Cyclic factorizability theories

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by

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Abstract

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Let Γ denote a finite group and R a commutative ring. Factorizability theories seek to describe similarities between the local structure of $R\Gamma$-modules M and N, where M and N are related by, for example, being isomorphic when tensored up with $\mathbb{Q}$.

In the first three chapters of this thesis, we define two families of factorizability theories, the invariance and coinvariance factorizability theories. We will consider three members of these families. We demonstrate that monomial invariance factorizability is equivalent to monomial factorizability as defined in [19]. We go on to consider the two cyclic cases. We demonstrate that the weak cyclic invariance factorizability theory is strict and is identical to the weak cyclic coinvariance factorizability theory. We also demonstrate that the strong cyclic invariance factorizability theory and the strong cyclic coinvariance factorizability theory are not identical but are equivalent.

In chapters 4 and 5, we discuss C.M.M. Γ-functors over R. Thus we find relations which can simplify the calculation of the invariance and coinvariance factorizability theories.

An index of the less well known definitions used in this thesis is included as an appendix.

Preface

Cyclic Factorizability Theories

Paul Glyn Jones

I would firstly like to thank my wife, Karen, for all her support and encouragement over the last four years, and thank my supervisor Steve Wilson for his expert help and guidance. I would also like to thank those friends in the department who have patiently endured sharing an office with me, including Michael, Karen, Helen, Mike, Mansour and Alan. Last, but not least, I would like to thank my family for their support throughout my education.

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Contents

Introduction

FOR a given finite group Γ and some commutative ring R, take M to be a finitely generated, right $R\Gamma$-module. We would like to be able to completely describe the local structure of M, however this is not in general possible. In specific cases the structure of M is known: for example suppose R is the ring of algebraic integers of some algebraic number field K, and take K' to be a Galois extension of K with ring of integers R', and take Γ to be the Galois group of K' over K. Results are then known regarding the structure of R' as an $R\Gamma$-module in certain cases, for example, if the algebraic number field is tame, its ring of integers is locally free. However, when we are in the situation of determining the structure of an arbitrary $R\Gamma$-module M there are few results, especially as in general there may be infinitely many different genera of $R\Gamma$-modules.

Rather than trying to describe the local structure of an $R\Gamma$-module M, factorizability theories look at how the local structure of a module M differs from the local structure of another module N, where M and N are related by, for example, being isomorphic when tensored up with $\mathbb{Q}$. Amongst the first examples of a factorizability theory was Frölich's *strict factorizability* of [6], which dealt only with $\mathbb{Z}\Gamma$-lattices. Other examples include *monomial factorizability*, detailed in [19], and theories introduced in [12]. In fact, in [12] it is shown that factorizability theories can yield results on the global structure of modules, and not just the local structure.

The first three chapters of this thesis are directly concerned with factorizability theories. We begin in the first chapter by providing a detailed definition of a factorizability theory. This definition will closely follow the definition given in [19]. In brief, a factorizability theory is a homomorphism ψ from some relative group, most commonly $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$, to some abelian group Ψ. The relative group $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$ consists of classes of triples $[M, f, N]$ with $f\colon M \otimes \mathbb{Q} \xrightarrow{\sim} N \otimes \mathbb{Q}$, and relations from composition of isomorphisms f, and the direct sum. However, in order to keep the definition sufficiently general to include each of the cases we will be interested in, we allow more general relative groups. Our definition of a relative group will be based upon the work of Heller, [9], which we reproduce in less generality here in order to make the proofs more transparent.

The work of [9] puts the relative group into an exact sequence made up of more familiar groups from $\mathcal{K}$-theory. This sequence will provide the definition of *strictness*—we go on to see that strictness is a useful minimum condition for a factorizability theory to satisfy. The homomorphism aspect of factorizability theories also provides a way of comparing factorizability theories, giving a concept of relative strength and equivalence of theories.

In chapter 1 we also present a detailed description of the monomial factorizability theory defined in [19]. In part this is done as an illustrative example of a factorizability theory; however more significantly this example will be shown to be equivalent to one of the invariance factorizability theories defined in chapter 2, and thus we will provide an alternative proof of the strictness of the monomial factorizability theory.

In chapters 2 and 3 we define two closely related families of factorizability theories, the *invariance* and *coinvariance* factorizability theories. These are defined by looking at $v_p|T^{\Delta}\tilde{e}_{\chi}|$ in the invariance case, and $v_p|T_{\Delta}\tilde{e}_{\chi}|$ in the coinvariance case, for various collections of characters χ of subgroups $\Gamma' \subseteq \Gamma$ with kernel Δ. We look at three particular collections of characters in detail. With the *monomial* case, we deal with all characters χ where $p \nmid |\Gamma' : \Delta|$ and go on to show that the monomial invariance factorizability theory is equivalent to monomial factorizability theory. With the *weak cyclic* case, we allow only those characters where Γ' is cyclic and $p \nmid |\Gamma' : \Delta|$; with the *strong cyclic* case, we allow only those characters where Γ' is of the form $C \rtimes G$ for C a cyclic p-group, and $p \nmid |\Gamma' : \Delta|$. It is from the consideration of these two cases that we obtain the title of this thesis.

In chapter 2 we deal exclusively with the invariance factorizability theories, demonstrating that the weak cyclic factorizability theory is strict, and monomial invariance factorizability theory is equivalent to monomial factorizability theory. We go on to examine the strong cyclic factorizability theory for $\Gamma = G_{q,p} = C_p \rtimes C_q$, where p and q are prime numbers with $q \mid p - 1$. In chapter 3 we compare these results with corresponding results for the coinvariance factorizability theories, showing that they are identical in the weak cyclic case, and nonidentical but still equivalent in the strong cyclic case.

We move on in chapters 4 and 5 to consider cohomological monomial Mackey Γ-functors (C.M.M. Γ-functors for short) over some ring R. In short, a C.M.M. Γ-functor $\mathcal{A}$ assigns an R-module $\mathcal{A}(\chi)$ to each of some family of characters χ of subgroups $\Gamma' \subseteq \Gamma$, together with a collection of R-module homomorphisms similar to induction, restriction and conjugation satisfying certain axioms. Our interest in C.M.M. Γ-functors over R stems from the fact that $M^{\Delta}\tilde{e}_{\chi}$ and $M_{\Delta}\tilde{e}_{\chi}$, encountered in the invariance and coinvariance factorizability theories, are both C.M.M. Γ-functors over $\mathbb{Z}_p$.

In chapter 4 the main result is that, given a direct sum relation of the form

$$\bigoplus_i \mathbb{Z}_p[\Gamma/\Delta_i]\tilde{e}_{\chi_i} \cong \bigoplus_j \mathbb{Z}_p[\Gamma/\Delta'_j]\tilde{e}_{\phi_j}$$

we necessarily have a direct sum relation

$$\bigoplus_i \mathcal{A}(\chi_i) \cong \bigoplus_j \mathcal{A}(\phi_j).$$

Thus via C.M.M. Γ-functors we have the possibility of simplifying calculations of the invariance and coinvariance factorizability theories by obtaining relations between the values of $v_p|T^{\Delta}\tilde{e}_{\chi}|$ in the invariance case, and $v_p|T_{\Delta}\tilde{e}_{\chi}|$ in the coinvariance case, for various characters χ. In chapter 5 we discuss how to identify all the relations of the form

$$\bigoplus_i \mathbb{Z}_p[\Gamma/\Delta_i]\tilde{e}_{\chi_i} = \bigoplus_j \mathbb{Z}_p[\Gamma/\Delta'_j]\tilde{e}_{\phi_j},$$

and look again at the metacyclic group $G_{q,p}$ as an example.

At the end of chapters 1 and 3 we include a section considering possible future developments of a number of results from this work. These sections include the more speculative and conjectural results. At the end of chapter 1, we look at "real" factorizability theories, that is, factorizability theories which are homomorphisms from $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R})$. Our approach is to derive such factorizability theories as a pushout of a factorizability theory from $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q})$.

In chapter three we demonstrated that the weak cyclic invariance and coinvariance factorizability theories were identical, and the strong cyclic invariance and coinvariance factorizability theories were equivalent but non-identical. We conclude chapter 3 by examining whether the invariance and coinvariance factorizability theories are equivalent for any stronger case.

An index to the less well known definitions used in this thesis is included as an appendix.

We begin with a number of definitions which will hold throughout this thesis.

0.1 Preliminaries

Throughout this thesis, we will use the following conventions.

Γ will always denote a finite group.

All modules will be *right* modules, unless otherwise stated. We denote the category of (right) modules over a ring R by $\mathrm{mod}(R)$. We denote the category of lattices over a ring R, i.e. finitely generated, freely generated (right) R-modules, by $\mathrm{lat}(R)$.

We denote the Grothendieck group (with respect to the direct sum) of permutation projective modules over a ring R, i.e. modules which are a direct summand of a permutation module over R, by $\mathrm{PP}(R)$.

For a prime number p, $\mathbb{Z}_p$ denotes the usual p-adic completion of $\mathbb{Z}$; likewise $\mathbb{Q}_p$ denotes the p-adic completion of $\mathbb{Q}$. $\overline{\mathbb{Q}}_p$ denotes the algebraic closure of $\mathbb{Q}_p$. For F an algebraic field extension of $\mathbb{Q}_p$, Ω_F denotes the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$.

Chapter 1

Definition of a Factorizability Theory

IN GENERAL, the structure of modules over a group ring may be very intricate. Results on the structure of modules tend to be confined to very specific questions, such as studying the structure of rings of algebraic integers as a module over the corresponding Galois group. For example, in this case it can be shown that if the algebraic number field is tame, its ring of integers is locally free. Another situation when we can look at the structure of modules is if there is only one (or at least, very few) different genera, for example with modules in mod(R) if R is of finite representation type.

Factorizability theories provide an alternative approach when we have no knowledge of the local structure of the modules. Rather than asking questions about the structure of a specific module, factorizability theories seek to describe aspects of the relationship between two modules, providing an equivalence relation called *factor equivalence* between modules. Usually we are considering aspects of the local structure of the modules concerned.

In this chapter we will define what we mean by a factorizability theory. In brief, a factorizability theory ψ assigns certain invariants to triples (M, f, N), where M and N are finitely generated Λ-modules ($\Lambda = \mathbb{Z}\Gamma$ or $\mathbb{Z}_p\Gamma$ for a finite group Γ), and f is a $\mathbb{Q}\Lambda$-isomorphism $f\colon M \otimes \mathbb{Q} \xrightarrow{\sim} N \otimes \mathbb{Q}$. By invariants, we mean that the value of ψ depends only on the isomorphism class $[M, f, N]$ of (M, f, N) in the fibre category, defined in the next section. The isomorphism classes $[M, f, N]$ form a group called the *relative group*; we consider some of the $\mathcal{K}$-theoretic properties of the relative group in this chapter. A factorizability theory ψ must also respect both the direct sum and (where defined) composition of triples, again, to be made precise later. Thus in essence a factorizability theory is a homomorphism from the relative group, and the techniques used to compare factorizability theories developed in the later sections stem from this fact.

We go on to give an example of a factorizability theory, namely the Monomial factorizability defined in [19]. Many of the other factorizability theories used to date (such as that used in [6], and the one alluded to in [12]) are shown in [19] to be equivalent to monomial factorizability; in the next chapter we will show that monomial factorizability is equivalent to one of the invariance factorizability theories there defined.

Much of the definition of a factorizability theory is based on [19]; the sequences from $\mathcal{K}$-theory rely on work from [9].

Let us begin by establishing some notation.

Definition 1.0.1. We will be dealing with two cases. They are the *global* case, where the group ring is $\mathbb{Z}\Gamma$, and the *local* case, where the group ring is $\mathbb{Z}_p\Gamma$ for a fixed prime p. If a part of the theory is true in both cases, we will use Λ to denote the group ring. We will use A to denote the algebra $\mathbb{Q}\Lambda$, that is, either $\mathbb{Q}\Gamma$ or $\mathbb{Q}_p\Gamma$.

1.1 Some results from $\mathcal{K}$-theory

The results of this section are based upon on the work of Heller [9]. In [9], however, the results are proven in far greater generality. We reproduce them here in a more specialised case in order to make the working more transparent, whilst still maintaining sufficient generality to cover a wide variety of module categories.

We begin by defining extensional module categories, and functors between extensional module categories. This will allow us to define the fibre category of such a functor, and hence the relative group. This relative group fits in to a (not necessarily exact) sequence, the Heller sequence, connecting the relative group to other well known $\mathcal{K}$-groups. We finally establish conditions on when the Heller sequence is exact, which will be satisfied by the cases we will mainly be interested in for the later chapters.

We want to be able to deal with mod(Λ) and lat(Λ), in both the local and global cases, with either all exact sequences or only those arising from the direct sum. Therefore we define:

Definition 1.1.1. An *extensional module category* $\mathcal{C} = (\mathcal{C}, \mathcal{E})$ consists of a category $\mathcal{C}$ which is a subcategory of mod(R) for some ring R, closed under direct sum, together with a category $\mathcal{E}$ whose objects are (some) short exact sequences of modules from $|\mathcal{C}|$, including at least those due to the direct sum. Note that this is a special case of Heller's "extensional category".

Note that both mod(Λ) and lat(Λ), in both the local and global cases, with either all exact sequences or only those arising from the direct sum, fit the above definition.

Definition 1.1.2. A functor $\mathcal{F}$ between two extensional module categories $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{C}', \mathcal{E}')$ (where $\mathcal{C}'$ is a subcategory of mod(R')) consists of a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ which preserves the exact sequences of $\mathcal{E}$. (Heller [9] allows a more arbitrary functor $\mathcal{F}^{\wedge}$ on the extensional structure, but our special case of extensional module categories allows us to fix this functor.)

We may now define

Definition 1.1.3. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ be a functor between two extensional module categories, where $\mathcal{C}'$ is a subcategory of mod(R'). The *fibre category* $\Phi(\mathcal{C}, \mathcal{F})$ is a category whose objects are triples (M, f, N), where M and N are objects in $\mathcal{C}$ and $f\colon \mathcal{F}(M) \xrightarrow{\sim} \mathcal{F}(N)$ is a R'-isomorphism. (The reader following [9] should note that Heller writes the triples in a different order.)

