a),(4.7b) and (4.8) are good approximations to the true black hole background solution. We therefore expect the results of the previous subsection to apply, and in the thin vortex limit the vortex will be given by the Nielsen-Olesen solution. Since $1 \ll E \ll M^{-1}$ the dilaton is also effectively massless as far as the string is concerned. (Although for a minimal dilaton mass of $m = 10^3 Gev$ this means black hole masses of rather less than 10^{11}g, and hence would require a primordial black hole.)

When the Schwarzschild radius E is much larger than the Compton wavelength of the dilaton $\frac{1}{M}$, i.e. $E \gg \frac{1}{M}$, the dilaton (and corrections to the geometry) are of order $\frac{Q^2}{M^2 r^4} \le O(\frac{1}{M^2 E^2})$ and hence we can regard the dilaton as being essentially fixed and the geometry as the Reissner-Nordstrøm one

$$ds^2 = \left(1 - \frac{2E}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2E}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (4.19)$$

which is now being extremal for $|Q| \simeq E$. We can now use the results of [13, 78] to conclude that in the thin vortex limit, the Nielsen-Olesen solution is a good approximation to the vortex, and for extremal black holes there is a flux expulsion phenomenon when the thickness of the string core becomes comparable to the black hole horizon scale.

We now consider black holes for which Schwarzschild radius is similar in scale to the Compton wavelength of the dilaton $EM \simeq 1$. In this case, there is no simple analytic form for the geometry, and we must estimate the correcting terms (4.14) from the equations of motion. We first note that if the charge of the black hole is small, the dilaton field will not differ much from its vacuum value, being of order Q^2/E^2 for $EM \simeq 1$. Therefore the interesting régime in which to analyse the vortex is close to the extremal limit, $QM = O(1)$. One of the interesting features of massive dilatonic black holes is that they possess a richer horizon structure than that of the massless dilatonic solutions. In particular, at $QM = e/2$, there is a phase transition in the types of extremal solutions possible. For $QM < e/2$ there is only one horizon, and the extremal solution corresponds to the singularity moving out to the black hole horizon (i.e. $C = 0$). For $QM > e/2$ an extremal solution similar to the Reissner-Nordstrøm one occurs, in that $\lambda = \lambda' = 0$ at the horizon. For $QM = e/2$, there is a special triply degenerate extremal solution, where λ, λ' and λ'' all vanish. For all values of QM however, the solutions do have the common feature that ϕ is decreasing (increasing) outside the horizon for the magnetic (electric) black hole, that λ monotonically increases from 0 to 1 outside the horizon, and finally that $C' \geq 1$ outside the horizon [80, 81]. Therefore one needs to estimate the $\mathcal{T}_i$ with all this in mind.

First note that

$$(C' + C\phi')^2 \leq |C'^2 - C^2\phi'^2| = \left|1 - \lambda CC' - M^2C^2\phi^2 - \frac{Q^2}{C^2e^{2\phi}}\right| \frac{1}{\lambda} \tag{4.20}$$

using (4.6c,4.6d), hence

$$\frac{\sigma^2}{C^2e^{2\phi}} \geq \mathcal{T}_1 \geq -\frac{\sigma^2}{C^2e^{2\phi}} \left|\lambda CC' + M^2C^2\phi^2 + \frac{Q^2}{C^2e^{2\phi}} - 1\right| \tag{4.21}$$

Then, using (4.6a,4.6c) one can show

$$\mathcal{T}_2 = \mathcal{T}_1 + \frac{\sigma^2}{C^2e^{2\phi}} \left[M^2C^2\phi(1-\phi) - 2\frac{Q^2}{C^2e^{2\phi}} + 2(a+1)\lambda C\phi'(C' + C\phi')\right] \tag{4.22}$$

(For the electric black hole, replace ϕ by $|\phi|$ except in the initial $\frac{\sigma^2}{C^2e^{2\phi}}$ term.) Since these terms, for reasonable a, and with the possible exception of $\frac{Q^2}{C^2e^{2\phi}}$, can be shown to be at most of order unity, the magnitude of the $\mathcal{T}_i$ boils down to the minimal value of Ce^{ϕ}.

In most cases, this quantity attains its minimum on the horizon, however, for a small subset of solutions (namely those close to extremal, for which the value of the dilaton on the horizon, ϕ_h, lies approximately in the range $[1 - 1/\sqrt{2}, 1]$), Ce^{ϕ} actually has its minimum outside the horizon, and the spacetime in the string frame has a wormhole-like structure [81].

