Affine Toda Field Theories on a half-line

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Abstract

This thesis is primarily concerned with the reflection factors of affine Toda field theories on the half-line $x \leq 0$. First, we consider the classical background configurations of low rank $a_r^{(1)}$ affine Toda theories with a boundary, constructed by the analytic-continuation of soliton solutions of the corresponding imaginary-coupling theories. We show that only a small subset of such solutions provide acceptable vacuum configurations. These are classified according to the integrable boundary conditions they obey and their classical reflection factors are considered.

We next consider the quantum theories, where we aim to provide evidence for or against exact reflection factors proposed in the literature. We do this by explicit calculation of the low-order coupling dependence of the reflection factors via perturbation theory. Two particular examples are considered in detail. The first is the $O(\beta^2)$ calculation for $a_2^{(1)}$ affine Toda field theory with the $- - -$ boundary condition. This will be a good example to study since it is the subject of many conjectured exact reflection factors and also demonstrates the renormalisation of the boundary potential required to retain quantum integrability.

The second example will be the $O(\beta^4)$ calculation for sinh-Gordon theory. In light of the added complexity of the higher-order calculation we consider only the Neumann boundary condition. Finally we look at the renormalisation of sinh-Gordon theory and its duality properties.

Declaration

The material presented in this thesis is based on research undertaken between October 1996 and September 1999. It has never previously been submitted for any degree at any university.

No claims for originality are made for the introductory chapter 1. The work in chapter 2 and in chapter 3 from 3.4 onwards (except where explicitly stated otherwise) was carried out in conjunction with my supervisor; the former is as yet unpublished (though currently in preprint form in [1]) whilst the latter has been published in [2]. The work in chapter 4 is all my own and is unpublished.

Acknowledgements

Firstly, I would like to thank my supervisor, Peter Bowcock for all the help given to me over the duration of this Ph.D. In addition, I gratefully acknowledge Ed Corrigan for helpful discussions and Patrick Dorey, in particular for his graduate lecture course on Exact S-matrices from which some of the introduction is drawn. Finally, I would also like to thank **Stan Bogle** and **Boardman!** for giving the motivation to continue.

This thesis is dedicated to "Rocket" McGuinn...

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Chapter 1

Introduction

1.1 Integrable models

Integrable systems are of great interest to field theorists since, in contrast to the Standard Model, they ought to be completely solvable without recourse to perturbation theory. Whilst at present the integrable models under study are far from describing these real-world interactions, it is hoped that the techniques developed in their study will perhaps one day aid the understanding of non-perturbative physics [3].

Moreover, in the short term the study of these integrable models can be of use in the modelling of physical systems, in areas ranging from solid state physics (in particular the Kondo model [4], which aims to describe the effect of impurities on the resistivity of a metal) to the study of monopoles [5]. In particular, integrable systems with one or more boundaries are of interest since many of these physical systems contain such boundaries. This thesis aims to extend the knowledge of these models by studying the *affine Toda field theories*, a large class of integrable systems in (1+1)-dimensions, restricted to the half-line $x \leq 0$. Affine Toda field theories have generated much interest since they are fairly general examples of relativistically invariant integrable models.

Before proceeding to the case of field theory, let us first recall what is meant by an integrable system in classical Hamiltonian mechanics. Such a system has the property that it is completely solvable; we can uniquely determine its trajectory in some, say $2N$-dimensional, phase-space. In order that this can be done, the system must satisfy some *integrability conditions.* These can be shown to be fulfilled when there exist N conserved charges K_i, which obey certain criteria: they must be functionally independent and be in *involution* — that is their Poisson bracket must be zero

$$\{K_i, K_j\}_{PB} = 0. \tag{1.1}$$

For the purposes of the present work, we shall provide the following definition for the integrability of a field theory. Since a field theory has an infinite number of degrees of freedom, then we assume that an infinite number of such conserved charges, all in involution, are required. In fact, such a definition turns out to be sufficient to give many desirable consequences. Two fundamental properties of such integrable field theories in (1+1)-dimensions are that (i) there is no particle production and (ii) the S-matrix is factorisable.

In order to show that these two properties hold it is necessary to consider the conserved charges of the system. The basic argument is that by using a conserved charge of spin $s > 1$ as a generator, we can shift the position of a particle by an amount dependent on the momentum of that particle. More precisely, a conserved charge P_s of Lorentz spin s transforms under a Lorentz transformation like s copies of the momentum k. Hence using such a conserved charge as a generator for transformations we can shift a particle at position x to a new position $x' = x + \alpha s k^{s-1}$. [1]

Particle production can shown to be disallowed by considering the conservation of these charges directly [6]; however an heuristic physical argument can also be given. Consider a process in which m particles collide and n particles are produced. Then the ability to shift particles with different momenta by differing amounts allows us to change from an acceptable situation, as demonstrated for a $2 \to 3$ particle process

[1] Consider a wavepacket, of the form $\psi(x) = \int_{-\infty}^{\infty} dk e^{-a^2(k-k_0)^2} e^{ik(x-x_0)}$. Now look at the effect of a transformation $e^{-i\alpha P_s}$; i.e. a transformation which uses one of the conserved charges P_s as a generator. This multiplies the wavepacket by an overall factor $e^{-i\alpha k^s}$. Since the integral is heavily weighted around $k = k_0$, then a Taylor expansion around this point quickly yields a shift in position of $\alpha s k_0^{s-1}$.

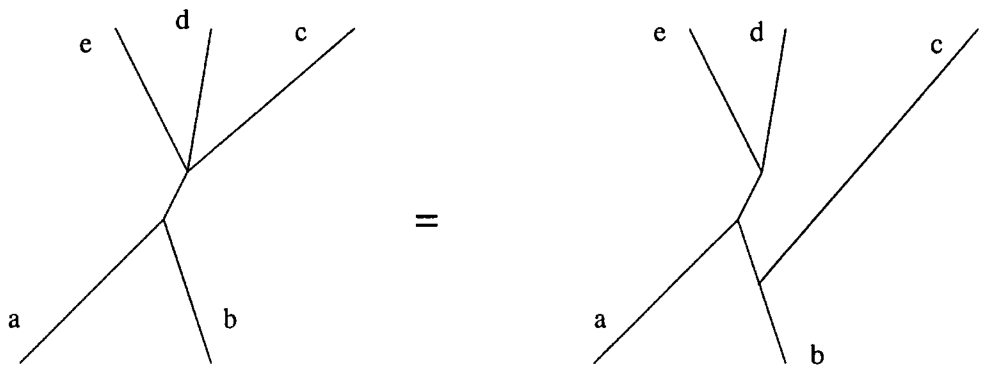

Figure 1.1: Particle production via a $2 \to 3$ process.

on the LHS of fig. 1.1, to an unacceptable one, shown schematically on the RHS. The RHS is not physically realistic as causality implies that the trajectories of the two incoming particles must not meet *after* the crossing of trajectories of an incoming and outgoing particle. In fact, the only case where it is not possible to perform this trick is when $n = m$ and the momenta of the incoming and outgoing particles are equal, as in this case the incoming and outgoing particles will be translated by the same amount. Therefore there is no particle production but an $m \to m$ particle process allowed.

The factorisability of the S-matrix follows by a similar argument. We can again use momentum-dependent translations to convert the $3 \to 3$ particle process on the RHS of fig. 1.2 to that on the LHS. This tells us that the three-particle S-matrix is the product of three two-particle S-matrices:

$$S_{abc}^{def} = S_{bc}^{b'c'} S_{ac'}^{a'f} S_{a'b'}^{de}. \tag{1.2}$$

Indeed this argument follows regardless of the number of particles present — all $m \to m$ particle S-matrices of the theory can be expressed as the products of two-particle S-matrices. This is a useful result since it will therefore suffice to determine only the two-particle scattering properties of the theory in order to obtain all scattering data.

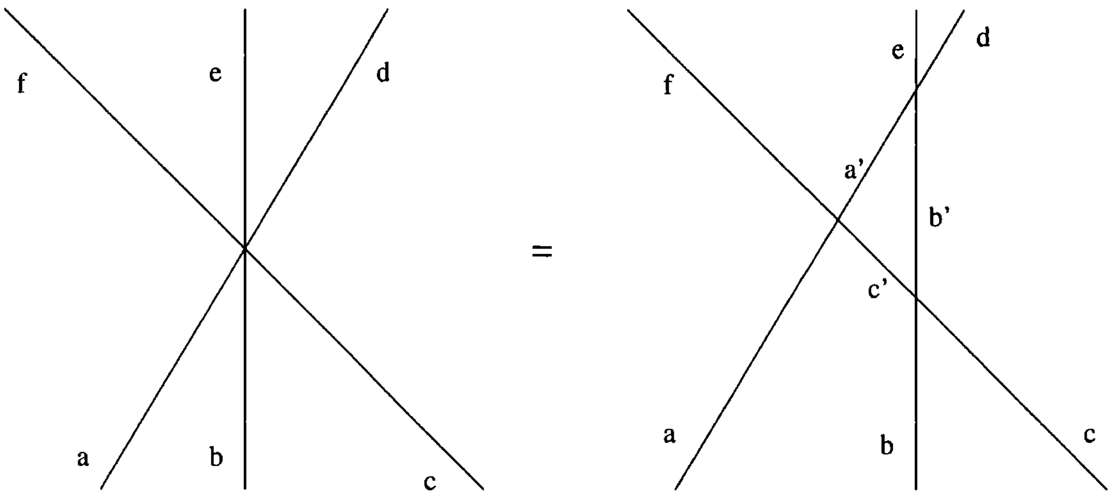

Figure 1.2: The factorisability of the S-matrix.

1.2 Affine Toda field theories

The affine Toda field theories are a class of integrable models in (1+1)-dimensions, built around some Lie algebra g of rank r, consisting of r real scalar fields ϕ^a (which we write succinctly as the vector ϕ). [2] We take the Lagrangian density (which we shall call $\mathcal{L}_0$) to be

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi - V \tag{1.3}$$

with the potential

$$V = \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i e^{\beta\alpha_i\cdot\phi}, \tag{1.4}$$

where the α_i, for $1 \leq i \leq r$, are the *simple roots* of g, and α_0 is given by

$$\alpha_0 = -\sum_{i=1}^{r} n_i\alpha_i. \tag{1.5}$$

This extra (or *affine*) root corresponds to the extra spot present in the Dynkin diagram of the affine Lie algebra $\hat{g}$ (examples of such Dynkin diagrams are given in figs. 1.3 and 1.4 for a_r and its associated affine algebra $a_r^{(1)}$) . A good review of Lie algebras can be found in [10] whilst their affine extensions are covered by [11]. The n_i are the 'marks' — characteristic integers for each algebra g — and we take conventionally $n_0 = 1$. The parameter m gives a mass-scale to the theory, which, for simplicity, it is usual to set to one.

[2]Good reviews of the ideas in this section can be found in [7–9].

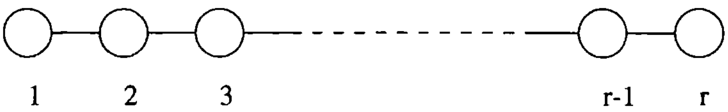

Figure 1.3: The Dynkin diagram for the Lie algebra a_r.

For the special case of the $a_r^{(1)}$ series of affine Toda field theories (which have rank r), the marks $n_i = 1\ \forall i$ so we have the Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{\beta^2}\sum_{i=0}^{r} e^{\beta\alpha_i\cdot\phi}. \tag{1.6}$$

The equations of motion corresponding to (1.3) and (1.6) are

$$\partial^2\phi + \frac{m^2}{\beta}\sum_{i=0}^{r} n_i\alpha_i e^{\beta\alpha_i\cdot\phi} = 0 \tag{1.7}$$

and

$$\partial^2\phi + \frac{1}{\beta}\sum_{i=0}^{r} \alpha_i e^{\beta\alpha_i\cdot\phi} = 0 \tag{1.8}$$

respectively.

We shall use (1.6) and (1.8) extensively later. However, for the moment let us retain the general forms (1.3) and (1.7).

Notice that if we rescale the field $\phi \rightarrow \frac{\phi}{\beta}$ then these equations become independent of the coupling constant β. Hence classically β is unimportant; in fact it shall frequently be convenient to set $\beta = 1$ for simplicity. Moreover, since the action S is given by

$$S = \frac{i}{\hbar}\int dx dt \mathcal{L} \tag{1.9}$$

then such a rescaling introduces a β^2 term into the denominator of the prefactor. So the classical limit can be found by taking the limit $\beta \rightarrow 0$ (which essentially then corresponds to the usual classical limit $\hbar \rightarrow 0$). We shall use this later when determining the classical limits of the exact quantum reflection factors.

Let us look at how to prove the classical integrability of this model, or in other words how to find an infinite set of conserved charges. We shall follow the *Lax pair* argument of [12], although the integrability was initially studied in [13–17]. We

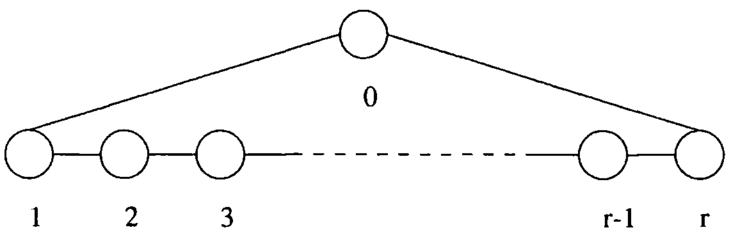

Figure 1.4: The Dynkin diagram for the affine Lie algebra $a_r^{(1)}$.

define the Lax pair to be a potential a_μ, in this case given by

$$\begin{aligned} a_0 &= \frac{H.\partial_x\phi}{2} + \sum_{i=0}^{r} \sqrt{m_i}(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2} \\ a_1 &= \frac{H.\partial_t\phi}{2} + \sum_{i=0}^{r} \sqrt{m_i}(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}. \end{aligned} \tag{1.10}$$

Here, H is a *Cartan subalgebra* of g and E_{α_i}, $E_{-\alpha_i}$ are the *root vectors* (sometimes called the *step-operators*) corresponding to the simple roots of the affine Lie algebra $\hat{g}$. [3] We define the m_i by

$$m_i = \frac{n_i\alpha_i^2}{8}, \tag{1.11}$$

where we use α_i^2 as a shorthand for $|\alpha_i|^2$. It is easy to show that the equations of motion for the affine Toda field theory can be reproduced by the zero curvature condition

$$[\partial_t + a_0, \partial_x + a_1] = \partial_t a_1 - \partial_x a_0 + [a_0, a_1] = 0, \tag{1.12}$$

for any value of the arbitrary parameter λ, using the Lie algebra relations (again see [10])

$$[H, E_{\pm\alpha_i}] = \pm\alpha_i E_{\pm\alpha_i} \text{ and } [E_{\alpha_i}, E_{-\alpha_i}] = \frac{2\alpha_i\cdot H}{\alpha_i^2}. \tag{1.13}$$

How does this help with the construction of an infinite set of conserved quantities? Let us define the path-ordered exponential

$$U(x_1, x_2; \lambda) = \mathrm{P}e^{\int_{x_1}^{x_2} a_1 dx}. \tag{1.14}$$

[3] Suppose that X_i is a basis for g, with the commutation relations $[X_i, X_j] = f_{ij}^k X_k$. Then we can also define a set of matrices, T_i, by $[T_i]_b^a = f_{ib}^a$ which must then satisfy $[T_i, T_j] = f_{ij}^k T_k$ by the Jacobi identity. Then for any $X = a^i X_i \in g$ we write $\mathrm{ad}(X) = a^i T_i$ and $\mathrm{ad}(X)Y = [X, Y]$. A Cartan subalgebra H is defined to be the algebra spanned by a maximal set of linearly independent commuting elements, h_i, in g — there are in fact r such elements in this set. We can then complete the basis for g by including the root vectors E_{α_i}; the eigenvectors of $\mathrm{ad}(h_i)$ with eigenvalue α_i: $\mathrm{ad}(h_i)E_{\alpha_i} = [h_i, E_{\alpha_i}] = \alpha_i E_{\alpha_i}$.

This satisfies the relation

$$\frac{d}{dt}U(x_1,x_2;\lambda) = U(x_1,x_2;\lambda)a_0(x_2) - a_0(x_1)U(x_1,x_2;\lambda) \tag{1.15}$$

and so if we define

$$Q(\lambda) = \mathrm{tr}U(-\infty,\infty;\lambda) \tag{1.16}$$

then, under the conditions $\partial_x\phi \to 0$ as $x_1 \to \pm\infty$ and $\phi(\infty) = \phi(-\infty)$ (which together imply that $a_0(\infty) = a_0(-\infty)$), $Q(\lambda)$ is conserved for all λ. Therefore an infinite number of conserved quantities can be constructed by the Taylor expansion of $Q(\lambda)$ around $\lambda = 0$, since all the coefficients of this power series must be themselves conserved.

Of course, we must also prove that these charges are in involution. This is more difficult and the argument shall not be reproduced here. It can be found in [16].

We shall just briefly note here that the conserved charges can be put in a more convenient form by performing a gauge transformation on the potential a_μ. This can be done to obtain the potential a_1 in the form

$$\tilde{a_1} = \lambda E_1 + \sum_{s\geq 1}\lambda^{-s}H_sI_0^{(s)} \tag{1.17}$$

(see [9]) where we define $E_{\pm 1} = \sum_{i=0}^{r} m_i E_{\pm\alpha_i}$. These $E_{\pm 1}$ lie in a Cartan subalgebra of g, spanned by H_i. The spin s takes values modulo h, the Coxeter number of the Lie algebra, defined by $h = \sum_{i=0}^{r} n_i$. In fact, both components of the potential now lie in the Cartan subalgebra (and hence commute), and so the zero curvature condition simply reads

$$\partial_t\tilde{a_1} = \partial_x\tilde{a_0}. \tag{1.18}$$

Then the integral of $\tilde{a_1}$ over the whole-line must be conserved, and thus we can see that so are

$$Q_s = \int_{-\infty}^{\infty} dx I_0^{(s)}. \tag{1.19}$$

These are the classical conserved charges of spin s. In a similar way we can obtain the conserved charges of the opposite spin, $-s$. [4]

[4] Whilst what we have discussed here relates to the classical theory, it is expected that integrability also extends to the quantum case. However, conservation of charges in the quantum theory is more involved [18, 19].

1.3 Affine Toda field theory on the whole-line

In this section we consider in more detail the properties of affine Toda field theory on the whole-line, $-\infty < x < \infty$. In particular, we shall look at the particle masses, three-point couplings and bulk S-matrices of the theory.

Let us consider the masses and three-point couplings of the theories first. These can easily be found by expanding the potential — they are given by the coefficients of the quadratic and cubic terms in the fields. Expanding (1.4) gives

$$\begin{aligned} V &= \frac{m^2}{\beta^2}\left[\sum_{i=0}^{r} n_i \left\{\beta\alpha_i\cdot\phi + \frac{\beta^2}{2}(\alpha_i\cdot\phi)^2 + \frac{\beta^3}{6}(\alpha_i\cdot\phi)^3 + O(\beta^4)\right\}\right] \\ &= \frac{1}{2}(M^2)^{ab}\phi_a\phi_b + C^{abc}\phi_a\phi_b\phi_c + O(\beta^2) \end{aligned} \qquad (1.20)$$

where the linear term in ϕ disappears since $\sum_{i=0}^{r} n_i\alpha_i = 0$ and we define the mass matrix M and three-point couplings C^{abc} by

$$(M^2)^{ab} = m^2\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b \qquad (1.21)$$

$$C^{abc} = \frac{m^2}{6}\beta\sum_{i=0}^{r} n_i\alpha_i^a\alpha_i^b\alpha_i^c. \qquad (1.22)$$

For consistency with later chapters, we use an alternative definition of the three-point coupling C^{abc} to much of the literature, which differs from ours by a factor of six. Note also that (1.21) can be written more succinctly as

$$M^2 = m^2\sum_{i=0}^{r} n_i\alpha_i\otimes\alpha_i. \qquad (1.23)$$

The masses can then be found by diagonalising this matrix. Let us see how this is done in the special case of $a_r^{(1)}$ theory, with which most of this thesis is concerned. We shall follow the argument given in [8].

Let us first of all define the *Cartan matrix* C of the Lie algebra g. It is given by

$$C_{ij} = \frac{2\alpha_i\cdot\alpha_j}{\alpha_j^2} \qquad (1.24)$$

and encodes the structure of the Dynkin diagram of g. For $a_r^{(1)}$ theory, we can easily compute the Cartan matrix from the results

$$\alpha_i^2 = 2 \text{ and } \alpha_i\cdot\alpha_{i+1} = -1 \;\; \forall i. \qquad (1.25)$$

Next, we need to find a suitable basis for the α_i's. Consider taking

$$\gamma_j^a = \omega^{aj} \tag{1.26}$$

for $j = 0, ..., r$ and $a = 1, ..., r$, and where $\omega = e^{\frac{2\pi i}{r+1}}$. Then it is not hard to show that

$$\gamma_i^* \cdot \gamma_j = (r+1)\delta_{ij} - 1. \tag{1.27}$$

Hence we can take a complex basis for the simple roots to be

$$\alpha_i = \frac{1}{\sqrt{r+1}}(\gamma_{i+1} - \gamma_i)^*; \tag{1.28}$$

if we use the inner product

$$(a, b) = a^* \cdot b \tag{1.29}$$

then these obey the properties required to reproduce the Dynkin diagram in fig. 1.4.

In addition, if we take a complex basis for ϕ in which we have the relation $(\phi^a)^* = \phi^{r+1-a}$ (in fact we show an example of how to construct such a basis explicitly in chapter 3), then it is not hard to show that we can write the Lagrangian as

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi^* \cdot \partial^\mu \phi - \frac{m^2}{\beta^2}\sum_{i=0}^{r} e^{\beta\alpha_i^* \cdot \phi}. \tag{1.30}$$

Since $(\alpha_i^* \cdot \phi)^* = (\alpha_i^* \cdot \phi)$ then we can write the mass term as

$$\frac{1}{2}m^2 \sum_{i=0}^{r} (\alpha_i^a)^* \alpha_i^b \phi_a^* \phi_b. \tag{1.31}$$

Indeed, it is handy to rename M as

$$(\tilde{M}^2)^{ab} = m^2 \sum_{i=0}^{r} (\alpha_i^a)^* \alpha_i^b, \tag{1.32}$$

and simply computing this mass squared matrix, we find that

$$\begin{aligned} (\tilde{M}^2)^{ab} &= \frac{m^2}{r+1}\sum_{i=0}^{r}(\gamma_{i+1}^* - \gamma_i^*)^a(\gamma_{i+1} - \gamma_i)^b \\ &= \begin{cases} 0 & \text{for} \quad a \neq b \\ 4m^2 \sin^2\left(\frac{a\pi}{r+1}\right) & \text{for} \quad a = b \end{cases} . \end{aligned} \tag{1.33}$$

So the mass squared matrix $\tilde{M}^2$ is diagonal and $\phi_a^* = \phi_{r+1-a}$ is the field associated with the conjugate particle to a. In fact, this is the form we expect the mass term to take, and hence the classical masses of the particles can be read off:

$$m_a = 2m \sin\left(\frac{a\pi}{r+1}\right). \tag{1.34}$$

Let us consider the vector whose elements are the particle masses:

$$\mathbf{m} = (m_1, m_2, ..., m_r). \tag{1.35}$$

If we order these masses correctly then we can obtain an eigenvector of the Cartan matrix associated with the Lie algebra g [8, 20, 21] (at least in the simply laced cases) so that

$$C\mathbf{m} = 4\sin^2\left(\frac{\pi}{2h}\right)\mathbf{m}. \tag{1.36}$$

The existence of this eigenvector makes it easy to unambiguously identify the various particles in the theory with spots on the associated (non-extended) Dynkin diagram. In the case of $a_r^{(1)}$ theory, the particles of types a and $r+1-a$ are conjugates to each other and have the same mass.

There also exists a relationship between the three-point couplings C^{abc} and the masses. Since we now have

$$\begin{aligned} C^{abc} &= \frac{m^2}{6}\beta\sum_{i=0}^{r}(\alpha_i^a)^*(\alpha_i^b)^*(\alpha_i^c)^* \\ &= \begin{cases} 0 & \text{for} \quad a+b+c \neq 0 \bmod r+1 \\ \frac{m^2\beta}{6\sqrt{r+1}}(\omega^a-1)(\omega^b-1)(\omega^c-1) & \text{for} \quad a+b+c = 0 \bmod r+1 \end{cases} \end{aligned} \tag{1.37}$$

then this can be easily rewritten as

$$|C^{abc}| = \frac{m^2\beta}{6\sqrt{r+1}}\left(2m_a m_b \sin\left(\frac{(a+b)\pi}{r+1}\right)\right) \tag{1.38}$$

when $a+b+c = 0 \bmod r+1$. This is proportional to the area of a triangle with sides m_a, m_b and m_c [22–24]. The internal angles of the triangle are $\overline{\theta}_{ab}^c$ and cyclic permutations of its indices, where we have defined

$$\overline{\theta} = \pi - \theta. \tag{1.39}$$

It is useful to use θ to represent the *rapidity* of a particle. We put the momentum k_a and energy ω_a of a particle a to be

$$k_a = m_a \sinh(\theta), \; \omega_a = m_a \cosh(\theta). \tag{1.40}$$

In fact, $i\theta_{ab}^c$ gives the relative rapidity required for the fusing processes $a+b \to \overline{c}$, and shall be used extensively later. It can be shown that [8]

$$\theta_{ab}^c = \begin{cases} \frac{a+b}{r+1}\pi & \text{for} \quad a+b+c = r+1 \\ \left(2-\frac{a+b}{r+1}\right)\pi & \text{for} \quad a+b+c = 2(r+1) \end{cases}. \tag{1.41}$$

Note that this fusing occurs at an imaginary rapidity difference since the integrability of the model tells us that there is no particle production.

It is possible to continue to look at higher order couplings of the theory. Indeed, where such non-zero couplings exist has been studied and is known as Dorey's rule [22, 23].

As noted previously, all scattering data can be found once we know the two-particle S-matrices of the theories. These were conjectured on the basis of knowledge of the properties which they must obey (which we shall see shortly) by various authors [8, 25–27]. [5] The conjectures were then checked using perturbative techniques. In fact, a parallel can be drawn between computation of the S-matrices and what is presently being done for the reflection factors (introduced in section 1.5).

Let us briefly review the arguments which lead to the conjectured exact S-matrices in [8]. The first property expected of the S-matrices is that they collapse to unity in the classical limit $\beta \to 0$. We also require that the S-matrices satisfy the constraints of unitarity

$$S_{ab}(\theta)S_{ab}(-\theta) = 1 \tag{1.42}$$

and crossing symmetry

$$S_{ab}(i\pi - \theta) = S_{b\bar{a}}(\theta). \tag{1.43}$$

Notice that these two conditions taken together imply that the S-matrix is $2\pi i$ periodic. It is helpful to set up some notation to aid with the determination of the S-matrices. We define the blocks

$$(x) = \frac{\sinh(\frac{\theta}{2} + \frac{i\pi x}{2h})}{\sinh(\frac{\theta}{2} - \frac{i\pi x}{2h})}. \tag{1.44}$$

These clearly obey (1.42) and have the desired periodicity.

Another property required of the S-matrices is that they obey the bootstrap equation. This is shown diagrammatically in fig. 1.5, and can be written algebraically as

$$S_{d\bar{c}}(\theta) = S_{da}(\theta - i\overline{\theta}^{b}_{ac})S_{db}(\theta + i\overline{\theta}^{a}_{bc}). \tag{1.45}$$

[5] Some new identities satisfied by the S-matrices have recently been discovered [28].

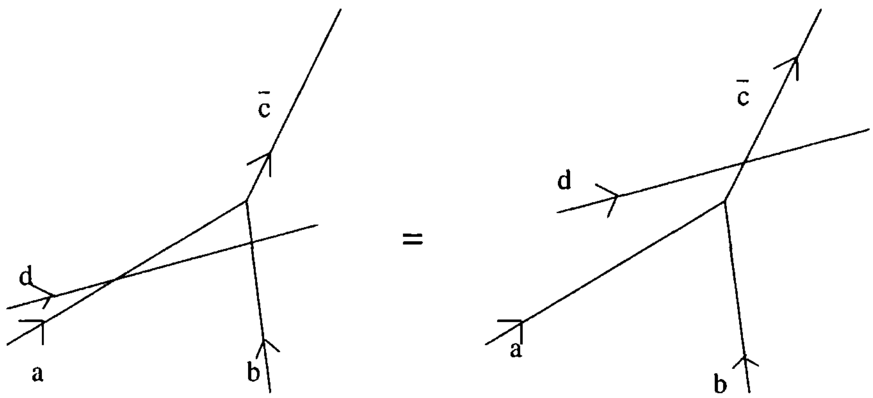

Figure 1.5: The bootstrap equation.

We also need to consider the pole structure of the S-matrices. There must be a pole to represent any possible fusing $a + b \to \bar{c}$; i.e. at $\theta = i\theta_{ab}^{c}$. This enables us to conjecture a minimal S-matrix for the theory.

Let us consider here only the case of $a_r^{(1)}$ affine Toda field theory: the other simply-laced cases can be found in [8] if required. We could guess the minimal S-matrix to be

$$S_{11} = (2) \tag{1.46}$$

but this does not have a classical limit of unity. We must at this stage guess the dependence of the theory on the coupling constant β. Taking

$$S_{11} = \frac{(2)}{(B)(2-B)} \tag{1.47}$$

where B is a function of β obeying $B \to 0$ as $\beta \to 0$, we satisfy all the requirements (1.42), (1.43) and (1.45). In this thesis we shall frequently concern ourselves with the notion of strong/weak coupling duality. A *duality* transformation is one where we send

$$\beta \to \frac{4\pi}{\beta}. \tag{1.48}$$

If we assume that the S-matrices remain unchanged under such a duality transformation (i.e. they are *self-dual*), we can suggest a possible form of $B(\beta)$ to be (this approach is backed up by [25, 29]):

$$B = \frac{\beta^2}{2\pi(1 + \frac{\beta^2}{4\pi})}. \tag{1.49}$$

Postulates for all the S-matrices of $a_r^{(1)}$ affine Toda field theory were constructed. These can be written succinctly if we define a new block

$$\{x\} = \frac{(x+1)(x-1)}{(x+1-B)(x-1+B)}. \tag{1.50}$$

Then the result is

$$S_{ab} = \prod_{\substack{p=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \{p\}. \tag{1.51}$$

At this stage, the S-matrices are only conjectures based upon some desired properties. Low-order perturbation theory [30–34] and numerical techniques [35] were then used to check these results.

It is useful before moving on to quickly note some of the properties of the block notation introduced. We have

$$(-x) = \frac{1}{(x)}, \quad (h+x) = \frac{1}{(h-x)} \tag{1.52}$$

and, consequently,

$$\{-x\} = \frac{1}{\{x\}}, \quad \{h+x\} = \frac{1}{\{h-x\}}. \tag{1.53}$$

In addition, for the purposes of checking agreement with the reflection bootstrap equations later, it is useful to note that

$$(x)_{\theta+iy}(x)_{\theta-iy} = (x-hy)(x+hy) \tag{1.54}$$

and

$$\{x\}(2\theta) = \frac{\left(\frac{x}{2}-\frac{1}{2}\right)\left(\frac{x}{2}+\frac{1}{2}\right)\left(h-\frac{x}{2}-\frac{1}{2}+\frac{B}{2}\right)\left(h-\frac{x}{2}+\frac{1}{2}-\frac{B}{2}\right)}{\left(h-\frac{x}{2}-\frac{1}{2}\right)\left(h-\frac{x}{2}+\frac{1}{2}\right)\left(\frac{x}{2}-\frac{1}{2}+\frac{B}{2}\right)\left(\frac{x}{2}+\frac{1}{2}-\frac{B}{2}\right)}. \tag{1.55}$$

1.4 Affine Toda field theory on the half-line

We discussed above the properties of affine Toda field theory defined on the whole line $\mathbb{R}$. But what would happen if we were to restrict the theory to some interval? This question was first addressed by Cherednik [36] and Sklyanin [37]; more recent reviews of the implications for affine Toda field theory can be found in [38–40]. The simplest case we can consider is when this interval is taken to be the half-line $x \leq 0$, or $(-\infty, 0]$. We shall be primarily considering in this thesis the $a_r^{(1)}$ series of affine Toda field theories defined on this interval.

