The toda equations and congruence in flag manifolds

Sijbrandij, Klass Rienk

The Toda Equations and Congruence in Flag Manifolds

Klaas Rienk Sijbrandij

Department of Mathematical Sciences
University of Durham
South Road
Durham DH1 3LE
England

A thesis presented for
the degree of
Doctor of Philosophy
September 2000

Abstract

The Toda Equations and Congruence in Flag Manifolds

Klaas Rienk Sijbrandij

This thesis is concerned with the 2-dimensional Toda equations and their geometric interpretation in form of τ-adapted maps into flag manifolds.

τ-adapted maps are not only of interest due to their relation with the Toda equations, but also for their adaption to the m-symmetric space structure of flag manifolds.

This thesis studies the congruence question for τ-adapted maps in flag manifolds. The main theorem of this thesis is a congruence theorem for τ-holomorphic maps $\psi : S^2 \rightarrow G/T$ of constant curvature, where G can be *any* compact simple Lie group.

It is supplemented by a congruence theorem for general τ-holomorphic maps $\psi : S^2 \rightarrow G/T$ if G has rank 2, and a number of congruence theorems for isometric τ-primitive $\psi : R^2 \rightarrow G/T$ of constant Kähler angle. The second group of congruence theorems is proved for the rank 2 case, as well as a selection of Lie groups with higher rank: $SU(4), SU(5), F_4, E_6, E_8, Sp(n)$.

Acknowledgements

Thanks are due to my two supervisors, the late Dr Lyndon Woodward, and Dr John Bolton. Without the support of the first, I might never have started work on this thesis, and without the encouragement of the second, I might never have finished it. Dr Woodward possessed a sadly rare quality: the capability to inspire. His ability to communicate the sheer beauty of the subject has enriched more than just this work. However, I am indebted to him not just for the enthusiasm and insight which he brought to supervisions, but also for his support in arranging financial assistence for the period of my research. I received funding from the Engineering and Physical Sciences Research Council, the Technische Universität Berlin, the Mathematics Department of Durham University, the Graduate Society and Durham University itself, and I am grateful to all of these institutions.

I would also like to thank my parents, for believing in something I could never completely explain to them, and never doubting its importance.

And finally, Caroline for her support, and for foregoing sleep, in order to see this through to the end.

Declaration

This thesis is the result of research carried out between October 1996 and September 1999, under the supervision of Dr L M Woodward and Dr J Bolton. It has not been submitted for any other degree, either at Durham University or any other institution.

Throughout this work all non-original material is accompanied by a reference to its source, made either for a section or a specific theorem. Chapters 5, 7 and 8 are composed either substantially or completely of original material.

Contents

Introduction

This thesis is concerned with the 2-dimensional Toda equations and their geometric interpretation in form of τ-adapted maps into flag manifolds.

The relation between Toda equations and these maps is as follows. Let G be a compact simple Lie group with Lie algebra $\mathbf{g}$ and maximal torus T. Then, under a non-singularity condition, τ-adapted maps into the flag manifold G/T can be lifted to maps into G, so called Toda frames. These Toda frames satisfy a special differential equation, and the integrability conditions for the frames are the Toda equations for the Lie algebra $\mathbf{g}$.

However, τ-adapted maps are not only of interest due to their relation with the Toda equations. A flag manifold G/T may not only be equipped with G-invariant structures such as a G-invariant metric and a G-invariant complex structure, it also has the structure of an m-symmetric space. It is the m-symmetric space structure that τ-adapted maps are, by their definition, adjusted to. τ-adapted maps have many interesting properties, for example strong harmonicity properties.

In the context of this thesis we will study the congruence question for τ-adapted maps in flag manifolds. First we will give a brief explanation what we understand under congruence.

Definition:

Let G be a group of transformations of a manifold M. Let S be a Riemann surface. Two maps $\psi, \tilde{\psi} : S \rightarrow M$ are called G-**congruent** if $\tilde{\psi} = g\psi$ for some $g \in G$.

This leads to the fundamental question that this thesis attempts to answer, namely what do we need to know about $\psi, \tilde{\psi}$ in order to decide whether they are congruent. Our solution to this problem consists of finding a set of invariants (as few as possible) and their geometrical interpretation such that if these invariants coincide for ψ and $\tilde{\psi}$ then we can conclude that ψ and $\tilde{\psi}$ are congruent.
A typical example of a congruence theorem would be the classical rigidity theorem for smooth maps in $\mathbf{R}^3$ where metric and 2nd fundamental form are the required invariants.

The main theorem of this thesis is a congruence theorem for τ-holomorphic $\psi : S^2 \rightarrow G/T$ of constant curvature, where G can be *any* compact simple Lie group. It is supplemented by a congruence theorem for general τ-holomorphic $\psi : S^2 \rightarrow G/T$ if G has rank 2 and a number of congruence theorems for isometric τ-primitive $\psi : R^2 \rightarrow G/T$ of constant curvature and Kähler angle. The second group of congruence theorems is proved for the rank 2 case as well as a selection of Lie groups with higher rank: $SU(4), SU(5), F_4, E_6, E_8, Sp(n)$.

The Thesis is structured as follows.

In Chapter 1 we give a brief overview of the aspects of harmonic sequences and congruence theorems for $\mathbf{CP}^n$. These harmonic sequences are used to lift maps $\mathbf{CP}^n$ to maps into the flag manifold $SU(n+1)/T^n$ and they give rise to a set of invariants which are related to the Toda equations and which determine these lifts up to congruence in $SU(n+1)/T^n$.

In Chapter 2 we investigate Toda equations of semisimple Lie algebras and their relation to lifts derived from harmonic sequences.

In Chapter 3 we introduce flag manifolds and their various structures.

In Chapter 4 we consider τ-adapted maps into G/T. We will look at two classes of τ-adapted maps, τ-primitive and τ-holomorphic maps. τ-adapted maps provide - via Toda frames - a geometric interpretation of solutions of Toda equations.

In Chapter 5 we sketch the proof of the constant curvature congruence theorem for τ-holomorphic S^2 in $SU(n+1)/T^n$, the motivation for subsequent generalisations.

In Chapter 6 we compute the induced metric of τ-adapted maps and their associated curves. Invariants which determine τ-adapted maps up to congruence are also introduced.

In Chapter 7 the main theorem is proved, constant curvature congruence for τ-holomorphic S^2 in G/T. We also prove a general congruence theorem for τ-holomorphic S^2 in G/T where G has rank two.

In Chapter 8 a collection of congruence theorems for isometric τ-primitive maps with constant Kähler angle is presented.

Additional supporting material can be found in the Appendices.

Chapter 1

Harmonic Sequences and Congruence Theorems in $\mathbf{CP}^n$

This Chapter is intended as a brief overview of the aspects of harmonic sequences and congruence theorems for $\mathbf{CP}^n$ *needed for this thesis. More details and all the proofs may be found in [BW1], [BPW] and [Sem] which will also serve as reference for this chapter.*

Starting from a harmonic map ϕ *into* $\mathbf{CP}^n$ *one can construct a sequence of harmonic maps (see [EW] for the original holomorphic case). Under certain conditions this sequence can then be used to lift* ϕ *to a map into the flag manifold* $SU(n+1)/T^n$. *We will also introduce some invariants which are related to the Toda equations and which will determine these lifts up to congruence in* $SU(n+1)/T^n$. *Finally we will consider a well-known congruence theorem in* $\mathbf{CP}^n$ *which will be used to prove our original congruence theorem in* $SU(n+1)/T^n$.

1.1 Harmonic maps

Definition 1.1 *Let $\phi : S \to M$ be a C^∞ map from a metric Riemann surface S to a Riemannian manifold M. ϕ is called* **harmonic** *if and only if* $\mathrm{tr}\nabla d\phi = 0$, *where ∇ is the connection on* $\mathrm{Hom}(TS, TM)$ *induced by the Levi-Civita-Connections on S and M by:* $(\nabla d\phi)(X, Y) := (\nabla_X d\phi)(Y) := \nabla_X(d\phi(Y)) - d\phi(\nabla_X Y)$.

For $M = \mathbf{CP}^n$ $d\phi$ may be extended to a complex linear map from the complexified tangent space $TS^{\mathbf{C}} = TS \otimes_{\mathbf{R}} \mathbf{C}$ to $T\mathbf{CP}^n$, again denoted by $d\phi$.

With z a local complex coordinate on S the **harmonicity condition** may be written as

$$(\nabla_{\frac{\partial}{\partial \bar z}} d\phi)(\frac{\partial}{\partial z}) = 0 \quad \text{or} \quad \nabla_{\frac{\partial}{\partial \bar z}}(d\phi(\frac{\partial}{\partial z})) = 0$$

as $\nabla_{\frac{\partial}{\partial \bar z}} \frac{\partial}{\partial z} = 0$ gives

$$0 = (\nabla_{\frac{\partial}{\partial \bar z}} d\phi)(\frac{\partial}{\partial z}) = \nabla_{\frac{\partial}{\partial \bar z}}(d\phi(\frac{\partial}{\partial z})) - d\phi(\underbrace{\nabla_{\frac{\partial}{\partial \bar z}} \frac{\partial}{\partial z}}_{=0}) = \nabla_{\frac{\partial}{\partial \bar z}}(d\phi(\frac{\partial}{\partial z})).$$

Eqivalently, we also have $\nabla_{\frac{\partial}{\partial z}}(d\phi(\frac{\partial}{\partial \bar z})) = 0$.

1.2 Construction of the harmonic sequence

Let S be Riemann surface and $\phi : S \to \mathbf{CP}^n$ be harmonic.

In this section we will construct from ϕ a sequence of harmonic maps $S \to \mathbf{CP}^n$

$$\ldots, \phi_{-2}, \phi_{-1}, \phi_0 = \phi, \phi_1, \phi_2, \ldots$$

and a sequence of complex line bundles over S

$$\ldots, L_{-2}, L_{-1}, L_0 = \phi^* L, L_1, L_2, \ldots$$

Here ϕ^*L denotes the pull-back of the **tautological line bundle** $L = \{(q, v) \in \mathbf{CP}^n \times \mathbf{C}^{n+1} : v \in p\}$. Let $L^\perp$ be the subbundle of the trivial bundle $\mathbf{CP}^n \times \mathbf{C}^{n+1}$ whose fibre at $q = [w]$ is $\{w\}^\perp_{\mathbf{C}}$ (w.r.t. the standard hermitian inner product $\langle \cdot, \cdot \rangle$ in $\mathbf{C}^{n+1}$):

$$\pi : L^\perp \to \mathbf{CP}^n, \quad \pi^{-1}(q) = \{w\}^\perp_{\mathbf{C}}.$$

We will also use the bijective correspondence between maps $\phi : S \to \mathbf{CP}^n$ and smooth complex line subbundles of $S \times \mathbf{C}^{n+1}$ given by $\phi \leftrightarrow \phi^*L$.

Let $\phi : S \to \mathbf{CP}^n$ be harmonic and let $L_0, L_0^\perp$ be the pullbacks via ϕ of $L, L^\perp$ resp. Due to the canonical identification $T\mathbf{CP}^n = \mathrm{Hom}(L, L^\perp)$ the derivative $d\phi$ may be regarded as a map $d\phi : TS^{\mathbf{C}} \otimes L_0 \to L_0^\perp$ defined by

$$d\phi(X \otimes s) = d\phi(X)s = \pi_{L_0^\perp}(Xs)$$

where X is a tanget vector field on S, $\pi_{L_0^\perp}$ denotes orthogonal projection into $L_0^\perp$, and the section s of L_0 is considered a $\mathbf{C}^{n+1}$-valued map on S.

Let $\partial_0 : TS^{1,0} \otimes L_0 \to L_0^\perp$ be the 1,0-part of $d\phi$ and $\bar{\partial}_0 : TS^{0,1} \otimes L_0 \to L_0^\perp$ be the 0,1-part of $d\phi$. If z is a local coordinate on S and s a section of L_0 we have

$$\partial_0(\frac{\partial}{\partial z})s = d\phi(\frac{\partial}{\partial z})s \quad \text{and} \quad \bar{\partial}_0(\frac{\partial}{\partial \bar{z}})s = d\phi(\frac{\partial}{\partial \bar{z}})s.$$

As $T^{1,0}S = \mathrm{span}_{\mathbf{C}}\{\frac{\partial}{\partial z}\}$ and $T^{0,1}S = \mathrm{span}_{\mathbf{C}}\{\frac{\partial}{\partial \bar{z}}\}$ we will define for simplicity

$$\partial_0 : L_0 \to L_0^\perp, \quad s \mapsto d\phi(\frac{\partial}{\partial z})s \quad \text{and} \quad \bar{\partial}_0 : L_0 \to L_0^\perp, \quad s \mapsto d\phi(\frac{\partial}{\partial \bar{z}})s.$$

A complex vector subbundle V of $S \times \mathbf{C}^{n+1}$ may be given a holomorphic structure for which a local section s is a **holomorphic section** iff $\frac{\partial s}{\partial \bar{z}}$ is orthogonal to V.

Therefore the harmonicity condition above is equivalent to the bundle map ∂_0 ($\bar{\partial}_0$) being holomorphic (anti-holomorphic).

If ∂_0 ($\bar{\partial}_0$) is not identically zero, i.e. ϕ is not anti-holomorphic (holomorphic), then the zeros of ∂_0 ($\bar{\partial}_0$) are isolated and there exists a unique complex line subbundle $L_1 \subset L_0^\perp$ with $\mathrm{Im}(\partial_0) \subseteq L_1$ ($L_{-1} \subset L_0^\perp$ with $\mathrm{Im}(\bar{\partial}_0) \subseteq L_{-1}$).
As ϕ is harmonic the bundle map $\partial_0 : L_0 \to L_1$, $s \mapsto d\phi(\frac{\partial}{\partial z})s$ is holomorphic and the bundle map $\bar{\partial}_0 : L_0 \to L_{-1}$, $s \mapsto d\phi(\frac{\partial}{\partial \bar{z}})s$ is anti-holomorphic. Also the maps $\phi_1, \phi_{-1} : S \to \mathbf{CP}^n$ corresponding to L_1, L_{-1} are again harmonic. Using induction, we obtain a sequence of line bundles

$$\cdots\ L_{-1} \underset{\bar{\partial}_0}{\overset{\partial_{-1}}{\rightleftarrows}} L_0 \underset{\bar{\partial}_1}{\overset{\partial_0}{\rightleftarrows}} L_1 \underset{\bar{\partial}_2}{\overset{\partial_1}{\rightleftarrows}} L_2\ \cdots$$

and the corresponding harmonic maps

$$\ldots, \phi_{-2}, \phi_{-1}, \phi_0 := \phi, \phi_1, \phi_2, \ldots$$

If for some $q \in \mathbf{Z}$ the map ϕ_q is holomorphic (anti-holomorphic), then $\bar{\partial}_q$ (∂_q) is identically zero, and the map ϕ_{q-1} (ϕ_{q+1}) cannot be defined. The sequence $\{\phi_p\}$ terminates at the left (right).

1.3 Local description of the harmonic sequence

Let z be a local complex coordinate on the Riemann surface S and let $\phi(z) = [f_0(z)]$ be a harmonic map into $\mathbf{CP}^n$ where f_0 is a nowhere zero holomorphic local section of L_0. Then

$$L_0 = \{(z,v) : z \in S, v \in [f_0(z)]\}, \qquad L_0^\perp = \{(z,v) : z \in S, v \in [f_0(z)]^\perp\}.$$

These are vector subbundles of the trivial $\mathbf{C}^{n+1}$-bundle over S and so each has a naturally induced connection. Also a section of L_0 may be regarded as a map $S \to \mathbf{C}^{n+1}$, in which case we may regard f_0 as a map into $\mathbf{C}^{n+1} \setminus \{0\}$.

The bundle map $\partial_0 : L_0 \to L_1$ is now given by $\partial_0 f_0 = \pi_{L_0^\perp}(\frac{\partial f_0}{\partial z}) =: f_1$

and $\bar{\partial}_0 : L_0 \to L_{-1}$is defined by $\overline{\partial}_0 f_0 = \pi_{L_0^\perp}(\frac{\partial f_0}{\partial \bar{z}}) = f_{-1}$.

Again, we can build a harmonic sequence $\phi_p(z) = [f_p(z)]$ where f_{p+1} is the part of $\frac{\partial f_p}{\partial z}$ which is orthogonal to f_p (w.r.t. $\langle \cdot, \cdot \rangle$):

$$\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{\partial}{\partial z} \log |f_p|^2 f_p = f_{p+1} + \frac{< \frac{\partial}{\partial z} f_p, f_p >}{|f_p|^2} f_p$$

We also obtain

$$\frac{\partial f_{p+1}}{\partial \bar{z}} = -\frac{|f_{p+1}|^2}{|f_p|^2} f_p$$

and, from the definiton, $f_{p+1} \perp f_p$ holds.

We therefore have

$$\partial_p : L_p \to L_{p+1}, \quad f_p \mapsto f_{p+1} \quad \text{and} \quad \bar{\partial}_p : L_p \to L_{p-1}, \quad f_p \mapsto -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}.$$

Recall that

$$d\phi(\frac{\partial}{\partial z}) = \partial_0 \quad \text{and} \quad d\phi(\frac{\partial}{\partial \bar{z}}) = \overline{\partial}_0,$$

so

$$|d\phi_p(\frac{\partial}{\partial z})|^2 = |\partial_p|^2 = \frac{|f_{p+1}|^2}{|f_p|^2} \quad \text{and} \quad |d\phi_p(\frac{\partial}{\partial \bar{z}})|^2 = |\bar{\partial}_p|^2 = \frac{|f_p|^2}{|f_{p-1}|^2}.$$

Lemma 1.2 *∂_0 is a holomorphic bundle map iff $\phi_0 = [f_0]$ is a harmonic map.*

Lemma 1.3 *If ϕ_0 is harmonic, then ϕ_1 is harmonic as well.*

Terminating harmonic sequences

Definition 1.4 *Let S be Riemann surface and $\phi : S \to \mathbf{CP}^n$ be a harmonic map. ϕ is called* **pseudo-holomorphic** *(or* **superminimal** *or* **totally isotropic***) if the harmonic sequence terminates.*

Assume that $\overline{\partial}_0 \equiv 0$, i.e. ϕ_0 is a holomorphic curve in $\mathbf{CP}^n$, and assume ϕ_0 is **linearly full**, i.e. $\mathrm{Im}\phi_0$ is not contained in a totally geodesic $\mathbf{CP}^k \subset \mathbf{CP}^n$. Let $\{\phi_p = [f_p]\}$ be the harmonic sequence of ϕ_0.

Then for $r > s$

$$\frac{\partial}{\partial z} < f_r, f_s > = < f_{r+1} + \frac{\partial}{\partial z} \log |f_r|^2 f_r, f_s > - < f_r, \frac{|f_s|^2}{|f_{s-1}|^2} f_{s-1} > .$$

Also note that $\overline{\partial}_0 \equiv 0$ implies that $\frac{\partial}{\partial \bar{z}} f_0$ and f_0 are parallel. This, together with $< f_{r+1}, f_r >= 0$, gives the result that any two elements of the sequence are orthogonal: $< f_r, f_s >= 0$ for $r \neq s$.

It follows that the harmonic sequence must terminate at the right hand end with an antiholomorphic curve ϕ_n as there are at most $n+1$ non-zero mutually orthogonal vectors in $\mathbf{C}^{n+1}$.

Definition 1.5 (cf. [BJRW], p.602, [Wo], p.167) *The line bundles $L_0, \ldots, L_n$ are called the* **Frenet frame** *of the holomorphic curve ϕ_0 as they are essentially the analogue of the Frenet frame of a real space curve.*

The Frenet frame of the holomorphic curve ϕ_0 is obtained via the harmonic sequence.

1.4 Γ- and U-invariants of the harmonic sequence

The γ-invariants

Let $\gamma_p := |d\phi_p(\frac{\partial}{\partial z})|^2 = \frac{|f_{p+1}|^2}{|f_p|^2}$ as above. This depends on ϕ_p, $\frac{\partial}{\partial z}$ but not on the choice of f_p. In fact, $\Gamma_p := \gamma_p|dz|^2$ is a globally defined form on S.

The integrability conditions $\frac{\partial^2}{\partial \bar{z}\partial z} f_p = \frac{\partial^2}{\partial z \partial \bar{z}} f_p$ for

$$\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{\partial}{\partial z}\log|f_p|^2 f_p \quad \text{and} \quad \frac{\partial f_p}{\partial \bar{z}} = -\frac{|f_p|^2}{|f_{p-1}|^2} f_{p-1}$$

are equivalent to

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log|f_p|^2 = \gamma_p - \gamma_{p-1},$$

i.e.

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \gamma_p = \gamma_{p+1} - 2\gamma_p + \gamma_{p-1}.$$

These are the **Toda equations** for $SU(\infty)$ in general and for $SU(n+1)$ if the sequence terminates (see Chapter 2).

The following Lemma is immediate from the above equations.

Lemma 1.6 *Any two consecutive γ-invariants determine all the γ-invariants.*

The U-invariants

Assume $p > q$ and let $u_{p,q} = \frac{<f_p, f_q>}{|f_q|^2}$. This is independent of the choice of f_p, and, in fact, $U_{p,q} = u_{p,q} dz^{p-q}$ is a well-defined $(p-q)$-form.

In the terminating case these invariants are identically zero, so we assume that we are not in this situation. Then

$$\begin{aligned} \frac{\partial}{\partial z} u_{p,q} &= u_{p,q} \frac{\partial}{\partial z} (\gamma_{p-1} \cdot \ldots \cdot \gamma_q) + u_{p+1,q} - u_{p,q-1} \\ \frac{\partial}{\partial \bar{z}} u_{p,q} &= \gamma_q u_{p,q+1} - \gamma_{p-1} u_{p-1,q} \end{aligned}$$

Corollary 1.7 *If some k consecutive elements of a harmonic sequence are mutually orthogonal then every k consecutive elements are mutually orthogonal.*

Corollary 1.8 *Every harmonic map $\phi : S^2 \to \mathbf{CP}^n$ is part of a Frenet frame.*

Relationship between the harmonic sequence of $\phi : S \to \mathbf{CP}^n$ and its complex conjugate $\tilde{\phi} := \overline{\phi} : S \to \mathbf{CP}^n$.

We will need this relation for the construction of some examples later.

Denote by $\{f_p\}$ the local sections for the harmonic sequence $\{\phi_p\}$. Define

$$\tilde{f}_p := (-1)^p \frac{\overline{f}_{-p}}{|f_{-p}|^2}.$$

Then it is obvious that $\text{span}\{\tilde{f}_0\} = \tilde{L}_0 = \overline{L}_0$ and it is easy to check that $\{\tilde{f}_p\}$ is in fact the sequence derived from $\tilde{f}_0$. Hence we have

$$\frac{\partial \tilde{f}_p}{\partial z} = \tilde{f}_{p+1} + \frac{\partial}{\partial z} \log |\tilde{f}_p|^2 \tilde{f}_p \quad \text{and} \quad \left\langle \frac{\partial \tilde{f}_0}{\partial \bar{z}}, \tilde{f}_0 \right\rangle = 0.$$

We also get the following relations between the metric invariants of ϕ and $\tilde{\phi}$:

- $\tilde{L}_p = \overline{L}_{-p}$.

- $\tilde{\Gamma}_p = \Gamma_{-(p+1)} : \tilde{\gamma}_p = \frac{|\tilde{f}_{p+1}|^2}{|\tilde{f}_p|^2} = \frac{|f_{-(p+1)}|^2}{|f_{-(p+1)}|^4} |f_{-p}|^2 = \frac{|f_{-p}|^2}{|f_{-p-1}|^2} = \gamma_{-(p+1)}$.

- $\tilde{U}_{p,0} = (-1)^p U_{0,-p}$.

1.5 Congruence Theorems

We have the following

Lemma 1.9 *(i) Every element of a Frenet frame is a weakly conformal harmonic map;*

(ii) If one element of the harmonic sequence is conformal then every element of that sequence is conformal.

Proof:

(i) Note that $\phi_p : S \rightarrow M$ where S is a Riemann surface and M is a Kähler manifold is weakly conformal iff $d\phi_p(\frac{\partial}{\partial z}) \perp d\phi_p(\frac{\partial}{\partial \bar{z}})$. Thus for $M = \mathbf{CP}^n$, ϕ_p is conformal iff $L_{p+1} \perp L_{p-1}$.

(ii) This is simply a consequence of Corollary 1.7 with k=3.

Definition 1.10 *Let g_p be the* **induced metric** *on S by ϕ_p, i.e.*

$$g_p(X,Y) := \Re \langle d\phi_p(X), d\phi_p(Y) \rangle \quad \forall X, Y \in TS.$$

Let $\omega(X,Y) = \langle X, JY \rangle$ be the **Kähler form** *on $\mathbf{CP}^n$ and dA_p be the area form on S (w.r.t. g_p and the orientation of S). Then at each point on S where ϕ_p is non-singular, we define the* **Kähler angle** *θ_p of ϕ_p by $\phi_p^*\omega = \cos\theta dA_p$. It is the angle between $d\phi_p(\frac{\partial}{\partial x})$ and $id\phi_p(\frac{\partial}{\partial y})$.*

Note 1.11 *If ϕ_p is conformal then its metric and Kähler angle are given by*

$$g_p = (\gamma_{p-1} + \gamma_p)|dz|^2, \quad \cos\theta_p = \frac{\gamma_p - \gamma_{p-1}}{\gamma_p + \gamma_{p-1}} \Rightarrow \tan^2\frac{\theta_p}{2} = \frac{\gamma_{p-1}}{\gamma_p}.$$

ϕ_p conformal implies $u_{p+1,p-1} = 0$.

Note 1.12 *The metric and Kähler angle of ϕ_p determine and are determined by $\Gamma_p, \Gamma_{p-1}, U_{p+1,p-1}$ ($\Gamma_p = \gamma_p|dz|^2, U_{p,q} = u_{p,q}|dz|^{p-q}$).*

Lemma 1.13 *The metric and Kähler angle of any element of a harmonic sequence determine the metric and Kähler angle of any other element of the sequence.*

Remark 1.14 *Using the differential equations for the U-invariants it may be shown that for $k \in \mathbf{N}$ the Γ-invariants together with $\{U_{2,0}, \ldots, U_{k,0}\}$ determine $\{U_{q+2,q}, \ldots, U_{q+k,q}\}$ $\forall q \in \mathbf{Z}$.*

Theorem 1.15 (Congruence Theorem for $\mathbf{CP}^n$ [BW1], p.372)

Let S be a connected Riemann surface. Let $\phi, \tilde{\phi} : S \to \mathbf{CP}^n$ be harmonic maps with $\Gamma_{-1} = \tilde{\Gamma}_{-1}$, $\Gamma_0 = \tilde{\Gamma}_0$. If either

(i) ϕ is pseudo-holomorphic, or

(ii) $\tilde{U}_{p,0} = U_{p,0}$ for $p = 2, \ldots, n+1$

then there exists a holomorphic isometry g of $\mathbf{CP}^n$ such that $\tilde{\phi} = g\phi$. If ϕ is linearly full then g is unique.

As a corollary to Theorem 1.15 we have the following extension theorem.

Theorem 1.16 (Extension Theorem, [BW1], p.373)

Let $\phi : S \to \mathbf{CP}^n$ be a harmonic map of a connected Riemann surface S and let $h : S \to S$ be a conformal diffeomorphism such that

(i) $h^*\Gamma_p = \Gamma_p$ for $p = 0, -1$, and

(ii) $h^*U_{p,0} = U_{p,0}$ for $p = 2, \ldots, n+1$.

Then there exists a holomorphic isometry g of $\mathbf{CP}^n$ such that $g\phi = \phi h$. If ϕ is linearly full then g is the unique holomorphic isometry with this property.

Remark 1.17 *This theorem is an "extension theorem" for the following reason: Assume that ϕ is bijective. Then h induces a diffeomorphism $\hat{h} : \phi(S) \to \phi(S)$, $\hat{h} = \phi h \phi^{-1}$. Extending h now means that $\exists g : \mathbf{CP}^n \to \mathbf{CP}^n$ such that $g|_{\phi(S)} = \hat{h} = \phi h \phi^{-1}$ or equally $g\phi = \phi h$.*

Chapter 2

The Toda equations

In this chapter we will investigate the 1- and 2-dimensional Toda equations of semisimple Lie algebras. Using harmonic sequences we will see that solutions to the 2-dimensional $su(n+1)$-Toda equations arise in a geometrical context from special maps into the flag manifold $SU(n+1)/T^n$. We will also introduce Toda frames whose integrability conditions are the Toda equations. The whole chapter is based on [BW2] and [Sem].

2.1 The 1-dimensional Toda Equations

Consider the following Hamiltonian dynamical system of particles of equal mass m joined by identical springs.

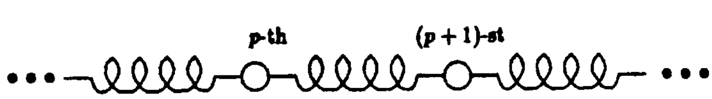

Figure 2.1: Springs

The equations of motion are

$$m\ddot{y}_p = f(y_{p+1} - y_p) - f(y_p - y_{p-1}).$$

if y_p denotes the displacement of the p^{th} mass. In the classical case we have $f(y) = \kappa y$, where κ is Hooke's constant.

We have the following interesting configurations:

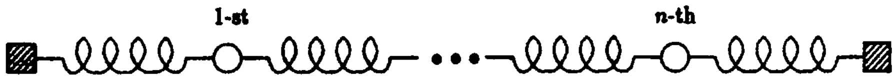

Figure 2.2: Finite or open case: $y_0 = y_{n+1} = 0$

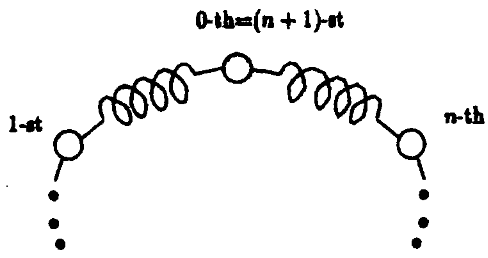

Figure 2.3: Periodic or affine case: $y_0 = y_{n+1}$

Figure 2.4: Infinite case

In the 1950s Fermi-Pasta-Ulam considered the case of a non-linear $f(y)$ and in 1967 Toda considered an exponential force $f(y) = ae^{\lambda y}$ with a, λ constants. This turned out to be a completely integrable Hamiltonian system. Let H be the Hamiltonian. Then we have for the first two configurations:

Open case $H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=0}^{n} e^{q_{j+1}-q_j} \qquad q_0 = q_{n+1} = 0.$

Periodic case $H = \frac{1}{2}\sum_{j=0}^{n} p_j^2 + \sum_{j=0}^{n} e^{q_{j+1}-q_j} \qquad q_0 = q_{n+1}.$

Here p, q are the momentum and position coordinates; $p_j = \frac{dq_j}{dt}$. The equations of motion are

$$\frac{\partial H}{\partial p_j} = \frac{dq_j}{dt} \quad \text{and} \quad \frac{\partial H}{\partial q_j} = -\frac{dp_j}{dt}$$

which give $\ddot{q}_j = e^{q_{j+1}-q_j} - e^{q_j - q_{j-1}}$.

In 1979 Adler, Kostant and Symes found that the Toda equations come from a Lie algebra formulation with equations corresponding to the case $g = su(n+1)$.

Let $\rho_i = q_i - q_{i-1}$. Then $\ddot{q}_i = e^{q_{i+1}-q_i} - e^{q_i - q_{i-1}}$ gives

$$\ddot{\rho}_i = e^{\rho_{i+1}} - e^{\rho_i} - (e^{\rho_i} - e^{\rho_{i-1}}) = e^{\rho_{i+1}} - 2e^{\rho_i} + e^{\rho_{i-1}}$$

or

$$\ddot{\rho}_i + (-1) \times e^{\rho_{i+1}} + 2 \times e^{\rho_i} + (-1) \times e^{\rho_{i-1}} = 0.$$

The factors before the exponential terms are exactly the entries of the (extended) Cartan matrix of $su(n+1)$: Let K be the Cartan matrix and $\hat{K}$ be the extended Cartan matrix of $\mathbf{g} = su(n+1)$ (see also Appendix B.3):

$$K = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \qquad \hat{K} = \left(\begin{array}{c|cccccc} 2 & -1 & & & & & -1 \\ \hline -1 & 2 & -1 & & & & \\ & -1 & 2 & & & & \\ & & & \ddots & & & \\ & & & & & 2 & -1 \\ -1 & & & & & -1 & 2 \end{array}\right).$$

Definition 2.1 *The* **open Toda equations** *are given by*

$$\ddot{\rho}_i + \sum_{j=1}^{n} K_{ij} e^{\rho_j} = 0$$

and the **affine Toda equations** *are given by*

$$\ddot{\rho}_i + \sum_{j=0}^{n} \hat{K}_{ij} e^{\rho_j} = 0.$$

Thus for every semisimple Lie algebra $\mathbf{g}$ the above gives a system of Toda equations via the (extended) Cartan matrix of $\mathbf{g}$. As in the $su(n+1)$ case this system is completely integrable.

It is interesting to see that for $su(n+1)$ the (extended) Dynkin diagram correponds exactly to the spring constellation.

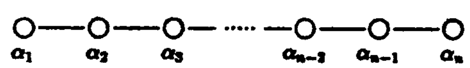

Figure 2.5: $su(n+1)$ Dynkin Diagram

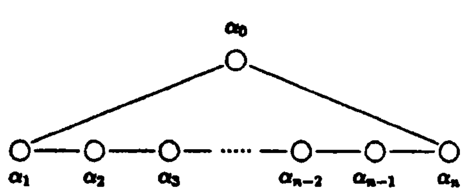

Figure 2.6: $su(n+1)$ Extended Dynkin Diagram

2.2 The 2-dimensional Toda Equations

For details about Lie algebras, Cartan matrix, root systems, etc. see [Sa] and [Se].