The morphisms of the fibre category are pairs (a, b) of R-homomorphisms $a\colon M \to M'$, $b\colon N \to N'$, such that the following square commutes:

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(a)} & \mathcal{F}(M') \\ f \downarrow \wr & & \wr \downarrow f' \\ \mathcal{F}(N) & \xrightarrow[\mathcal{F}(b)]{} & \mathcal{F}(N') \end{array}$$

1.1.4. Remarks. We may sensibly construct exact sequences in $\Phi(\mathcal{C}, \mathcal{F})$: a sequence

$$0 \to (M', f', N') \xrightarrow{(a',b')} (M, f, N) \xrightarrow{(a,b)} (M'', f'', N'') \to 0$$

is thought of as exact if

$$0 \to M' \xrightarrow{a'} M \xrightarrow{a} M'' \to 0$$

and

$$0 \to N' \xrightarrow{b'} N \xrightarrow{b} N'' \to 0$$

are exact, since necessarily the sequences

$$0 \to \mathcal{F}(M)' \xrightarrow{\mathcal{F}(a')} \mathcal{F}(M) \xrightarrow{\mathcal{F}(a)} \mathcal{F}(M)'' \to 0$$

and

$$0 \to \mathcal{F}(N)' \xrightarrow{\mathcal{F}(b')} \mathcal{F}(N) \xrightarrow{\mathcal{F}(b)} \mathcal{F}(N)'' \to 0$$

are also exact.

When $\mathcal{F}$ is an unconditionally exact functor, as is the case when $\mathcal{F} = \otimes\mathbb{Q}$, the diagram

$$\begin{array}{ccccccc}
\mathcal{F}(\ker(a)) & \hookrightarrow & \mathcal{F}(M) & \xrightarrow{\mathcal{F}(a)} & \mathcal{F}(M') & \twoheadrightarrow & \mathcal{F}(\operatorname{coker}(a)) \\
{\scriptstyle g}\downarrow{\scriptstyle \wr} & & {\scriptstyle f}\downarrow{\scriptstyle \wr} & & {\scriptstyle f'}\downarrow{\scriptstyle \wr} & & {\scriptstyle g'}\downarrow{\scriptstyle \wr} \\
\mathcal{F}(\ker(b)) & \hookrightarrow & \mathcal{F}(N) & \xrightarrow[\mathcal{F}(b)]{} & \mathcal{F}(N') & \twoheadrightarrow & \mathcal{F}(\operatorname{coker}(b))
\end{array}$$

necessarily commutes, where g is the restriction of f and g' is the map induced from f', and therefore we can sensibly talk of the kernel and cokernel of a morphism (a, b).

Definition 1.1.5. The *relative group* $\mathcal{K}_0(\mathcal{C}, \mathcal{F}) = \mathcal{K}_0^{\mathcal{E}}(\mathcal{C}, \mathcal{F})$ (written additively) is the abelian group generated by isomorphism classes $[M, f, N]$ of triples (M, f, N) from the fibre category $\Phi(\mathcal{C}, \mathcal{F})$, with relations from exact sequences in the fibre category, and relations from composition of triples, that is,

$$[M, f, N] + [M', f', N'] = [M, f' \circ f, N'] \text{ whenever } N = M'.$$

Note that where $\mathcal{E}$ consists only of sequences due to the direct sum, the relations from exact sequences are precisely

$$[M, f, N] + [M', f', N'] = [M \oplus M', f \oplus f', N \oplus N'].$$

We have the following useful facts about the identity element in the relative group:

Lemma 1.1.6. *1. In $\mathcal{K}_0^{\mathcal{E}}(\mathcal{C}, \mathcal{F})$, $[M, 1, M] = 0$.*

2. In $\mathcal{K}_0^{\mathcal{E}}(\mathcal{C}, \mathcal{F})$, if $f\colon M \xrightarrow{\sim} N$ is an R-isomorphism, then

$$[M, \mathcal{F}(f), N] = 0.$$

Proof. 1. $[M,1,M]+[M,f,N]=[M,f,N]$ by composition rule.

2. The square

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow[1]{\sim} & \mathcal{F}(M) \\ {\scriptstyle 1}\downarrow\wr & & \wr\downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathcal{F}(M) & \xrightarrow[\mathcal{F}(f)]{\sim} & \mathcal{F}(N) \end{array}$$

commutes, therefore $(1,f)$ is an isomorphism in $\Phi(\mathcal{C},\mathcal{F})$, and therefore

$$[M,\mathcal{F}(f),N]=[M,1,M]=0$$

as required. □

Lemma 1.1.7. *All elements of the relative group $\mathcal{K}_0^{\mathcal{E}}(\mathcal{C},\mathcal{F})$ are expressible in the form $[M,f,N]$, where (M,f,N) is a triple in $\Phi(\mathcal{C},\mathcal{F})$.*

Proof. Clearly all elements of $\mathcal{K}_0^{\mathcal{E}}(\mathcal{C},\mathcal{F})$ are expressible as finite sums and differences of elements of the form $[M,f,N]$. However,

$$-[M,f,N]=[N,f^{-1},M]$$

since $[M,f,N]+[N,f^{-1},M]=[M,1,M]=0$ by composition rule, and

$$[M,f,N]+[M',f',N']=[M\oplus M',f\oplus f',N\oplus N'],$$

so all elements of $\mathcal{K}_0^{\mathcal{E}}(\mathcal{C},\mathcal{F})$ are of the required form. □

Now that we have established our definitions, we state without proof a key result due to Heller:

Proposition 1.1.8. The Heller Sequence *(see Heller, [9], 4.1, 4.4, 5.1, 5.2, or Bass [1]). For* $\mathcal{F}\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C}',\mathcal{E}')$ *a cofinal functor between two extensional module categories, consider the sequence*

$$\mathcal{K}_1(\mathcal{C}) \xrightarrow{[\mathcal{F}]} \mathcal{K}_1(\mathcal{C}') \xrightarrow{\partial} \mathcal{K}_0(\mathcal{C},\mathcal{F}) \xrightarrow{\delta} \mathcal{K}_0(\mathcal{C}) \xrightarrow{[\mathcal{F}]} \mathcal{K}_0(\mathcal{C}')$$

where $\partial\colon [\overline{M},\alpha] \mapsto [M,\alpha,M]$ *(with* M *chosen so that there exists* $\overline{N}$ *with* $\mathcal{F}\colon M \mapsto \overline{M}\oplus\overline{N}$*, and necessarily* $[\overline{M},\alpha] = [\overline{M}\oplus\overline{N},\alpha\oplus 1]$*) and* $\delta\colon [M,f,N] \mapsto [N]-[M]$.

1. *This sequence is a chain complex.*
2. *If* $\mathcal{F}$ *is a fibration then the sequence is exact at* $\mathcal{K}_0(\mathcal{C})$.
3. *If* $\mathcal{F}$ *is a fibration and the sequences of* $\mathcal{E}'$ *split then the sequence is exact at* $\mathcal{K}_0(\mathcal{C})$ *and* $\mathcal{K}_0(\mathcal{C},\mathcal{F})$.
4. *If* $\mathcal{F}$ *is a fibration and the sequences of* $\mathcal{E}$ *split then the sequence is exact everywhere.*

1.1.9. Remark. For a definition of *fibration* see [9], section 4. For now, we observe without proof:

Lemma 1.1.10. *Let* $\mathcal{F}\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C}',\mathcal{E}')$ *be a functor between two extensional module categories.*

1. *If the exact sequences in* $\mathcal{E}'$ *split and* $\mathcal{F}$ *is cofinal, then* $\mathcal{F}$ *is a fibration.*
2. *If for every* $B \in \mathcal{E}'$, $B = \mathcal{F}(A)$ *for some* $A \in \mathcal{E}$ *then* $\mathcal{F}$ *is a fibration.*

In addition to the exact sequence results, there are several other homomorphisms between the $\mathcal{K}$-groups which we shall need.

We will then introduce the concept of "strictness"—a useful condition for a factorizability theory to satisfy.

In order to motivate the definitions of factorizability and strictness, we will manipulate the Heller sequence of proposition 1.1.8. This manipulation can be found clearly set out in [19], and therefore we will present without proof an overview of the relevant points. We begin by truncating the five-term Heller sequence on the right to a four term exact sequence, with the final map surjective. We go on to replace the $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma))$ by an alternative, isomorphic group which will prove more convenient for later proofs. We have a map from $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma))$ to Ψ, namely $\psi \circ \tilde{\partial}$. The definition of "strictness" is that the kernel of this map lies in the "units" of $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma))$, and we go on to discuss what elements are units in each of the groups isomorphic to $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma))$.

We start by establishing some notation.

Notation 1.2.1. Throughout this section, A will denote the $\mathbb{Q}$-algebra $\mathbb{Q}\Gamma$ (the global case) or the $\mathbb{Q}_p$-algebra $\mathbb{Q}_p\Gamma$ (the local case), and Λ will denote the order $\mathbb{Z}\Gamma$ or $\mathbb{Z}_p\Gamma$, as appropriate. We will denote $\mathrm{Z}(\mathbb{Q}\Gamma)$ (the centre of $\mathbb{Q}\Gamma$) by C, and $\mathrm{Z}(\mathbb{Q}_p\Gamma)$ by C_p. The maximal order in C (respectively C_p) we will denote by $\mathcal{O}_C$ (respectively $\mathcal{O}_{C_p}$).

Definition 1.2.2. Let $\mathcal{F}\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ be a functor between two extensional module categories. A *factorizability theory* is a homomorphism

$$\psi\colon \mathcal{K}_0^{\mathcal{E}}(\mathcal{C}, \mathcal{F}) \to \Psi$$

where Ψ is some abelian group.

1.2.3. Remark. In most cases, we will have in mind a specific functor between extensional module categories $\mathcal{F}\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$, for example, we could consider $\otimes\mathbb{Q}\colon \operatorname{mod}(\Lambda) \to \operatorname{mod}(A)$ with relations only from direct sums (when the relative group would be denoted $\mathcal{K}_0^{\oplus}(\operatorname{mod}(\Lambda), \otimes\mathbb{Q})$) or with relations from all short exact sequences (when the relative group would be denoted $\mathcal{K}_0(\operatorname{mod}(\Lambda), \otimes\mathbb{Q})$), or possibly $\otimes\mathbb{Q}\colon \operatorname{lat}(\Lambda) \to \operatorname{mod}(A)$ giving the groups $\mathcal{K}_0^{\oplus}(\operatorname{lat}(\Lambda), \otimes\mathbb{Q})$ or $\mathcal{K}_0(\operatorname{lat}(\Lambda), \otimes\mathbb{Q})$. Note that by lemma 1.1.10 these are all fibrations.

The first step of our manipulation of the Heller sequence of proposition 1.1.8 is to truncate the sequence on the right.

Definition 1.2.4. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ be a functor between extensional module categories. We define $\tilde{\mathcal{K}}_0(\mathcal{C})_{\mathcal{F}}$ to be the kernel of $[\mathcal{F}]\colon \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{C}')$. In most situations it will be clear which particular $\mathcal{F}$ we have in mind, and we will denote $\tilde{\mathcal{K}}_0(\mathcal{C})_{\mathcal{F}}$ by $\tilde{\mathcal{K}}_0(\mathcal{C})$.

Lemma 1.2.5. *Keeping the above notation, if $\mathcal{F}$ is a fibration then $\tilde{\mathcal{K}}_0(\mathcal{C})$ is the image of $\mathcal{K}_0(\mathcal{C}, \mathcal{F})$ in $\mathcal{K}_0(\mathcal{C})$. Then we have a new sequence*

$$\mathcal{K}_1(\mathcal{C}) \xrightarrow{[\mathcal{F}]} \mathcal{K}_1(\mathcal{C}') \xrightarrow{\partial} \mathcal{K}_0(\mathcal{C}, \mathcal{F}) \xrightarrow{\delta} \tilde{\mathcal{K}}_0(\mathcal{C}),$$

not necessarily exact at $\mathcal{K}_1(\mathcal{C}')$ or $\mathcal{K}_0(\mathcal{C}, \mathcal{F})$, with ∂ and δ as before.

Proof. By proposition 1.1.8, if $\mathcal{F}$ is a fibration then the Heller sequence is exact at $\mathcal{K}_0(\mathcal{C})$. The result then follows from proposition 1.1.8. Note that this new sequence is exact at $\mathcal{K}_1(\mathcal{C}')$ or $\mathcal{K}_0(\mathcal{C},\mathcal{F})$ precisely when the Heller sequence is exact at $\mathcal{K}_1(\mathcal{C}')$ or $\mathcal{K}_0(\mathcal{C},\mathcal{F})$. □

We now focus our attention on a specific choice of $\mathcal{F}$. Throughout what follows, we will work with $[\otimes\mathbb{Q}]\colon \operatorname{mod}(\Lambda) \to \operatorname{mod}(A)$, with only those exact sequences due to the direct sum. However, much of what follows works equally well for the other choices of $\mathcal{F}$ mentioned in remark 1.2.3.

1.2.6. Remark. For a given group Γ, for almost all primes p (in particular whenever $p \nmid |\Gamma|$), $\mathbb{Z}_p\Gamma$ is a maximal order in $\mathbb{Q}_p\Gamma$. When this happens, $[\otimes\mathbb{Q}]\colon \mathcal{K}_0(\operatorname{mod}(\mathbb{Z}_p\Gamma)) \to \mathcal{K}_0(\operatorname{mod}(\mathbb{Q}_p\Gamma))$ is injective, and therefore $\tilde{\mathcal{K}}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma)) = \{0\}$. See [17], theorem 7.6 for details.

We define what we mean by the idèles and unit idèles of an arbitrary $\mathbb{Z}$-order Λ' contained in a semisimple $\mathbb{Q}$-algebra A'.

Definition 1.2.7. The *finite unit idèles* of Λ', denoted $\mathcal{U}_{\mathbb{Z}}(\Lambda')$ are defined to be $\prod_p(\Lambda'_p)^\times$, where Λ'_p is the localisation at p, the product running over all finite primes p.