The Lagrangian density (1.3) is now defined on the the half-line, and in addition we may have some boundary potential $\mathcal{B}$ defined at $x = 0$:

$$\mathcal{L} = \mathcal{L}_0(\phi)\theta(-x) - \mathcal{B}(\phi)\delta(x). \tag{1.56}$$

Far from the boundary we have the usual bulk Lagrangian and hence the particle masses, three-point couplings and bulk S-matrices are the same as the whole-line case discussed in the previous section.

Varying the Lagrangian (1.56) we see that this is equivalent to usual equations of motion (though now of course for $x \leq 0$) and the boundary condition

$$\partial_x \phi|_{x=0} = -\frac{\partial \mathcal{B}}{\partial \phi}. \tag{1.57}$$

However, not all boundary conditions will retain the classical integrability of the theory. In this section, we shall review the arguments used to determine the boundary conditions which may be imposed whilst keeping the theory integrable.

A proof of the form of the boundary conditions which are consistent with integrability uses a Lax pair argument like that in section 1.2 but adapted for use on the half-line. This can be found in [12, 41]; however, it is considerably more involved than the whole-line argument and we shall not discuss it here. We shall, however, note the results, which give the possible integrable boundary conditions to be

$$\partial_x \phi|_{x=0} = \frac{1}{\beta} \sum_{i=0}^{r} A_i \alpha_i \sqrt{\frac{2n_i}{\alpha_i^2}} e^{\beta \alpha_i \cdot \phi / 2}. \tag{1.58}$$

The parameters A_i are severely restricted in the values they can take. Full details are given in [12]; however it is useful to note that in the simply-laced cases we are only allowed to take either $|A_i| = 1 \ \forall i$, or the Neumann boundary condition

$$\partial_x \phi|_{x=0} = 0. \tag{1.59}$$

It is easy to refer to the former boundary conditions by listing the signs of the A_i in turn; e.g. $A_i = 1 \ \forall i$ shall be referred to as $+ + +...+$. Our notation here is at odds with some of the literature on the subject, which takes the opposite sign or a factor of two compared to our A_i.

The exception to this is the sinh-Gordon case, i.e. $a_1^{(1)}$ theory. This is the only simply-laced case for which the integrable boundary conditions have continuous degrees of freedom. For any σ_0 and σ_1, the boundary condition

$$\partial_x \phi|_{x=0} = -\frac{\sqrt{2}}{\beta}\left(\sigma_1 e^{\beta\phi/\sqrt{2}} - \sigma_0 e^{-\beta\phi/\sqrt{2}}\right) \tag{1.60}$$

retains the integrability of sinh-Gordon theory [42–44].

An argument supporting the integrability of the boundary conditions (1.58) and (1.59) can be constructed by considering the boundary conditions necessary to conserve the various spin charges on the half-line. Since we have now lost translational invariance, there is no chance of conserving momentum on the half-line. However, it may be possible to conserve combinations of spin ± 2, ± 3 charges and so on. Let us consider how to conserve a combination of the spin ± 2 charges, closely following [45].

An ansatz for the densities corresponding to these on the whole-line is given by

$$T_{\pm 3} = \frac{1}{3} A_{abc} \partial_\pm \phi_a \partial_\pm \phi_b \partial_\pm \phi_c + B_{ab} \partial_\pm^2 \phi_a \partial_\pm \phi_b \tag{1.61}$$

where we use light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(t \pm x)$. In addition, A_{abc} is taken to be completely symmetric whilst B_{ab} is anti-symmetric. To construct the conserved quantities, we need the relation

$$\partial_\mp T_{\pm 3} = \partial_\pm \Theta_{\pm 1} \tag{1.62}$$

to reproduce the equations of motion (1.7). We therefore take

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_\pm \phi_a V_b \tag{1.63}$$

with the constraint

$$A_{abc} V_a + B_{ab} V_{ac} + B_{ac} V_{ab} = 0. \tag{1.64}$$

Here V is as usual the bulk potential and its subscripts imply differentiation with respect to ϕ with that subscript, e.g. $V_b = \frac{\partial V}{\partial \phi_b}$. Multiplying this by $\alpha_j^b \alpha_k^c$ and putting in the form of V tells us that

$$\frac{1}{\beta} A_{ijk} + B_{ij} C_{ik} + B_{ik} C_{ij} = 0 \tag{1.65}$$

where $A_{ijk} = A_{abc}\alpha_i^a\alpha_j^b\alpha_k^c$, $B_{ij} = B_{ab}\alpha_i^a\alpha_j^b$ and $C_{ij} = \alpha_i \cdot \alpha_j$.

Back in terms of the Minkowski coordinates x and t, (1.62) is

$$\partial_t(T_{+3} - \Theta_{-1} \pm (T_{-3} - \Theta_{-1})) = \partial_x(T_{+3} + \Theta_{+1} \mp (T_{-3} + \Theta_{-1})) \tag{1.66}$$

so the quantity

$$\int_{-\infty}^{0} dx(T_{+3} + \Theta_{+1} - T_{-3} - \Theta_{-1}) - \Sigma_2 \tag{1.67}$$

will be conserved on the half-line if

$$T_{+3} + \Theta_{+1} - T_{-3} - \Theta_{-1} = \partial_t\Sigma_2 \tag{1.68}$$

at the point $x = 0$. In order to achieve (1.68), we can show that we require the constraints

$$A_{abc}\mathcal{B}_a + 2B_{ab}\mathcal{B}_{ac} + 2B_{ac}\mathcal{B}_{ab} = 0 \tag{1.69}$$

and

$$\frac{1}{3}A_{abc}\mathcal{B}_a\mathcal{B}_b\mathcal{B}_c + 2B_{ab}V_a\mathcal{B}_b = 0. \tag{1.70}$$

We now only need make a comparison between the first of these constraints and (1.64) to see that the boundary term must be of the form

$$\mathcal{B} = \frac{m^2}{\beta^2}\sum_{i=0}^{r} 2A_i e^{\beta\alpha_i\cdot\phi/2}. \tag{1.71}$$

The second constraint is slightly harder to analyse. We eliminate A_{abc} using the condition (1.65). Then, writing $e_i = e^{\beta\alpha_i\cdot\phi/2}$ and using our result (1.71) allows (1.70) to be written

$$\frac{1}{6}\sum_{ijk}(B_{ij}C_{ik} + B_{ik}C_{ij})A_iA_jA_ke_ie_je_k = \sum_{ij} B_{ij}A_je_i^2e_j. \tag{1.72}$$

Comparing coefficients of the exponentials on either side reveals that the A_i take the values

$$\text{either } A_i = 0 \ \ \forall i \qquad \text{or } A_i^2 = 1 \ \ \forall i. \tag{1.73}$$

In fact, these conclusions have been checked for the conservation of higher spin charges (± 3 and ± 4) and lead to the same conditions (1.71) and (1.73), as implied by the Lax pair argument on the half-line.

What has been discussed above only relates to the classical integrability of the theory. Integrability of the quantum theory is an on-going area for study and in fact we shall see in later chapters that it leads to some additional subtleties.

1.5 Reflection factors

Suppose that we have determined the static background solution ϕ_{vac} and consider now a small, time-dependent, perturbation around that solution. We put

$$\phi(x,t) = \phi_{vac}(x) + \epsilon(x,t). \tag{1.74}$$

This perturbation must then obey the equation of motion (from (1.7))

$$\partial^2\epsilon(x,t) + m^2\sum_{i=0}^{r} n_i\alpha_i(\alpha_i\cdot\epsilon(x,t))e^{\beta\alpha_i\cdot\phi_{vac}} = 0. \tag{1.75}$$

If we consider the limit of this equation asymptotically far from the boundary, then $\phi_{vac} \to 0$ and so the asymptotic limit of $\epsilon(x,t)$ must obey

$$\partial^2\epsilon(x,t) + M^2\epsilon(x,t) = 0. \tag{1.76}$$

A useful solution to this is, for example

$$\epsilon(x,t) = \rho_a I_a e^{ik_a x - i\omega_a t} \tag{1.77}$$

where ρ_a is an eigenvector of the mass-squared matrix (1.23). In fact, this is of the form of a right-moving particle of mass m_a, albeit asymptotically far from the boundary, and the energy ω_a and momentum k_a obey $\omega_a^2 - k_a^2 = m_a^2$.

What will happen when this particle collides with the boundary? It is reasonable to assume that some particle, say of type b, will be elastically reflected. In fact, integrability tells us that we can only obtain rearrangements among mass-degenerate particles — for example between a particle and its conjugate. Moreover, for the theories we shall be considering, we expect that the particle type will be conserved. This is a consequence of the ability to find spin-even charges which are preserved (see section 1.4) as it is these which distinguish between particles and their conjugates [46]. We shall, however, initially keep our discussion to the general case where the two particles, although mass-degenerate, need not be the same. So a full scattering solution on the half-line has the asymptotic form as $x \to -\infty$

$$\epsilon(x,t) = I^a(\rho_a e^{ikx} + \rho_b K_a^b e^{-ikx})e^{-i\omega t}, \tag{1.78}$$

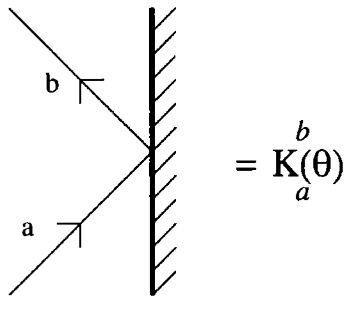

Figure 1.6: Reflection of a particle from the boundary.

where of course $\omega^2 - k^2 = m^2$. The reflection factor associated with an incoming particle of type a and outgoing particle of type b is therefore given by K_a^b. This process is depicted in fig. 1.6.

Applying a little thought to the physical processes occurring in the theory (remembering of course the existence of the conserved charges which allow momentum-dependent translations) gives us several additional algebraic constraints which must be satisfied by the reflection factors $K_a^b(\theta)$. The first of these is the boundary Yang-Baxter equation. This tells us that the scattering of two particles from the boundary is not dependent on the time at which they scatter from each other. This is shown diagrammatically in fig. 1.7. We write the condition algebraically as

$$\sum_{b',c',d',d''} K_a^{d''}(\theta_a) S_{bd''}^{b'd'}(\theta_a+\theta_b) K_{b'}^{c'}(\theta_b) S_{c'd'}^{cd}(\theta_b-\theta_a)$$
$$= \sum_{a',a'',b',c'} S_{ab}^{a'b'}(\theta_b-\theta_a) K_{b'}^{c'}(\theta_b) S_{a'c'}^{a''c}(\theta_a+\theta_b) K_{a''}^{d}(\theta_a). \quad (1.79)$$

This equation places constraints on the forms of the reflection matrices $K_a^b(\theta)$. However, if we take the S-matrices and reflection factors to be diagonal (i.e. conservation of particle type at the boundary), then this relation is satisfied automatically.

Another condition which must be considered is that of boundary unitarity. We simply require

$$K_a^c(\theta) K_c^b(-\theta) = \delta_a^b. \quad (1.80)$$

Finally, another important constraint is the boundary bootstrap equation, which for diagonal scattering is shown diagrammatically [6] in fig. 1.8. This can be written

[6] An interesting alternative argument that the classical reflection factors must satisfy

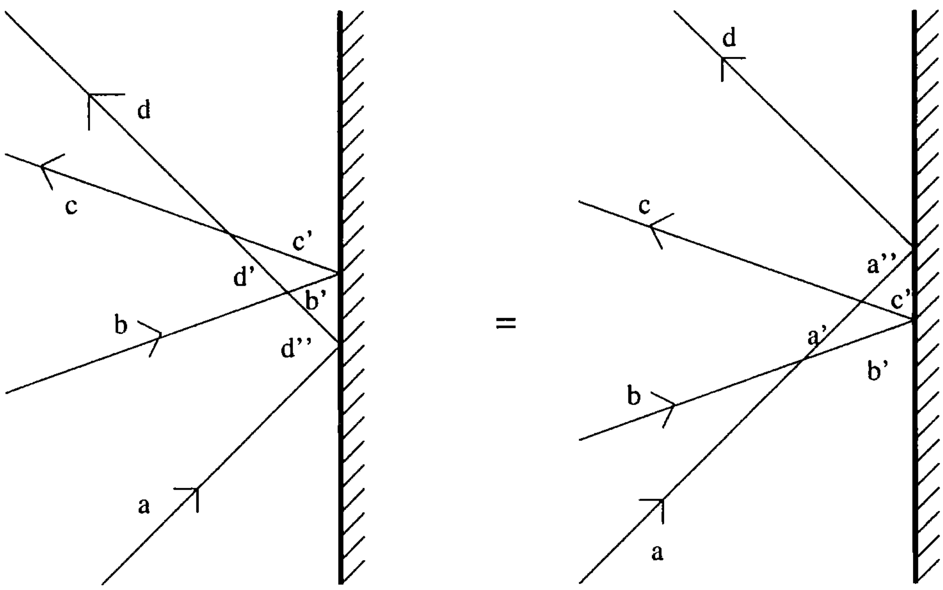

Figure 1.7: The boundary Yang-Baxter equation.

algebraically as

$$K_c(\theta) = K_a(\theta + i\overline{\theta}_{ac}^{b})S_{ab}(2\theta + i\overline{\theta}_{ac}^{b} - i\overline{\theta}_{bc}^{a})K_b(\theta - i\overline{\theta}_{bc}^{a}). \tag{1.81}$$

In the following section we shall see that this is a very useful condition.

We shall now proceed to examine some of the techniques used to give insight into the possible exact reflection factors of affine Toda field theory. However, it is not the purpose of this section to provide a full discussion of the work in this area, and it shall be necessary to restrict ourselves to looking at only a small proportion of the literature on the subject. Other examples of techniques used to make headway in this difficult area can be found in [47, 48].

1.5.1 Reflection factors from the boundary bootstrap equation

This section will look at some examples of work which have attempted to solve the reflection bootstrap equation algebraically. Fring and Köberle [49, 50] have carried out much work in this area. However, here we shall instead take a brief look at the

this equation is given in [41].

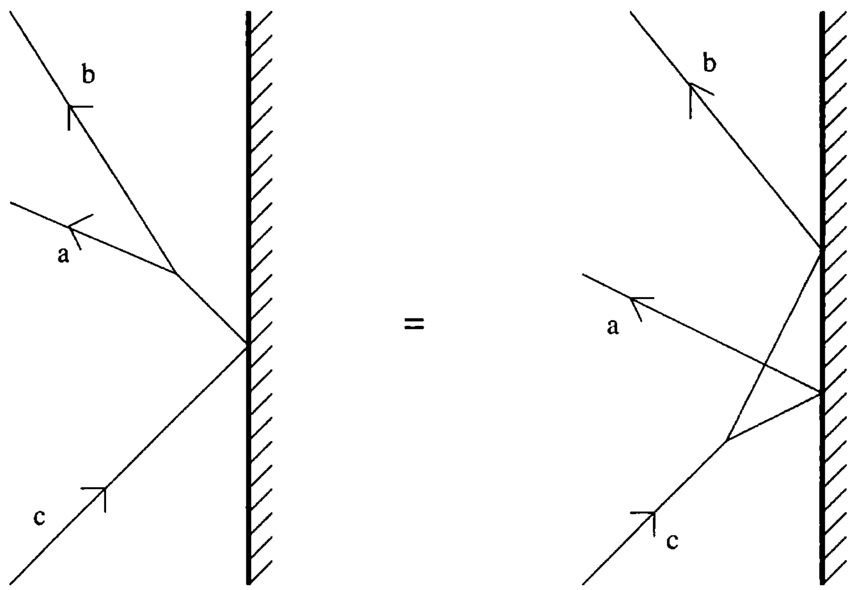

Figure 1.8: The boundary bootstrap equation.

work of Sasaki [51] and Kim et al. [52].

Sasaki considered the solution of the reflection bootstrap equations on a case by case basis. His methodology was to write down the various reflection bootstrap equations corresponding to the fusings of the theory, before rearranging these (making use of the S-matrix bootstrap equation (1.45)), to obtain an equation in only one K_a. A solution to this could then be found. Using this result, the process was continued to find formulae for all the reflection factors of the theory.

It should be noted that many of the postulates given by Sasaki give different reflection factors corresponding to a particle and its conjugate. We will consider only the charge conjugation even solutions. In particular, for the case $a_2^{(1)}$, he finds two such possible reflection factors;

$$K_1 = K_2 = \frac{(1)^2(2)}{(\frac{B}{2})(1-\frac{B}{2})(1+\frac{B}{2})(2-\frac{B}{2})} \tag{1.82}$$

and

$$K_1 = K_2 = -\frac{(1+\frac{B}{2})(2-\frac{B}{2})(2+\frac{B}{2})(3-\frac{B}{2})}{(1)(2)^2}. \tag{1.83}$$

It is not difficult to verify that these do indeed obey the reflection bootstrap equation using the relations given in section 1.3.

Kim et al. [52] also considered the construction of solutions to the boundary bootstrap equation using a different method. Here, the substitution

$$J_a(\theta) = \sqrt{\frac{K_a(\theta)}{K(i\pi+\theta)}} = \frac{K_a(\theta)}{\sqrt{S_{aa}(2\theta)}} \tag{1.84}$$

was made to simplify the bootstrap equation. Solutions using this technique were given for the simply-laced series of affine Toda field theories. Looking again at $a_2^{(1)}$ theory, Kim finds the results

$$K_1 = K_2 = \frac{(1)^2(2)}{(\frac{B}{2})(1-\frac{B}{2})(1+\frac{B}{2})(2-\frac{B}{2})} \tag{1.85}$$

which is the same as the result (1.82) above.

There are various ambiguities in the solution of the reflection bootstrap equation. In other words, multiplicative factors exist which allow consistent reflection factors to be created from previous solutions. These are known as the *CDD factors.* One such ambiguity noticed by Sasaki is that of multiplication of K_a by a factor $\prod_{b=1}^{r} S_{ab}$. Using the S-matrix bootstrap equation, it is not hard to check that this also yields a result which obeys the reflection bootstrap.

Other authors have found additional ambiguities. Taking the result from Fring and Köberle [50] for the specific case of $a_2^{(1)}$ theory, there is a factor of the form

$$\frac{(\frac{1}{2}+\frac{B}{2})(\frac{3}{2}-\frac{B}{2})(\frac{3}{2}+\frac{B}{2})(\frac{5}{2}-\frac{B}{2})}{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}. \tag{1.86}$$

Again we can check that a solution constructed in such a way will obey the constraints.

We have seen, therefore, that various possible reflection factors can be constructed. However, it should be noted that these results do not give us the reflection factor

$$K_1 = K_2 = \frac{(2+\frac{B}{2})}{(\frac{B}{2})(2)} \tag{1.87}$$

which also obeys the bootstrap equation and has a classical limit of unity. We shall see later that this is one of the most likely contenders for an exact reflection of the theory with Neumann boundary condition. In fact, all the reflection factors considered by Fring, Köberle, Sasaki and Kim are self-dual whilst (1.87) is not.

Another drawback to these analyses is that they assume a classical limit of unity. Since the only boundary condition (as we shall see in section 2.5) for which this is true is the Neumann condition $\partial_x \phi = 0$, then these particular solutions give no insight into the possible exact reflection factors corresponding to the other boundary conditions.

1.5.2 Reflection factors from minimality

Other criteria have also been used to suggest possible exact reflection factors. One example we shall look at is found in [46].

The principal used here is simply *minimality* — that is containing as few poles as possible in the physical strip $0 \leq \mathrm{Im}(\theta) \leq \pi$ — coupled with an analysis of which poles, if any, *are* expected to be present. Such poles correspond to boundary bound states of the theory. We shall summarise the argument used to suggest reflection factors for various boundary conditions of $a_2^{(1)}$ affine Toda field theory.

Consider the boundary condition $A_i = 1 \ \forall i$ first. The classical reflection factor here is

$$K_1 = K_2 = -(1)(2) \tag{1.88}$$

as we shall see in section 2.5. Now we expect the quantum factor to contain a fixed pole (i.e. not dependent on the coupling β) at $\theta = \frac{i\pi}{3}$ corresponding to a boundary bound state. Hence we can guess

$$K_1 = K_2 = -\frac{(1)(2+\frac{B}{2})}{(\frac{B}{2})}. \tag{1.89}$$

This obeys the bootstrap equation and has the correct classical limit and pole structure.

Moving on to the boundary condition $A_i = -1$, we now have the classical reflection factor

$$K_1 = K_2 = -\frac{1}{(1)(2)}. \tag{1.90}$$

The quantum reflection factor suggested now is

$$K_1 = K_2 = \frac{(3-\frac{B}{2})}{(1-\frac{B}{2})(2)} \tag{1.91}$$

(there are now no poles in the physical strip corresponding to bound states). Once more this satisfies the bootstrap equation along with having the correct classical limit.

The asymmetric boundary condition $A_1 = -1$, $A_0 = A_2 = 1$ is more difficult to analyse. The classical reflection factor in this case is

$$K_1 = K_2 = \frac{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}{(1)(2)(3)}. \tag{1.92}$$

In [46] this is found by performing a linear expansion of the field around the background configuration. In fact, the technique employed there shall be followed closely in section 2.5 when we consider the case of the $A_1 = 1$, $A_0 = A_2 = -1$ boundary condition; therefore we shall not repeat it here.

Given the classical limit (1.92) we need to suggest a quantum reflection factor. The simplest possible solution would be

$$K_1 = K_2 = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)^2\left(\frac{5}{2}\right)\frac{(3-\frac{B}{2})}{(1-\frac{B}{2})(2)}; \tag{1.93}$$

however other consistent reflection factors exist.

1.5.3 Reflection factors from breather reflection matrices

Another technique which has been used to try to determine the exact reflection factors is the analytic-continuation of breather reflection factors from the imaginary-coupling theory. We shall discuss the approach taken by Gandenberger in [53] for the $a_2^{(1)}$ theory.

Imaginary-coupling Toda theory, which is obtained by putting $\beta \rightarrow i\beta$ in our real-coupling equations, has been extensively studied and has differing properties to that of the real-coupling theory. In this case the Hamiltonian is complex, so we do not in general expect to have real energies and momenta. In addition, the theory contains degenerate vacua since if we make the transformation

$$\phi \rightarrow \phi + i\lambda, \tag{1.94}$$

where $\lambda \cdot \alpha_i = 2\pi\kappa_i$, $\kappa_i \in \mathbb{Z}$, then one vacuum solution is transformed into another. This implies the existence of soliton solutions, which interpolate between the different vacua. Indeed, it has been shown that these do exist; a large class of these has been written down in a particularly useful form by Hollowood [54, 55] and by other authors [56–61]. Moreover, it turns out that the energy of these soliton solutions is real. [7] A good review of these properties can be found in [62]. In fact, these solitonic solutions shall be of considerable use in section 2, and we shall defer a more detailed discussion until then. Let us, however, reveal a few of their properties here.

The integrable boundary conditions for the imaginary-coupling theory are the same as those of the real-coupling theory. Their treatment, however, is somewhat different; different signs of the parameters A_i are easily related by transformations of the field of the type (1.94).

It is a well known fact that in general when a soliton of type a (again, we refer the reader to section 2) is reflected from a boundary of type (1.58), it returns as a soliton of type $\overline{a}$. [8] This initially appears therefore to have little relevance to the real-coupling theory. However, it is possible to construct a breather solution, consisting of a bound state of solitons of types a and $\overline{a}$, which can be reflected from the boundary. It is clear that this must be unchanged by reflection. It has been suggested that there is a correspondence between these breather solutions in the imaginary-coupling theory and the particles of the real-coupling theory [67, 68]. In particular, it was shown [69] that the S-matrices corresponding to the lowest breathers in imaginary-coupling theory and the particles in real-coupling theory are indeed the same.

In addition, it turns out to be easier to calculate the reflection factors of these breather solutions. Gandenberger's method was therefore to propose the analytically-continued breather reflection matrices as the reflection factors for particles in the real-coupling theory.

The calculation of the breather reflection factors is involved and we shall not repeat it here. However, for the discussions in sections 2 and 3 it is useful to note the

[7] Some problems may arise in certain cases; issues of stability have been discussed in [63–65].

[8] A new class of boundary conditions which reflect a soliton back to itself has recently been discovered by Delius [66].

results. For $a_2^{(1)}$ theory, the proposed reflection factors are (with, as usual, the reflection factors for the two particles being equal)

$$K^{N,1} = \frac{(2+\frac{B}{2})}{(2)(\frac{B}{2})} \tag{1.95}$$

$$K^{N,2} = -\frac{(1)(3-\frac{B}{2})}{(1-\frac{B}{2})} \tag{1.96}$$

$$K^{d} = \frac{(3-\frac{B}{2})}{(2)(1-\frac{B}{2})}. \tag{1.97}$$

The factors $K^{N,1}$ and $K^{N,2}$ are so labelled as they have a classical limit of 1, corresponding to the Neumann boundary condition. K^d is dual to the $K^{N,1}$ reflection factor. In fact, the analysis has been extended in [70] to cover the whole $a_r^{(1)}$ series of affine Toda field theories. Moreover, an additional $a_2^{(1)}$ reflection factor, corresponding to the $++-$ boundary condition, is found there. It is

$$K^{++-} = \frac{(\frac{1}{2}-\frac{B}{2})(\frac{3}{2}-\frac{B}{2})(\frac{3}{2}+\frac{B}{2})(\frac{5}{2}+\frac{B}{2})(3-\frac{B}{2})}{(1-\frac{B}{2})(2)}. \tag{1.98}$$

Of course, these exact reflection factors assume that analytic-continuation of breather reflection factors is indeed a valid technique for obtaining the reflection factors of real-coupling theory. It shall be the purpose of section 3, where perturbation theory is used to calculate the one-loop order quantum correction to the classical reflection factor, to provide a partial check on all these results.

Chapter 2

Background field configurations and classical reflection factors

2.1 Introduction

In this section we shall look at several aspects of the classical theory. First we shall give a Bogomol'nyi argument which restricts the maximum values of the boundary parameters A_i, introduced in section 1.4. For values of A_i greater than these limits the theory will not have energies bounded below and hence will not be stable. This argument shall be presented in a general form which is applicable to any affine Toda field theory. We shall then restrict our attention to the $a_r^{(1)}$ series. The classical background solutions of these theories corresponding to different boundary conditions will be calculated before proceeding to determine the associated classical reflection factors.

Let us begin with a little background work. It will be useful to consider the solitonic solutions of the imaginary-coupling theories, and to see how these relate to the background solutions of real-coupling Toda theory. To do this, we need to translate the equations of motion (1.7) into the language of tau functions. This is effected by making the transformation

$$\phi = -\sum_{i=0}^{r} \frac{2\alpha_i}{\alpha_i^2} \ln(\tau_i). \tag{2.1}$$

It is worth noting a few consequences of this formula. First, it is clear that equation (2.1) cannot completely specify the $r+1$ tau functions, since there are only r components of ϕ; indeed we can send

$$\tau_i \to \tau_i(f(x,t))^{m_i} \tag{2.2}$$

where

$$\sum_{i=0}^{r} m_i \frac{2\alpha_i}{\alpha_i^2} = 0, \tag{2.3}$$

without affecting (2.1). We can partially remove this ambiguity by fixing a particular form of the resulting equations of motion for the tau functions. (1.7) becomes in the tau function language

$$\sum_{i=0}^{r} \left\{ \frac{2\alpha_i}{\alpha_i^2}\frac{1}{\tau_i^2}\left[\dot{\tau}_i^2 - \tau_i\ddot{\tau}_i - \tau_i'^2 + \tau_i\tau_i''\right] + \frac{n_i\alpha_i}{2}\prod_{j=0}^{r}\tau_j^{-K_{ij}} \right\} = 0 \tag{2.4}$$

where $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$ is the *extended* Cartan matrix (i.e. defined for $0 \le i,j \le r$). Let us define here the *fundamental weights* associated with g; these obey the relations

$$\lambda_i \cdot \alpha_j = \delta_{ij} \tag{2.5}$$

for $1 \le i,j \le r$. Hence taking the inner product of (2.4) with the fundamental weight λ_i, we obtain, multiplying through by $\alpha_i^2\tau_i^2/2$,

$$\begin{aligned} &\left[\dot{\tau}_i^2 - \tau_i\ddot{\tau}_i - \tau_i'^2 + \tau_i\tau_i''\right] + \frac{n_i\alpha_i^2}{2}\prod_{j=0}^{r}\tau_j^{2\delta_{ij}-K_{ij}} - \\ \frac{n_i\alpha_i^2\tau_i^2}{\alpha_0^2\tau_0^2}&\left\{\left[\dot{\tau}_0^2 - \tau_0\ddot{\tau}_0 - \tau_0'^2 + \tau_0\tau_0''\right] + \frac{n_0\alpha_0^2}{2}\prod_{j=0}^{r}\tau_j^{2\delta_{0j}-K_{0j}}\right\} = 0 \end{aligned} \tag{2.6}$$

where now $1 \le i \le r$. To fix a particular form of these equations of motion, we simply take $f(x,t)$ to insist that

$$\dot{\tau}_0{}^2 - \tau_0\ddot{\tau}_0 - \tau_0'^2 + \tau_0\tau_0'' + \frac{n_0\alpha_0^2}{2}\prod_{j=0}^{r}\tau_j^{2\delta_{0j}-K_{0j}} = \tau_0^2\frac{n_0\alpha_0^2}{2} \tag{2.7}$$

so that we have the equations of motion (now for all $0 \le i \le r$)

$$\tau_i\ddot{\tau}_i - \dot{\tau}_i{}^2 - \tau_i\tau_i'' + \tau_i'^2 = \left(\prod_{j=0}^{r}\tau_j^{2\delta_{ij}-K_{ij}} - \tau_i^2\right)\frac{n_i\alpha_i^2}{2}. \tag{2.8}$$

It is not hard to show that we are still free to do a transformation of the form (2.2) for any f that satisfies $\partial_\mu\partial^\mu \ln(f) = 0$. For time independent solutions this implies that

$$f = e^{ax+b} \tag{2.9}$$

and we can complete the 'gauge fixing' by demanding that $\tau_i \to 1$ as $x \to -\infty$. With this choice of tau functions all the static solutions to the real Toda equations can be described in terms of r parameters, one real parameter for each node on the (non-extended) Dynkin diagram corresponding to a real fundamental representation, and one complex parameter for each pair of complex conjugate nodes. We shall see this explicitly when we come to look at the solitonic solutions of the imaginary-coupling theory.

For the theory defined on the whole line the bulk energy, corresponding to the Lagrangian (1.3), is obviously

$$E = \int_{-\infty}^{\infty} dx \left(\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + \sum_{i=0}^{r} n_i (e^{\alpha_i \cdot \phi} - 1) \right). \tag{2.10}$$

Since $\prod_{i=0}^{r} e^{n_i \alpha_i \cdot \phi} = 1$ then $\sum_{i=0}^{r} n_i (e^{\alpha_i \cdot \phi} - 1) \geq 0$ and so the bulk-energy is manifestly positive for real-coupling constant. However we must modify this formula if we wish to consider the theory on the half-line, by restricting the above integral to the physical region $x \leq 0$ and adding the boundary potential term. In other words, the energy for the theory on the half-line is given by

$$E = \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + \sum_{i=0}^{r} n_i (e^{\alpha_i \cdot \phi} - 1) \right) + \mathcal{B}(\phi)|_{x=0}. \tag{2.11}$$

Remember that it was shown in section 1.4 that to preserve the integrability of the theory we must take the boundary potential to have the form

$$\mathcal{B}(\phi) = -2 \sum_{i=0}^{r} A_i \sqrt{\frac{2n_i}{\alpha_i^2}} e^{\alpha_i \cdot \phi / 2} \tag{2.12}$$

where the constants A_i are severely restricted in the values they may take. However, an important point is that A_i can be positive as well as negative, and so whilst the theory in the bulk has positive energy, it is not clear whether the boundary potential term can destabilise the theory. To get a large negative boundary contribution to the energy we need $\alpha_i \cdot \phi$ to become large, which in turn produces a positive contribution to the gradient terms in the energy. Which of these two competing factors wins is determined by the value of A_i. Hence we want to determine the maximum value of A_i for the positive bulk energy to win out and for the theory to have energy bounded below.

To find this result we shall use an approach based on generalising the stability argument for the sinh-Gordon model given in [46] which uses the idea of a Bogomol'nyi-like bound. This argument shall be reviewed in the following section before explicitly extending it to the case of $a_2^{(1)}$. We can then finally extend the analysis to cover all other affine Toda field theories.