We begin therefore by estimating Ce^{ϕ} on the horizon. Starting with the magnetic black hole and evaluating (4.6a,4.6c) at the horizon, using the properties of the dilaton and metric functions, one can readily obtain the following inequalities for C_h:

$$C_h^2 \le \frac{Qe^{-\phi_h}}{M\sqrt{\phi_h}} \tag{4.23a}$$

$$\frac{1}{2M^2\phi_h^2}\left[1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}}\right] \le C_h^2 \le \frac{1}{2M^2\phi_h^2}\left[1 + \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}}\right] \tag{4.23b}$$

Hence

$$C_h^2 e^{2\phi_h} \ge \frac{Q^2}{2M^2Q^2\phi_h^2 e^{-2\phi_h}}\left[1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2\phi_h}}\right] \ge Q^2 \tag{4.24}$$

If Ce^{ϕ} is minimised at the horizon, then clearly $\mathcal{T}_i = O(Q^{-2}) = O(E^{-2})$, and the thin vortex approximation is satisfied. If Ce^{ϕ} is not minimised at the horizon, then we note that the value of ϕ required is of order unity [81], hence $\mathrm{O}(Ce^{\phi}|_{\min}) > \mathrm{O}(C_h) \simeq \mathrm{O}(Q)$, and so $\mathcal{T}_i = \mathrm{O}(E^{-2})$ in this case as well. Therefore magnetic black holes always admit a thin vortex approximation.

For the electric black hole, the inequalities (4.23) are still valid, provided we replace ϕ_h by $|\phi_h|$. This however means that $C_h^2 e^{2\phi_h} = C_h^2 e^{-2|\phi_h|}$, and hence

$$\frac{e^{-2|\phi_h|}}{2M^2\phi_h^2}\left[1 - \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2|\phi_h|}}\right] \le C_h^2 e^{-2|\phi_h|}$$

$$\le \mathrm{Min}\left\{\frac{Qe^{-3|\phi_h|}}{M\sqrt{|\phi_h|}},\ \frac{e^{-2|\phi_h|}}{2M^2\phi_h^2}\left[1 + \sqrt{1 - 4M^2Q^2\phi_h^2 e^{-2|\phi_h|}}\right]\right\} \tag{4.25}$$

which gives no satisfactory bound on $C_h^2 e^{2\phi_h}$, as might have been expected, given the massless electric black hole. We therefore suspect that electric black holes are closer to their massless counterparts, in that unless $QM > e/2$, (so that $|\phi_h| < 1$), nearly extremal electric black holes will have no analytic thin vortex approximation for the vortex.

To sum up, in this section we have shown that in a wide variety of cases, the Nielsen-Olesen solution gives a good approximation to the thin vortex solution in the presence of a black hole. The only situation in which it does not give an adequate description is that of near extremal electric black holes where the dilaton is either massless, or has a low mass. In this case, a full numerical study is required[1]. Since none of these arguments rest on the fact that the string must thread the black hole, we may conclude, as in [11], that these arguments can be used to construct strings terminating on black holes.

4.4 Gravitating strings

In this section we consider the gravitational back-reaction of a thin vortex on the spacetime geometry and dilaton, using the same method as in [11,74,78], i.e., expanding the equation of motion in powers of ϵ, the gravitational strength of the string, which is assumed small. Before starting, it is worth asking what sort of solutions we expect to obtain: for the Einstein string a known asymptotic metric is the AFV metric. One expects that the generalisation of the AFV metric is the dilatonic black hole metric (4.4), with either a massive or massless dilaton, with a conical slice removed. As we will see, because of our choice of the arbitrary coupling parameter a, the actual set of solutions obtained is somewhat more complex.

We begin by considering the most general static axially symmetric metric

$$ds^2 = e^{2\psi}dt^2 - e^{2(\gamma-\psi)}\left(d\zeta^2 + d\rho^2\right) - \alpha^2 e^{-2\psi}\,d\varphi^2 \tag{4.26}$$

where ψ, γ, α are functions of ζ and ρ and the coordinates are given in "vortex units".