Let $\mathbf{g}$ be a simple Lie algebra of rank ℓ with (extended) Cartan matrix $K = (K_{ij})$, $i, j = (0), 1, \ldots, \ell$.

Let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a set of simple roots and let $-\alpha_0 = m_1\alpha_1 + \ldots m_\ell\alpha_\ell$ be the highest root. Set $m_0 = 1$, so $\sum_0^\ell m_j\alpha_j = 0$.

Definition 2.2 *The 2-dimensional* **open g-Toda equations** *are the non-linear elliptic system of partial differential equations given by*

$$2\Delta\Omega + \sum_{j=1}^{\ell} m_j e^{2\alpha_j(\Omega)} H_{\alpha_j} = 0,$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\Omega : U \to$ **it** *is a smooth map of an open subset* U *of* $\mathbf{R}^2$ *into the purely imaginary part of* $\mathbf{t}^{\mathbf{C}}$, *the complexified Cartan subalgebra of* $\mathbf{g}$, *and* $H_\alpha = \frac{2\alpha^\sharp}{\kappa(\alpha^\sharp, \alpha^\sharp)}$ *is the coroot to the root* α *(see Appendix B.2). The 2-dimensional* **affine g-Toda equations** *are given by*

$$2\Delta\Omega + \sum_{j=0}^{\ell} m_j e^{2\alpha_j(\Omega)} H_{\alpha_j} = 0.$$

This system is also completely integrable (see [G] for an excellent account of the modern theory of integrable systems) and we will show next that this formulation corresponds to the Toda equations of Section 2.1 with $\frac{d^2}{dt^2}$ replaced by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Claim 2.3 *The Toda equations* $2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)} H_{\alpha_j} = 0$ *may be written as*

$$\Delta \log \eta_i + \sum K_{ij}\eta_j = 0 \quad i = 1, \ldots, \ell$$

where (K_{ij}) *is the (extended) Cartan matrix of* $\mathbf{g}^{\mathbf{C}}$ *and* $\eta_j := m_j e^{2\alpha_j(\Omega)}$.

Note that with $\rho_i := \log \eta_i$ this is exactly the form of the Toda equations in Section 2.1 ($m_i = 1$ for all $i = 0, ...n$ in the $su(n+1)$-case). The η_i will be discussed in more detail in chapter 6.2.

Proof: We have $\mathbf{t}_{\mathbf{C}}^* = \text{span}\{\alpha_1, \ldots, \alpha_\ell\}$. Hence for $H \in \mathbf{t}_{\mathbf{C}}$ we have

$$H = 0 \quad \Longleftrightarrow \quad \alpha_i(H) = 0 \quad \forall i = 1, \ldots, \ell.$$

Therefore $2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)} H_{\alpha_j} = 0$ iff

$$\alpha_i(2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)} H_{\alpha_j}) = 0 \quad \forall i = 1, \ldots, \ell.$$

Using the linearity of α_i and its independence of $z, \bar{z}$ we get

$$0 = \alpha_i(2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)} H_{\alpha_j}) \quad = \quad \Delta 2\alpha_i(\Omega) + \sum m_j e^{2\alpha_j(\Omega)} \alpha_i(H_{\alpha_j})$$

Now $H_\alpha = \frac{2}{\kappa(\alpha^\sharp, \alpha^\sharp)} \alpha^\sharp$ and $\alpha_i(\alpha_j^\sharp) = \kappa(\alpha_i^\sharp, \alpha_j^\sharp)$ by definition of $\alpha^\sharp$ (see Appendix B.2). With $\eta_i = m_i e^{2\alpha_i(\Omega)}$ we see that $\Delta 2\alpha_i(\Omega) = \Delta \log \eta_i$. Thus the Toda equations are equivalent to

$$\Delta \log \eta_i + \sum \eta_j \frac{2\kappa(\alpha_i^\sharp, \alpha_j^\sharp)}{\kappa(\alpha_j^\sharp, \alpha_j^\sharp)} = 0$$

However the Cartan matrix is defined to be $K_{ij} = \frac{2\kappa(\alpha_i^\sharp, \alpha_j^\sharp)}{\kappa(\alpha_j^\sharp, \alpha_j^\sharp)}$. Hence the Toda equations are

$$\Delta \log \eta_i + \sum K_{ij} \eta_j = 0$$

□

2.3 Geometric Interpretation of the 2-dimensional su(n + 1)-Toda equations

In this section we will see how the harmonic sequence of a harmonic map $\phi : S \to \mathbf{CP}^n$ provides solutions to the Toda equations and gives rise to a map $\psi : S \to SU(n+1)/T^n$.

Suppose $\phi : S \to \mathbf{CP}^n$ is a linearly full harmonic map. We then have a harmonic sequence $\{\phi_p\}$, $\phi_p = [f_p]$, defined by

- $\frac{\partial f_p}{\partial z} = f_{p+1} + \frac{\partial}{\partial z} \log |f_p|^2 f_p = f_{p+1} + \frac{<\frac{\partial}{\partial z} f_p, f_p>}{|f_p|^2} f_p$
- $\frac{\partial f_{p+1}}{\partial \bar{z}} = -\frac{|f_{p+1}|^2}{|f_p|^2} f_p$
- $f_{p+1} \perp f_p$

Put $|f_p|^2 = e^{2\omega_p}$ (assuming that f_p does not vanish). Then from the basic equations of the harmonic sequence, the integrability condition $\frac{\partial^2}{\partial z \partial \bar{z}} f_p = \frac{\partial^2}{\partial \bar{z} \partial z} f_p$, and using $\frac{\partial^2}{\partial z \partial \bar{z}} \log |f_p|^2 = \gamma_p - \gamma_{p-1}$ we deduce that

$$2\frac{\partial^2 \omega_p}{\partial z \partial \bar{z}} = e^{2(\omega_{p+1} - \omega_p)} - e^{2(\omega_p - \omega_{p-1})},$$

i.e.

$$2\frac{\partial^2 (\omega_p - \omega_{p-1})}{\partial z \partial \bar{z}} - e^{2(\omega_{p-1} - \omega_{p-2})} + 2e^{2(\omega_p - \omega_{p-1})} - e^{2(\omega_{p+1} - \omega_p)} = 0$$

Thus $\omega_p - \omega_{p-1}$ satisfies the Toda equations and we can see how the harmonic sequence is related to the Toda equations. In general we have infinitely many equations for infinitely many unknowns: $su(\infty)$-Toda equations.

We will now concentrate on the two simplest cases

(1) Superminimal (or pseudo-holomorphic) case:

$\phi : S \to \mathbf{CP}^n$ is an element of the Frenet frame of a holomorphic curve.

$$L_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_1 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_2 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} \dots \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_n$$

Figure 2.7: $L_0, \dots, L_n$ mutually orthogonal

(2) Orthogonally periodic case: $\phi_{n+1+p} = \phi_p$ for all p. Further assumption: $L_0, \dots, L_n$ are mutually orthogonal.

$$\begin{array}{ccccc} & & L_0 & & \\ & \partial \nearrow\swarrow \bar\partial & & \bar\partial \nwarrow\searrow \partial & \\ L_n & & & & L_1 \\ \partial \uparrow\downarrow \bar\partial & & & & \bar\partial \uparrow\downarrow \partial \\ L_{n-1} & & & & L_2 \\ & \partial \nwarrow\searrow \bar\partial & & \bar\partial \nearrow\swarrow \partial & \\ & & \dots & & \end{array}$$

Figure 2.8: $L_p, \dots, L_{n+p}$ mutually orthogonal - circle

A lift to $SU(n+1)/T^n$

Let $\mathcal{F} = \{V_1 \subset \dots \subset V_{n+1} = \mathbf{C}^{n+1} \ : \ V_k$ vector subspace of $\mathbf{C}^{n+1}$ of dimension $k\}$ be the **manifold of full flags**.

Then, from the Orbit-Stabilizer Theorem, it is easy to see that $\mathcal{F} = SU(n+1)/T^n = U(n+1)/T^{n+1}$.

We now use the harmonic sequence of ϕ to define the lift

$\psi : S \to \mathcal{F} = SU(n+1)/T^n$ by $\psi = (V_1, \ldots, V_{n+1})$, where $V_1 = L_0, V_2 = L_0 \oplus L_1, \ldots, V_k = L_0 \oplus \ldots \oplus L_{k-1}, \ldots$

We will see in chapter 4 that ψ is τ-adapted and has a number of interesting properties. For example, if ϕ is holomorphic the lift ψ will be holomorphic as well and we have the following correspondence

$$
\begin{array}{ccc}
\{\psi : S \to SU(n+1)/T^n \ \tau\text{-holomorphic}\} & \longleftrightarrow & \{\phi : S \to \mathbf{CP}^n \text{ holomorphic}\} \\
\psi & \mapsto & \pi\psi \\
(\phi_0| \ldots |\phi_n) & \leftarrow & \phi = \phi_0
\end{array}
$$

2.4 Toda frames

Away from singularities there locally exists a moving frame $E : U \to SU(n+1)$ from an open subset U of S given by $E = (e_0| \ldots |e_n)$, $e_p = \frac{1}{\det(\frac{f_0}{|f_0|}|\cdots|\frac{f_n}{|f_n|})^{1/(n+1)}} \frac{f_p}{|f_p|}$. The normalising factor is needed to get $E \in SU(n+1)$ rather than $E \in U(n+1)$.

Then

$$
\begin{aligned}
\frac{\partial e_p}{\partial z} &= e^{\omega_{p+1} - \omega_p} e_{p+1} + \frac{\partial \omega_p}{\partial z} e_p \\
\frac{\partial e_p}{\partial \bar{z}} &= -e^{\omega_p - \omega_p} e_{p-1} - \frac{\partial \omega_p}{\partial \bar{z}} e_p
\end{aligned}
$$

and these equations can be expressed as

$$
\begin{aligned}
E^{-1} \frac{\partial E}{\partial z} &= \frac{\partial \Omega}{\partial z} + e^{\Omega} B_0 e^{-\Omega} \\
E^{-1} \frac{\partial E}{\partial \bar{z}} &= -\frac{\partial \Omega}{\partial \bar{z}} + e^{-\Omega} B_0 e^{\Omega}
\end{aligned}
$$

where $\Omega = \mathrm{diag}(\omega_0, \ldots, \omega_n)$ and

$$B_0(\text{open case}) = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \quad \text{and} \quad B_0(\text{affine case}) = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

The integrability conditions for this frame E are the Toda equations (see chapter 4 for details).

Using the differential equation above one can show that for a disc-like open set U, solutions of the Toda equations correspond to special moving frames: **Toda frames**.

Given a frame $E : U \to SU(n+1)$ we get a map $\psi := \pi E : U \to SU(n+1)/T^n$ where π denotes the canonical projection. These maps are precisely the ones which arise from harmonic sequences of maps into $\mathbf{CP}^n$ in the pseudo-holomorphic / orthogonally periodic cases.

Chapter 3

Flag Manifolds

In this chapter we will introduce flag manifolds G/H and their properties. Flag manifolds may be desribed by parabolic subalgebras and can be equipped with G-invariant metrics and G-invariant complex structures. They also have an m-symmetric space structure which is the crucial geometric property in the context of this thesis.

3.1 Flag manifolds - definition, examples and Lie algebraic description

The main reference for this Section is Burstall-Rawnsley [BR], Chapter 4.

Definition 3.1 ([FH], p.95) *A* **flag** *is a sequence of subspaces of a fixed vector space, each properly contained in the next; it is a* **complete** *flag if the dimension of each subspace is one larger larger than that of the preceding subspace, and a* **partial** *flag otherwise.*

Definition 3.2 ([BH], p.39) *A* **flag manifold** *is a homogeneous space of the form G/H where G is a compact Lie group and H is the centralizer of a torus in G. Note that H is therefore of maximal rank.*

Example 3.3 (Flag manifolds G/H are manifolds of flags)

(i) $G = SU(n+1)$, $H = T^n$. Then H is its own centralizer and G/T is the manifold of full flags $G/T = \{V_1 \subset V_2 \subset \ldots \subset V_n \subset \mathbf{C}^{n+1}\}$ where V_j is a subspace of $\mathbf{C}^{n+1}$ of dimension j.

(ii) $G = SU(n+1)$, $H = S(U(r) \times U(n+1-r))$. H is the centralizer of

$$S^1 = \left\{ \left(\begin{array}{c|c} e^{i\theta} I_r & 0 \\ \hline 0 & e^{i\phi} I_{n-r+1} \end{array} \right) \mid r\theta + (n+1-r)\phi = 0 \right\}.$$

Here $G/H = Gr_r(\mathbf{C}^{n+1}) = \{V_r \subset \mathbf{C}^{n+1}\}$.

(iii) $G = SO(2n)$ or $SO(2n+1)$, $H = T^n$. Here the corresponding flag manifold is $\{V_1 \subset V_2 \subset \ldots \subset V_n \subset \mathbf{C}^{2n}$ or $\mathbf{C}^{2n+1}\}$ where V_j is an j-dimensional isotropic subspace of $\mathbf{C}^{2n}$ or $\mathbf{C}^{2n+1}$, i.e. $\langle v, v\rangle = 0\ \forall v \in V_j$.

(iv) $SO(2n)/U(n)$, $U(n) = \{A \in SO(2n) \mid AJ = JA\}$ = centralizer of $\{\cos\theta I + \sin\theta J\}$ ($= S^1$). This flag manifold is the space of all orthogonal complex structures on $\mathbf{R}^{2n}$.

Lie algebraic description of flag manifolds - parabolic subalgebras and subgroups

We will investigate the structure of G/H by looking at the corresponding infinitesimal situation, i.e. Lie algebras. This will give an alternative definition for a flag manifold, and we will see that for each H as above there exists a parabolic subgroup P of $G^{\mathbf{C}}$ such that $G/H = G^{\mathbf{C}}/P$.

Let $\mathbf{g}$ be a compact real form of a semisimple complex Lie algebra $\mathbf{g}^{\mathbf{C}}$. Let $\mathbf{t}$ be a Cartan subalgebra of $\mathbf{g}$. Consider the usual decomposition of $\mathbf{g}^{\mathbf{C}}$ given by a choice of simple roots $\alpha_1, \ldots, \alpha_\ell$ ($\ell = \operatorname{rank} \mathbf{g}$). We have

$$\mathbf{g}^{\mathbf{C}} = \mathbf{t}^{\mathbf{C}} \oplus \underbrace{\sum_{\alpha\in\Delta^+} \mathbf{g}^{\alpha} \oplus \sum_{\alpha\in\Delta^+} \mathbf{g}^{-\alpha}}_{\swarrow\ cx.conj.\ \searrow} .$$

Definition 3.4 *A subalgebra* $\mathbf{b}$ *of* $\mathbf{g}^{\mathbf{C}}$ *is a* **Borel subalgebra** *if it is a maximal solvable subalgebra of* $\mathbf{g}^{\mathbf{C}}$*, where a subalgebra* $\mathbf{c}$ *of* $\mathbf{g}^{\mathbf{C}}$ *is* **solvable** *if its* **derived series** $\{\mathcal{D}^k\mathbf{c}\}$*, defined by* $\mathcal{D}^1\mathbf{c} = [\mathbf{c}, \mathbf{c}]$ *and* $\mathcal{D}^k\mathbf{c} = [\mathcal{D}^{k-1}\mathbf{c}, \mathcal{D}^{k-1}\mathbf{c}]$*, terminates in the sense that* $\mathcal{D}^k\mathbf{c} = \{0\}$ *for some* k.

A subalgebra $\mathbf{p}$ *of* $\mathbf{g}^{\mathbf{C}}$ *is a* **parabolic subalgebra** *if it contains a Borel subalgebra.*

Each subset S of the set $\{\alpha_1, \ldots, \alpha_\ell\}$ of simple roots determines a further decomposition of $\mathbf{g}^{\mathbf{C}}$ as follows. Let $T(S)$ be the set of positive roots which are linear combinations of roots in S.

Then

$$\mathbf{g}^{\mathbf{C}} = \underbrace{\overbrace{\mathbf{t}^{\mathbf{C}} \oplus \sum_{\alpha\in T(S)} \mathbf{g}^{\alpha} \oplus \sum_{\alpha\in T(S)} \mathbf{g}^{-\alpha}}^{\mathbf{h}^{\mathbf{C}} \text{ where } \mathbf{h} = \mathbf{g}\cap p_S} \oplus \overbrace{\sum_{\beta\in\Delta^+\setminus T(S)} \mathbf{g}^{\beta}}^{\mathbf{n} = \text{nilradical of } p_S}}_{\text{parabolic subalgebra } p_S \text{ determined by } S} \oplus \sum_{\beta\in\Delta^+\setminus T(S)} \mathbf{g}^{-\beta}$$

Note that $\mathbf{h}^{\mathbf{C}}$ is the complexification of a real subalgebra because it is invariant under complex conjugation, and that $\mathbf{h}$ is the centralizer of the toral Lie subalgebra $\{X \in \mathbf{t} \mid \alpha(X) = 0\ \forall \alpha \in S\}$. Also the bigger S is, the bigger the corresponding parabolic subalgebra is.

Example 3.5 *(i) If* $S = \{\alpha_1, \ldots, \alpha_\ell\}$ *then* $p_S = \mathbf{g}^{\mathbf{C}}$ = *centralizer of* $\{0\}$. *This is the largest parabolic subalgebra.*

(ii) If $S = \emptyset$ *then* $p_\emptyset = \mathbf{t}^{\mathbf{C}} \oplus \sum_{\beta\in\Delta^+} \mathbf{g}^\beta$. *It is a Borel subalgebra (smallest parabolic subalgebra).* $\mathbf{h} = \mathbf{t}$ = *centralizer of* $\mathbf{t}$. *Any two Borel subalgebras are conjugate.*

(iii) If $|S| = \ell - 1$, *the corresponding* p_S *is a* **maximal parabolic subalgebra**. *The corresponding* $\mathbf{h}$ *is the centralizer of a* 1*-dimensional toral subalgebra.*

Lemma 3.6 ([BR]) *Let* $G^{\mathbf{C}}$ *be a connected semi-simple complex Lie group. A* **parabolic subgroup** *of* $G^{\mathbf{C}}$ *is a complex Lie subgroup which is the normaliser of a parabolic subalgebra of* $\mathbf{g}^{\mathbf{C}}$. *A* **flag manifold** *is a homogeneous space of the form* $G^{\mathbf{C}}/P$ *with* P *a parabolic subgroup.*

Theorem 3.7 ([BR]) *(i) If* $\mathbf{h}$ *is the centralizer of a torus in* $\mathbf{g}$ *then* $\mathbf{h} = \mathbf{g} \cap p_S$ *for some parabolic subalgebra* p_S.

(ii) On group level $G/H = G^{\mathbf{C}}/P$.

(iii) $G^{\mathbf{C}}/P$ *is compact iff* P *is parabolic.*

Note 3.8 *If* $S = \emptyset$ *then* $p_\emptyset = \mathbf{t}^{\mathbf{C}} \oplus \sum_{\alpha\in\Delta^+} \mathbf{g}^\alpha$ *and the corresponding flag manifold is* G/T.

3.2 G-invariant metrics on flag manifolds

The following holds for any homogeneous space G/H, not just flag manifolds.

Definition 3.9 *Let G/H be a homogeneous space. Let dL_g be the differential of left-translation L_g by g. Then a metric $\langle\cdot,\cdot\rangle$ on G/H is G-**invariant** if $\forall g, k \in G$*

$$\langle X, Y\rangle_{kH} = \langle dL_g X, dL_g Y\rangle_{gkH} \quad \forall X, Y \in T_{kH}(G/H).$$

Remark 3.10 *For all $g \in G$ left-translation is an isometry w.r.t. any G-invariant metric.*

Denote the base point $eH = H \in G/H$ by o. Any G-invariant metric on G/H can be constructed by defining an $\mathrm{Ad}(H)$-invariant inner product on T_oG/H and then moving it around via left-translation. The metric on T_oG/H has to be $\mathrm{Ad}(H)$-invariant so that its left-translation is well-defined.

Proposition 3.11 ([G] p.16-17)
$\{\mathrm{Ad}(H)\text{-invariant } \textit{inner products on } T_oG/H\} \stackrel{1:1}{\longleftrightarrow} \{G\textit{-invariant metrics on } G/H\}$.

3.3 Complex structures on flag manifolds

In this section we will construct G-invariant complex structures on flag manifolds G/H. If $\mathbf{g} = \mathbf{h} + \mathbf{m}$ with $\mathbf{m} \equiv T_oG/H$ we therefore need an $\mathrm{ad}(\mathbf{h})$-invariant complex structure J on $\mathbf{m}$, i.e. we need an $\mathrm{ad}(\mathbf{h})$-invariant complex subspace V of $\mathbf{m}^{\mathbf{C}}$ such that $\mathbf{m}^{\mathbf{C}} = V \oplus \overline{V}$. We will see that for G/T we can take $V = \mathbf{m}^+ = \sum_{\alpha\in\Delta^+} g^\alpha$ where Δ^+ denotes a choice of positive roots. The main reference for this section is Borel-Hirzebruch [BH].

In order to classify G-invariant (almost) complex structures we need to investigate some real adjoint representation theory.

Real adjoint representation theory for compact Lie groups

Let G be a compact Lie group, $\mathbf{g}$ its Lie algebra, T a maximal torus in G with Lie algebra $\mathbf{t}$.

The representation $\mathrm{Ad} : T \to \mathrm{Aut}(\mathbf{g})$ of T in $\mathbf{g}$ is fully reducible and there exists a direct sum decomposition of $\mathbf{g}$ into irreducible $\mathrm{Ad}(T)$ submodules

$\mathbf{g} = \mathbf{t} \oplus a_1 \oplus \ldots \oplus a_m$ such that

(i) $\mathrm{Ad}(T).a_k = a_k$

(ii) $\dim a_k = 2$

(iii) For $h \in T$ $\mathrm{Ad}(h)|_{a_k}$ can be represented by $\begin{pmatrix} \cos a_k(h) & -\sin a_k(h) \\ \sin a_k(h) & \cos a_k(h) \end{pmatrix}$.

Simply choose an ONB for a_k with respect to an $\mathrm{Ad}(T)$-invariant inner product on $\mathbf{g}$. Note that $a_k : T \to \mathbf{R}/2\pi\mathbf{Z}$ is a homomorphism, so in particular $a_k(e) = 0$.

(iv) Let $\tilde{\alpha}_k := d_e a : \mathbf{t} \to \mathbf{R}$. The $\pm\tilde{\alpha}_k$ are called the **(infinitesimal) roots** of G w.r.t. T. In the literature the roots are usually $\frac{\tilde{\alpha}_k}{2\pi}$ instead of $\tilde{\alpha}_k$. The adjoint representation of $\mathbf{t}$ on $\mathbf{g}$ gives rise to the same direct sum decomposition of $\mathbf{g}$: $\mathbf{g} = \mathbf{t} \oplus a_1 \oplus \ldots \oplus a_m$. For $H \in \mathbf{t}$ $\mathrm{ad}_H|_{a_k}$ may be represented by $\begin{pmatrix} 0 & -\tilde{\alpha}_k \\ \tilde{\alpha}_k & 0 \end{pmatrix}$

which can be seen by differentiating the $\mathrm{Ad}(\exp(sH))$ with respect to s at 0.

(v) Let $\alpha_k = i\tilde{\alpha}_k$. These are the standard roots w.r.t. the adjoint representation of $\mathbf{t}^{\mathbf{C}}$ in $\mathbf{g}^{\mathbf{C}}$. Let $\mathbf{g}^{\mathbf{C}} = \mathbf{t}^{\mathbf{C}} \oplus a_1^{\mathbf{C}} \oplus \ldots \oplus a_m^{\mathbf{C}}$ be the complexification of $\mathbf{g}$. Then $a_k^{\mathbf{C}} = \mathbf{g}^{\alpha_k} \oplus \mathbf{g}^{-\alpha_k}$, where $\mathbf{g}^{\alpha} = \{X \in \mathbf{g}^{\mathbf{C}} \mid \mathrm{ad}_H(X) = \alpha(H)X \ \forall H \in \mathbf{t}\}$. Conversely we have $a_k = \mathbf{g} \cap (\mathbf{g}^{\alpha_k} \oplus \mathbf{g}^{-\alpha_k})$.

Complementary roots

Let G be a compact, connected, semisimple Lie group, $\ell = \mathrm{rank}\, G$, H a proper closed connected subgroup of the same rank ℓ, and T a maximal torus of H, i.e. we have $T \leq H < G$. Thus we get the decomposition of $\mathbf{h}^{\mathbf{C}}$ into irreducible modules (with respect to the adjoint action of T on $\mathbf{h}$):

$$\mathbf{h} = \mathbf{t} \oplus a_1 \oplus \ldots \oplus a_n.$$

We also have

$$\mathbf{g} = \mathbf{t} \oplus a_1 \oplus \ldots \oplus a_m.$$

Hence $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ splits as

$$\mathbf{g} = \overbrace{\mathbf{t} \oplus a_1 \oplus \ldots \oplus a_n}^{\mathbf{h}} \oplus \overbrace{a_{n+1} \oplus \ldots \oplus a_m}^{\mathbf{m}}$$

where the a_k have been suitably numbered.

The $2(m-n)$ roots $\pm\tilde{\alpha}_{n+1}, \ldots, \pm\tilde{\alpha}_m$ are called **complementary roots** (see [BH], p.464).

Almost complex structures on homogeneous spaces

Let G be a compact, connected, semisimple Lie group, $\ell = \text{rank } G$, H a proper closed connected subgroup of the same rank ℓ and T a maximal torus of H, so $T \leq H < G$.

Definition 3.12 *A* **G-invariant almost complex structure** *on G/H is an almost complex structure J on G/H such that $J = L_g J L_{g^{-1}}$ for all $g \in G$ where L_g denotes the differential of left action by g. Hence the following diagram commutes*

$$\begin{array}{ccc} T(G/H) & \xrightarrow{J} & T(G/H) \\ \downarrow L_g & & \downarrow L_g \\ T(G/H) & \xrightarrow{J} & T(G/H) \end{array}$$

and for $X \in T_{[x]}G/H$ we have

$$L_g J_{[x]} X = J_{[gx]} L_g X.$$

Proposition 3.13 *There is a one-to-one correspondence between*

(1) G-invariant almost complex structures J on G/H, and

(2) complex structures J_o on T_oG/H which commute with the isotropy group, i.e.
$\text{Ad}^{G/H}(h) J_o = J_o \text{Ad}^{G/H}(h)$ *for all $h \in H$.*

Proof: For details about the isotropy representation see Appendix B.7.

(1) $\Rightarrow$ (2) Let J be a G-invariant almost complex structure on G/H. Then $J_o = J_{[e]}$ is a complex structure on the tangent space T_oG/H. Next note that L_h maps T_oG/H into itself. Thus from the definition of G-invariance, $L_g J = J L_g$ $\forall g \in G$, it follows in particular that $L_h J_o = J_o L_h$ $\forall h \in H$. Now

recall the definiton of the isotropy representation $\mathrm{Ad}^{G/H}(h) = L_h|_o$ to get $\mathrm{Ad}^{G/H}(h)J_o = J_o\mathrm{Ad}^{G/H}(h)$ for all $h \in H$.

(2) $\Rightarrow$ (1) Now let J_o be a complex structure on T_oG/H which commutes with the isotropy group. Define an almost complex structure J on G/H by $J_{[g]} = L_gJ_oL_{g^{-1}}$.

Claim: J is well-defined.

Proof: Let $[g] = [g']$. Then there is an $h \in H$ such that $g' = gh$. Thus, using $L_hJ_o = J_oL_h$, we get

$$J_{[g']} = L_{g'}J_oL_{g'^{-1}} = L_{gh}J_oL_{(gh)^{-1}} = L_gL_hJ_oL_{h^{-1}}L_{g^{-1}} = L_gJ_oL_{g^{-1}} = J_{[g]},$$

i.e. J is well-defined.

Claim: J is G-invariant.

Proof: We have to show that $L_gJ_{[x]} = J_{[gx]}L_g$ for all $[x] \in G/H$ and all $g \in G$:

$$J_{[gx]}L_g = L_{gx}J_oL_{(gx)^{-1}}L_g = L_gL_xJ_oL_{x^{-1}}L_{g^{-1}}L_g = L_gJ_{[x]}.$$

Hence J is G-invariant.

□

We will now describe all possible almost complex structures on G/H. From the proposition above it is sufficient to find all complex structures J_o on T_oG/H which commute with the isotropy group $\mathrm{Ad}(H)$.

Since G/H is reductive we have $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, and we can identify T_oG/H with the Lie subspace $\mathbf{m}$ and $\mathrm{Ad}^{G/H}(h) : T_oG/H \to T_oG/H$ with $\mathrm{Ad}(h) : \mathbf{m} \to \mathbf{m}$.

The G-invariant almost complex structures on G/H are described by the following theorem.

Theorem 3.14 *There exists a 1-1 correspondence between G-invariant almost complex structures on G/H and splittings of $T_o^{\mathbf{C}}G/H$ into* $\mathrm{Ad}(H)$*-invariant subspaces $T_o^{1,0}G/H = \sum \mathbf{g}^{\epsilon_k\alpha_k}$, $T_o^{0,1}G/H = \sum \mathbf{g}^{-\epsilon_k\alpha_k}$ with $\epsilon_k \in \{\pm 1\}$ and $\{\pm\alpha_k \mid k = n+1,\ldots,m\}$ the set of complementary roots.*

Corollary 3.15 *There are $2^{\frac{1}{2}\mathrm{dim}G/H}$ different G-invariant almost complex structures on G/H.*

Proof of the Corollary:

Let $\mathrm{dim}H = \ell + 2n$, $\mathrm{dim}G = \ell + 2m$. Then $m - n = \frac{1}{2}\mathrm{dim}G/H$ and there are 2 choices for each ϵ_k, $k = n+1,\ldots,m$. $\square$

Proof of Theorem 3.14:

We have to find all complex structures on $\mathbf{m}$ which commute with elements $\mathrm{Ad}(h)$ of the isotropy group. Let J be a complex structure on $\mathbf{m}$ commuting with $\mathrm{Ad}(H)$. We will now determine which properties J has.

Claim: J commutes with the adjoint representation of $\mathbf{g}$ in $\mathbf{g}$

Proof: Since J is G-invariant, we have $\mathrm{Ad}(h)J = J\mathrm{Ad}(h)$ $\forall h \in H$. We want to show $\mathrm{ad}(H)J = J\mathrm{ad}(H)$ for all $H \in \mathbf{h}$ which follows from $\mathrm{Ad}(h)J = J\mathrm{Ad}(h)$ by diffentiation. More explicitly, let $H \in \mathbf{h}$ be arbitrary and $h(t) = \exp(tH)$ the corresponding curve in H with tangent vector H at the origin. Then

$$\mathrm{ad}(H)J = \frac{d}{dt}|_0 \mathrm{Ad}(h(t))J = \frac{d}{dt}|_0 J\mathrm{Ad}(h(t)) = J\mathrm{ad}(H).$$

This relation can also be seen from the following diagram

$$\begin{array}{ccc}
\mathbf{h} & \stackrel{\mathrm{ad}(\cdot)J}{\longrightarrow} & T_I\mathrm{Aut}(\mathbf{g}) \equiv \mathrm{End}(\mathbf{g}) \\
\cap & & \cap \\
TH & \stackrel{d(\mathrm{Ad}(\cdot)J)}{\longrightarrow} & T\mathrm{Aut}(\mathbf{g}) \\
\downarrow & & \downarrow \\
H & \stackrel{\mathrm{Ad}(\cdot)J}{\longrightarrow} & \mathrm{Aut}(\mathbf{g}) \\
\mathrm{id} \updownarrow & & \updownarrow \mathrm{id} \\
H & \stackrel{J\mathrm{Ad}(\cdot)}{\longrightarrow} & \mathrm{Aut}(\mathbf{g}) \\
\uparrow & & \uparrow \\
TH & \stackrel{d(J\mathrm{Ad}(\cdot))}{\longrightarrow} & T\mathrm{Aut}(\mathbf{g}) \\
\cup & & \cup \\
\mathbf{h} & \stackrel{J\mathrm{ad}(\cdot)}{\longrightarrow} & T_I\mathrm{Aut}(\mathbf{g}) \equiv \mathrm{End}(\mathbf{g})
\end{array}$$

□

We have $\mathbf{m} = a_{n+1} \oplus \ldots \oplus a_m$ with the complementary root spaces $a_{n+1}, \ldots, a_m$.

Recall $a_k = \mathbf{g} \cap (\mathbf{g}^{\alpha_k} \oplus \mathbf{g}^{-\alpha_k})$.

It will prove useful to consider the complexification of $\mathbf{m}$ to determine the almost complex structures. Recall that J extends canonically to a complex structure on $\mathbf{m}^{\mathbf{C}}$, also denoted by J.

Claim: J leaves the complementary complex root spaces $\mathbf{g}^{\alpha}$ invariant ($\alpha \in \{\pm\alpha_{n+1}, \ldots, \pm\alpha_m\}$).

Proof: This follows immediately from $\mathrm{ad}_H J = J\mathrm{ad}_H$ for all $H \in \mathbf{h}$. Let $X \in \mathbf{g}^\alpha$. We will show $JX \in \mathbf{g}^\alpha$:

$$\mathrm{ad}_H JX = J\mathrm{ad}_H X = J\alpha(H)X = \alpha(H)JX.$$

Alternatively, let $X \in \mathbf{g}^\alpha \setminus \{0\}$. Suppose $JX \in \mathbf{g}^\beta$. Then for all $H \in \mathbf{h}$

$$\begin{aligned} e^{\beta(H)}JX &= \exp(\mathrm{ad}_H).JX = \mathrm{Ad}(\exp H).JX \\ &= J\mathrm{Ad}(\exp H).X = J\exp(\mathrm{ad}_H).X \\ &= Je^{\alpha(H)}X = e^{\alpha(H)}JX \end{aligned}$$

which implies $\alpha = \beta$. □

Claim: J leaves the complementary root spaces a_k invariant ($k = n+1, \ldots, m$).