2.2 The stability of sinh-Gordon and $a_2^{(1)}$ affine Toda field theory on the half-line

For the sinh-Gordon model the parameters σ_i appearing in the boundary potential (1.60) are known to be unconstrained by integrability. However, if we would like the Hamiltonian describing the theory on the half-line (2.11) to have energies bounded from below, we must have that $\sigma_i \leq 1$ as shown in [46]. This result relied on being able to write the energy for the sinh-Gordon theory in a Bogomol'nyi form. Explicitly we have that the energy

$$\begin{aligned} E &= \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + (e^{\sqrt{2}\phi} + e^{-\sqrt{2}\phi} - 2)\right) \\ &\quad -(2\sigma_1 e^{\phi/\sqrt{2}} + 2\sigma_0 e^{-\phi/\sqrt{2}})|_{x=0} \end{aligned} \tag{2.13}$$

can be rewritten as

$$\begin{aligned} E &= \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi - \sqrt{2}e^{\phi/\sqrt{2}} + \sqrt{2}e^{-\phi/\sqrt{2}})^2 + 2\partial_x(e^{\phi/\sqrt{2}} + e^{-\phi/\sqrt{2}})\right) \\ &\quad -(2\sigma_1 e^{\phi/\sqrt{2}} + 2\sigma_0 e^{-\phi/\sqrt{2}})|_{x=0} \end{aligned} \tag{2.14}$$

$$\begin{aligned} &= \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi - \sqrt{2}e^{\phi/\sqrt{2}} + \sqrt{2}e^{-\phi/\sqrt{2}})^2\right) \\ &\quad -(2(\sigma_1 - 1)e^{\phi/\sqrt{2}} + 2(\sigma_0 - 1)e^{-\phi/\sqrt{2}})|_{x=0}. \end{aligned} \tag{2.15}$$

The integral is non-negative, as is the boundary term if $\sigma_i \leq 1$. Furthermore we can show that this condition is *necessary* for stability by taking the field to be of the form

$$e^{\phi/\sqrt{2}} = \frac{1 + de^{2x}}{1 - de^{2x}}, \tag{2.16}$$

where d is a constant which must be taken to be less than one for the solution to be non-singular in the region $x < 0$. This is the analytic-continuation of the sine-

Gordon kink to the real-coupling theory, and it satisfies the Bogomol'nyi equation

$$\partial_x \phi - \sqrt{2} e^{\phi/\sqrt{2}} + \sqrt{2} e^{-\phi/\sqrt{2}} = 0 \tag{2.17}$$

so that the only contributions to the energy come from the boundary terms at $x = 0$ in (2.15). By choosing the constant d to be close to one, we can take the field at the boundary, $\phi(0,t)$, to be as large as we like, and hence take the energy to be as negative as we like if $(\sigma_1 - 1) > 0$. Similarly we can show that $\sigma_0 \leq 1$ is necessary for stability by considering the solution obtained by taking $\phi \to -\phi$ in the above.

The aim is now to generalise this result to affine Toda field theories based on other algebras by finding a Bogomol'nyi-like form

$$E = \int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_t \phi_i)^2 + \frac{1}{2}(\partial_x \phi_i - W_i)^2 + \partial_x W(\phi_i) \right) + \mathcal{B}(\phi_i) \tag{2.18}$$

for the energy in these cases too. Comparing this to (2.13), we see that we can write the energy in this form provided that

$$\sum_{i=1}^{r} \frac{1}{2} W_i^2 = V = \sum_{i=0}^{r} n_i (e^{\alpha_i \cdot \phi} - 1) \tag{2.19}$$

and

$$\partial_x W = \sum_{i=1}^{r} W_i \partial_x \phi_i. \tag{2.20}$$

From the second equation (2.20) we deduce that

$$W_i = \frac{\partial W}{\partial \phi_i} \tag{2.21}$$

so that the W_i must satisfy the integrability condition

$$\frac{\partial W_i}{\partial \phi_j} = \frac{\partial W_j}{\partial \phi_i}. \tag{2.22}$$

As usual, a solution of the Bogomol'nyi equations $\partial_x \phi_i = W_i$ will automatically be a (static) solution to the Toda field equations. In fact, these are the equations for the $(1+0)$-dimensional Toda molecule:

$$-\partial_x^2 \phi + \frac{m}{\beta} \sum_{i=0}^{r} n_i \alpha_i e^{\beta \alpha_i \cdot \phi} = 0. \tag{2.23}$$

The integrability of the Toda molecule can be used to show that any solution to these r second order equations must also satisfy r first order equations. The Toda molecule consists of r degrees of freedom, and integrability can be taken in the strict Liouville sense; that there exist r conserved quantities which Poisson-commute with each other. Note that now x plays the rôle of time in this system. By using the equation of motion (2.23) to eliminate higher derivatives, the conserved quantities (which we label H_i) can be written as functions of ϕ_i and the momenta $p_i = \partial_x \phi_i$. Thus we have

$$H_i(\phi_i, p_i) = \gamma_i. \tag{2.24}$$

We shall be interested in solutions for which $\phi_i, p_i \to 0$ as the 'time' $x \to -\infty$, since these are the solutions which will have finite energy in the $(1+1)$-dimensional affine Toda field theory. For such solutions the constant $\gamma_i = H_i(0,0)$. An example of one of the conserved charges is the energy of the $(1+0)$-dimensional system

$$H_1 = \sum_{i=1}^{r} \frac{1}{2} p_i^2 - V = \sum_{i=1}^{r} \frac{1}{2} p_i^2 - \sum_{i=0}^{r} n_i (e^{\alpha_i \cdot \phi} - 1) = 0. \tag{2.25}$$

This, together with the other $r-1$ equations from conserved charges, gives us r equations which can be used to solve for p_i in terms of the ϕ_i. Let us write these solutions as

$$p_i = \partial_x \phi_i = W_i(\phi). \tag{2.26}$$

For these equations to satisfy the criteria to be Bogomol'nyi equations for the Toda system we need to show that the conditions (2.19) and (2.22) hold. The first condition follows immediately from (2.25) and (2.26). The second condition follows from a result in classical mechanics that any expressions derived from Poisson commuting quantities must also be Poisson commuting (see for instance [71]). Thus we must have that

$$\{p_i - W_i, p_j - W_j\}_{PB} = \frac{\partial W_j}{\partial \phi_i} - \frac{\partial W_i}{\partial \phi_j} = 0. \tag{2.27}$$

In fact, this procedure can be explicitly performed in the case of the affine Toda theory based on the algebra $a_2^{(1)}$. For this case it is convenient to use variables

$$u_i = e^{\lambda_i \cdot \phi} \tag{2.28}$$

$$\pi_i = \lambda_i \cdot \partial_x \phi. \tag{2.29}$$

The fundamental weights for $a_2^{(1)}$ are

$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) \tag{2.30}$$

$$\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2); \tag{2.31}$$

these relations can be inverted to find the α_i's in terms of the weights if desired.

In terms of these new variables u_1 and u_2 the two conserved quantities (H_1 is the energy as defined in (2.25)) can be written

$$H_1 = \pi_1^2 + \pi_2^2 - \pi_1\pi_2 - \frac{u_1^2}{u_2} - \frac{u_2^2}{u_1} - \frac{1}{u_1u_2} + 3 = 0 \tag{2.32}$$

$$H_2 = \pi_1\pi_2^2 - \pi_1^2\pi_2 - \frac{u_2^2}{u_1}\pi_1 + \frac{u_1^2}{u_2}\pi_2 - \frac{\pi_2}{u_1u_2} + \frac{\pi_1}{u_1u_2} = 0 \tag{2.33}$$

which can be obtained by using the Lax pair which describes the theory.

Using the first of these equations to give us an expression for π_2^2, we can solve the second equation for π_2. We find

$$\pi_2 = \frac{\pi_1}{u_1^3 - 1}\left\{(3 + \pi_1^2)u_1u_2 - (2 + u_1^3)\right\} \tag{2.34}$$

which can then be substituted back into the first equation to find π_1. This gives us several possible solutions for π_1; however using the fact that the momenta must be real (since ϕ is real), we see that the relevant solutions for π_1 and π_2 are

$$\pi_1 = (u_1 - 1)\sqrt{\frac{1 + u_1 + u_2}{u_1u_2}} \tag{2.35}$$

$$\pi_2 = (u_2 - 1)\sqrt{\frac{1 + u_1 + u_2}{u_1u_2}}. \tag{2.36}$$

Using appropriate changes of basis we know that W is defined by the equations

$$\frac{\partial W}{\partial u_1}u_1 = 2\pi_1 - \pi_2 \tag{2.37}$$

$$\frac{\partial W}{\partial u_2}u_2 = 2\pi_2 - \pi_1 \tag{2.38}$$

which are satisfied by

$$W = 2\sqrt{\frac{(1 + u_1 + u_2)^3}{u_1u_2}}. \tag{2.39}$$

From the Bogomol'nyi argument it now follows that the energy of any field configuration on the half-line is bounded below by

$$E \geq [W]_{-\infty}^{0} + \mathcal{B}|_{x=0} \tag{2.40}$$

$$= -6\sqrt{3} + 2\sqrt{\frac{(1 + u_1 + u_2)^3}{u_1u_2}} - 2A_1\frac{u_1}{\sqrt{u_2}} - 2A_2\frac{u_2}{\sqrt{u_1}} - 2A_0\frac{1}{\sqrt{u_1u_2}} \tag{2.41}$$

$$
\begin{aligned}
&= -6\sqrt{3} + 2\frac{\sqrt{(1+u_1+u_2)^3} - \sqrt{u_1^3} - \sqrt{u_2^3} - 1}{\sqrt{u_1 u_2}} \\
&\quad -2(A_1-1)\frac{u_1}{\sqrt{u_2}} - 2(A_2-1)\frac{u_2}{\sqrt{u_1}} - 2(A_0-1)\frac{1}{\sqrt{u_1 u_2}} \qquad (2.42)
\end{aligned}
$$

where we use u_1 and u_2 to mean their values at $x = 0$. For real ϕ, $u_i \geq 0$ and it is easy to show that the second term is bounded below in this region. On the other hand, the last terms are also non-negative for

$$A_i \leq 1. \qquad (2.43)$$

These conditions ensure classical stability. Indeed it can be shown that these conditions are also necessary.

2.3 Bogomol'nyi equations and stability for other affine Toda field theories on the half-line

Let us proceed now to the case of a general affine Toda field theory. In principle we could follow the same steps as for $a_2^{(1)}$ and $a_1^{(1)}$ but in practice we are unable to invert the conserved quantities to find explicit relations between for the momenta in terms of the fields ϕ. However, we can use our knowledge of (analytically-continued) static solutions of the imaginary-coupling Toda equations to circumvent this difficulty. Any static solution must necessarily obey the conservation laws for the Toda molecule, and so in turn must satisfy the Bogomol'nyi equations (2.26), and saturate the Bogomol'nyi bound. It therefore follows that the energy density for such a solution can be written as a total derivative of some function W. But it has been known for some time in the context of imaginary-coupling Toda theories that this is indeed the case, and the explicit formula for W is given in terms of tau functions as

$$W = -2\sum_{i=0}^{r} \frac{2}{\alpha_i^2}\frac{\tau_i'}{\tau_i}. \qquad (2.44)$$

One can now consider the map between the parameters in the tau functions and the value of the corresponding static soliton r-component field ϕ on the boundary which is obtained by putting $x = 0$ in (2.1). We shall simply assume that imposing the condition that the field is free of singularities in the physical region and tends to zero as $x \to -\infty$ renders this map invertible so that we can write the parameters in terms of the values of the fields of the corresponding solitons at $x = 0$. In this way we can view W as a function of ϕ as we did for $a_2^{(1)}$ and $a_1^{(1)}$. With this in mind, it is possible to write the energy on the half-line of *any* field configuration in the form (2.18) so that the energy is bounded below by

$$E \geq E_{bound} = W(\phi)|_{x=0} - W(\phi)|_{x=-\infty} + \mathcal{B}(\phi)|_{x=0} \tag{2.45}$$

i.e. the energy of the static soliton configuration with a value of ϕ that coincides with that of our arbitrary solution at $x = 0, -\infty$ at some point in time.

To prove stability of affine Toda field theory on the half-line, we must show that the right hand side of (2.45) is bounded from below. This can be written

$$E_{bound} = -2\sum_{i=0}^{r}\left(\frac{2}{\alpha_i^2}\frac{\tau_i'}{\tau_i} + A_i\sqrt{\frac{2n_i}{\alpha_i^2}}\prod_{j=0}^{r}\left|\sqrt{\tau_j^{-K_{ij}}}\right|\right) \tag{2.46}$$

where we have used the fact that $W(\phi)|_{x=-\infty}$ vanishes since $\tau_i'(x) \to 0$ as $x \to -\infty$. If the energy is not bounded below, we can tune the parameters defining the tau functions to make E_{bound} arbitrarily negative. Naïvely this can occur in two ways: either one of the tau functions becomes very large, or else becomes zero in the denominator. In fact the first possibility never arises, since if one tunes the parameters to make the tau function large, the quotients appearing in E_{bound} tend to a finite limit. So if the energy is unbounded below we must have a soliton solution with one or more tau functions vanishing at the boundary, and giving $E_{bound} = -\infty$.

Our approach is therefore to consider the function E_{bound} for such soliton solutions with singularities at the boundary, and determine whether the residue of the pole in x is positive (corresponding to infinite positive energy) or negative. If the residue is always positive (or zero) then E_{bound} must be bounded below. To analyse the behaviour near the origin we write

$$\tau_i = a_i x^{y_i} + O(x^{y_i+1}) \tag{2.47}$$

with of course $y_i \geq 0$ and $a_i \neq 0$ for all i. Let $S \subset 0, 1, \ldots r$ be the set of i for which $y_i > 0$, i.e. for which the corresponding tau function τ_i vanishes at $x = 0$. For

$i \in S$ the equation of motion (2.8) contains a term of the form x^{2y_i-2}; comparing its coefficients we find that

$$y_i \prod_{j=0}^{r} a_j^{K_{ij}} = \frac{n_i \alpha_i^2}{2} \tag{2.48}$$

and

$$\sum_{j=0}^{r} K_{ij} y_j = 2. \tag{2.49}$$

Note that the second equation implies that not all the tau functions can vanish simultaneously, or in other words $S \neq \{0, 1, ...r\}$ since then we could write

$$0 = \sum_{i,j} n_i \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} y_j = \sum_{i,j} n_i K_{ij} y_j = 2 \sum_i n_i = 2h \tag{2.50}$$

which is clearly not true. Also from both equations we see that for $i \in S$, then for x close to 0,

$$e^{\alpha_i \cdot \phi} = \prod_{j=0}^{r} \tau_j^{-K_{ij}} \sim \prod_{j=0}^{r} (a_j x^{y_j})^{-K_{ij}} = \frac{2y_i}{n_i \alpha_i^2 x^2}. \tag{2.51}$$

Thus the poles in E_{bound} at $x = 0$ have the form

$$E_{bound} \sim -\frac{2}{x} \sum_{i \in S} \left(\frac{2y_i}{\alpha_i^2} - A_i \frac{2\sqrt{y_i}}{\alpha_i^2} \right) + O(1) \tag{2.52}$$

where we have used that $e^{\alpha_i \cdot \phi} > 0$ and $x < 0$ to identify the correct sign for the last term. We shall see equation (2.49) again later; however here it is sufficient to note that we can always arrange for $y_i = 1$ by arranging that only τ_i vanishes. Then we see that E_{bound} will be bounded below if and only if

$$A_i \leq 1, \tag{2.53}$$

generalising the result we found explicitly for $a_2^{(1)}$ and $a_1^{(1)}$ to all other affine algebras. This result is interesting since it tells us that all the simply-laced theories with integrable boundary conditions are stable. However, we only have marginal stability when any of the boundary parameters $A_i = 1$.

2.4 Vacuum soliton solutions to the Toda field equations

In this section we elaborate on the results of [46, 72] looking for static solutions to the $a_r^{(1)}$ Toda equations which satisfy the integrable boundary equations. This work is heavily dependent on the techniques developed in [72], so let us first of all recall some of the notation used there. In the language of the tau functions introduced in section 2.1, the boundary conditions can be conveniently written (specialising to the $a_r^{(1)}$ series of affine Toda field theory where $n_i = 1$ and $\alpha_i^2 = 2$)

$$\frac{\tau_i'}{\tau_i} + A_i e^{\alpha_i \cdot \phi/2} = C \tag{2.54}$$

where

$$C = \left(\frac{\tau_0'}{\tau_0} + A_0 e^{\alpha_0 \cdot \phi/2} \right). \tag{2.55}$$

These conditions can be easily obtained from (1.57) and (2.12) by taking the inner products with the fundamental weights λ_i defined earlier. Using this definition of C, the equations of motion imply at the boundary $x = 0$ (substituting (2.54) into the equations of motion)

$$\ddot{\tau}_i - \frac{\dot{\tau}_i^{\,2}}{\tau_i} - \tau_i'' + 2C\tau_i' - (C^2 - 1)\tau_i|_{x=0} = 0 \tag{2.56}$$

which simplifies to

$$\tau_i'' - 2C\tau_i' + (C^2 - 1)\tau_i|_{x=0} = 0 \tag{2.57}$$

in the static case. Notice, however, that to obtain this equation we have squared all the coefficients A_i and so we no longer know which boundary conditions a given solution of this will obey. We will show that it is possible to recover this information later.

The parameter C turns out to be proportional to the energy of the field. Since the energy

$$E = E_{bulk} + E_{boundary} = -2\sum_{i=0}^{r} \left(\frac{\tau_i'}{\tau_i} + A_i e^{\alpha_i \cdot \phi/2} \right) \tag{2.58}$$

then

$$E = -2\sum_{i=0}^{r} n_i C = -2hC. \tag{2.59}$$

We shall therefore refer to C as the 'energy parameter'.

We wish to find the lowest energy (i.e. highest C) static background solutions to these equations. These will depend, in general, on the boundary conditions imposed. As noted in section 1.5, for the imaginary-coupling theory a large set of solutions was found by Hollowood [54]; these were subsequently used for the background solutions of real-coupling theory by Bowcock [72]. These solutions typically involved pairs of analytically-continued solitons. However, the analysis could not determine whether solutions (beyond those containing a single pair of solitons) had any singularities in the physical region $x < 0$. Such singularities in the physical region have infinite energy and are hence unacceptable. Nor was it clear whether there were other solutions to the equations, potentially with lower energy and therefore the 'true' vacuum solutions. Here we carry the analysis a little further, utilising the singularity analysis of the previous section. In fact, extensive numerical work indicates that generally the vacuum solutions of [72] with more than one pair of constituent solitons *do* contain singularities in the physical region. Moreover, we shall see that even for critical values of the soliton parameters where it is possible to place all such singularities on the boundary, we cannot obtain acceptable vacuum solutions. However, a number of what we shall call 'exceptional' solutions are found which contain multiple solitons yet which provide acceptable vacuum solutions for the $a_r^{(1)}$ theories up to $r = 5$.

Let us first of all review Hollowood's work here. He uses the Hirota method to find the multi-soliton solutions of the imaginary-coupling theory. To do this we first of all define the field in terms of the tau functions just discussed. The Hirota method tells us to expand the tau functions in terms of some arbitrary parameter, ϵ. We take

$$\tau_j = 1 + \epsilon\tau_j^{(1)} + \epsilon^2\tau_j^{(2)} + ... \tag{2.60}$$

This series can be truncated for an N soliton solution by setting $\tau_j^{(a)} = 0$ for all $a > N$. We then need to solve the equation of motion (2.8) at each order in ϵ to obtain the multi-soliton solutions. Consider as an example the single soliton solution. Putting

$$\tau_j = 1 + \epsilon\tau_j^{(1)} \tag{2.61}$$

for $a_r^{(1)}$ we obtain the equations

$$\ddot{\tau}_j^{(1)} - \tau_j^{(1)\prime\prime} = \left(\tau_{j+1}^{(1)} + \tau_{j-1}^{(1)} - 2\tau_j^{(1)}\right) \tag{2.62}$$

and

$$\tau_{j-1}^{(1)}\tau_{j+1}^{(1)} = \tau_j^{(1)^2}. \tag{2.63}$$

We can solve these by taking

$$\tau_j^{(1)} = e^{\sigma(x-vt)+\xi}\omega^{aj} \tag{2.64}$$

where as usual $\omega = e^{\frac{2\pi i}{r+1}}$, and we have the relation $\sigma^2(1-v^2) = m_a^2$.

The same idea can be followed to obtain the higher order soliton solutions. In fact, it was shown [54] that the tau functions for an N-soliton solution can be written as:

$$\tau_j(x,t) = \sum_{\mu_1=0}^{1} \dots \sum_{\mu_N=0}^{1} \exp\left(\sum_{p=1}^{N} \mu_p(\Phi_p + \ln\omega^{a_p j}) + \sum_{1\le p<q\le N} \mu_p\mu_q \ln A^{(a_p a_q)}\right) \tag{2.65}$$

where $\Phi_p = \sigma_p(x - v_p t) + \xi_p$. Again we have the mass-shell condition for each soliton $\sigma^2(1-v_p^2) = m_{a_p}^2$, which in the static case implies $\sigma = m_{a_p}$ since we require asymptotically $\tau_j \to 1$ as $x \to -\infty$. We shall see later that it is also useful to set $\xi_p = \ln(d_p) + i\chi_p$, where d_p is referred to as the position of the soliton p.

The interaction between solitons a_p and a_q is given by the interaction constant

$$A^{(a_p a_q)} = -\frac{(\sigma_p - \sigma_q)^2 - (\sigma_p v_p - \sigma_q v_q)^2 - 4\sin^2\frac{\pi}{r+1}(a_p - a_q)}{(\sigma_p + \sigma_q)^2 - (\sigma_p v_p + \sigma_q v_q)^2 - 4\sin^2\frac{\pi}{r+1}(a_p + a_q)}. \tag{2.66}$$

From this we note that when we interact one soliton with another of the same type and velocity, we find that the interaction constant $A^{(a_p a_q)}$ vanishes. Thus adding two solitons of the same type into a static solution is merely equivalent to having a single soliton of this type at a different position. Hence we need only consider background solutions which contain at most one of each type of soliton.

It is convenient to make the change of basis

$$\tau_j = \sum_k T_k \omega^{kj} \tag{2.67}$$

which splits the tau functions up into 'charge sectors' T_k. This gives us a similar equation to before but now for T_k:

$$T_k'' - 2CT_k' + (C^2 - 1)T_k = 0 \tag{2.68}$$

at $x = 0$. We now have a simpler set of equations: a soliton of type a resides in T_a while its interaction with another soliton b resides in T_{a+b} and so on.

It is required that the field ϕ be real everywhere on the interval $(-\infty, 0]$. Thus the tau functions must be real and non-negative on this interval. Reality imposes the condition that each soliton must appear with its conjugate, i.e. type a with type $r+1-a$; the only exception to this being where we have what we call a single 'middle' soliton. The middle soliton has $a = \frac{r+1}{2}$ when r is odd, and hence if we take $\chi = 0, \pi$ we obtain a real tau function. However the restriction imposed by insisting that the tau function be non-negative on the interval is more subtle and will be considered case by case below, where we look at the candidate background configurations in turn.

2.4.1 Flat background

Looking at the boundary conditions (2.54) and remembering (1.5), it is clear that the flat background solution $\phi = 0$ is only valid when

$$0 = \sum_{i=0}^{r} A_i \alpha_i. \tag{2.69}$$

This is only true when all of the A_i are the same sign. In addition the value of C is found to be ± 1 for the $\pm\pm..\pm$ boundary conditions.

2.4.2 Single Middle Soliton

As we noted above, in the case $r =$ odd there is a self-conjugate soliton associated with the central spot of the Dynkin diagram, i.e. of type $a = \frac{r+1}{2}$. If we consider a background configuration consisting only of such a static soliton, we find;

$$\tau_j = 1 + (-1)^j d e^{2x} \tag{2.70}$$

where we have taken $\chi = 0$ for simplicity (the other option $\chi = \pi$ merely swaps the rôle of the odd and even tau functions). It is apparent from this equation that we must take $d \leq 1$ in order that none of the tau functions are negative in the physical region. In fact, this is the analytic-continuation of the sine-Gordon kink which was used in section 2.2.

Let us consider the restrictions imposed by the equations of motion and boundary conditions. Using (2.68) for the $\frac{r+1}{2}$ charge sector gives us

$$d e^{mx}(4 - 4C + C^2 - 1)|_{x=0} = 0, \tag{2.71}$$

which implies that C can be 1 or 3. The zero charge sector tells us that $C^2 = 1$ so the only consistent solution is $C = 1$. However, we have no further restrictions on the parameter d.

Since

$$e^{\alpha_i \cdot \phi} = \frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}, \tag{2.72}$$

then the boundary conditions imply that

$$\frac{1-(-1)^j d}{1+(-1)^j d} = A_j \left| \frac{1-(-1)^j d}{1+(-1)^j d} \right| \tag{2.73}$$

utilising the value of C found above. For $d < 1$, the modulus sign is irrelevant and we obtain the result that all the A_i must be positive. In the special case $d = 1$, all the odd tau functions vanish at the boundary. Consider a vanishing tau function, τ_j. Since we can write the $j-1$ and $j+1$ boundary conditions as

$$C - \frac{\tau'_{j-1}}{\tau_{j-1}} = A_{j-1}\frac{1}{\tau_{j-1}}\sqrt{\tau_j \tau_{j-2}} \tag{2.74}$$

and

$$C - \frac{\tau'_{j+1}}{\tau_{j+1}} = A_{j+1}\frac{1}{\tau_{j+1}}\sqrt{\tau_j \tau_{j+2}} \tag{2.75}$$

respectively, then clearly in this case we can take A_{j-1} and A_{j+1} to be of arbitrary sign. This gives us a variety of consistent boundary conditions. It is worth noting also that since $\tau'_j < 0$ and $\tau_j > 0$ for x near to but below 0, then the boundary condition $A_j e^{\alpha_i \cdot \phi/2} = C - \frac{\tau'_j}{\tau_j} > 0$ for $x \to 0$ and hence if τ_j vanishes at $x = 0$ then $A_j = 1$. Hence when $d = 1$, we can fit boundary conditions of the form $a{+}a{+}a{+}a{+}..$ where "a" denotes an arbitrary choice of sign. Taking instead $\chi = \pi$ allows us to fit the boundary conditions of the form $+a+a+a+\ldots$

2.4.3 Two soliton solutions

The single middle soliton was very much a special case: let us now consider the case where we have a soliton, type a, and its conjugate, type $\bar{a} = r + 1 - a$. Then the tau functions become;

$$\tau_j = 1 + 2d\cos\left(\chi + \frac{2\pi j a}{r+1}\right)e^{m_a x} + A^{(a\bar{a})}d^2 e^{2m_a x}. \tag{2.76}$$

Let us again look at what the charge sector equations of motion (2.68) tell us. Consider first charge sector a. This gives us (in an exactly analogous way to the single soliton case of the previous section)

$$C_{\pm} = m \pm 1. \tag{2.77}$$

Now consider the equations for charge sector zero. Substituting in these values of C, we find

$$d^2\left(1-\frac{m_a^2}{4}\right)\left\{4m_a^2 - 4m_a(m_a \pm 1) + (m_a^2 \pm 2m_a)\right\} + (m_a^2 \pm 2m_a) = 0 \tag{2.78}$$

which tells us that

$$d = \frac{2}{|2 \mp m_a|}. \tag{2.79}$$

since d must be positive. So this time there are two possible solutions corresponding to two different energies $C_{\pm}$. It is easy to show that these two solutions are parity inverses of each other and in fact the lower energy solution C_+ has singularities in the physical region. Thus in this case it is the C_- solution that we require.

Working with the boundary conditions (2.54) as before it can be shown that the coefficients A_j are given by

$$A_j = -\text{sign}\left(\cos\left(\frac{\chi}{2}+\frac{\pi(j+1)a}{r+1}\right)\cos\left(\frac{\chi}{2}+\frac{\pi(j-1)a}{r+1}\right)\right). \tag{2.80}$$

Note that a shift of the parameter χ by $2\pi/(r+1)$ merely cyclically permutes the boundary conditions obeyed. However, by choosing χ so that $\cos(\frac{\chi}{2}+\frac{\pi ja}{r+1}) = 0$ we can ensure that $\tau_j \to 0$ as $x \to 0$. So as before this allows us to take the signs of A_{j-1} and A_{j+1} to be arbitrary. Therefore there are a number of boundary conditions consistent with a two-soliton solution which is singular at the boundary.

2.4.4 Multi-soliton solutions

We first consider the case developed in [72] where we assume that there is no 'wrap-around' or 'overlap' of the charge sectors, by which we mean the highest occupied charge sector $Q_{max} = \Sigma_p a_p \leq \frac{r+1}{2}$. In other words, the interaction between all the solitons (not including conjugates) always resides alone in the Q_{max} charge sector.

Consider the case where we have an even number of constituent solitons first, i.e. the middle soliton $\frac{r+1}{2}$ is not present. Then an analogous calculation to the two soliton case leads to the expressions for C and d_p:

$$C_{\pm} = \sum_{p=0}^{N} m_{a_p} \pm 1 \tag{2.81}$$

$$d_p = \prod_{r \neq p} \frac{m_{a_r} + m_{a_p}}{|m_{a_r} - m_{a_p}|} \frac{2}{2 \mp m_{a_p}}. \tag{2.82}$$

For the case where we have $2N+1$ solitons, the calculation is similar excepting that we only obtain one possible solution [72], where

$$C = \sum_{p=0}^{N} m_{a_p} + 1. \tag{2.83}$$

The soliton positions are the same as (2.82) (with the product now running over all solitons including the middle one) excepting for the middle soliton itself whose position is unconstrained.

Extensive numerical work with 3, 4, and 5 soliton solutions suggests that, under this regime, the only solutions which are non-zero in the region $(-\infty, 0)$ occur when a maximal number of tau functions vanish at the boundary $x = 0$. However, solutions in these cases appear to have problems. Consider a solution which contains at least two consecutive tau functions which are singular at the origin. In the language of section 2.3, this means that the set S contains at least two neighbouring nodes. But look at (2.49) for $a_r^{(1)}$; it gives

$$2y_i - y_{i+1} - y_{i-1} = 2 \tag{2.84}$$

and from this it is straightforward to show that y_i, the order of zero of τ_i at $x = 0$ must be greater than one. However for $i \in S$, both sides of the boundary condition (2.54)

$$C - \frac{\tau_i'}{\tau_i} = A_i \sqrt{\frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}} \tag{2.85}$$

go as $1/x$ as $x \to 0$. Hence, since C is finite, the coefficients of these leading order terms must match. This requires (using (2.48))

$$y_i = A_i \sqrt{y_i} \tag{2.86}$$

which cannot be satisfied for $y_i > 1$. This tells us that only non-consecutive tau functions tending to zero at the origin are allowed. This result is important since numerical investigation suggests that, under the non-overlapping charge sector regime, it is not possible to find multi-soliton solutions which are both regular in the physical region and contain only non-consecutive tau functions which vanish at the origin. However, it seems that this is a difficult hypothesis to prove.

Let us show that it is true in the very simplest case of a soliton / conjugate pair, of mass m, together with the middle soliton in $a_3^{(1)}$ affine Toda field theory. In this case we only have two free parameters in the tau functions. The angle χ_1 takes values in the range $-\pi \leq \chi_1 \leq \pi$ whilst the position of the middle soliton, d_2, can take any real value (negative values of d_2 correspond to positive ones with χ_2 shifted by π). We have only one possible solution for the energy, with $C = m + 1$. It is not difficult to show that in order to keep the values of the tau functions non-negative at the origin we must restrict d_2 to be in the range

$$-\frac{2+m}{2-m} \leq d_2 \leq \frac{2+m}{2-m} \tag{2.87}$$

(the even tau functions give the lower bound, the odd tau functions the upper).