The *finite idèles* of A', denoted $\mathcal{J}_{\mathbb{Z}}(A')$ are defined to be $\coprod_p(A'_p)^\times \cdot \mathcal{U}_{\mathbb{Z}}(\Lambda')$. They are thus in effect those elements of $\prod_p(A'_p)^\times$ where all but finitely many components lie in the appropriate Λ'_p.

In the local case, $\mathcal{K}_1(\operatorname{mod}(\mathbb{Q}_p\Gamma))$ is isomorphic by the reduced norm to $C^\times = \mathrm{Z}(\mathbb{Q}_p\Gamma)^\times$, as we will discuss further at 1.2.17. Thus we have a new exact sequence

$$\mathcal{K}_1^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma)) \xrightarrow{\mathrm{Nrd}\circ[\otimes\mathbb{Q}]} C^\times \xrightarrow{\tilde{\partial}_p^{\oplus}} \mathcal{K}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma),\otimes\mathbb{Q}) \xrightarrow{\delta_p^{\oplus}} \tilde{\mathcal{K}}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma))$$

where $\tilde{\partial}_p^{\oplus}\colon \alpha \mapsto [\mathbb{Z}_p\Gamma, \beta, \mathbb{Z}_p\Gamma]$, β chosen so that $\mathrm{Nrd}(\beta) = \alpha$.

In the global case it can similarly be shown that $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}\Gamma))$ is isomorphic by the reduced norm to $C^{\times +} \overset{\text{def}}{=} \mathrm{Nrd}(\mathbb{Q}\Gamma^{\times}) \subseteq C^{\times}$, giving us the exact sequence

$$\mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \overset{\mathrm{Nrd}\circ[\otimes\mathbb{Q}]}{\longrightarrow} C^{\times +} \overset{\tilde{\partial}^{\oplus}}{\longrightarrow} \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \overset{\delta^{\oplus}}{\twoheadrightarrow} \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)).$$

However, we wish to derive a different sequence in the global case. We begin by taking the coproduct over all primes p of the previous sequence to obtain the exact sequence

$$\coprod_p \mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma)) \overset{\mathrm{Nrd}\circ[\otimes\mathbb{Q}]}{\longrightarrow} \coprod_p C_p^{\times} \overset{\coprod_p \tilde{\partial}_p^{\oplus}}{\longrightarrow} \coprod_p \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$$
$$\overset{\coprod_p \delta_p^{\oplus}}{\twoheadrightarrow} \coprod_p \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma)).$$

As has already been observed, for almost all p, $\mathbb{Z}_p\Gamma$ is a maximal order in $\mathbb{Q}_p\Gamma$ and in this case $\mathrm{Nrd}\colon \mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma)) \overset{\sim}{\longrightarrow} \mathcal{O}_{C_p}^{\times}$. Therefore, we may change the first term of the above exact sequence from a coproduct to a product provided we change the second term from $\coprod_p C_p^{\times}$ to $\coprod_p C_p^{\times} \cdot \prod_p \mathcal{O}_{C_p}^{\times} = \mathcal{J}_{\mathbb{Z}}(C)$ and in turn replace the map $\coprod_p \tilde{\partial}_p^{\oplus}$ by $\prod_p \tilde{\partial}_p^{\oplus}$ in order to maintain exactness.

Lemma 1.2.8. *We have by [11], theorem 3.1, an isomorphism*

$$\lambda\colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \overset{\sim}{\longrightarrow} \coprod_p \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$$

$$[M, f, N] \longmapsto \{[M_p, f \otimes 1, N_p]\}.$$

Adjusting the maps accordingly, we have a new exact sequence

Theorem 1.2.9. *The sequence*

$$\coprod_p \mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma)) \xrightarrow{\mathrm{Nrd}\circ[\otimes\mathbb{Q}]} \mathcal{J}_{\mathbb{Z}}(C) \xrightarrow{\tilde{\partial}^{\oplus}} \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q})$$

$$\xrightarrow{\delta^{\oplus}} \coprod_p \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma))$$

is exact, where $\mathrm{Nrd}\circ[\otimes\mathbb{Q}]$ *sends* $\mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma))$ *to* $C_p^{\times}$ *via* $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$*, and* $\delta^{\oplus}$ *maps* $[M, f, N]$ *to* $\{[N_p] - [M_p]\}$.

Definition 1.2.10. Another consequence of lemma 1.2.8 is that, by choosing a local theory $\psi_p\colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \Psi_p$ for each p, we may construct a global theory $\psi = \coprod_p \psi_p \circ \lambda$ with invariant group $\coprod_p \Psi_p$. Such a ψ is said to be *locally defined.*

Let us now bring in the factorizability theory ψ. The definitions of when an element of Ψ, or a triple $[M, f, N]$ is factorizable, are connected to the map from $\mathcal{J}_{\mathbb{Z}}(C)$ in the global case, and the map from $C_p^{\times}$ in the local case. Denoting either of these groups by $\mathcal{J}$, and denoting by $\mathcal{U}$ the group $\mathcal{U}_{\mathbb{Z}}(\mathcal{O}_C)$ in the global case and $\mathcal{O}_{C_p}^{\times}$ in the local case, we have the following situation:

$$\begin{array}{ccc} \mathcal{U} \subseteq \mathcal{J} & \xrightarrow{\tilde{\partial}^{\oplus}} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q}) \\ & & \Big\downarrow \psi \\ & & \Psi \end{array}$$

Definition 1.2.11. For a factorizability theory ψ, $x \in \Psi$ is said to be *factorizable*, with factorization α, if $x = (\psi \circ \tilde{\partial}^{\oplus})(\alpha)$, for some $\alpha \in \mathcal{J}$.

Definition 1.2.12. For a factorizability theory ψ, a triple $[M, f, N]$ lying in $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q})$ is said to be *factorizable* if $\psi([M, f, N])$ is factorizable.

Definition 1.2.13. For a factorizability theory ψ, two modules M and N are said to be *factor equivalent* (written $M \wedge_\psi N$) if there exists a triple $[M, f, N] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q})$ which is factorizable.

It is worth noting that, although the definition of factor equivalence appears to rely upon a choice of isomorphism f, it is in fact independant of this choice, since

Lemma 1.2.14. *If $[M, f, N]$ has ψ-factorization α and $f'\colon M\otimes\mathbb{Q} \to N\otimes\mathbb{Q}$ is any other A-isomorphism then $[M, f', N]$ is factorizable with factorization $\alpha \cdot \mathrm{Nrd}(f^{-1} \circ f')$.*

Proof. In $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q})$, we have

$$\begin{aligned}[M, f', N] - [M, f, N] &= [M, f', N] + [N, f^{-1}, M] \\ &= [M, f^{-1} \circ f', M] \\ &= \tilde{\partial}^{\oplus}(\mathrm{Nrd}(f^{-1} \circ f'))\end{aligned}$$

therefore $[M, f', N]$ has ψ-factorization $\alpha \cdot \mathrm{Nrd}(f^{-1} \circ f')$. □

Definition 1.2.15. A factorizability theory ψ is said to be *strict* if the kernel of $\psi \circ \tilde{\partial}^{\oplus}$ is contained in $\mathcal{U}$.

1.2.16. Remark. It is worth noting that if ψ is locally defined, and each of the ψ_p are strict, then ψ is also strict.

Lemma 1.2.17. *The following diagram commutes, with the bottom three* *groups isomorphic:*

$$
\begin{array}{ccccc}
 & & \mathbb{Q}_p\Gamma^\times & & \\
 & \swarrow & \downarrow \scriptstyle{\mathrm{Nrd}} & \searrow \scriptstyle{\mathrm{Det}} & \\
\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma)) & \xrightarrow[\text{[Nrd]}]{\sim} & \mathrm{Z}(\mathbb{Q}_p\Gamma)^\times = C_p^\times & \xrightarrow{\sim} & \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)
\end{array}
$$

where the maps are as follows. An element β in $\mathbb{Q}_p\Gamma^\times$ is mapped to the class $[\beta\times, \mathbb{Q}_p\Gamma]$ in $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ where $\beta\times$ means multiplication on the left by β. The isomorphism from $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ to $C_p^\times$ sends the class $[\beta\times, \mathbb{Q}_p\Gamma]$ to the reduced norm of β, $\alpha = \mathrm{Nrd}(\beta)$. The isomorphism from $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ to $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$ sends the class $[\beta\times, \mathbb{Q}_p\Gamma]$ in $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ to $\tilde{\alpha} = \mathrm{Det}(\beta)$, that is, $\tilde{\alpha}$ where $\tilde{\alpha}(\chi) = \det_{\mathbb{Q}_p}(\beta\times\colon V_\chi \to V_\chi)$ with V_χ is a $\mathbb{Q}_p\Gamma$-module with character χ. For the isomorphism $C_p^\times$ to $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$, we do the following. An element $x \in C_p^\times$ maps to $f_x \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$, where $f_x(\chi) = \rho(x)$ and ρ is a representation of $\mathbb{Q}_p\Gamma$ of character χ.

Proof. For the right-hand triangle, see [7] section II, particularly lemma 1.6. For the left-hand triangle, see [1] chapter 5, §9. In particular the isomorphism follows from the exact sequence

$$0 \to \mathcal{SK}_1(\mathbb{Q}_p\Gamma) \to \mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma)) \overset{[\mathrm{Nrd}]}{\twoheadrightarrow} \mathrm{Z}(\mathbb{Q}_p\Gamma)^\times$$

and the fact that $\mathcal{SK}_1(\mathbb{Q}_p\Gamma) = 0$ by [1], chapter 5, theorem 9.7. □

Of these three groups, $\mathrm{Z}(\mathbb{Q}_p\Gamma)^\times$ is the easiest group to define what is meant by a unit, but it is the third group $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$ in which we will do most of the work.

Definition 1.2.18. $Z(\mathbb{Q}_p\Gamma)^\times = C_p^\times \cong \bigoplus_i K_i$ for some fields K_i, and so we define the *units* of $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ to be those $[\beta\times, \mathbb{Q}_p\Gamma]$ where the corresponding $\alpha \in \prod \mathcal{O}_{K_i}^\times = \mathcal{O}_{C_p}^\times$.

Proposition 1.2.19. *The following three are equivalent:*

1. $[\beta\times, \mathbb{Q}_p\Gamma]$ *is a unit in* $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$,
2. $\tilde{\alpha}(\chi) \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^\times$ *for all* $\chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$,
3. $v_p(\tilde{\alpha}(\chi)) = 0$ *for all* $\chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$.

Proof. $1 \Leftrightarrow 2$ is clear from the description of the isomorphism. $2 \Leftrightarrow 3$ is clear from $x \in \mathcal{O}_{\overline{\mathbb{Q}}_p}^\times \Leftrightarrow v_p(x) = 0$. □

Proposition 1.2.20. *For a local theory* ψ_p, ψ_p *is strict if and only if*

$$(\psi_p \circ \tilde{\partial}^\oplus)([\beta\times, \mathbb{Q}_p\Gamma]) = 0 \qquad \textit{implies} \qquad v_p(\tilde{\alpha}(\chi)) = 0 \quad \forall \chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}.$$

Proof. ψ_p is strict if and only if the kernel of $\psi_p \circ \tilde{\partial}^\oplus$ is contained in $\mathcal{O}_{C_p}^\times$, which is true if and only if

$$(\psi_p \circ \tilde{\partial}^\oplus)([\beta\times, \mathbb{Q}_p\Gamma]) = 0 \qquad \text{implies} \qquad [\beta\times, \mathbb{Q}_p\Gamma] \text{ is a unit in } \mathcal{K}_1(\mathbb{Q}_p\Gamma)$$

which is true if and only if

$$(\psi_p \circ \tilde{\partial}^\oplus)([\beta\times, \mathbb{Q}_p\Gamma]) = 0 \qquad \text{implies} \qquad v_p(\tilde{\alpha}(\chi)) = 0 \quad \forall \chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$$

by proposition 1.2.19. □

1.3 Comparing factorizability theories

In this section we will consider ways of comparing and combining factorizability theories for the same relative group. We will consider two different ways of comparing factorizability theories. The first approach is due to the homomorphism aspect of factorizability theories—one theory is said to be stronger than another if the kernel in the relative group is smaller. This will be the approach we will use in the following chapters. The second approach is to consider which elements of the relative group are factorizable—one theory is stronger than another if, for triples in the relative group, factorizability by the first theory implies factorizability by the second theory.

Finally we will look at ways of combining factorizability theories, and look at the relative strength of the combined theory and the original theories.

Definitions 1.3.1. Consider two factorizability theories ψ, ϕ from the same relative group to groups of invariants Ψ, Φ respectively. If $\ker(\psi) = \ker(\phi)$ then we say that ψ is *equivalent to* ϕ, written $\psi \sim \phi$.

If $\ker(\psi) \subseteq \ker(\phi)$ then we say that ψ is *stronger than or equivalent to* ϕ, written $\psi \succcurlyeq \phi$.

If $\ker(\psi) \subsetneq \ker(\phi)$ then we say that ψ is *stronger than* ϕ, written $\psi \succ \phi$.

$\preccurlyeq$ and $\prec$ are similarly defined.

We consider now whether or not M being ψ-factor equivalent to N implies M is ϕ-factor equivalent to N. We have

Proposition 1.3.2. *If $\psi \sim \phi$, then $[M, f, N]$ is ψ-factorizable if and only if $[M, f, N]$ is ϕ-factorizable.*

If $\psi \succcurlyeq \phi$, then $[M, f, N]$ being ψ-factorizable implies $[M, f, N]$ is ϕ-factorizable.

A similar statement is true for $\succ$; for $\preccurlyeq$ and $\prec$ we get similar statements by using the fact that $\psi \preccurlyeq \phi \Leftrightarrow \phi \succcurlyeq \psi$.

Proof. The first statement is a simple consequence of the second statement, by using $\psi \sim \phi$ if and only if $\psi \succcurlyeq \phi$ and $\phi \succcurlyeq \psi$.