Proof: Let $X \in a_k = \mathbf{g} \cap (\mathbf{g}_k^\alpha \oplus \mathbf{g}^{-\alpha_k})$.

(i) $J : \mathbf{m} \to \mathbf{m}$ implies $JX \in \mathbf{m} \subset \mathbf{g}$.

(ii) $X = X_+ + X_-$ with $X_\pm \in \mathbf{g}^{\pm\alpha_k}$. From the previous claim we also have $JX_\pm \in \mathbf{g}^{\pm\alpha_k}$. Thus $JX = JX_+ + JX_- \in \mathbf{g}_k^\alpha \oplus \mathbf{g}^{-\alpha_k}$.

It now follows from (i) and (ii) that $JX \in a_k$. □

Claim: On each of the complementary root spaces a_k ($k = n+1, \ldots, m$) there are only two different complex structures which commute with the isotropy group.

Proof: Consider the complexification $a_k^{\mathbf{C}} = \mathbf{g}^{\alpha_k} \oplus \mathbf{g}^{-\alpha_k}$ of a_k. The extension of $J : a_k \to a_k$ to $J : a_k^{\mathbf{C}} \to a_k^{\mathbf{C}}$ has 1-dimensional $\pm i$ eigenspaces $a_k^{1,0}$ and $a_k^{0,1}$ which are invariant under J. Next note that a 1-dimensional complex space which is invariant under J has J acting as multiplication by $\pm i$. However, by the claim above, J leaves both $\mathbf{g}^{\alpha_k}$ and $\mathbf{g}^{-\alpha_k}$ invariant, thus acting by multiplication of $\pm i$. Since all space considered are 1-dimensional, it follows that eiher

- $\mathbf{g}^{\alpha_k} = Eig(i) = a_k^{1,0}$ and $\mathbf{g}^{-\alpha_k} = Eig(-i) = a_k^{0,1}$ or
- $\mathbf{g}^{\alpha_k} = Eig(-i) = a_k^{0,1}$ and $\mathbf{g}^{-\alpha_k} = Eig(i) = a_k^{1,0}$.

These are the only possibilities for a splitting of $a_k^{\mathbf{C}}$ which in turn determines the complex structure J. □

For all $k = n+1, \ldots, m$ let $\epsilon_k \in \{\pm 1\}$ be such that

$$a_k^{1,0} = \mathbf{g}^{\epsilon_k \alpha_k} \quad \text{and} \quad a_k^{0,1} = \mathbf{g}^{-\epsilon_k \alpha_k}.$$

The $\{\epsilon_k \alpha_k \mid k = n+1, \ldots, m\}$ are called the **roots of the almost complex structure** and determine J completely.

The splittings on each $a_k^{\mathbf{C}}$ into $\pm i$ eigenspaces of J determine a direct sum decomposition of $T_o^{\mathbf{C}} G/H = \mathbf{m}^{\mathbf{C}}$ into the $\pm i$ eigenspaces of J:

$$T_o^{\mathbf{C}} G/H = T_o^{1,0} G/H \oplus T_o^{0,1} G/H$$

with

$$T_o^{1,0} G/H = \sum_{k=n+1}^{m} \mathbf{g}^{\epsilon_k \alpha_k} \qquad T_o^{0,1} G/H = \sum_{k=n+1}^{m} \mathbf{g}^{-\epsilon_k \alpha_k}$$

The spaces $T_o^{1,0} G/H$ and $T_o^{0,1} G/H$ are invariant under the isotropy group $\mathrm{Ad}(H)$ and hence determine a direct sum decomposition

$$T^{\mathbf{C}} G/H = T^{1,0} G/H \oplus T^{0,1} G/H$$

with $T^{1,0}_{[x]}G/H := L_x T^{1,0}_o G/H$ and $T^{0,1}_{[x]}G/H := L_x T^{0,1}_o G/H$. On the other hand, we can define a G-invariant almost complex structure on G/H by choosing the roots for an almost complex structure, i.e. the space $T^{1,0}_o G/H$.

This gives a 1-1 correspondence

$$\{G\text{-invariant a.cx. structures on } G/H\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} T^{\mathbf{C}}_o G/H = \underbrace{\sum \mathbf{g}^{\epsilon_k \alpha_k}}_{T^{1,0}_o G/H} \oplus \underbrace{\sum \mathbf{g}^{-\epsilon_k \alpha_k}}_{T^{0,1}_o G/H}; \\ T^{1,0}_o G/H,\ T^{1,0}_o G/H \ \mathrm{Ad}(H)\text{-invariant} \end{array} \right\}.$$

The complex isomorphism $Z \mapsto Z + \bar{Z}$ from $T^{1,0}_o G/H$ to $T_o G/H$ gives $T_o G/H$ an $\mathrm{Ad}(H)$-invariant complex structure J. This completes the proof of Theorem 3.14.

Complex structures on flag manifolds

The question whether a G-invariant almost complex structure on G/H comes from a complex structure is answered by the following theorem.

Theorem 3.16 ([BH], p. 499) *The almost complex structure on G/H determined by $T^{1,0}_o G/H = \sum \mathbf{g}^{\epsilon_k \alpha_k}$ is integrable iff $p = \mathbf{h}^{\mathbf{C}} \oplus \sum \mathbf{g}^{\epsilon_k \alpha_k}$ is a Lie algebra.*

Corollary 3.17 *Flag manifolds allow G-invariant complex structures.*

Proof:

For a flag manifold $G/H = G^{\mathbf{C}}/P$ we have the direct sum decomposition

$$\mathbf{g}^{\mathbf{C}} = \underbrace{\overbrace{\mathbf{t}^{\mathbf{C}} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{\alpha} \oplus \sum_{\alpha \in T(S)} \mathbf{g}^{-\alpha}}^{\mathbf{h}^{\mathbf{C}}} \oplus \overbrace{\sum_{\beta \in \Delta^+ \setminus T(S)} \mathbf{g}^{\beta}}^{\mathbf{n}}}_{\text{parabolic subalgebra } p_S} \oplus \sum_{\beta \in \Delta^+ \setminus T(S)} \mathbf{g}^{-\beta}$$

where p_S is the parabolic subalgebra determinig G/H, therefore a Lie algebra. Thus the almost complex structure on the flag manifold G/H given by $T^{1,0}_o G/H =$

$\sum_{\beta\in\Delta^+\setminus T(S)} \mathbf{g}^\beta$ is integrable. $\square$

Theorem 3.18 *The number of different G-invariant complex structures on G/T is $|W(G)|$ where $W(G)$ is the Weyl group.*

Sketch of Proof: Let Δ^+ be *any* choice of positive roots. Then ($S = \emptyset$)

$$\mathbf{g}^{\mathbf{C}} = \mathbf{t}^{\mathbf{C}} \oplus \sum_{\beta\in\Delta+} \mathbf{g}^\beta \oplus \sum_{\beta\in\Delta+} \mathbf{g}^{-\beta}$$

and $T_o^{1,0}G/T = \sum_{\beta\in\Delta^+} \mathbf{g}^\beta$, $T_o^{0,1}G/T = \sum_{\beta\in\Delta^+} \mathbf{g}^{-\beta}$ define a G-invariant complex structure on G/T. Now for each $w \in W(G)$ the set $w(\Delta^+)$ gives another system of positive roots. Hence the number of different G-invariant complex structures on G/T is $|W(G)|$. $\square$

The next theorem states that certain homogeneous complex manifolds must be flag manifolds.

Theorem 3.19 ([BH], p.501) *Let H be a connected subgroup of the compact Lie group G with* rank H = rank G. *Then G/H allows a complex structure iff H is the centralizer of a torus in G i.e. iff G/H is a flag manifold.*

Note 3.20 *If Δ^+ is any choice of positive roots, then $T_o^{1,0}G/T = \sum_{\alpha\in\Delta^+} \mathbf{g}^\alpha$ and $T_o^{0,1}G/T = \sum_{\alpha\in\Delta^+} \mathbf{g}^{-\alpha}$ define a G-invariant complex structure on G/T. A map $\psi : S \to G/H$ from a Riemann surface S with lift $F : S \supset U \to G$ is holomorphic iff $F^{-1}dF(\partial_z) \in \mathbf{t}^{\mathbf{C}} \oplus \sum_{\alpha\in\Delta^+} \mathbf{g}^\alpha$.*

3.4 The m-symmetric space structure of G/T

The main references for this section are Burstall-Rawnsley [BR] and Bolton-Woodward [BW2]. See also section 5 of Salamon [Sal] for a treatment of 3-symmetric spaces.

Definition 3.21 *An* m**-symmetric space** *is a Riemannian manifold* M *such that for each* $p \in M$ *there exists an isometry* $\tau_p : M \to M$ *of order* m $(\tau_p^m = I)$, *such that* p *is an isolated fixed point and the map*

$$M \to \mathrm{Isom}(M), \quad p \mapsto \tau_p$$

is smooth.

In order to define and describe the m-symmetric space structure of G/T we will need a special element of the Lie algebra $\mathbf{g}$ called the canonical element which will be described below.

The canonical element of G/T

Definition 3.22 *Recall that if* α *is a root then* $\alpha(X) \in i\mathbf{R}$ $\forall X \in \mathbf{t}$. *If* $\alpha_1, \ldots, \alpha_\ell$ *are the simple roots, let* $\xi_1, \ldots, \xi_\ell \in \mathbf{t}$ *be such that* $\alpha_k(\xi_j) = i\delta_{kj}$. *If* p_S *is the parabolic subalgebra determined by the subset* $S \subseteq \{\alpha_1, \ldots, \alpha_\ell\}$ *let*

$$\xi = \sum_{\{j \ | \ \alpha_j \notin S\}} \xi_j \in \mathbf{t}.$$

ξ *is called the* **canonical element**.

Lemma 3.23 ([BR]) *The canonical element has the following properties.*

(a) $\xi \in$ *centre of* $\mathbf{h}$ = *torus centralized by* $\mathbf{h}$.

(b) *The eigenvalues of* ad_ξ *lie in* $i\mathbf{Z}$.

(c) *For* $r \in \mathbf{Z}$ *let* $\mathbf{g}_r$ *be the* ir *eigenspace of* ad_ξ. *Then* $p_S = \sum_{r\geq 0} \mathbf{g}_r$. *Also* $n^{(r)} = \sum_{j\geq r} \mathbf{g}_j$ *where* $n^{(r)}$ *is defined inductively by* $n^{(1)} = n$, $n^{(2)} = [n,n]$, $n^{(3)} = [n, n^{(2)}], \ldots$. *This is called the* **central descending series**. *Property* (c) *determines* ad_ξ *and since* $\mathbf{g}$ *has zero centre determines* ξ.

Example 3.24 *For* G/T *we have* $S = \emptyset$, $p_\emptyset$ = *lower triangular matrices.* $\mathbf{h}^{\mathbf{C}}$ *is the set of diagonal matrices and* n *consists of strictly lower triangular matrices.*

$$p_\emptyset = \mathbf{h}^{\mathbf{C}} \oplus n = \left\{ \left(\begin{array}{c|c|c|c} 0 & & & \\ \hline \star & \ddots & & \\ \hline \vdots & \ddots & \ddots & \\ \hline \star & \cdots & \star & 0 \end{array} \right) + \left(\begin{array}{c|c|c|c} \star & & & \\ \hline & \ddots & & \\ \hline & & \ddots & \\ \hline & & & \star \end{array} \right) \right\}.$$

Thus with the choice of simple roots $\alpha_1, \ldots, \alpha_\ell$ *from Appendix B.10 we have* $\xi_j = i\left\{ \left(\begin{array}{c|c} -I_j & 0 \\ \hline 0 & 0 \end{array} \right) - \frac{j}{n+1} I_{n+1} \right\}$ *and hence* $\xi = \sum_{j=1}^{\ell} \xi_j = i\{\mathrm{diag}(0,1,2,\ldots,n) - \frac{1}{2} n I_{n+1}\}$.
Therefore $\mathrm{ad}_\xi(X) = [\xi, X] = i[\mathrm{diag}(0,1,2,\ldots,n), X]$.

m-symmetric space structure

We will now construct the symmetry of order m at each point.

Theorem 3.25 ([Ji], p. 455) *The order* m *of symmetry of* G/T *is given via the highest root. Let*

$$m = 1 + \textit{height of highest root} = 1 + m_1 + \ldots + m_\ell$$

where $-\alpha_0 = m_1\alpha_1 + \ldots m_\ell\alpha_\ell$ *is the highest root. The* m_i *are non-negative integers and* $\{\alpha_1, \ldots, \alpha_\ell\}$ *a set of simple roots.* G/T *is not homeomorphic to the underlying manifold of an* k*-symmetric space for* $k = 2, \ldots, m-1$.

Example 3.26 $G = SU(n+1)$. *According to Appendix B.10 the highest root is given by* $-\alpha_0 = \alpha_1 + \ldots + \alpha_n$, *i.e.* $m_i = 1\ \forall\, i = 1, \ldots, n$. *Therefore* $SU(n+1)/T^n = U(n+1)/T^{n+1}$ *is an* $m = n+1$ *symmetric space. The canonical element of* T/G *is* $\xi = i\{\mathrm{diag}(0, 1, 2, \ldots, n) - \frac{1}{2}nI_{n+1}\}$.

We will now define the automorphism of G/T which defines the m-symmetric structure of G/T.

Definition 3.27 ([BW2], p.74) *Let* ξ *be the canonical element and let* $h = \exp(\frac{2\pi}{m}\xi) \in T$. *Define the inner automorhpism* $\tau = i_h : G \to G$ *by* $\tau(g) = hgh^{-1}$. τ *is called the* **Coxeter automorphism**.

Example 3.28 *For* $SU(n+1)$ *we have*

$$\tau(g) = \mathrm{diag}(1, \mu, \ldots, \mu^n)\ g\ \mathrm{diag}(1, \bar{\mu}, \ldots, \bar{\mu}^n),$$

where $\mu = e^{\frac{2\pi i}{m}} = e^{\frac{2\pi i}{n+1}}$.

Lemma 3.29 (Properties of the Coxeter automorphism)

- τ *has order* m.
- $G^\tau = T$, *i.e. the fixed point set of* τ *is* T.

  Example: If m *is smaller then given by the Theorem 3.25, e.g.* $m = 2$ *for* $SU(n+1)$ *with* $n \geq 2$ *we have*

$$\tau(g) = \mathrm{diag}(1, -1, \ldots, \pm 1)\ g\ \mathrm{diag}(1, -1, \ldots, \pm 1)$$

and hence $G^\tau = S(U([\frac{n+1}{2}]+1) \times U([\frac{n+1}{2}])) \neq T$.

- *For all* $[x] \in G/T$ τ *induces a map* $\tau_{[x]} : G/T \to G/T$ *of order* m *where* $[x]$ *is an isolated fixed point.*

  Let $\tau_o([g]) = [\tau(g)]$.

$$\begin{array}{ccc} G & \xrightarrow{\tau} & G \\ \downarrow \pi & & \downarrow \pi \\ G/T & \xrightarrow{\tau_o} & G/T \end{array}$$

Then by the above $o = eT = T$ *is an isolated fixed point of* τ_o. *Define now* $\tau_{[x]} = \ell_{[x]} \circ \tau_{[x]} \circ \ell_{[x]^{-1}}$ *where* ℓ *denotes left translation in* G/T.

$$\begin{array}{ccc} G/T & \xrightarrow{\tau_{[x]}} & G/T \\ \uparrow \ell_{[x]} & & \uparrow \ell_{[x]} \\ G/T & \xrightarrow{\tau_o} & G/T \end{array}$$

Then $[x]$ *is an isolated fixed point of* $\tau_{[x]}$. $\tau_{[x]}$ *has the same order as* τ_o: $\operatorname{ord} \tau_{[x]} = m$.

- *If* G/T *is equipped with a* G*-invariant metric then* $(G/T, \{\tau_{[x]}\})$ *is an* m*-symmetric space.*

Canonical decomposition of $T^{\mathbf{C}}G/T$

We now investigate the canonical decomposition induced by the derivative of the Coxeter automorphism $d\tau = \operatorname{Ad}(h)$.

Lemma 3.30 (Properties of $d\tau$**)**

- $\operatorname{Ad}(h)$ *has order* m
- $\operatorname{Ad}(h) : \mathbf{g}^{\mathbf{C}} \to \mathbf{g}^{\mathbf{C}}$ *has* m*-th roots of unity as eigenvalues:* μ^k *with* $\mu = e^{\frac{2\pi i}{m}}$.

- $\mathbf{g}^{\mathbf{C}}$ *splits into the direct sum of the* μ^k *eigenspaces of* $\mathrm{Ad}(h)$.

$$\mathbf{g}^{\mathbf{C}} = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_{m-1}$$

where $\mathcal{M}_k$ *is the* μ^k *eigenspace.*

- $\mathcal{M}_r$ *is the direct sum of eigenspaces of roots of height* $s = r \bmod m$.

- $\mathcal{M}_0 = \mathbf{t}^{\mathbf{C}}$

- $\mathcal{M}_1 = \oplus_0^\ell \mathbf{g}^{\alpha_k}$: *Let* $X \in \mathbf{g}^{\alpha_k}$. *Then* $\mathrm{Ad}(h).X = \exp \frac{2\pi}{m}\mathrm{ad}_\xi.X = e^{\frac{2\pi}{m}\alpha_k(\xi)}X = e^{\frac{2\pi i}{m}}X = \mu X$ *since* $\alpha_k(\xi_j) = i\delta_{jk}$.

- $[\mathcal{M}_r, \mathcal{M}_s] = \mathcal{M}_{r+s}$.

- $[\mathcal{M}_0, \mathcal{M}_k] = \mathcal{M}_k$ *ensures that* $[\mathcal{M}_k]_{gT}$ *in Notation 3.32 is well defined.*

For the relation between τ-adapted maps and Toda equations (see Chapter 2) the existence of a special element of $\mathcal{M}_1$ is required.

Definition 3.31 *An element* $\xi \in \mathcal{M}_1$ *is called* **cyclic** *if*

$$\xi = \sum_{k=0}^{\ell} a_k X_{\alpha_k} \quad \text{with} \quad a_k \in \mathbf{C} \setminus \{0\} \ \ \forall\, k.$$

Notation 3.32 *Denote by* $[\mathcal{M}_k]$ *the vector bundle over* G/T *obtained by left translating* $\mathcal{M}_k$, *i.e.* $[\mathcal{M}_k]_{gT} = L_g(\mathcal{M}_k) \subset T^{\mathbf{C}}_{gT}(G/T)$.

From the above we therefore have

$$T^{\mathbf{C}}(G/T) = \bigoplus_{k=1}^{m-1} [\mathcal{M}_k] \quad \text{and} \quad [\mathcal{M}_1] = \bigoplus_{j=0}^{\ell} [\mathbf{g}^{\alpha_j}].$$

Example 3.33 *For $sl(n+1, \mathbf{C}) = su(n+1)^{\mathbf{C}}$ we have $m = n+1$. Therefore it splits into the direct sum of the μ^k eigenspaces $\mathcal{M}_k$ where $\mu = e^{\frac{2\pi i}{n+1}}$*

$$sl(n+1, \mathbf{C}) = \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \ldots \oplus \mathcal{M}_n.$$

Represented as matrices we have

$$\mathcal{M}_0 = \left(\begin{array}{c|c|c|c|c} \star & & & & \\ \hline & \ddots & & & \\ \hline & & \ddots & & \\ \hline & & & \ddots & \\ \hline & & & & \star \end{array}\right), \mathcal{M}_1 = \left(\begin{array}{c|c|c|c|c} 0 & & & & \star \\ \hline \star & 0 & & & \\ \hline & \ddots & \ddots & & \\ \hline & & \ddots & 0 & \\ \hline & & & \star & 0 \end{array}\right),$$

$$\mathcal{M}_2 = \left(\begin{array}{c|c|c|c|c} 0 & & & \star & \\ \hline & 0 & & & \star \\ \hline \star & & \ddots & & \\ \hline & \ddots & & 0 & \\ \hline & & \star & & 0 \end{array}\right), \ldots, \mathcal{M}_n = \left(\begin{array}{c|c|c|c|c} 0 & \star & & & \\ \hline & 0 & \ddots & & \\ \hline & & \ddots & \ddots & \\ \hline & & & 0 & \star \\ \hline \star & & & & 0 \end{array}\right)$$

Chapter 4

τ-adapted maps and Toda equations

In this chapter we will consider τ-adapted maps into G/T. These maps are adapted to the m-symmetric space structure of G/T and have a number of interesting geometric properties. We will then look at two classes of τ-adapted maps, namely τ-primitive and τ-holomorphic maps satisfying a non-singularity / holomorphicity condition. It will be seen that τ-adapted maps provide - via Toda frames - a geometric interpretation of solutions of Toda equations. Finally, we will introduce invariants which determine τ-adapted maps up to congruence. The main references for this chapter are [BW2] and [BPW]. The concept of τ-primitive maps was first introduced in [BW4] and [BP] (simply called primitive maps in [BP]). A good account of (τ-)primitive maps and their relation ot harmonic maps may be found in [G].

4.1 τ-adapted maps

In this section we look at maps from Riemann surfaces into flag manifolds which are adapted to the m-symmetric space structure.

Definition 4.1 *Let S be a Riemann surface and let G/T be a flag manifold equipped with some G-invariant metric. Recall that G/T is an m-symmetric space with symmetry τ of order m at each point of G/T. A conformal immersion $\psi : S \to G/T$ is called* τ-**adapted** *if, for each $p \in S$, the symmetry $\tau_{\psi(p)}$ maps $d\psi_p(T_pS)$ into itself by rotation through $\frac{2\pi}{m}$.*

Note 4.2 *Since $\tau : \mathbf{g} \to \mathbf{g}$ is an automorphism of order m it gives rise to the following splitting*

$$\mathbf{g}^{\mathbf{C}} = \mathcal{M}_0 \oplus \ldots \oplus \mathcal{M}_{m-1}$$

where $\mathcal{M}_k$ is the μ^k-eigenspace ($\mu = e^{\frac{2\pi i}{m}}$). Because $\mathcal{M}_0 = \mathbf{t}^{\mathbf{C}}$ and $\mathbf{g}^{\mathbf{C}} = \mathbf{t}^{\mathbf{C}} \oplus \mathbf{m}^{\mathbf{C}}$ we get for the complexified tangent bundle of G/T

$$T^{\mathbf{C}}(G/T) = [\mathbf{m}^{\mathbf{C}}] = [\mathcal{M}_1] \oplus \ldots \oplus [\mathcal{M}_{m-1}]$$

where $[\mathcal{M}_k]$ denotes the vector bundle over G/T obtained by left translating $\mathcal{M}_k$. Hence τ-adapted means

$$d\psi(T^{1,0}S) \subseteq [\mathcal{M}_1],$$

i.e.

$$\tau d\psi(\frac{\partial}{\partial z}) = \mu d\psi(\frac{\partial}{\partial z}).$$

Definition 4.3 *Let K be a closed subgroup of G containing T. Then a smooth map $\psi : S \to G/K$ is* **equiharmonic** *if it is harmonic with respect to* any *G-invariant* *metric on G/K.*

Theorem 4.4 ([BW2], [B]) *Let $\psi : S \to G/T$ be τ-adapted and let K be any closed subgroup of G with $T \subseteq K$. Denote the natural projection by $\pi : G/T \to G/K$. Then $\pi \circ \psi : S \to G/K$ is equiharmonic and in particular ψ is equiharmonic.*

Corollary 4.5 *The conformal map ψ is a harmonic conformal immersion and hence its image is a minimal surface.*

Definition 4.6 *Let S be a Riemann surface and let G/T be a flag manifold with G-invariant metric. Choosing a set of positive roots for* $\mathbf{g}$ *gives rise to a complex structure on G/T given by $T^{1,0}G/T = \sum_{\alpha \in \Delta^+} \mathbf{g}^{\alpha}$ (see Chapter 3.3). Thus S and G/T are complex manifolds. A conformal immersion $\psi : S \to G/T$ is called* **τ-primitive** *if it is τ-adapted and if $d\psi(T^{1,0}S)$ contains a cyclic element. ψ is called* **τ-holomorphic** *if it is τ-adapted and holomorphic.*

4.2 Toda equations are the integrability condition for Toda frames

Details about Toda frame

Definition 4.7 *A local frame*

$$F : S \supseteq U \to G$$

is called a **Toda frame** *if there is a complex coordinate* $z : U \to \mathbf{C}$ *and a smooth map* $\Omega : U \to \mathfrak{t}$ *such that*

$$F^{-1}\partial_z F = \partial_z \Omega + \mathrm{Ad}(\exp \Omega).B \in \mathcal{M}_0 \oplus \mathcal{M}_1,$$

where $B = \sum_{0,1}^{\ell} \sqrt{m_j} X_{\alpha_j} \in \mathcal{M}_1$. *The* $\{\alpha_1, \ldots, \alpha_\ell\}$ *are a set of simple roots,* $-\alpha_0 = \sum m_j \alpha_j$ *is the highest root and* $\{X_\alpha\}$ *is a set of Cartan-Weyl generators. If* $j = 1, \ldots, \ell$ *the frame is called an* **open Toda frame**, *if* $j = 0, \ldots, \ell$ *it is called an* **affine Toda frame**.

Claim 4.8 *We have*

$$F^{-1}\partial_z F = \partial_z \Omega + \sum_j \sqrt{m_j} e^{\alpha_j(\Omega)} X_{\alpha_j}$$

For the proof of Claim 4.8 we will need the following Lemma.

Lemma 4.9 ([He], p.128)

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad} X).$$

In other words the following diagram commutes, where $\exp : \mathrm{End}(\mathbf{g}) \to \mathrm{Aut}(\mathbf{g})$ *is given by* $A \mapsto \sum \frac{1}{m!} A^m$.

$$\begin{array}{ccc} \mathbf{g} & \xrightarrow{\mathrm{ad}} & \mathrm{End}(\mathbf{g}) \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}(\mathbf{g}) \end{array}$$

This follows from the naturality of the exponential map.

Proof of Claim 4.8:

First we will show that for $X_\alpha \in \mathbf{g}^\alpha$ we have $\mathrm{Ad}(\exp \Omega).X_\alpha = e^{\alpha(\Omega)} X_\alpha$.

Since $\mathrm{ad}(\Omega).X_\alpha = \alpha(\Omega)X_\alpha$ we have $\mathrm{ad}(\Omega)^n.X_\alpha = \alpha(\Omega)^n X_\alpha$. Thus

$$\begin{aligned}\mathrm{Ad}(\exp\Omega).X_\alpha &= \exp(\mathrm{ad}(\Omega)).X_\alpha = \sum_n \frac{1}{n!}\mathrm{ad}(\Omega)^n.X_\alpha \\ &= \sum_n \frac{1}{n!}\alpha(\Omega)^n X_\alpha = e^{\alpha(\Omega)}X_\alpha.\end{aligned}$$

By linearity it is now clear that

$$\mathrm{Ad}(\exp\Omega).B = \mathrm{Ad}(\exp\Omega).(\sum\sqrt{m_j}X_{\alpha_j}) = \sum\sqrt{m_j}\mathrm{Ad}(\exp\Omega).X_{\alpha_j} = \sum\sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j}$$

Thus

$$F^{-1}\partial_z F = \partial_z\Omega + \mathrm{Ad}(\exp\Omega).B = \partial_z\Omega + \sum_j \sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j}.$$

□

Toda equations are integrability conditions

Claim 4.10 *The integrability conditions for the Toda frame are the Toda equations* $2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)}H_{\alpha_j} = 0$ *where* $H_\alpha = \frac{2\alpha^\sharp}{\kappa(\alpha^\sharp,\alpha^\sharp)}$.

Proof:

For a Toda frame we have

$$\begin{aligned}F^{-1}\partial_z F &= \partial_z\Omega + \sum\sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j} = A_0 + A_1, \quad A_i \in \mathcal{M}_i \\ F^{-1}\partial_{\bar z} F &= -\partial_{\bar z}\Omega - \sum\sqrt{m_j}e^{\alpha_j(\Omega)}X_{-\alpha_j} = \bar A_0 + \bar A_1, \quad \bar A_i \in \mathcal{M}_{-i}\end{aligned}$$

Let $A = F^{-1}\partial_z F$ and $C = F^{-1}\partial_{\bar z}F$, i.e. $\partial_z F = FA$ and $\partial_{\bar z}F = FC$. Taking derivatives gives

$$\partial_{\bar z}\partial_z F = \partial_{\bar z}(FA) = (\partial_{\bar z}F)A + F\partial_{\bar z}A = F(CA + \partial_{\bar z}A)$$

and

$$\partial_z \partial_{\bar z} F = \partial_z (FC) = (\partial_z F)C + F\partial_z C = F(AC + \partial_z C).$$

Now using the integrability condition

$$\frac{\partial^2 F}{\partial z \partial \bar z} = \frac{\partial^2 F}{\partial \bar z \partial z}.$$

we get

$$F(CA + \partial_{\bar z} A) = F(AC + \partial_z C)$$

or

$$\partial_{\bar z} A - \partial_z C = [A, C].$$

Since $A = A_0 + A_1$ and $C = \bar A_0 + \bar A_1$ for the Toda frame F this becomes

$$\begin{aligned} \partial_{\bar z}(A_0 + A_1) - \partial_z(\bar A_0 + \bar A_1) &= [A_0 + A_1, \bar A_0 + \bar A_1] \\ &= [A_0, \bar A_1] + [A_1, \bar A_1] + [A_1, \bar A_0] \in \mathcal{M}_{-1} \oplus \mathcal{M}_0 \oplus \mathcal{M}_1. \qquad (\star) \end{aligned}$$

Note that this expression is real and that $\mathcal{M}_0$ is abelian, so $[A_0, \bar A_0] = 0$.

For the $\mathcal{M}_0$ part we have

$$\frac{\partial A_0}{\partial \bar z} - \frac{\partial \bar A_0}{\partial z} = [A_1, \bar A_1]$$

where $A_0, A_1, \bar A_0, \bar A_1$ are given by

$$\begin{aligned} A_0 &= \partial_z \Omega \\ A_1 &= \sum \sqrt{m_j} e^{\alpha_j(\Omega)} X_{\alpha_j} \\ \bar A_0 &= -\partial_{\bar z} \Omega \\ \bar A_1 &= -\sum \sqrt{m_j} e^{\alpha_j(\Omega)} X_{-\alpha_j}. \end{aligned}$$

Therefore

$$\frac{\partial A_0}{\partial \bar{z}} - \frac{\partial \bar{A}_0}{\partial z} = \partial_{\bar{z}}\partial_z\Omega - (-\partial_z\partial_{\bar{z}}\Omega) = 2\Delta\Omega$$

and

$$[A_1, \bar{A}_1] = [\sum \sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j}, -\sum \sqrt{m_k}e^{\alpha_k(\Omega)}X_{-\alpha_k}] = -\sum m_j e^{2\alpha_j(\Omega)}H_{\alpha_j}$$

as $[X_{\alpha_i}, X_{-\alpha_j}] = \delta_{ij}H_{\alpha_i}$.

Thus

$$2\Delta\Omega = -\sum m_j e^{2\alpha_j(\Omega)}H_{\alpha_j}$$

or

$$2\Delta\Omega + \sum m_j e^{2\alpha_j(\Omega)}\frac{2}{\kappa(\alpha_j^\sharp, \alpha_j^\sharp)}\alpha_j^\sharp = 0.$$

□

Remark 4.11 *We have seen already in Chapter 2 that the Toda equations may be expressed in terms of η-invariants.*

4.3 $\psi : S \to G/T$ τ-holomorphic $\Longleftrightarrow$ $\exists$ open Toda frame $F : U \to G$

In order to show the correspondence between τ-holomorphic maps and Toda frames we will need the following claim.

Claim 4.12 *Let $B = \sum_1^\ell \sqrt{m_j} X_{\alpha_j} \in \mathcal{M}_1$ and let $A = \sum_1^\ell a_j X_{\alpha_j} \in \mathcal{M}_1$ be non-singular, i.e. $a_j \neq 0\ \forall j = 1, \dots \ell$. Then there exists an $\Xi \in \mathbf{t}^{\mathbf{C}}$ such that*

$$\mathrm{Ad}(\exp \Xi).B = A$$

Proof: As in the proof of Claim 4.8 we have for any $\Xi \in \mathbf{t}^{\mathbf{C}}$

$$\mathrm{Ad}(\exp \Xi).B = \sum_{j=1}^{\ell} \sqrt{m_j} e^{\alpha_j(\Xi)} X_{\alpha_j}.$$

We want to determine Ξ such that $\mathrm{Ad}(\exp \Xi).B = A = \sum_1^\ell a_j X_{\alpha_j}$, i.e.

$$a_j = \sqrt{m_j} e^{\alpha_j(\Xi)} \quad \forall j = 1, \dots, \ell$$

or

$$\alpha_j(\Xi) = \log \frac{a_j}{\sqrt{m_j}} \quad \forall j = 1, \dots, \ell.$$

Since $\{\alpha_1, \dots, \alpha_\ell\}$ is a basis for $(\mathbf{t}^{\mathbf{C}})^*$ this linear system can be solved uniquely (w.r.t. the chosen branch of the logarithm) to give the required Ξ. For this Ξ, $\mathrm{Ad}(\exp \Xi).B = A$.

□

Lemma 4.13 (c.f. [BW2], p.77, and [BPW], p.126)

$\psi : S \to G/T$ is τ-holomorphic iff there exists an open Toda frame $F : U \to G$.

Proof: Let $\psi : S \to G/T$ be τ-holomorphic and let $z : U \to \mathbf{C}$ be a complex coordinate on a simply connected open subset U of S. Recall that a frame $F : U \to G$ is a Toda frame if there exists a smooth map $\Omega : U \to i\mathbf{t}$ such that

$$F^{-1}\partial_z F = \partial_z \Omega + \mathrm{Ad}(\exp \Omega).B \in \mathcal{M}_0 \oplus \mathcal{M}_1,$$

where $B = \sum \sqrt{m_j} X_{\alpha_j} \in \mathcal{M}_1$. We will now construct maps F and Ω satisfying this relation.