Now consider the bulk energy, which is as noted previously given by

$$E_{bulk} = -2\left[\frac{\tau_i'}{\tau_i}\right]_{-\infty}^{0} . \tag{2.88}$$

We expect $-2\tau_i'/\tau_i$ to be a monotonically increasing function of x. In addition, its value at $x = -\infty$ is zero. Hence we know that there must be a singularity in the physical region $x \leq 0$ if we find that its value at the boundary is negative. A graph of an energy density which demonstrates this point, is given in fig. (2.1). Let us reparametrise the position of the middle soliton by putting $d_2 = \mu\frac{2+m}{2-m}$ (so that μ is restricted to lie in the range $-1 \leq \mu \leq 1$). Putting $r = 3$ and so on into the bulk energy (2.88), we obtain

$$E_{bulk} = \frac{-2((1-\mu)^2 + 2\cos^2\chi_1 \sin^2\chi_1(\sqrt{2}(1-\mu^2) - 2\mu^2) + 4\mu\cos^2\chi_1)}{\cos^2\chi_1\sin^2\chi_1(1-\mu^2)}. \tag{2.89}$$

In fact, it is not too hard to show that this function is non-positive for all values of χ_1 and μ. So there must always be a singularity somewhere in the physical region

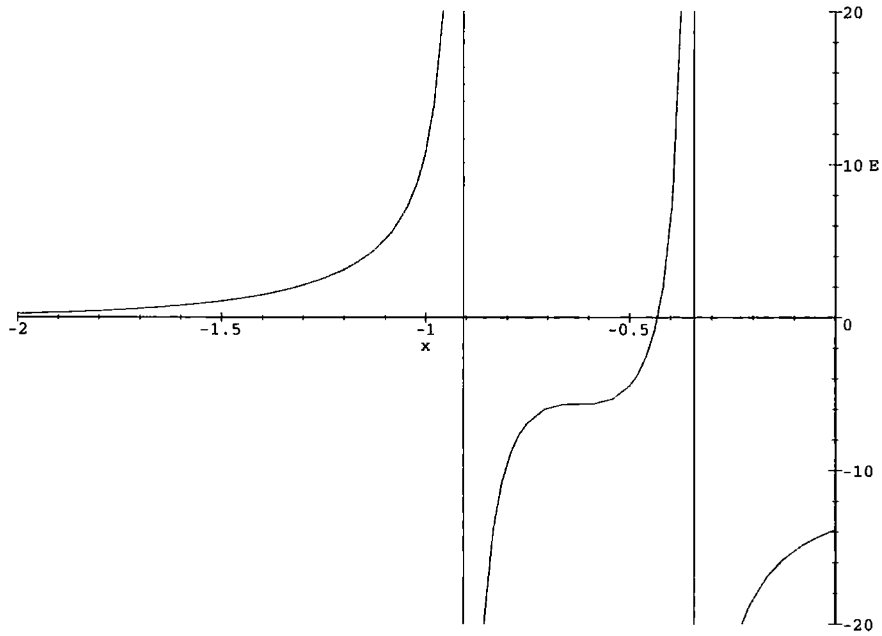

Figure 2.1: The energy density of an unacceptable background solution, showing singularities in the physical region.

$(-\infty, 0]$. The best we can hope to do is to place this singularity at the boundary $x = 0$. In fact, this creates an acceptable solution energy-wise since in this case the infinite parts of the bulk and boundary energies exactly cancel. Let us choose parameters so that τ_0 vanishes at the boundary. Since

$$\tau_0|_{x=0} = 4\frac{(1+\cos\chi_1)(1+\mu)}{2-m} \tag{2.90}$$

then we can make this vanish by putting either $\mu = -1$ or $\cos\chi_1 = -1$. The former makes all even tau functions vanish. However, comparing the resulting derivatives τ_0' and τ_2' at the boundary, we find

$$\tau_0'|_{x=0} = 4\frac{-2\cos\chi_1}{2-\sqrt{2}} \tag{2.91}$$

$$\tau_2'|_{x=0} = 4\frac{2\cos\chi_1}{2-\sqrt{2}}. \tag{2.92}$$

In order that there are no singularities in the physical region it is necessary that both of these are non-positive. Hence the only acceptable solution here is to put $\chi_1 = \frac{\pi}{2}$. But since

$$\tau_1|_{x=0} = 4\frac{(1+\cos(\frac{\pi}{2}+\chi_1))(1-\mu)}{2-m} \tag{2.93}$$

then this is exactly the choice of χ_1 which makes τ_1 vanish. Hence three consecutive tau functions now vanish — τ_0, τ_1 and τ_2.

Consider now what happens when we choose to make τ_0 vanish by putting $\cos\chi_1 = -1$. Then look at the derivative

$$\tau_0'|_{x=0} = -2m^2 \frac{\mu - 1}{2 - m}. \tag{2.94}$$

Again we need this to be non-positive which implies that $\mu \geq 1$. But this lower bound of μ is exactly the value which makes all odd tau functions vanish (so again for this value we have consecutive tau functions vanishing — this time τ_3, τ_0 and τ_1), whilst for any μ greater than this value the odd tau functions become negative at the origin.

This proof is of course specific to the $r = 3$ case with three solitons; however, it is expected that the results will hold in general although proofs of this sort rapidly become extremely difficult.

2.4.5 Overlapping charge sectors

It is however sometimes possible to obtain acceptable multi-soliton solutions in cases where the charge sectors overlap, i.e. when $Q_{max} > \frac{r+1}{2}$. The first case where this can be done is $r = 4$. Consider a solution containing all the solitons of the theory. Then look at the charge sectors:

Charge sector	Soliton combinations
-2	$\bar{2}$ $\bar{2}1\bar{1}$ 12
-1	$\bar{1}$ $\bar{1}2\bar{2}$ $1\bar{2}$
0	0 $1\bar{1}$ $2\bar{2}$ $1\bar{1}2\bar{2}$
1	1 $12\bar{2}$ $\bar{1}2$
2	2 $21\bar{1}$ $\overline{12}$

Now as usual we must apply the charge-sector equations (2.68). We have seen in the non-overlapping case that the highest occupied charge sector yields an equation solely in C. Here, however, we obtain an equation in d_1 and C which must be solved simultaneously with the other charge sector equations. The three equations are:

$$\begin{aligned} C^2 - 1 &+ d_1^2 A^{1\bar{1}}[(C - 2m_1)^2 - 1] + d_2^2 A^{2\bar{2}}[(C - 2m_2)^2 - 1] \\ &+ d_1^2 d_2^2 (A^{12} A^{1\bar{2}})^2 A^{1\bar{1}} A^{2\bar{2}}[(C - 2(m_1 + m_2))^2 - 1] = 0 \end{aligned} \tag{2.95}$$

$$\begin{aligned}(C-m_1)^2-1 &+ d_2^2A^{12}A^{1\bar{2}}A^{2\bar{2}}[(C-(m_1+2m_2))^2-1]\\ &+ d_2e^{i(\chi_2-2\chi_1)}A^{1\bar{2}}[(C-(m_1+m_2))^2-1]=0 \qquad (2.96)\end{aligned}$$

and

$$\begin{aligned}(C-m_2)^2-1 &+ d_1^2A^{12}A^{1\bar{2}}A^{1\bar{1}}[(C-(m_2+2m_1))^2-1]\\ &+ d_1e^{-i(\chi_1+2\chi_2)}A^{12}[(C-(m_1+m_2))^2-1]=0. \qquad (2.97)\end{aligned}$$

If we allow $e^{i(\chi_2-2\chi_1)}$ and $e^{-i(\chi_1+2\chi_2)}$ to take any complex values then the solutions to these equations are as before. However, consider the case where we choose χ_1 and χ_2 so that these exponentials are real. Then there may exist other real solutions to the equations. We shall call these the 'exceptional' solutions.

To find them, we must solve the three equations simultaneously for d_1, d_2 and C, with the restriction that d_1 and d_2 are positive. [1] The latter two equations are quadratic in d_2 and d_1 respectively and so the positions of the solitons can be found easily in terms of C. We can then determine the parameter C from the remaining equation. The only solution consistent with all these criteria (the symbolic algebra package *MAPLE V* was invaluable in finding this) gives

$$\begin{aligned}C &= m_1+m_2-\sqrt{5}\\ d_1 &= -\left(9\sqrt{5}+25\right)m_1+22+14\sqrt{5}\\ d_2 &= -\left(115+51\sqrt{5}\right)(m_2-m_1)+82+38\sqrt{5} \qquad (2.98)\end{aligned}$$

when we take $e^{i(\chi_2-2\chi_1)}=1$ and $e^{-i(\chi_1+2\chi_2)}=-1$. The energy parameter $C\simeq 0.8416$, which lies between the energies of the $1\bar{1}$ and $2\bar{2}$ two-soliton vacuum configurations. We can once again determine the boundary condition which this solution obeys. What is remarkable is that in this case, this exceptional solution is regular in the physical region, has non-consecutive zeroes of the tau functions at $x=0$, and fits exactly those boundary conditions not covered by the non-exceptional solutions. Specifically, we find that the boundary conditions covered are of the form $+a+a\,a$ in the usual notation, with cyclic permutations again being allowed by the shifts $\chi_1\to\chi_1+\frac{2\pi}{5}$ and $\chi_2\to\chi_2+\frac{4\pi}{5}$.

[1] An acceptable solution can be found from one where one or more of the positions are negative by considering different values of χ_1 and χ_2.

We can also use this technique for the case $r = 5$, where greater difficulties with finding solutions to the four simultaneous equations are encountered. Simplifications can be made by a rescaling of the position parameters of the three types of soliton. We do this so that a parity inversion is effected by a simple mapping $d_p \to \frac{1}{d_p}$ for each position. We can achieve this by defining

$$\tilde{d_p} = d_p \prod_{q \neq p} A_{pq} \tag{2.99}$$

where the product is over all solitons and their conjugates. After doing this and a great deal of manipulation using *MAPLE V*, we obtain some interesting results. A single acceptable solution, obeying the criteria developed above and corresponding to boundary conditions not covered by the non-exceptional solutions is again found. In this case, we shall only quote the energy of the solution, which is

$$C = \sqrt{3} + 2 - \sqrt{2(2^{1/3} + 2^{2/3} + 2)} \simeq 0.6184; \tag{2.100}$$

it lies between the $1\bar{1}$ and $2\bar{2}$ two-soliton solutions of the $a_5^{(1)}$ theory.

2.4.6 Vacuum Solution Results

Before presenting our results for the static background solutions of the $a_r^{(1)}$ series of affine Toda field theories up to $r = 5$ below, let us quickly recap the allowed background solutions. Of the non-exceptional solutions, there are few permitted background configurations. In the cases $r =$ even, the only acceptable background solutions contain either no solitons or a soliton/conjugate soliton pair. When $r =$ odd we are also allowed the case of a single middle soliton. In any of these cases, the singularity may reside at the boundary $x = 0$, allowing additional choices in the boundary conditions obeyed. However, when $r \geq 4$, these configurations do not span all the possible boundary conditions. To find the vacuum solutions for these remaining boundary conditions, it is necessary to consider the 'exceptional' solutions. In the cases considered these exceptional solutions completely cover the boundary conditions not covered by the non-exceptional configurations. It is expected that similar results will extend beyond $r = 5$ although a proof is lacking.

It should be noted that, in many cases, more than one possible background configuration fits with a particular boundary condition. When this occurs, the vacuum solution is of course that background configuration which has lowest energy.

Our results up to $r = 5$ are tabulated in table 2.1. They are ordered within each theory with decreasing C, or equivalently, increasing energy.

2.5 Classical Scattering Solutions

We now turn to the next stage of the procedure: the evaluation of the classical reflection factors K_a associated with each integrable boundary condition. We recall that K_a is the factor which encodes the reflection of a particle from the boundary $x = 0$ (we assume for the time being that a particle a is reflected back to itself). We can determine this factor using the soliton solutions of this chapter; this approach was developed in [72]. Recall from section 1.5 that a diagonal scattering solution $\epsilon(x,t)$ has the form, asymptotically far from the boundary,

$$\epsilon(x,t) \to \rho_a e^{-i\omega t}(e^{ikx} + K_a e^{-ikx}), \tag{2.101}$$

consisting of the superposition of incoming and outgoing states. Solutions of this form which obey the boundary conditions can be generated by considering non-static two soliton solutions, where now

$$\Phi(x,t) = \sigma(x - vt) \tag{2.102}$$

is defined by

$$\sigma = \pm ik \tag{2.103}$$

and

$$\sigma v = i\omega. \tag{2.104}$$

Hence we must add two solitons, both of type a, to the background configuration and determine K_a by insisting that the resulting tau functions satisfy

$$\ddot{\tau}_i - \tau_i'' + 2C\tau_i' - (C^2 - 1)\tau_i|_{x=0} = 0 \tag{2.105}$$

Boundary Condition	Solitons	C	Singular at $x=0$
$+++$	none	1	no
$-++$ $--+$	$1\bar{1}$	$\sqrt{3}-1$	no yes
$---$	none	-1	no

$r=2$

Boundary Condition	Solitons	C	Singular at $x=0$
$++++$	none	1	no
$-+++$ $-+-+$	2	1	yes yes
$--++$ $---+$	$1\bar{1}$	$\sqrt{2}-1$	no yes
$----$	none	-1	no

$r=3$

Boundary Condition	Solitons	C	Singular at $x=0$
$+++++$	none	1	no
$-++++$ $-+-++$	$2\bar{2}$	$\frac{1}{\sqrt{2}}\sqrt{5+\sqrt{5}}-1$	no yes
$--+++$ $--+-+$	$1\bar{1}2\bar{2}$	$\sqrt{5+2\sqrt{5}}-\sqrt{5}$	yes yes
$---++$ $----+$	$1\bar{1}$	$\frac{1}{\sqrt{2}}\sqrt{5-\sqrt{5}}-1$	no yes
$-----$	none	-1	no

$r=4$

Boundary Condition	Solitons	C	Singular at $x=0$
$++++++$	none	1	no
$-+++++$ $-+-+++$ $-+-+-+$	3	1	yes yes yes
$-++-++$ $--++++$ $--++-+$ $--+-++$ $--+--+$	$2\bar{2}$	$\sqrt{3}-1$	no yes yes yes yes
$---+++$ $---+-+$	$1\bar{1}2\bar{2}3$	$\sqrt{3}+2-\sqrt{2(2^{1/3}+2^{2/3}+2)}$	yes yes
$----++$ $-----+$	$1\bar{1}$	0	no yes
$------$	none	-1	no

$r=5$

Table 2.1: Constituent solitons and energy parameters C for the vacuum solutions of $a_r^{(1)}$ affine Toda field theory on a half-line with integrable boundary conditions, given up to $r=5$.

or of course equivalently, from the charge sector point of view,

$$\ddot{T}_i - T_i'' + 2CT_i' - (C^2 - 1)T_i|_{x=0} = 0. \tag{2.106}$$

It is then very simple to determine the values of K_a corresponding to different boundary conditions.

Although they have already been set out in [72], let us briefly restate the results for the classical reflection factors in the cases which have non-exceptional vacuum solutions. Solving the equation for the highest occupied charge sector reveals that

$$K_a = \frac{2ik + m_a^2}{2ik - m_a^2} \prod_{j=1}^{N} \frac{A^{ab_j}(k)}{A^{ab_j}(-k)} \tag{2.107}$$

where a is the particle being scattered and b_j are the solitons present in the background. N as usual refers to the number of soliton species (not including conjugates) present in the background. The only exception to this is the flat background case $++\ldots+$, where

$$K_a = \frac{2ik - m_a^2}{2ik + m_a^2}. \tag{2.108}$$

It has been checked [72] that these reflection factors obey the classical reflection bootstrap equation. We shall show an explicit example of how this works in section 2.5.1.

In fact, we shall see that this is not the full story; there is a little more subtlety involved when we consider the cases where one or more of the background tau functions vanish at $x = 0$. Now it is not so clear that a linear perturbation in the field $\phi \to \phi + \epsilon$ and in the tau functions $\tau_i \to \tau_i + \epsilon_i$ are equivalent. However, with certain restrictions on the ϵ_i (namely that if $\tau_i \to 0$ as $x \to 0$ then $\epsilon_i \to 0$ as $x \to 0$ as quickly or faster in order to result in a finite perturbation ϵ) we can achieve the same results. Look again at the boundary conditions

$$C - \frac{\tau_i'}{\tau_i} = A_i \sqrt{\frac{\tau_{i-1}\tau_{i+1}}{\tau_i^2}}. \tag{2.109}$$

Now if we take i so that τ_{i+1} goes to zero whilst τ_{i-1} and τ_i do not, then clearly we require the term linear in ϵ in τ_{i+1} to go to zero at $x = 0$. In fact, this is almost the same restriction as the requirement that the perturbation ϵ is finite.

This requirement gives us another equation which must be satisfied by the reflection factor K.

It is also useful to consider the case where we take the parameters $e^{i\chi}$ to be real, so that the vacuum solution tau functions are symmetric about τ_0 (i.e. $\tau_i = \tau_{r+1-i}$). In other words, we look at a particular example of the set of boundary conditions which are related by a cyclic permutation. Let us illustrate our point by considering the case of $a_2^{(1)}$ for the boundary conditions $+--$ or $++-$, which contain two static solitons in the background solution. Here, let us take $\chi = \pi$ so that the $\tau_0 \to 0$ as $x \to 0$ and $\tau_1 = \tau_2$ in the vacuum solution. Hence at $O(\epsilon)$ the right-hand sides of the two boundary conditions

$$C - \frac{\tau_1'}{\tau_1} = A_1\sqrt{\frac{\tau_0\tau_2}{\tau_1^2}} \tag{2.110}$$

and

$$C - \frac{\tau_2'}{\tau_2} = A_2\sqrt{\frac{\tau_0\tau_1}{\tau_2^2}} \tag{2.111}$$

(both evaluated at $x = 0$) must be equal up to a possible sign difference between A_1 and A_2. Hence we expect

$$A_1\frac{\tau_1'}{\tau_1} = A_2\frac{\tau_2'}{\tau_2} \tag{2.112}$$

up to $O(\epsilon)$. In actual fact, for $A_1 = -A_2$, this equation is satisfied identically for all K and hence the same scattering data as we found before for the non-singular background also obeys this boundary condition. However, for $A_1 = A_2$ we require both sides of the equation to vanish. Imposing this condition leads to a somewhat unaesthetic non-diagonal scattering solution.

As we saw in section 1.5, diagonal scattering is expected in $a_r^{(1)}$ as the conservation of spin even charges (which distinguish between particles and their conjugates) implies that a particle should be reflected back into itself. Moreover, this is implied by the full reflection bootstrap equation given in [42]. We can see this by the following argument. Look again at the reflection bootstrap equation depicted in fig. 1.8, and consider the non-diagonal case. Suppose that $c = 1$. Then the LHS implies that c can be reflected as a particle of type 1 or 2. This particle will then decompose into two particles a and b, either of type 1 or 2, but in any case with $a = b$. However, the RHS implies that c will split first into two particles of type 2, which are then each reflected as types 1 or 2. In this case there is no requirement that $a = b$. This tells us that the scattering must be either diagonal (as generally believed for real-coupling affine Toda field theory) or completely off-diagonal. Hence we rule out the unaesthetic non-diagonal scattering solution for the case $A_1 = A_2$ as expected.

These rather unsatisfying results have been backed up using a direct method used by Corrigan et al. [46]. This shall be summarised below. Let us introduce an ansatz for the background solution $\phi^{(-1)}$ to the $-+-$ case (cyclic permutations ensure that this will have the same reflection factor solutions as the $+--$ case). We take

$$\phi^{(-1)} = \alpha_1 \rho \tag{2.113}$$

— a choice which is consistent with the symmetry of the boundary conditions. Then the equations of motion (1.7) and the boundary condition (1.58) become:

$$\rho'' = e^{2\rho} - e^{-\rho} \qquad x < 0 \tag{2.114}$$

$$\rho' = e^{\rho} + e^{-\rho/2} \qquad x = 0. \tag{2.115}$$

(2.114) is the equation of motion for Bullough-Dodd theory, $a_2^{(2)}$, for which the solution is well-known. Integrating this equation with $\rho \to 0$ as $x \to -\infty$ yields

$$(\rho')^2 = e^{2\rho} + 2e^{-\rho} - 3. \tag{2.116}$$

and using this result at $x = 0$ along with the boundary condition (2.115) gives an equation which can be solved for $e^{\rho/2}$. We find that at $x = 0$ we need

$$e^{\rho/2} = -1, \frac{1}{2}, \mathrm{or} \rho \to \infty. \tag{2.117}$$

To obtain this result we have again squared out the sign contained in the boundary condition. We therefore need to check which of these solutions corresponds to our case $-+-$. The first solution is invalid anyway since it implies a complex field ρ. Consider the second possibility. The appropriate solution to the Bullough-Dodd equation is

$$e^{-\rho} = 1 + \frac{3/2}{\sinh^2\left(\frac{\sqrt{3}}{2}(x - x_0)\right)}. \tag{2.118}$$

Now we need to impose the boundary condition (2.115) along with $e^{\rho/2} = \frac{1}{2}$. It is not hard to show that to do this we require $x_0 < 0$. However, this means that there is a singularity at $x = x_0$, i.e. in the physical region $x < 0$, which is of course unacceptable.

Now look at the third case, $e^{\rho/2} \to \infty$. The necessary Bullough-Dodd solution is now

$$e^{-\rho} = 1 - \frac{3/2}{\cosh^2\left(\frac{\sqrt{3}}{2}(x - x_0)\right)}. \tag{2.119}$$

We can check that we have agreement with (2.115)

$$\frac{d}{dx} e^{-\rho} = -1 \tag{2.120}$$

and no singularities in the physical region if we take x_0 to be defined by

$$e^{\sqrt{3}x_0} = 2 + \sqrt{3}. \tag{2.121}$$

It is thus this solution that we require. In fact, this is exactly the solution we obtained in section 2.4 from the two-soliton background.

We now want to linearise the field around the background solution; we expand it in powers of β to obtain $\phi = \frac{1}{\beta}\phi^{(-1)} + \phi^{(0)} + O(\beta)$. Taking a real basis for the roots of $a_2^{(1)}$ to be

$$\alpha_1 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{3}{2}} \end{pmatrix} \text{ and } \alpha_0 = -\alpha_1 - \alpha_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}} \end{pmatrix} \tag{2.122}$$

we find the equations of motion for $\phi^{(0)}$

$$\begin{aligned} \partial^2 \phi^{(0)} &= -\sum_{i=0}^{r} \alpha_i e^{\alpha_i \cdot \phi^{(-1)}} \alpha_i \cdot \phi^{(0)} \\ &= -\begin{pmatrix} 2e^{2\rho} + e^{-\rho} & 0 \\ 0 & 3e^{-\rho} \end{pmatrix} \phi^{(0)} \end{aligned} \tag{2.123}$$

and boundary conditions at $x = 0$

$$\begin{aligned} \partial_x \phi^{(0)} &= \frac{1}{2} \sum_{i=0}^{r} A_i \alpha_i e^{\alpha_i \cdot \phi^{(-1)}/2} \alpha_i \cdot \phi^{(0)} \\ &= \frac{1}{2} \begin{pmatrix} 2e^{\rho} - e^{-\rho/2} & 0 \\ 0 & -3e^{-\rho/2} \end{pmatrix} \phi^{(0)}. \end{aligned} \tag{2.124}$$

We can solve the first order equations for both the first and second channels using the same methods as [46]. The second channel (which we call $\phi_2^{(0)}$) is the easier of the two so it shall be considered first. Substituting in the value of ρ at the boundary gives the boundary condition

$$\partial_x \phi_2^{(0)} = 0. \tag{2.125}$$

In addition, the equation of motion from (2.119) and (2.123) becomes, where we define $\phi_2^{(0)} = e^{-i\omega t}\Phi_2^{(0)}$

$$\partial_x^2 \Phi_2^{(0)} = \left(-\omega^2 + 3 - \frac{9}{2\cosh^2\left(\frac{\sqrt{3}}{2}(x - x_0)\right)} \right) \Phi_2^{(0)}. \tag{2.126}$$

Further defining $z = \frac{\sqrt{3}}{2}x$ and $\lambda^2 = \frac{4}{3}(\omega^2 - 3)$ allows us to write this as

$$\partial_z^2 \Phi_2^{(0)} = \left(-\lambda^2 - \frac{6}{\cosh^2(z - z_0)} \right) \Phi_2^{(0)} \tag{2.127}$$

where of course $z_0 = \frac{\sqrt{3}}{2}x_0$. In fact, the solution to this equation (which is the case of a reflectionless potential (e.g. [73])) is given by

$$\Phi_2^{(0)} = a\left(\frac{d}{dz} - 2\tanh(z - z_0)\right)\left(\frac{d}{dz} - \tanh(z - z_0)\right) e^{i\lambda z} + \text{complex conjugate.} \tag{2.128}$$

The reflection factor for this channel can be found by taking the ratios of the left and right-moving waves, in the asymptotic limit $x \to -\infty$. We therefore obtain

$$K_{\text{channel 2}} = \frac{a^*(i\lambda - 2)(i\lambda - 1)}{a(i\lambda + 2)(i\lambda + 1)} \tag{2.129}$$

since as $z \to -\infty$, $\tanh(z - z_0) \to -1$. It only remains to determine the ratio a^*/a from the boundary condition (2.125). Substituting in (2.128) quickly gives

$$\frac{a^*}{a} = \frac{3i\lambda^3 + 3\sqrt{3}\lambda^2 + 6i\lambda + 4\sqrt{3}}{3i\lambda^3 - 3\sqrt{3}\lambda^2 + 6i\lambda - 4\sqrt{3}}. \tag{2.130}$$

Finally we put $\lambda = \frac{2}{\sqrt{3}}k$ and we can see that our result is

$$K_{\text{channel 2}} = \frac{(k + i\frac{\sqrt{3}}{2})(k + i\sqrt{3})(2ik^3 + 3k^2 + 3ik + 3)}{(k - i\frac{\sqrt{3}}{2})(k - i\sqrt{3})(2ik^3 - 3k^2 + 3ik - 3)}. \tag{2.131}$$

This does not appear to be a good reflection factor: we normally expect poles in the momentum k to occur at purely imaginary values. Moreover, this reflection factor cannot be written in terms of the blocks (x).

What happens in the first channel? This is more difficult to analyse. Again, the techniques used are analogous to those in [46]. First, we note that the boundary condition (2.124) this time gives

$$\partial_x \phi_1^{(0)} = e^{\rho} \phi_1^{(0)}. \tag{2.132}$$

The difference here is that since $\rho \to \infty$ as $x \to 0$ then we must consider the asymptotics of this boundary condition.

The equation of motion is

$$\partial^2 \phi_1^{(0)} = -(2e^{2\rho} + e^{-\rho})\phi_1^{(0)} \tag{2.133}$$

which, by making the same definitions as for the second channel, we can write as

$$\partial_z^2 \Phi_1^{(0)} = \left(-\lambda^2 + \frac{4r}{q^2}\right) \Phi_1^{(0)}. \tag{2.134}$$

Here, r and q are functions of $E = e^{2(z-z_0)}$ given by

$$r = 6E(E^4 + 4E^3 - 6E^2 + 4E + 1) \tag{2.135}$$

$$q = (E+1)(E^2 - 4E + 1). \tag{2.136}$$

The solution is of the form

$$\Phi_1^{(0)} = \frac{p}{q} e^{i\lambda z} + \text{ complex conjugate} \tag{2.137}$$

where p, a function of E, must be determined. By substitution into (2.134) we can find that

$$p = a((i\lambda-1)(i\lambda-2)E^3 - 3(i\lambda+1)(i\lambda-2)E^2 - 3(i\lambda-1)(i\lambda+2)E + (i\lambda+1)(i\lambda+2)). \tag{2.138}$$

Once again we require the boundary condition (2.132) to deduce the ratio a^*/a. We must be careful as both sides of the boundary condition contain a singularity at $x = 0$; comparing the asymptotic behaviour on each side, we eventually obtain

$$\frac{a^*}{a} = \frac{i\lambda + \sqrt{3}}{i\lambda - \sqrt{3}}. \tag{2.139}$$

Finally, it only remains for us to notice that $E = e^{2(z-z_0)} \to 0$ as $x \to -\infty$ so the reflection factor is given by

$$\begin{aligned} K_{\text{channel 1}} &= \frac{(ik - \frac{\sqrt{3}}{2})(ik - \sqrt{3})(ik + \frac{3}{2})}{(ik + \frac{\sqrt{3}}{2})(ik + \sqrt{3})(ik - \frac{3}{2})} \\ &= \frac{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}{(1)(2)(3)}. \end{aligned} \tag{2.140}$$

Notice that this is the same as the reflection factor associated with the boundary condition $+-+$. However the reflection data for the second channel was different. In fact, the scattering data found in this way is exactly the same as that found using the tau function techniques and the condition (2.112). This can be seen by rewriting the scattering data in a basis corresponding to the particle eigenstates. Hence we find that there is no consistent classical scattering data for the boundary condition $+--$.

It is not difficult to extend the argument to include *all* boundary conditions in $a_r^{(1)}$ affine Toda field theory which require singular vacuum solutions consisting of two static solitons. When the singularity is placed at the boundary it can be shown that, for any $i \in S$, we obtain an analogous equation to (2.112), involving the boundary parameters A_{i-1} and A_{i+1}. This relation is only solved for diagonal scattering if we takes these two parameters to be opposite. This rules out consistent scattering data for a large subset of the integrable boundary conditions.

Whilst it would appear that there is no problem with singular background solutions containing only one static middle soliton (here, the signs of the (say) odd i boundary parameters remain arbitrary since the $O(\epsilon)$ term of τ_i'/τ_i vanishes), we also find that the 'exceptional' solutions of $a_4^{(1)}$ and $a_5^{(1)}$ have problems. Consider the $a_4^{(1)}$ exceptional case. Taking $\chi_1 = \chi_2 = 0$ we again obtain tau functions such that $\tau_i = \tau_{r+1-i}$. In this case, it is τ_1 and τ_4 which vanish at the origin. Hence the same argument as before tells us that we must choose K so that the linear perturbations of these two tau functions must also vanish at the origin. It is however not too difficult to check that since the incoming and outgoing parts of these tau functions are not complex conjugates of each other, then the reflection factor K which results is non-unitary. In addition, the conditions resulting from these two criteria do not yield the same reflection factor. Hence in these cases it seems that no diagonal scattering solutions are possible which go as ρ_a as $x \to -\infty$. We expect, of course, that non-diagonal scattering solutions may exist, although as before these cannot obey the reflection bootstrap equation.

This analysis suggests that not all of the boundary conditions allowed by integrability are consistent with classical scattering. The ones that do obey certain criteria. These boundary conditions are the ones which contain either none, one or two static solitons in their background configurations, and, in the latter case, where a singularity is present at the boundary, the signs of the boundary parameters on either side of a vanishing tau function must be opposite.

2.5.1 Classical Scattering Results

The results for the reflection factors for r up to 5 are given in table 2.2.

As noted previously, it was shown in [72] that all reflection factors of the form (2.107) obey the classical reflection bootstrap equation. Instead of reproducing that argument, we shall give here an explicit example of how this works. Since, in the classical case, the S-matrix is unity then (1.81) becomes

$$K^c(\theta_c) = K^a(\theta_c + i\overline{\theta}^b_{ac})K^b(\theta_c - i\overline{\theta}^a_{bc}) \tag{2.141}$$

where θ^b_{ac} are the fusing angles given by the relation (1.41).

Consider the data for $r = 4$ for the $- - - + +$ boundary condition. Look at the coupling $11 \to 2$. Then the angle $\overline{\theta}^1_{13} = \frac{\pi}{5}$. Since $K_1 = -\frac{(\frac{5}{2})^2}{(\frac{1}{2})(1)(4)(\frac{9}{2})}$ then it is easy to check that

$$\begin{aligned} K_1(\theta + i\overline{\theta}^1_{13})K_1(\theta - i\overline{\theta}^1_{13}) &= \frac{(\frac{3}{2})^2(\frac{7}{2})^2}{(-\frac{1}{2})(\frac{3}{2})(0)(2)(3)(5)(\frac{7}{2})(\frac{11}{2})} \\ &= -\frac{(\frac{1}{2})(\frac{3}{2})(\frac{7}{2})(\frac{9}{2})}{(2)(3)} \end{aligned} \tag{2.142}$$

using the block relationships given in section 1.3. So this example does indeed satisfy the relation (2.141).