For example, the massless dilaton black hole in axisymmetric coordinates is

$$\alpha_0 = \rho \tag{4.27a}$$

$$e^{2\psi_0} = \frac{R_+ + R_- - 2\Delta}{R_+ + R_- + 4E - 2\Delta} \tag{4.27b}$$

$$e^{2\gamma_0} = \frac{(R_+ + R_-)^2 - 4\Delta^2}{4R_+R_-} \tag{4.27c}$$

$$e^{\pm 2\phi_0} = \frac{R_+ + R_- + 4E + 2\Delta}{R_+ + R_- + 4E - 2\Delta} \tag{4.27d}$$

where

$$R_\pm^2 = \rho^2 + [\zeta \pm \Delta]^2 = [r - 2E + \Delta \pm \Delta\cos\theta]^2 \tag{4.28}$$

[1] Such a study was performed by Moderski and Rogatko [84] for '$a = -1$', and they do indeed observe the flux expulsion phenomenon.

with $\Delta = E - \frac{Q^2}{2E}$. This is obtained by using the coordinate transformation

$$\zeta = \left(r - E - \frac{Q^2}{2E}\right)\cos\theta \tag{4.29a}$$

$$\rho^2 = \left(r - \frac{Q^2}{E}\right)(r - 2E)\sin^2\theta \,. \tag{4.29b}$$

In these coordinates the (relevant) equations of motion are:

$$\alpha_{,\zeta\zeta} + \alpha_{,\rho\rho} = -\sqrt{-g}\left[\mathcal{T}^\zeta_\zeta + \mathcal{T}^\rho_\rho\right] \tag{4.30a}$$

$$(\alpha\psi_{,\zeta})_{,\zeta} + (\alpha\psi_{,\rho})_{,\rho} = -\frac{1}{2}\sqrt{-g}\left[\mathcal{T}^\zeta_\zeta + \mathcal{T}^\rho_\rho + \mathcal{T}^\varphi_\varphi - \mathcal{T}^t_t\right] \tag{4.30b}$$

$$\gamma_{,\zeta\zeta} + \gamma_{,\rho\rho} = -\frac{\sqrt{-g}}{\alpha}\mathcal{T}^\varphi_\varphi - \left[(\psi_{,\rho})^2 + (\psi_{,\zeta})^2\right] \tag{4.30c}$$

$$4e^{-2(\gamma-\psi)}\left[\frac{1}{\alpha}(\alpha\phi_{,\rho})_{,\rho} + \frac{1}{\alpha}(\alpha\phi_{,\zeta})_{,\zeta}\right] - 4M^2\phi - 2e^{-2\phi}F^2$$
$$+\epsilon\left(-4(1+a)\hat{T}^t_t - 2\left(\hat{T}^\rho_\rho + \hat{T}^\varphi_\varphi\right) + 2\left(\hat{T}^t_t - \hat{T}^\zeta_\zeta\right)\right) = 0 \tag{4.30d}$$

$$\nabla_\mu\left[e^{-2\phi}F^{\mu\nu}\right] = 0 \tag{4.30e}$$

with

$$\mathcal{T}^\rho_\rho = \epsilon\,\hat{T}^\rho_\rho + 2S^\rho_\rho + K^\rho_\rho \tag{4.31}$$

where S_{ab}, $\hat{T}_{ab}$ and K_{ab} are the energy momentum tensors for the dilaton, the vortex and the Maxwell field respectively with

$$\hat{T}^\zeta_\zeta + \hat{T}^\rho_\rho = 2\,e^{(2+2a)\phi+2\psi}\frac{X^2P^2}{\alpha^2} + \frac{1}{2}(X^2-1)^2e^{(4+2a)\phi} \tag{4.32a}$$

$$\hat{T}^\varphi_\varphi = -\frac{X^2P^2}{\alpha^2}e^{(2+2a)\phi+2\psi} + e^{(2+2a)\phi-2(\gamma-\psi)}\left[(X_{,\rho})^2 + (X_{,\zeta})^2\right]$$
$$+\frac{1}{4}(X^2-1)^2e^{(4+2a)\phi} - \frac{\beta}{\alpha^2}e^{4\psi-2\gamma-2a\phi}\left[(P_{,\rho})^2 + (P_{,\zeta})^2\right] \tag{4.32b}$$

$$\hat{T}^t_t = e^{2\psi+(2+2a)\phi}\left[e^{-2\gamma}(X_{,\rho})^2 + e^{-2\gamma}(X_{,\zeta})^2 + \frac{X^2P^2}{\alpha^2}\right]$$
$$+\frac{1}{4}(X^2-1)^2e^{(4+2a)\phi} + \frac{\beta}{\alpha^2}e^{4\psi-2\gamma+2a\phi}\left[(P_{,\rho})^2 + (P_{,\zeta})^2\right] \tag{4.32c}$$