Let F be any local framing of $\psi : U \to G/T$. Since ψ is τ-adapted we have

$$F^{-1}\partial_z F = A_0 + A_1 \in \mathcal{M}_0 \oplus \mathcal{M}_1.$$

We need a map Ω such that $A_1 = \mathrm{Ad}(\exp \Omega).B$ and, in general, we will have to regauge F in order to achieve this.

Construction of Ω:

Since ψ is τ-holomorphic, A_1 is non-singular except for a finite number of points ([BW2], p.76) and varies smoothly with z. Hence we can apply Claim 4.12 with a unique branch of the logarithm on the simply connected domain U (possibly reduced to exclude singular points) to find a smooth map $\Xi : U \to \mathbf{t}^{\mathbf{C}}$ such that

$$\mathrm{Ad}(\exp \Xi).B = A_1.$$

We can write Ξ as the sum of its real and imaginary part:

$$\Xi = \Lambda + \Omega,$$

where $\Lambda = \frac{1}{2}(\Xi + \bar{\Xi}) = \bar{\Lambda}$ is the real and $\Omega = \frac{1}{2}(\Xi - \bar{\Xi}) = -\bar{\Omega}$ is the imaginary part.

Regauging F so that it satisfies the Toda frame differential equation:

Because we need a frame F with $A_1 = \mathrm{Ad}(\exp \Omega).B$ we will now regauge the frame F from above by $\exp \Lambda$. Let $\tilde{F} = F \exp \Lambda$. Since Λ is a map into $\mathbf{t}$ we find that $\exp \Lambda \in T$. Thus $\psi = \pi F = \pi \tilde{F}$, so $\tilde{F}$ is a local frame for ψ. Now

$$\tilde{A}_0 + \tilde{A}_1 \quad = \quad \tilde{F}^{-1}\partial_z \tilde{F} = (\exp \Lambda)^{-1} F^{-1} \left(\frac{\partial F}{\partial z} \exp \Lambda + F \frac{\partial \Lambda}{\partial z} \exp \Lambda \right)$$

$$= (\exp\Lambda)^{-1}(A_0 + A_1)\exp\Lambda + \frac{\partial\Lambda}{\partial z}$$

as $\mathrm{Ad}(\exp\Lambda)$ acts trivially on the tangent space of $\mathbf{t}$

$$\begin{aligned} &= \frac{\partial\Lambda}{\partial z} + \mathrm{Ad}((\exp\Lambda)^{-1}).(A_0 + A_1) \\ &= \frac{\partial\Lambda}{\partial z} + \mathrm{Ad}((\exp(-\Lambda)).(A_0 + A_1) \\ &= \frac{\partial\Lambda}{\partial z} + A_0 + \mathrm{Ad}((\exp(-\Lambda)).A_1, \end{aligned}$$

so

$$\tilde{A}_0 = A_0 + \frac{\partial\Lambda}{\partial z}$$

and

$$\begin{aligned} \tilde{A}_1 &= \mathrm{Ad}((\exp(-\Lambda)).A_1 \\ &= \mathrm{Ad}((\exp(-\Lambda)).\mathrm{Ad}(\exp\Xi).B \\ &= \mathrm{Ad}((\exp(-\Lambda)).\mathrm{Ad}(\exp(\Lambda + \Omega).B \\ &= \mathrm{Ad}(\exp\Omega).B \end{aligned}$$

For simplicity denote the new $\tilde{F}$ by F again. We then have

$$F^{-1}\partial_z F = A_0 + A_1 = A_0 + \mathrm{Ad}(\exp\Omega).B \in \mathcal{M}_0 \oplus \mathcal{M}_1$$

with a smooth map $\Omega : U \to i\mathbf{t}$ (i.e. mapping into the purely imaginary part of $\mathbf{t}^{\mathbf{C}}$). So it only remains to prove that

$$A_0 = \partial_z\Omega.$$

From $A_1 = \mathrm{Ad}(\exp\Omega).B$ we get

$$\partial_{\bar{z}}A_1 = [\partial_{\bar{z}}\Omega, A_1].$$

On the other hand the integrability conditions give (c.f. equation ($\star$))

$$\partial_{\bar{z}} A_1 = [A_1, \bar{A}_0].$$

Thus

$$[A_1, \partial_{\bar{z}}\Omega + \bar{A}_0] = 0,$$

i.e. $\partial_{\bar{z}}\Omega + \bar{A}_0$ is in the centralizer of A_1 $\forall z$.

But the centralizer of A_1 is a Cartan subalgebra orthogonal to $\mathcal{M}_0$. Since $\partial_{\bar{z}}\Omega + \bar{A}_0 \in \mathcal{M}_0$ this yields $\partial_{\bar{z}}\Omega + \bar{A}_0 = 0$ or, taking the complex conjugate,

$$A_0 = -\overline{\partial_{\bar{z}}\Omega} = -\partial_z\bar{\Omega} = \partial_z\Omega$$

since $\bar{\Omega} = -\Omega$.

Therefore F is the required Toda frame. □

A similar theorem also holds for τ-primitive maps. See [BPW].

Theorem 4.14 ([BPW], p.126) *$\psi : S \to G/T$ τ-primitive $\Longleftrightarrow$ $\exists$ affine Toda frame $F : U \to G$.*

Chapter 5

A congruence theorem for $SU(n+1)/T^n$

In this chapter we will sketch the proof of the constant curvature congruence theorem for τ-holomorphic S^2 in $SU(n+1)/T^n$. It was the first congruence theorem obtained during the course of research for this thesis and it will serve as a motivation for the subsequent generalisations in chapters 7 and 8.

5.1 The Veronese sequence and congruence theorems

Definition 5.1 ([BJRW] p.608) *Let $\phi : S^2 \to \mathbf{CP}^n$ be the holomorphic embedding defined by*

$$\phi([z_0, z_1]) = [z_0^n, \sqrt{\binom{n}{1}} z_0^{n-1} z_1, \dots, \sqrt{\binom{n}{k}} z_0^{n-k} z_1^k, \dots, z_1^n],$$

where $[z_0, z_1] \in \mathbf{CP}^1 = S^2$. *Alternatively, in terms of the holomorphic coordinate* $z = z_0/z_1$ *on* S^2 *we may write*

$$\phi(z) = [1, \sqrt{\binom{n}{1}} z, \ldots, \sqrt{\binom{n}{k}} z^k, \ldots, z^n].$$

ϕ *is called the* **Veronese embedding**.

Let $\phi_0, \ldots, \phi_n$ *be the harmonic sequence of* $\phi = \phi_0$. *We call* $\phi_0, \ldots, \phi_n$ *the* **Veronese sequence**. *For the specific form of the* ϕ_p *and more further information see [BJRW], p.609.*

For the Veronese embedding and sequence we have the following two remarkable theorems.

Theorem 5.2 ([Ri]) *The Veronese embedding is of constant curvature and, up to holomorphic isometries of* $\mathbf{CP}^n$, *is the only such linearly full holomorphic curve.*

Theorem 5.3 ([BJRW] p.611) *Let* $\phi : S^2 \to \mathbf{CP}^n$ *be a linearly full conformal immersion of constant curvature. Then, up to a holomorphic isometry of* $\mathbf{CP}^n$, *the harmonic sequence determined by* ϕ *is the Veronese sequence.*

5.2 A congruence theorem for τ-holomorphic $\psi : S^2 \to SU(n+1)/T^n$

We will prove the following theorem.

Theorem 5.4 *Let* $\psi : S^2 \to SU(n+1)/T^n$ *be a* τ*-holomorphic map with induced metric of constant curvature. Then* ψ *is congruent to the Veronese sequence.*

As a corollary we get the following congruence theorem for τ-holomorphic maps of constant curvature.

Theorem 5.5 *Let $\psi, \tilde{\psi} : S^2 \to SU(n+1)/T^n$ be τ-holomorphic maps with induced metrics of constant curvature. Then ψ and $\tilde{\psi}$ are congruent to each other.*

Sketch of Proof of Theorem 5.4:

Through the following steps we will see that a τ-holomorphic ψ can be assigned a set of invariants which in turn determine ψ up to congruence (weak congruence theorem). The induced metric of ψ can be expressed as the sum of these γ-invariants, and the associated curves ψ_j have metric $\gamma_j|dz|^2$. Using a factorisation theorem, we will then see that if ψ is of constant curvature then so are the ψ_j. However, if the ψ_j are of constant curvature then the γ-invariants of ψ coincide with those of the Veronese sequence. Therefore ψ is congruent to the Veronese sequence by the weak congruence theorem which concludes the proof.

τ-holomorphic maps and their γ-invariants:

From Chapter 2 and Chapter 4.3 we have the following correspondence

$$\begin{array}{ccc} \{\psi : S \to SU(n+1)/T^n \ \tau\text{-holomorphic}\} & \longleftrightarrow & \{\phi : S \to \mathbf{CP}^n \text{ holomorphic}\} \\ \psi & \mapsto & \pi\psi \\ (\phi_0|\ldots|\phi_n) & \leftarrow & \phi = \phi_0 \end{array}$$

Let $\psi : S^2 \to SU(n+1)/T^n$ be τ-holomorphic. Then, by the above correspondence, ψ gives rise to a harmonic sequence $[f_0], \ldots, [f_n]$ ($\psi = (f_0|\ldots|f_n)$). The γ-invariants for the harmonic sequence are given by $\gamma_p = \frac{|f_{p+1}|^2}{|f_p|^2}$. From the definition of the harmonic sequence we have

$$\begin{aligned}
\partial_z \partial_{\bar z} \log |f_p|^2 &= \gamma_p - \gamma_{p-1} \\
\partial_z \partial_{\bar z} \log \gamma_p &= \gamma_{p-1} - 2\gamma_p + \gamma_{p-1} \\
\partial_z f_p &= f_{p+1} + \partial_z \log |f_p|^2 f_p \\
\partial_{\bar z} f_p &= -\gamma_{p-1} f_{p-1}
\end{aligned}$$

Thus every τ-holomorphic $\psi : S^2 \to SU(n+1)/T^n$ can be assigned the set of γ-invariants: $\gamma_0, \ldots, \gamma_{n-1}$.

A weak congruence theorem:

Let $\psi, \tilde\psi : S^2 \to SU(n+1)/T^n$ be τ-holomorphic maps whose γ-invariants coincide, i.e. $\gamma_j = \tilde\gamma_j \ \forall\ j$. Then $\pi\psi$ and $\pi\tilde\psi$ are both holomorphic maps into $\mathbf{CP}^n$ with $\gamma_{-1} = \tilde\gamma_{-1} = 0$ and $\gamma_0 = \tilde\gamma_0$. Thus, by Theorem 1.15, $\pi\psi$ and $\pi\tilde\psi$ are congruent in $\mathbf{CP}^n$ and there exists a $g \in SU(n+1)$ such that $\pi\psi = [g]\pi\tilde\psi = \pi g\tilde\psi$ ($[g] \in PU(n+1)$). From the above correspondence we get $\psi = g\tilde\psi$ (lift to τ-holomophic maps). Therefore the γ-invariants determine τ-holomophic maps up to congruence.

The metric of ψ and its associated curves ψ_j:

The induced metric of ψ is $ds^2 = \sum \gamma_j |dz|^2$ (see chapter 6.1 with metric coefficients $k_j = 1$ and $\eta_j = \gamma_{j-1}$).

Consider the projections

$$\pi_j : SU(n+1)/T^n \to G_{j+1}(\mathbf{C}^{n+1}) = SU(n+1)/S(U(j+1)\times U(n-j)) \subset P(\bigwedge^{j+1} \mathbf{C}^{n+1})$$

given by $\pi_j(gT) = [g_0 \wedge \ldots \wedge g_j] \quad (g = (g_0|\ldots|g_n))$.

Let $[f_0], \ldots, [f_n]$ be the Frenet frame for $\phi = \pi\psi : S^2 \to \mathbf{CP}^n$. Let $\hat{F} = (\frac{f_0}{|f_0|}|\ldots|\frac{f_n}{|f_n|}) \in U(n+1)$ and $\alpha = \frac{1}{\det \hat{F}^{1/(n+1)}}$. Then $F = \alpha\hat{F} \in SU(n+1)$ is the Toda lift for ψ.

Therefore the j-th associated curve

$$\psi_j = \pi_j\psi : S^2 \to G_{j+1}(\mathbf{C}^{n+1}) = SU(n+1)/S(U(j+1)\times U(n-j)) \subset P(\bigwedge^{j+1} \mathbf{C}^{n+1})$$

is given by

$$F = (\alpha\frac{f_0}{|f_0|}, \ldots, \alpha\frac{f_n}{|f_n|}) \mapsto [\alpha\frac{f_0}{|f_0|} \wedge \ldots \wedge \alpha\frac{f_j}{|f_j|}] = [f_0 \wedge \ldots \wedge f_j].$$

$$\begin{array}{ccc} S^2 & \xrightarrow{\psi} & SU(n+1)/T^n \\ & \overset{\psi_j}{\searrow} & \pi_j \downarrow \\ & & G_{j+1}(\mathbf{C}^{n+1}) \end{array}$$

Claim: The metric induced by ψ_j is $ds_j^2 = \gamma_j|dz|^2$.

Proof:

$$\partial_z(f_0 \wedge \ldots \wedge f_j) = (\star)f_0 \wedge \ldots \wedge f_j + \underbrace{f_0 \wedge \ldots \wedge f_{j-1} \wedge f_{j+1}}_{\text{orthog. to plane } f_0\wedge\ldots\wedge f_j}$$

The change of this plane orthogonal to the plane $f_0 \wedge \ldots \wedge f_j$ is

$$\frac{|f_0 \wedge \ldots \wedge f_{j-1} \wedge f_{j+1}|^2}{|f_0 \wedge \ldots \wedge f_j|^2} = \frac{|f_0|^2\cdots|f_{j-1}|^2|f_{j+1}|^2}{|f_0|^2\cdots|f_j|^2} = \frac{|f_{j+1}|^2}{|f_j|^2} = \gamma_j$$

Thus $ds_j^2 = \gamma_j|dz|^2$. But also

$$\Delta\log|f_0\wedge\ldots\wedge f_j|^2 = \Delta\log|f_0|^2\cdots|f_j|^2 = (\gamma_0 - 0) + (\gamma_1 - \gamma_0) + \ldots(\gamma_j - \gamma_{j-1}) = \gamma_j.$$

Hence $ds_j^2 = \gamma_j|dz|^2 = \Delta\log|f_0 \wedge \ldots \wedge f_j|^2|dz|^2$.

If ψ has constant curvature then the ψ_j have also constant curvature:

From above we have

$$\gamma_0+\ldots+\gamma_{n-1} = \Delta\log|f_0|^2+\ldots+\Delta\log|f_0\wedge\ldots\wedge f_{n-1}|^2 = \Delta\log|f_0|^2\cdots|f_0\wedge\ldots\wedge f_{n-1}|^2$$

f_0 may be chosen to be a polynomial in z for $\psi : S^2 \to SU(n+1)/T^n$ (both S^2 and $SU(n+1)/T^n$ are algebraic varieties).

From

$$|f_0 \wedge f_1 \wedge \cdots \wedge f_j|^2 = |f_0 \wedge f_0' \wedge \cdots \wedge f_0^{(j)}|^2$$

it follows that $p_j := |f_0 \wedge f_1 \wedge \cdots \wedge f_j|^2$ is a real polynomial in $z, \bar{z}$.

Now let $\psi : S^2 \to SU(n+1)/T^n$ be of constant curvature. Then

$$ds^2 = (\gamma_0 + \ldots + \gamma_{n-1})|dz|^2 = \frac{c}{(1+z\bar{z})^2}|dz|^2.$$

With $\gamma_j = \Delta\log|f_0 \wedge \ldots \wedge f_j|^2 = \Delta\log p_j$ we get

$$\Delta\log p_0\cdots p_{n-1} = \gamma_0 + \ldots + \gamma_{n-1} = \frac{c}{(1+z\bar{z})^2} = c\Delta\log(1+z\bar{z}).$$

Applying the prime factorisation argument used in the proof of Lemma 7.5, we obtain

$$\gamma_j = \frac{c_j}{(1+z\bar{z})^2} \qquad \forall j.$$

Consequently, the associated curves ψ_j are of constant curvature.

ψ is congruent to the Veronese sequence:

From above $\phi := \psi_0 : S^2 \to \mathbf{CP}^n$ is of constant curvature. Thus by Theorem 5.3 the harmonic sequence determined by ϕ, i.e. the τ-holomorphic lift ψ of ϕ, is congruent to the Veronese sequence. □

Chapter 6

Induced metric of τ-adapted maps and associated curves

In this chapter we will compute the induced metric of τ-adapted maps $\psi : S \to G/T$ and their associated curves. We will then introduce the η-invariants and will derive different expressions for them. These were needed to establish the relation between the different forms of Toda equations (c.f. Chapter 2.2) and will be crucial in the proof of the constant curvature congruence theorem.

6.1 The induced metric by ψ on S

Let $\langle \cdot, \cdot \rangle$ be a G-invariant inner product on G/T and denote the norm induced by $\langle \cdot, \cdot \rangle$ by $| \cdot |_{G/T}$.

Let the complex structure on G/T be given by $T_o^{1,0}G/T = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ where Δ^+ is a choice of positive roots.

We will show that the metric induced by τ-adapted/holomorphic ψ is given by

$$ds^2 = \sum_{j=0,1}^{\ell} k_j \eta_j |dz|^2$$

where the η_j are invariants of ψ to be defined below, and the k_j are real constants depending on the G-invariant metric on G/T.

We will compute ds_p^2 with the help of a local lift $F : U \to G$ of ψ. In order to do this consider the following commutative diagrams. Let $p \in S$ be fixed and denote left multiplication in G by ℓ and in G/T by L.

Then

$$\begin{array}{ccccc} & & G & \xrightarrow{\ell_{F(p)^{-1}}} & G \\ & \overset{F}{\nearrow} & \downarrow \pi & & \downarrow \pi \\ S \supset U & \xrightarrow{\psi} & G/T & \xrightarrow{L_{F(p)^{-1}}} & G/T \end{array}$$

induces on the tangent bundles

$$\begin{array}{ccccc} & & T^{\mathbf{C}}G & \xrightarrow{d\ell_{F(p)^{-1}}} & T^{\mathbf{C}}G \\ & \overset{dF}{\nearrow} & \downarrow d\pi & & \downarrow d\pi \\ T^{\mathbf{C}}U & \xrightarrow{d\psi} & T^{\mathbf{C}}G/T & \xrightarrow{dL_{F(p)^{-1}}} & T^{\mathbf{C}}G/T \end{array}$$

Since $\langle \cdot, \cdot \rangle$ is G-invariant, we have

$$\langle d\psi(\partial_z|_p), d\psi(\partial_z|_p) \rangle_{\psi(p)} = \left\langle dL_{F(p)^{-1}} d\psi(\partial_z|_p), dL_{F(p)^{-1}} d\psi(\partial_z|_p) \right\rangle_o$$

But from the commutative diagram above we get

$$dL_{F(p)^{-1}} d\psi(\partial_z|_p) = d\pi(F^{-1} dF(\partial_z|_p) = d\pi(F^{-1}\partial_z F)$$

or, alternatively,

$$d\psi(\partial_z|_p) = dL_{F(p)} d\pi(F^{-1} dF(\partial_z|_p) = dL_{F(p)} d\pi(F^{-1}\partial_z F)$$

Note that by construction

$$F(p)^{-1}\partial_z F(p) \in \mathbf{g}^{\mathbf{C}}$$

For simplicity we will from now on omit the particular point p, so we have

$$d\psi(\partial_z) = dL_F d\pi(F^{-1}dF(\partial_z)) = dL_F d\pi(F^{-1}\partial_z F).$$

Since ψ is τ-adapted we have

$$d\psi(\partial_z) \in [\mathcal{M}_1]$$

Thus

$$d\pi(F^{-1}\partial_z F) \in \mathcal{M}_1,$$

so

$$F^{-1}\partial_z F \in \mathbf{t}^{\mathbf{C}} \oplus \mathcal{M}_1 = \mathcal{M}_0 \oplus \mathcal{M}_1.$$

Let

$$F^{-1}\partial_z F = A_0 + A_1$$

where $A_i \in \mathcal{M}_i$. Thus

$$d\pi(F^{-1}\partial_z F) = d\pi(A_0 + A_1) = d\pi(A_1)$$

and hence

$$d\psi(\partial_z) = dL_F d\pi(F^{-1}\partial_z F) = dL_F d\pi(A_1).$$

Denote the projection of $A \in \mathbf{g}^{\mathbf{C}}$ onto a subspace $\mathbf{k}$ by $A^{\mathbf{k}}$.

So

$$A_1 = A_1^{\mathbf{g}^{\alpha_0}} + A_1^{\mathbf{g}^{\alpha_1}} + \ldots A_1^{\mathbf{g}^{\alpha_\ell}}$$

and

$$A_1^{\mathbf{g}^+} = A_1^{\mathbf{g}^{\alpha_1}} + \ldots + A_1^{\mathbf{g}^{\alpha_\ell}}, \quad A_1^{\mathbf{g}^-} = A_1^{\mathbf{g}^{\alpha_0}}$$

where $A_1^{\mathfrak{g}^+} \in T_o^{1,0}G/T$ and $A_1^{\mathfrak{g}^-} \in T_o^{0,1}G/T$.

A similar calculation gives

$$d\psi(\partial_{\bar z}) = dL_F d\pi(F^{-1}\partial_{\bar z}F) = dL_F d\pi(\bar A_1),$$

and splitting

$$\bar A_1 = \bar A_1^{\mathfrak{g}^{-\alpha_0}} + \bar A_1^{\mathfrak{g}^{-\alpha_1}} + \ldots \bar A_1^{\mathfrak{g}^{-\alpha_\ell}}$$

into 1, 0-part and 0, 1-part gives

$$\bar A_1^{\mathfrak{g}^+} = \bar A_1^{\mathfrak{g}^{-\alpha_0}} \in T_o^{1,0}G/T \quad \text{and} \quad \bar A_1^{\mathfrak{g}^-} = \bar A_1^{\mathfrak{g}^{-\alpha_1}} + \ldots + \bar A_1^{\mathfrak{g}^{-\alpha_\ell}} \in T_o^{0,1}G/T.$$

Example 6.1 *For the $SU(n+1)/T^n$ case we have*

$$A_1 = \left(\begin{array}{c|c|c|c|c} 0 & & & & \star \\ \hline \star & 0 & & & \\ \hline & \ddots & \ddots & & \\ \hline & & \ddots & 0 & \\ \hline & & & \star & 0 \end{array}\right), \bar A_1 = \left(\begin{array}{c|c|c|c|c} 0 & \star & & & \\ \hline & 0 & \ddots & & \\ \hline & & \ddots & \ddots & \\ \hline & & & 0 & \star \\ \hline \star & & & & 0 \end{array}\right)$$

so that

$$A_1^{\mathfrak{g}^{\alpha_0}} = \left(\begin{array}{c|c|c|c|c} 0 & & & & \star \\ \hline & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \\ \hline & & & & 0 \end{array}\right), \; A_1^{\mathfrak{g}^{\alpha_1}} = \left(\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline \star & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \\ \hline & & & & 0 \end{array}\right);$$

$$A_1^{\mathfrak{g}^{\alpha_2}} = \left(\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & 0 & & & \\ \hline & \star & \ddots & & \\ \hline & & & 0 & \\ \hline & & & & 0 \end{array}\right), \ldots, A_1^{\mathfrak{g}^{\alpha_n}} = \left(\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \\ \hline & & & \star & 0 \end{array}\right).$$

and

$$\bar{A}_1^{\mathfrak{g}^{-\alpha_0}} = \left(\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & 0 & & & \\ \hline & & \ddots & & \\ \hline & & & 0 & \\ \hline \star & & & & 0 \end{array}\right)$$

Definition 6.2 *Let $\psi : S \to G/T$ be τ-adapted. Let $\{X_{\alpha_j}\}$ be a set of Cartan-Weyl generators. With the notation as above let*

$$|X_{\alpha_j}|^2_{G/T}\eta_j := |A_1^{\mathfrak{g}^{\alpha_j}}|^2_{G/T}, \quad j = 0, \ldots, \ell.$$

The η_j are called **η-invariants** *of ψ.*

We will see in section 6.2 that the η_j are indeed invariants only depending on the choice of Cartan-Weyl generators.

Lemma 6.3 *Let $\psi : S \to G/T$ be τ-adapted/holomorphic. Then the induced metric on S is given by*

$$ds^2 = \sum_{j=0/1}^{\ell} k_j \eta_j |dz|^2$$

where $k_j = |X_{\alpha_j}|^2_{G/T} = \left\langle X_{\alpha_j}, X_{\alpha_j} \right\rangle \ \forall j$.

Proof:

We have

$$\begin{aligned}
|d\psi(\partial_z)^{1,0}|^2_{G/T} &= |dL_F d\pi(F^{-1}\partial_z F)^{1,0}|^2_{G/T} \\
&= |dL_F d\pi(A_1^{\mathfrak{g}^+})|^2_{G/T} \\
&= |d\pi(A_1^{\mathfrak{g}^+})|^2_{G/T} \qquad \text{as the metric is } G\text{-invariant} \\
&= |A_1^{\mathfrak{g}^+}|^2_{G/T} \qquad \text{identifying } \mathbf{m}^{\mathbf{C}} \text{ with } T_oG/T \text{ via } d\pi \\
&= |A_1^{\mathfrak{g}^{\alpha_1}} + \ldots + A_1^{\mathfrak{g}^{\alpha_\ell}}|^2_{G/T} \\
&= |A_1^{\mathfrak{g}^{\alpha_1}}|^2_{G/T} + \ldots + |A_1^{\mathfrak{g}^{\alpha_\ell}}|^2_{G/T} \qquad \text{Lemma B.13} \\
&= k_1\eta_1 + \ldots + k_\ell\eta_\ell
\end{aligned}$$

and

$$\begin{aligned}
|d\psi(\partial_{\bar z})^{1,0}|^2_{G/T} &= |dL_F d\pi(F^{-1}\partial_{\bar z} F)^{1,0}|^2_{G/T} \\
&= |dL_F d\pi(\bar A_1^{\mathfrak{g}^+})|^2_{G/T} \\
&= |d\pi(\bar A_1^{\mathfrak{g}^+})|^2_{G/T} \\
&= |\bar A_1^{\mathfrak{g}^+}|^2_{G/T} \\
&= |\bar A_1^{\mathfrak{g}^{-\alpha_0}}|^2_{G/T} \\
&= |A_1^{\mathfrak{g}^{\alpha_0}}|^2_{G/T} \qquad \text{as complex conjugation preserves lengths} \\
&= k_0\eta_0.
\end{aligned}$$

Therefore, using the usual identification of $T(G/T)$ with $T^{1,0}G/T$, the induced metric is given by $ds^2 = \sum_j k_j\eta_j|dz|^2$. Also since $d\psi(\partial_z) \perp d\psi(\partial_{\bar z})$ we see that ψ is conformal. Also note that the $k_j \in \mathbf{R}^+$ depend on the choice of G-invariant metric but the η_j do not, see Corollary 6.9. □

Corollary 6.4 *The Kähler angle θ is given by*

$$\tan^2\frac{\theta}{2} = \frac{k_0\eta_0}{\sum_{j=1}^{\ell} k_j\eta_j}.$$

Remark 6.5 *If ψ is τ-holomorphic we have $|d\psi(\partial_z)^{0,1}|^2_{G/T} = |d\psi(\partial_{\bar{z}})^{1,0}|^2_{G/T} = 0$, so $k_0\eta_0 = 0$. Thus $ds^2 = \sum_{j=1}^{\ell} k_j\eta_j|dz|^2$ for τ-holomorphic ψ.*

6.2 The η-invariants

Lemma 6.6 *The η-invariants are left invariant by left translation, i.e. if $\tilde{\psi} = g\psi$ for $g \in G$ then $\tilde{\eta}_j = \eta_j$ $\forall j = 0, \ldots, \ell$.*

Proof:

Let $\tilde{\psi} = g\psi$. Then if F is a Toda lift for ψ, $\tilde{F} = gF$ is a Toda frame for $\tilde{\psi}$. Then $\tilde{F}^{-1}\partial_z\tilde{F} = F^{-1}\partial_z F$ and hence $\tilde{A}_0 = A_0$ and $\tilde{A}_1 = A_1$ (terminology as in section 6.1). Since the Cartan-Weyl generators remain unchanged we see from definition 6.2 that $\tilde{\eta}_j = \eta_j$ for all $k = 0, \ldots, \ell$.

Lemma 6.7 (c.f. [BW1]) *For all $j = 0, \ldots, \ell$ is $H_j := \eta_j|dz|^2$ a globally defined 2-form.*

Lemma 6.8 *For τ-primitive / τ-holomorphic ψ the η-invariants may be expressed as*

$$\eta_j = m_j e^{2\alpha_j(\Omega)} \quad \forall j = 0/1, \ldots, \ell.$$

Proof: Let F be a Toda frame and ψ be τ-primitive / τ-holomorphic. Then

$$A_1^{\mathfrak{g}^{\alpha_j}} = \sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j},$$

so

$$\begin{aligned}|X_{\alpha_j}|^2_{G/T}\eta_j &= |A_1^{\mathfrak{g}^{\alpha_j}}|_{G/T} = |\sqrt{m_j}e^{\alpha_j(\Omega)}X_{\alpha_j}|^2_{G/T} \\ &= m_j e^{2\alpha_j(\Omega)}|X_{\alpha_j}|^2_{G/T} = |X_{\alpha_j}|^2_{G/T} m_j e^{2\alpha_j(\Omega)}\end{aligned}$$

and hence

$$\eta_j = m_j e^{2\alpha_j(\Omega)}.$$

□

Corollary 6.9 *The η-invariants are independent of the particular choice of G-invariant metric on G/T.*

6.3 Induced metrics of associated curves.

For details about fundamental representations see [FH].

Let P_j be the maximal parabolic subgroup with maximal parabolic subalgebra $\mathbf{p}_S$ determined by $S = \{\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_\ell\}$ (c.f. chapter 3). Let $\psi_j : S \to G^{\mathbf{C}}/P_j = G/H_j$ be the j**-th associated curve** given by

$$\begin{array}{ccc} S & \xrightarrow{\psi} & G/T \\ & \psi_j \searrow & \downarrow pr_j \\ & & G/H_j \end{array}$$

If F is a Toda frame for ψ then we have

$$\begin{array}{ccc} & & G \\ & \overset{F}{\nearrow} & \downarrow \pi_j \\ S \supset U & \overset{\psi_j}{\longrightarrow} & G/H_j \end{array}$$

Lemma 6.10 *The metric and Kähler angle induced by ψ_j are given by*

$$ds_j^2 = (k_0\eta_0 + k_j\eta_j)|dz|^2 \qquad \tan^2\frac{\theta_j}{2} = \frac{k_0\eta_0}{k_j\eta_j}.$$

For τ-holomorphic ψ we obtain $ds_j^2 = k_j\eta_j|dz|^2$.

Proof:

Similarly to the calculation in section 6.1, we find

$$d\psi_j(\partial_z) = dL_F d\pi_j(F^{-1}\partial_z F) = dL_F d\pi_j(A_1)$$

and

$$d\psi_j(\partial_{\bar{z}}) = dL_F d\pi_j(F^{-1}\partial_{\bar{z}} F) = dL_F d\pi_j(\bar{A}_1).$$

Now $|\cdot|_{G/H_j}$ is given by restricting $|\cdot|_{G/T}$ to $T(G/H_j)$. Therefore

$$|d\psi_j(\partial_z)^{1,0}|^2_{G/T} = |d\pi_j(A_1^{\mathfrak{g}^+})|^2_{G/T} = |A_1^{\mathfrak{g}_j^{\alpha}}|^2_{G/T} = k_j\eta_j$$

and

$$|d\psi_j(\partial_{\bar{z}})^{1,0}|^2_{G/T} = |d\pi_j(\bar{A}_1^{\mathfrak{g}^+})|^2_{G/T} = |\bar{A}_1^{\mathfrak{g}^+}|^2_{G/T} = |\bar{A}_1^{\mathfrak{g}^{-\alpha_0}}|^2_{G/T} = |A_1^{\mathfrak{g}^{\alpha_0}}|^2_{G/T} = k_0\eta_0$$

which proves the assertion. $\square$

Lemma 6.11 *For $j = 1, \ldots, \ell$ let V_j be the j-th fundamental representation space and let $|\cdot|_{V_j}$ be a Hermitian metric on V_j. If ψ is τ-holomorphic, there exist holomorphic functions $F_j : U \subset S \to V_j$ such that*

$$\eta_j = \Delta \log |F_j|^2_{V_j}.$$

Furthermore, the F_j may be chosen so that $p_j(z, \bar{z}) := |F_j|^2_{V_j}$ is a polynomial in $z, \bar{z}$.

Proof: Let v_j be the lowest weight vector in V_j. The orbit of the v_j is given by

$$\vartheta_j : G \to V_j \setminus \{0\}, \quad F \mapsto F.v_j.$$

Let $i_j : G/H_j \hookrightarrow \mathbf{P}(V_j)$ be the embedding given by the lowest weight vector in $v_j \in V_j$. Define

$$\hat{\psi}_j : S \to \mathbf{P}(V_j), \quad \hat{\psi}_j = i_j \psi_j,$$

so that locally $\hat{\psi}_j = [F v_j]$. The following diagram commutes

$$\begin{array}{ccccc} & & G & \xrightarrow{\vartheta_j} & V_j \setminus \{0\} \\ & \overset{F}{\nearrow} & \downarrow \pi_j & & \downarrow \pi_{V_j} \\ S \supset U & \xrightarrow{\psi_j} & G/H_j & \xrightarrow{i_j} & \mathbf{P}(V_j) \end{array}$$

Finally let $\{\lambda_j\}$ be the **fundamental weights** given by $\lambda_j(H_{\alpha_k}) = \delta_{jk}$ and define $\hat{F}_j := e^{-\lambda_j(\Omega)} F v_j : U \to V_j, \quad \hat{F}_j(z) = e^{-\lambda_j(\Omega(z))} F(z) v_j.$

Claim: $\hat{F}_j$ is holomorphic for τ-holomorphic ψ.