Before we finally leave this section, let us quickly review what happens in the sinh-Gordon case. This theory will be the subject of section 3.3 and chapter 4 and

Boundary Condition	K_1, $K_{\bar{1}}$
$+++$	$-(1)(2)$
$-++$	$-\frac{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}{(1)(2)}$
$--+$	not diagonal
$---$	$-\frac{1}{(1)(2)}$

$r = 2$

Boundary Condition	K_1, $K_{\bar{1}}$	K_2
$++++$ $-+++$ $-+-+$	$-(1)(3)$	$-(2)(2)$
$--++$	$\frac{(2)^2}{(1)(3)}$	$-\frac{(1)^2(3)^2}{(2)^2}$
$---+$	not diagonal	not diagonal
$----$	$-\frac{1}{(1)(3)}$	$-\frac{1}{(2)^2}$

$r = 3$

Boundary Condition	K_1, $K_{\bar{1}}$	K_2, $K_{\bar{2}}$
$+++++$	$-(1)(4)$	$-(2)(3)$
$-++++$	$-\frac{(\frac{1}{2})(\frac{3}{2})(\frac{7}{2})(\frac{9}{2})}{(1)(4)}$	$-\frac{(\frac{3}{2})(\frac{5}{2})^2(\frac{7}{2})}{(2)(3)}$
$-+-++$	not diagonal	not diagonal
$--+++$ $--+-+$	not diagonal	not diagonal
$---++$	$-\frac{(\frac{5}{2})^2}{(\frac{1}{2})(1)(4)(\frac{9}{2})}$	$-\frac{(\frac{1}{2})(\frac{3}{2})(\frac{7}{2})(\frac{9}{2})}{(2)(3)}$
$----+$	not diagonal	not diagonal
$-----$	$-\frac{1}{(1)(4)}$	$-\frac{1}{(2)(3)}$

$r = 4$

Boundary Condition	K_1, $K_{\bar{1}}$	K_2, $K_{\bar{2}}$	K_3
$++++++$ $-+++++$ $-+-+++$ $-+-+-+$	$-(1)(5)$	$-(2)(4)$	$-(3)^2$
$-++-++$ $--++++$	$\frac{(2)(4)}{(1)(5)}$	$-\frac{(1)(3)^2(5)}{(2)(4)}$	$-\frac{(2)^2(4)^2}{(3)^2}$
$--++-+$ $--+-++$ $--+--+$	not diagonal	not diagonal	not diagonal
$---+++$ $---+-+$	not diagonal	not diagonal	not diagonal
$----++$	$-\frac{(3)^2}{(1)^2(5)^2}$	1	$-\frac{(1)^2(5)^2}{(3)^2}$
$-----+$	not diagonal	not diagonal	not diagonal
$------$	$-\frac{1}{(1)(5)}$	$-\frac{1}{(2)(4)}$	$-\frac{1}{(3)^2}$

$r = 5$

Table 2.2: Classical reflection factors for $a_r^{(1)}$ affine Toda field theory with integrable boundary conditions, given up to $r = 5$.

we will need to know the associated classical reflection factor; this has not been covered in the present analysis. Restricting ourselves to the case where the two boundary parameters σ_0 and σ_1 in (1.60) are equal, the theory has a flat background solution. It is easy to solve the equations of motion together with the boundary conditions in this case; we obtain a classical reflection factor dependent on the boundary parameter σ:

$$K = \frac{ik + 2\sigma}{ik - 2\sigma}. \tag{2.143}$$

There is, in this case, no reflection bootstrap equation to obey since there are no fusings of the sinh-Gordon particle.

2.6 Conclusions

In this chapter we have presented a Bogomol'nyi argument for restrictions on the boundary parameters of affine Toda field theory on a half-line. This extends what has previously been achieved for the special case of sinh-Gordon theory. We found that, in the simply-laced cases, all the integrable boundary conditions lead to stable theories; however, where $A_i = 1$ we are only on the boundary of stability.

We have also considered the vacuum solutions for low rank $a_r^{(1)}$ affine Toda field theories in order to attempt to shed some light on what happens in general. We have found many unusual characteristics of these theories. Firstly, these vacuum configurations were found to be unexpected in their complexity. Indeed, there is scope for further work to see if a more enlightening argument leading to the background solutions relating to various boundary conditions can be found. In addition, it was found that although acceptable vacuum solutions can be found for all the integrable boundary conditions, not all of these appear to admit classical scattering data consistent with the reflection bootstrap equation. It would be interesting to see if other methods back up these results.

Chapter 3

The quantum reflection factor of $a_2^{(1)}$ affine Toda field theory to $O(\beta^2)$

3.1 Introduction

We saw in chapter 1 that there are many conjectured exact reflection factors for the affine Toda field theories on a half-line. This chapter aims to use perturbation theory to give supporting evidence for one or more of these for the particular case of $a_2^{(1)}$ affine Toda field theory (discussed in the subsections of 1.5). We shall do this by the perturbative calculation of terms in the expansion of the exact reflection factor as a power series in β. Essentially this amounts to the calculation of contributions to the two-point function arising at the order under consideration. We shall see shortly how this works explicitly.

Kim [74–76] has used such techniques for the simply-laced affine Toda field theories in the case of the Neumann boundary condition $\partial_x\phi|_{x=0} = 0$. Perturbative work on sinh-Gordon theory for more general boundary conditions has also been performed by Corrigan [77]. In the following two sections we shall briefly review these two areas of work. These will form much of the basis for chapter 4, which will study sinh-Gordon theory with the Neumann boundary condition at two-loop order, or $O(\beta^4)$. We shall then move on to discuss the reflection factor of $a_2^{(1)}$ affine Toda

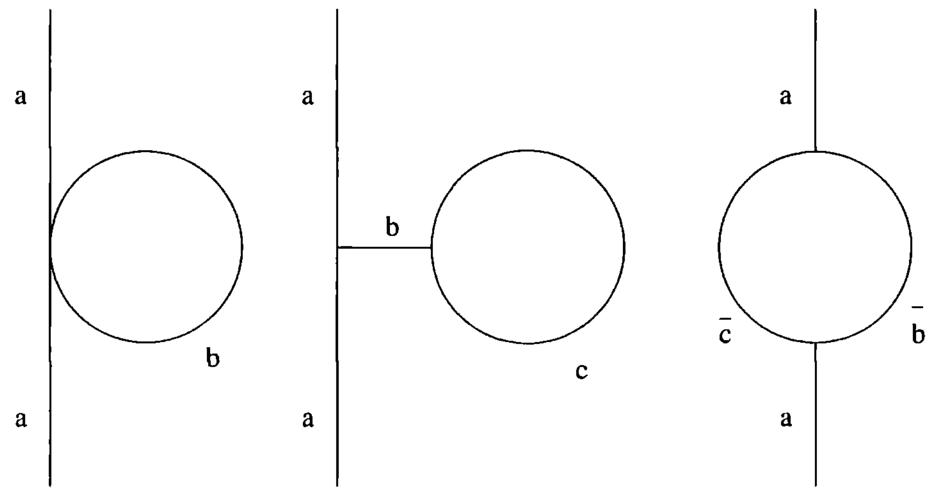

Figure 3.1: Types I, II and III diagrams respectively.

field theory with the $---$ boundary condition, which shall be calculated to $O(\beta^2)$. This was chosen as it has a flat background $\phi = 0$ and would seem to be the next simplest case after the sinh-Gordon case of [77].

3.2 The Neumann boundary condition

Kim considers the case of Neumann boundary condition for the simply-laced series of affine Toda field theories. We shall see that the Neumann case allows considerable simplifications to be made in the calculations, stemming from the fact that it has no boundary potential (i.e. $\mathcal{B} = 0$) as well as a classical reflection factor of unity. Let us consider only the $a_r^{(1)}$ cases since they are of primary interest here.

Throughout the reconstruction of Kim's argument we shall use our own notation where relevant. Kim uses the Green's function for the half-line case which comes from the sum of two whole-line Green's functions, the second of which can be considered as coming from the image point reflected in the boundary. The whole-line Green's function is, for some particle of type a,

$$G_a(x,t;x',t') = \int \frac{d^2p}{(2\pi)^2} \frac{i}{p^2 - m_a^2 + i\epsilon} e^{-i\omega(t'-t)} e^{ik(x'-x)} \tag{3.1}$$

where we use the usual shorthand notation $d^2p = d\omega dk$ and $p^2 = \omega^2 - k^2$. So we take the Green's function for the theory on the half-line to be

$$G_a(x,t;x',t') = \int \frac{d^2p}{(2\pi)^2} \frac{i}{p^2 - m_a^2 + i\epsilon} e^{-i\omega(t'-t)} (e^{ik(x'-x)} + e^{-ik(x+x')}). \tag{3.2}$$

In fact, a more rigorous argument supporting this form of the Green's function is given in [77].

We expect the two-point Green's function to be, to all orders in β,

$$\begin{aligned} G_a(x,t;x',t') &= \int \frac{d^2p}{(2\pi)^2} \frac{i}{p^2 - m_a^2 + i\epsilon} e^{-i\omega(t'-t)} (e^{ik(x'-x)} + K^q(k) e^{-ik(x+x')}) \\ &= \int \frac{d\omega}{2\pi} \frac{1}{2\hat{k}} e^{-i\omega(t'-t)} (e^{i\hat{k}(x'-x)} + K^q(\hat{k}) e^{-i\hat{k}(x+x')}). \end{aligned} \tag{3.3}$$

Hence by calculating the $O(\beta^2)$ contribution to the two-point function and reading off the coefficient of $e^{-i\hat{k}(x+x')}$ we find the $O(\beta^2)$ term in the quantum reflection factor K^q.

There are three possible diagrams, shown in fig. 3.1, which can contribute to this correction. [1] However, diagram II in fact gives no contribution since the relevant vertex factors in $a_r^{(1)}$ with a flat background are zero. In addition, since we have zero boundary potential then all vertices must be located in the bulk region $x < 0$. Let us look at the diagram I case first. The incoming and outgoing particles are of type a whilst the loop particle is of some type b. So the amplitude associated with this diagram is

$$-iS_{a\bar{a}b\bar{b}} C^{a\bar{a}b\bar{b}} \int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 G_a(x,t;x_1,t_1) G_b(x_1,t_1;x_1,t_1) G_a(x_1,t_1;x',t') \tag{3.4}$$

where $S_{a\bar{a}b\bar{b}}$ is the associated symmetry factor and $C^{a\bar{a}b\bar{b}}$ is the vertex factor for the four-point coupling. Substituting in the Green's functions this becomes

$$\begin{aligned} -iS_{a\bar{a}b\bar{b}} C^{a\bar{a}b\bar{b}} \int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 \int \frac{d^2p d^2p' d^2p_1}{(2\pi)^6} \frac{i}{p^2 - m_a^2 + i\epsilon} \frac{i}{p'^2 - m_a + i\epsilon} \\ \frac{i}{p_1^2 - m_b^2 + i\epsilon} e^{-i\omega(t_1-t)} e^{-i\omega'(t'-t_1)} (e^{ik(x_1-x)} + e^{-ik(x+x_1)}) \\ (e^{ik'(x'-x_1)} + e^{-ik'(x'+x_1)})(1 + e^{-2ik_1x_1}). \end{aligned} \tag{3.5}$$

[1] Recall that we generate such Feynman diagrams and their associated symmetry factors from Wick contractions stemming from the expansion of $< p'|\mathrm{T}\left\{\exp\left(\frac{i}{\hbar}\int dxdt\mathcal{L}_I\right)\right\}|p >$, where $\mathcal{L}_I$ is the interaction Lagrangian. A full discussion of this can be found in any quantum field theory text, e.g. [78].

The last bracket contains a divergence. We remove this by an infinite renormalisation of the bulk potential (i.e. addition of some infinite counter-term $I_2\beta^2\phi^2$ to the bulk potential), effectively removing the "1" from this bracket. We shall see more of this procedure later. The t_1 integral produces a delta function which sets the two external energies ω and ω' equal. In addition, we can extend the range of the x_1 integration (the integrand remains unchanged under the transformation $x_1 \to -x_1$ so the integral over the half-line is half that over the full-line) and hence this integral too yields a delta function. Hence we obtain

$$
\begin{aligned}
&-iS_{a\bar{a}b\bar{b}}C^{a\bar{a}b\bar{b}}\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{\omega^2-k'^2-m_a^2+i\epsilon}e^{-i\omega(t'-t)}\\
&(e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'})\int\frac{d^2p_1}{2\pi}\frac{i}{p_1^2-m_b^2+i\epsilon}\delta(k+k'+2k_1).
\end{aligned}
\tag{3.6}
$$

Now defining $\hat{k_a} = \sqrt{\omega^2-m_a^2}$ the final k and k' integrations yield (choosing the direction for closure of the complex contours appropriately)

$$
-iS_{a\bar{a}b\bar{b}}C^{a\bar{a}b\bar{b}}\frac{1}{2\hat{k_a}}\left(\frac{1}{4\sqrt{\hat{k_a}^2+m_b^2}}+\frac{1}{4m_b}\right)\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k_a}}e^{-i\omega(t'-t)}e^{-i\hat{k_a}(x+x')}\right\}. \tag{3.7}
$$

Now let us consider the type III diagram. The contribution is

$$
\begin{aligned}
-S_{abc}(C^{abc})^2\int_{-\infty}^0 dx_1dx_2\int_{-\infty}^{\infty}dt_1dt_2G_a(x,t;x_1,t_1)G_b(x_1,t_1;x_2,t_2)&\\
G_c(x_1,t_1;x_2,t_2)G_a(x_2,t_2;x',t')&
\end{aligned}
\tag{3.8}
$$

where $a+b+c \equiv 0 \bmod(r+1)$. The S_{abc} is now the symmetry factor associated with diagram III and as usual C^{abc} are the relevant vertex factors. Substituting in the Green's functions gives

$$
\begin{aligned}
&-S_{abc}(C^{abc})^2\int_{-\infty}^0 dx_1dx_2\int_{-\infty}^{\infty}dt_1dt_2\int\frac{d^2pd^2p'd^2p_1d^2p_2}{(2\pi)^8}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{p'^2-m_a^2+i\epsilon}\\
&\frac{i}{p_1^2-m_b^2+i\epsilon}\frac{i}{p_2^2-m_c^2+i\epsilon}e^{-i\omega(t_1-t)}e^{-i\omega'(t'-t_2)}e^{-i\omega_1(t_2-t_1)}e^{-i\omega_2(t_2-t_1)}\\
&(e^{ik(x_1-x)}+e^{-ik(x+x_1)})(e^{ik'(x'-x_2)}+e^{-ik'(x'+x_2)})(e^{ik_1(x_2-x_1)}+e^{-ik_1(x_1+x_2)})\\
&(e^{ik_2(x_2-x_1)}+e^{-ik_2(x_1+x_2)})
\end{aligned}
\tag{3.9}
$$

Again we integrate over t_1 and t_2 to give delta functions in the energies (i.e. the ω's) and by extending the range of the x_1 and x_2 integrations we can similarly obtain delta functions in the momenta (the k's). So we obtain

$$-\frac{1}{4}S_{abc}(C^{abc})^2\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{\omega^2-k'^2-m_a^2+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+e^{-ikx})$$
$$(e^{ik'x'}+e^{-ik'x'})\int\frac{d^2p_1dk_2}{2\pi}\frac{i}{p_1^2-m_b^2+i\epsilon}\frac{i}{(\omega-\omega_1)^2-k_2^2-m_c^2+i\epsilon}\delta(k+k_1+k_2)$$
$$(\delta(k'+k_1+k_2)+\delta(k'+k_1-k_2)+\delta(k'-k_1+k_2)+\delta(k'-k_1-k_2)) \quad (3.10)$$

or, since we can change the sign of k',

$$-\frac{1}{2}S_{abc}(C^{abc})^2\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{\omega^2-k'^2-m_a^2+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+e^{-ikx})$$
$$(e^{ik'x'}+e^{-ik'x'})\int\frac{d^2p_1dk_2}{2\pi}\frac{i}{p_1^2-m_b^2+i\epsilon}\frac{i}{(\omega-\omega_1)^2-k_2^2-m_c^2+i\epsilon}$$
$$\delta(k+k_1+k_2)(\delta(k'+k_1+k_2)+\delta(k'+k_1-k_2)). \quad (3.11)$$

Carrying out the k_2 integral yields

$$-\frac{1}{2}S_{abc}(C^{abc})^2\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{\omega^2-k'^2-m_a^2+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+e^{-ikx})$$
$$(e^{ik'x'}+e^{-ik'x'})\int\frac{d^2p_1}{2\pi}\frac{i}{p_1^2-m_b^2+i\epsilon}\frac{i}{(\omega-\omega_1)^2-(k_1-k')^2-m_c^2+i\epsilon}$$
$$(\delta(k+k')+\delta(k-k'+2k_1)). \quad (3.12)$$

Notice that the first delta function inside the last bracket implies the existence of a double pole in the external propagator. This must be removed by the addition of a finite mass counter-term. Such renormalisations shall be considered in greater detail later; however here we shall introduce a useful argument which allows us to quickly see the effect of such a renormalisation.

If we consider the two-point function on the whole-line, then we want this to be given to all orders in β by (3.1). But diagram III for the whole-line theory gives a contribution

$$-S_{abc}(C^{abc})^2\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-m_a^2+i\epsilon}\frac{i}{\omega^2-k'^2-m_a^2+i\epsilon}e^{-i\omega(t'-t)}e^{ikx}e^{ik'x'}$$
$$\int\frac{d^2p_1}{2\pi}\frac{i}{p_1^2-m_b^2+i\epsilon}\frac{i}{(\omega-\omega_1)^2-(k_1-k')^2-m_c^2+i\epsilon}\delta(k+k'). \quad (3.13)$$

Hence we need to introduce counter-terms which will remove this term on the whole-line. If we assume that this can be done, then it is not hard to see that the same counter-terms will, for the theory on the half-line, remove the equivalent terms to

(3.13) in (3.12). Hence we shall assume that it is possible to do this. Let us then look at the second delta function. The normal techniques of complex integration give us the final type III result

$$-S_{abc}(C^{abc})^2\frac{i}{8\hat{k}_a}\left(\frac{1}{m_c}\frac{1}{(\omega+m_c)^2-\hat{k}_a^2-m_b^2}+\frac{1}{m_b}\frac{1}{(\omega-m_b)^2-\hat{k}_a^2-m_c^2}\right.$$
$$\left.+\frac{1}{\sqrt{\hat{k}_a^2+m_c^2}}\frac{1}{(\omega+\sqrt{\hat{k}_a^2+m_c^2})^2-m_b^2}+\frac{1}{\sqrt{\hat{k}_a^2+m_b^2}}\frac{1}{(\omega-\sqrt{\hat{k}_a^2+m_b^2})^2-m_c^2}\right)$$
$$\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k}_a}e^{-i\omega(t'-t)}e^{-i\hat{k}_a(x+x')}\right\}. \tag{3.14}$$

The correction to the reflection factor now consists of the sum of the contributions from (3.7) and (3.14). We obtain

$$\begin{aligned}\delta K_a &= -\sum_{b,c}\frac{i}{8\hat{k}_a}\left\{S_{a\bar{a}b\bar{b}}C^{a\bar{a}b\bar{b}}\left(\frac{1}{\sqrt{\hat{k}_a^2+m_b^2}}+\frac{1}{m_b}\right)\right.\\ &+ S_{abc}(C^{abc})^2\left(\frac{1}{m_c}\frac{1}{m_a^2+m_c^2-m_b^2+2m_c\omega}+\frac{1}{m_b}\frac{1}{m_a^2+m_b^2-m_c^2-2m_b\omega}\right.\\ &+ \frac{1}{\sqrt{\hat{k}_a^2+m_c^2}}\frac{1}{2\omega(\omega+\sqrt{\hat{k}_a^2+m_c^2})+m_c^2-m_a^2-m_b^2}\\ &+ \left.\left.\frac{1}{\sqrt{\hat{k}_a^2+m_b^2}}\frac{1}{2\omega(\omega-\sqrt{\hat{k}_a^2+m_b^2})+m_b^2-m_a^2-m_c^2}\right)\right\}.\end{aligned} \tag{3.15}$$

In particular, let us briefly state the results for sinh-Gordon ($a_1^{(1)}$ theory) and $a_2^{(1)}$ theory. The former has only one particle (type 1) with mass $m_1 = 2$. The vertex factors are $C^{111} = 0$ and $C^{1111} = \beta^2/3$, and the symmetry factor $S_{1111} = 12$. So

$$\delta K_1 = -\frac{i\beta^2}{2\hat{k}}\left(\frac{1}{\omega}+\frac{1}{2}\right). \tag{3.16}$$

For the case of $a_2^{(1)}$, as we shall see repeated in the results of section 3.3, we have two particles (conjugate to each other), and

$$m_1 = m_2 = \sqrt{3},\ C^{111} = \frac{\beta}{2},\ C^{1122} = \frac{3\beta^2}{4}. \tag{3.17}$$

The symmetry factors this time are

$$S_{111} = 18 \text{ and } S_{1122} = 4. \tag{3.18}$$

Hence we obtain

$$\delta K_1 = \delta K_2 = -\frac{\sqrt{3}i\beta^2(\omega+\sqrt{3})}{4\hat{k}(2\omega+\sqrt{3})}. \tag{3.19}$$

This result shall be considered in section 3.7.

3.3 Sinh-Gordon theory

Let us now review the calculations of Corrigan in [77]. These consider the $O(\beta^2)$ quantum correction to the classical reflection factor of sinh-Gordon theory. However, the difference between these calculations and those of Kim comes from the boundary conditions considered. As discussed in section 1.4, sinh-Gordon theory permits a large class of integrable boundary conditions, parametrised by two continuous degrees of freedom. However, only those where $\sigma_0 = \sigma_1$ allow the flat background solution $\phi = 0$. It was this class of boundary conditions that were considered in [77].

Let us write down the sinh-Gordon Lagrangian (as usual found from (1.3)):

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\beta^2}(e^{\beta\sqrt{2}\phi} + e^{-\beta\sqrt{2}\phi}). \tag{3.20}$$

For the flat background case we must take the boundary potential to be

$$\mathcal{B} = \frac{2\sigma}{\beta^2}\left(e^{\beta\phi/\sqrt{2}} + e^{-\beta\phi/\sqrt{2}}\right). \tag{3.21}$$

It is useful for perturbation theory (in order to read off the vertex factors) to expand the bulk and boundary potentials as power series in the coupling constant β. We write them as

$$V = 2\phi^2 + \beta^2\frac{1}{3}\phi^4 + \beta^4\frac{1}{45}\phi^6 + O(\beta^6) \tag{3.22}$$

and

$$\mathcal{B} = \sigma\phi^2 + \beta^2\frac{\sigma}{24}\phi^4 + \beta^4\frac{\sigma}{1440}\phi^6 + O(\beta^6) \tag{3.23}$$

respectively.

Now we must write down the Green's function to be used. This was derived in [77] to be

$$G(x,t;x't') = \int \frac{d^2p}{(2\pi)^2}\frac{i}{p^2-4+i\epsilon}e^{-i\omega(t'-t)}(e^{ik(x'-x)} + K(k)e^{-ik(x+x')}). \tag{3.24}$$

$K(k)$ is the reflection factor of the classical theory, determined in section 2.5.1 and given in (2.143). The mass of the sinh-Gordon particle is of course 2 giving the 4 in the propagator.

Sinh-Gordon theory on a flat background contains no cubic terms in the Lagrangian. Hence the only diagram which must be considered here is that of type I in fig. 3.1. However, it *is* now necessary to calculate terms arising from the boundary potential — in this case a diagram of type I but with the vertex located at $x = 0$. Let us consider this latter (and simpler) diagram first. The integral is

$$-\frac{i\sigma\beta^2}{2}\int_{-\infty}^{\infty} dt_1 G(x,t;0,t_1)G(0,t_1;0,t_1)G(0,t_1;x',t') \tag{3.25}$$

(now explicitly including the vertex and symmetry factors associated with this diagram) and this is equivalent to

$$\begin{aligned}-\frac{i\sigma\beta^2}{2}\int_{-\infty}^{\infty} dt_1 \int \frac{d^2p d^2p' d^2p_1}{(2\pi)^6}\frac{i}{p^2-4+i\epsilon}\frac{i}{p'^2-4+i\epsilon}\frac{i}{p_1^2-4+i\epsilon}e^{-i\omega(t_1-t)} \\ e^{-i\omega'(t'-t_1)}(e^{-ikx}+Ke^{-ikx})(e^{ik'x'}+K'e^{-ik'x'})(1+K_1).\end{aligned} \tag{3.26}$$

Throughout this chapter and chapter 4 we shall be using the convenient notation where $K' = K(k')$, $K_1 = K(k_1)$, $\hat{K} = K(\hat{k})$ and so on, except where explicitly stated otherwise.

Look at the last bracket of the above integral. It is clear that this again contains a divergence. We can remove this by an infinite renormalisation of the boundary potential, i.e. since $1+K_1 = 2+\frac{4\sigma}{ik_1-2\sigma}$ then we add a counter-term $I_2^{bndry}\beta^2\phi^2$ where

$$I_2^{bndry} = -\frac{\sigma}{4}\int\frac{d^2p_1}{(2\pi)^2}\frac{i}{p_1^2-4+i\epsilon}. \tag{3.27}$$

This is exactly right to cancel off the entirety of the divergent part.

Hence we now wish to compute the integral

$$\begin{aligned}-\frac{i\sigma\beta^2}{2}\int_{-\infty}^{\infty} dt_1 \int \frac{d^2p d^2p' d^2p_1}{(2\pi)^6}\frac{i}{p^2-4+i\epsilon}\frac{i}{p'^2-4+i\epsilon}\frac{i}{p_1^2-4+i\epsilon}e^{-i\omega(t_1-t)} \\ e^{-i\omega'(t'-t_1)}(e^{-ikx}+Ke^{-ikx})(e^{ik'x'}+K'e^{-ik'x'})\frac{4\sigma}{ik_1-2\sigma}.\end{aligned} \tag{3.28}$$

Integrating over t_1 to produce a delta function and then over ω' gives

$$\begin{aligned}-\frac{i\sigma\beta^2}{2}\int\frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-4+i\epsilon}\frac{i}{\omega^2-k'^2-4+i\epsilon}e^{-i\omega(t'-t)}e^{-ikx-ik'x'}(1+K)(1+K') \\ \int\frac{d^2p_1}{(2\pi)^2}\frac{i}{p_1^2-4+i\epsilon}\frac{4\sigma}{ik_1-2\sigma}.\end{aligned} \tag{3.29}$$

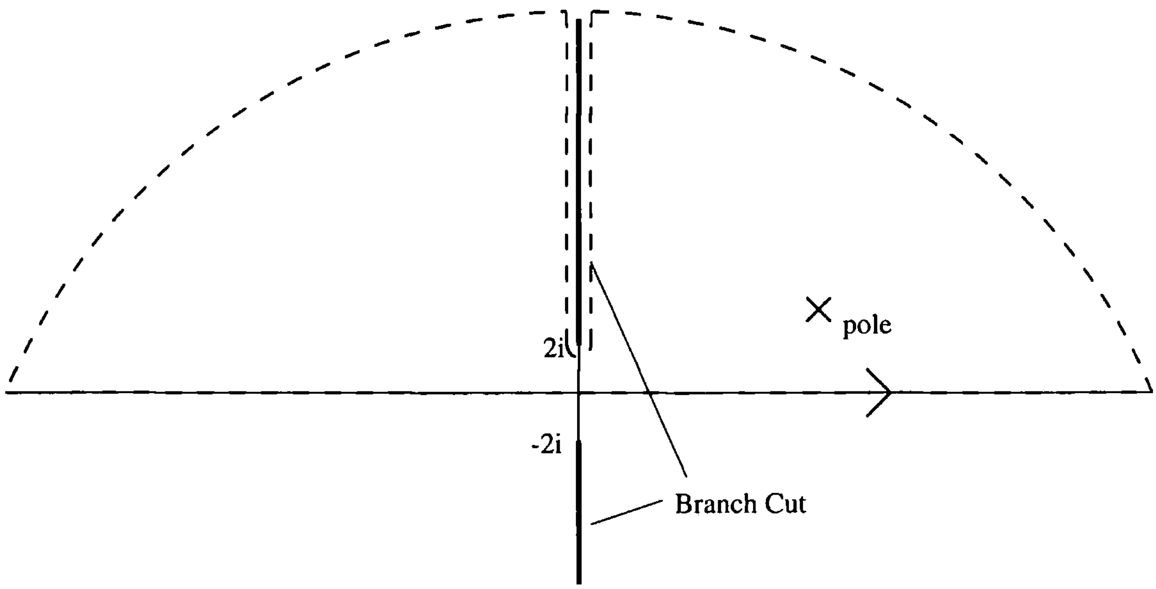

Figure 3.2: Integrating around the branch cut in the upper half-plane.

Let us look now at the p_1 integral alone. Integrating over ω_1 yields

$$\int \frac{dk_1}{2\pi} \frac{1}{2\sqrt{k_1^2+4}} \frac{4\sigma}{ik_1 - 2\sigma}. \tag{3.30}$$

However, this still leaves us with the task of computing the k_1 integral. To do this, it is easiest (particularly in the cases of later calculations) to perform a contour integration closed in the upper half-plane. In doing this we need to negotiate the branch cut, which runs from $2i$ to $i\infty$, as shown in fig. 3.2. Then in general we obtain two contributions to the result — one from any poles lying within the contour, and one from the integral around the branch cut. Consider the latter contribution first. This is (noting the factor of two which arises from integrating over both sides of the branch cut)

$$2\int_{2i}^{i\infty} \frac{dk_1}{2\pi} \frac{1}{2\sqrt{k_1^2+4}} \frac{4\sigma}{ik_1 - 2\sigma}. \tag{3.31}$$

If we now make the change of variables $k_1 = iy$ then we obtain

$$-2\int_{2}^{\infty} \frac{dy}{2\pi} \frac{1}{2\sqrt{y^2-4}} \frac{4\sigma}{y+2\sigma} \tag{3.32}$$

which can be calculated using the result

$$\int_m^{\infty} dy \frac{1}{\sqrt{y^2-m^2}} \frac{1}{y+m\zeta} = \frac{1}{m\sqrt{1-\zeta^2}} \left(\frac{\pi}{2} - \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{3.33}$$

to give

$$\frac{1}{2\pi}\frac{-2\sigma}{\sqrt{1-\sigma^2}}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{\sigma}{\sqrt{1-\sigma^2}}\right)\right). \tag{3.34}$$

The last thing we need to do is to perform the k and k' integrations. Both of these give multiplicative factors of $\frac{1}{2\hat{k}}(1+\hat{K})$, where $\hat{k}=\sqrt{\omega^2-4}$. However, it is clear from (3.3) that one of the factors $\frac{1}{2\hat{k}}$ is absorbed into the propagator. So finally, with a little manipulation, the contribution to the reflection factor can be written as

$$\frac{i\beta^2\cos^2(a\pi)a}{4\hat{k}\sin(a\pi)}(1+\hat{K})^2 \tag{3.35}$$

where, as in [77] we have used a more convenient boundary parameter a defined by $\sigma=\cos(a\pi)$. This is taken to lie in the range $0\le a\le 1$.

In addition, a pole may be present in the upper half-plane. This pole resides at $k_1=-2i\sigma$ and so only need be included if $\sigma<0$; when present it yields an additional contribution

$$-\frac{i\beta^2\cos^2(a\pi)}{4\hat{k}\sin(a\pi)}(1+\hat{K})^2. \tag{3.36}$$

Hence we obtain the results

$$\frac{i\beta^2\cos^2(a\pi)a}{4\hat{k}\sin(a\pi)}(1+\hat{K})^2 \quad \text{and} \quad \frac{i\beta^2\cos^2(a\pi)(a-1)}{4\hat{k}\sin(a\pi)}(1+\hat{K})^2 \tag{3.37}$$

for a in the ranges $0\le a\le\frac{1}{2}$ and $\frac{1}{2}<a\le 1$ respectively. We can check that these results are related correctly by looking at the integrand. Putting $\sigma\to-\sigma$ in (3.30) and making the transformation $k_1\to-k_1$ leaves the integrand unchanged; including the vertex factors, which are dependent on σ, means that we expect the total result to be odd in σ. Similarly the results in (3.37) are opposite for a and $1-a$ respectively.