and

$$K_{ab} = 2e^{-2\phi}F_{ai}F_b{}^i - \frac{1}{2}g_{ab}e^{-2\phi}F^2 \tag{4.33}$$

where

$$K^\zeta_\zeta + K^\rho_\rho = 0 \tag{4.34a}$$

$$K^t_t = -K^\varphi_\varphi = -\frac{1}{2}e^{-2\phi}|F^2| \,. \tag{4.34b}$$

We begin by examining the effect of the dilatonic vortex threading the Schwarzschild black hole since the lack of electromagnetic charge considerably simplifies the problem. First note that to O(ϵ) the geometry is unaffected by the dilaton, and only reacts to the vortex energy-momentum. The metric is therefore given by the results in [11], giving

$$ds^2 = \left(1 - \frac{2\tilde{E}}{\tilde{r}_s}\right) d\tilde{t}^2 - \left(1 - \frac{2\tilde{E}}{\tilde{r}_s}\right)^{-1} d\tilde{r}_s^2 - \tilde{r}_s^2\, d\theta^2 - \tilde{r}_s^2\, (1 - A)^2\, e^{-2D} \sin^2\theta\, d\varphi^2 \quad (4.35)$$

where the time, t, has been rescaled to the proper time at asymptotic infinity, $\tilde{t} = e^{\frac{D}{2}}\, t$, etc. This metric is clearly that of a Schwarzschild black hole with renormalised mass $\tilde{E} = e^{\frac{D}{2}}\, E$, with a deficit angle of $2\pi(A + D) = 2\pi\epsilon\hat{\mu}$ (independent of the radial stresses), and an apparent conical singularity which is of course smoothed out by the vortex. When the radial stresses do not vanish ($\beta \neq 1$) there is a red/blue-shift of time between infinity and the core of the string [11].

We now calculate up to O(ϵ), the back reaction of the vortex on the dilaton. We use the spherical, Schwarzschild, coordinates for simplicity. We first look for the general coordinate dependence on the dilaton corrections, ϕ_1. For that we take Nielsen-Olesen solution for the vortex, up to corrections of order $O(E^{-2})$.

$$X = X_0(R) + O(E^{-2}) \quad (4.36)$$

$$P = P_0(R) + O(E^{-2}) \quad (4.37)$$

to write the source term in the dilaton equation (4.30d) that become:

$$4\epsilon(a+1)\left[X'^2\left(1 - \frac{2E}{r}\sin^2\theta\right) + \frac{X^2P^2}{R^2}\right] + \epsilon(a+2)\left(X^2 - 1\right)^2 + 4\epsilon a\beta\left[\frac{P'^2}{R^2}\left(1 - \frac{2E}{r}\sin^2\theta\right)\right]. \quad (4.38)$$

In and near the core $\sin\theta \ll 1$ and $X \sim X_0(R)$ and $P \sim P_0(R)$ while outside the core the correcting terms $\frac{E}{r}\sin^2\theta$ may not be negligible but now X_0 and P_0 fall off rapidly [30]. Therefore, as up to negligible corrections the source term is only a function of R one shall look whether one can assume a form $\phi_1 = \epsilon f_s(R)$ where f_s is the pure dilatonic cosmic string solution given in Section 3.1.2 of chapter 3 with

$$\left(1 - \frac{2E}{r}\sin^2\theta\right)\left[f_s'' + \frac{f_s'}{R}\right] - M^2 f_s + \frac{1}{2}(\mathcal{P}_{0R}(R) + \mathcal{P}_{0\varphi}(R)) - (1 + a)\mathcal{E}_0(R) = 0\ . \quad (4.39)$$

For $M^2 = 0$, this equation is clearly satisfied to order O(E^{-2}) since the dilaton is logarithmic outside the core. For $M^2 \neq 0$, the situation is slightly more subtle. If the Compton wavelength of the dilaton is much greater, or much less, than the Schwarzschild radius, then the equation is valid, since the dilaton will either be qualitatively massless, or at its vacuum value near the horizon. However, for $M^{-1} \simeq E$, this analytic approximation will not hold, and the functional form of the dilaton will be modified in the vicinity of the horizon. In all cases however, this approximation holds for large radius.

This shows that the vortex switches on a non-vanishing dilaton field on the horizon of the black hole, $\phi = \epsilon f_s(2E\sin\theta)$, which means that there is an effective dilatonic charge for the massless dilaton of $\mathcal{D}_1 = 2E(a+1)\epsilon\hat{\mu}$, in other words, the charge generated by a fragment of cosmic string of length $2E$. In this sense, the system behaves very much as if it can "see" the fragment of string behind the event horizon.