Proof: We will show $\partial_{\bar{z}} \hat{F}_j = 0$.

$$\partial_{\bar{z}} F v_j = F F^{-1} \partial_{\bar{z}} F v_j$$

$$
\begin{aligned}
&= F(\overline{A}_0 + \overline{A}_1)v_j \\
&= F\overline{A}_0 v_j + F\overline{A}_1 v_j \\
&= F(-\lambda_j(\overline{A}_0)v_j) + F\overline{A}_1 v_j \\
&= -\lambda_j(\overline{A}_0)Fv_j + F\overline{A}_1^{\mathfrak{g}^{-\alpha_0}} v_j
\end{aligned}
$$

note that $\overline{A}_1^{\mathfrak{g}^{-\alpha_k}} v_j = 0$ for all $k = 1, \ldots, \ell$ since v_j is the lowest weight vector

$$
= -\lambda_j(\overline{A}_0)Fv_j \quad \text{for } \tau\text{-holomorphic } \psi
$$

Assume now that F is a Toda frame. Then $A_0 = \partial_z \Omega$, so $\overline{A_0} = -\partial_{\bar{z}}\Omega$.

Thus

$$
\begin{aligned}
\partial_{\bar{z}}\hat{F}_j &= \partial_{\bar{z}}(e^{-\lambda_j(\Omega)}Fv_j) \\
&= \partial_{\bar{z}}e^{-\lambda_j(\Omega)}Fv_j + e^{-\lambda_j(\Omega)}\partial_{\bar{z}}Fv_j \\
&= -\partial_{\bar{z}}\lambda_j(\Omega)e^{-\lambda_j(\Omega)}Fv_j + e^{-\lambda_j(\Omega)}(-\lambda_j(\overline{A}_0)Fv_j + F\overline{A}_1^{\mathfrak{g}^{-\alpha_0}} v_j) \\
&= -\partial_{\bar{z}}\lambda_j(\Omega)e^{-\lambda_j(\Omega)}Fv_j + e^{-\lambda_j(\Omega)}(-\lambda_j(-\partial_{\bar{z}}\Omega)Fv_j + F\overline{A}_1^{\mathfrak{g}^{-\alpha_0}} v_j) \\
&= -\partial_{\bar{z}}\lambda_j(\Omega)e^{-\lambda_j(\Omega)}Fv_j + e^{-\lambda_j(\Omega)}(\partial_{\bar{z}}\lambda_j(\Omega)Fv_j + F\overline{A}_1^{\mathfrak{g}^{-\alpha_0}} v_j) \\
&= e^{-\lambda_j(\Omega)}F\overline{A}_1^{\mathfrak{g}^{-\alpha_0}} v_j \\
&= 0 \quad \text{for } \tau\text{-holomorphic } \psi
\end{aligned}
$$

Thus $\hat{F}_j$ is holomorphic.

Next note that $\mathbf{P}(V_j)$ and S are projective varieties, so $[\hat{F}_j]$ may be expressed in terms of polynomials. Hence there exists a polynomial $h_j : U \to \mathbf{C}$ such that $|\hat{F}_j \frac{h_j}{|Fv_j|_{V_j}}|_{V_j}^2$ is a polynomial in $z, \bar{z}$, so define $F_j := \hat{F}_j \frac{h_j}{|Fv_j|_{V_j}}$. Thus

$$
|F_j|_{V_j}^2 = \left|\hat{F}_j \frac{h_j}{|Fv_j|_{V_j}}\right|_{V_j}^2 = \left|e^{-\lambda_j(\Omega)}Fv_j \frac{h_j}{|Fv_j|_{V_j}}\right|_{V_j}^2 = e^{-2\lambda_j(\Omega)}|h_j|^2 \left|\frac{Fv_j}{|Fv_j|_{V_j}}\right|_{V_j}^2 = e^{-2\lambda_j(\Omega)}|h_j|^2
$$

and hence

$$-2\lambda_j(\Omega) = \log |F_j|^2_{V_j} - \log |h_j|^2.$$

Therefore

$$\begin{aligned} -2\Delta\lambda_j(\Omega) &= \Delta \log |F_j|^2_{V_j} - \Delta \log |h_j|^2 \\ &= \Delta \log |F_j|^2_{V_j} \quad \text{as } \Delta \log |h|^2 = 0 \text{ for holomorphic } h. \end{aligned}$$

The positive simple roots are related with the fundamental weights via the Cartan matrix K

$$\alpha_i = \sum K_{ij}\lambda_j \quad \text{or} \quad \lambda_j = \sum K^{-1}_{ij}\alpha_i.$$

Expresssed as matrix equation we have

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix} = K \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_\ell \end{pmatrix}.$$

Now from the Toda equations we have

$$\Delta 2\alpha_i(\Omega) + \sum m_j e^{2\alpha_j(\Omega)} K_{ij} = 0, \quad i = 1, \ldots, \ell,$$

and hence ($\eta_j = m_j e^{2\alpha_j(\Omega)}$)

$$\Delta 2\alpha_i(\Omega) = -\sum \eta_j K_{ij}, \quad i = 1, \ldots, \ell.$$

Applying the inverse of the Cartan matrix now gives

$$-\eta_j = \sum K^{-1}_{ij} \Delta 2\alpha_i(\Omega) = 2\Delta \left(\sum K^{-1}_{ij}\alpha_i(\Omega)\right) = 2\Delta\lambda_j(\Omega).$$

Thus

$$\eta_j = -2\Delta\lambda_j(\Omega) = \Delta \log |F_j|^2_{V_j}.$$

□

Chapter 7

Congruence theorems for S^2 in G/T

In this chapter we will prove the constant curvature congruence theorem for τ-holomorphic S^2 in G/T. At first we will prove a weak congruence theorem, namely that the η-invariants determine τ-adapted maps up to congruence. Then we will investigate the case when all associated curves ψ_j of a τ-holomorphic map $\psi : S^2 \to G/T$ are of constant curvature themselves. Next, using a prime factorisation argument, we will prove that ψ being of constant curvature implies that the ψ_j are of constant curvature as well. This then results in the constant curvature theorem. Finally we will prove a general congruence theorem (without the constant curvature condition but an additional assumption on the metric) for τ-holomorphic S^2 in G/T where G has rank two.

7.1 A weak congruence theorem

Theorem 7.1 (Weak congruence theorem) *Let G be a compact simple Lie group and T its maximal torus. Let $\psi, \tilde{\psi} : S \to G/T$ be τ-adapted maps. Then ψ and $\tilde{\psi}$ are congruent by an isometry $A \in G$, $\tilde{\psi} = A\psi$ iff their η-invariants coincide, $\eta_k = \tilde{\eta}_k$ $\forall k$.*

Proof: Let $\pi : G \to G/T$ be the canonical projection.

Locally ψ and $\tilde{\psi}$ have Toda frames $F, \tilde{F} : U \to G$ satisfying

$$F^{-1}\partial_z F = \partial_z \Omega + e^{\Omega} B e^{-\Omega}$$

and

$$\tilde{F}^{-1}\partial_z \tilde{F} = \partial_z \tilde{\Omega} + e^{\tilde{\Omega}} B e^{-\tilde{\Omega}}.$$

where $\Omega, \tilde{\Omega} : U \to i\mathbf{t}$ are smooth maps and $B = \sum \sqrt{m_j} X_{\alpha_j}$.

However, since the η-invariants coincide, it follows that $\Omega = \tilde{\Omega}$. Thus

$$F^{-1}\partial_z F = \partial_z \Omega + e^{\Omega} B e^{-\Omega} = \partial_z \tilde{\Omega} + e^{\tilde{\Omega}} B e^{-\tilde{\Omega}} = \tilde{F}^{-1}\partial_z \tilde{F}.$$

Claim: $F = A\tilde{F}$ with $A \in G$ constant.

Proof: Let $A = F\tilde{F}^{-1}$. We need to show that A is constant, i.e. $\partial_z A = \partial_{\bar{z}} A = 0$.

Using $F^{-1}\partial_z F = \tilde{F}^{-1}\partial_z \tilde{F}$ we get

$$\begin{aligned} \partial_z(F\tilde{F}^{-1}) &= (\partial_z F)\tilde{F}^{-1} + F\partial_z(\tilde{F}^{-1}) \\ &= (\partial_z F)\tilde{F}^{-1} + F(-\tilde{F}^{-1}(\partial_z \tilde{F})\tilde{F}^{-1}) \\ &= (\partial_z F)\tilde{F}^{-1} - FF^{-1}(\partial_z F)\tilde{F}^{-1} \end{aligned}$$

$$= (\partial_z F)\tilde{F}^{-1} - (\partial_z F)\tilde{F}^{-1}$$

$$= 0.$$

However, since F and $\tilde{F}$ are both real, it also follows that

$$\partial_{\bar{z}}(F\tilde{F}^{-1}) = 0.$$

Hence A is constant.

It now follows that $F = A\tilde{F}$ and hence locally $\psi = A\tilde{\psi}$ with A dependent on the open set U: $A = A_U$. However, whenever two open sets U and V overlap, then $A_U = A_V$. Hence $\psi = A\tilde{\psi}$ globally with $A = const.$

7.2 Calculations for constant curvature ψ_j

The following Lemma shows that there is only *one* possibility for all ψ_j to be of constant curvature.

Lemma 7.2 *Let the ψ_j be the maps induced by the fundamental representations. Suppose they all have constant curvature, i.e. $\eta_j = \frac{r_j}{(1+z\bar{z})^2}$ $\forall j$ with r_j constant. Then $r_j = c_j$ $\forall j$ where the $c_j \in \mathbf{N}$ are given by*

$$\begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix} = K^{-1} \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}.$$

Proof: Let $\eta_j = \frac{r_j}{(1+z\bar{z})^2}$ $\forall j$. From Claim 2.3 we know that the Toda equations may be written as

$$\begin{pmatrix} \Delta \log \eta_1 \\ \vdots \\ \Delta \log \eta_\ell \end{pmatrix} = -K \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_\ell \end{pmatrix},$$

where K is the Cartan matrix.

Now $\eta_j = \frac{r_j}{(1+z\bar{z})^2}$ yields

$$\begin{aligned} \Delta \log \eta_j &= -2\partial_z \partial_{\bar{z}} \log(1+z\bar{z}) \\ &= -2\partial_z \frac{z}{1+z\bar{z}} \\ &= -2\left(\frac{1}{1+z\bar{z}} - \frac{z\bar{z}}{(1+z\bar{z})^2}\right) \\ &= \frac{-2}{(1+z\bar{z})^2}. \end{aligned}$$

Thus

$$\begin{pmatrix} -2 \\ \vdots \\ -2 \end{pmatrix} = -K \begin{pmatrix} r_1 \\ \vdots \\ r_\ell \end{pmatrix},$$

i.e.

$$\begin{pmatrix} r_1 \\ \vdots \\ r_\ell \end{pmatrix} = K^{-1} \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} =: \begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}.$$

Therefore $r_j = c_j$ $\forall j$. □

Corollary 7.3 *In the above case the metric induced by ψ is $ds^2 = \frac{c}{(1+z\bar{z})^2}|dz|^2$ with $c = \sum k_j c_j$ where $k_j = |X_{\alpha_j}|^2$.*

7.3 Irreducible polynomials

To prove the congruence theorem for S^2 we will need the following Lemma about irreducible polynomials.

Lemma 7.4 *Let $\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_n$ be polynomials in $z, \bar{z}$ such that*

$$\sum_{i=1}^{n} \frac{\sigma_i}{\tau_i} = 0$$

and $gcd(\sigma_i, \tau_i) = 1$ for all i and $gcd(\tau_i, \tau_j) = 1$ for all $i \neq j$. Then $\sigma_i = 0$ for all $i = 1, \dots, n$.

Proof: We will prove the lemma by induction. For $n = 1$ the assertion is clear. Let now the assertion be true for n. We want to show that this is also the case for $n + 1$. Let

$$\sum_{i=1}^{n+1} \frac{\sigma_i}{\tau_i} = 0.$$

Then

$$\frac{\sigma_{n+1}}{\tau_{n+1}} = -\sum_{i=1}^{n} \frac{\sigma_i}{\tau_i}.$$

and hence

$$\sigma_{n+1} \prod_{i=1}^{n} \tau_i = -\tau_{n+1} \sum_{i=1}^{n} \sigma_i \prod_{i \neq j} \tau_j.$$

Therefore τ_{n+1} divides $\sigma_{n+1} \prod_{i=1}^{n} \tau_i$. However, neither σ_{n+1} nor $\tau_1, \dots, \tau_n$ have common factors with τ_{n+1}. Thus $\sigma_{n+1} = 0$ and hence $\sigma_i = 0$ for all $i = 1, \dots, n + 1$ as the assertion is true for n. □

7.4 Computing the η-invariants of ψ

The next Lemma shows that if ψ has constant curvature then all ψ_j induced by the fundamental representations have constant curvature.

Lemma 7.5 *If the induced metric is of constant curvature $ds^2 = \frac{c}{(1+z\bar{z})^2}|dz|^2$ then*

$$\eta_j = \frac{c_j}{(1+z\bar{z})^2} \quad \forall j,$$

where the $c_j \in \mathbf{N}$ are given by

$$\begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix} = K^{-1} \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}.$$

Proof: Recall that the metric is given in terms of the η-invariants by

$$ds^2 = \sum k_j \eta_j |dz|^2$$

with $k_j = |X_{\alpha_j}|^2$. The k_j depend on the choice of G-invariant metric on G/T. We will show $\eta_j = \frac{c_j}{(1+z\bar{z})^2}$

Recall $\eta_j = \Delta \log |F_j|^2_{V_j} = \Delta \log p_j$ where p_j is a real polynomial in $z, \bar{z}$. Let $p_j(z,\bar{z}) = (1+z\bar{z})^{c_j}\varphi_j$ where φ_j is a polynomial that has no common factors with $1+z\bar{z}$. Then

$$\eta_j = \Delta \log p_j = \Delta \log(1+z\bar{z})^{c_j} + \Delta \log \varphi_j = \frac{c_j}{(1+z\bar{z})^2} + \Delta \log \varphi_j$$

Since ψ is of constant curvature we have

$$\sum k_j \eta_j |dz|^2 = ds^2 = \frac{c}{(1+z\bar{z})^2}|dz|^2.$$

Thus

$$\frac{k_j c_j}{(1+z\bar{z})^2} + \sum k_j \Delta \log \varphi_j = \frac{c}{(1+z\bar{z})^2}.$$

Let $\pi_1, \ldots, \pi_N$ be the prime factors of $\prod_{j=1}^{\ell} \varphi_j$. Then $\varphi_j = \pi_1^{r_{j1}} \cdots \pi_N^{r_{jN}}$ with $r_{ji} \in \mathbf{N}_0$, so

$$\Delta \log \varphi_j = \sum_{i=1}^{N} r_{ji} \frac{\pi_i \Delta \pi_i - \partial_z \pi_i \partial_{\bar{z}} \pi_i}{\pi_i^2}.$$

Hence

$$\frac{-c + \sum_j k_j c_j}{(1+z\bar{z})^2} + \sum_{j,i} k_j r_{ji} \frac{\pi_i \Delta \pi_i - \partial_z \pi_i \partial_{\bar{z}} \pi_i}{\pi_i^2} = \frac{-c + \sum_j k_j c_j}{(1+z\bar{z})^2} + \sum_i \left(\sum_j k_j r_{ji} \right) \frac{\pi_i \Delta \pi_i - \partial_z \pi_i \partial_{\bar{z}} \pi_i}{\pi_i^2} = 0$$

Since $\pi_i \Delta \pi_i - \partial_z \pi_i \partial_{\bar{z}} \pi_i$ and π_i^2 are coprime it follows by Lemma 7.4 that

$$\sum_j k_j r_{ji} = 0.$$

However, all k_j are strictly positive, hence all r_{ji} have to be zero. Thus $\varphi_j \equiv d_j \in \mathbf{R}$ and $p_j = d_j (1+z\bar{z})^{c_j}$. It follows also from Lemma 7.4 that $\sum_j k_j c_j = c$.

For the η-invariants we finally get

$$\eta_j = \Delta \log p_j = \Delta \log (1+z\bar{z})^{c_j} + \Delta \log d_j = \frac{c_j}{(1+z\bar{z})^2}.$$

□

7.5 The constant curvature congruence theorem

Theorem 7.6 *Let G be a compact simple Lie group and T its maximal torus. Let $\psi, \tilde{\psi} : S^2 \to G/T$ be τ-holomorphic maps of constant curvature with same induced metric. Then ψ and $\tilde{\psi}$ are congruent by a holomorphic isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof: Let $\psi, \tilde{\psi} : S^2 \to G/T$ be of constant curvature. By Lemma 7.5 the respective η-invariants are

$$\eta_j = \frac{c_j}{(1+z\bar{z})^2}$$

and

$$\tilde{\eta}_j = \frac{c_j}{(1+z\bar{z})^2}$$

for all $j = 1, \ldots, \ell$. Thus $\eta_j = \tilde{\eta}_j\ \forall j$. By Theorem 7.1 ψ and $\tilde{\psi}$ are congruent. □

Example 7.7 *For $su(\ell+1)$ the curvature constants are as follows. The inverse of the cartan matrix is given by ([OV], p.295)*

$$K_{ij}^{-1} = \frac{1}{\ell+1}\begin{cases} i(\ell+1-j) & : \ i \le j \\ (\ell+1-i)j & : \ i > j. \end{cases}$$

Using the formula in Lemma 7.2 the constant curvature constants c_i, $i = 1, \ldots, \ell$, may be computed as $c_i = \sum_j K_{ij}^{-1} 2 = 2\sum_j K_{ij}^{-1}$:

$$\begin{aligned}
c_i &= 2\sum_j K_{ij}^{-1} \\
&= \frac{2}{\ell+1}\left(\sum_{j=1}^{i-1}(\ell+1-i)j + \sum_{j=i}^{\ell} i(\ell+1-j)\right) \\
&= \frac{2}{\ell+1}\left(\sum_{j=1}^{i-1}(\ell+1-i)j + \sum_{j=1}^{\ell+1-i} ij\right) \\
&= \frac{2}{\ell+1}\left((\ell+1-i)j\frac{i(i-1)}{2} + i\frac{(\ell+1-i)(\ell+2-i)}{2}\right) \\
&= \frac{i(\ell+1-i)}{\ell+1}(i-1+\ell+2-i) \\
&= i(\ell+1-i).
\end{aligned}$$

Thus

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{\ell-1} \\ c_\ell \end{pmatrix} = \begin{pmatrix} \ell \\ 2(\ell-1) \\ \vdots \\ (\ell-1)2 \\ \ell \end{pmatrix}.$$

The constant c is given as $c = \sum_j k_j c_j$. Choose a G-invariant metric such that $k_j = 1$ for all j. Then

$$\begin{aligned} c &= \sum_j j(\ell+1-j) = \sum_j j(\ell+1) - j^2 \\ &= (\ell+1)\frac{\ell(\ell+1)}{2} - \frac{(\ell)(\ell+1)(2\ell+1)}{6} \\ &= \frac{\ell(\ell+1)}{6}(3\ell+3-2\ell-1) = \frac{\ell(\ell+1)(\ell+2)}{6} \end{aligned}$$

Note that the curvature of a τ-holomorphic constant curvature S^2 is strictly positive. It is given by

$$K = \frac{4}{k_1 c_1 + \ldots k_\ell c_\ell}.$$

7.6 A general congruence theorem for rank 2 Lie groups

Recall $\eta_j = \Delta \log |F_j|^2_{V_j} = \Delta \log p_j$ where p_j is a real polynomial in $z, \bar{z}$. Then

$$\eta_j = \Delta \log p_j$$

Let $\pi_1, \ldots, \pi_N$ be the normalised prime factors of $\prod_{j=1}^{\ell} p_j$. Then $p_j = a_j \pi_1^{r_{j1}} \cdots \pi_N^{r_{jN}}$ with $r_{jk} \in \mathbf{N}_0$, $a_j \in \mathbf{R}$.

For a general congruence theorem we need to be able to determine the r_{jk} from the prime factors of ds^2.

Let $K_j := \{k \mid r_{jk} \neq 0\} \subseteq \{1, \ldots, N\}$, so

$$p_j = a_j \prod_{k \in K_j} \pi_k^{r_{jk}}.$$

So we get for the η-invariants

$$\eta_j = \Delta \log p_j = \sum_{k \in K_j} r_{jk} \Delta \log \pi_k = \sum_{k \in K_j} r_{jk} \frac{\pi_k \Delta \pi_k - \partial_z \pi_k \partial_{\bar{z}} \pi_k}{\pi_k^2}$$

or, alternatively,

$$\eta_j = \sum_{k \in K_j} r_{jk} \frac{(\pi_k \Delta \pi_k - \partial_z \pi_k \partial_{\bar{z}} \pi_k) \prod_{n \in K_j \setminus \{k\}} \pi_n^2}{\prod_{n \in K_j} \pi_n^2}.$$

Define

$$q_j := \sum_{k \in K_j} r_{jk} (\pi_k \Delta \pi_k - \partial_z \pi_k \partial_{\bar{z}} \pi_k) \prod_{n \in K_j \setminus \{k\}} \pi_n^2.$$

Then, for all $n \in K_j$, q_j and π_n are coprime, $(q_j, \pi_n) = 1$, and

$$\deg q_j \leq \left(\sum_{n \in K_j} 2 \deg \pi_n \right) - 2.$$

Also

$$\eta_j = \frac{q_j}{\prod_{n \in K_j} \pi_n^2}.$$

So

$$\Delta \log \eta_j = \Delta \log q_j - \sum_{n \in K_j} 2 \Delta \log \pi_n$$

or changing the index

$$\Delta \log \eta_i = \Delta \log q_i - \sum_{n \in K_i} 2 \Delta \log \pi_n.$$

From the Toda equations we also have

$$\Delta \log \eta_i = -\sum_{j=1}^{\ell} K_{ij}\eta_j$$

or

$$\begin{pmatrix} \Delta \log \eta_1 \\ \vdots \\ \Delta \log \eta_\ell \end{pmatrix} = -K \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_\ell \end{pmatrix}.$$

Hence

$$\Delta \log q_i - \sum_{n\in K_i} 2\Delta \log \pi_n = \Delta \log \eta_i = -\sum_{j=1}^{\ell} K_{ij} \sum_{k\in K_j} r_{jk}\Delta \log \pi_k,$$

so

$$\begin{aligned} \Delta \log q_i &= \sum_{n\in K_i} 2\Delta \log \pi_n - \sum_{j=1}^{\ell}\sum_{k\in K_j} K_{ij}r_{jk}\Delta \log \pi_k \\ &= \sum_{k\in K_i} 2\Delta \log \pi_k - \sum_{j=1}^{\ell}\sum_{k=1}^{N} K_{ij}r_{jk}\Delta \log \pi_k \\ &= \sum_{k\in K_i} 2\Delta \log \pi_k - \sum_{k=1}^{N}\sum_{j=1}^{\ell} K_{ij}r_{jk}\Delta \log \pi_k \\ &= \sum_{k\in K_i} (2 - \sum_{j=1}^{\ell} K_{ij}r_{jk})\Delta \log \pi_k + \sum_{k\in K_i^c} (-\sum_{j=1}^{\ell} K_{ij}r_{jk})\Delta \log \pi_k. \end{aligned}$$

Thus

$$\Delta \log q_i = \Delta \log \frac{\prod_{n\in K_i} \pi_n^2}{\prod_{j=1}^{\ell}\prod_{k\in K_j} \pi_k^{K_{ij}r_{jk}}}.$$

On the other hand we know that q_i has no common factors with $\prod_{k\in K_i} \pi_k$. Thus for a suitable holomorphic function g_i

$$q_i = |g_i|^2 \prod_{k\in K_i^c} \pi_k^{s_{ik}}, \quad s_{ik} \in \mathbf{N}_0$$

and hence

$$\Delta \log q_i = \sum_{k\in K_i^c} s_{ik}\Delta \log \pi_k.$$

Comparing this with the last sum expression for $\log q_i$ above we get

$$\sum_{k\in K_i^c} s_{ik}\Delta\log\pi_k = \sum_{k\in K_i}(2-\sum_{j=1}^{\ell}K_{ij}r_{jk})\Delta\log\pi_k + \sum_{k\in K_i^c}(-\sum_{j=1}^{\ell}K_{ij}r_{jk})\Delta\log\pi_k$$

or

$$\sum_{k\in K_i}(2-\sum_{j=1}^{\ell}K_{ij}r_{jk})\Delta\log\pi_k + \sum_{k\in K_i^c}(-s_{ik}-\sum_{j=1}^{\ell}K_{ij}r_{jk})\Delta\log\pi_k = 0.$$

Therefore for all $i = 1, \dots, \ell$

$$2-\sum_{j=1}^{\ell}K_{ij}r_{jk} = 0 \quad \forall k\in K_i \text{ and } s_{ik} = -\sum_{j=1}^{\ell}K_{ij}r_{jk} \quad \forall k\in K_i^c. \qquad (\star)$$

From the first equation it follows that if all p_i have the same prime factors, i.e. if $K_i = \{1,\dots,N\}$ $\forall i = 1,\dots,\ell$ then the r_{ik} are uniquely determined and given via the Cartan matrix as follows.

For all $k \in \bigcap_{i=1}^{\ell} K_i$ we have

$$\begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix} = K \begin{pmatrix} r_{1k} \\ \vdots \\ r_{\ell k} \end{pmatrix},$$

i.e.

$$\begin{pmatrix} r_{1k} \\ \vdots \\ r_{\ell k} \end{pmatrix} = K^{-1} \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}.$$

So if all p_i have the same prime factors (c.f. Theorem 7.6) this would give a general congruence theorem. However, in general the p_i have different prime factors as can be seen in Example A.2.

The rank 2 case

Theorem 7.8 *Let G be a compact simple Lie group of rank two and T its maximal torus. Let $\psi, \tilde{\psi} : S^2 \to G/T$ be τ-holomorphic maps with same induced metric. If $k_1 := |X_{\alpha_1}|^2 \neq |X_{\alpha_2}|^2 =: k_2$ w.r.t. the G-invariant metric on G/T then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof:

1. Simplification of $(\star)$

For the rank two case we can simplify the above equations $(\star)$ as follows. We have

$$\{1, \ldots, N\} = K_2^c \cup (K_1 \cap K_2) \cup K_1^c.$$

Therefore $2 - \sum_{j=1}^{2} K_{ij} r_{jk} = 0 \quad \forall k \in K_i$ becomes two sets of equations. For both i we have

$$2 - (K_{i1} r_{1k} + K_{i2} r_{2k}) = 0 \quad \forall k \in K_1 \cap K_2$$

as before. However, $(\star)$ simplifies for $i = 1$ to

$$2 - K_{11} r_{1k} = 0 \quad \forall k \in K_2^c$$

and for $i = 2$ to

$$2 - K_{22} r_{2k} = 0 \quad \forall k \in K_1^c.$$

Also $s_{ik} = -\sum_{j=1}^{2} K_{ij} r_{jk}$ $\forall k \in K_i^c$ becomes

$$s_{1k} = -K_{12} r_{2k} \quad \forall k \in K_1^c$$

and

$$s_{2k} = -K_{21} r_{1k} \quad \forall k \in K_2^c.$$

Since $K_{11} = K_{22} = 2$ we get

$$r_{1k} = 1 \; \forall k \in K_2^c \quad \text{and} \quad r_{2k} = 1 \; \forall k \in K_1^c$$

For $k \in K_1 \cap K_2$ however we find

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix},$$

so

$$\begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

or

$$\begin{pmatrix} r_{1k} \\ r_{2k} \end{pmatrix} = \frac{1}{\det K} \begin{pmatrix} K_{22} & -K_{12} \\ -K_{21} & K_{11} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \frac{1}{4 - K_{12}K_{21}} \begin{pmatrix} 4 - 2K_{12} \\ 4 - 2K_{21} \end{pmatrix} \quad \forall k \in K_1 \cap K_2.$$

In particular $r_{jk} = r_{jk'} \; \forall \; k, k' \in K_1 \cap K_2$.

Note that all constants r_{jk} for $k = 1, \ldots, N$ are uniquely determined by K_1 and K_2.

The above expression for the r_{jk} yields for the s-constants

$$s_{1k} = -K_{12} \; \forall k \in K_1^c \quad \text{and} \quad s_{2k} = -K_{21} \; \forall k \in K_2^c.$$

Therefore

$$\Delta \log q_1 = -K_{12} \sum_{k \in K_1^c} \Delta \log \pi_k \quad \text{and} \quad \Delta \log q_2 = -K_{21} \sum_{k \in K_2^c} \Delta \log \pi_k$$

2. Metric calculations

The metric in the rank 2 case is given by

$$ds^2 = (k_1\eta_1 + k_2\eta_2)|dz|^2.$$

Therefore if the two τ-holomorphic curves $\psi, \tilde{\psi}$ have the same metric we have

$$k_1\eta_1 + k_2\eta_2 = k_1\tilde{\eta}_1 + k_2\tilde{\eta}_2.$$

Now

$$\eta_j = \sum_{k\in K_j} r_{jk}\Delta \log \pi_k = \sum_{k=1}^{N} r_{jk}\Delta \log \pi_k$$

so using Lemma 7.4 we get

$$k_1 r_{1k} + k_2 r_{2k} = k_1\tilde{r}_{1k} + k_2\tilde{r}_{2k} \quad \forall k = 1, \dots, N.$$

Define $\kappa := \frac{k_2}{k_1} \neq 1$ by assumption. Then

$$r_{1k} + \kappa r_{2k} = \tilde{r}_{1k} + \kappa\tilde{r}_{2k} \quad \forall k = 1, \dots, N.$$

Summing up gives

$$\begin{aligned}
\sum_{k=1}^{N}(r_{1k} + \kappa r_{2k}) &= \sum_{k\in K_2^c} r_{1k} + \sum_{k\in K_1\cap K_2} (r_{1k} + \kappa r_{2k}) + \sum_{k\in K_1^c} \kappa r_{2k} \\
&= \sum_{k\in K_2^c} 1 + \sum_{k\in K_1\cap K_2} \frac{1}{4 - K_{12}K_{21}}(4 - 2K_{12} + \kappa(4 - 2K_{21})) + \sum_{k\in K_1^c} \kappa \\
&= |K_2^c| + |K_1 \cap K_2|\frac{4 - 2K_{12} + \kappa(4 - 2K_{21})}{4 - K_{12}K_{21}} + \kappa|K_1^c|.
\end{aligned}$$

Setting $N_1 = |K_2^c|$, $N_2 = |K_1 \cap K_2|$, and $N_3 = |K_1^c|$ we get

$$\sum_{k=1}^{N}(r_{1k} + \kappa r_{2k}) = N_1 + N_2\frac{4 - 2K_{12} + \kappa(4 - 2K_{21})}{4 - K_{12}K_{21}} + \kappa N_3.$$

Note that

$$|K_2^c| + |K_1 \cap K_2| + |K_1^c| = N_1 + N_2 + N_3 = N.$$

A similar computation for the $\tilde{\eta}$ gives

$$\sum_{k=1}^{N} \tilde{r}_{1k} + \kappa \tilde{r}_{2k} = \tilde{N}_1 + \tilde{N}_2 \frac{4 - 2K_{12} + \kappa(4 - 2K_{21})}{4 - K_{12}K_{21}} + \kappa \tilde{N}_3.$$

Thus

$$N_1 + N_2 \frac{4 - 2K_{12} + \kappa(4 - 2K_{21})}{4 - K_{12}K_{21}} + \kappa N_3 = \tilde{N}_1 + \tilde{N}_2 \frac{4 - 2K_{12} + \kappa(4 - 2K_{21})}{4 - K_{12}K_{21}} + \kappa \tilde{N}_3.$$

Now $N_3 = N - N_1 - N_2$ and $\tilde{N}_3 = N - \tilde{N}_1 - \tilde{N}_2$. So

$$N_1(1 - \kappa) + N_2 \frac{4 - 2K_{12} + \kappa(4 - 2K_{21}) - \kappa(4 - K_{12}K_{21})}{4 - K_{12}K_{21}} + \kappa N =$$
$$\tilde{N}_1(1 - \kappa) + \tilde{N}_2 \frac{4 - 2K_{12} + \kappa(4 - 2K_{21}) - \kappa(4 - K_{12}K_{21})}{4 - K_{12}K_{21}} + \kappa N,$$

i.e.

$$N_1(1-\kappa)+N_2 \frac{4 - 2K_{12} + \kappa(K_{12} - 2)K_{21}}{4 - K_{12}K_{21}} = \tilde{N}_1(1-\kappa)+\tilde{N}_2 \frac{4 - 2K_{12} + \kappa(K_{12} - 2)K_{21}}{4 - K_{12}K_{21}}.$$

We will now show $N_i = \tilde{N}_i$ and $K_i = \tilde{K}_i$ which then gives the congruence theorem.

3. Calculations for specific Lie groups

We will conclude the proof for the Lie group G_2 (the calculations for $SU(3)$ and $SO(5)$ are completely analogous).