We shall now consider the type I diagram in the bulk. The integral this time is

$$-4i\beta^2\int_{-\infty}^{0}dx_1\int_{-\infty}^{\infty}dt_1 G(x,t;x_1,t_1)G(x_1,t_1;x_1,t_1)G(x_1,t_1;x',t') \tag{3.38}$$

or

$$-4i\beta^2\int_{-\infty}^{0}dx_1\int_{-\infty}^{\infty}dt_1\int\frac{d^2p\,d^2p'\,d^2p_1}{(2\pi)^6}\frac{i}{p^2-4+i\epsilon}\frac{i}{p'^2-4+i\epsilon}\frac{i}{p_1^2-4+i\epsilon}e^{-i\omega(t_1-t)}$$
$$e^{-i\omega'(t'-t_1)}(e^{ik(x_1-x)}+Ke^{-ik(x+x_1)})(e^{ik'(x'-x_1)}+K'e^{-ik'(x_1+x')})(1+K_1e^{-2ik_1x_1}). \tag{3.39}$$

Again a divergence is present in the above integral. This time an infinite mass renormalisation on the bulk is required: i.e. addition of a counter-term of the form $I_2\beta^2\phi^2$ to the bulk potential. The value is

$$I_2 = -2\int \frac{d^2p_1}{(2\pi)^2}\frac{i}{p_1^2-4+i\epsilon}. \tag{3.40}$$

This once more has the effect of removing the "1" from the final bracket. Performing the t_1 and ω' integrals in the usual way allows us to write (3.39) as

$$\int \frac{d^2pdk'}{(2\pi)^3}\frac{i}{p^2-4+i\epsilon}\frac{i}{\omega^2-k'^2-4+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+Ke^{-ikx})(e^{ik'x'}+K'e^{-ik'x'})$$
$$\int_{-\infty}^{0} dx_1 \int \frac{d^2p_1}{(2\pi)^2}(-4i\beta^2)\frac{i}{p_1^2-4+i\epsilon}\frac{ik_1+2\sigma}{ik_1-2\sigma}e^{-ix_1(k+k'+2k_1)}. \tag{3.41}$$

We now need to integrate over x_1. No range extension method can be used this time since putting $x_1 \to -x_1$ does not leave the integrand invariant. Hence we must use another trick. We know that

$$\int_{-\infty}^{0} dx_1 e^{(-i\lambda+\rho)x_1} = \frac{i}{\lambda+i\rho} \tag{3.42}$$

if ρ is a small positive constant. So let us introduce such a ρ into our integrand in order to enable us to perform the integration; we can then take the limit as $\rho \to 0$ at the end of the calculation. So, looking at the x_1 and p_1 integrals alone, we have

$$\int_{-\infty}^{0} dx_1 \int \frac{d^2p_1}{(2\pi)^2}(-4i\beta^2)\frac{i}{p_1^2-4+i\epsilon}\frac{ik_1+2\sigma}{ik_1-2\sigma}e^{-ix_1(k+k'+2k_1+i\rho)}$$
$$= \int \frac{d^2p_1}{(2\pi)^2}(-4i\beta^2)\frac{i}{p_1^2-4+i\epsilon}\frac{ik_1+2\sigma}{ik_1-2\sigma}\frac{i}{k+k'+2k_1+i\rho}. \tag{3.43}$$

We can now do the ω_1 integral to leave

$$\int \frac{dk_1}{2\pi}(-4i\beta^2)\frac{1}{2\sqrt{k_1^2+4}}\frac{ik_1+2\sigma}{ik_1-2\sigma}\frac{i}{k+k'+2k_1+i\rho}. \tag{3.44}$$

Now we just need to carry out the k_1 integral in the same way as before. We must first split the integrand into partial fractions, before considering the branch cut and pole pieces in turn. Let us write (3.44) in the form

$$\int \frac{dk_1}{2\pi}(-4i\beta^2)\frac{1}{2\sqrt{k_1^2+4}}\frac{i}{i(k+k')+4\sigma}\left\{\frac{4\sigma}{k_1+2i\sigma}+\frac{i(k+k')-4\sigma}{2k_1+k+k'+i\rho}\right\}. \tag{3.45}$$

Then the branch cut contributions this time are

$$\frac{-4i\beta^2}{i(k+k')+4\sigma}\left\{\frac{\cos(a\pi)a}{\sin(a\pi)}+\frac{i(k+k')-4\sigma}{2\sqrt{4+(k+k')^2}}\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{i\frac{k+k'}{2}}{\sqrt{4+\left(\frac{k+k'}{2}\right)^2}}\right)\right)\right\}. \tag{3.46}$$

Let us call the result above I.

The k and k' integrations this time give four terms. These consist merely of giving the values of k and k' at their poles at $\pm\hat{k}$, i.e.

$$I|_{k=-\hat{k},k'=-\hat{k}}+\hat{K}I|_{k=-\hat{k},k'=\hat{k}}+\hat{K}I|_{k=\hat{k},k'=-\hat{k}}+\hat{K}^2I|_{k=\hat{k},k'=\hat{k}}. \tag{3.47}$$

Any other poles in k and k' yield exponentially decreasing terms as we take the limits $x,x'\to-\infty$. So we finally have the result, with a little manipulation,

$$-\frac{i\beta^2\hat{K}}{\hat{k}}\left\{\frac{\hat{k}^2a}{\sin(a\pi)(\hat{k}^2+4\sigma^2)}-\frac{1}{4}+\frac{1}{2\omega}\right\} \tag{3.48}$$

for the branch cut integral.

Again the only possible pole is at $k_1=-2i\sigma$. When it occurs, it yields a contribution to the k_1 integral of

$$-\frac{\cos(a\pi)}{\sin(a\pi)}\frac{-4i\beta^2}{i(k+k')+4\sigma} \tag{3.49}$$

which, using the pole values of k and k' again gives a total contribution

$$\frac{i\beta^2\hat{k}\hat{K}}{\sin(a\pi)(\hat{k}^2+4\sigma^2)}. \tag{3.50}$$

Adding all these results together, and with a little more manipulation, we find an $O(\beta^2)$ contribution to the sinh-Gordon reflection factor of

$$-\frac{i\beta^2\hat{K}}{\hat{k}}\left\{\frac{\hat{k}^2a\sin(a\pi)}{\hat{k}^2+4\sigma^2}-\frac{1}{4}+\frac{1}{2\omega}\right\} \tag{3.51}$$

for the range $0\le a\le\frac{1}{2}$ and

$$-\frac{i\beta^2\hat{K}}{\hat{k}}\left\{\frac{\hat{k}^2(a-1)\sin(a\pi)}{\hat{k}^2+4\sigma^2}-\frac{1}{4}+\frac{1}{2\omega}\right\} \tag{3.52}$$

for the range $\frac{1}{2}<a\le 1$.

3.4 $a_2^{(1)}$ affine Toda field theory

Let us now proceed to repeat this calculation for the case of $a_2^{(1)}$ affine Toda field theory. As we saw, this has been done in the case of the Neumann boundary condition by Kim; however, no perturbative results for any of the other integrable boundary conditions are known. The bulk Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi \cdot \partial^\mu\phi - \frac{1}{\beta^2}\sum_{i=0}^{2} e^{\beta\alpha_i\cdot\phi}. \tag{3.53}$$

We shall be considering the boundary condition $A_i = -1$, i.e.

$$\partial_x\phi|_{x=0} = -\frac{1}{\beta}\sum_{i=0}^{2}\alpha_i e^{\beta\alpha_i\cdot\phi/2}. \tag{3.54}$$

Since this allows a flat background solution $\phi = 0$, the vertex factors are not dependent on the vertex position x and hence the calculation is significantly simpler than the case of asymmetric boundary conditions. [2]

We shall take the real basis for the α_i's given in (2.122). In order to calculate the vertex factors for $a_2^{(1)}$ affine Toda field theory, we need to expand the bulk and boundary potentials in terms of the coupling constant β. To do this, it is first useful to write the field ϕ in the basis given by the mass eigenvectors, ρ_1 and ρ_2, corresponding to the two particles. We can do this by looking at the asymptotic form of the tau function solutions given in chapter 2, whence;

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} (-1+i\sqrt{3})\Phi - (1+i\sqrt{3})\overline{\Phi} \\ -(\sqrt{3}+i)\Phi + (-\sqrt{3}+i)\overline{\Phi} \end{pmatrix} \tag{3.55}$$

where Φ and $\overline{\Phi}$ are the fields corresponding to the particle (type 1) and its conjugate (type 2) respectively. In fact, this is exactly the complex basis which we used to determine the masses in section 1.3. We then find the bulk and boundary potentials respectively;

$$V = 3\Phi\overline{\Phi} + \frac{1}{2}\beta(\Phi^3 + \overline{\Phi}^3) + \frac{3}{4}\beta^2\Phi^2\overline{\Phi}^2 + O(\beta^3) \tag{3.56}$$

and from (3.54)

$$\mathcal{B} = \frac{3}{2}\Phi\overline{\Phi} + \frac{1}{8}\beta(\Phi^3 + \overline{\Phi}^3) + \frac{3}{32}\beta^2\Phi^2\overline{\Phi}^2 + O(\beta^3). \tag{3.57}$$

[2] A calculation for a non-trivial background is currently being undertaken by Chenaghlou et al. for the case of sinh-Gordon theory.

The Green's function for the $a_2^{(1)}$ case is exactly analogous to that used in the previous section:

$$G(x,t;x',t') = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 - m^2 + i\epsilon} e^{-i\omega(t'-t)} (e^{ik(x'-x)} + K(k)e^{-ik(x+x')}) \qquad (3.58)$$

where $K(k)$ is of course now the classical reflection factor associated with this boundary condition of $a_2^{(1)}$ affine Toda field theory. From the previous chapter, we know this to be

$$K(k) = \frac{2ik + m^2}{2ik - m^2}. \qquad (3.59)$$

The masses of the two particles in $a_2^{(1)}$ are both $m = \sqrt{3}$; we shall in general leave the calculations in terms of m rather than substituting in these values.

Note that we obtain different contributions to the quantum reflection factor from diagrams where none, one or both of the vertices are located on the boundary $x = 0$. There are thus five different diagrams, two of type I and three of type III, which contribute to the two-point function and hence to the quantum correction to the reflection factor.

3.5 The Calculations

Let us now proceed to the calculations used to determine the one-loop contribution to the two-point function. We shall consider the two types of Feynman diagram separately in the two following subsections.

3.5.1 Type I diagrams

There are two possible type I diagrams: one where the vertex is in the bulk region $x < 0$, and one where the vertex is situated on the boundary $x = 0$ itself. Let us consider the latter first. We are required to compute the integral:

$$I^I_{bndry} = -\frac{3}{8} i\beta^2 \int_{-\infty}^{\infty} dt_1 G(x,t;0,t_1)G(0,t_1;0,t_1)G(0,t_1;x',t'). \qquad (3.60)$$

Included here are the vertex and combinatorial factors arising in the four point interaction.

After integrating over t_1, which generates a delta function, and using this delta function to integrate over ω', we find

$$\begin{aligned} I^I_{bndry} &= -\frac{3}{8}i\beta^2 \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{\omega^2 - k'^2 - m^2 + i\epsilon} e^{-i\omega(t'-t)} \\ & e^{-ikx - ik'x'}(1+K)(1+K')\left\{ \int \frac{d^2p_1}{(2\pi)^2} \frac{i}{p_1^2 - m^2 + i\epsilon}(1+K_1)\right\}. \end{aligned} \tag{3.61}$$

Notice that the p_1 integral separates from the others and is divergent due to the $(1 + K_1)$ term. This can be removed by an infinite renormalisation of the boundary potential. In fact, in a similar way to before, we put

$$1 + K_1 = 2 + \frac{2m^2}{2ik_1 - m^2}, \tag{3.62}$$

so we can perform a minimal subtraction of the divergent part to leave $\frac{2m^2}{2ik_1 - m^2}$ by adding a suitable counter term. Then by integrating over ω_1 using a contour integral (closing the contour into the lower half-plane), we pick up a pole at

$$\omega_1 = \hat{\omega}_1 = \sqrt{k_1^2 + m^2} \tag{3.63}$$

which leaves us with the task of finding the integral

$$\int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \frac{1}{2\hat{\omega}_1} \frac{2m^2}{2ik_1 - m^2}. \tag{3.64}$$

As before, this can be achieved by performing a contour integral closed in the upper half plane, negotiating the branch cuts which now run from im to $i\infty$. Again we obtain in general two parts to the integral: branch cut integrals, which can be evaluated using the result (3.33), and residues coming from poles enclosed by the contour.

Consider the integral we need to compute. There are no poles in the upper half-plane so we only need consider the branch cut contribution. Making the change $k_1 = iy$ we obtain

$$\int_m^{\infty} \frac{dy}{\pi} \frac{1}{\hat{\omega}_1} \frac{m^2}{-2(y + m(\frac{m}{2}))} = -\frac{m}{2\pi\sqrt{1 - \frac{m^2}{4}}} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\frac{m}{2}}{\sqrt{1 - \frac{m^2}{4}}} \right) \right) = -\frac{m}{6} \tag{3.65}$$

where in the last step we use the value of the mass $m = \sqrt{3}$. Finally, integrating over the remaining k and k' integrals picks up poles at $k = k' = \hat{k} \equiv \sqrt{\omega^2 - m^2}$ and we obtain the result

$$\begin{aligned} I^I_{bndry} &= \frac{3i}{8}\beta^2 \int \frac{d\omega}{2\pi} \frac{1}{4\hat{k}^2} e^{-i\hat{k}(x+x')} e^{-i\omega(t'-t)} (1+\hat{K})^2 \frac{m}{6} \\ &= \beta^2 \int \frac{d\omega}{2\pi} \frac{1}{2\hat{k}} e^{-i\hat{k}(x+x')} e^{-i\omega(t'-t)} \frac{-im\hat{k}}{2(2i\hat{k}-3)^2}. \end{aligned} \tag{3.66}$$

Now let us consider the case where the vertex is located in the bulk section. The contribution this time is

$$I^I_{bulk} = -3i\beta^2 \int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 G(x,t;x_1,t_1) G(x_1,t_1;x_1,t_1) G(x_1,t_1;x',t'). \tag{3.67}$$

Again we have a divergence similar to that in (3.5) and after an infinite mass renormalisation (which again simply removes the "1" from the bracket $(1 + K_1 e^{-2ik_1x_1})$), and integration using the delta function in ω', this becomes

$$\begin{aligned} -3i\beta^2 \int_{-\infty}^{0} dx_1 \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{\omega^2 - k'^2 - m^2 + i\epsilon} e^{-i\omega(t'-t)} (e^{ikx} + Ke^{-ikx}) \\ (e^{ik'x'} + K'e^{-ik'x'}) \left\{ \int \frac{d^2p_1}{(2\pi)^2} K_1 e^{-ix_1(k+k'+2k_1)} \frac{i}{p_1^2 - m^2 + i\epsilon} \right\}. \end{aligned} \tag{3.68}$$

We can use the same trick as in section 3.3 — the introduction of a small positive number ρ into the exponential — to perform the x_1 integral. Consider only the integrations over internal momenta and energies. We need to find

$$\int \frac{d^2p_1}{(2\pi)^2} \frac{i}{p_1^2 - m^2 + i\epsilon} K_1 \frac{i}{(k + k' + 2k_1) + i\rho}. \tag{3.69}$$

As usual, integration over ω_1 picks up the pole at $\hat{\omega_1}$ and we can complete the integration over k_1 by decomposing into partial fractions and using the usual contour integration method. We must then perform the k and k' integrations. This is done as before by integrating along a complex contour closed in such a direction that the exponential factors of each term decay to zero on the complex part of the contour. As we noted previously, the only poles we pick up are simply $\pm\hat{k}$ since it can be checked that all other poles have a finite imaginary part and hence the exponentials decay to zero if we take the limits $x, x' \to -\infty$.

It turns out to be simpler if, knowing this fact, we simplify the integrand of the k_1 integral by substituting in the poles of the k and k' integration *first*, i.e. if we define the previous integrand to be I, then the new integrand is

$$\frac{1}{4\hat{k}^2} e^{-i\hat{k}(x+x')} (I|_{k=-\hat{k},k'=-\hat{k}} + \hat{K} I|_{k=-\hat{k},k'=\hat{k}} + \hat{K} I|_{k=\hat{k},k'=-\hat{k}} + \hat{K}^2 I|_{k=\hat{k},k'=\hat{k}}). \tag{3.70}$$

We can then restrict our attention to that part of the integrand which is even in k_1 (since the k_1 integral runs from $-\infty$ to ∞), and by making the substitutions $\omega = m\cosh(\theta)$ and $e^\theta = y$ the integrand can be made simpler still. This is necessary as it permits some cancellations in the integrand, without which the calculation is too large to be practicable.

The answer then consists of a sum of contributions from integrals along branch cuts, residues arising from poles with finite imaginary part, and residues from poles which are only infinitesimally shifted from the real axis. In the first two cases, the infinitesimals present in the integrand are insignificant to the calculation. We can therefore reduce the calculation to a manageable size by setting all infinitesimals to zero, and simplifying the integrand, before calculation of these two contributions. However, in the case of poles with infinitesimal imaginary part, no such simplification can be made.

Hence the integration over the branch cuts and poles with finite imaginary part becomes

$$\int \frac{d\omega}{2\pi} \int \frac{dk_1}{2\pi} \frac{1}{2\hat{k}} e^{-i\omega(t'-t)} e^{-i\hat{k}(x+x')} \frac{1}{\hat{\omega}_1} \frac{54i\beta^2(y^2-1)}{(2i\hat{k}-3)^2(2k_1+3i)(2k_1-3i)\sqrt{3}y}. \tag{3.71}$$

We integrate using the same method as before, but this time we encounter a pole at $k_1 = \frac{3}{2}i$ and hence must include the contribution arising from its residue. Summing the contributions gives a term in the quantum reflection factor of

$$\frac{i\beta^2(y^2-1)}{(2i\hat{k}-3)^2 y}. \tag{3.72}$$

The integral over the poles which have been shifted off the real axis by an infinitesimal amount involves considering the residues of poles in the upper half-plane of

$$\frac{9\sqrt{3}(3-8k_1^2y^2+3y^2-12ik_1y^2+3y^4)(2k_1-3i)(y^2-1)}{2y^2(3y^2-3+2\sqrt{3}k_1y+i\sqrt{3}\rho y)(3y^2-3-2\sqrt{3}k_1y-i\sqrt{3}\rho y)(2k_1+i\rho)(2k_1+3i)} + (k_1 \to -k_1) \tag{3.73}$$

which yield a reflection factor contribution

$$-\frac{3i\beta^2}{4}\frac{(y^2+y+1)(y^2-y+1)(y-1)}{(2i\hat{k}-3)^2y(y^2+1)(y+1)}. \tag{3.74}$$

The reason for leaving the factor $\frac{1}{(2i\hat{k}-3)^2}$ in terms of $\hat{k}$ rather than y will become clear shortly.

3.5.2 Type III diagrams

In the case of the type III diagrams, there are three possible configurations - we can have none, one, or both of the vertices located on the boundary. The simplest of these is the last — the boundary-boundary case;

$$I^{III}_{bndry-bndry} = -\frac{9}{32}\beta^2 \int_{-\infty}^{\infty} dt_1 dt_2 G(x,t;0,t_1)G(0,t_1;0,t_2)G(0,t_1;0,t_2)G(0,t_2;x',t'). \tag{3.75}$$

Let us consider this case in detail since it is instructive for performing the later integrals, which are more tedious. Putting in the form of the Green's function, we obtain

$$-\frac{9}{32}\beta^2 \int_{-\infty}^{\infty} dt_1 dt_2 \int \frac{d^2p d^2p' d^2p_1 d^2p_2}{(2\pi)^8} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{p'^2 - m^2 + i\epsilon} \frac{i}{p_1^2 - m^2 + i\epsilon} \frac{i}{p_2^2 - m^2 + i\epsilon} e^{-i\omega(t_1-t) - i\omega_1(t_2-t_1) - i\omega_2(t_2-t_1) - i\omega'(t'-t_2)} e^{-ikx - ik'x'}(1+K)(1+K')(1+K_1)(1+K_2). \tag{3.76}$$

As before, integrating over the t_1 and t_2 gives us delta functions which enable integration over ω' and ω_2. However, this means that we must not only set $\omega' = \omega$ as before but we also have $\omega_2 = \omega - \omega_1$. Substituting in the form of K_1 and K_2, we find that we must perform the integral

$$-\frac{9}{32}\beta^2 \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2 - m^2 + i\epsilon} \frac{i}{\omega^2 - k'^2 - m^2 + i\epsilon} e^{i\omega(t-t')} e^{-ikx - ik'x'}(1+K)(1+K') \int \frac{d^2p_1 dk_2}{(2\pi)^3} \frac{i}{\omega_1^2 - k_1^2 - m^2 + i\epsilon} \frac{i}{(\omega-\omega_1)^2 - k_2^2 - m^2 + i\epsilon} \frac{4ik_1}{2ik_1 - m^2} \frac{4ik_2}{2ik_2 - m^2}. \tag{3.77}$$

Consider the second integral — the one over the internal momenta. Integrating over ω_1 (again closing the contour downwards) we pick up two poles, at $\sqrt{k_1^2 + m^2}$ and $\sqrt{k_2^2 + m^2} + \omega$. Hence we obtain

$$\int \frac{dk_1 dk_2}{(2\pi)^2} \frac{-16k_1k_2}{(2ik_1 - m^2)(2ik_2 - m^2)} \left(\frac{1}{2\sqrt{k_1^2 + m^2}} \frac{i}{(\omega - \sqrt{k_1^2 + m^2})^2 - k_2^2 - m^2 + i\epsilon} + \frac{1}{2\sqrt{k_2^2 + m^2}} \frac{i}{(\omega + \sqrt{k_2^2 + m^2})^2 - k_1^2 - m^2 + i\epsilon} \right). \tag{3.78}$$

Notice however that we can simply exchange the indices 1 and 2 on the second term giving us;

$$\int \frac{dk_1 dk_2}{(2\pi)^2} \frac{1}{2\hat{\omega}_1} \frac{i}{(\omega - \hat{\omega}_1)^2 - k_2^2 - m^2 + i\epsilon} \frac{-16k_1k_2}{(2ik_1 - m^2)(2ik_2 - m^2)} + (\omega \to -\omega) \tag{3.79}$$

where as before $\hat{\omega_1} \equiv \sqrt{k_1^2 + m^2}$.

Now integrating this over k_2 (closed in the upper half-plane) gives a residue due to the pole at $\hat{k_2} \equiv \sqrt{(\omega - \hat{\omega_1})^2 - m^2}$. So we are left with the integral

$$\int \frac{dk_1}{2\pi} \frac{1}{\hat{\omega_1}} \frac{-4k_1}{(2ik_1 - m^2)(2i\hat{k_2} - m^2)} + (\omega \to -\omega). \tag{3.80}$$

This can be easily decomposed into two pieces: that which contains odd powers of $\hat{k_2}$ and that which does not. Moreover, if we again throw away all terms which are odd in k_1, we are left with

$$\int \frac{dk_1}{2\pi} \frac{1}{\hat{\omega_1}} \left(\frac{-8im^2k_1^2}{(4k_1^2 + m^4)(4\hat{k_2}^2 + m^4)} + \frac{16k_1^2\hat{k_2}}{(4k_1^2 + m^4)(4\hat{k_2}^2 + m^4)} \right) + (\omega \to -\omega). \tag{3.81}$$

The first term in the above can be handled as before. The second term, however, is difficult to deal with since it generates elliptic integrals [79, 80].

At this stage it is worth doing a little analysis of the properties expected of the results. Unitarity implies that the quantum reflection factor, K^q is a pure phase, i.e. of the form $e^{i\chi}$. Suppose that the classical reflection factor is $K = e^{i\chi_0}$. Then the quantum reflection factor, to order β^2, is given by the expansion of $e^{i(\chi_0 + \beta^2\chi_1)}$ (χ_0 and χ_1 some functions of k). So we obtain

$$K^q = e^{i(\chi_0+\beta^2\chi_1)} = e^{i\chi_0}(1 + i\beta^2\chi_1 + O(\beta^4)) = K + iK\chi_1\beta^2 + O(\beta^4). \tag{3.82}$$

Hence we are looking for a β^2 correction which is what we shall term "completely imaginary with respect to the phase of the classical reflection factor K", i.e. its argument is $\arg K + \frac{\pi}{2}$. Notice that the phase of $(1 + K)^2$ is the same as that of K, and this is exactly the prefactor we obtain from the k and k' integrals. So there is good physical justification for assuming that all the completely real parts (which in every case are the "elliptic" parts) of the integrals will eventually, though perhaps only after summation of terms from all diagrams, vanish. This assumption shall make the job of calculating these integrals significantly simpler.

Hence ignoring the real part of this integral, and using the expression for $\hat{k_2}$, we obtain

$$\int \frac{dk_1}{2\pi} \frac{1}{\hat{\omega_1}} \frac{-8im^2k_1^2}{(4k_1^2 + m^4)(4\hat{k_2}^2 + m^4)} + (\omega \to -\omega) \tag{3.83}$$

$$= \int \frac{dk_1}{2\pi}\frac{1}{\hat{\omega}_1}\frac{-16ik_1^2m^2(4\omega^2+4k_1^2+m^4)}{(4k_1^2+m^4)((4\omega^2+4k_1^2+m^4)^2-64\omega^2(k_1^2+m^2))}. \tag{3.84}$$

This integral can be computed as before and is found to contribute $\tanh^{-1}$ terms to the $O(\beta^2)$ correction. These are unexpected since they cannot be obtained from the expansion of any K composed of blocks of the type (x). However, they are found to exactly cancel with equivalent terms from the other two type III diagrams.

The other two type III contributions are

$$I^{III}_{bulk-bndry} = -\frac{9}{4}\beta^2\int_{-\infty}^{0}dx_1\int_{-\infty}^{\infty}dt_1dt_2G(x,t;x_1,t_1)G(x_1,t_1;0,t_2)$$
$$G(x_1,t_1;0,t_2)G(0,t_2;x',t') \tag{3.85}$$

and

$$I^{III}_{bulk-bulk} = -\frac{9}{2}\beta^2\int_{-\infty}^{0}dx_1dx_2\int_{-\infty}^{\infty}dt_1dt_2G(x,t;x_1,t_1)G(x_1,t_1;x_2,t_2)$$
$$G(x_1,t_1;x_2,t_2)G(x_2,t_2;x',t'). \tag{3.86}$$

In a similar way to before, we find the necessary k_1 integrals corresponding to these to be

$$I^{III}_{bulk-bndry} = -\frac{9}{4}\beta^2\int\frac{dk_1}{2\pi}\frac{1}{\hat{\omega}_1}\frac{4ik_1}{(k+k_1+\hat{k}_2+i\rho)(2ik_1-m^2)(2i\hat{k}_2-m^2)}+(\omega\to-\omega) \tag{3.87}$$

and

$$\begin{aligned} I^{III}_{bulk-bulk} &= -\frac{9}{2}\beta^2\int\frac{dk_1}{2\pi}\left\{\frac{1}{4\hat{\omega}_1\hat{k}_2}\frac{1}{k_1+\hat{k}_2+k+i\rho}\left(\frac{1}{k'-k_1-\hat{k}_2+i\rho}\right.\right.\\ &+ K_1\frac{1}{k_1-\hat{k}_2+k'+i\rho}+\hat{K}_2\frac{1}{\hat{k}_2-k_1+k'+i\rho}+K_1\hat{K}_2\left.\frac{1}{k_1+\hat{k}_2+k'+i\rho}\right)\\ &+ \frac{1}{2\hat{\omega}_1}\frac{1}{(\omega-\hat{\omega}_1)^2-(k'-k_1)^2-m^2}\frac{1}{k+k'+i\rho}\\ &+ \left.\frac{1}{2\hat{\omega}_1}K_1\frac{1}{(\omega-\hat{\omega}_1)^2-(k_1+k')^2-m^2}\frac{1}{2k_1+k+k'+i\rho}\right\}\\ &+ (\omega\to-\omega) \end{aligned} \tag{3.88}$$

before separation of the real and imaginary parts relative to the phase. The ubiquitous k and k' integrations take the same form as in (3.68).

Let us consider the fifth term of the integral (3.88), whose analysis is quite subtle. It is clear that this term contains a double pole in the external momentum k and hence we need to perform some *finite* mass renormalisation [81, 82]. In fact, we could use an argument of the type used in section 3.2 to discount this term but instead it is perhaps more enlightening to show how this works explicitly. A finite mass renormalisation is equivalent to adding a term of the form $a\Phi\overline{\Phi}$, where a is a constant, to the original Lagrangian in order to cancel out this double pole. The contribution arising from such a counter-term is

$$-ai\int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 G(x,t;x_1,t_1)G(x_1,t_1;x',t') \tag{3.89}$$

which can be manipulated to give

$$-ai\int \frac{d^2p dk'}{(2\pi)^3} e^{-i\omega(t'-t)} \frac{i}{\omega^2-k^2-m^2+i\epsilon}\frac{i}{\omega^2-k'^2-m^2+i\epsilon}\frac{i}{k+k'+i\rho} (e^{ikx}+Ke^{-ikx})(e^{ik'x'}+K'e^{-ik'x'}). \tag{3.90}$$

Now look at the double pole term in our integral. This has broadly the above form, but instead of the constant a, we have a function, $F(\omega,k') = \int dk_1 f(k_1,k',\omega)$ say, of ω and k'. Looking at its precise form in (3.88), and discarding all odd terms in k_1, we can see that f is also even in k'. Hence by substituting $\omega^2 = \hat{k}^2 + m^2$ into this function and Taylor expanding the result around $k'^2 = \hat{k}^2$ we obtain

$$f(k_1,k',\omega) = f_0(k_1,\hat{k}) + f_1(k_1,\hat{k})(k'^2-\hat{k}^2) + O((k'^2-\hat{k}^2)^2). \tag{3.91}$$

In fact, integrating over the k_1, the first term in the above expansion gives a constant independent of $\hat{k}$. So we can indeed cancel this contribution by adding in an equivalent finite mass renormalisation term. Computing the value of the renormalisation required we find that

$$F_0 = \int \frac{dk_1}{2\pi} f_0(k_1,\hat{k}) = \frac{-1}{4\sqrt{3}}\beta^2 \tag{3.92}$$

which is exactly the same mass renormalisation as that obtained in the full-line case [81], as we would of course expect.

The second term in the expansion (3.91) is more interesting. This gives us

$$\int \frac{d^2p dk'}{(2\pi)^3} i \frac{i}{\omega^2-k^2-m^2+i\epsilon}\frac{i}{k+k'+i\rho}(e^{ikx}+Ke^{-ikx})(e^{ik'x'}+K'e^{-ik'x'})F_1(\hat{k}) \tag{3.93}$$

where $F_1(\hat{k}) = \int dk_1 f_1(k_1,\hat{k})$. By integrating over k' first and then k we obtain

$$\int \frac{d\omega}{2\pi}\frac{1}{2\hat{k}} iF_1(\hat{k})(e^{i\hat{k}(x'-x)} + \hat{K}e^{-i\hat{k}(x+x')}). \tag{3.94}$$

The value of $F_1(\hat{k})$ can be calculated and it is found that, as must be the case, it is simply a number, not dependent on $\hat{k}$. The value is

$$F_1 = \left(-\frac{1}{36\sqrt{3}} - \frac{1}{12\pi}\right)\beta^2. \tag{3.95}$$

Notice that this term has changed our coefficient of $e^{i\hat{k}(x'-x)}$. We must perform a wavefunction renormalisation in order to return the coefficient of this term to unity, or in other words to cancel out the first term in (3.94). This can be done by rescaling Φ and $\overline{\Phi}$ by

$$\Phi\overline{\Phi} \to (1 - F_1)\Phi\overline{\Phi}. \tag{3.96}$$

This rescales the propagator by the same amount, cancelling the entirety of (3.94). It can thus be seen that renormalisation allows us to completely discard the fifth term of (3.88). Notice also that the rescaling (3.96) changes the boundary potential, so we must add in an equivalent term to cancel out this contribution.

The calculations from here are tedious and it is worth noting that as before, considerable simplifications can be made by using the values of k and k' given by their poles and simplifying the integrand. By adding all type III integrands, discarding the elliptic parts (which it can be checked are always completely real with respect to the phase of K), and considering only those parts even in k_1, we finally obtain

$$\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t'-t)} e^{-i\hat{k}(x+x')} \frac{1}{2\hat{k}} \frac{-i(2y^2+y+2)(y^2+3y+1)(y-1)}{(2i\hat{k}-3)^2 4y(y^2+1)(y+1)}. \tag{3.97}$$

Adding this result to those of the type I integrals yields a total contribution

$$\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t'-t)} e^{-i\hat{k}(x+x')} \frac{1}{2\hat{k}} \frac{-i(4y^2+5y+4)(y-1)}{(2i\hat{k}-3)^2 4y(y+1)} \tag{3.98}$$

and hence the naïve β^2 correction to the reflection factor is

$$K^{\beta^2} = \frac{-i(4y^2+5y+4)(y-1)}{(2i\hat{k}-3)^2 4y(y+1)}\beta^2. \tag{3.99}$$

In the next section we shall show that it is necessary to carry out a further finite renormalisation in order to make sense of this result.