Moving to the charged black holes, first note that the existence of a dilatonic vortex breaks the electromagnetic duality invariance via the presence of the $\mathcal{E}_0$ etc. terms in (4.30d) which only vanish for $\beta = -a = 1$. We will therefore have to consider electric and magnetic black holes separately. First we note that the function appearing in the analytic thin vortex approximation is now $\sigma = \alpha e^{\phi-\psi}$. In order for this approximation to hold, the equations of motion for X and P imply that

$$\sigma_{,i}^2 = e^{2\phi}e^{2(\gamma-\psi)} + O(E^{-1}) \tag{4.40a}$$

$$\sigma_{,ii} + \frac{\sigma_{,i}\alpha_{,i}}{\alpha} + 2(a+1)\sigma_{,i}\phi_{,i} = \frac{e^{2\phi}e^{2(\gamma-\psi)}}{\sigma} + O(E^{-1}) \tag{4.40b}$$

throughout the core of the string. Applying this to the energy-momentum tensor for the vortex (4.32a)-(4.32c), gives

$$\begin{aligned}\hat{T}^t_t &= e^{(4+2a)\phi}\left[\frac{X^2P^2}{\sigma^2} + \frac{1}{4}(X^2-1)^2 + e^{-2(\gamma+\phi-\psi)}\left(X_{,i}^2 + \frac{\beta}{\sigma^2}P_{,i}^2\right)\right]\\ &\simeq e^{(4+2a)\phi}\mathcal{E}_0(\sigma)\end{aligned} \tag{4.41a}$$

$$\hat{T}^\varphi_\varphi \simeq -e^{(4+2a)\phi}\mathcal{P}_{0\varphi}(\sigma) \tag{4.41b}$$

$$\hat{T}^\zeta_\zeta + \hat{T}^\rho_\rho \simeq e^{(4+2a)\phi}[\mathcal{E}_0(\sigma) - \mathcal{P}_{0R}(\sigma)]\,. \tag{4.41c}$$

In these coordinates the equations of motion (4.30a)-(4.30e) become

$$\alpha_{,ii} = -\sqrt{-g}\left[2M^2\phi^2 + \epsilon e^{(4+2a)\phi}\left(\mathcal{E}_0(\sigma) - \mathcal{P}_{0R}(\sigma)\right)\right] \tag{4.42a}$$

$$(\alpha\psi_{,i})_{,i} = \frac{1}{2}\sqrt{-g}\left[2M^2\phi^2 - e^{-2\phi}|F^2| - \epsilon e^{(4+2a)\phi}(\mathcal{P}_{0R}(\sigma) + \mathcal{P}_{0\varphi}(\sigma))\right] \tag{4.42b}$$

$$\gamma_{,ii} = -\psi_{,i}^2 - \phi_{,i}^2 - \frac{\sqrt{-g}}{\alpha}\left[M^2\phi^2 - \frac{1}{2}e^{-2\phi}|F^2| - \epsilon e^{(4+2a)\phi}\mathcal{P}_{0\varphi}(\sigma)\right] \tag{4.42c}$$

$$\begin{aligned}(\alpha\phi_{,i})_{,i} &= \sqrt{-g}\Bigg[M^2\phi + \frac{1}{2}e^{-2\phi}F^2 \\ &\quad -\epsilon e^{(4+2a)\phi}\left(\frac{1}{2}\left(\mathcal{P}_{0R}(\sigma) + \mathcal{P}_{0\varphi}(\sigma)\right) + (1+a)\mathcal{E}_0(\sigma)\right)\Bigg] \qquad (4.42\text{d}) \\ 0 &= \left[e^{-2\phi}\alpha F_i^{\ \mu}\right]_{,i} \qquad (4.42\text{e})\end{aligned}$$

(where $i = \rho, \zeta$, and the summation convention applies). Hence we see that the source terms in the Einstein equations consist of terms which are functions of the original spherical r-coordinate, and the vortex function, σ.