The Cartan matrix for G_2 is given by

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Thus

$$r_{1k} = 1 \ \forall k \in K_2^c \quad \text{and} \quad r_{2k} = 1 \ \forall k \in K_1^c$$

and for $k \in K_1 \cap K_2$ we have

$$r_{1k} = 6 \quad \text{and} \quad r_{2k} = 10 \quad \forall k \in K_1 \cap K_2.$$

For the s-constants we have

$$s_{1k} = 1 \ \forall k \in K_1^c \quad \text{and} \quad s_{2k} = 3 \ \forall k \in K_2^c,$$

so

$$\Delta \log q_1 = \sum_{k \in K_1^c} \Delta \log \pi_k \quad \text{and} \quad \Delta \log q_2 = 3 \sum_{k \in K_2^c} \Delta \log \pi_k.$$

The metric equations are, as before,

$$r_{1k} + \kappa r_{2k} = \tilde{r}_{1k} + \kappa \tilde{r}_{2k} \quad \forall k = 1, \ldots, N$$

and their sum yields

$$N_1(1 - \kappa) + N_2(6 + 9\kappa) = \tilde{N}_1(1 - \kappa) + \tilde{N}_2(6 + 9\kappa).$$

The metric equations give for all $k \in K_1 \cap K_2$

$$6 + \kappa 10 = \tilde{r}_{1k} + \kappa \tilde{r}_{2k}.$$

However, if $k \notin \tilde{K}_1 \cap \tilde{K}_2$ then $\tilde{r}_{1k} + \kappa \tilde{r}_{2k}$ equals 1 or κ, depending on whether $k \in \tilde{K}_2^c$ or $k \in \tilde{K}_1^c$ which results in a contradiction. Therefore $k \in K_1 \cap K_2$ implies $k \in \tilde{K}_1 \cap \tilde{K}_2$, so by symmetry

$$K_1 \cap K_2 = \tilde{K}_1 \cap \tilde{K}_2.$$

It follows that $N_2 = \tilde{N}_2$ and $N_1 = \tilde{N}_1$.

Now

$$1 = \tilde{r}_{1k} + \kappa\tilde{r}_{2k} \text{ equals 1 or } \kappa \text{ depending on whether } k \in \tilde{K}_2^c \text{ or } k \in \tilde{K}_1^c.$$

Again we get

$$\tilde{K}_2^c = K_2^c \quad \text{and} \quad \tilde{K}_1^c = K_1^c.$$

and hence

$$\tilde{K}_1 = K_1 \quad \text{and} \quad \tilde{K}_2 = K_2.$$

Therefore

$$\eta_j = \sum_{k\in K_j} r_{jk}\Delta\log\pi_k = \sum_{k\in\tilde{K}_j} r_{jk}\Delta\log\pi_k = \sum_{k\in\tilde{K}_j} \tilde{r}_{jk}\Delta\log\pi_k = \tilde{\eta}_j \quad j = 1,2$$

as the r_{jk} are uniquely determined by K_1, K_2. Thus ψ and $\tilde{\psi}$ are congruent by the weak congruence theorem. □

Remark 7.9 *It might be interesting to investigate the following. Let $\psi, \tilde{\psi}$ be τ-holomorphic with same induced metric. If the $\{k_1, \ldots, k_\ell\}$ are symmetric in the sense of $k_i = k_{\ell+1-i}$ $\forall i$, does there exist an isometry g such that either*

- $\tilde{\psi} = g\psi$ *or*
- $\tilde{\psi} = g\bar{\psi}$ *or*
- $\tilde{\psi} = \overline{g\psi}$ *?*

Also, if the $\{k_1, \ldots, k_\ell\}$ are not symmetric, is then $\tilde{\psi} = g\psi$ for some isometry g?

Chapter 8

Characterisation of isometric τ-primitive maps $\psi : \mathbf{R}^2 \to G/T$ with constant Kähler angle

In this chapter we give a collection of congruence theorems for isometric τ-primitive maps $\psi : \mathbf{R}^2 \to G/T$ with constant Kähler angle for different Lie groups. Although it was not possible within the scope of this thesis to prove the most general version of this theorem for all Lie groups G, the approach for each Lie group is illustrated quite explicitly, so that it might be possible to solve the problem for all Lie groups in the future. The essential idea is to use the Toda equations and the expressions for metric and Kähler angle to find and solve polynomial equations for the η-invariants.

8.1 General calculations

Give $\mathbf{R}^2$ the standard flat metric and fix a coordinate system z on $\mathbf{R}^2$ such that $d\tilde{s}^2 = ds^2 = c|dz|^2$ $(c \in \mathbf{R}^+)$.

Claim 8.1 *Let $\psi : \mathbf{R}^2 \to G/T$ be an isometric τ-primitive map with constant Kähler angle. Then η_0 is constant.*

Proof: Let $ds^2 = c|dz|^2$ as above and $\tan^2 \frac{\theta}{2} = d \in \mathbf{R}$.

Then

$$\sum_{j=1}^{\ell} k_j \eta_j = c - k_0 \eta_0$$

and from Corollary 6.4 we also have

$$\tan^2 \frac{\theta}{2} = \frac{k_0 \eta_0}{\sum_{j=1}^{\ell} k_j \eta_j}.$$

Therefore

$$d = \frac{k_0 \eta_0}{c - k_0 \eta_0}$$

or

$$k_0 \eta_0 = d(c - k_0 \eta_0) \iff \eta_0 = \frac{cd}{k_0(1+d)}.$$

Hence η_0 is constant. □

Claim 8.2

$$\Delta \log \prod_{i=0}^{\ell} \eta_i^{m_i} = 0$$

Proof: From the Toda equations we have

$$\Delta \log \eta_i = - \sum_{j=0}^{\ell} \hat{K}_{ij} \eta_j$$

and from the singularity of the extended Cartan matrix $\hat{K}$ we have

$$\sum_{i=0}^{\ell} m_i \hat{K}_{ij} = 0$$

from Claim B.8. Thus

$$\begin{aligned}
\Delta \log \prod_{i=0}^{\ell} \eta_i^{m_i} &= \sum_{i=0}^{\ell} m_i \Delta \log \eta_i \\
&= \sum_{i=0}^{\ell} m_i \sum_{j=0}^{\ell} -\hat{K}_{ij}\eta_j \\
&= -\sum_{j=0}^{\ell} \left(\sum_{i=0}^{\ell} m_i \hat{K}_{ij} \right) \eta_j \\
&= -\sum_{j=0}^{\ell} 0 \cdot \eta_j = 0.
\end{aligned}$$

□

Definition 8.3 *A function h satisfying $\Delta h \geq 0$ in a domain D is called* **subharmonic**. *If $\Delta h \leq 0$ so that $-h$ is subharmonic, h is called* **superharmonic**.

Theorem 8.4 (Liouville's Theorem, [PW], p.130) *If h is subharmonic in the whole x, y-plane except possibly at the origin and if h is uniformly bounded above, then h is constant.*

Claim 8.5 *Let $h : \mathbf{R}^2 \to \mathbf{R}$ be bounded and $\Delta h =$ constant. Then h is constant.*

Proof: This is a direct consequence of Liouville's Theorem. Let $c = \Delta h$, $c \in \mathbf{R}$. If $c \leq 0$ the h is subharmonic and Theorem 8.4 yields that h is constant. If $c \geq 0$ then h is superharmonic, so Theorem 8.4 applied to $-h$ gives that $-h$ and thus h

is constant. □

Claim 8.6 *If all η-invariants of an isometric τ-primitive map $\psi : \mathbf{R}^2 \to G/T$ are constant then $(\eta_0, \ldots, \eta_\ell)$ is a multiple of $\mathbf{n}$, where* $\mathrm{Ker}\hat{K} = \mathrm{span}\{\mathbf{n}\}$.

Proof: Let the η-invariants be constant. Then $\Delta \log \eta_i = 0$ for all $i = 0, \ldots, \ell$. Thus

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta \log \eta_0 \\ \vdots \\ \Delta \log \eta_\ell \end{pmatrix} = -\hat{K} \begin{pmatrix} \eta_0 \\ \vdots \\ \eta_\ell \end{pmatrix},$$

i.e. $(\eta_0, \ldots, \eta_\ell) \in \mathrm{Ker}\hat{K}$. Now $\dim \mathrm{Ker}\hat{K} = 1$ since rank $\hat{K} = \ell$ which finishes the proof. □

Example 8.7 *For $G = SU(n+1)$ we have $\mathbf{n} = (1, \ldots, 1)$ and* $\mathrm{Ker}\hat{K} = \mathrm{span}\{\mathbf{n}\}$.

Corollary 8.8 *If all η-invariants of an isometric τ-primitive map $\psi : \mathbf{R}^2 \to G/T$ are constant then ψ has constant Kähler angle θ given by*

$$\tan^2 \frac{\theta}{2} = \frac{k_0 n_0}{\sum_{j=1}^{\ell} k_j n_j}.$$

where $\mathbf{n} = (n_0, \ldots, n_\ell)$ spans $\mathrm{Ker}\hat{K}$.

Proof:

This is a direct consequence of Corollary 6.4 in conjuction with Claim 8.6. □

Remark 8.9 *This corollary will be used implicitly in the proofs of the congruence theorems of this chapter as follows.*

Let $\psi, \tilde{\psi}$ be isometric τ-primitive maps with constant Kähler angles given by $\tan^2 \frac{\theta}{2} = d \in \mathbf{R}$ *and* $\tan^2 \frac{\tilde{\theta}}{2} = \tilde{d} \in \mathbf{R}$. *The constant curvature and Kähler angle conditions imply that all η-invariants are constant. Thus $d = \tan^2 \frac{\theta}{2} = \tan^2 \frac{\tilde{\theta}}{2} = \tilde{d}$ and consequently $\eta_0 = \tilde{\eta}_0$ (c.f. Claim 8.1) which then implies $\eta_j = \tilde{\eta}_j \ \forall\, j$.*

8.2 The rank two case

Theorem 8.10 *Let G be a compact simple Lie group of rank two and T its maximal torus. Let $\psi, \tilde{\psi} : \mathbf{R}^2 \to G/T$ be isometric τ-primitive maps with constant Kähler angle. Then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof: We will show that constant curvature metric and constant Kähler angle determine the η-invariants of a τ-primitive map completely. Thus the weak congruence theorem (Theorem 7.1) gives the required result.

Let $ds^2 = c|dz|^2$ and $\tan^2 \frac{\theta}{2} = d \in \mathbf{R}$. Then

$$k_1\eta_1 + k_2\eta_2 = c - k_0\eta_0 \quad \text{where} \quad \eta_0 = \frac{cd}{k_0(1+d)}.$$

Thus

$$k_1\eta_1 + k_2\eta_2 = c - \frac{cd}{1+d} = \frac{c}{1+d}.$$

Claim 8.2 gives $\Delta \log \eta_0^{m_0}\eta_1^{m_1}\eta_2^{m_2} = 0$. From this, and the fact that η_0 is constant, we get $\Delta \log \eta_1^{m_1}\eta_2^{m_2} = 0$. From Claim 8.5 it now follows that

$$\eta_1^{m_1}\eta_2^{m_2} = a \in \mathbf{R}.$$

However

$$\eta_1 = \frac{c}{k_1(1+d)} - \frac{k_2}{k_1}\eta_2.$$

which means that η_2 satisfies the following polynomial equation

$$(\frac{c}{k_1(1+d)} - \frac{k_2}{k_1}\eta_2)^{m_1}\eta_2^{m_2} = a.$$

By continuity η_2 is constant, so η_1 is constant as well. From Claim 8.6 it follows that (η_0, η_1, η_2) is a multiple of $\mathbf{n}$. Since $\eta_0 = \frac{cd}{k_0(1+d)}$ it follows that (η_0, η_1, η_2) is uniquely determined, so we can apply the weak congruence theorem and reach the desired result. □

8.3 Congruence theorem for the $SU(n+1)$-case

In this section we will prove a constant curvature and Kähler angle congruence theorem for $SU(4)$ and $SU(5)$ under the additonal assumption that $k_j = 1 \ \forall \ j$. As far as possible the proof is done for general $SU(n+1)$ and we hope that these parts might be useful in a future attempt to prove the general $SU(n+1)$-case.

Initial calculations

Let G/T be equipped with a G-invariant metric such that $k_j = 1$ for all j. If ψ is of constant curvature and Kähler angle we know $\eta_0 = const$ and $\sum_1^n \eta_k = c - \eta_0 = const$.

From $\eta_0 = const$ and the Toda equation

$$0 = \Delta \log \eta_0 = -2\eta_0 + \eta_1 + \eta_n$$

we get

$$\eta_1 + \eta_n = 2\eta_0 = const.$$

We want to show that $\eta_1\eta_n = const$ to deduce that η_1 and η_n are constant. This would then imply that all η_k are constant.

Let $n = 2m - 1$ or $= 2m$ and let $H(r_1, \ldots, r_m) := \prod_{k=1}^{m}(\eta_k\eta_{n+1-k})^{r_k}$. We have to find real constants $r_1, \ldots, r_m$ such that $\Delta \log H = const$. From this we can then deduce that H is constant, and if this would be the case for $r_1 \neq 0$ and $r_2, \ldots, r_m = 0$, we would get $\Delta \log \eta_1\eta_n = const$, and hence $\eta_1\eta_n = const$ as required.

$$\begin{aligned}
\Delta \log H(r_1, \ldots, r_m) &= \Delta \log \prod_{k=1}^{m} (\eta_k\eta_{n+1-k})^{r_k} \\
&= \sum_{k=1}^{m} r_k \Delta \log(\eta_k\eta_{n+1-k}) \\
&= \sum_{k=1}^{m} r_k\{(\eta_{k-1} - 2\eta_k + \eta_{k+1}) + (\eta_{n-k} - 2\eta_{n+1-k} + \eta_{n+2-k})\} \\
&= \sum_{k=1}^{m} r_k\{(\eta_{k-1} + \eta_{n+2-k}) - 2(\eta_k + \eta_{n+1-k}) + (\eta_{k+1} + \eta_{n-k})\} \\
&= \sum_{k=1}^{m} r_k\{(\eta_{k-1} + \eta_{n+1-(k-1)}) - 2(\eta_k + \eta_{n+1-k}) + (\eta_{k+1} + \eta_{n+1-(k+1)})\} \\
&= \sum_{k=1}^{m} r_k(a_{k-1} - 2a_k + a_{k+1})
\end{aligned}$$

where

$$a_k := \eta_k + \eta_{n+1-k} \quad k = 0, \ldots, n.$$

Note that $a_0 = 2\eta_0 = a_1$ and $a_k = a_{n+1-k}$. In particular we have

- $a_{m+1} = a_{n+1-(m+1)} = a_{2m-m-1} = a_{m-1}$ and $a_m = 2\eta_m$ for $n + 1 = 2m$, and
- $a_{m+1} = a_{n+1-(m+1)} = a_{2m+1-m-1} = a_m$ for $n + 1 = 2m + 1$.

Sorting the above expression w.r.t. the a_k gives

$$\begin{aligned}
\Delta \log H(r_1, \ldots, r_m) &= \sum_{k=1}^{m} r_k(a_{k-1} - 2a_k + a_{k+1}) \\
&= \sum_{k=0}^{m-1} r_{k+1}a_k - 2\sum_{k=1}^{m} r_k a_k + \sum_{k=1}^{m+1} r_{k-1}a_k \\
&= r_1 a_0 + r_2 a_1 + \sum_{k=2}^{m-1} r_{k+1}a_k \\
&\quad -2r_1 a_1 - 2\sum_{k=2}^{m-1} r_k a_k - 2r_m a_m \\
&\quad + \sum_{k=2}^{m-1} r_{k-1}a_k + r_{m-1}a_m + r_m a_{m+1} \\
&= r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-1}(r_{k-1} - 2r_k + r_{k+1})a_k \\
&\quad +(r_{m-1} - 2r_m)a_m + r_m a_{m+1}
\end{aligned}$$

Our aim is to find real numbers $r_1, \ldots, r_m$ such that this is constant. To simplify this expression for $\Delta \log H$ we will use $\sum_0^n \eta_k = c$ for $n+1 = 2m$ and $n+1 = 2m+1$ separately.

$SU(2m+1)$ calculations

For $n+1 = 2m+1$ using $a_{m+1} = a_m$ we get

$$\Delta \log H(r_1, \ldots, r_m) = r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-1}(r_{k-1} - 2r_k + r_{k+1})a_k + (r_{m-1} - r_m)a_m.$$

Also

$$c = \sum_0^n \eta_k = \frac{1}{2}a_0 + \sum_1^m a_k$$

so

$$a_m = c - \frac{1}{2}a_0 - \sum_1^{m-1} a_k.$$

Thus our equation becomes

$$r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-1}(r_{k-1} - 2r_k + r_{k+1})a_k + (r_{m-1} - r_m)(c - \frac{1}{2}a_0 - \sum_{1}^{m-1} a_k)$$
$$= (r_{m-1} - r_m)c + (r_1 - \frac{1}{2}(r_{m-1} - r_m))a_0 + (r_2 - 2r_1 - r_{m-1} + r_m)a_1$$
$$+ \sum_{k=2}^{m-1}(r_{k-1} - 2r_k + r_{k+1} - r_{m-1} + r_m)a_k$$

This will be constant if

$$r_{k-1} - 2r_k + r_{k+1} - r_{m-1} + r_m = 0 \quad \forall k = 2, \ldots, m-1$$

or

$$r_k - 2r_{k+1} + r_{k+2} - r_{m-1} + r_m = 0 \quad \forall k = 1, \ldots, m-2$$

Now r_{m-1} and r_m are free variables which determine $r_1, \ldots, r_{m-2}$. In order to see this we will write the above equations as equations with the r_k terms on the left hand side for $k = 1, \ldots, m-2$ and the r_{m-1}, r_m terms on thcoe right hand side.

$$r_k - 2r_{k+1} + r_{k+2} = r_{m-1} - r_m \quad \forall k = 1, \ldots, m-4$$

$$r_{m-3} - 2r_{m-2} = -r_m \quad (k = m-3)$$

and

$$r_{m-2} = 3r_{m-1} - 2r_m \quad (k = m-2).$$

This can be written as a matrix equation

$$\begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ r_{m-2} \end{pmatrix} = r_{m-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 3 \end{pmatrix} + r_m \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ -2 \end{pmatrix}$$

Now the inverse of this matrix is

$$\begin{pmatrix} 1 & 2 & 3 & \dots & \dots & m-2 \\ & 1 & 2 & 3 & \dots & m-3 \\ & & \ddots & \ddots & & \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ r_{m-2} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \dots & \dots & m-2 \\ & 1 & 2 & 3 & \dots & m-3 \\ & & \ddots & \ddots & & \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix} \left(r_{m-1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 3 \end{pmatrix} + r_m \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ -2 \end{pmatrix} \right).$$

Now

$$\begin{pmatrix} 1 & 2 & 3 & \dots & \dots & m-2 \\ & 1 & 2 & 3 & \dots & m-3 \\ & & \ddots & \ddots & & \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{m-3-1} i + 3(m-1-1) \\ \sum_{i=1}^{m-3-2} i + 3(m-1-2) \\ \sum_{i=1}^{m-3-3} i + 3(m-1-3) \\ \vdots \\ \sum_{i=1}^{m-3-(m-3)} i + 3(m-1-(m-3)) \\ \sum_{i=1}^{m-3-(m-2)} i + 3(m-1-(m-2)) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{(m-3-1)(m-2-1)}{2} + 3(m-1-1) \\ \frac{(m-3-2)(m-2-2)}{2} + 3(m-1-2) \\ \frac{(m-3-3)(m-2-3)}{2} + 3(m-1-3) \\ \vdots \\ \frac{(m-3-(m-3))(m-2-(m-3))}{2} + 3(m-1-(m-3)) \\ \frac{(m-3-(m-2))(m-2-(m-2))}{2} + 3(m-1-(m-2)) \end{pmatrix}$$

$$= \sum_{k=1}^{m-2} \left((\frac{(m-3-k)(m-2-k)}{2} + 3(m-1-k) \right) e_k$$

$$= \sum_{k=1}^{m-2} \frac{(m-k)(m-k+1)}{2} e_k.$$

and

$$\begin{pmatrix} 1 & 2 & 3 & \dots & \dots & m-2 \\ & 1 & 2 & 3 & \dots & m-3 \\ & & \ddots & \ddots & & \\ & & & 1 & 2 & 3 \\ & & & & 1 & 2 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \\ -2 \end{pmatrix}$$

$$= -\begin{pmatrix} \sum_{i=1}^{m-2-1} i + 2(m-1-1) \\ \sum_{i=1}^{m-2-2} i + 2(m-1-2) \\ \sum_{i=1}^{m-2-3} i + 2(m-1-3) \\ \vdots \\ \sum_{i=1}^{m-2-(m-3)} i + 2(m-1-(m-3)) \\ \sum_{i=1}^{m-2-(m-2)} i + 2(m-1-(m-2)) \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{(m-2-1)(m-1-1)}{2} + 2(m-1-1) \\ \frac{(m-2-2)(m-1-2)}{2} + 2(m-1-2) \\ \frac{(m-2-3)(m-1-3)}{2} + 2(m-1-3) \\ \vdots \\ \frac{(m-2-(m-3))(m-1-(m-3))}{2} + 2(m-1-(m-3)) \\ \frac{(m-2-(m-2))(m-1-(m-2))}{2} + 2(m-1-(m-2)) \end{pmatrix}$$

$$= -\sum_{k=1}^{m-2} \left(\left(\frac{(m-2-k)(m-1-k)}{2} + 2(m-1-k) \right) e_k \right.$$

$$= \sum_{k=1}^{m-2} -\frac{(m-k+2)(m-k-1)}{2} e_k .$$

Thus

$$r_k = r_{m-1}\frac{(m-k)(m-k+1)}{2} - r_m\frac{(m-k+2)(m-k-1)}{2} \quad \forall k = 1,\dots,m-2.$$

Note that $r_{m-1} = r_m = 1$ gives $r_k = \frac{(m-k)(m-k+1)}{2} - \frac{(m-k+2)(m-k-1)}{2} = 1$ which corresponds to the trivial solution, i.e. $\Delta \log \eta_1\eta_2\cdots\eta_n = 0$.

Claim 8.11 *If $\psi : \mathbf{R}^2 \to SU(2m+1)/T$ is of constant curvature and Kähler angle, and the metric coefficients k_j all coincide, then*

$$\tan^2\frac{\theta}{2} = \frac{1}{2m}.$$

Proof:

First note that if $\Delta \log H(r_1,\dots,r_m) = const$ then $\Delta \log H(r_1,\dots,r_m) = 0$. Thus for constants r_k as above we have

$$(r_{m-1} - r_m)c + (r_1 - \frac{1}{2}(r_{m-1} - r_m))a_0 + (r_2 - 2r_1 - r_{m-1} + r_m)a_1 = 0.$$

Noting that $a_1 = a_0$ and setting $r_m = 1$ and $r_{m-1} = 0$ this becomes

$$-c + (r_1 + \frac{1}{2} + r_2 - 2r_1 + 1)a_0 = 0$$

or

$$\begin{aligned}
c &= (-r_1 + r_2 + \frac{3}{2})a_0 \\
&= (\frac{(m-1+2)(m-1-1)}{2} - \frac{(m-2+2)(m-2-1)}{2} + \frac{3}{2})a_0 \\
&= (\frac{(m+1)(m-2)}{2} - \frac{m(m-3)}{2} + \frac{3}{2})a_0 \\
&= (\frac{2m-2}{2} + \frac{3}{2})a_0 \\
&= \frac{2m+1}{2}a_0.
\end{aligned}$$

Thus

$$\eta_0 = \frac{1}{2}a_0 = \frac{c}{2m+1}.$$

Comparing this with

$$\eta_0 = \frac{cd}{1+d}$$

where $\tan^2\frac{\theta}{2} = d$, gives

$$\frac{cd}{1+d} = \frac{c}{2m+1},$$

so

$$d = \frac{1}{2m}.$$

□

This was to be expected because we have in general

$$\tan^2\frac{\theta}{2} = \frac{k_0\eta_0}{\sum_{j=1}^{\ell} k_j\eta_j}$$

and we are aiming to show that $(\eta_0, \ldots, \eta_{2m}) = r(1, \ldots, 1)$ so that with $k_i = k_j$ $\forall i, j$, we would get

$$\tan^2\frac{\theta}{2} = \frac{r}{\sum_{j=1}^{2m} r} = \frac{1}{2m}.$$

Knowing the Kähler angle now gives a nicer expression for η_0. Recall that

$$\eta_0 = \frac{cd}{1+d}.$$

Now $d = \frac{1}{2m}$, so

$$\eta_0 = \frac{cd}{1+d} = \frac{c\frac{1}{2m}}{1+\frac{1}{2m}} = \frac{c}{2m+1}.$$

Also

$$a_1 = a_0 = \frac{2c}{2m+1}$$

and

$$\sum_{k=2}^{m} a_k = c - a_1 - \frac{1}{2}a_0 = c - \frac{3c}{2m+1} = \frac{(2m-2)c}{2m+1},$$

which also was to be expected.

Theorem 8.12 *Let $SU(5)/T$ be equipped with a G-invariant metric such that the metric coefficients satisfy $k_j = 1$. Let $\psi, \tilde{\psi} : \mathbf{R}^2 \to SU(5)/T$ be isometric τ-primitive maps with constant Kähler angle. Then the Kähler angle satisfies* $\tan^2 \frac{\theta}{2} = \frac{1}{4}$ *and ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof: From the above we obtain $a_2 = c - a_1 - \frac{1}{2}a_0 = \frac{2c}{5} = const.$ Thus $H(r_1, r_2) = const$ for all $r_1, r_2 \in \mathbf{R}$ which gives that η_1 and η_4 are constant. However, if two consecutive η-invariants (in this case η_0 and η_1) are constant, then all of them are constant which gives the congruence theorem by Claim 8.6 together with the weak congruence theorem. □

$SU(2m)$ calculations

For $n+1 = 2m$ using $a_{m+1} = a_{m-1}$ we get

$$\begin{aligned}
\Delta \log H(r_1, \ldots, r_m) &= r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-2}(r_{k-1} - 2r_k + r_{k+1})a_k \\
&\quad + (r_{m-2} - 2r_{m-1} + r_m + r_m)a_{m-1} + (r_{m-1} - r_m)a_m \\
&= r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-2}(r_{k-1} - 2r_k + r_{k+1})a_k \\
&\quad + (r_{m-2} - 2r_{m-1} + 2r_m)a_{m-1} + (r_{m-1} - 2r_m)a_m.
\end{aligned}$$

From the constant curvature condition we get

$$c = \sum_0^n \eta_k = \frac{1}{2}a_0 + \sum_1^{m-1} a_k + \frac{1}{2}a_m,$$

so

$$a_m = 2c - a_0 - 2\sum_1^{m-1} a_k. \qquad (\star)$$

Thus our equation becomes

$$\begin{aligned}
&r_1 a_0 + (r_2 - 2r_1)a_1 + \sum_{k=2}^{m-2}(r_{k-1} - 2r_k + r_{k+1})a_k \\
&\quad +(r_{m-2} - 2r_{m-1} + 2r_m)a_{m-1} + (r_{m-1} - 2r_m)(2c - a_0 - 2\sum_1^{m-1} a_k) \\
= &\quad 2(r_{m-1} - 2r_m)c + (r_1 - r_{m-1} + 2r_m)a_0 + (r_2 - 2r_1 - 2(r_{m-1} - 2r_m))a_1 \\
&\quad + \sum_{k=2}^{m-2}(r_{k-1} - 2r_k + r_{k+1} - 2(r_{m-1} - 2r_m))a_k \\
&\quad +(r_{m-2} - 2r_{m-1} + 2r_m - 2(r_{m-1} - 2r_m))a_{m-1} \\
= &\quad 2(r_{m-1} - 2r_m)c + (r_1 - r_{m-1} + 2r_m)a_0 + (r_2 - 2r_1 - 2(r_{m-1} - 2r_m))a_1 \\
&\quad + \sum_{k=2}^{m-2}(r_{k-1} - 2r_k + r_{k+1} - 2(r_{m-1} - 2r_m))a_k \\
&\quad +(r_{m-2} - 4r_{m-1} + 6r_m)a_{m-1}
\end{aligned}$$

This will be constant if

$$r_{k-1} - 2r_k + r_{k+1} - 2(r_{m-1} - 2r_m) = 0 \quad \forall k = 2, \ldots, m-2$$

and

$$r_{m-2} - 4r_{m-1} + 6r_m = 0 \quad (m \geq 3).$$

Note that $(1, \ldots, 1, \frac{1}{2})$ is a solution as $\Delta \log \eta_1 \cdots \eta_{2m} = 0$.

Theorem 8.13 *Let $SU(4)/T$ be equipped with a G-invariant metric such that the metric coefficients satisfy $k_j = 1$. Let $\psi, \tilde{\psi} : \mathbf{R}^2 \to SU(4)/T$ be isometric τ-*

primitive maps with constant Kähler angle. Then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.

Proof: From equation ($\star$) above we obtain $a_2 = 2c - a_0 - 2a_1 = const.$ Thus $H(r_1, r_2) = const$ for all $r_1, r_2 \in \mathbf{R}$ which gives that η_1 and η_3 are constant. However, if two consecutive η-invariants are constant, then all of them are constant. This gives the congruence theorem by Claim 8.6 together with the weak congruence theorem. □

8.4 A Congruence theorem for E_8

As before we will try to give the proof for the E_8 congruence theorem in its most general form, in order to have the opportunity to improve the result in the future.

Theorem 8.14 *Let E_8/T be equipped with a G-invariant metric such that the metric coefficients satisfy $k_j = 1$. Let $\psi, \tilde{\psi} : \mathbf{R}^2 \to E_8/T$ be isometric τ-primitive maps with constant Kähler angle. Then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$ and curvature and Kähler angle are given by $c = \sum_0^8 m_j = 30$ and* $\tan^2 \frac{\theta}{2} = \frac{1}{29}$.

Proof: The affine Toda equations of E_8 can be read off from the extended Dynkin diagram (for details see [CSM], p.17-22).

From Claim 8.1 we know that $\eta_0 = \frac{cd}{k_0(1+d)}$ is constant ($ds^2 = c|dz|^2$ and $\tan^2 \frac{\theta}{2} = d$).

We will now compute $\eta_1, \ldots, \eta_8$ w.r.t. η_0.

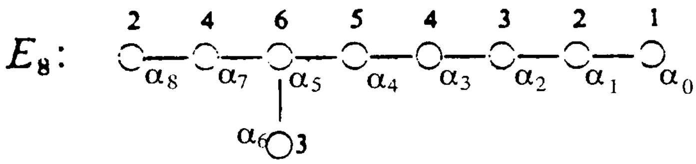

Figure 8.1: Extendend Dynkin diagram of E_8

$$0 = \Delta \log \eta_0 = \eta_1 - 2\eta_0 \iff \eta_1 = 2\eta_0$$

$$0 = \Delta \log \eta_1 = \eta_0 - 2\eta_1 + \eta_2 \iff \eta_2 = 3\eta_0$$

$$0 = \Delta \log \eta_2 = \eta_1 - 2\eta_2 + \eta_3 \iff \eta_3 = 4\eta_0$$

$$0 = \Delta \log \eta_3 = \eta_2 - 2\eta_3 + \eta_4 \iff \eta_4 = 5\eta_0$$

$$0 = \Delta \log \eta_4 = \eta_3 - 2\eta_4 + \eta_5 \iff \eta_5 = 6\eta_0$$

$$0 = \Delta \log \eta_5 = \eta_4 - 2\eta_5 + \eta_6 + \eta_7 \iff \eta_6 + \eta_7 = 7\eta_0$$

Now use the fact that the induced metric is of constant curvature:

$$c = \sum_{j=0}^{8} k_j\eta_j = \sum_{j=0}^{5} k_j\eta_j + k_6\eta_6 + k_7\eta_7 + k_8\eta_8.$$

With $\eta_6 = 7\eta_0 - \eta_7$ and $\eta_j = (j+1)\eta_0$ for $j = 0, \ldots, 5$ this becomes

$$c = \sum_{j=0}^{5}(j+1)k_j\eta_0 + k_6(7\eta_0 - \eta_7) + k_7\eta_7 + k_8\eta_8 = (\sum_{j=0}^{5}(j+1)k_j + 7k_6)\eta_0 + (k_7 - k_6)\eta_7 + k_8\eta_8,$$

so

$$\begin{aligned}
(k_7 - k_6)\eta_7 + k_8\eta_8 &= c - (\sum_{j=0}^{5}(j+1)k_j + 7k_6)\eta_0 \\
&= c - (\sum_{j=0}^{5}(j+1)k_j + 7k_6)\frac{cd}{k_0(1+d)} \\
&= c\Big(k_0 + k_0 d - (\sum_{j=0}^{5}(j+1)k_j + 7k_6)d\Big)\frac{1}{k_0(1+d)} \\
&= \Big(k_0 - (\sum_{j=1}^{5}(j+1)k_j + 7k_6)d\Big)\frac{c}{k_0(1+d)}.
\end{aligned}$$

Note that the left hand side of this equation is constant since the right hand is.

Claim: If $c = \sum_0^8 \eta_j$ then constant curvature and Kähler angle determine congruence.