For the second statement we do the following. $\psi \succcurlyeq \phi$ implies that $\ker(\psi) \subseteq \ker(\phi)$. Suppose $[M, f, N]$ is ψ-factorizable, with factorization α. Then $[M, f, N] - \tilde{\partial}^{\oplus}(\alpha)$ is an element of $\ker(\psi)$, and therefore an element of $\ker(\phi)$. Therefore $\phi([M, f, N]) = \phi \circ \tilde{\partial}^{\oplus}(\alpha)$ and $[M, f, N]$ is ϕ-factorizable. □

1.3.3. Remark. The concept of a theory being stronger than another imposes a partial ordering on factorizability theories. There are two extreme theories: ψ_{id} which is the identity map, and ψ_0 the zero map. For any factorizability theory ψ, it is clear that we have $\psi_{\text{id}} \succcurlyeq \psi \succcurlyeq \psi_0$. Also, for any two theories ψ and ϕ, we can find a weakest theory which is stronger than or equivalent to both ψ and ϕ, namely,

$$\psi \oplus \phi \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q}) \to \Psi \oplus \Phi$$
$$x \mapsto (\psi(x), \phi(x)).$$

Clearly $\ker(\psi \oplus \phi) = \ker(\psi) \cap \ker(\phi)$, and therefore $\psi \oplus \phi$ is stronger than both ψ and ϕ, but no weaker theory can be stronger than both ψ and ϕ.

Where $\Phi = \Psi$, we could also consider factorizability theories of the form

$$\psi + \phi \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q}) \to \Psi$$
$$x \mapsto \psi(x) + \phi(x).$$

Necessarily $\psi + \phi \preccurlyeq \psi \oplus \phi$, since if $x \in \ker(\psi) \cap \ker(\phi)$, then $(\psi + \phi)(x) = 0$. However in general we cannot say any more than this, for example in the case when $\phi(x) = -\psi(x)$ for all $x \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q})$ we have $\psi + \phi \sim \psi_0$, and in the case when $\psi(x) = 0$ for all $x \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\Lambda), \otimes\mathbb{Q})$ we have $\psi + \phi \sim \psi$.

1.4 An example of a factorizability theory

The following example of a factorizability theory, Monomial factorizability, is taken from [19]. In [19] it is shown to be equivalent to the factorizability theories hinted at in [12], and when restricted to $\mathrm{lat}(\mathbb{Z}\Gamma)$, equivalent to Fröhlich's "strict factorizability" of [6].

We choose this as an example because we will show it to be equivalent to one of the invariance factorizability theories described in chapter 2. In [19] it is also shown that the monomial factorizability theory as defined below is in fact strict. However, the approach taken there is fairly technical. Instead, we will use the equivalence to one of the invariance factorizability theories to prove strictness.

We begin by defining μ-monomial representations. In what follows, F will denote a field of characteristic 0, B a subring of F, and μ a torsion subgroup of $B^\times$.

Definitions 1.4.1. A *μ-monomial representation* of Γ is a right $\mu \times \Gamma$-set S such that $\mu \times \{1\}$ acts freely on S and S consists of only finitely many orbits.

We define $\mathrm{Mon}(\Gamma, \mu)$ to be the Grothendieck group with respect to disjoint union of μ-monomial representations.

If T is a μ-monomial representation of a subgroup Γ' of Γ, we define the *induced representation* $T{\uparrow}_{\Gamma'}^{\Gamma} \stackrel{\text{def}}{=} T \times_{\Gamma'} \Gamma$.

For Γ' a subgroup of Γ, and $\chi\colon \Gamma' \to \mu$, we define the μ-monomial representation of Γ' *afforded by* χ (denoted by μ_χ) to be a copy of μ, with Γ' acting via χ.

It can be shown (see [19], 7.3 and preceding) that any indecomposable μ-monomial representation is isomorphic to one of the form $\mu_\chi{\uparrow}_{\Gamma'}^{\Gamma}$, $\mathrm{Mon}(\Gamma, \mu)$ is free on the classes of indecomposable μ-monomial representations, and therefore $\mathrm{Mon}(\Gamma, \mu)$ is generated by the classes $[\mu_\chi{\uparrow}_{\Gamma'}^{\Gamma}]$.

Definitions 1.4.2. Let S be a μ-monomial representation. Then we define the *B-linearization* of S (denoted $B_\mu[S]$) to be the $B\Gamma$-module $\mathbb{Z}S \otimes_{\mathbb{Z}\mu} B$.

We denote $B_\mu[\mu_\chi]$ by B_χ.

For any $B\Gamma$-module M, and any homomorphism $\chi\colon \Gamma' \to \mu$, we define $M^\chi \stackrel{\text{def}}{=} \{m \in M : m\gamma = m\chi(\gamma), \forall \gamma \in \Gamma'\}$.

An alternative description of M^χ is from the natural isomorphism

$$\mathrm{Hom}_{B\Gamma}(B_\chi\uparrow_{\Gamma'}^{\Gamma}, M) \xrightarrow{\sim} M^\chi$$
$$f \longmapsto f((1,1)\otimes 1).$$

Definitions 1.4.3. Let p be a prime number. We define $F\{p\}$ to be the maximal unramified extension of $\mathbb{Q}_p$. We define $\mathcal{O}\{p\}$ to be the ring of integers in $F\{p\}$. We define $\mu^{(p)}$ to be the roots of unity of $\mathcal{O}\{p\}$ of order prime to p. We define $G\{p\} = \mathrm{Gal}(F\{p\}/\mathbb{Q}_p)$.

We are now in a position to define the locally unramified monomial factorizability $\psi_{\mathrm{mon}}\colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \to \mathcal{M}$. It is a locally defined theory (see definition 1.2.10), with local theories $\psi_{\mathrm{mon},p}$ and local groups of invariants $\mathcal{M}_p$ for each p. $\psi_{\mathrm{mon}} = \coprod_p \psi_{\mathrm{mon},p} \circ \lambda$ by lemma 1.2.8, and $\mathcal{M} = \coprod_p \mathcal{M}_p$.

We define $\mathcal{M}_p = \mathrm{Hom}_{G\{p\}}(\mathrm{Mon}(\Gamma, \mu^{(p)}), \mathcal{I}(\mathcal{O}\{p\}))$.

We define the local theory as

$$\psi_{\mathrm{mon},p}\colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \mathcal{M}_p$$
$$[T] \mapsto \psi_{\mathrm{mon},p}([T])$$

where

$$\psi_{\mathrm{mon},p}([T])\colon [S] \mapsto \left|\mathrm{Hom}_{\mathcal{O}\{p\}\Gamma}(\mathcal{O}\{p\}_{\mu^{(p)}}[S], [T\otimes_{\mathbb{Z}_p}\mathcal{O}\{p\}])\right|_{\mathcal{O}\{p\}} \in \mathcal{I}(\mathcal{O}\{p\}).$$

1.4.4. Remark. Since $\mathrm{Mon}(\Gamma, \mu^{(p)})$ is generated by the classes $[\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}]$, $\psi_{\mathrm{mon},p}$ is determined by its action on these representations:

$$\psi_{\mathrm{mon},p}([T])\colon [\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}] \mapsto \left|(T\otimes_{\mathbb{Z}_p}\mathcal{O}\{p\})^\chi\right|_{\mathcal{O}\{p\}}.$$

1.5 Real factorization

In the previous section, and in the following chapters, we discuss local and global factorizability theories where the functor $\mathcal{F}$ is the tensor product $\otimes\mathbb{Q}$. However, there are occasions when it is more "sensible" to use a different functor, for example $\otimes\mathbb{R}$. By "sensible", we mean there is a single sensible choice of isomorphism between modules $M \otimes \mathbb{R}$ and $N \otimes \mathbb{R}$, but no single choice for an isomorphism between $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$. In this section we will briefly investigate this case. We begin with an illustrative example of a situation where the "sensible" choice of isomorphism is between $M \otimes \mathbb{R}$ and $N \otimes \mathbb{R}$, not $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$. We then move on to discuss a possible approach towards defining a "real" factorizability theory rather than a "rational" factorizability theory. Our approach will be to construct the pushout of a "rational" factorizability theory, and determine some of its properties. In order to do this, we will introduce a series of lemmas to manipulate pushouts and pullbacks. We will also make further use of the Heller sequence, this time with the functor $\otimes\mathbb{R}$.

We begin with our example. Our isomorphism will arise from the proof of the Dirichlet unit theorem, [8] theorem 37. For details of this theorem and the derivation of the isomorphism in question, see for example [8], IV.4. We will present here an overview of the construction.

Notation 1.5.1. Throughout this section, F denotes an algebraic number field. F has s distinct real embeddings, and $2t$ distinct complex embeddings (i.e. embeddings in $\mathbb{C}$ which do not lie wholly in $\mathbb{R}$). Recall that the complex embeddings must occur in complex conjugate pairs, hence the even integer $2t$. We define $r = s + t - 1$—the Dirichlet rank. U_F denotes the group of units in F. μ_F denotes the torsion subgroup of U_F, which is therefore the group of roots of unity lying in F.

We state without proof

Theorem 1.5.2. The Dirichlet Unit Theorem *(see for example [8] theorem 37). Keeping the above notation,*

$$U_F \cong \mu_F \times \mathbb{Z}^r.$$

Although we will not prove this theorem ourselves, we will provide an outline of the proof in order to derive our desired isomorphism.

Definition 1.5.3. Let $W = \mathbb{R}^{s+t}$. For $i = 1, \ldots, s$, let σ_i denote the distinct real embeddings of F. For $i = s+1, \ldots, s+t$, let σ_i denote one from each distinct conjugate pair of complex embeddings of F. We define the logarithmic map to be the group homomorphism

$$\begin{aligned} l\colon U_F &\to W \\ u &\mapsto \sum_{i=1}^{s} \log|\sigma_i(u)|e_i + \sum_{i=s+1}^{s+t} 2\log|\sigma_i(u)|e_i \end{aligned}$$

where $\{e_i\}$ is the usual canonical basis of W. $\mathcal{H}$ will denote the hyperplane

$$\{\sum_{i=1}^{s+t} a_i e_i : \sum_{i=1}^{s} a_i + 2\sum_{i=s+1}^{s+t} a_i = 0\} \subset W.$$

Note that necessarily $\operatorname{Im} l \subseteq \mathcal{H}$. In fact, we also have that $\operatorname{Im} l$ is a discrete subgroup of W: to see this, we would demonstrate that only a finite number of points of $\operatorname{Im} l$ lie in a ball of radius x about the origin, for any $x \in \mathbb{R}$. So $\operatorname{Im} l$ is a lattice in $\mathcal{H}$ and therefore of rank at most $s + t - 1 = r$. Also, we can see that $\ker l = \mu_F$.

The final stage of the proof would be to define a set of fundamental units $\{u_i : i = 1, \dots, s+t\}$, with $\log|\sigma_j(u_i)| > 0$ if $i = j$ and $\log|\sigma_j(u_i)| < 0$ if $i \neq j$. Then by looking at the matrix with entries $\log|\sigma_j(u_i)|$ and determining that this matrix has rank at least r, the Dirichlet Unit Theorem is proven.

The individual maps $\log|\sigma_i(\cdot)|$ can be thought of as the valuations at the infinite primes. For any element $x \in \mathcal{O}_F^\times$, necessarily

$$\sum_{i=1}^{s} \log|\sigma_i(x)| + 2 \sum_{i=s+1}^{s+t} \log|\sigma_i(x)| = 0.$$

We define S_∞ to be the set of these valuations at the infinite primes, and consider $\mathbb{Z}S_\infty$. Define

$$\begin{aligned} \sigma' \colon \mathbb{Z}S_\infty &\to \mathbb{Z} \\ \sum_{i=1}^{s+t} a_i\sigma_i &\mapsto \sum_{i=1}^{s} a_i + 2 \sum_{i=s+1}^{s+t} a_i, \end{aligned}$$

and $\Delta S_\infty = \ker \sigma'$. Clearly ΔS_∞ is a full lattice in $\mathcal{H}$. Then, as a corollary to the Dirichlet Unit Theorem, we have

Corollary 1.5.4.

$$l \otimes 1 \colon U_F \otimes \mathbb{R} \xrightarrow{\sim} \Delta S_\infty \otimes \mathbb{R}.$$

From now on, suppose that F is Galois. Putting $\Gamma = \mathrm{Gal}(F/\mathbb{Q})$, it is well known that U_F is a $\mathbb{Z}\Gamma$-module. Also, the action of Γ permutes the infinite primes, so ΔS_∞ is also a $\mathbb{Z}\Gamma$-module.

Thus

Proposition 1.5.5.

$$[U_F, l \otimes 1, \Delta S_\infty] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}).$$

This fact is a key motivation for looking at real factorizability theories. $U_F \cong \mu_F \times \mathbb{Z}^r$, and ΔS_∞ is a full lattice in $\mathcal{H}$ and is hence isomorphic to $\mathbb{Z}^r$, and therefore $U_F \otimes \mathbb{Q} \cong \Delta S_\infty \otimes \mathbb{Q}$ as $\mathbb{Q}$-modules; further, since there is an isomorphism $U_F \otimes \mathbb{R} \cong \Delta S_\infty \otimes \mathbb{R}$ which respects the Γ action, there is necessarily an isomorphism $U_F \otimes \mathbb{Q} \cong \Delta S_\infty \otimes \mathbb{Q}$ as $\mathbb{Q}\Gamma$-modules. However, there is no single obvious candidate for this isomorphism. We know by lemma 1.2.14 that our choice of isomorphism would not alter the factorizability of our triple; however, it would be preferable to use our obvious choice of isomorphism.

We now turn our attention to considering one particular approach to constructing real factorizability theories. Our approach will be to take a rational factorizability theory and construct the pushout.

Definition 1.5.6. Suppose $\psi \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \to \Psi$ is a factorizability theory. We define a real factorizability theory $\psi_{\mathbb{R}}$, and its target group $\Psi_{\mathbb{R}}$

by the pushout diagram

$$\begin{array}{ccc} \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma),\otimes\mathbb{Q}) & \xrightarrow{\psi} & \Psi \\ {\scriptstyle [\otimes\mathbb{R}]}\downarrow & & \downarrow{\scriptstyle [\otimes\mathbb{R}]} \\ \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma),\otimes\mathbb{R}) & \xrightarrow[\psi_{\mathbb{R}}]{} & \Psi_{\mathbb{R}} \end{array}$$

To simplify calculations, we include here a series of lemmas concerning pushouts.