To zeroth order we have the background solutions (4.27a)-(4.27c) and using [78] as a guide, we guess that the perturbed solution takes the form:

$$\alpha = \alpha_0\left(1 + e^{2(a+1)\phi_0}b(\sigma)\right) \qquad (4.43\text{a})$$

$$\psi = \psi_0 + e^{2(a+1)\phi_0}d(\sigma) \qquad (4.43\text{b})$$

$$\gamma = \gamma_0 + e^{2(a+1)\phi_0}g(\sigma) \qquad (4.43\text{c})$$

$$\phi = \phi_0 + e^{2(a+1)\phi_0}f(\sigma) \qquad (4.43\text{d})$$

$$A_\mu = A_{0\mu}\left(1 + e^{2(a+1)\phi_0}q(\sigma)\right) \qquad (4.43\text{e})$$

Inputting these into the equation of motion gives, after some algebra, and to order $O(E^{-2})$:

$$b'' + \frac{2b'}{\sigma} = -\epsilon[\mathcal{E}_0 - \mathcal{P}_{0R}] \qquad (4.44\text{a})$$

$$d'' + \frac{d}{\sigma} = \frac{\epsilon}{2}[\mathcal{P}_{0R} + \mathcal{P}_{0\varphi}] \qquad (4.44\text{b})$$

$$g'' = \epsilon\mathcal{P}_{0\varphi} \qquad (4.44\text{c})$$

$$f'' + \frac{f'}{\sigma} = M^2 f + (a+1)\epsilon\mathcal{E}_0 - \frac{\epsilon}{2}\left(\mathcal{P}_{0R} + \mathcal{P}_{0\varphi}\right) \qquad (4.44\text{d})$$

$$(\sigma^3 q_M')' = 2\sigma^2\left(b' + 2f' - 2d'\right) \qquad (4.44\text{e})$$

$$q_e'' + \frac{q_E'}{\sigma} = 0 \qquad (4.44\text{f})$$

where the subscripts M and E indicate the magnetic and electric corrections respectively. Note that these equations are valid only in the vicinity of the core, and only to $O(E^{-2})$; outside the core, where the terms no longer involve the vortex core, and are typically of order E^2/r^4, the equations differ depending on whether the dilaton is massive or massless, and whether the black hole is electrically or magnetically charged.

These equations are readily integrated to obtain for the leading order correction

$$\begin{aligned}
b &= -A(\sigma) + \frac{B(\sigma)}{\sigma} + b_0 && \text{(4.45a)}\\
d &= \frac{1}{2}D(\sigma) + d_0 && \text{(4.45b)}\\
g &= D(\sigma) + g_0 && \text{(4.45c)}\\
f &= f_s(\sigma) + f_0 =_{M=0} -\frac{1}{2}D(\sigma) + f_0 + (a+1)\int_0^\sigma \frac{A(\sigma)+D(\sigma)}{\sigma} && \text{(4.45d)}\\
q_M &= b + 2(f-d) + q_{0M} + \frac{1}{\sigma^2}\int_0^\sigma \left[B(\sigma) + 2\sigma(a+1)\left(A(\sigma)+D(\sigma)\right)\right] && \text{(4.45e)}\\
q_E &= q_{0E} && \text{(4.45f)}
\end{aligned}$$

where the integration constants are fixed in part by the desired boundary conditions. and in part by the equations of motion outside the core.

Focusing on the string threading the black hole. and transforming back to spherically symmetric coordinates (4.4). we see that for the black hole with a massless dilaton the solution outside the vortex core becomes

$$\begin{aligned}
ds^2 &= \left(1 + D\left(1-\tfrac{Q^2}{Er}\right)^{-(a+1)}\right) \times \left[\left(1-\tfrac{2E}{r}\right)dt^2 - \left(1-\tfrac{2E}{r}\right)^{-1}dr^2\right.\\
&\quad \left. -r\left(r-\tfrac{Q^2}{E}\right)\left(d\theta^2 + \left(1-\epsilon\hat{\mu}\left(1-\tfrac{Q^2}{Er}\right)^{-(a+1)}\right)^2 d\varphi^2\right)\right] && \text{(4.46a)}\\
e^{\pm 2\phi} &= \left(1-\frac{Q^2}{Er}\right)\left(1 \pm 2\epsilon f(\sigma)\left(1-\frac{Q^2}{Er}\right)^{\pm(a+1)}\right) && \text{(4.46b)}\\
A_\nu &= \begin{cases} \frac{Q}{r}\,\partial_\nu t & \text{electric.}\\ Q(1-\cos\theta)[1-(A_\infty + D_\infty - 4\epsilon(a+1)\pi\hat{\mu}\ln(r\sin\theta))]\,\partial_\nu\varphi & \text{magnetic} \end{cases} && \text{(4.46c)}
\end{aligned}$$

where the two roots in (4.46b) correspond to electric and magnetic black holes respectively. This allows us to quantify precisely the limits of validity of our approximation. If $a \neq -1$, then it is easy to see that at very large distances, the strong effect of the vortex on the dilaton means that our simple form of the perturbation is no longer valid. Across the horizon there is an additional dilaton flux switched on, and we see that in spite of the fact that the thin vortex solution works for an extremal magnetic black hole, the back reaction for $a > -1$ is badly behaved at the horizon.