Proof: If $k_6 = k_7$ it follows that η_8 is constant. But then

$$0 = \Delta \log \eta_8 = \eta_7 - 2\eta_8 \quad \Longleftrightarrow \quad \eta_7 = 2\eta_8,$$

so η_7 is constant, and from $\eta_6 + \eta_7 = 7\eta_0$ it then follows that also η_6 is constant. Therefore all η-invariants are constant, from which the congruence theorem follows. We will now compute all η-invariants in the case that η_8 is constant.

$$0 = \Delta \log \eta_8 = \eta_7 - 2\eta_8 \quad \Longleftrightarrow \quad \eta_7 = 2\eta_8$$

$$0 = \Delta \log \eta_7 = \eta_8 - 2\eta_7 + \eta_5 \quad \Longleftrightarrow \quad \eta_5 = 3\eta_8$$

Since also $\eta_5 = 6\eta_0$ we find

$$\eta_8 = 2\eta_0 \quad \text{and} \quad \eta_7 = 2\eta_8 = 4\eta_0.$$

From $\eta_6 + \eta_7 = 7\eta_0$ we finally get

$$\eta_6 = 3\eta_0.$$

Therefore the η-invariants are given by

$$(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8) = \eta_0(1, 2, 3, 4, 5, 6, 3, 4, 2).$$

Note 8.15 *Note that these are exactly the coefficients of the simple roots giving the linear combination of the maximal root. This was also the case for $SU(n+1)$. It might be interesting to investigate whether this is a general rule.*

The metric is given by

$$c = (\sum_{j=0}^{8} m_j k_j)\eta_0,$$

so

$$\eta_0 = \frac{c}{\sum_{j=0}^{8} m_j k_j}.$$

Thus

$$\begin{aligned}(\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8) &= \frac{c}{\sum_{j=0}^{8} m_j k_j}(1, 2, 3, 4, 5, 6, 3, 4, 2) \\ &= \frac{c}{\sum_{j=0}^{8} m_j k_j}(m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8)..\end{aligned}$$

Finally, we find for the Kähler angle

$$\tan^2\frac{\theta}{2} = \frac{k_0\eta_0}{c - k_0\eta_0} = \frac{k_0\eta_0}{\sum_{j=0}^{8} m_j k_j \eta_0 - k_0\eta_0} = \frac{k_0}{\sum_{j=1}^{8} m_j k_j}$$

and if we assume $k_j = 1$ for all j then $c = \sum_0^8 m_j = 30$ and $\tan^2\frac{\theta}{2} = \frac{1}{29}$, and thus the proof is complete. □

8.5 A Congruence theorem for E_6

Theorem 8.16 *Let E_6/T be equipped with a G-invariant metric such that the metric coefficients satisfy $k_j = 1$. Let $\psi, \tilde{\psi} : \mathbf{R}^2 \to E_6/T$ be isometric τ-primitive maps with constant Kähler angle. Then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof: As before we can read off the affine Toda equations of E_6 from the extended Dynkin diagram.

We will now use the Toda equations to see which η-invariants are constant.

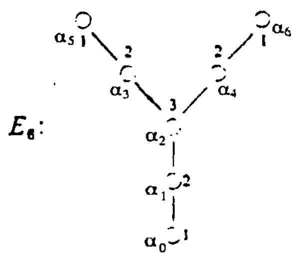

Figure 8.2: Extendend Dynkin diagram of E_6

$$0 = \Delta \log \eta_0 = \eta_1 - 2\eta_0 \iff \eta_1 = 2\eta_0$$

$$0 = \Delta \log \eta_1 = \eta_0 - 2\eta_1 + \eta_2 \iff \eta_2 = 3\eta_0$$

$$0 = \Delta \log \eta_2 = \eta_1 - 2\eta_2 + \eta_3 + \eta_4 \iff \eta_3 + \eta_4 = 4\eta_0$$

Now $\eta_0 = const$ implies that also η_1, η_2 and $\eta_3 + \eta_4$ are constant. However, if we assume that $c = \sum_0^6 \eta_j$ we also find that $\eta_5 + \eta_6$ is constant.

It follows that

$$\Delta \log \eta_0^{r_0} \eta_1^{r_1} \eta_2^{r_2} (\eta_3\eta_4)^{r_3} (\eta_5\eta_6)^{r_4} = const(=0) \quad \forall r_0, \ldots, r_4$$

Thus

$$\eta_3\eta_4 = const \quad \text{and} \quad \eta_5\eta_6 = const.$$

which implies that $\eta_3, \ldots, \eta_6$ are all constant. Consequently all η-invariants are constant and hence uniquely determined. This proves the theorem. □

8.6 A Congruence theorem for F_4 and $Sp(\ell)$

Theorem 8.17 *Let $G = F_4$ or $Sp(\ell)$. Let G/T be equipped with any G-invariant metric and let $\psi, \tilde{\psi} : \mathbf{R}^2 \to G/T$ be isometric τ-primitive maps with constant Kähler angle. Then ψ and $\tilde{\psi}$ are congruent by an isometry $g \in G$, $\tilde{\psi} = g\psi$.*

Proof:

As before, constant curvature and Kähler angle imply $\eta_0 = const$. Since the extended Dynkin diagrams of F_4 and $Sp(\ell)$ have no ramifications, it is clear that $\eta_0 = const$ gives $\eta_1 = const$, and hence $\eta_j = const \ \forall\, j$ as in this case two consecutive η-invariants determine all η-invariants. Therefore Claim 8.6 together with the weak congruence theorem finish the proof. □

8.7 No Congruence theorem for E_7

The only Lie group for which it was not possible to find a congruence theorem within this setting was E_7, due to the particular form of its extended Dynkin diagram. It is hoped that, using some of the ideas developed in this thesis, it will be possible in the future to find a congruence theorem in this case as well.

Appendix A

Computations for maps into $SU(3)/T^2$

A.1 Computing the Frenet frame of S^1-symmetric holomorphic maps $S^2 \to \mathbf{CP}^2$

Let $\phi : S^2 \to CP^2$ be holomorphic and S^1-symmetric. There then exists a holomorphic coordinate z on S^2 such that ϕ can be expressed as

$$\phi(z) = [a, bz^k, cz^\ell]$$

with $a, b, c \in \mathbf{R}^+$ and $k, \ell \in \mathbf{N}$, $k < \ell$. (See [BW3] for details.)

We now compute the Frenet frame of ϕ. Let

$$f_0(z) = f(z) = \begin{pmatrix} a \\ bz^k \\ cz^\ell \end{pmatrix}.$$

Then $\phi = [f_0]$. To compute f_1 and f_2 observe that for the harmonic sequence

$$f_i' = f_{i+1} + \frac{\partial \log |f_i|^2}{\partial z} f_i$$

holds. In general we have

$$\frac{\partial \log |g|^2}{\partial z} = \frac{\partial \log(\bar{g} \cdot g)}{\partial z} = \frac{1}{|g|^2} \frac{\partial(\bar{g} \cdot g)}{\partial z} = \frac{1}{|g|^2} \left(\bar{g} \cdot \frac{\partial g}{\partial z} + \overline{\frac{\partial g}{\partial \bar{z}}} \cdot g \right) = \frac{1}{|g|^2} (\bar{g} \cdot g' + \bar{g}' \cdot g)$$

but for holomorphic g this simplifies to

$$\frac{\partial \log |g|^2}{\partial z} = \frac{1}{|g|^2} \bar{g} \cdot \frac{\partial g}{\partial z} = \frac{1}{|g|^2} \bar{g} \cdot g'.$$

Computation of f_1

Let

$$A(z) = |f_0|^2 = a^2 + b^2 |z|^{2k} + c^2 |z|^{2\ell}.$$

The derivative of f_0 is

$$f'(z) = \begin{pmatrix} 0 \\ kbz^{k-1} \\ \ell cz^{\ell-1} \end{pmatrix}.$$

Thus

$$\frac{\partial \log |f|^2}{\partial z} = \frac{1}{|f|^2} \bar{f} \cdot f' = \frac{1}{|f|^2} \begin{pmatrix} a \\ b\bar{z}^k \\ c\bar{z}^\ell \end{pmatrix} \cdot \begin{pmatrix} 0 \\ kbz^{k-1} \\ \ell cz^{\ell-1} \end{pmatrix}$$

$$
\begin{aligned}
&= \frac{1}{|f|^2}(0 + kb^2 z^{k-1}\bar{z}^k + \ell c^2 z^{\ell-1}\bar{z}^\ell) \\
&= \frac{1}{|f|^2}(kb^2|z|^{2(k-1)}\bar{z} + \ell c^2|z|^{2(\ell-1)}\bar{z}) \\
&= \frac{1}{|f|^2}(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z}
\end{aligned}
$$

From

$$f_1 = f' - \frac{\partial \log |f|^2}{\partial z} f$$

we hence get for the components of f_1

$$
\begin{aligned}
(f_1)_1 &= 0 - \frac{1}{|f|^2}(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z}a \\
&= -\frac{a(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z}}{|f|^2} \\
(f_1)_2 &= kbz^{k-1} - \frac{1}{|f|^2}(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z}bz^k \\
&= \frac{kbz^{k-1}|f|^2 - (kb^2|z|^{2k} + \ell c^2|z|^{2\ell})bz^{k-1}}{|f|^2} \\
&= \frac{(k|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})bz^{k-1}}{|f|^2} \\
(f_1)_3 &= \ell cz^{\ell-1} - \frac{1}{|f|^2}(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z}cz^\ell \\
&= \frac{\ell cz^{\ell-1}|f|^2 - (kb^2|z|^{2k} + \ell c^2|z|^{2\ell})cz^{\ell-1}}{|f|^2} \\
&= \frac{(\ell|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})cz^{\ell-1}}{|f|^2}
\end{aligned}
$$

Thus for $z \neq 0$

$$f_1(z) = \frac{1}{|f|^2}\begin{pmatrix} -a(kb^2|z|^{2(k-1)} + \ell c^2|z|^{2(\ell-1)})\bar{z} \\ (k|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})bz^{k-1} \\ (\ell|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})cz^{\ell-1}) \end{pmatrix} = \frac{1}{z|f|^2}\begin{pmatrix} -a(kb^2|z|^{2k} + \ell c^2|z|^{2\ell}) \\ b(k|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})z^k \\ c(\ell|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})z^\ell) \end{pmatrix}$$

Note that this expression extends continuously to $f_1(0) = 0$ as required.

Let further

$$B(z) = kb^2|z|^{2k} + \ell c^2|z|^{2\ell}.$$

Then

$$f_1(z) = \frac{1}{zA}\begin{pmatrix} -aB \\ b(kA-B)z^k \\ c(\ell A - B)z^\ell \end{pmatrix}.$$

Computation of f_2

Next note that f_2 has to be orthogonal to both, f_0 and f_1. Using this orthogonality relation we can compute f_2 up to a factor (consisting of a meromorphic function).

Let

$$f_2 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

It must satisfy

$$f_2 \cdot \overline{f_0} = 0 \quad \text{and} \quad f_2 \cdot \overline{f_1} = 0 \quad (1).$$

Thus

$$f_2 \cdot \overline{f_1} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot \begin{pmatrix} a \\ b\bar{z}^k \\ c\bar{z}^\ell \end{pmatrix} = \alpha a + \beta b \bar{z}^k + \gamma c \bar{z}^\ell \overset{!}{=} 0$$

and

$$f_2 \cdot \overline{f_0} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot \begin{pmatrix} -aB \\ b(kA-B)\bar{z}^k \\ c(\ell A - B)\bar{z}^\ell \end{pmatrix} = -\alpha a B + \beta b(kA-B)\bar{z}^k + \gamma c(\ell A - B)\bar{z}^\ell \overset{!}{=} 0. \quad (2)$$

Hence $(1) \times B + (2)$ gives

$$\beta b k A \bar{z}^k + \gamma c \ell A \bar{z}^\ell = 0,$$

so

$$\beta = -\frac{c\ell}{bk}\bar{z}^{\ell-k}\gamma.$$

Putting this into (1) gives

$$\alpha = -\frac{1}{a}\left(-\frac{c\ell}{k}\bar{z}^{\ell}\gamma + \gamma c\bar{z}^{\ell}\right) = \frac{c(\ell-k)}{ak}\bar{z}^{\ell}\gamma.$$

Therefore

$$f_2 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \gamma \begin{pmatrix} \frac{c(\ell-k)}{ak}\bar{z}^{\ell} \\ -\frac{c\ell}{bk}\bar{z}^{\ell-k} \\ 1 \end{pmatrix},$$

i.e. f_2 is a multiple of

$$g := \begin{pmatrix} bc(\ell-k)\bar{z}^{\ell} \\ -ac\ell\bar{z}^{\ell-k} \\ abk \end{pmatrix}.$$

We have

$$f_2 = \lambda g$$

where λ is a function of $z, \bar{z}$. We will now compute λ to determine f_2.

From the constuction of the harmonic sequence we know that

$$\frac{\partial f_2}{\partial \bar{z}} = -\frac{|f_2|^2}{|f_1|^2}f_1.$$

Taking the complex conjugate of this and taking the dot-product with f_1 gives

$$\overline{\frac{\partial f_2}{\partial \bar{z}}} \cdot f_1 = -\frac{|f_2|^2}{|f_1|^2}\overline{f_1} \cdot f_1 = -|f_2|^2. \quad (\star)$$

Since

$$\overline{\frac{\partial f_2}{\partial \bar{z}}} = \overline{\frac{\partial \lambda}{\partial \bar{z}}}g + \lambda\overline{\frac{\partial g}{\partial \bar{z}}}$$

and $f_1 \perp f_2$, i.e. $f_1 \perp g$ we get

$$\overline{\frac{\partial f_2}{\partial \bar{z}}} \cdot f_1 = \bar{\lambda}\overline{\frac{\partial g}{\partial \bar{z}}} \cdot f_1.$$

On the other hand

$$|f_2|^2 = |\lambda|^2|g|^2 = \lambda\bar{\lambda}|g|^2,$$

so we get from $(\star)$

$$\lambda = -\frac{1}{|g|^2}\overline{\frac{\partial g}{\partial \bar{z}}} \cdot f_1.$$

With

$$f_1(z) = \frac{1}{z|f|^2}\begin{pmatrix} -aB \\ b(kA - B)z^k \\ c(\ell A - B)z^\ell \end{pmatrix}.$$

and

$$C := |g|^2 = |\begin{pmatrix} bc(\ell - k)\bar{z}^\ell \\ -ac\ell\bar{z}^{\ell-k} \\ abk \end{pmatrix}|^2 = a^2b^2k^2 + a^2c^2\ell^2|z|^{2(\ell-k)} + b^2c^2(\ell - k)^2|z|^{2\ell}$$

we hence get for λ

$$\lambda = -\frac{1}{|g|^2}\overline{\frac{\partial g}{\partial \bar{z}}} \cdot f_1.$$

Now

$$\frac{\partial g}{\partial \bar{z}} = \begin{pmatrix} bc(\ell - k)\ell\bar{z}^{\ell-1} \\ -ac(\ell - k)\ell\bar{z}^{\ell-k-1} \\ 0 \end{pmatrix},$$

so

$$\overline{\frac{\partial g}{\partial \bar{z}}} = \begin{pmatrix} bc(\ell - k)\ell z^{\ell-1} \\ -ac(\ell - k)\ell z^{\ell-k-1} \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned}
\overline{\frac{\partial g}{\partial \bar{z}}} \cdot f_1 &= \begin{pmatrix} bc(\ell-k)\ell z^{\ell-1} \\ -ac(\ell-k)\ell z^{\ell-k-1} \\ 0 \end{pmatrix} \cdot \frac{1}{z|f|^2} \begin{pmatrix} -aB \\ b(kA-B)z^k \\ c(\ell A - B)z^\ell \end{pmatrix} \\
&= \frac{1}{z|f|^2}(-abc(\ell-k)\ell B z^{\ell-1} - abc(\ell-k)\ell(kA-B)z^k z^{\ell-k-1}) \\
&= -\frac{abc}{|f|^2}(\ell-k)\ell k A z^{\ell-2} = -abc(\ell-k)k\ell z^{\ell-2},
\end{aligned}$$

so

$$\lambda = \frac{1}{|g|^2} abc(\ell-k)k\ell z^{\ell-2}$$

and finally

$$\begin{aligned}
f_2 &= \lambda g = \frac{abc(\ell-k)k\ell z^{\ell-2}}{|g|^2} \begin{pmatrix} bc(\ell-k)\bar{z}^\ell \\ -ac\ell \bar{z}^{\ell-k} \\ abk \end{pmatrix} \\
&= \frac{abc(\ell-k)k\ell z^{\ell-2}}{a^2b^2k^2 + a^2c^2\ell^2|z|^{2(\ell-k)} + b^2c^2(\ell-k)^2|z|^{2\ell}} \begin{pmatrix} bc(\ell-k)\bar{z}^\ell \\ -ac\ell\bar{z}^{\ell-k} \\ abk \end{pmatrix}.
\end{aligned}$$

Therefore the Frenet frame of $\phi(z) = [a, bz^k, c^\ell]$ is given by

$$f_0(z) = \begin{pmatrix} a \\ bz^k \\ cz^\ell \end{pmatrix}, \quad f_1(z) = \frac{1}{z|f|^2} \begin{pmatrix} -a(kb^2|z|^{2k} + \ell c^2|z|^{2\ell})\bar{z} \\ b(k|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})z^k \\ c(\ell|f|^2 - kb^2|z|^{2k} - \ell c^2|z|^{2\ell})z^\ell \end{pmatrix}$$

and

$$f_2(z) = \frac{abc(\ell-k)k\ell z^{\ell-2}}{a^2b^2k^2 + a^2c^2\ell^2|z|^{2(\ell-k)} + b^2c^2(\ell-k)^2|z|^{2\ell}} \begin{pmatrix} bc(\ell-k)\bar{z}^\ell \\ -ac\ell\bar{z}^{\ell-k} \\ abk \end{pmatrix}$$

or

$$f_0(z) = \begin{pmatrix} a \\ bz^k \\ cz^\ell \end{pmatrix}, \quad f_1(z) = \frac{1}{zA} \begin{pmatrix} -aB \\ b(kA - B)z^k \\ c(\ell A - B)z^\ell \end{pmatrix}$$

and

$$f_2(z) = \frac{abc(\ell - k)k\ell z^\ell}{z^2 C} \begin{pmatrix} bc(\ell - k)\bar{z}^\ell \\ -ac\ell \bar{z}^{\ell - k} \\ abk \end{pmatrix}.$$

Lift of ϕ to $SU(3)$

A local lift of ϕ is given in terms of the Frenet frame by

$$F(z) = \frac{1}{\left(\det\left(\frac{f_0}{|f_0|}\middle|\frac{f_1}{|f_1|}\middle|\frac{f_2}{|f_2|}\right)\right)^{1/3}} \left(\frac{f_0}{|f_0|}\middle|\frac{f_1}{|f_1|}\middle|\frac{f_2}{|f_2|}\right),$$

thus

$$\begin{aligned} F(z) &= \begin{pmatrix} \frac{1}{K}a & -\frac{1}{L}a(kb^2|z|^{2k} + \ell c^2|z|^{2\ell}) & \frac{1}{M}(\ell - k)bc\bar{z}^\ell \\ \frac{1}{K}bz^k & \frac{1}{L}b(ka^2 - (\ell - k)c^2|z|^{2\ell})z^k & -\frac{1}{M}\ell ac\bar{z}^{\ell - k} \\ \frac{1}{K}cz^\ell & \frac{1}{L}c(\ell a^2 + (\ell - k)b^2|z|^{2k})z^\ell & \frac{1}{M}kab \end{pmatrix} \\ &= \left(\begin{array}{c|c|c} \frac{1}{K}a & -\frac{1}{L}aB & \frac{1}{M}(\ell - k)bc\bar{z}^\ell \\ \frac{1}{K}bz^k & \frac{1}{L}b(kA - B)z^k & -\frac{1}{M}\ell ac\bar{z}^{\ell - k} \\ \frac{1}{K}cz^\ell & \frac{1}{L}c(\ell A - B)z^\ell & \frac{1}{M}kab \end{array}\right) \end{aligned}$$

where $K(z), L(z)$ and $M(z)$ are normalising factors such that $F \in SU(3)$.

A.2 η-invariants of $\psi : S^2 \to SU(3)/T^2$

Lemma A.1 *Let $\psi : S^2 \to SU(3)/T^2$ be the τ-holomorphic curves obtained from lifting the holomorphic S^1-symmetric map*

$$\phi : S^2 \to CP^2, \quad z \mapsto \phi(z) = [a, bz^k, cz^\ell].$$

Then the η-invariants of ψ are given by

$$\begin{aligned} \eta_1 &= \frac{b^2c^2(\ell-k)^2|z|^{2(\ell+k-1)} + a^2c^2\ell^2|z|^{2(\ell-1)} + a^2b^2k^2|z|^{2(k-1)}}{(a^2+b^2|z|^{2k}+c^2|z|^{2\ell})^2} \\ \eta_2 &= \frac{(a^2+b^2|z|^{2k}+c^2|z|^{2\ell})a^2b^2c^2k^2\ell^2(\ell-k)^2|z|^{2(\ell+k-3)}}{(b^2c^2(\ell-k)^2|z|^{2(\ell+k-1)} + a^2c^2\ell^2|z|^{2(\ell-1)} + a^2b^2k^2|z|^{2(k-1)})^2}. \end{aligned}$$

Proof:

Recall that the harmonic sequence of ϕ gives rise to γ-invariants. The γ-invariants of ϕ are related to the η-invariants of the lift by $\eta_p = \gamma_{p-1}$ and we will hence compute γ_0, γ_1 for ϕ.

Computing the γ-invariants of ϕ:

Let

$$f(z) = f_0(z) = \begin{pmatrix} a \\ bz^k \\ cz^\ell \end{pmatrix}.$$

Then

$$f'(z) = \begin{pmatrix} 0 \\ bkz^{k-1} \\ c\ell z^{\ell-1} \end{pmatrix} \quad \text{and} \quad f''(z) = \begin{pmatrix} 0 \\ bk(k-1)z^{k-2} \\ c\ell(\ell-1)z^{\ell-2} \end{pmatrix}.$$

Now

$$\gamma_0 = \frac{|f_1|^2}{|f_0|^2} \quad \text{and} \quad \gamma_1 = \frac{|f_2|^2}{|f_1|^2}.$$

In order not to have to compute the Frenet frame of f (which is rather complicated) we observe the following.

$$|f_0|^2|f_1|^2 = |f_0 \wedge f_1|^2 = |f \wedge f'|^2,$$

so

$$\gamma_0 = \frac{|f_1|^2}{|f_0|^2} = \frac{|f_0|^2|f_1|^2}{|f_0|^4} = \frac{|f \wedge f'|^2}{|f|^4}$$

and

$$|f_0|^2|f_1|^2|f_2|^2 = |f_0 \wedge f_1 \wedge f_2|^2 = |f \wedge f' \wedge f''|^2,$$

so

$$\gamma_1 = \frac{|f_2|^2}{|f_1|^2} = \frac{|f_0|^2|f_1|^2|f_2|^2}{|f_0|^2|f_1|^4} = \frac{|f \wedge f' \wedge f''|^2}{|f \wedge f'|^2|f_1|^2} = \frac{|f|^2}{|f|^2}\frac{|f \wedge f' \wedge f''|^2}{|f \wedge f'|^2|f_1|^2} = \frac{|f|^2|f \wedge f' \wedge f''|^2}{|f \wedge f'|^4}.$$

We have

$$|f|^2 = a^2+b^2|z|^{2k}+c^2|z|^{2\ell} = A+Bx^k+Cx^\ell \quad \text{where } x := |z|^2,\ A = a^2, B = b^2 \text{ and } C = c^2.$$

For the cross-product we get

$$f \wedge f' = \begin{pmatrix} bc(\ell - k)z^{\ell+k-1} \\ -ac\ell z^{\ell-1} \\ abkz^{k-1} \end{pmatrix},$$

so

$$\begin{aligned} |f \wedge f'|^2 &= b^2c^2(\ell-k)^2|z|^{2(\ell+k-1)} + a^2c^2\ell^2|z|^{2(\ell-1)} + a^2b^2k^2|z|^{2(k-1)} \\ &= BCMx^{\ell+k-1} + ACLx^{\ell-1} + ABKx^{k-1} \quad \text{where } M = (\ell-k)^2, K = k^2 \text{ and } L = \ell^2. \end{aligned}$$

The determinant is

$$f \wedge f' \wedge f'' = \det \begin{pmatrix} a & 0 & 0 \\ bz^k & bkz^{k-1} & bk(k-1)z^{k-2} \\ cz^\ell & c\ell z^{\ell-1} & c\ell(\ell-1)z^{\ell-2} \end{pmatrix} = abck\ell(\ell-k)z^{k+\ell-3},$$

so

$$|f \wedge f' \wedge f''|^2 = a^2b^2c^2k^2\ell^2(\ell-k)^2|z|^{2(\ell+k-3)}|z|^{2(k+\ell-3)} = ABCKLMx^{k+\ell-3}.$$

Thus

$$\begin{aligned} \eta_1 = \gamma_0 &= \frac{|f \wedge f'|^2}{|f|^4} = \frac{BCMx^{\ell+k-1} + ACLx^{\ell-1} + ABKx^{k-1}}{(A + Bx^k + Cx^\ell)^2} \\ &= \frac{b^2c^2(\ell-k)^2|z|^{2(\ell+k-1)} + a^2c^2\ell^2|z|^{2(\ell-1)} + a^2b^2k^2|z|^{2(k-1)}}{(a^2 + b^2|z|^{2k} + c^2|z|^{2\ell})^2} \end{aligned}$$

and

$$\begin{aligned} \eta_2 = \gamma_1 &= \frac{|f|^2|f \wedge f' \wedge f''|^2}{|f \wedge f'|^4} = \frac{(A + Bx^k + Cx^\ell)ABCKLMx^{k+\ell-3}}{(BCMx^{\ell+k-1} + ACLx^{\ell-1} + ABKx^{k-1})^2} \\ &= \frac{(a^2 + b^2|z|^{2k} + c^2|z|^{2\ell})a^2b^2c^2k^2\ell^2(\ell-k)^2|z|^{2(\ell+k-3)}}{(b^2c^2(\ell-k)^2|z|^{2(\ell+k-1)} + a^2c^2\ell^2|z|^{2(\ell-1)} + a^2b^2k^2|z|^{2(k-1)})^2} \end{aligned}$$

□

A.3 Example of two non-congruent τ-holomorphic curves of the same metric

Example A.2 *Let $\psi, \tilde{\psi} : S^2 \to SU(3)/T^2$ be the τ-holomorphic curves obtained from lifting the holomorphic, S^1-symmetric maps*

$$\phi : S^2 \to CP^2, \quad z \mapsto \phi(z) = [1, z, z^3]$$

and

$$\tilde{\phi} : S^2 \to CP^2, \quad z \mapsto \phi(z) = [1, 3z^2, 2z^3].$$

Let $SU(3)/T^2$ be equipped with a G-invariant metric such that $|X_{\alpha_1}| = |X_{\alpha_2}|$. Then $\psi, \tilde{\psi}$ have the same induced metric but are not *congruent.*

Proof: We will assume that $|X_{\alpha_1}| = |X_{\alpha_2}| = 1$. Then the induced metrics of $\psi, \tilde{\psi}$ are $ds^2 = \eta_1 + \eta_2$ and $d\tilde{s}^2 = \tilde{\eta}_1 + \tilde{\eta}_2$. We will show that $\eta_1 = \tilde{\eta}_2 \neq \tilde{\eta}_1 = \eta_2$. Thus the metrics coincide but the η-invariants do not, i.e. ψ and $\tilde{\psi}$ are not congruent. From Lemma A.1 we know that the η-invariants of the lift of $[a, bz^k, cz^\ell] : S^2 \to \mathbf{CP}^2$ are given by

$$\begin{aligned} \eta_1 &= \frac{b^2c^2(\ell-k)^2x^{\ell+k-1} + a^2c^2\ell^2x^{\ell-1} + a^2b^2k^2x^{k-1}}{(a^2 + b^2x^k + c^2x^\ell)^2} \\ \eta_2 &= \frac{(a^2 + b^2x^k + c^2x^\ell)a^2b^2c^2k^2\ell^2(\ell-k)^2x^{\ell+k-3}}{(b^2c^2(\ell-k)^2x^{\ell+k-1} + a^2c^2\ell^2x^{\ell-1} + a^2b^2k^2x^{k-1})^2}, \end{aligned}$$

where $x = |z|^2$.

For ψ we have $a = 1, b = 1, c = 1$ and $k = 1, \ell = 3$, so

$$\begin{aligned} \eta_1 &= \frac{4x^3 + 9x^2 + 1}{(1 + x + x^3)^2} \\ \eta_2 &= \frac{(1 + x + x^3)36x}{(4x^3 + 9x^2 + 1)^2}. \end{aligned}$$

For $\tilde{\psi}$ on the other hand we have $a = 1, b = 3, c = 2$ and $k = 2, \ell = 3$, so

$$\begin{aligned} \tilde{\eta}_1 &= \frac{36x^4 + 36x^2 + 36x}{(1 + 9x^2 + 4x^3)^2} = \frac{36x(x^3 + x + 1)}{(1 + 9x^2 + 4x^3)^2} \\ \tilde{\eta}_2 &= \frac{(1 + 9x^2 + 4x^3)36^2x^2}{(36x^4 + 36x^2 + 36x)^2} = \frac{1 + 9x^2 + 4x^3}{(x^3 + x + 1)^2}. \end{aligned}$$

Thus

$$\tilde{\eta}_1 = \eta_2 \quad \text{and} \quad \tilde{\eta}_2 = \eta_1,$$

so ψ and $\tilde{\psi}$ have indeed the same induced metric, but as $\eta_1 \neq \tilde{\eta}_1$ they cannot be congruent by the Weak Congruence Theorem (Theorem 7.1). □

Remark A.3 *The above example comes from the following fact. If $[f] : S \to CP^n$ is a linearly full holomorpic curve with Frenet frame $f_0, \ldots, f_n$ then $[\tilde{f}] = [\bar{f}_n]$ is also a holomorphic curve and since $\tilde{\gamma}_p = \gamma_{n-p-1}$ it follows that*

$$\gamma_0 + \ldots + \gamma_{n-1} = \tilde{\gamma}_0 + \ldots + \tilde{\gamma}_{n-1}.$$

Since $\gamma_0 \neq \tilde{\gamma}_0$ in general $[f], [\tilde{f}]$ are not congruent. However the corresponding maps into $SU(n+1)/T^n$ have the same induced metric. Thus the metric is not enough to determine τ-holomorphic maps into G/T up to congruence. The chosen $\tilde{\phi}$ above is $[\bar{f}_2]$ for $f(z) = (1, z, z^3)$.

Appendix B

Basic background material

For details about Lie algebras, adjoint representations, root spaces, Cartan matrices, etc. see [Bau], [BtD], [Se], [Sa] and [FH].

B.1 Killing form

Let **g** be a cx. Lie algebra. The **Killing form** on **g** is a complex-valued, bilinear form given by $\kappa(X, Y) = \text{tr}(\text{ad}X \circ \text{ad}Y)$.

The Killing form of a simple Lie algebra **g** is non-degenerate. It is also $\text{Ad}(G)$-invariant ([He], p.131). However, this thesis relies only on its $\text{Ad}(T)$-invariance.

B.2 Properties of roots

Definition B.1 ([Bau], p.110) *Let* **h** *be a maximal toral subalgebra of a complex semisimple Lie algebra* **g** *and let* $\mathbf{h}^*$ *be the dual space of* **h**. *The element* $h_\alpha = \alpha^\sharp$

defined by

$$\alpha(H) = \kappa(h_\alpha, H) = \kappa(\alpha^\sharp, H)$$

is called the **star vector** *or* **root vector**.

Theorem B.2 ([Bau], p.110) *Let* $\mathbf{h}$ *be a maximal toral subalgebra of a complex semisimple Lie algebra* $\mathbf{g}$. *Let* $\mathbf{h}^*$*be the dual space of* $\mathbf{h}$.

(i) *The root system* Δ *spans the dual space* $\mathbf{h}^*$.

(ii) *Let* $\alpha \in \Delta$, *that is* $\alpha \neq 0$ *and* $\mathbf{g}^\alpha \neq 0$. *Then* $-\alpha$ *is also a root. Hence* $\alpha \in \Delta$ *implies* $-\alpha \in \Delta$. *II*

(iii) *For* $\alpha \in \Delta$, $x \in \mathbf{g}^\alpha$ *and* $y \in \mathbf{g}^{-\alpha}$ *the commutator is given by*

$$[x, y] = \kappa(x, y)h_\alpha = \kappa(x, y)\alpha^\sharp.$$

(iv) *For* $\alpha \in \Delta$ *the subspace* $[\mathbf{g}^\alpha, \mathbf{g}^{-\alpha}]$ *is one-dimensional an it is spanned by the star vector* $h_\alpha = \alpha^\sharp$.

(v) *Let* α *be a root. Then*

$$\kappa(\alpha^\sharp, \alpha^\sharp) = \alpha(\alpha^\sharp) \neq 0.$$

(vi) *Let* $\alpha \in \Delta$ *and* E_α *an arbitrary non-zero element in the root space* $\mathbf{g}^\alpha$. *Then there exists a non-zero element* F_α *in* $\mathbf{g}^{-\alpha}$ *such that the set* $\{E_\alpha, F_\alpha, H_\alpha\}$, *where* H_α *is defined by*

$$H_\alpha = [E_\alpha, F_\alpha],$$

spans a three-dimensional simple Lie algebra denoted by S_α. *The Lie algebra* S_α *is isomorphic to the Lie algebra* $sl(2, \mathbf{C})$.

(vii) *For each* $\alpha \in \Delta$ *there is a special choice of vectors* $X_{\pm\alpha} \in \mathbf{g}^{\pm\alpha}$ *and* $H_\alpha \in \mathbf{t}$ *such that the set* $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ *spans the three-dimensional simple* $sl(2, \mathbf{C})$ *Lie algebra* S_α. *The* $\{X_\alpha, X_{-\alpha}, H_\alpha\}$ *are called* **Cartan-Weyl generators** *and satisfy*

- $[X_\alpha, X_{-\beta}] = \delta_{\alpha\beta} H_\alpha$
- $[H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}$.

(viii) *The vector* H_α *satisfies*

$$H_\alpha = \frac{2h_\alpha}{\kappa(h_\alpha, h_\alpha)} = \frac{2\alpha^\sharp}{\kappa(\alpha^\sharp, \alpha^\sharp)},$$

$$H_\alpha = -H_{-\alpha}$$

and

$$\alpha(H_\alpha) = 2.$$

H_α *is called a* **coroot**.

(ix) *The Cartan-Weyl generators satisfy* $\kappa(H_\alpha, H_\alpha) = 2\kappa(X_\alpha, X_{-\alpha})$.