Definition 1.5.7. For the commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[h]{} & D \end{array}$$

we label the kernels and cokernels of each map, and maps induced from f, g, h and k, as in the diagram

$$\begin{array}{ccccccc} & & K_g & \xrightarrow{\mu_2} & K_k & & \\ & & {\scriptstyle i_g}\downarrow & & \downarrow{\scriptstyle i_k} & & \\ K_f & \xrightarrow{i_f} & A & \xrightarrow{f} & B & \xrightarrow{\pi_f} & C_f \\ {\scriptstyle \mu_1}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle \nu_1} \\ K_h & \xrightarrow[i_h]{} & C & \xrightarrow[h]{} & D & \xrightarrow[\pi_h]{} & C_h \\ & & {\scriptstyle \pi_g}\downarrow & & \downarrow{\scriptstyle \pi_k} & & \\ & & C_g & \xrightarrow[\nu_2]{} & C_k & & \end{array}$$

which is commutative with all horizontal and vertical four term sequences exact. We also consider the (not necessarily exact) three-term sequence

$$A \xrightarrow{\theta} B \oplus C \xrightarrow{\phi} D$$

where $\theta\colon a \mapsto (f(a), g(a))$ and $\phi\colon (b,c) \mapsto k(b) - h(c)$. Note that the square being commutative ensures that $\mathrm{im}(\theta) \subseteq \ker(\phi)$.

Lemma 1.5.8. *Consider the situation of definition 1.5.7. Then ν_1 is injective and μ_1 is surjective if and only if* $\mathrm{im}(\theta) = \ker(\phi)$.

Proof. $\Rightarrow$: Suppose $(b,c) \in B \oplus C$ is such that $\phi(b,c) = 0$. Then $k(b) = h(c)$, but since $\pi_h(h(c)) = 0$ by exactness of our commutative diagram, and ν_1 is injective, $\pi_f(b) = 0$. Therefore, by exactness of our commutative diagram, there must exist $a \in A$ such that $f(a) = b$. So $h(g(a)) = h(c)$, and so $g(a) = c + i_h(k_1)$, some $k_1 \in K_h$. But then, by the surjectivity of μ_1, there must exist $k_0 \in K_f$ such that $\mu_1(k_0) = k_1$. Consider $a_0 = a - i_f(k_0)$. By exactness, $f(a_0) = f(a) = b$, and by commutivity, $g(a_0) = c + i_h(k_1) - i_h(k_1) = c$, so $\theta(a_0) = (b,c)$.

$\Leftarrow$: First we shall show that ν_1 is injective, i.e. if $x \in C_f$ is such that if $\nu_1(x) = 0$ then $x = 0$. Choose $b \in B$ such that $\pi_f(b) = x$, and let $d = k(b)$. Then $\pi_h(d) = 0$ so there exists $c \in C$ with $h(c) = d$. But now we have $\phi(b,c) = 0$, so there exists an $a \in A$ such that $f(a) = b$. So, by exactness, $\pi_f(f(a)) = k = 0$.

Now we show that μ_1 is surjective. Let $x \in K_h$. Then $h(i_h(x)) = \phi(0, i_h(x)) = 0$, and so there exists $a \in A$ with $f(a) = 0$, and hence there exists $a_0 \in K_f$ with $i_f(a_0) = a$. $i_h(\mu_1(a_0)) = i_h(x)$, and therefore $\mu_1(a_0) = x$ by the injectivity of i_h. □

Lemma 1.5.9. *Consider the situation of definition 1.5.7. Then μ_1 is injective if and only if θ is injective.*

Proof. $\Rightarrow$: Consider $a \in A$ such that $f(a) = 0$ and $g(a) = 0$. Then there exists $x \in K_f$ with $i_f(x) = a$, whose image in K_h is zero, by the injectivity of i_h. So, since μ_1 is injective, $x = 0$ and hence $a = 0$.

$\Leftarrow$: Suppose we have $x \in K_f$ such that $\mu_1(x) = 0$. Then $f(i_f(x)) = 0$ by exactness, and $g(i_f(x)) = i_h(\mu_1(x)) = 0$ by commutivity. So $\theta(i_f(x)) = (0, 0)$ and therefore $i_f(x) = 0$ by the injectivity of θ. Therefore $x = 0$. □

Lemma 1.5.10. *Consider the situation of definition 1.5.7. Then ν_1 is surjective if and only if ϕ is surjective.*

Proof. $\Rightarrow$: Let $d \in D$. Consider $b \in B$ such that $\nu_1(\pi_f(b)) = \pi_h(d)$. Then $\pi_h(d) = \pi_h(k(b))$, so $d - k(b) = h(c)$, for some $c \in C$. But now $d = k(b) - h(c)$, so $d = \phi(b, c)$.

$\Leftarrow$: Let $x \in C_h$, and choose $d \in D$ such that $\pi_h(d) = x$. Then there exists $(b, c) \in B \oplus C$ such that $\phi(b, c) = d$, that is, $d = k(b) - h(c)$. But $\pi_h(h(c)) = 0$. Therefore $\pi_h(k(b)) = \nu_1(\pi_f(b)) = \pi_h(d) = x$. □

Proposition 1.5.11. *Consider the situation of definition 1.5.7. Then*

1. *the commutative square is a pushout iff $K_f \twoheadrightarrow K_h$ and $C_f \cong C_h$, iff $K_g \twoheadrightarrow K_k$ and $C_g \cong C_k$,*

2. *the commutative square is a pullback iff $K_f \cong K_h$ and $C_f \hookrightarrow C_h$, iff $K_g \cong K_k$ and $C_g \hookrightarrow C_k$,*

3. *the commutative square is a cartesian square iff* $K_f \cong K_h$ *and* $C_f \cong C_h$*, iff* $K_g \cong K_k$ *and* $C_g \cong C_k$*.*

Proof. The first "iff" of the first one is a consequence of lemmas 1.5.8 and 1.5.10. The first "iff" of the second one is a consequence of lemmas 1.5.8 and 1.5.9. In both cases, the second "iff" follows from the fact that the sequence

$$A \xrightarrow{\theta} B \oplus C \xrightarrow{\phi} D$$

is exact, or has injective or surjective maps, exactly when the sequence

$$A \to C \oplus B \to D$$

is exact, or has injective or surjective maps, and thus we can swap rows for columns in each of lemmas 1.5.8, 1.5.9 and 1.5.10. The third one is now clear. □

Corollary 1.5.12. *Consider the three squares*

$$S1: \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[h]{} & D \end{array} \qquad S2: \begin{array}{ccc} B & \xrightarrow{f'} & B' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k'} \\ D & \xrightarrow[h']{} & D' \end{array} \qquad S3: \begin{array}{ccc} A & \xrightarrow{f' \circ f} & B' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k'} \\ C & \xrightarrow[h' \circ h]{} & D' \end{array}$$

then

1. *if S1 and S2 are pushouts, so is S3,*

2. *if S1 and S2 are pullbacks, so is S3,*

3. *if S1 and S2 are cartesian squares, so is S3.*

Proof. By repeated applications of proposition 1.5.11, using the fact that the composition of two injective maps is injective, and so on. $\square$

We now turn back to our consideration of factorizability theories for $\otimes\mathbb{R}$. Our next step will be to consider the Heller sequences for $\otimes\mathbb{Q}$ and $\otimes\mathbb{R}$. We start by looking at the right-hand end of the sequences.

Lemma 1.5.13. *The homomorphisms*

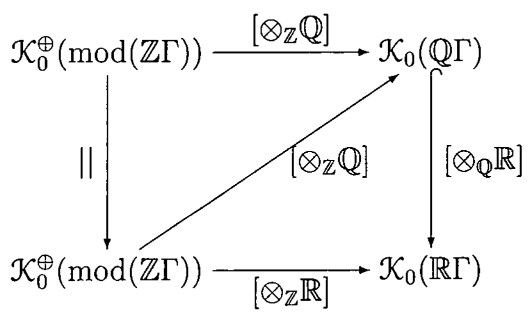
$$[\otimes_{\mathbb{Z}}\mathbb{Q}] \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \to \mathcal{K}_0(\mathbb{Q}\Gamma)$$

and

$$[\otimes_{\mathbb{Z}}\mathbb{R}] \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \to \mathcal{K}_0(\mathbb{R}\Gamma)$$

have the same kernel, $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma))$.

Proof. These homomorphisms can be linked by the following diagram, which is clearly commutative:



Therefore, since the map $\mathcal{K}_0(\mathbb{Q}\Gamma) \to \mathcal{K}_0(\mathbb{R}\Gamma)$ is injective, the horizontal maps must both have the same kernel, namely $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma))$. $\square$

We can similarly look at the left-hand end of the sequences.

Lemma 1.5.14. *The homomorphisms*

$$[\otimes_{\mathbb{Z}}\mathbb{Q}]\colon \mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \to \mathcal{K}_1(\mathbb{Q}\Gamma)$$

and

$$[\otimes_{\mathbb{Z}}\mathbb{R}]\colon \mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \to \mathcal{K}_1(\mathbb{R}\Gamma)$$

have the same kernel.

Proof. These homomorphisms can be linked by the following diagram, again clearly commutative:

$$\begin{array}{ccc}
\mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) & \xrightarrow{[\otimes_{\mathbb{Z}}\mathbb{Q}]} & \mathcal{K}_1(\mathbb{Q}\Gamma) \\
\Big\downarrow \| & \nearrow [\otimes_{\mathbb{Z}}\mathbb{Q}] & \Big\downarrow [\otimes_{\mathbb{Q}}\mathbb{R}] \\
\mathcal{K}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) & \xrightarrow[{[\otimes_{\mathbb{Z}}\mathbb{R}]}]{} & \mathcal{K}_1(\mathbb{R}\Gamma)
\end{array}$$

Again, since the map $\mathcal{K}_1(\mathbb{Q}\Gamma) \to \mathcal{K}_1(\mathbb{R}\Gamma)$ is injective, the horizontal maps must both have the same kernel. $\square$

We denote the kernel of either of these homomorphisms by $\tilde{\mathcal{K}}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma))$.

Putting these two lemmas together, and looking at the Heller sequences for $\otimes\mathbb{Q}$ and $\otimes\mathbb{R}$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
\tilde{\mathcal{K}}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) & \xhookrightarrow{[\otimes_{\mathbb{Z}}\mathbb{Q}]} & \mathcal{K}_1^{\oplus}(\mathbb{Q}\Gamma) & \xrightarrow{\tilde{\partial}^{\oplus}} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) & \xrightarrow{\delta^{\oplus}} & \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) \\
\Big\downarrow \| & & \Big\downarrow [\otimes_{\mathbb{Q}}\mathbb{R}] & & \Big\downarrow [\otimes_{\mathbb{Q}}\mathbb{R}] & & \Big\downarrow \| \\
\tilde{\mathcal{K}}_1^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma)) & \xhookrightarrow[{[\otimes_{\mathbb{Z}}\mathbb{R}]}]{} & \mathcal{K}_1^{\oplus}(\mathbb{R}\Gamma) & \xrightarrow[\tilde{\partial}^{\oplus}]{} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}) & \xrightarrow[\delta^{\oplus}]{} & \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma))
\end{array}$$

Lemma 1.5.15. *The square of definition 1.5.6 is cartesian, with the two vertical maps injective, i.e.,*

$$\begin{array}{ccc}
\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) & \xrightarrow{\psi} & \Psi \\
{\scriptstyle [\otimes\mathbb{R}]}\big\downarrow & & \big\downarrow{\scriptstyle [\otimes\mathbb{R}]} \\
\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}) & \xrightarrow[\psi_{\mathbb{R}}]{} & \Psi_{\mathbb{R}}
\end{array}$$

Proof. By proposition 1.5.11, part 3, we see that the commutative square

$$\begin{array}{ccc}
\mathcal{K}_1^{\oplus}(\mathbb{Q}\Gamma) & \xrightarrow{\tilde{\partial}^{\oplus}} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \\
{\scriptstyle [\otimes_{\mathbb{Q}}\mathbb{R}]}\big\downarrow & & \big\downarrow{\scriptstyle [\otimes_{\mathbb{Q}}\mathbb{R}]} \\
\mathcal{K}_1^{\oplus}(\mathbb{R}\Gamma) & \xrightarrow[\tilde{\partial}^{\oplus}]{} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R})
\end{array}$$

is cartesian, and therefore the homomorphism

$$[\otimes_{\mathbb{Q}}\mathbb{R}] \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \to \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R})$$

is injective. Therefore, since

$$\begin{array}{ccc}
\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) & \xrightarrow{\psi} & \Psi \\
{\scriptstyle [\otimes\mathbb{R}]}\big\downarrow & & \big\downarrow{\scriptstyle [\otimes\mathbb{R}]} \\
\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}) & \xrightarrow[\psi_{\mathbb{R}}]{} & \Psi_{\mathbb{R}}
\end{array}$$

is a pushout, and one of the vertical maps is injective, both the vertical maps must be injective. Finally, by proposition 1.5.11, since the kernels of the vertical maps are isomorphic (since they are trivial), and the cokernels are isomorphic (since the square is a pushout), the square is in fact cartesian. □

Proposition 1.5.16. *The homomorphisms*

$$[\otimes_{\mathbb{Q}}\mathbb{R}] \colon \mathcal{K}_1^{\oplus}(\mathbb{Q}\Gamma) \to \mathcal{K}_1^{\oplus}(\mathbb{R}\Gamma)$$

$$[\otimes_{\mathbb{Q}}\mathbb{R}] \colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) \to \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R})$$

$$[\otimes_{\mathbb{Q}}\mathbb{R}] \colon \Psi \to \Psi_{\mathbb{R}}$$

all have trivial kernels, and cokernels isomorphic to $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Q}\Gamma), \otimes_{\mathbb{Q}}\mathbb{R})$.