For $a = -1$, none of these problems arise, and we simply have a gentle shift in the value of the dilaton generated by the radial stresses of the vortex, $\phi(\infty) \to \phi(\infty) - \frac{1}{2}\epsilon D(\infty)$, which can be either positive, negative or zero depending on whether β is greater than, less than, or equal to unity. Note that this shift has the same sign for both magnetic and electric black holes, so that if the dilaton is increased in magnitude for an electric black hole, it is decreased in magnitude for the magnetic one, and vice versa. For $\beta = 1$, the only fields affected by the vortex are the $g_{\varphi\varphi}$ component of the metric, and the magnetic potential.

4.5 Conclusions.

To summarise, we have provided analytic arguments to show that a vortex can sit on a black hole horizon in dilatonic gravity, much the same as in Einstein gravity, the crucial difference being that for near extremal electrically charged black holes, the thin vortex approximation ceases to hold, and the flux starts to expel, however, this can be viewed as a consequence of the vanishing area of the horizon. For the case of massive dilatonic black holes, the thin vortex approximation holds in a range of cases, the only exception being near extremal electrically charged black holes for a small dilaton mass. We should also point out that these arguments can be used to paint a global vortex onto the dilatonic black hole, since a global vortex is obtained by setting $P = 1$, $\beta \to \infty$. However, in this case, we might expect the gravitational back reaction to be problematic, given the nature of the Einstein global string metric, which is not only non-asymptotically locally flat, but also time dependent [42].

The gravitational back reaction of the vortex was found assuming the validity of thin vortex approximation. The spacetime was found to be approximately conical to leading order, however, if the dilaton is massless, and if $a \neq -1$, there are long range effects on the geometry, which is not precisely conical.

Chapter 5

Conclusions

In this final chapter, we aim to summarise the main results of the other chapters.

The cosmic strings which have attracted the most attention are those produced at GUT scales. Such strings generate large angle cosmic microwave background anisotropies of roughly the observed order of magnitude. Their gravitational effects are weak and their static spacetimes are non-singular. It is also possible that cosmic strings appear at other thermal phase transitions. In chapter 2 we showed that in particular when the order of magnitude of the critical temperature is above the Planck mass, any such static spacetimes must be singular, while for scales below the Planck mass static cosmic strings are non-singular for a critical Higgs coupling. For other Higgs couplings, due to the lack of analytical methods a full numerical study would be required. Although, as shown in that chapter, one suspects that for Higgs masses below the boson vector mass static strings are non-singular when formed at scales below the Planck mass, and are singular when formed at scales above that mass.

In chapter 3 we showed that in the presence of a dilaton the gravitational field of cosmic strings is surprisingly close to that of an Einstein cosmic string on cosmological distance scales. For a massless dilaton we showed that there are long range dilatonic effects on the very large scale and the spacetime is not asymptotically locally flat in the Einstein and string frames. On scales of cosmological interest, however, the metric is conical in the Einstein frame, and conformally conical in the string frame. Meanwhile for a massive dilaton and apart from distances of order the compton wavelength of the dilaton, the long-range structure of the string is as for Einstein gravity, and so the metric asymptotes a conical metric, in both string and Einstein frames. The main exception to these results is for the special value where the coupling of the dilaton to the vortex fields is canonical ($a = -1$) for which there are no long range effects (other than the deficit angle) and the dilaton merely shifts the value of the dilaton between the core and infinity by a constant. For a critical Higgs coupling this constant vanishes and there is no effect at all on the dilaton. The dilatonic string is then the same as the Einstein one. The Bogomolnyi bounds for the dilatonic string were also considered and it was shown that again $a = -1$ is a special value for which the Bogomolnyi bounds are saturated.

Finally the effects of these strings on test particles were also explored. For a massive dilaton and in the string frame geodesics are essentially the same as for the Einstein self gravitating string, i.e., all non-static trajectories escape to infinity. The same conclusions hold for $a = -1$. For a massless dilaton and $a \neq -1$ photons escape to infinity and are infinitely redshifted ($a > -1$) or blueshifted ($a < -1$) while for massive particles trajectories are bound for $a > -1$ or escape to infinity for $a < -1$. In the Einstein frame photon trajectories are identical to that of the string frame while all massive particle trajectories are bound. Although cosmic strings are predicted by many unified theories of fundamental forces there is no firm indication that cosmic strings of any type do actually exist, but equally they are far from being ruled out and in some contexts they provide a very natural explanation for the observations [27].