3.6 Interpretation

We now wish to test whether our answer obeys the reflection bootstrap equation (1.81). At $O(\beta^2)$, this can be written for $a_2^{(1)}$ affine Toda field theory (taking $K_a^q = K_a + K_a^{\beta^2} + O(\beta^4)$ and $S_{ab} = 1 + S_{ab}^{\beta^2} + O(\beta^4)$):

$$\begin{aligned} K_2^{\beta^2}(\theta) &= K_1^{\beta^2}\left(\theta + i\frac{\pi}{3}\right) K_1\left(\theta - i\frac{\pi}{3}\right) + K_1\left(\theta + i\frac{\pi}{3}\right) K_1^{\beta^2}\left(\theta - i\frac{\pi}{3}\right) \\ &+ K_1\left(\theta + i\frac{\pi}{3}\right) S_{11}^{\beta^2}(2\theta) K_1\left(\theta - i\frac{\pi}{3}\right). \end{aligned} \tag{3.100}$$

Here, we use subscripts to denote particle type.

Let us assume that the quantum reflection factors for particles 1 and 2 are equal (and hence that $K_1^{\beta^2} = K_2^{\beta^2}$), as would seem sensible since they have equal classical limits and identical calculations for the $O(\beta^2)$ correction. Then it is found that the $O(\beta^2)$ correction (3.99) calculated above does not obey the reflection bootstrap equation. This would appear to be a very severe problem, since it would imply that the theory is not quantum integrable. However, let us consider adding a finite counter term of the form $\alpha\beta^2\Phi\overline{\Phi}$, where α is some coefficient, into the boundary potential. It can quickly be shown that this yields a contribution

$$-i\alpha\beta^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-i\hat{k}(x+x')} \frac{1}{2\hat{k}} 4\sqrt{3} \frac{y^2-1}{y(2i\hat{k}-3)^2}. \tag{3.101}$$

Notice that this does not change the *form* of the propagator, but merely adds another contribution to the $O(\beta^2)$ correction to K. Hence we have a freedom to add in such a counter term and change our $O(\beta^2)$ result by the according amount;

$$4\sqrt{3}i\alpha \frac{y^2-1}{y(2i\hat{k}-3)^2}\beta^2. \tag{3.102}$$

Suppose that we call our result (3.99) of the perturbative calculation f, and the correction (3.102) above αg. Now suppose that $f + \alpha g$ obeys the bootstrap. Then we can find α by rearranging (3.100), i.e.

$$\alpha = \frac{f(\theta) - K(\theta + i\frac{\pi}{3}) f(\theta - i\frac{\pi}{3}) - K(\theta - i\frac{\pi}{3}) f(\theta + i\frac{\pi}{3}) - K(\theta + i\frac{\pi}{3}) S_{11}^{\beta^2}(2\theta) K(\theta - i\frac{\pi}{3})}{K(\theta + i\frac{\pi}{3}) g(\theta - i\frac{\pi}{3}) - K(\theta - i\frac{\pi}{3}) g(\theta + i\frac{\pi}{3}) - g(\theta)}. \tag{3.103}$$

If this gives α as a number (as opposed to a function of y) then we know that we can satisfy the reflection bootstrap equation in this way. Moreover, it is clear that at most one such α exists, and thus we cannot obtain more than one consistent reflection factor by means of different boundary renormalisations. In fact, using our β^2 correction (3.99), we find $\alpha = \frac{1}{16\sqrt{3}}$, which gives our total $O(\beta^2)$ correction (i.e. $f + \alpha g$) as

$$\frac{-3i(y^3-1)}{4y(y+1)(2i\hat{k}-3)^2}\beta^2. \tag{3.104}$$

This correction satisfies the reflection bootstrap equations (3.100) to second order in β, and is the main result of this section.

The idea that a finite renormalisation of the boundary parameters A_i is necessary to retain integrability of $a_n^{(1)}$ affine Toda field theory is not new. Penati et al. [83, 84] have discussed the renormalisation of $a_2^{(1)}$ and their results agree qualitatively with those presented here.

We can now move on to considering the possible candidates for the exact form of the reflection factor. A possible exact reflection factor (postulated in [46] for the boundary condition being considered here) which has the correct classical limit, obeys the reflection bootstrap equation (1.81), and appears to be minimal, is

$$K^q = \frac{(3-\frac{B}{2})}{(1-\frac{B}{2})(2)} \tag{3.105}$$

with $B(\beta)$ as defined in (1.49).

In fact, as we saw in section 1.5.3, further evidence that this is indeed the correct reflection factor was provided by Gandenberger [53] using a method based on analytically-continued breather reflection matrices in the imaginary-coupling theory. Moreover, we find that expanding (3.105) in powers of β gives the same $O(\beta^2)$ correction as (3.104) above. Thus our perturbative answer is in agreement with the exact reflection factor (3.105) found by other methods and is a highly non-trivial check of these results.

There are of course other exact reflection factors which obey the bootstrap, have the correct classical limit, and the same $O(\beta^2)$ quantum correction. These can be obtained by multiplying the minimal K matrix by CDD factors. We saw examples of such factors in section 1.5, but they will be examined in greater detail here. Similar ambiguities occur for the bulk S-matrix, where they have been resolved by a careful consideration of the poles on the physical strip that extra factors would introduce. Such an analysis applied to the reflection factors is beyond the scope of the present work. Instead, we will analyse the possible forms of such ambiguities and discuss their duality properties in the following section.

We shall consider additional factors of the form

$$F_{C,D} = \frac{(1-C)(1+C)(2-C)(2+C)}{(1-D)(1+D)(2-D)(2+D)} \tag{3.106}$$

where C and D are two functions of β which tend to the same limit as $\beta \to 0$. This has classical limit

$$\lim_{\beta\to 0} F_{C,D} = 1 \tag{3.107}$$

and satisfies

$$F_{C,D}(\theta) = F_{C,D}\left(\theta - i\frac{\pi}{3}\right) F_{C,D}\left(\theta + i\frac{\pi}{3}\right) \tag{3.108}$$

so that a new candidate reflection factor, formed by multiplying any previous solution by $F_{C,D}$, will also be a solution to the reflection bootstrap equation. By choosing the functions C and D carefully we can ensure that the $O(\beta)$ term vanishes (needed in order to fit with perturbation theory which predicts that there be no $O(\beta)$ term). The necessary condition is

$$\frac{dC}{d\beta}(0) = \frac{dD}{d\beta}(0). \tag{3.109}$$

We can also make the $O(\beta^2)$ term disappear. For the case where C and $D \to 0$ as $\beta \to 0$, (3.109) is a sufficient condition.

For simplicity, let us consider only cases where C and D take the form $\frac{n}{2} \pm \frac{B}{2}$ where n is an integer. Whilst there are of course many other possible forms for C and D, it is blocks of the type $(\frac{n}{2} \pm \frac{B}{2})$ which are most commonly postulated to make up the exact reflection factors. For these cases we have $\frac{dC}{d\beta}(0) = \frac{dD}{d\beta}(0) = 0$ and find that there are four fundamental factors from which all others can be generated. These are given in table 3.1.

Hence we are free to multiply our previous solution (3.105) by any prefactor consisting of powers of F_1, F_2, F_3 or F_4, in order to give us a new solution to the reflection bootstrap equations (1.81). Moreover, if the prefactor consists only of powers of F_1 and F_3 then this new solution will not be distinguishable from our previous solution

Factor	Form	$O(\beta^2)$ term	Classical limit of dual
$F_1 = F(\frac{B}{2},0)$	$\frac{(1-\frac{B}{2})(1+\frac{B}{2})(2-\frac{B}{2})(2+\frac{B}{2})}{(1)^2(2)^2}$	0	$\frac{(3)}{(1)(2)}$
$F_2 = \frac{F(\frac{B}{2},0)}{F(2+\frac{B}{2},2)}$	$\frac{(\frac{B}{2})(1-\frac{B}{2})(1+\frac{B}{2})^2(2-\frac{B}{2})^2(2+\frac{B}{2})(3-\frac{B}{2})}{(1)^3(2)^3(3)}$	$\frac{iy(y^4+1)}{2(y^6-1)}$	1
$F_3 = \frac{F(\frac{5}{2}+\frac{B}{2},\frac{5}{2})^2}{F(\frac{3}{2}+\frac{B}{2},\frac{3}{2})}$	$\frac{(\frac{1}{2}-\frac{B}{2})(\frac{1}{2}+\frac{B}{2})(\frac{3}{2}-\frac{B}{2})^2(\frac{3}{2}+\frac{B}{2})^2(\frac{5}{2}-\frac{B}{2})(\frac{5}{2}+\frac{B}{2})}{(\frac{1}{2})^2(\frac{3}{2})^4(\frac{5}{2})^2}$	0	$(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})$
$F_4 = F(\frac{5}{2}+\frac{B}{2},\frac{5}{2})$	$\frac{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}{(\frac{1}{2}+\frac{B}{2})(\frac{3}{2}-\frac{B}{2})(\frac{3}{2}+\frac{B}{2})(\frac{5}{2}-\frac{B}{2})}$	$\frac{-iy(y^2-1)\sqrt{3}}{6(y^4-y^2+1)}$	1

Table 3.1: Factors with which to generate new solutions to the reflection bootstrap equations.

by its $O(\beta^2)$ correction. The differences between these solutions leads us naturally into the question of duality.

3.7 Duality

As we saw in section 1.3, the affine Toda theories possess a non-perturbative weak-strong coupling duality, whereby the bulk S-matrix is left invariant under $\beta \to 4\pi/\beta$. It is plausible that the theory with a boundary shares this symmetry. One way that this could be realised is if K itself is self-dual; we saw in section 1.5.1 that this was advocated by Kim [74]. It is also possible however that the symmetry is realised in a more subtle manner, and that under duality a theory with one boundary condition is mapped to a theory with a second boundary condition.

Let us suppose that the correct exact reflection factor is that given by (3.105). The dual of this minimal reflection factor is

$$\frac{(2+\frac{B}{2})}{(\frac{B}{2})(2)}. \tag{3.110}$$

First, note that this obeys the reflection bootstrap equation, as it clearly must as the scattering matrix S is self-dual. We can now ask whether this corresponds to any known reflection factor. Since the classical limit is unity, this cannot correspond to the reflection factor associated with any of the boundary conditions (1.58), but it could correspond to the reflection factor associated with the Neumann boundary condition. Kim's perturbative calculation for the Neumann boundary condition which we reviewed in section 3.2 determined an $O(\beta^2)$ correction to the classical reflection factor (3.19) which is in agreement with (3.110). Kim, however, concentrates on the assumption that the reflection factor must be self-dual, and hence proposes a different exact form from the above. However, it is interesting to note that these results are consistent with the $---$ and Neumann boundary conditions being related by a duality transformation.

How would this conclusion be changed if we were to consider a non-minimal reflection factor? Suppose we multiply the minimal solution (3.105) by F_1, F_1^{-1} and F_1F_3, leading to the reflection factors

$$\frac{(1+\frac{B}{2})(2-\frac{B}{2})(2+\frac{B}{2})(3-\frac{B}{2})}{(1)^2(2)^3}, \tag{3.111}$$

$$\frac{(1)^2(2)(3-\frac{B}{2})}{(1-\frac{B}{2})^2(1+\frac{B}{2})(2-\frac{B}{2})(2+\frac{B}{2})}, \tag{3.112}$$

and

$$\frac{(\frac{1}{2})^2(\frac{3}{2})^4(\frac{5}{2})^2(1+\frac{B}{2})(2-\frac{B}{2})(2+\frac{B}{2})(3-\frac{B}{2})}{(\frac{1}{2}-\frac{B}{2})(\frac{1}{2}+\frac{B}{2})(\frac{3}{2}-\frac{B}{2})^2(\frac{3}{2}+\frac{B}{2})^2(\frac{5}{2}-\frac{B}{2})(\frac{5}{2}+\frac{B}{2})(1)^2(2)^3}. \tag{3.113}$$

respectively. These cannot be distinguished from each other and from (3.105) by the $O(\beta^2)$ term alone.

Look first at (3.111). It is easy to see that this is self-dual. On the other hand, (3.112) transforms into

$$\frac{(1)^2(2)(2+\frac{B}{2})}{(\frac{B}{2})^2(1+\frac{B}{2})(2-\frac{B}{2})(3-\frac{B}{2})} \tag{3.114}$$

which in the classical limit becomes

$$(1)(2)(3). \tag{3.115}$$

This is the classical reflection factor associated with the boundary condition where all the $A_i = 1$ i.e. $+++$. Finally, taking the classical limit of the dual to (3.113) we obtain

$$\frac{(\frac{1}{2})(\frac{3}{2})^2(\frac{5}{2})}{(1)(2)(3)} \tag{3.116}$$

which is the classical reflection factor associated with the $-++$ boundary condition and its cyclical permutations, as we saw in the previous chapter. Moreover, consistent reflection factors can be constructed whose duals do not correspond to *any* of the integrable boundary conditions. So even given the $O(\beta^2)$ correction to the classical reflection factor, we still cannot conclusively determine the duality properties of the exact quantum reflection factors.

We saw above that it is possible to suppose that the boundary conditions are related in pairs by a duality transformation. On the other hand, it may be that the reflection factors are self-dual, or have no duality symmetries. Indeed, even if we knew all the $O(\beta^2)$ corrections for all different boundary conditions, and could postulate consistent reflection factors which related these in pairs under duality transformations, it would still always be possible to generate self-dual reflection factors by use of F_1 and F_3 which would be equally valid. From another perspective, however, it seems that if we knew both the $O(\beta^2)$ correction and the mappings between the various boundary conditions under duality transformations, we should be able to pin down the exact quantum reflection factor of the theory.

What we have not considered is the use of more general C and D functions. By multiplying the minimal reflection factor by some $F_{C,D}$ with appropriate choices of C and D it may be possible to create more factors which fit with the perturbative answer. However we shall leave this possibility for future analysis.

3.8 Conclusions

The perturbative result above has been useful in that it has provided further strong evidence in support of the exact quantum reflection factor (3.105). This (and indeed the result found by Kim for the Neumann boundary condition [74]) is in agreement with the hypothesis that the $---$ and Neumann boundary conditions are related by a duality transformation. However, it should not be ignored that there exist other exact reflection factors, albeit non-minimal, which are also in agreement with our perturbative result, and have vastly different duality properties.

In order to place further bounds on the form of the exact reflection factor one could consider extending the perturbative calculations to $O(\beta^4)$ or higher; indeed, this shall be attempted in the following chapter for the case of sinh-Gordon theory with the Neumann boundary condition. It is expected that for $a_2^{(1)}$ this could involve prohibitively laborious computations. However, this would be expected to place strong constraints on the exact reflection factor since the ambiguities F_1 and F_3 contain non-zero $O(\beta^4)$ terms. Alternatively, it may also be possible to restrict possibilities for the reflection factors by considering the associated boundary bound states [85] and pole structures present, and this would be an interesting possibility for future work.

Chapter 4

The $O(\beta^4)$ calculation for sinh-Gordon theory

4.1 Introduction

In the previous chapter we saw that the calculation of the $O(\beta^2)$ term in the quantum reflection factor of $a_2^{(1)}$ affine Toda field theory with the $---$ boundary condition still left questions unanswered regarding its exact form. In this chapter we aim to consider the next order calculation to see if this sheds further light on the non-perturbative conjectures. Whilst it might have been interesting to continue with the calculation for $a_2^{(1)}$ theory, it would seem wise to first of all consider the simplest case to first of all get a feel for the techniques required. Indeed, it is found that the two-loop calculation is considerably more difficult than the one-loop case. Therefore in order to try to render the calculation tractable we will consider here quantum sinh-Gordon theory with the Neumann boundary condition. In this case perturbation theory is relatively simple for a number of reasons. Firstly, we still have a flat background solution (since $\sigma_0 = \sigma_1 = 0$ in (1.60)) and thus again have no vertices of odd-order to consider. In addition, the Green's function takes the simple form (3.2), as the classical reflection factor is unity. Also, since the boundary potential $\mathcal{B} = 0$, then all vertices are located in the bulk region.

However, this calculation is an interesting one to consider in its own right. It allows us to provide a check on the formula for the reflection factors suggested by Ghoshal

[86]. There, the reflection factors of breathers present in sine-Gordon theory were determined. These can be analytically-continued to give a formula for the reflection factors of the sinh-Gordon particle. The result is

$$K = \frac{(1)(2-\frac{B}{2})(1+\frac{B}{2})}{(1-E(\beta))(1+E(\beta))(1-F(\beta))(1+F(\beta))}, \tag{4.1}$$

where the functions E and F depend (in a way which has yet to be fully determined) on the boundary parameters. Furthermore, an analysis of the values of E and F was carried out in the specific case of the Neumann boundary condition. It was expected that in this case the reflection factor takes the simple form

$$K = \frac{(1+\frac{B}{2})}{(\frac{B}{2})(1)}, \tag{4.2}$$

corresponding to $E = 1 - \frac{B}{2}$ and $F = 0$.

The calculation in section 3.3 (performed originally by Corrigan [77]) provided a perturbative check on these results to $O(\beta^2)$. We extend this analysis to the next order to see if this gives us any further insight.

4.2 The Calculations

We saw in section 3.3 how this calculation was performed up to one-loop order. Hence we have already seen the Lagrangian (3.20) and the necessary expansion of the bulk potential as a power series in β, given in (3.22). So we can now start the $O(\beta^4)$ calculation. There are four diagrams which contribute at this order; they are given in fig. 4.1.

Before moving on to the calculations, let us write down the prefactors associated with each of these diagrams. These come from the combinatorial factors (which are calculated as usual by considering the Wick contractions) and the various vertex factors. We thus find that the prefactors for the four diagrams are -16, -16, $-\frac{32}{3}$ and $-2i$ respectively.

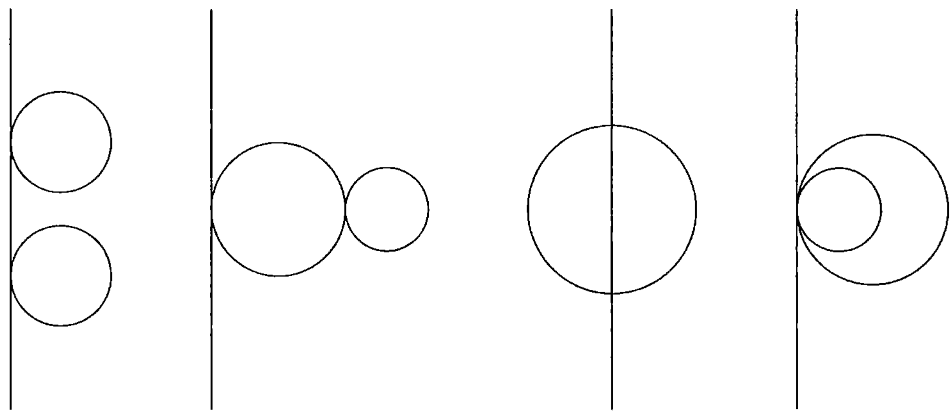

Figure 4.1: Types I, II, III and IV diagrams respectively.

4.2.1 The type I diagram

Writing down the integrals in the usual way, we see that for the type I diagram we need to calculate

$$-16\beta^4 \int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G(x,t;x_1,t_1)G(x_1,t_1;x_1,t_1)G(x_1,t_1;x_2,t_2)$$
$$G(x_2,t_2;x_2,t_2)G(x_2,t_2;x',t'). \quad (4.3)$$

Substituting in the propagators (3.24), this becomes;

$$-16\beta^4 \int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 \int \frac{d^2p d^2p' d^2p_1 d^2p_2 d^2p_3}{(2\pi)^{10}} \frac{i}{p^2-4+i\epsilon} \frac{i}{p'^2-4+i\epsilon}$$
$$\frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_2^2-4+i\epsilon} \frac{i}{p_3^2-4+i\epsilon} e^{-i\omega(t_1-t)} e^{-i\omega_2(t_2-t_1)} e^{-i\omega'(t'-t_2)}$$
$$\left(e^{ik(x_1-x)}+e^{-ik(x+x_1)}\right)\left(e^{ik'(x'-x_2)}+e^{-ik'(x'+x_2)}\right)\left(e^{ik_2(x_2-x_1)}+e^{-ik_2(x_1+x_2)}\right)$$
$$(1+e^{-2ik_1x_1})(1+e^{-2ik_3x_2}). \quad (4.4)$$

As we would expect (since it is a tadpole type diagram) this integral contains divergences. To remove these we need to perform an infinite mass renormalisation. In fact, it is not too difficult to see that the counter-term required to remove these divergent parts is exactly that already included to regularise the $O(\beta^2)$ correction. In other words, if we denote the counter-term interaction (coming from the term $I_2\beta^2\phi^2$, with I_2 defined in (3.40)) by a cross, then the divergent parts in (4.4) are cancelled by contributions arising from replacing either, or both, of the bubbles on diagram I by this cross. Since

$$\int_{-\infty}^{0} dx_1 dx_2 \int \frac{d^2p_1 d^2p_3}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_3^2-4+i\epsilon}(1+e^{-2ik_1x_1})(1+e^{-2ik_3x_2})$$
$$= \int_{-\infty}^{0} dx_1 dx_2 \int \frac{d^2p_1 d^2p_3}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_3^2-4+i\epsilon} \Big\{e^{-2ik_1x_1}e^{-2ik_3x_2} - 1$$
$$+(1+e^{-2ik_1x_1}) + (1+e^{-2ik_3x_2})\Big\} \qquad (4.5)$$

then it is not hard to see that the last three terms on the RHS are removed by these contributions, and hence we are simply left with the finite part

$$\int_{-\infty}^{0} dx_1 dx_2 \int \frac{d^2p_1 d^2p_3}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_3^2-4+i\epsilon} e^{-2ik_1x_1}e^{-2ik_3x_2}. \qquad (4.6)$$

We are now merely left with the task of computing this integral. We use the same techniques as in section 3.2. By extending the range of the x_1 and x_2 integrals to the whole line we are able to generate delta functions in the momenta. We can then continue as normal to give

$$-\frac{i\beta^4}{\hat{k}} \left\{\int \frac{d\omega}{2\pi} \frac{1}{2\hat{k}} e^{-i\omega(t'-t)} e^{-i\hat{k}(x'+x)}\right\} \int \frac{dk_1}{2\pi} \frac{1}{\omega^2-(\hat{k}-2k_1)^2-4+i\epsilon}$$
$$\left\{\frac{1}{\sqrt{k_1^2+4}\sqrt{(k_1-\hat{k})^2+4}} + \frac{1}{k_1^2+4}\right\} \qquad (4.7)$$

with the usual $\hat{k} = \sqrt{\omega^2-4}$. The first term in the second bracket generates elliptic integrals as well as pole pieces. We shall for the moment ignore such terms; they shall be treated separately in section 4.2.5. The second term can be computed using the usual contour integral, giving a contribution to the reflection factor of

$$-\frac{\beta^4}{16\omega^2}\left(\frac{1}{2\hat{k}^2}(\hat{k}^2+8) + \frac{i}{\hat{k}}\right). \qquad (4.8)$$

4.2.2 The type II diagram

The required integral for this diagram is

$$-16\beta^4 \int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G(x,t;x_1,t_1)G(x_1,t_1;x_2,t_2)G(x_1,t_1;x_2,t_2)$$
$$G(x_2,t_2;x_2,t_2)G(x_1,t_1;x',t'). \qquad (4.9)$$

This integral contains an infinite piece which is again removed by the $O(\beta^2)$ infinite mass counter term, this time by replacing the bubble by a cross.

Using the normal integration techniques again, we can obtain;

$$-2\beta^4 \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2-4+i\epsilon} \frac{i}{\omega^2-k'^2-4+i\epsilon} e^{-i\omega(t'-t)} (e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'})$$
$$\int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2+4}} \frac{i}{k_1^2-(k_1-k-k')^2+i\epsilon} \left\{ \frac{1}{\sqrt{\frac{(k+k')^2}{4}+4}} + \frac{1}{\sqrt{\frac{(2k_1-k-k')^2}{4}+4}} \right\}. \quad (4.10)$$

Notice that this equation appears to contain a double pole in the external propagators. We can see this if we simply substitute in the poles $k=-k'=\pm\hat{k}$ in the second integral. We therefore expect to remove this double pole with a finite mass renormalisation. We saw in the previous chapter that such a renormalisation adds a term of the form

$$-2i\alpha \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2-4+i\epsilon} \frac{i}{\omega^2-k'^2-4+i\epsilon} \frac{i}{k+k'+i\epsilon} e^{-i\omega(t'-t)}$$
$$(e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'}) \quad (4.11)$$

or, equivalently,

$$-i\alpha \int \frac{d^2p dk'}{(2\pi)^2} \frac{i}{p^2-4+i\epsilon} \frac{i}{\omega^2-k'^2-4+i\epsilon} \delta(k+k') e^{-i\omega(t'-t)}$$
$$(e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'}) \quad (4.12)$$

to the $O(\beta^4)$ correction. Note that (4.11) is very similar to our case above — we must simply take

$$\alpha = -\frac{1}{2} \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2+4}} \frac{1}{k_1+i\epsilon} \left\{ \frac{1}{2} + \frac{1}{\sqrt{k_1^2+4}} \right\} \quad (4.13)$$

to cancel off the double pole in (4.10).

In actual fact, the α given above can be calculated and is found to give zero; no finite mass renormalisation is in fact required. This is exactly as we expect since such a term is not required by the renormalisation for the whole-line. However, for the purposes of our calculation it is easier to add in such a (zero) counter term to permit some cancellations in the integrand.

We can now substitute in the values of k and k' at their poles. The cases $k=k'=\pm\hat{k}$ give

$$-\frac{\beta^4}{2\hat{k}^2}\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k}}e^{-i\omega(t'-t)}e^{-i\hat{k}(x'+x)}\right\}\int\frac{dk_1}{2\pi}\frac{i}{k_1-\hat{k}+i\epsilon}$$
$$\left\{\frac{1}{\omega\sqrt{k_1^2+4}}+\frac{1}{\sqrt{k_1^2+4}\sqrt{(k_1-\hat{k})^2+4}}\right\}. \tag{4.14}$$

Note that the second term in the second bracket again generates elliptic integrals, whilst the first term is a branch cut integral — these shall both be discussed later. However, we *will* look here at the poles in k_1 present in the latter. In fact, we will see in section 4.2.6 that in order to calculate the branch cut integrals it is easiest to make the integrand even in k_1. So the first term becomes

$$-\frac{\beta^4}{4\hat{k}^2}\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k}}e^{-i\omega(t'-t)}e^{-i\hat{k}(x'+x)}\right\}\int\frac{dk_1}{2\pi}\frac{1}{\omega\sqrt{k_1^2+4}}\left\{\frac{i}{k_1-\hat{k}+i\epsilon}-\frac{i}{k_1+\hat{k}-i\epsilon}\right\} \tag{4.15}$$

which gives a pole contribution of

$$-\frac{\beta^4}{4\omega^2\hat{k}^2}. \tag{4.16}$$

We can find the contribution coming from the cases $k=-k'=\pm\hat{k}$ by Taylor expanding about $k=-k'$. This yields;

$$-\frac{i\beta^4}{4\hat{k}}\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k}}e^{-i\omega(t'-t)}e^{-i\hat{k}(x'+x)}\right\}\int\frac{dk_1}{2\pi}\left\{\frac{1}{\sqrt{k_1^2+4}(k_1+i\rho)^2}\right.$$
$$\left.+\frac{4}{(k_1^2+4)(k_1+i\rho)^2}-\frac{8}{(k_1^2+4)^2(k_1+i\rho)^2}\right\}. \tag{4.17}$$

The first term in the above contains a branch cut integral. This can be worked out in the usual way, giving

$$\frac{i\beta^4}{16\pi\hat{k}}. \tag{4.18}$$

The other terms only contain pole contributions. These give

$$\frac{i\beta^4}{64\hat{k}}. \tag{4.19}$$

Indeed, we can back up this technique by considering the $k=\pm k'$ terms from the start and performing the calculation with no renormalisation or Taylor expansions. This of course yields the same results, as we would expect.

4.2.3 The type III diagram

The integral corresponding to diagram III is

$$-\frac{32\beta^4}{3}\int_{-\infty}^{0} dx_1 dx_2 \int_{-\infty}^{\infty} dt_1 dt_2 G(x,t;x_1,t_1)G(x_1,t_1;x_2,t_2)G(x_1,t_1;x_2,t_2)$$
$$G(x_1,t_1;x_2,t_2)G(x_2,t_2;x',t'). \quad (4.20)$$

This diagram does not contain any tadpole type parts, and no infinite renormalisation is required this time. We proceed as normal until getting two terms:

$$-\frac{16\beta^4}{3}\int \frac{d^2p dk'}{(2\pi)^2}\frac{i}{p^2-4+i\epsilon}\frac{i}{\omega^2-k'^2-4+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'})$$
$$\delta(k+k')\int\frac{d^2p_1 d^2p_2 d^2p_3}{(2\pi)^4}\frac{i}{p_1^2-4+i\epsilon}\frac{i}{p_2^2-4+i\epsilon}\frac{i}{p_3^2-4+i\epsilon}$$
$$\delta(k_1+k_2+k_3-k')\delta(\omega_1+\omega_2+\omega_3-\omega) \quad (4.21)$$

and

$$-16\beta^4\int \frac{d^2p dk'}{(2\pi)^3}\frac{i}{p^2-4+i\epsilon}\frac{i}{\omega^2-k'^2-4+i\epsilon}e^{-i\omega(t'-t)}(e^{ikx}+e^{-ikx})(e^{ik'x'}+e^{-ik'x'})$$
$$\int\frac{d^2p_1 d^2p_2 d^2p_3}{(2\pi)^3}\frac{i}{p_1^2-4+i\epsilon}\frac{i}{p_2^2-4+i\epsilon}\frac{i}{p_3^2-4+i\epsilon}\delta(k_1+k_2-k_3-k')$$
$$\delta(k_1+k_2+k_3+k)\delta(\omega_1+\omega_2+\omega_3-\omega). \quad (4.22)$$

The first term clearly appears to contain a double pole in the external propagator. To cope with this, we should expand

$$\int\frac{d^2p_1 d^2p_2 d^2p_3}{(2\pi)^4}\frac{i}{p_1^2-4+i\epsilon}\frac{i}{p_2^2-4+i\epsilon}\frac{i}{p_3^2-4+i\epsilon}\delta(k_1+k_2+k_3-k')$$
$$\delta(\omega_1+\omega_2+\omega_3-\omega) \quad (4.23)$$

around $k'^2=\hat{k}^2$ (the integral is even in k') as we did for the type III diagram of $a_2^{(1)}$ theory. The zeroth order term in this expansion will then, assuming it turns out to be independent of $\hat{k}$, be cancelled by a finite mass renormalisation (like (4.12)) whilst the first order will be removed by a wavefunction renormalisation. Higher order terms in the expansion are irrelevant since they integrate to zero. We have yet to explicitly show that these two integrals are indeed independent of $\hat{k}$, but this is expected to be the case. As considered in section 3.2, we can argue this by considering the renormalisation which keeps the correct form of the whole-line propagator. In this case it is easy to show that the whole-line contribution to the internal integrals of diagram III is exactly double that of expression (4.21) and thus the counter-terms required to renormalise the half-line theory are exactly those required by the theory on the whole-line.

Now let us consider the second integral (4.22). Continuing in the normal way, we obtain

$$-2\beta^4 \int \frac{d^2p dk'}{(2\pi)^3} \frac{i}{p^2 - 4 + i\epsilon} \frac{i}{\omega^2 - k'^2 - 4 + i\epsilon} e^{-i\omega(t'-t)} (e^{ikx} + e^{-ikx})(e^{ik'x'} + e^{-ik'x'})$$
$$\int \frac{dk_1}{2\pi} \left\{ \frac{1}{\hat{\omega}_1 \hat{\omega}_2} \frac{i}{(\omega - \hat{\omega}_1 - \hat{\omega}_2)^2 - \hat{\omega}^2 + i\epsilon} + \frac{2}{\hat{\omega}_1 \hat{\omega}} \frac{i}{(\hat{\omega}_1 - \hat{\omega} - \omega)^2 - \hat{\omega}_2{}^2 + i\epsilon} \right.$$
$$\left. + \frac{1}{\hat{\omega}_1 \hat{\omega}} \frac{i}{(\hat{\omega}_1 + \hat{\omega} + \omega)^2 - \hat{\omega}_2{}^2 + i\epsilon} \right\} \tag{4.24}$$

where for convenience we have used the notation

$$\hat{\omega} = \sqrt{\frac{(k+k')^2}{4} + 4},\ \hat{\omega}_1 = \sqrt{k_1^2 + 4} \text{ and } \hat{\omega}_2 = \sqrt{\left(\frac{k'-k}{2} - k_1\right)^2 + 4}. \tag{4.25}$$

The first term in the above is in part elliptic. We need to separate this part from the normal branch cut parts in an analogous manner to the previous chapter.