B.3 Cartan matrix, highest root and extended Cartan matrix

The Killing form $\kappa : \mathbf{g} \times \mathbf{g} \to \mathbf{C}$ gives rise to a metric on $\mathbf{h}^*$. There is a bijective correspondence

$$\lambda : \mathbf{h} \to \mathbf{C} \stackrel{1:1}{\longleftrightarrow} \kappa(\lambda^\sharp, \cdot) : \mathbf{h} \to \mathbf{C}.$$

Definition B.3 ([Bau], p.121) *Let* $\mathbf{g}$ *be a complex semisimple Lie algebra. Define*

$$\langle \cdot, \cdot \rangle : \mathbf{h}^* \times \mathbf{h}^* \to \mathbf{C} \quad (\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$$

with $\langle \lambda, \mu \rangle = \kappa(\lambda^\sharp, \mu^\sharp)$.

Definition B.4 ([He], p.459, [Bau], p.144) *The* **Cartan matrix** $K = (K_{ij})_{i,j=1}^{\ell}$ *of a semisimple Lie algebra* $\mathbf{g}$ *is defined by*

$$K_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \frac{2\kappa(\alpha_i^\sharp, \alpha_j^\sharp)}{\kappa(\alpha_j^\sharp, \alpha_j^\sharp)}$$

Definition B.5 ([Bau], p.146) *Let* α *be a root with expansion* $\alpha = \sum_1^\ell n_i \alpha_i$ *w.r.t. the set of positive roots* Δ^+. *Then the sum of the coefficients* n_i *is denoted by*

$$\mathrm{ht}\alpha := \sum_1^\ell n_i$$

and it is called the **height** *of* α.

Lemma B.6 ([Bau], p.146) *The root system* Δ *of a finite-dimensional complex semisimple Lie algebra contains a unique root*

$$\theta = \sum_1^\ell m_i \alpha_i$$

with $\mathrm{ht}\theta > \mathrm{ht}\alpha$ *for all* $\alpha \neq \theta$ *in* Δ. *The root* θ *is called the* **highest root**.

Definition B.7 *The* **extended Cartan matrix** $\hat{K} = (\hat{K}_{ij})_{i,j=0}^{\ell}$ *of a semisimple Lie algebra* $\mathbf{g}$ *is defined by*

$$\hat{K}_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \frac{2\kappa(\alpha_i^\sharp, \alpha_j^\sharp)}{\kappa(\alpha_j^\sharp, \alpha_j^\sharp)}.$$

Here $-\alpha_0 = \sum_1^\ell m_i \alpha_i$ *is the highest root.*

Note that for $i, j = 1, \ldots \ell$ this definiton coincides with the definition of the Cartan matrix. Thus the extended Cartan matrix $\hat{K}$ contains the Cartan matrix K.

Claim B.8 . *The extended Cartan matrix $\hat{K}$ is singular and satisfies*

$$\sum_{i=0}^{\ell} m_i \hat{K}_{ij} = 0 \quad \forall j = 0, \ldots, \ell.$$

In other words, addding up all rows with their multiplicities gives the zero row vector.

Proof: First let $j \neq 0$. Then

$$\hat{K}_{0j} = -\sum_{k=1}^{\ell} m_k K_{kj} = -\sum_{k=1}^{\ell} m_k \hat{K}_{kj},$$

so

$$\sum_{i=0}^{\ell} m_i \hat{K}_{ij} = \hat{K}_{0j} + \sum_{i=1}^{\ell} m_i \hat{K}_{ij} = -\sum_{k=1}^{\ell} m_k \hat{K}_{kj} + \sum_{i=1}^{\ell} m_i \hat{K}_{ij} = 0.$$

Now let $j = 0$. Since

$$\hat{K}_{i0} = \frac{2 \langle \alpha_i, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle}$$

and

$$\hat{K}_{00} = \frac{2 \langle \alpha_0, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle} = -\sum_{k=1}^{\ell} m_k \frac{2 \langle \alpha_k, \alpha_0 \rangle}{\langle \alpha_0, \alpha_0 \rangle} = -\sum_{k=1}^{\ell} m_k \hat{K}_{k0}$$

we get

$$\sum_{i=0}^{\ell} m_i \hat{K}_{i0} = \hat{K}_{00} + \sum_{i=1}^{\ell} m_i \hat{K}_{i0} = -\sum_{k=1}^{\ell} m_k \hat{K}_{k0} + \sum_{i=1}^{\ell} m_i \hat{K}_{i0} = 0.$$

□

B.4 Complexification of Lie groups

A short treatment of this can be found in [PS], p.13, and [G], p.8.

For the complexification of a vector spaces see B.6. Using this process, we find that any (abstract) Lie algebra $\mathbf{g}$ has a complexification $\mathbf{g}^C = \mathbf{g} \otimes_{\mathbf{R}} \mathbf{C}$.

Definition B.9 ([PS], p.13) *Let G be a real Lie group and $\mathbf{g}$ be its Lie algebra. A complex Lie group G^C with $\mathbf{g}^C = \mathbf{g} \otimes_{\mathbf{R}} \mathbf{C}$ as its Lie algebra is called a* **complexification** *of G if it contains G as a subgroup.*

Remark B.10 ([PS], p.13) *A complexification of a Lie group does not need to exist. However, if G is compact, then it does possess a complexification G^C: every compact Lie group can be embedded in some $U(n)$. The complexification of $U(n)$ is $GL(n, \mathbf{C})$ and G^C can be realised as a subgroup of $GL(n, \mathbf{C})$. This group G^C is unique up to isomorphism and will be refered to as* **the** *complexification of G. Hence the complexification of S^1 is C^*. Other possible complexifications such as $\mathbf{C}/\mathbf{Z}^2 \cong S^1 \times S^1$ cannot arise as complex subgroups of a general linear group.*

Example B.11

Lie Group G	$GL(n, \mathbf{R})$	$U(n)$	$SL(n, \mathbf{R})$
Complexification G^C	$GL(n, \mathbf{C})$	$GL(n, \mathbf{C})$	$SL(n, \mathbf{C})$

Remark B.12 *As can be seen from the examples it is possible to have $G_1^C = G_2^C$ but $G_1 \neq G_2$.*

B.5 Orthogonal root spaces

Lemma B.13 *The root spaces* $\mathbf{g}^\alpha$ *are orthogonal to each other w.r.t. any* $\mathrm{Ad}(T)$*-invariant hermitian metric on* $T_o^{1,0}G/T = \sum_{\alpha\in\Delta^+}\mathbf{g}^\alpha$*:* $\mathbf{g}^\alpha \perp \mathbf{g}^\beta$ *if* $\alpha \neq \beta$.

Proof:

Let $\langle\cdot,\cdot\rangle$ be an $\mathrm{Ad}(T)$-invariant hermitian metric on $T_o^{1,0}G/T = \sum_{\alpha\in\Delta^+}\mathbf{g}^\alpha$. Suppose there are $X_\alpha \in \mathbf{g}^\alpha$, $X_\beta \in \mathbf{g}^\beta$ with $\langle X_\alpha, X_\beta\rangle \neq 0$. We will show $\alpha = \beta$. Recall $\mathrm{Ad}(\exp H) = e^{\mathrm{ad}H}$. Thus for all $X \in \mathbf{g}^\alpha$ and all $H \in \mathbf{t}$ we have $\mathrm{Ad}(\exp H).X = e^{\mathrm{ad}H}.X = e^{\alpha(H)}X$. Now since $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(T)$-invariant we have the following equalities for all $H \in \mathbf{t}$.

$$\begin{aligned}
\langle X_\alpha, X_\beta\rangle &= \langle \mathrm{Ad}(\exp H).X_\alpha, \mathrm{Ad}(\exp H).X_\beta\rangle \\
&= \left\langle e^{\mathrm{ad}H}.X_\alpha, e^{\mathrm{ad}H}.X_\beta\right\rangle \\
&= \left\langle e^{\alpha(H)}X_\alpha, e^{\beta(H)}X_\beta\right\rangle \\
&= e^{\overline{\alpha(H)}+\beta(H)}\langle X_\alpha, X_\beta\rangle \\
&= e^{-\alpha(H)+\beta(H)}\langle X_\alpha, X_\beta\rangle \qquad \text{as } \alpha : \mathbf{t} \to i\mathbf{R}.
\end{aligned}$$

Since $\langle X_\alpha, X_\beta\rangle \neq 0$ by assumption we have $e^{\beta(H)-\alpha(H)} = 1$, i.e. $\beta(H)-\alpha(H) \in 2\pi i\mathbf{Z}$ $\forall H \in \mathbf{t}$.

Since $\beta - \alpha$ is a linear map it is continuous, so $\beta(H) - \alpha(H) \equiv c \in 2\pi i\mathbf{Z}$.

But now $\beta(0) - \alpha(0) = 0$ implies $c = 0$. Thus $\beta = \alpha$ and hence $\mathbf{g}^\alpha \perp \mathbf{g}^\beta$ if $\alpha \neq \beta$.

B.6 Complex structures on vector spaces

Definition B.14 *Let V be a $2n$-dimensional real vector space. A* **complex structure** *on V is an endomorphism $J : V \to V$ such that $J^2 = -I$.*

Note B.15 *V must be even-dimensional since* $(\det J)^2 = \det J^2 = \det(-I) = (-1)^m$ *where m is the dimension of V.*

Definition B.16 *Let V be a real n-dimensional vector space. The* **complexification** *of V is the complex vector space $V^C = V \otimes_{\mathbf{R}} \mathbf{C} = \{X + iY \mid X, Y \in V\}$. If $\{v^k\}$ is a basis of V, then $\{v^k \otimes 1\}$ is a basis of V^C (since $\mathbf{C} = \mathrm{span}_{\mathbf{C}}\{1\}$).*

Lemma B.17 ([Wi], p.154) *Let V be a $2n$-dimensional real vector space with complex structure J. Let $V^{\mathbf{C}}$ be the complexification of V. Then the complex structure J of V extends canonically to a complex structure $\tilde{J}$ of $V^{\mathbf{C}}$, $\tilde{J}^2 = -1$. $\tilde{J}$ has $\pm i$ as eigenvalues and corresponding eigenspaces*

$$\begin{aligned} V^{1,0} &= \mathrm{Eig}(i) = \{Z \in V^C \mid JZ = iZ\} = \{X - iJX \mid X \in V\} \\ V^{0,1} &= \mathrm{Eig}(-i) = \{Z \in V^C \mid JZ = -iZ\} = \{X + iJX \mid X \in V\}. \end{aligned}$$

$V^{\mathbf{C}}$ splits (w.r.t. the complex structure J) into an orthogonal direct sum of these eigenspaces: $V^{\mathbf{C}} = V^{1,0} \oplus V^{0,1}$.

As $\overline{V^{1,0}} = V^{0,1}$ and $\overline{V^{0,1}} = V^{1,0}$ we have the correspondence

$$V^{1,0} \overset{cx.\ conj.}{\longleftrightarrow} V^{0,1}.$$

Complex conjugation with respect to V is a real linear isomorphism.

Knowing $V^{1,0}, V^{0,1}$ one can reconstruct the original complex structure as follows. Define $J : V^{\mathbf{C}} \to V^{\mathbf{C}}$ by

$$JZ = \begin{cases} iZ & : \; Z \in V^{1,0} \\ -iZ & : \; Z \in V^{0,1}. \end{cases}$$

J leaves $V = \{Z + \bar{Z} \mid Z \in V^{1,0}\}$ invariant:

$$J(Z + \bar{Z}) = iZ - i\bar{Z} = Z + \overline{iZ} \in V.$$

So if $X = Z + \bar{Z} \in V$ then $JX = i(Z - \bar{Z})$. We have recaptured the map $J : V \to V$.

A complex vector space V is canonically isomorphic to $V^{1,0}$

Claim: The map $\phi : (V^{1,0}, i) \to (V, J)$ given by $Z \mapsto Z + \bar{Z}$ is an isomorphism of complex vector spaces. The inverse of ϕ is obtained by taking the $(1,0)$-part of $X = X^{1,0} + X^{0,1} = \pi^{1,0}(X) + \pi^{0,1}(X)$: $\phi^{-1}(X) = \pi^{1,0}(X) = \frac{1}{2}X - iJ\frac{1}{2}X$.

Proof:

- $\phi(Z_1 + Z_2) = \phi(Z_1) + \phi(Z_2)$
- $\phi(iZ) = iZ + \overline{iZ} = iZ - i\bar{Z} = JZ + J\bar{Z} = J(Z + \bar{Z}) = J\phi(Z)$.

Almost complex manifolds

Definition B.18 ([Wi], p.157) *An* **almost complex structure** *on a real differentiable manifold M is a tensor field J which at every point $x \in M$ is an* *endomorphism of the tangent space* T_xM *such that*

$$J^2 = -\mathrm{Id}.$$

A manifold with such a structure is called an **almost complex manifold**.

Definition B.19 *An almost complex structure is called* **integrable** *if it comes from a complex structure on* M.

B.7 The isotropy representation

Definition B.20 ([G], p.16, [BH], p.462) *Let* G/H *be a homogeneous space. The* **isotropy representation** *of* H *on* T_oG/H *is the homomorphism*

$$\mathrm{Ad}^{G/H} : H \to \mathrm{Aut}(T_oG/H)$$

defined by

$$\mathrm{Ad}^{G/H}(h).X = L_h(X) \quad \forall X \in T_oG/H,$$

where $\ell_h : G/H \to G/H$ *is left translation* $\ell_h([g]) = [hg](= [hgh^{-1})$ *and* $L_h = \ell_{h*} : T(G/H) \to T(G/H)$ *is its differential. The group* $\{\mathrm{Ad}^{G/H}(h) \mid h \in H\}$ *is called the* **linear isotropy group**.

The relation between the standard adjoint representation and the isotropy representation can be seen in the commutative diagram below.

$$
\begin{array}{ccccccccc}
\mathbf{g} & \rightarrow & \rightarrow & \xrightarrow{\mathrm{Ad}(h)} & & \rightarrow & \rightarrow & \mathbf{g} \\
\cap & & & & & & & \cap \\
TG & \rightarrow & \rightarrow & \xrightarrow{(i_h)_*} & & \rightarrow & \rightarrow & TG \\
\downarrow & \searrow & & & & & \swarrow & \downarrow \\
\downarrow & & G & \xrightarrow{i_h} & G & & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & & \downarrow \\
\downarrow & & G/H & \xrightarrow{\ell_h} & G/H & & & \downarrow \\
\downarrow & \nearrow & & & & & \nwarrow & \downarrow \\
T(G/H) & \rightarrow & \rightarrow & \xrightarrow{L_h} & & \rightarrow & \rightarrow & T(G/H) \\
\cup & & & & & & & \cup \\
T_oG/H & \rightarrow & \rightarrow & \xrightarrow{\mathrm{Ad}^{G/H}(h)} & & \rightarrow & \rightarrow & T_oG/H
\end{array}
$$

Here $i_h : G \rightarrow G$ is the standard inner automorphism given by $i_h(x) = hxh^{-1}$ $\forall x \in G$.

If we denote the projection $\mathbf{g} \rightarrow T_oG/H$ by $[\cdot]$ then $\mathrm{Ad}^{G/H}(h).[X] = [\mathrm{Ad}(h).X]$ $\forall X \in \mathbf{g}$ ($[X] \in T_oG/H$).

Recall that for a reductive homogeneous space $M = G/H$ there exists a subspace $\mathbf{m}$ of $\mathbf{g}$ such that $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ and $\mathrm{Ad}(h).\mathbf{m} \subseteq \mathbf{m}$ $\forall h \in H$.

If G (or H) is compact, then G/H is reductive.

For a reductive homogeneous space we see that

$$\mathrm{Ad}^{G/H}(h) : T_oG/H \to T_oG/H$$

can be identified with

$$\mathrm{Ad}(h)|_{\mathbf{m}} : \mathbf{m} \to \mathbf{m},$$

as can be seen from the commutative diagram below and the fact that the projection $\mathbf{g} = \mathbf{h} \oplus \mathbf{m} \to \mathbf{m}$ is bijective if restricted to $\mathbf{m}$.

$$\begin{array}{ccc} \mathbf{g} = \mathbf{h} \oplus \mathbf{m} & \stackrel{\mathrm{Ad}(h)}{\longrightarrow} & \mathbf{g} = \mathbf{h} \oplus \mathbf{m} \\ \downarrow & & \downarrow \\ T_oG/H \cong \mathbf{m} & \stackrel{\mathrm{Ad}^{G/H}(h)}{\longrightarrow} & T_oG/H \cong \mathbf{m} \end{array}$$

More generally we have

Proposition B.21 ([G], p.16) *Assume that G/H is reductive. Let $h \in H$, and let $X \in \mathbf{h}$, $Y \in \mathbf{m}$. Then we have*

$$\mathrm{Ad}^{G/H}(h).(X, Y) = (\mathrm{Ad}^H(h).X, \mathrm{Ad}^{G/H}(h).Y).$$

B.8 Facts about real harmonic maps

The proof of the congruence theorem makes use of a factorisation argument. In order to apply this we need the following Lemma.

Lemma B.22 *Let $g(z, \bar{z})$ be a real-valued function such that*

$$\Delta \log g = \partial_z \partial_{\bar{z}} \log g = 0.$$

Then $g(z, \bar{z}) = |h(z)|^2$ with $h(z)$ holomorphic.

Proof: Since $\log g$ is harmonic, it follows that it is the real part of a holomorphic function f: $\log g = \Re(f)$. Then $g = \exp(\log g) = \exp(\Re f)$. Now let $h(z) = \exp(\frac{1}{2}f)$. Then h is holomorphic and

$$|h|^2 = h\bar{h} = \exp(\frac{1}{2}f)\exp(\frac{1}{2}\bar{f}) = \exp(\frac{1}{2}(f+\bar{f})) = \exp(\Re f) = g.$$

□

B.9 Root spaces of $sl(n+1, \mathbf{C})$

Recall that the standard Cartan subalgebra of $sl(n, \mathbf{C})$ is the space of diagonal matrices with zero-trace.

The roots of $sl(n+1, \mathbf{C})$ are

$$\{\alpha_{ij} := \sigma_i - \sigma_j \mid i \neq j; i, j = 0, \ldots, n\}$$

where

$$\sigma_i(\mathrm{diag}(y_0, \ldots, y_n)) = y_i, \quad i = 0, \ldots, n.$$

Let

$$\{\alpha_{ij} = \sigma_i - \sigma_j \mid i > j; i, j = 0, \ldots, n\}$$

be the positive roots and choose

$$\alpha_j := \alpha_{j,j-1} = \sigma_j - \sigma_{j-1}, \quad j = 1, \ldots, n,$$

to be the positive simple roots. Then

$$\alpha_{ij} = \alpha_i + \alpha_{i-1} + \ldots + \alpha_{j+1}$$

for all positive roots $(i > j)$ and

$$\alpha_{ij} = -\alpha_{ji} = -(\alpha_j + \alpha_{j-1} + \ldots + \alpha_{i+1})$$

for all negative roots $(i < j)$.

Computing the Cartan-Weyl basis and root space

Let

$$E_{ij} = (\delta_{ik}\delta_{j\ell})_{k,\ell=0,\ldots,n} \quad i \neq j, i, j = 0, \ldots, n.$$

This is a matrix with a 1 in the i-th row and j-th column and zeros everywhere else.

$$E_{ij} = \begin{pmatrix} & j & \\ & & \\ & & \\ i & 1 & \\ & & \end{pmatrix}$$

Claim B.23 *The root spaces are*

$$\mathbf{g}^{\alpha_{ij}} = \text{span}_{\mathbf{C}}\{E_{ij}\} \quad i \neq j; i, j = 0, \ldots, n.$$

Proof: Recall that the adjoint representation is given by

$$\text{ad}_H : \mathbf{g} \to \mathbf{g}, \quad X \mapsto \text{ad}_H(X) = [H, X].$$

As the root spaces are one-dimensional it suffices to show that

$$E_{ij} \in \mathbf{g}^{\alpha_{ij}} = \{X \in sl(n+1, \mathbf{C}) \mid \text{ad}_H(X) = \alpha_{ij}(H)X \ \forall H \in \mathbf{h}\}$$

where $\mathbf{h}$ = {diagonal matrices with zero-trace} is the standard Cartan subalgebra of $sl(n+1,\mathbf{C})$. Let $H = \mathrm{diag}(y_0,\ldots,y_n) =\in \mathbf{h}$. Then

$$HE_{ij} = (y_k\delta_{k\ell})_{k,\ell=0,\ldots,n}(\delta_{i\ell}\delta_{jm})_{\ell,m=0,\ldots,n} = y_iE_{ij},$$

so for the k,m-th component of HE_{ij} we have

$$[HE_{ij}]_{km} = \sum_{\ell=0}^{n} y_k\delta_{k\ell}\delta_{i\ell}\delta_{jm} = y_k\delta_{ik}\delta_{jm} = y_i\delta_{ik}\delta_{jm}.$$

Also

$$E_{ij}H = (\delta_{ik}\delta_{j\ell})_{k,\ell=0,\ldots,n}(y_\ell\delta_{\ell m})_{\ell,m=0,\ldots,n} = y_jE_{ij},$$

so for the k,m-th component of $E_{ij}H$ we have

$$[E_{ij}H]_{km} = \sum_{\ell=0}^{n} \delta_{ik}\delta_{j\ell}y_\ell\delta_{\ell m} = y_j\delta_{ik}\delta_{jm}.$$

Thus

$$\begin{aligned}
\mathrm{ad}_H(E_{ij}) &= [H,E_{ij}] = HE_{ij} - E_{ij}H \\
&= (y_i\delta_{ik}\delta_{jm} - y_j\delta_{ik}\delta_{jm})_{k,m=0,\ldots,n} = (y_i - y_j)E_{ij} \\
&= (\sigma_i(H) - \sigma_j(H))E_{ij} = (\sigma_i - \sigma_j)(H)E_{ij} \\
&= \alpha_{ij}(H)E_{ij},
\end{aligned}$$

so $E_{ij} \in \mathbf{g}^{\alpha_{ij}}$ and as $\dim \mathbf{g}^{\alpha_{ij}} = 1$ we have $\mathbf{g}^{\alpha_{ij}} = \mathrm{span}_{\mathbf{C}}\{E_{ij}\}$.

□

For the root space we get the following picture in terms of matrices

$$\left(\begin{array}{c|c|c|c|c}
0 & -\alpha_{1,0} & -\alpha_{2,0} & \dots & -\alpha_{n,0} \\ \hline
\alpha_{1,0} & 0 & -\alpha_{2,1} & \dots & -\alpha_{n,1} \\ \hline
\alpha_{2,0} & \alpha_{2,1} & \ddots & \ddots & \vdots \\ \hline
\vdots & \vdots & \ddots & 0 & -\alpha_{n,n-1} \\ \hline
\alpha_{n,0} & \alpha_{n,1} & \dots & \alpha_{n,n-1} & 0
\end{array}\right)$$

So in terms of the positive simple roots, we have the corresponding root spaces at the following positions.

$$\left(\begin{array}{c|c|c|c|c}
0 & -\alpha_1 & -(\alpha_1+\alpha_2) & \dots & -(\alpha_1+\dots+\alpha_n) \\ \hline
\alpha_1 & 0 & -\alpha_2 & \dots & -(\alpha_2+\dots+\alpha_n) \\ \hline
\alpha_1+\alpha_2 & \alpha_2 & \ddots & \ddots & \vdots \\ \hline
\vdots & \vdots & \ddots & 0 & -\alpha_n \\ \hline
\alpha_1+\dots+\alpha_n & \alpha_2+\dots+\alpha_n & \dots & \alpha_n & 0
\end{array}\right)$$

The highest root is now $-\alpha_0 = \sigma_n - \sigma_0 = \alpha_1 + \dots + \alpha_n$. Its height is n.

Hence we can write the root spaces as follows:

$$\left(\begin{array}{c|c|c|c|c}
0 & -\alpha_1 & -(\alpha_1+\alpha_2) & \dots & \alpha_0 \\ \hline
\alpha_1 & 0 & -\alpha_2 & \dots & \alpha_0+\alpha_1 \\ \hline
\alpha_1+\alpha_2 & \alpha_2 & \ddots & \ddots & \vdots \\ \hline
\vdots & \vdots & \ddots & 0 & -\alpha_n \\ \hline
-\alpha_0 & -\alpha_0+\alpha_1 & \dots & \alpha_n & 0
\end{array}\right)$$

B.10 Representations, weights and lowest weight vector for $SU(n+1)$

Recall that the maximal torus of $SU(n+1)$ is given by

$$T = \{\mathrm{diag}(e^{i\theta_0}, \ldots, e^{i\theta_n}) \mid \theta_j \in \mathbf{R}, \theta_0 + \ldots \theta_n = 0\}.$$

The Cartan subalgebra of the Lie algebra $su(n+1)$ is then

$$\mathbf{t} = \{\mathrm{diag}(iy_0, \ldots, iy_n) \mid y_j \in \mathbf{R}, y_0 + \ldots + y_n = 0\}$$

and from B.9 the roots of $sl(n+1, \mathbf{C}) = su(n+1)^{\mathbf{C}}$ are

$$\sigma_j - \sigma_k, \quad j \neq k.$$

For the set of positive simple roots we take $\alpha_1, \ldots, \alpha_n$, where $\alpha_j = \sigma_j - \sigma_{j-1}$.

Then

$$\begin{aligned} \alpha_1 + 2\alpha_2 + \ldots + n\alpha_n &= (\sigma_1 - \sigma_0) + 2(\sigma_2 - \sigma_1) + \ldots n(\sigma_n - \sigma_{n-1}) \\ &= -(\sigma_0 + \ldots + \sigma_{n-1}) + n\sigma_n \\ &= -(-\sigma_n) + n\sigma_n \quad \text{as } y_0 + \ldots + y_{n-1} = -y_n \\ &= (n+1)\sigma_n \end{aligned}$$

and

$$\begin{aligned} \alpha_{k+1} + \alpha_{k+2} + \ldots + \alpha_n &= (\sigma_{k+1} - \sigma_k) + (\sigma_{k+2} - \sigma_{k+1}) + \ldots (\sigma_n - \sigma_{n-1}) \\ &= \sigma_n - \sigma_k \end{aligned}$$

so

$$\sigma_k = \frac{1}{n+1}(\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n) - (\alpha_{k+1} + \alpha_{k+2} + \ldots + \alpha_n)$$

Remark B.24 *The highest root is* $-\alpha_0 = \sigma_n - \sigma_0 = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

The irreducible representations of $SU(n+1)$ are given by $\bigwedge^k V$, $k = 0, \ldots, n+1$, where $V \cong \mathbf{C}^{n+1}$ denotes the standard representation. If $\{e_0, \ldots, e_n\}$ denotes the standard unitary basis for $\mathbf{C}^{n+1}$ then

$$\{e_{i_1} \wedge \ldots \wedge e_{i_k} \mid 0 \leq i_1 < \ldots < i_k \leq n\}$$

is a unitary basis for $\bigwedge^k \mathbf{C}^{n+1}$. Moreover, since $\mathrm{diag}(e^{i\theta_0}, \ldots, e^{i\theta_n})e_j = e^{i\theta_j}e_j$ the restriction of the action of $SU(n+1)$ on $\bigwedge^k V$ to the maximal torus is given by

$$\mathrm{diag}(e^{i\theta_0}, \ldots, e^{i\theta_n}).e_{i_1} \wedge \ldots \wedge e_{i_k} = e^{i(\theta_{i_1} + \ldots \theta_{i_k})}e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

We recall that if

$$\bigwedge^k \rho : SU(n+1) \to GL(\bigwedge^k \mathbf{C}^{n+1})$$

denotes the representation $\bigwedge^k V$, then its differential

$$d\bigwedge^k \rho : su(n+1) \to gl(\bigwedge^k \mathbf{C}^{n+1})$$

defines the action of $su(n+1)$ on $\bigwedge^k V$ which is given by

$$\mathrm{diag}(iy_0, \ldots, iy_n).e_{i_1} \wedge \ldots \wedge e_{i_k} = i(y_{i_1} + \ldots y_{i_k})e_{i_1} \wedge \ldots \wedge e_{i_k}.$$

Since

$$(\sigma_{i_1} + \ldots + \sigma_{i_k})(\mathrm{diag}(iy_0, \ldots, iy_n)) = i(y_{i_1} + \ldots y_{i_k})$$

we see immediately that the weights of this representation are

$$\sigma_{i_1} + \ldots + \sigma_{i_k}, \quad 0 \leq i_1 < \ldots < i_k \leq n,$$

and the corresponding weight spaces are

$$V_{i_1,\ldots,i_k} := \mathrm{span}_{\mathbf{C}}\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}.$$

In terms of the positive simple roots we have that

$$\sigma_{i_1}+\ldots+\sigma_{i_k} = \frac{k}{n+1}(\alpha_1+2\alpha_2+\ldots+n\alpha_n)-(\alpha_{i_1+1}+\ldots+\alpha_n)-\ldots-(\alpha_{i_k+1}+\ldots+\alpha_n).$$

From this it is clear that the lowest weight is

$$\sigma_0+\ldots+\sigma_{k-1} = \frac{k}{n+1}(\alpha_1+2\alpha_2+\ldots+n\alpha_n)-(\alpha_1+\ldots+\alpha_n)-\ldots-(\alpha_k+\ldots+\alpha_n),$$

with corresponding lowest weight vector $e_0 \wedge \ldots \wedge e_{k-1}$.

The stabilizer of $[e_0 \wedge \ldots \wedge e_{k-1}] \in \mathbf{P}(\bigwedge^k \mathbf{C}^{n+1})$ is $S(U(k) \times U(n+1-k))$ and the orbit is $G_k(\mathbf{C}^{n+1}) = SU(n+1)/S(U(k) \times U(n+1-k))$.

Bibliography

[Bau] G.G.A. Bäuerele, E.A. de Kerf: Lie Algebras Part I - Finite and Infinite Dimensional Lie Algebras and Applications in Physics, North Holland 1990

[BH] A.Borel and F.Hirzebruch: Characteristic classes and homogeneous spaces I, Amer. J. Math. **80** (1958) 458-538

[BJRW] J.Bolton,G.R.Jensen,M.Rigoli, and L.M.Woodward: On Conformal Minimal Immersions of S^2 into $\mathbf{CP}^n$, Math. Ann. 279, 599-620 (1988)

[B] M.Black: harmonic Maps into Homogeneous Spaces, Longman, 1991

[BP] F.E. Burstall and F. Pedit: Harmonic maps via Adler-Kostant-Symes theory, Eds. A.P.Fordy and J.C.Wood, Vieweg, 1994, 221-272

[BPW] J.Bolton, F.Petit and L.M.Woodward: Minimal surfaces and the affine Toda field model, J. Reine Angew. Math. **459** (1995), 119-150

[BW1] J.Bolton and L.M.Woodward: Congruence theorems for harmonic maps from a Riemann surface into $\mathbf{CP}^n$ and S^n, J. Lond. Math. Soc. (2) **45** (1992), 363-376

[BW2] J.Bolton and L.M.Woodward: Some geometrical aspects of the 2-dimensional Toda equations, Geometry, Topology and Physics, 1997

[BW3] J.Bolton and L.M.Woodward: On the Simon conjecture for Minimal Immersions with S^1-symmetry, Math.Z. **200** (1988) 111-121

[BW4] J.Bolton and L.M.Woodward: The affine Toda equations and minimal surfaces, Eds. A.P.Fordy and J.C.Wood, Vieweg, 1994, 59-82 Math.Z. **200** (1988) 111-121

[BtD] Theodor Bröcker, Tammo tom Dieck: Representations of Compact Lie Groups, Springer-Verlag GTM 98, 1985

[BR] Francis Burstall, John Rawnsley: Twistor Theory for Riemannian Symmetric Spaces, Lecture Notes in Math. 1424, Springer-Verlag, 1990

[CSM] Roger Carter, Graeme Segal, Ian MacDonald: Lectures on Lie Groups and Lie Algebras, Cambridge University Press, LMSST 32, 1995

[EW] J.Eells and J.C.Wood: Harmonic maps from surfaces to complex projective spaces, Adv. in Math. **49** (1983), 217-263

[FH] W.Fulton and J.Harris: Representation Theory, Springer-Verlag, 1991

[G] Martin A. Guest: Harmonic Maps, Loop Groups and Integrable Systems, LMSST 38, Cambridge University Press, 1997

[He] Sigurdur Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press 1978

[Ji] J. Alfredo Jimenez: Addendum to "Existence of Hermitian n-Symmetric Spaces and of Non-commutative Naturally Reductive Spaces", Math. Z. **197**, 455-456 (1988)

[OV] A.L.Onishchik, E.B.Vinberg: Lie Groups and Algebraic Groups, Springer-Verlag 1990

[PS] A.Pressley and G.B.Segal : Loop Groups, Oxford University Press, 1986

[PW] Murray H. Protter, Hans F. Weinberger: Maximum Principles in Differential Equations, Prentice-Hall, 1967

[Ri] M.Rigoli: A rigidity result for holomorphic immersions of surfaces in $\mathbf{CP}^n$, Proc. Am. Math. Soc. 93, 317-320 (1985)

[Sa] Hans Samelson: Notes on Lie Algebras, Universitext, Springer-Verlag 1990

[Sal] Simon Salamon: Harmonic and Holomorphic Maps, Geometry Seminar "Luigi Bianchi" II - 1984, Lecture Notes in Math. 1164, Springer-Verlag (1986), 161-224.

[Se] Jean-Pierre Serre: Complex Semisimple Lie Algebras, Springer-Verlag, 1987

[Sem] Bolton et al.: Seminar Notes on Almost Complex Curves, University of Durham, 1997

[Wi] T.J. Willmore: Riemannian Geometry, Oxford University Press, 1993

[Wo] Jon G. Wolfson: Harmonic Sequences and Harmonic Maps of Surfaces into Complex Grassmann Manifolds, J. Diff. Geom. **27**, 161-178 (1988)

Index