Proof. The three homomorphisms are all injective and therefore have trivial kernels. Consider the commutative diagram (with non-exact rows):

$$\begin{array}{ccccc}
\mathcal{K}_1(\mathbb{Q}\Gamma) & \xrightarrow{\tilde{\partial}^{\oplus}} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) & \xrightarrow{\psi} & \Psi \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{K}_1(\mathbb{R}\Gamma) & \xrightarrow[\tilde{\partial}^{\oplus}]{} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}) & \xrightarrow[\psi_{\mathbb{R}}]{} & \Psi_{\mathbb{R}}
\end{array}$$

Both the squares are cartesian, and therefore the three vertical maps all have isomorphic cokernels by proposition 1.5.11. We can say precisely what this cokernel is (up to isomorphism) from the Heller sequence for $\otimes_{\mathbb{Q}}\mathbb{R}$:

$$\mathcal{K}_1(\mathbb{Q}\Gamma) \hookrightarrow \mathcal{K}_1(\mathbb{R}\Gamma) \to \mathcal{K}_0(\mathrm{mod}(\mathbb{Q}\Gamma), \otimes_{\mathbb{Q}}\mathbb{R}) \to \mathcal{K}_0(\mathbb{Q}\Gamma) \hookrightarrow \mathcal{K}_0(\mathbb{R}\Gamma)$$

which, since the final map is injective, yields a short exact sequence

$$\mathcal{K}_1(\mathbb{Q}\Gamma) \hookrightarrow \mathcal{K}_1(\mathbb{R}\Gamma) \twoheadrightarrow \mathcal{K}_0(\mathrm{mod}(\mathbb{Q}\Gamma), \otimes_{\mathbb{Q}}\mathbb{R}).$$

Hence the three homomorphisms in question each have cokernel isomorphic to $\mathcal{K}_0(\mathrm{mod}(\mathbb{Q}\Gamma), \otimes_{\mathbb{Q}}\mathbb{R})$. □

The final comment we make on real factorizability theories relates to strictness.

Proposition 1.5.17. *If ψ is strict, then $\psi_{\mathbb{R}}$ is also strict.*

Proof. From the commutative diagram (with non-exact rows)

$$\begin{array}{ccccc}
\mathcal{K}_1(\mathbb{Q}\Gamma) & \xrightarrow{\tilde{\partial}^{\oplus}} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{Q}) & \xrightarrow{\psi} & \Psi \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_1(\mathbb{R}\Gamma) & \xrightarrow[\tilde{\partial}^{\oplus}]{} & \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}\Gamma), \otimes\mathbb{R}) & \xrightarrow[\psi_{\mathbb{R}}]{} & \Psi_{\mathbb{R}}
\end{array}$$

as both squares are cartesian, by corollary 1.5.12 the following square is also cartesian:

$$\begin{array}{ccc}
\mathcal{K}_1(\mathbb{Q}\Gamma) & \xrightarrow{\psi \circ \tilde{\partial}^{\oplus}} & \Psi \\
\downarrow & & \downarrow \\
\mathcal{K}_1(\mathbb{R}\Gamma) & \xrightarrow[\psi_{\mathbb{R}} \circ \tilde{\partial}^{\oplus}]{} & \Psi_{\mathbb{R}}
\end{array}$$

Therefore $\psi\circ\tilde{\partial}^{\oplus}$ and $\psi_{\mathbb{R}}\circ\tilde{\partial}^{\oplus}$ both have the same kernel, by proposition 1.5.11. Clearly the units in $\mathcal{K}_1^{\oplus}(\mathbb{Q}\Gamma)$ map into the units in $\mathcal{K}_1^{\oplus}(\mathbb{R}\Gamma)$. Therefore if ψ is strict then $\ker(\psi\circ\tilde{\partial}^{\oplus})$ is contained in the units of $\mathcal{K}_1^{\oplus}(\mathbb{Q}\Gamma)$ hence $\ker(\psi_{\mathbb{R}}\circ\tilde{\partial}^{\oplus})$ is contained in the units of $\mathcal{K}_1^{\oplus}(\mathbb{R}\Gamma)$ and $\psi_{\mathbb{R}}$ is strict. □

Chapter 2

The Invariance Factorizability Theories

IN THIS chapter we will define a large family of factorizability theories, namely the invariance factorizability theories. We will also show how one particular invariance factorizability theory is equivalent to the monomial factorizability theory, described in chapter 1.

In brief, the invariance factorizability theories involve calculating the value of various generalised indices corresponding to each of some collection of triples (Γ', χ, Δ), where χ is a character of $\Gamma' \subseteq \Gamma$ with kernel Δ. The different invariance factorizability theories originate from looking at different collections of triples (Γ', χ, Δ). The generalised indices calculated are as follows. For the class of a triple $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$, we firstly find the elements invariant under Δ, $[T^{\Delta}]$ (hence the name "invariance"), and then these fixed elements hit by various idempotents $[T^{\Delta}\tilde{e}_{\chi}]$. We take the generalised index of this triple $|T^{\Delta}\tilde{e}_{\chi}|$, and then the p-adic valuation of this number.

These invariance factorizability theories "nest" inside each other in terms of strength. We go on to prove that the weakest of the cases considered, the weak cyclic case, is strict. Consequently all the stronger invariance factorizability theories will also be strict. We establish that the monomial factorizability theory of section 1.4 is equivalent to one of these stronger invariance factorizability theories, and hence demonstrate its strictness.

Although $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$ is infinitely generated in general, for certain choices of Γ (such as those for which $\mathbb{Z}_p\Gamma$ has finite representation type) $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$ is of finite dimension. In this case, we can explicitly work out the kernel of an invariance factorizability theory. We go on to do this for the case of the metacyclic group $G_{q,p}$, and will build on this example in the next chapter when we will show that the invariance and coinvariance factorizability theories (to be defined in the next chapter) are different in general, but are identical in the weak cyclic case and equivalent in the strong cyclic case.

2.1 Invariance factorizability theories

In this section we define the invariance factorizability theories. As we mentioned before, the invariance factorizability theories involve calculating the p-adic valuation of various generalised indices, $v_p|T^{\Delta}\tilde{e}_{\chi}|$, corresponding to each of some collection of triples (Γ', χ, Δ), where χ is a character of $\Gamma' \subseteq \Gamma$

with kernel Δ.

We begin this section by establishing suitability conditions and notation for the triples (Γ', χ, Δ), and defining the idempotents $\tilde{e}_\chi$. We go on to consider some functorial aspects of the invariance factorizability theories. In particular we show how the invariance factorizability theories can be thought of as "factoring through" functors of the form $M \mapsto M^\Delta \tilde{e}_\chi$ (or more properly, homomorphisms between $\mathcal{K}$-groups induced from such functors). We also consider using $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$ in place of $\mathcal{K}_1(\mathrm{mod}(\mathbb{Q}_p\Gamma))$. These ideas will considerably simplify calculations involving invariance factorizability theories. We conclude this section by considering aspects of the relative strength of invariance factorizability theories.

We begin by establishing some notation.

Definitions 2.1.1. Consider a triple (Γ', χ, Δ) where Γ' is a subgroup of Γ and $\chi\colon \Gamma' \to \overline{\mathbb{Q}}_p^\times$ is a homomorphism with kernel Δ such that $p \nmid |\Gamma' : \Delta|$. We define uRep_p to be the set of all such triples. We define $\mathbb{Q}_p[\chi]$ to be the smallest subfield of $\overline{\mathbb{Q}}_p$ into which χ maps, that is, $\mathbb{Q}_p$ with the values of $\chi(\Gamma')$ added.

2.1.2. Remarks. The triple (Γ', χ, Δ) is completely determined by χ alone. However, keeping the Γ' and Δ avoids confusion, especially when there are a number of different χ's in play. Also, note that the requirement that $p \nmid |\Gamma' : \Delta|$ ensures that $\mathbb{Q}_p[\chi]$ is an unramified extension of $\mathbb{Q}_p$ and χ is thus an unramified representation of Γ', hence the name uRep_p.

Definitions 2.1.3. For a triple $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$, we define two idempotents in $\mathbb{Q}_p[\Gamma'/\Delta]$:

$$e_\chi = \frac{1}{|\Gamma'/\Delta|} \sum_{\bar{\gamma} \in \Gamma'/\Delta} \chi(\bar{\gamma})\bar{\gamma}^{-1}$$

and

$$\tilde{e}_\chi = \sum_{\chi' \in \mathrm{orb}(\chi)} e_{\chi'}$$

where $\mathrm{orb}(\chi)$ is the orbit of χ under the action of $\mathrm{Gal}(\mathbb{Q}_p[\chi]/\mathbb{Q}_p)$. We also define

$$\tilde{\chi} = \sum_{\chi' \in \mathrm{orb}(\chi)} \chi'.$$

Note that necessarily $\tilde{e}_\chi = e_{\tilde{\chi}}$.

For a triple $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$ and a right $\mathbb{Z}_p\Gamma$-module $M \in \mathrm{mod}(\mathbb{Z}_p\Gamma)$ we have that M^Δ is a $\mathbb{Z}_p[\Gamma'/\Delta]$-module, and therefore $M^\Delta\tilde{e}_\chi$ is a $\mathbb{Z}_p[\Gamma'/\Delta]$-module. For $[M, f, N] \in \mathcal{K}_0^\oplus(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$, we may therefore compute the generalised index $|M^\Delta\tilde{e}_\chi, f, N^\Delta\tilde{e}_\chi| \in <p>$, where in this triple f denotes the restriction of f to the module $M^\Delta\tilde{e}_\chi \otimes_\mathbb{Z} \mathbb{Q}$. We are now in a position to define:

Definition 2.1.4. Let $S \subseteq \mathrm{uRep}_p$. The *$S$-invariance factorizability theory* ψ_p^S is a local factorizability theory

$$\psi_p^S \colon \mathcal{K}_0^\oplus(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \mathrm{Map}(S, \mathbb{Z})$$
$$[M, f, N] \mapsto \,:\, (\Gamma', \chi, \Delta) \mapsto v_p(|M^\Delta\tilde{e}_\chi, f, N^\Delta\tilde{e}_\chi|).$$

If we choose a collection $\underline{S} = \{S_p : S_p \subseteq \mathrm{uRep}_p\}$ of local theories, then we may define a global theory $\psi^{\underline{S}}$, a locally defined factorizability theory with local theories $\psi_p^{S_p}$.

2.1.5. Remark. For $(\Gamma', \chi, \Delta) \in S$, consider the functor

$$\mathcal{F}_{(\Gamma',\chi,\Delta)} \colon \operatorname{mod}(\mathbb{Z}_p\Gamma) \to \operatorname{mod}(\mathbb{Z}_p)$$
$$M \mapsto M^{\Delta}\tilde{e}_{\chi}.$$

This induces a functor

$$\mathcal{F}'_{(\Gamma',\chi,\Delta)} \colon \operatorname{mod}(\mathbb{Q}_p\Gamma) \to \operatorname{mod}(\mathbb{Q}_p)$$
$$M \mapsto M^{\Delta}\tilde{e}_{\chi}.$$

By proposition 1.1.11, these functors in turn induce a homomorphism

$$[\mathcal{F}_{(\Gamma',\chi,\Delta)}] \colon \mathcal{K}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \mathcal{K}_0(\operatorname{mod}(\mathbb{Z}_p), \otimes\mathbb{Q}).$$

The map ψ_p^S can be viewed as this homomorphism composed with the generalised index and the p-adic valuation:

$$\psi_p^S([M, f, N]) \colon (\Gamma', \chi, \Delta) \to v_p|[\mathcal{F}_{(\Gamma',\chi,\Delta)}]([M, f, N])|.$$

In preparation for our discussion on the strictness of these theories, we observe

Lemma 2.1.6. *For* $\alpha \in \operatorname{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$*, and a triple* (Γ', χ, Δ) *lying in* $S \subseteq \mathrm{uRep}_p$*, we have*

$$(\psi_p^S \circ \tilde{\partial}_p^{\oplus})(\alpha) \colon (\Gamma', \chi, \Delta) \mapsto v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})).$$

Proof. Under the isomorphism

$$\operatorname{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times}) \cong \mathcal{K}_1(\mathbb{Q}_p\Gamma)$$

let α correspond to $[\beta\times, \mathbb{Q}_p\Gamma]$, where $\beta \in \mathbb{Q}_p\Gamma^\times$. Let $\mathcal{F} = \mathcal{F}_{(\Gamma',\chi,\Delta)}$ be as in remark 2.1.5. We shall compute $(\psi_p^S \circ \tilde{\partial}_p^\oplus)([\beta\times, \mathbb{Q}_p\Gamma])((\Gamma', \chi, \Delta))$.

$$\begin{aligned}(\psi_p^S \circ \tilde{\partial}_p^\oplus)([\beta\times, \mathbb{Q}_p\Gamma])((\Gamma', \chi, \Delta)) &= \psi_p^S([\mathbb{Z}_p\Gamma, \beta\times, \mathbb{Z}_p\Gamma]) \\ &= v_p\big|[\mathcal{F}]([\mathbb{Z}_p\Gamma, \beta\times, \mathbb{Z}_p\Gamma])\big|.\end{aligned}$$

Now,

$$\begin{aligned}[\mathcal{F}]([\mathbb{Z}_p\Gamma, \beta\times, \mathbb{Z}_p\Gamma]) &= [\mathbb{Z}_p\Gamma^\Delta\tilde{e}_\chi, \beta\times, \mathbb{Z}_p\Gamma^\Delta\tilde{e}_\chi] \\ &= \tilde{\partial}_p([\beta\times, \mathbb{Q}_p\Gamma^\Delta\tilde{e}_\chi]) \\ &= \tilde{\partial}_p \circ [\mathcal{F}]([\beta\times, \mathbb{Q}_p\Gamma])\end{aligned}$$

where

$$\tilde{\partial}_p\colon \mathcal{K}_1(\mathbb{Q}_p) \to \mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q})$$

is the usual map and, in the last equation,

$$[\mathcal{F}]\colon \mathcal{K}_1(\mathbb{Q}_p\Gamma) \to \mathcal{K}_1(\mathbb{Q}_p)$$

is the homomorphism induced from $\mathcal{F}$. Therefore the square

$$\begin{array}{ccc} \mathcal{K}_1(\mathbb{Q}_p\Gamma) & \xrightarrow{\tilde{\partial}_p^\oplus} & \mathcal{K}_0^\oplus(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \\ {\scriptstyle [\mathcal{F}]}\downarrow & & \downarrow{\scriptstyle [\mathcal{F}]} \\ \mathcal{K}_1(\mathbb{Q}_p) & \xrightarrow[\tilde{\partial}_p]{} & \mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q}) \end{array}$$

commutes.