The other terms contain branch cut integrals and pole contributions. We must be particularly careful in looking at the poles in the integrand in order to determine on which side of the real line they lie. Doing this we find the contributions coming from poles at $k_1 = \pm\hat{k}$ and $k_1 = 0$ to be

$$-\frac{3\beta^4}{4\omega^2 \hat{k}^2}(2 + \omega^2) \tag{4.26}$$

whilst there is also an extra contribution, arising from a pole at $k_1 = 2i$ in the first term of (4.24) when $k = k' = \pm\hat{k}$, which gives

$$\frac{i\beta^4}{8\omega^2 \hat{k}}. \tag{4.27}$$

In addition, this term also contains the integral

$$\frac{i\beta^4}{2\omega\hat{k}} \int \frac{dk_1}{2\pi} \frac{1}{(k_1^2 + 4)^{\frac{3}{2}}}, \tag{4.28}$$

which (by putting $k_1 = 2\sinh\eta$) gives the result

$$\frac{i\beta^4}{8\pi\omega\hat{k}}. \tag{4.29}$$

4.2.4 The type IV diagram

The required integral here is

$$-2i\beta^4 \int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 G(x,t;x_1,t_1)G(x_1,t_1;x_1,t_1)G(x_1,t_1;x_1,t_1) \\ G(x_1,t_1;x',t') \tag{4.30}$$

which becomes

$$-2i\beta^4 \int_{-\infty}^{0} dx_1 \int_{-\infty}^{\infty} dt_1 \int \frac{d^2p d^2p' d^2p_1 d^2p_2}{(2\pi)^8} \frac{i}{p^2-4+i\epsilon} \frac{i}{p'^2-4+i\epsilon} \\ \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_2^2-4+i\epsilon} e^{-i\omega(t_1-t)} e^{-i\omega'(t'-t_1)} (e^{ik(x_1-x)} + e^{-ik(x+x_1)}) \\ (e^{ik'(x'-x_1)} + e^{-ik'(x'+x_1)})(1+e^{-2ik_1x_1})(1+e^{-2ik_2x_1}) \tag{4.31}$$

when the form of the propagator is used.

To cancel the divergences we must add in two extra infinite renormalisations, this time at $O(\beta^4)$. In a similar way to section 4.2.1, let us write;

$$\int_{-\infty}^{0} dx_1 \int \frac{d^2p_1 d^2p_2}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_2^2-4+i\epsilon} \left\{(1+e^{-2ik_1x_1})(1+e^{-2ik_2x_1})\right\} \\ = \int_{-\infty}^{0} dx_1 \int \frac{d^2p_1 d^2p_2}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_2^2-4+i\epsilon} \left\{e^{-2ik_1x_1} e^{-2ik_2x_1} - 1 \right. \\ \left. +(1+e^{-2ik_1x_1}) + (1+e^{-2ik_2x_1})\right\}. \tag{4.32}$$

The last two terms on the RHS integrate to the same value, and are cancelled by a renormalisation of the ϕ^4 coefficient, say $I_4\beta^4\phi^4$. In fact, the coefficient is

$$I_4 = -\frac{1}{3} \int \frac{d^2p_1}{(2\pi)^2} \frac{i}{p_1^2-4+i\epsilon}. \tag{4.33}$$

It is interesting that

$$I_4 = \frac{1}{6} I_2. \tag{4.34}$$

so that these coefficients of ϕ^2 and ϕ^4 are in the same ratio as those in bulk sinh-Gordon potential

$$V = \frac{1}{\beta^2}(e^{\beta\sqrt{2}\phi} + e^{-\beta\sqrt{2}\phi} - 2). \tag{4.35}$$

The other renormalisation required (to remove the second term of the RHS of (4.32)) is one of the ϕ^2 coefficient. This time we must add a counter-term $J_2\beta^4\phi^2$, say, where

$$J_2 = \int \frac{d^2p_1 d^2p_2}{(2\pi)^4} \frac{i}{p_1^2-4+i\epsilon} \frac{i}{p_2^2-4+i\epsilon}. \tag{4.36}$$

Then standard techniques give us:

$$-\frac{i\beta^4}{8\hat{k}}\left\{\int\frac{d\omega}{2\pi}\frac{1}{2\hat{k}}e^{-i\omega(t'-t)}e^{-i\hat{k}(x'+x)}\right\}\int\frac{dk_1}{2\pi}\left\{\frac{1}{\sqrt{k_1^2+4}\sqrt{(k_1-\hat{k})^2+4}}+\frac{1}{k_1^2+4}\right\}. \tag{4.37}$$

Again the first term is elliptic, and will be discussed in section 4.2.5. The second term is simple to analyse, giving a contribution to K of

$$-\frac{i\beta^4}{32\hat{k}}. \tag{4.38}$$

4.2.5 Elliptic Parts

Consider a k_1 integral of type

$$\int\frac{dk_1}{2\pi}\frac{1}{\hat{\omega}_1\hat{\omega}_2}f(k_1) \tag{4.39}$$

where the function $f(k_1)$ only contains poles in k_1, and we use the notation

$$\hat{\omega_1}=\sqrt{k_1^2+4}\text{ and }\hat{\omega_2}=\sqrt{(k_1-\hat{k})^2+4}. \tag{4.40}$$

Putting $k_1 \to iy$ in the above allows this to be written as the sum of elliptic integrals along the two branch cuts, and pole parts from $f(k_1)$. The contour used is shown in fig. 4.2. In addition, any infinitesimal parts present in the poles of $f(k_1)$ (at least in the cases we shall be considering) will be irrelevant to the calculation of the elliptic branch cut integrals. Using this, we show below that the elliptic integrals from the four diagrams all cancel.

From the first diagram, we have

$$-\frac{i\beta^4}{\hat{k}}\int\frac{dk_1}{2\pi}\frac{1}{\hat{\omega_1}\hat{\omega_2}}\frac{1}{\omega^2-(\hat{k}-2k_1)^2-4+i\epsilon}$$

Writing this as partial fractions and shifting one of the k_1 integrals, we obtain

$$\frac{i\beta^4}{2\hat{k}^2}\int\frac{dk_1}{2\pi}\frac{1}{\hat{\omega_1}\hat{\omega_2}}\frac{1}{k_1-\hat{k}-i\epsilon}. \tag{4.41}$$

This gives a pole contribution (integrating around the upper half-plane) of

$$-\frac{\beta^4}{4\omega\hat{k}^2}. \tag{4.42}$$

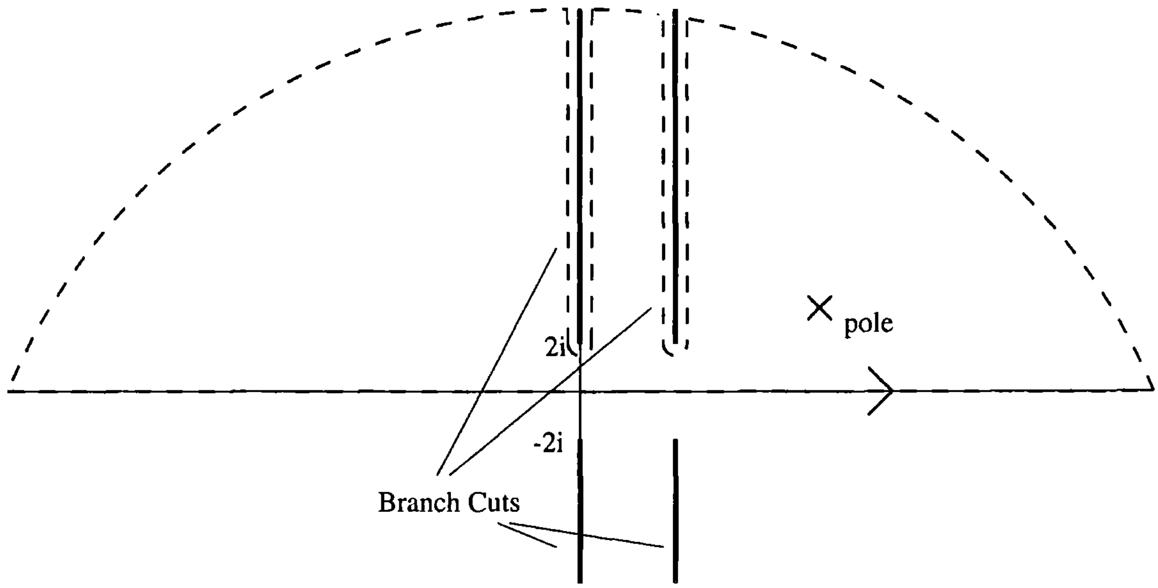

Figure 4.2: Integrating around the two branch cuts in the upper half-plane.

Similarly from the second diagram, we have

$$\frac{-i\beta^4}{4\hat{k}^2}\int\frac{dk_1}{2\pi}\frac{1}{\hat{\omega}_1\hat{\omega}_2}\frac{1}{k_1-\hat{k}+i\epsilon}, \tag{4.43}$$

which gives a zero pole contribution. Putting the infinitesimals in the above integrands to zero it is clear that these two elliptic parts exactly cancel.

Diagram III is more difficult to analyse. We must split off the elliptic part from the normal branch cut parts. It can then be shown that the elliptic part, ignoring infinitesimals, becomes

$$\frac{i\beta^4}{2\hat{k}}\int\frac{dk_1}{2\pi}\frac{1}{4\hat{\omega}_1\hat{\omega}_2}. \tag{4.44}$$

This exactly cancels with the elliptic part from diagram IV, which is minus the above. These integrals contain no pole parts to consider.

4.2.6 Branch Cut integrals

These appear from diagrams II and III. To find them we separate off the parts which are odd in ω_1 (in the case of the first term of (4.24) we require the terms which are odd in *either* ω_1 *or* ω_2; we can perform a change of variables which switches the labels 1 and 2) and make all integrals even in k_1. With a little manipulation, we find a total from all diagrams of

$$\beta^4 \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2+4}} \frac{i}{(k_1-\hat{k})(k_1+\hat{k})} \left\{ \frac{1}{2\omega\hat{k}} + \frac{\omega}{8\hat{k}} \right\} \tag{4.45}$$

which integrates along the branch cut to

$$\frac{i\beta^4(\omega^2+4)\theta}{4\omega^2\hat{k}^2}. \tag{4.46}$$

This does not appear to be a very satisfactory result. It is not periodic in θ and hence is not a desirable term to remain in the $O(\beta^4)$ correction.

4.3 Results

Let us now proceed to sum the above contributions to the $O(\beta^4)$ correction. The contributions to K fall into three distinct parts. Firstly, there is the real part. Contributions to this term come from (4.8), (4.16), (4.26) and (4.42) to give

$$-\frac{\beta^4}{32\omega^2\hat{k}^2}(25\omega^2+8\omega+60). \tag{4.47}$$

This real part causes us problems, as we shall shortly see.

Secondly, there is the imaginary part which contains π in the denominator. There are two contributions to this term, given in (4.18) and (4.29). These add up to give

$$\frac{i(2+\omega)}{16\pi\omega\hat{k}}. \tag{4.48}$$

Finally, the third distinct part of the answer is that which is imaginary but has no π denominator. Adding the terms from (4.8), (4.19), (4.27) and (4.38), gives

$$\frac{i\beta^4}{8\omega^2\hat{k}} - \frac{i\beta^4}{16\omega^2\hat{k}} + \frac{i\beta^4}{64\hat{k}} - \frac{i\beta^4}{32\hat{k}} \tag{4.49}$$

which becomes

$$-\frac{i\beta^4\hat{k}}{64\omega^2}. \tag{4.50}$$

In addition, we recall that there is also the undesirable θ contribution (4.46) to this part.

We can write the total answer therefore (adding all the contributions (4.47), (4.48) and (4.50)), plus the non-periodic piece (4.46) as a total $O(\beta^4)$ correction of

$$\left(-\frac{(2+\omega)^2}{32\hat{k}^2\omega^2} - \frac{i\hat{k}}{64\omega^2} + \frac{i(2+\omega)}{16\pi\omega\hat{k}}\right)\beta^4 - \frac{1}{8\omega^2\hat{k}^2}(6\omega^2+\omega+14-2(\omega^2+4)i\theta)\beta^4. \quad (4.51)$$

Let us consider the properties we expect the result to have. As in the previous chapter, unitary implies that the reflection factor be a pure phase, $e^{i\chi}$, where χ is a function of β^2. So putting

$$\chi = \chi_0 + \beta^2\chi_2 + \beta^4\chi_4 + O(\beta^6) \quad (4.52)$$

we expand

$$e^{i\chi} = e^{\chi_0}(1 + i\beta^2\chi_2 + \beta^4(i\chi_4 - \frac{1}{2}\chi_2^2) + O(\beta^6)). \quad (4.53)$$

Comparison with the classical result tells us $e^{\chi_0} = 1$ whilst the $O(\beta^2)$ result from the previous chapter, (3.51), says

$$\chi_2 = -\frac{\omega+2}{4\omega\hat{k}}. \quad (4.54)$$

Hence we expect that the real part of the $O(\beta^4)$ term be given by

$$-\frac{\beta^4}{2}\frac{(\omega+2)^2}{16\omega^2\hat{k}^2} \quad (4.55)$$

which is the real part in the first bracket of (4.51). Hence the extra real contribution, found in the second bracket of (4.51), causes problems as it spoils this property of the result.

Let us consider Ghoshal's postulated reflection factor (4.2). Expanding this as a power series in β we find that this proposed exact reflection factor has exactly the same $O(\beta^4)$ correction as that part of our perturbative result (4.51) contained within the first set of brackets. We may note here that the imaginary parts (both with and without the π denominator) of our perturbative answer and Ghoshal's exact reflection factor are equal, modulo the non-periodic piece proportional to θ.

4.4 Renormalisation

We saw above that our result is not in agreement with the postulated exact reflection factor. In section 3 we obtained a superficially similar result, yet we were able, by means of a finite renormalisation of the boundary potential, to obtain an answer which satisfied the boundary bootstrap equation and fitted with one of the postulated exact reflection factors. In the present case, there is no boundary bootstrap equation to help us but it worth looking at the possible renormalisations which can be undertaken and their effect on the perturbative answer.

Let us briefly summarise the renormalisations we have used so far. Infinite mass renormalisations have been used to control the divergences in the integrals. These have turned out to be of just the forms we would expect. In section 4.2.2 it initially seemed as if some extra *finite* mass renormalisation was required — however, this turned out to have zero coefficient. In addition, some finite mass and wavefunction renormalisation seem to be required in the computation of the type III integral. The coefficients of these have not been determined as they are the same as the whole-line case.

However, we have not considered any renormalisations of the boundary potential. It is plausible that, as in section 3.4, these could have an important part to play. We will examine here what would happen in the present case if such a renormalisation were included.

Let us consider a renormalisation of the two boundary parameters, σ_0 and σ_1. In fact, it is simplest to look here at the effects of a renormalisation which retains the equality of these two parameters. We shall calculate the contributory terms from such a renormalisation using a method suggested by Bowcock. Clearly the physical meaning of a reflection factor must be independent of the renormalisation scheme used. In other words, two reflection factors — one with and one without renormalisation — can be viewed as the same assuming that the former contains the "true" coupling constants of the theory while the latter contains the "bare" coupling constants. So if we write the "bare" reflection factor as $K_{bare}(\sigma)$ then the renormalised reflection factor, $K_{ren}(\sigma)$, is given by

$$\begin{aligned} K_{ren}(\sigma) &= K_{bare}(\sigma + \mu\beta^2 + \zeta\beta^4 + O(\beta^6)) \\ &= K_{bare}(\sigma) + \frac{\partial K_{bare}}{\partial \sigma}(\mu\beta^2 + \zeta\beta^4) + \frac{1}{2}\frac{\partial^2 K_{bare}}{\partial \sigma^2}\mu^2\beta^4 + O(\beta^6) \quad (4.56) \end{aligned}$$

assuming of course that K_{bare} (and the boundary parameter σ) is a function of β^2.

Hence we can easily find the extra contributions to the reflection factor given by such counter-terms if we know the dependence of the reflection factor on σ at lower orders. From section 3.3 we know that, up to $O(\beta^2)$ (and with the condition that $\sigma \geq 0$);

$$K_{bare}(\sigma) = \frac{i\hat{k}+2\sigma}{i\hat{k}-2\sigma} - \frac{i}{\hat{k}}\frac{i\hat{k}+2\sigma}{i\hat{k}-2\sigma}\left\{\frac{\hat{k}^2 a\sin(a\pi)}{\hat{k}^2+4\sigma^2} - \frac{1}{4} + \frac{1}{2\omega}\right\}\beta^2. \tag{4.57}$$

So the $O(\beta^2)$ counter-term contribution to sinh-Gordon theory with the Neumann boundary condition is given by

$$\left.\frac{\partial K_{bare}}{\partial\sigma}\right|_{\sigma=0}\mu\beta^2 = \left.\frac{\partial}{\partial\sigma}\frac{i\hat{k}+2\sigma}{i\hat{k}-2\sigma}\right|_{\sigma=0}\mu\beta^2 = -\frac{4i}{\hat{k}}\mu\beta^2. \tag{4.58}$$

Similarly the $O(\beta^4)$ term can be found from (4.56) and is

$$-\frac{4i}{\hat{k}}\zeta\beta^4 - \frac{8}{\hat{k}^2}\mu^2\beta^4 + \left\{\frac{i}{\pi\hat{k}} - \frac{\omega+2}{\omega\hat{k}^2}\right\}\mu\beta^4. \tag{4.59}$$

These terms could equally well have been calculated explicitly by adding in appropriate boundary counter-terms and considering their associated Feynman diagrams. However this approach lets us calculate all possible renormalisations in a simple and succinct way.

The real part of (4.59) is

$$-\frac{\omega+2}{\omega\hat{k}^2}\mu\beta^4 - \frac{8}{\hat{k}^2}\mu^2\beta^4, \tag{4.60}$$

whilst by adding in such a renormalisation the $O(\beta^2)$ term of the reflection factor has become

$$-\frac{i\beta^2(\omega+2)}{4\omega\hat{k}} - \frac{4i}{\hat{k}}\mu\beta^2. \tag{4.61}$$

Hence if the non-renormalised solution obeyed the unitarity condition then (4.60) is exactly the extra real part we require to maintain unitarity, for any value of μ. Conversely, we cannot obtain a solution which obeys unitarity from one that does not by means of such a renormalisation.

It would be interesting to consider possible renormalisations which do *not* retain the equality $\sigma_0 = \sigma_1$. However, in this case we do not know the general form of the reflection factor up to the $O(\beta^2)$ correction, so this possibility is not one which can be addressed at present.

4.5 Discussion

The result found above was not in agreement with several expected properties of the $O(\beta^4)$ correction. Firstly, the real part was not of the correct form to agree with unitarity. Secondly, the correction contained a term proportional to θ. This does not obey the required periodicity property of the reflection factor. Whilst a similar term exists in the reflection factor of supersymmetric sinh-Gordon theory [88], it is the opinion of the author that in the present case we can only conclude that some subtlety has been missed by our calculation. At present the reason for this is not known.

Some possible, if unaesthetic, explanations may be given. The discrepancy may arise if some extra renormalisation of the theory needs to be undertaken. One idea might be a renormalisation of the boundary potential using derivative counter-terms, which give momentum dependence, although these seem both difficult to justify and unlikely to generate the required contributions. We also noted in the previous section that we have not considered renormalisations which do not retain the equality of the two boundary parameters σ_0 and σ_1. This would imply some breaking of the $\mathbb{Z}_2$ symmetry by the quantum theory. If this were to occur, it would not only imply the existence of boundary counter-terms of odd-order in the field, but may also mean that the vacuum field configuration undergoes some small perturbation near the boundary. This could cause the existence of position dependent bulk counter-terms. A more detailed analysis would be required to see if this could indeed occur. Although these ideas may not seem attractive, they do at least give some areas for future study.

We noted before that the imaginary part of our perturbative answer was the same (disregarding the piece proportional to θ) as that of Ghoshal's exact reflection factor. This may hint that the full perturbative answer is in agreement with Ghoshal's result, though this is of course far from conclusive.

It is perhaps interesting to know, however, what we could deduce had a feasible $O(\beta^4)$ correction been produced by the calculation. Hence for the purposes of the remaining discussion, we shall consider what a perturbative result in agreement with

Ghoshal's factor (4.2) would have told us. However, we should not forget that we have *not* (perhaps as yet) obtained such a correction.

Consider (4.1). Putting

$$E = e_0 + e_1\frac{\beta^2}{4\pi} + e_2\left(\frac{\beta^2}{4\pi}\right)^2 + O(\beta^6) \text{ and } F = f_0 + f_1\frac{\beta^2}{4\pi} + f_2\left(\frac{\beta^2}{4\pi}\right)^2 + O(\beta^6) \quad (4.62)$$

and looking at the classical limit shows us that in the Neumann case we must have $e_0 = 1$ and $f_0 = 0$. In fact, it is expected [87] that, for the flat background case, $F = 0$ to all orders; however we shall here leave F unconstrained. Then our perturbative result for the $O(\beta^2)$ order contribution tells us that

$$\frac{i(\omega-2)}{4\omega\hat{k}} + \frac{i}{2\hat{k}}e_1 = -\frac{i(\omega+2)}{4\omega\hat{k}} - \frac{4i}{\hat{k}}\mu \quad (4.63)$$

or

$$e_1 = -1 - 8\mu. \quad (4.64)$$

So we proceed to the next order. The real part of course satisfies the equations automatically so no constraints are placed on the coefficients of E and F. Let us consider then the imaginary part. This is found to be

$$\frac{i\hat{k}}{64\omega^2}(2f_1^2 - 1) + \frac{i}{16\pi\omega\hat{k}}(\omega(2e_2 - 1) + 2). \quad (4.65)$$

Comparing this with the contribution which agrees with Ghoshal, plus the renormalisation pieces

$$-\frac{i\hat{k}}{64\omega^2} - \frac{4i}{\hat{k}}\zeta + \frac{i(2+\omega)}{16\pi\omega\hat{k}} + \frac{i}{\pi\hat{k}}\mu \quad (4.66)$$

yields the results

$$f_1^2 = -\frac{128\omega^2\zeta}{\hat{k}^2} \text{ and } e_2 = 1 + 8\mu. \quad (4.67)$$

No information is given about f_2 by this order in perturbation theory. Note that the first expression gives f_1 as a function of θ. This is not an attractive possibility as we would prefer all θ dependence to reside implicitly in the block forms (x). Hence we assume that the only acceptable solution is to take $\zeta = 0$.

Note that the values for e_1 and e_2 are consistent with E of the form

$$E = 1 - (1 + 8\mu)\frac{B}{2}. \quad (4.68)$$

The interesting point to note here is that we cannot distinguish between the possibilities for the boundary renormalisation μ. Whilst $\mu = 0$ is the most attractive option, and corresponds to the postulated exact Neumann reflection factor (4.2), there would be no reason from the perturbative point of view (at least to $O(\beta^4)$) to choose this result.

4.6 Duality

We now turn to discuss in this section how the various reflection factors present in the theory might be related under a duality transformation. This was considered at length in [77] but no solution in which these boundary conditions were related was found. In particular we aim to draw attention to the possibilities provided by the boundary renormalisations and their effect on duality properties.

The dual of the Neumann reflection factor (4.2) (given by sending $B \to 2 - B$ as before) is

$$\frac{(2-\frac{B}{2})}{(1-\frac{B}{2})(1)} \tag{4.69}$$

which, as shown in [77], has the same classical limit as the reflection factor belonging to the $\sigma = 1$ boundary condition. But how are the other values of σ related under a duality transformation? Remember that in section 3.3 we determined the $O(\beta^2)$ corrections (3.51) and (3.52) to the sinh-Gordon reflection factor. Consider also the extra contribution which appears if we include some finite renormalisation of the boundary potential. From (4.56) we know that the contribution from such at counter-term at $O(\beta^2)$ is

$$\frac{\partial}{\partial\sigma}\frac{i\hat{k}+2\sigma}{i\hat{k}-2\sigma}\mu\beta^2 = \frac{4i\hat{k}\mu\beta^2}{(i\hat{k}-2\sigma)^2}. \tag{4.70}$$

Addition of this term to (3.51) (i.e. the case $\sigma \geq 0$) yields

$$-\frac{i\beta^2}{\hat{k}}\left\{\left(\frac{1}{\omega+2}-\frac{1}{\omega}\right)+\frac{\hat{k}^2\left(a+\frac{4\mu}{\sin(a\pi)}\right)\sin(a\pi)}{\hat{k}^2+4\sigma^2}\right\}. \tag{4.71}$$

As we did in the previous section we can use (4.71) to find the expansions of E and

F in (4.1). To $O(\beta^2)$, $F = 0$ but we can show that

$$E = 2a - \left(a + \frac{4\mu}{\sin(a\pi)}\right)\frac{\beta^2}{2\pi} + O(\beta^4). \tag{4.72}$$

If we assume that β only appears as part of the function B then we can guess

$$E = 2a - \left(2a + \frac{8\mu}{\sin(a\pi)}\right)\frac{B}{2}. \tag{4.73}$$

Of course, μ can be a function of a so let us write

$$E = 2a - (2a + 2\alpha(a))\frac{B}{2} \tag{4.74}$$

where $\alpha(a) = \frac{4\mu}{\sin(a\pi)}$.

This gives us added scope for trying to work out how boundary conditions can be related under duality transformations. Let us first assume that Ghoshal's formula for the reflection factor for the Neumann boundary condition is correct, or equivalently that $\alpha(\frac{1}{2}) = 0$. Moreover, we expect that if we perform two duality transformations, we return to the first reflection factor. The first dual (denoted by a star) gives

$$E^* = 2a - (2a + 2\alpha(a))\left(1 - \frac{B}{2}\right) = -\left(2\alpha(a) - (2a + 2\alpha(a))\frac{B}{2}\right) \tag{4.75}$$

whilst the second clearly returns this to its original form. Now let us suppose that (4.75) is of the same form as (4.74). Clearly for this to be true we require

$$\alpha(\alpha(a)) = a; \tag{4.76}$$

a function which is its own inverse. One example of a simple function which obeys these criteria is

$$\alpha(a) = \frac{1}{2} - a. \tag{4.77}$$

This would give

$$E = 2a - \frac{B}{2} \text{ and } E^* = -\left((1 - 2a) - \frac{B}{2}\right) \tag{4.78}$$

and hence the boundary condition dual to a would be $a^* = \frac{1}{2} - a$. This duality is shown in the region $0 \leq a \leq \frac{1}{2}$ in fig. 4.3.

Similarly, for the cases $\sigma < 0$, we can obtain

$$E = 2a - (2a - 2 + 2\alpha(a))\frac{B}{2} \tag{4.79}$$

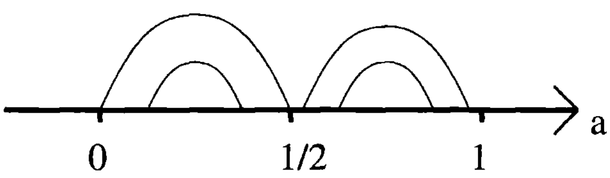

Figure 4.3: One example of how the boundary renormalisation allows us to relate the integrable boundary conditions of sinh-Gordon theory under a duality transformation.

with the dual

$$E^* = -\left(2\alpha(a) - 2 - (2a - 2 + 2\alpha(a))\frac{B}{2}\right). \tag{4.80}$$

Now we require the condition

$$\alpha(\alpha(a) - 1) = a + 1. \tag{4.81}$$

Possible examples here would be

$$\alpha(a) = a + 1, \tag{4.82}$$

which makes all these boundary conditions self-dual, or

$$\alpha(a) = -a + \xi \tag{4.83}$$

which gives the duality structure as $a^* = \xi - 1 - a$ where ξ is some constant. It might seem sensible to put $\xi = \frac{1}{2}$ so that the function $\alpha(a)$ is continuous across the Neumann boundary condition. This would mean that $a^* = -\frac{1}{2} - a \equiv \frac{3}{2} - a$. Such duality is shown in fig. 4.3. However, the duality properties of the $\sigma = -1$ boundary condition remain unexplained as this would imply it to be dual to the Neumann case, which is already dual to $a = 0$. There are of course many other possible ways in which the duality relations could hold; we only mention one here as an example.

It is beyond the scope of this discussion to consider whether any of these duality structures is consistent with the physical properties and other known facts of sinh-Gordon theory. However, it would be particularly interesting to compare the possibilities suggested by this section with the recent work of Corrigan and Delius

[87], who use a different method to give insight into the renormalisation and duality properties of sinh-Gordon theory. Their work suggests that

$$E = 2a\left(1 - \frac{B}{2}\right) \tag{4.84}$$

and hence the dual is $E^* = 2a\frac{B}{2}$. Therefore the boundary conditions are not related under duality transformations. In addition, the classical limit of every dual is $-\frac{1}{(1)^2}$, which is the classical reflection factor associated with the $\sigma = 1$ boundary condition.

4.7 Conclusions

In this chapter we have tried to conduct a check of the validity of the exact reflection factor for the Neumann boundary condition of sinh-Gordon theory proposed by Ghoshal. However, we were unable to determine an $O(\beta^4)$ correction which was consistent with the physical properties expected. It would therefore be of great interest to know why this is the case. In addition, an analysis of the possible renormalisations and their effects on the duality properties of the theories was conducted. This generated some interesting possibilities which may warrant further investigation, particularly in the light of new investigations into sinh-Gordon theory and its duality.

Chapter 5

Conclusions and further work

This thesis has been concerned with various aspects of the reflection factors of $a_r^{(1)}$ affine Toda field theory. A systematic methodology has been used; first the vacuum solutions of the theories with various boundary conditions were found, before considering the classical reflection factors and finally the quantum reflection factors. For the study of the latter, the approach taken has been to use perturbation theory to calculate the low-order coupling dependence of the reflection factors. This allows us to provide some kind of a check on work proposing exact reflection factors based upon more sophisticated yet conjectured principles.

Many interesting results have been thrown up by the present work. Firstly the vacuum solutions found in chapter 2 were unexpected in their complexity yet precisely covered all the integrable boundary conditions. It would be interesting to see if other techniques generate the same background solutions and if the 'exceptional' solutions can be understood in a simple way. We also saw that the non-existence of classical scattering solutions for certain boundary conditions was predicted.

In addition, the necessity for a renormalisation of the boundary potential in chapter 3 to preserve the integrability of $a_2^{(1)}$ theory with the $- - -$ boundary condition was in agreement with the work of Penati et al. For sinh-Gordon theory, studied in chapter 4, such renormalisations may have the scope to change the duality properties of the theories. The work there also suggests that looking at higher orders in perturbation theory may not pin this ambiguity down, though this would be an interesting area for further work. Indeed it would also be interesting to consider the renormalisation of sinh-Gordon theory with general symmetric boundary conditions, and any restrictions imposed by higher-order calculations. There have been recent conjectures concerning the dependence of the reflection factor on the boundary parameters which could be checked by perturbation theory. This is a matter for future work.

Another area for further study would be the duality structure of general $a_r^{(1)}$ theories. For $a_2^{(1)}$, it is expected that under a duality transformation the various boundary conditions are related but as yet this structure has not been conclusively determined. Proceeding to the next order in perturbation theory would be expected to shed further light on this. However, work by Corrigan and Delius seems to suggest that for sinh-Gordon theory, the integrable boundary conditions are *not* related by duality transformations. Again perhaps work on higher order perturbation theory might add weight to this result.

We also saw in chapter 4 that perturbation theory did not predict an $O(\beta^4)$ term in the quantum reflection factor which was in agreement with the physical properties expected. Suggestions were given there as to how this problem could be tackled. It would seem that there is much scope for further work in all these areas before we can fully determine these subtle properties of affine Toda field theories on the half-line.

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