Damour and Vilenkin [60] have explored the impact of a massive dilaton on string networks. They conclude that a TeV mass dilaton was incompatible with a GUT string network. Although one should renormalise their calculations by factors of $(a + 1)$ which means that their conclusion is only valid if a is not close to -1. If the strings couple to the dilaton directly ($a = -1$), then such constraints are identical to those derived in Einstein gravity. However, if the string couples with a different from -1, then the constraints of Damour and Vilenkin [60] apply, and a "*low*"(i.e. close to electroweak) mass for the dilaton rules out the cosmic string scenario of galaxy formation. This scenario is also ruled out by current observations of the cosmic microwave background anisotropies and galaxy clustering as shown by Pen et. al. [85]. In particular they showed that those observations do not favor models with global strings. In fact there is a serious conflict between standard scaling defect models and the current observational data [86]. This conflict can be expressed in terms of the "b_{100} *problem*", where b_{100} is the bias on scales of $100h^{-1}Mpc$ (with $0 < h < 0.5$). Current theoretical and experimental results indicate that the actual value of b_{100} is close to unity but the standard defect models require $b_{100} \approx 5$ to reconcile the predictions for the density field fluctuations with the observed galaxy distribution. This problem is likely to have a significant impact on the understanding of the origin of cosmic structure. The defect models are examples of models of structure formation seeded by perturbations of homogeneous universes via causal processes in the standard big bang (SBB). With the demise of the standard defect models, the question arises whether any plausible sbb causal model exists and if it doesn't then this is a very strong evidence for an inflationary origin of cosmic structure [86].

In the presence of an axion we showed in chapter 3 that the situation is quite different from the dilatonic strings and the gravitational field of axionic-dilatonic strings is not close to that of an Einstein cosmic string but instead asymptotically approaches the dilatonic Melvin magnetic universe in the presence of an axion. The long range effects of the dilaton and of the axion exclude the existence of local string solutions. The axion is strongly damped to zero while the dilaton has very strong asymptotic effects similar to those for the dilatonic string for massless dilaton.

Finally as infinitely extended objects strings can provide hairs for black holes being another example of physical systems where due to inclusion of matter fields in the horizon the non-hair theorem is not verified. In particular, in reference [11], it was shown that an Abelian Higgs vortex can act as an hair for the Schwarzschild black hole. Then Chamblin et al [77] generalised the analysis of that reference, [11], to the Reissner-Nordstrom black hole in Einstein-Maxwell theory. They found expulsion of the vortex from the extremal black hole even for thin vortices. However the work of Bonjour et al [13, 78] shows that a flux expulsion only occurs for black holes of size comparable to the string core. The thin Abelian Higgs vortex is also an hair for non-extreme electrically charged black holes with a massless dilaton in low energy string theory as first shown in reference [12]. Finally in reference [84] it is also shown the flux expulsion of the Abelian Higgs vortex from extremal electrically charged black holes with a massless dilaton. Therefore in chapter 4 we extend the work of reference [11] to strings and black holes in dilatonic gravity. Using analytic arguments we showed that the Nielsen-Olesen is hair for non-extreme charged black holes in the presence of a dilaton that can be either massless or massive. Near extremal electrically charged black holes, the thin vortex approximation ceases to hold, and the flux starts to expel. The gravitational back reaction of the vortex was found and the spacetime was found to be conical except when the dilaton is massless and $a \neq -1$. It was also shown that the effect of the dilaton on the horizon is to generate an aditional charge. For $a = -1$, the vortex can be used to smooth out the conical singularities of the dilatonic C- metrics as proven in [79].

The transition from penetration to expulsion can be viewed as a phase transition on the horizon of the black hole. The order parameter is the value of the Higgs field on the horizon and the phase transition takes place when we vary the inverse size of the horizon $(\frac{1}{E})$ [78]. It might be interesting to study this phase transition, e.g., study the critical exponents near the transition point, such as $X \sim |E - E_c|^b$ (and see, e.g., how b varies, or not, at different points on the horizon).

Another interesting question to be explored is the study of the geodesics and of the interactions between the black hole and the dilatonic cosmic string.

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