Next consider the diagram

$$\begin{array}{ccc} \mathcal{K}_1(\mathbb{Q}_p) & \xrightarrow{\tilde{\partial}_p} & \mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q}) \\ \Big\downarrow \det & & \Big\downarrow v_p|\cdot| \\ \mathbb{Q}_p^\times & \xrightarrow[v_p]{} & \mathbb{Z} \end{array}$$

and consider $[\beta'\times, \mathbb{Q}_p] \in \mathcal{K}_1(\mathbb{Q}_p)$. Its image under $\tilde{\partial}_p$ is $[\mathbb{Z}_p, \beta'\times, \mathbb{Z}_p]$ which maps to $v_p|\mathbb{Z}_p, \beta'\times, \mathbb{Z}_p|$. But this equals $v_p(\det(\beta'\times))$, so this square commutes. From these diagrams, we see

$$\begin{aligned}(\psi_p^S \circ \tilde{\partial}_p^{\oplus})([\beta\times, \mathbb{Q}_p\Gamma])((\Gamma', \chi, \Delta)) &= v_p|[\mathcal{F}] \circ \tilde{\partial}_p^{\oplus}([\beta\times, \mathbb{Q}_p\Gamma])| \\ &= v_p|\det{}_{\mathbb{Q}_p}([\beta\times, \mathbb{Q}_p\Gamma^{\Delta}\tilde{e}_\chi])|.\end{aligned}$$

Now,

$$\mathbb{Q}_p\Gamma^{\Delta}\tilde{e}_\chi = \mathbb{Q}_p\Gamma/\Delta\tilde{e}_\chi = \mathbb{Q}_p\Gamma \otimes_{\mathbb{Q}_p\Gamma'} \mathbb{Q}_p\Gamma'/\Delta\tilde{e}_\chi.$$

$\mathbb{Q}_p\Gamma'/\Delta\tilde{e}_\chi$ has character $\tilde{\chi}$, since

$$\begin{aligned}\mathbb{Q}_p\Gamma'/\Delta\tilde{e}_\chi \otimes \overline{\mathbb{Q}}_p &= \overline{\mathbb{Q}}_p\Gamma'/\Delta\tilde{e}_\chi \qquad \text{as } \Gamma'/\Delta \text{ is abelian} \\ &= \bigoplus_{\chi' \in \mathrm{orb}(\chi)} \overline{\mathbb{Q}}_p\Gamma'/\Delta e_{\chi'}\end{aligned}$$

and $\overline{\mathbb{Q}}_p\Gamma'/\Delta e_{\chi'}$ has character χ'. So $\mathbb{Q}_p\Gamma^{\Delta}\tilde{e}_\chi$ has character $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\tilde{\chi})$. χ and $\tilde{\chi}$ are abelian characters and since the induced character of an abelian character can be written independently of the handedness of the module (see for example the formula in [14], page 686), $\mathrm{Ind}_{\Gamma'}^{\Gamma}(\tilde{\chi})$ equals $\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma}$. So

$$(\psi_p^S \circ \tilde{\partial}_p^{\oplus})(\alpha)((\Gamma', \chi, \Delta)) = v_p \det([\beta\times, V_{\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma}}]),$$

where $V_{\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma}}$ is a left module with character $\tilde{\chi}\big\uparrow_{\Gamma'}^{\Gamma}$. So

$$(\psi_p^S \circ \tilde{\partial}_p^{\oplus})(\alpha)((\Gamma', \chi, \Delta)) = v_p \mathrm{Det}_{\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma}}(\beta) = v_p(\alpha(\tilde{\chi}\big\uparrow_{\Gamma'}^{\Gamma}))$$

as required. □

Corollary 2.1.7. *For* $[\beta\times, \mathbb{Q}_p\Gamma] \in \mathcal{K}_1(\mathbb{Q}_p\Gamma)$*, and a triple* (Γ', χ, Δ) *lying in* $S \subseteq \mathrm{uRep}_p$*, we have*

$$(\psi_p^S \circ \tilde{\partial}_p^{\oplus})([\beta\times, \mathbb{Q}_p\Gamma])\colon ((\Gamma', \chi, \Delta)) \mapsto v_p \det{}_{\mathbb{Q}_p}([\beta\times, \mathbb{Q}_p\Gamma^{\Delta}\tilde{e}_{\chi}]).$$

One particularly useful aspect of these theories is how they relate to each other in terms of strength:

Proposition 2.1.8. *If* $S' \subseteq S \subseteq \mathrm{uRep}_p$ *then* $\psi_p^S \succcurlyeq \psi_p^{S'}$.

Proof. If

$$\psi_p^S([M, f, N])\colon (\Gamma', \chi, \Delta) \mapsto 0 \qquad \forall (\Gamma', \chi, \Delta) \in S,$$

then clearly

$$\psi_p^{S'}([M, f, N])\colon (\Gamma', \chi, \Delta) \mapsto 0 \qquad \forall (\Gamma', \chi, \Delta) \in S'.$$

Therefore $\ker(\psi_p^S) \subseteq \ker(\psi_p^{S'})$, and $\psi_p^S \succcurlyeq \psi_p^{S'}$. □

We define a few useful subsets of uRep_p.

Definitions 2.1.9. We define

$$\mathcal{C}_0 = \mathcal{C}_{0p} = \{(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p : \Gamma' \text{ is cyclic}\}.$$

We will refer to the factorizability theories obtained using this subset of uRep_p as the *weak cyclic* case.

We define

$$\mathcal{C} = \mathcal{C}_p = \{(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p : \Gamma' = C_{p^k} \rtimes G, \text{ where } k \geq 0 \text{ and } p \nmid |G|\}.$$

We will refer to the factorizability theories obtained using this subset of uRep_p as the *strong cyclic* case.

We define

$$\mathcal{M} = \mathcal{M}_p = \mathrm{uRep}_p.$$

We will refer to the factorizability theories obtained using this subset of uRep_p as the *monomial* case.

2.1.10. Remark. Since $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{M}$, we have that $\psi_p^{\mathcal{C}_0} \preccurlyeq \psi_p^{\mathcal{C}} \preccurlyeq \psi_p^{\mathcal{M}}$ and hence $\underline{\psi^{\mathcal{C}_0}} \preccurlyeq \underline{\psi^{\mathcal{C}}} \preccurlyeq \underline{\psi^{\mathcal{M}}}$.

2.2 Strictness of the weak cyclic case

In this section we shall prove that the weak cyclic factorizability theory $\psi_p^{\mathcal{C}_0}$ is strict. We will use the Hom-version of the group $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$, that is, the group $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$. Our approach will be as follows. By proposition 1.2.20 we know that $\psi_p^{\mathcal{C}_0}$ is strict if and only if

$$(\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus})(\alpha) = 0 \qquad \text{implies} \qquad v_p(\alpha(\chi)) = 0 \quad \forall \chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p},$$

where $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$. Suppose such an α is in the kernel of $\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus}$. Then obviously $(\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus})(\alpha) = 0$. By using lemma 2.1.6, this will allow us to establish that $v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})) = 0$ for the χ's in the $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$. We go on to demonstrate that if $v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})) = 0$ for these χ, then $v_p(\alpha(\chi)) = 0$ for all $\chi \in \mathrm{R}_{\Gamma,\mathbb{Q}_p}$, hence $v_p(\alpha(\chi)) = 0$ for all $\chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$, and hence $\psi_p^{\mathcal{C}_0}$ is strict.

Lemma 2.2.1. *Suppose* $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$, *and* $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$ *lies in the kernel of* $\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus}$. *Then* $v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})) = 0$.

Proof. Consider $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$. By lemma 2.1.6, we know that

$$(\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus})(\alpha) \colon (\Gamma', \chi, \Delta) \mapsto v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})).$$

So if α is in the kernel of $\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus}$ then $v_p(\alpha(\tilde{\chi}\uparrow_{\Gamma'}^{\Gamma})) = 0$. □

We introduce the following definition in order to simplify the notation of the following lemma.

Definition 2.2.2. Let e be an idempotent in $\mathbb{Q}_p[\Gamma/\Delta]$. We define $\phi(e)$ to be the character of $\mathbb{Q}_p\Gamma^{\Delta}e$. In particular, for a triple (Γ', χ, Δ), $\phi(\tilde{e}_\chi) = \tilde{\chi}\uparrow_{\Gamma'}^{\Gamma}$.

Lemma 2.2.3. *Let* Γ *be a cyclic group, and* χ *an irreducible character in* $\mathrm{R}_{\Gamma,\mathbb{Q}_p}$. *Then there exist integers* m_i *such that*

$$\chi = \sum_{(\Gamma_i', \chi_i, \Delta_i) \in \mathcal{C}_0} m_i \phi(\tilde{e}_{\chi_i}).$$

Proof. Let $|\Gamma| = p^r n$, where $p, n \in \mathbb{Z}$, $p \nmid n$. So

$$\Gamma = C_{p^r n} = C_{p^r} \times C_n.$$

Then

$$\mathbb{Q}_p\Gamma = \mathbb{Q}_p C_{p^r} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p C_n,$$

and we can write $e_\chi = e_p e_n$ with e_p, e_n indecomposable idempotents in $\mathbb{Q}_p C_{p^r}$, $\mathbb{Q}_p C_n$ respectively.

In fact, since $p \nmid n$, it follows that $e_n \in \mathbb{Z}_p C_n$. It also has image $e_n \Gamma'$ in $\mathbb{Z}_p[\Gamma/\Gamma']$ for all subgroups Γ' of Γ. Also, we have a character $\chi_n \colon C_n \to \mathbb{Z}_p$ such that $e_n = e_{\chi_n} = \tilde{e}_{\chi_n}$.

Now, there are $r+1$ possibilities for e_p. Either $e_p = e_{C_{p^s}} - e_{C_{p^{s+1}}}$ for $s = 0, 1, \ldots, r-1$, or $e_p = e_{C_{p^r}}$. Since χ is the character of $\mathbb{Q}_p \Gamma e_p e_n$, $\chi = \phi(e_p e_n)$. Hence χ equals either $\phi(e_{C_{p^s}} e_n) - \phi(e_{C_{p^{s+1}}} e_n)$ or $\phi(e_{C_{p^r}} e_n)$.

In conclusion we note that $\phi(e_{C_{p^s}} e_n) = \phi(e_n C_{p^s})$ where $e_n C_{p^s}$ is an idempotent in $\mathbb{Z}_p[\Gamma/C_{p^s}]$, and so $\phi(e_{C_{p^s}} e_n) = \phi(\tilde{e}_{\chi_N})$ where $\chi_N = \chi_n \big\uparrow_{C_n}^{C_{p^s} \times C_n}$ and $(C_{p^s} \times C_n, \chi_N, C_{p^s} \times \ker(\chi_n)) \in \mathcal{C}_0$. □

Corollary 2.2.4. *Let χ be an irreducible character in $\mathrm{R}_{\Gamma, \mathbb{Q}_p}$, with Γ not necessarily cyclic. Then there exist integers m_i, n with $n \neq 0$ such that*

$$n\chi = \sum_{(\Gamma'_i, \chi_i, \Delta_i) \in \mathcal{C}_0} m_i \tilde{\chi}_i \big\uparrow_{\Gamma'_i}^{\Gamma}.$$

Proof. By the Artin induction theorem, χ is a rational sum of characters induced from cyclic subgroups of Γ, and hence

$$n\chi = \sum_j a_j \xi_j \big\uparrow_{\Gamma'_j}^{\Gamma}$$

where ξ_j is an irreducible character of Γ'_j and $a_j \in \mathbb{Z}$. Now

$$\xi_j = \sum_{(\Gamma'_{i,j}, \chi_{i,j}, \Delta_{i,j}) \in \mathcal{C}_0} m_{i,j} \phi(\tilde{e}_{\chi_{i,j}}),$$

and $\phi(\tilde{e}_{\chi_{i,j}}) = \tilde{\chi}_{i,j} \big\uparrow_{\Gamma'_{i,j}}^{\Gamma'_j}$. So

$$\xi_j \big\uparrow_{\Gamma'_j}^{\Gamma} = \sum_{(\Gamma'_{i,j}, \chi_{i,j}, \Delta_{i,j}) \in \mathcal{C}_0} m_{i,j} \tilde{\chi}_{i,j} \big\uparrow_{\Gamma'_{i,j}}^{\Gamma'_j} \big\uparrow_{\Gamma'_j}^{\Gamma} = \sum_{(\Gamma'_{i,j}, \chi_{i,j}, \Delta_{i,j}) \in \mathcal{C}_0} m_{i,j} \tilde{\chi}_{i,j} \big\uparrow_{\Gamma'_{i,j}}^{\Gamma}$$

and the result follows. □

Lemma 2.2.5. *Let χ be an irreducible character in $\mathrm{R}_{\Gamma,\mathbb{Q}_p}$, and let α lie in the kernel of $\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus}$. Then $v_p(\alpha(\chi)) = 0$.*

Proof. We have that

$$n\chi = \sum_{(\Gamma_i', \chi_i, \Delta_i) \in \mathcal{C}_0} m_i \tilde{\chi}_i \uparrow_{\Gamma_i'}^{\Gamma}$$

and therefore

$$nv_p(\alpha(\chi)) = \sum_{(\Gamma_i', \chi_i, \Delta_i) \in \mathcal{C}_0} m_i v_p(\alpha(\tilde{\chi}_i \uparrow_{\Gamma_i'}^{\Gamma})).$$

But we know that if $\alpha \in \ker(\psi_p^{\mathcal{C}_0} \circ \tilde{\partial}_p^{\oplus})$ then $v_p(\alpha(\tilde{\chi}_i \uparrow_{\Gamma_i'}^{\Gamma})) = 0$. Therefore $v_p(\alpha(\chi)) = 0$. □

Lemma 2.2.6. *Let $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$ and let $\chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$. Suppose $\chi_1, \ldots, \chi_n$ are the distinct characters in the orbit of χ under the action of $\Omega_{\mathbb{Q}_p}$, and let $\phi = \sum_{i=1}^{n} \chi_i$. Then $nv_p(\alpha(\chi)) = v_p(\alpha(\phi))$.*

Proof.

$$v_p(\alpha(\phi)) = v_p(\alpha(\sum_{i=1}^{n} \chi_i)) = \sum_{i=1}^{n} v_p(\alpha(\chi_i)).$$

Now, $\chi_i = \chi^{\omega_i}$ for some $\omega_i \in \Omega_{\mathbb{Q}_p}$, and $\alpha(\chi_i) = \alpha(\chi^{\omega_i}) = \alpha(\chi)^{\omega_i}$. Therefore, as α is invariant under the action of $\Omega_{\mathbb{Q}_p}$, $v_p(\alpha(\chi_i)) = v_p(\alpha(\chi))$ for all i and $nv_p(\alpha(\chi)) = v_p(\alpha(\phi))$. □

2.2.7. Remark. ϕ above is clearly a $\mathbb{Q}_p$-valued character, since its values are invariant under $\Omega_{\mathbb{Q}_p}$. Therefore some non-zero rational multiple of ϕ must be a $\mathbb{Q}_p$-character.

Theorem 2.2.8. *Let χ be an irreducible character in $\mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$, and let α lie in the kernel of $\psi_p^{\mathcal{C}o} \circ \tilde{\partial}_p^{\oplus}$. Then $v_p(\alpha(\chi)) = 0$.*

Proof. By lemma 2.2.6 we know $v_p(\alpha(\chi)) = nv_p(\alpha(\phi))$ for $n \in \mathbb{Z}$ and ϕ some $\mathbb{Q}_p$-valued character of Γ. Hence using the above remark, we know $v_p(\alpha(\chi)) = nv_p(\alpha(\phi))$ for $n \in \mathbb{Q}$ and ϕ some $\mathbb{Q}_p$-character of Γ. But we have $\phi = \sum_i \phi_i$ for ϕ_i irreducible $\mathbb{Q}_p$-characters of Γ, with $v_p(\phi_i) = 0$ by lemma 2.2.5. Hence $v_p(\alpha(\chi)) = n\sum_i v_p(\alpha(\phi_i)) = 0$. □

Thus we have proved

Theorem 2.2.9. $\psi_p^{\mathcal{C}o}$ *is strict.*

Proof. We have shown that if α lies in the kernel of $\psi_p^{\mathcal{C}o} \circ \tilde{\partial}_p^{\oplus}$ then $v_p(\alpha(\chi)) = 0$ for all $\chi \in \mathrm{R}_{\Gamma,\overline{\mathbb{Q}}_p}$, thus by proposition 1.2.20 the theorem is proved. □

2.3 Equivalence of monomial factorizability to $\psi_p^{\mathcal{M}}$

In this section we will establish the equivalence between the monomial factorizability theory $\psi_{\mathrm{mon},p}$ of section 1.4 and the monomial invariance factorizability theory $\psi_p^{\mathcal{M}}$. In [19], $\psi_{\mathrm{mon},p}$ was shown to be a strict theory. We include here an alternative proof as a corollary to the equivalence between $\psi_{\mathrm{mon},p}$ and $\psi_p^{\mathcal{M}}$.

We will use the notation of section 1.4, and the construction of monomial factorizability as set out in that section.

As noted in section 1.4, $\mathrm{Mon}(\Gamma, \mu^{(p)})$ is generated by the classes $[\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}]$ and $\psi_{\mathrm{mon},p}$ is determined by its action on these representations, namely

$$\psi_{\mathrm{mon},p}([T])\colon [\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}] \mapsto \left|(T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\chi}\right|_{\mathcal{O}\{p\}}.$$

Lemma 2.3.1. *Let* $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$. *We have a 1 to 1 correspondence between the* $[\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}]$ *generating* $\mathrm{Mon}(\Gamma, \mu^{(p)})$ *and* $(\Gamma', \chi, \Delta) \in \mathcal{M}$. *Under this correspondence,*

$$\left|(T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\chi}\right|_{\mathcal{O}\{p\}} = 1 \quad \Leftrightarrow \quad v_p\left|(T \otimes \mathcal{O}\{p\})^{\Delta} e_\chi\right|_{\mathcal{O}\{p\}} = 0.$$

Proof. Consider the triple $(\Gamma', \chi, \Delta) \in \mathcal{M}$. This χ is clearly a homomorphism $\chi\colon \Gamma' \to \mu^{(p)}$ and thus there exists a $[\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}]$ corresponding to each (Γ', χ, Δ) in $\mathcal{M}$. Conversely, for each $[\mu_\chi^{(p)}\uparrow_{\Gamma'}^{\Gamma}]$ (the classes of which we know generate $\mathrm{Mon}(\Gamma, \mu^{(p)})$) there exists (Γ', χ, Δ) in $\mathcal{M}$, since the order of χ must be prime to p.

Under the equivalence above, clearly

$$\left|(T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\chi}\right|_{\mathcal{O}\{p\}} = \left|(T \otimes \mathcal{O}\{p\})^{\Delta} e_\chi\right|_{\mathcal{O}\{p\}}.$$

It is also clear that

$$\left|(T \otimes \mathcal{O}\{p\})^{\Delta} e_\chi\right|_{\mathcal{O}\{p\}} = 1 \quad \Leftrightarrow \quad v_p\left|(T \otimes \mathcal{O}\{p\})^{\Delta} e_\chi\right|_{\mathcal{O}\{p\}} = 0.$$

Thus the lemma is proved. □

Lemma 2.3.2. *Consider* $(\Gamma', \chi, \Delta) \in \mathcal{M}$. *Suppose* χ' *lies in the orbit of* χ *under the action of* $\mathrm{Gal}(\mathbb{Q}_p[\chi]/\mathbb{Q}_p)$. *Then*

$$|(T \otimes \mathcal{O}\{p\})^{\Delta} e'_\chi| = |(T \otimes \mathcal{O}\{p\})^{\Delta} e_\chi|.$$

Proof. Suppose for concreteness that $\chi' = \chi^{\omega}$, where $\omega \in \mathrm{Gal}(\mathbb{Q}_p[\chi]/\mathbb{Q}_p)$. Thus ω acts on $\mathcal{O}\{p\}$. Write T' for $(T \otimes \mathcal{O}\{p\})^{\Delta}$, and write $(M', f', N') = T'$. Thus M' and N' are $\mathcal{O}\{p\}\Gamma'/\Delta$-modules and ω acts via the $\mathcal{O}\{p\}$. Clearly $M' \cong (M')^{\omega}$ and $N' \cong (N')^{\omega}$, hence $[T'] = [(T')^{\omega}]$. Now,

$$(e_{\chi})^{\omega} = \frac{1}{|\Gamma'/\Delta|} \sum_{\gamma \in \Gamma'/\Delta} \chi(\gamma)^{\omega} \gamma^{-1} = e_{\chi'}.$$

Therefore

$$[T' e_{\chi}] = [(T' e_{\chi})^{\omega}] = [(T')^{\omega} e_{\chi'}] = [T' e_{\chi'}].$$

Therefore

$$v_p |T' e_{\chi'}| = v_p |T' e_{\chi}|$$

as required. □

Lemma 2.3.3. *Consider* $(\Gamma', \chi, \Delta) \in \mathcal{M}$. *Then*

$$v_p \big| (T \otimes \mathcal{O}\{p\})^{\Delta} e_{\chi} \big|_{\mathcal{O}\{p\}} = 0 \quad \Leftrightarrow \quad v_p |T^{\Delta} \tilde{e}_{\chi}| = 0.$$

Proof.

$$\begin{aligned} v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} \tilde{e}_{\chi} \big|_{\mathcal{O}\{p\}} &= \sum_{\chi' \in \mathrm{orb}(\chi)} v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} e_{\chi'} \big|_{\mathcal{O}\{p\}} \\ &= |\mathrm{orb}(\chi)| v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} e_{\chi} \big|_{\mathcal{O}\{p\}} \end{aligned}$$

by lemma 2.3.2, where $\mathrm{orb}(\chi)$ is the orbit of χ under $\mathrm{Gal}(\mathbb{Q}_p[\chi]/\mathbb{Q}_p)$. Therefore

$$\begin{aligned} & v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} \tilde{e}_{\chi} \big|_{\mathcal{O}\{p\}} = 0 \\ \Leftrightarrow \quad & |\mathrm{orb}(\chi)| v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} e_{\chi} \big|_{\mathcal{O}\{p\}} = 0 \\ \Leftrightarrow \quad & v_p \big| (T \otimes_{\mathbb{Z}_p} \mathcal{O}\{p\})^{\Delta} e_{\chi} \big|_{\mathcal{O}\{p\}} = 0. \end{aligned}$$

The lemma then follows from the fact that the functor

$$\otimes\mathcal{O}\{p\}\colon \operatorname{mod}(\mathbb{Z}_p\Gamma) \to \operatorname{mod}(\mathcal{O}\{p\}\Gamma)$$

is flat. □

Therefore we have proved

Theorem 2.3.4.

$$\psi_p^{\mathcal{M}} \sim \psi_{\mathrm{mon},p}.$$

2.3.5. Remark. As we remarked before, $\psi_{\mathrm{mon},p}$ is known to be strict; for a direct proof of this, see [19], sections 7 and 8. However, the strictness of $\psi_{\mathrm{mon},p}$ also follows as a corollary to this theorem.

Corollary 2.3.6. *$\psi_{\mathrm{mon},p}$ is strict.*

Proof. $\psi_p^{\mathcal{M}} \succcurlyeq \psi_p^{\mathcal{C}_0}$ by proposition 2.1.8, and $\psi_p^{\mathcal{C}_0}$ is strict. The corollary follows. □

2.4 The metacyclic group $G_{q,p}$

We know that $\psi_p^{\mathcal{C}}$ is strict by proposition 2.1.8 and theorem 2.2.9. In this section we will explicitly calculate the values of $\psi_p^{\mathcal{C}}$ for $\mathcal{K}_0^{\oplus}(\operatorname{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$, and thus explicitly identify the kernel. Here, $G_{q,p}$ denotes the metacyclic group $C_p \rtimes C_q$, with p, q prime numbers and $q \mid p-1$. In the next chapter we will define the coinvariance factorizability theories. In that chapter, we will calculate the values of the corresponding coinvariance factorizability theory, and thus explicitly identify its kernel. Using the results of this section we will then be able to prove that the invariance and coinvariance factorizability theories take different values, but are still equivalent when restricted to $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$.

Our approach in this section will be to begin by identifying a base for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$, by identifying a full set of nonisomorphic irreducible $\mathbb{Z}_pG_{q,p}$-lattices and using the exactness at $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$ of the Heller sequence of proposition 1.1.8. We will then identify the triples in $\mathcal{C}$. Thus by calculating $v_p(|T^{\Delta}\tilde{e}_{\chi}|)$ for each $(\Gamma', \chi, \Delta) \in \mathcal{C}$ and each $[T]$ in our generating set for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$, we will have expressed $\psi_p^{\mathcal{C}}$ as a linear map. Identifying the kernel of this map is then a matter of row reducing the matrix of this linear transformation.

We begin by establishing some notation.

Notation 2.4.1. p and q will denote fixed rational primes, with $q \mid p-1$. Consider the cyclic groups $C_p = <\sigma>$ and $C_q = <\tau>$, of orders p and q respectively. θ will denote a primitive qth root of 1 in $\mathbb{Z}_p$. r will denote a qth root of 1 in $(\mathbb{Z}_p/p\mathbb{Z}_p)^{\times}$, with $\theta \equiv r \pmod{p}$. We define the metacyclic group $G_{q,p} = C_p \rtimes C_q$, where C_q acts on C_p as $\sigma^{\tau} = \sigma^{r}$.

$\mathbb{Z}_p^{(i)}$ will denote a copy of $\mathbb{Z}_p$ on which τ acts as multiplication by θ^i. In general, we will denote $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)}$ by $M^{(i)}$ for any $\mathbb{Z}_pG_{q,p}$-module M. $\zeta = \zeta_p$ will denote a primitive pth root of 1. v_p will denote the usual p-adic valuation. R will denote the ring $\mathbb{Z}_p[\zeta]$, with field of fractions K. P

will denote the unique maximal ideal $(1-\zeta)R$. ϵ will denote the usual augmentation map from $\mathbb{Z}_p C_p$ to $\mathbb{Z}_p$; the corresponding maps from $\mathbb{Z}_p^{(i)} C_p$ to $\mathbb{Z}_p^{(i)}$ we will denote by $\epsilon^{(i)}$. π will denote the usual projection map from $\mathbb{Z}_p C_p$ to R, where $\pi(\sigma) = \zeta$; the corresponding maps from $\mathbb{Z}_p^{(i)} C_p$ to $R^{(i)}$ we will denote by $\pi^{(i)}$.

Note that $G_{2,p} = D_p$, the usual dihedral group of order $2p$.

$G_{q,p}$ is known to be of finite representation type, since we may write down a list of the distinct irreducible $\mathbb{Z}_p G_{q,p}$-lattices (upon which $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ is freely generated). For one method of obtaining these lattices, see [4], p.750.

Lemma 2.4.2. *There are $3q$ distinct irreducible $\mathbb{Z}_p G_{q,p}$-lattices. They are:*

$$\mathbb{Z}_p^{(i)}, \qquad i = 0, \ldots, q-1$$

which have character χ_i, with $\chi_i(\sigma) = 1$ and $\chi_i(\tau) = \theta^i$;

$$P^i, \qquad i = 0, \ldots, q-1$$

which all have the same q dimensional character χ where

$$\chi(\sigma) = \mathrm{diag}(\zeta, \zeta^r, \ldots, \zeta^{r^{q-1}}), \qquad \chi(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix};$$

and

$$V_i = \mathbb{Z}_p^{(i)} C_p, \qquad i = 0, \ldots, q-1$$

with character $\chi + \chi_i$.

Proof. See [4], p.750 and [3], p.335. □

We observe at this point that

Lemma 2.4.3. *$R^{(i)} \cong P^i$ as $\mathbb{Z}_p G_{q,p}$-modules, the isomorphism being given by $x \mapsto (\sum_{j=0}^{q-1} \theta^{-j}\zeta^{r^j})^i x$.*

Proof. It is sufficient to show that $R^{(1)} \cong P$; the result will then follow from the fact that $(R^{(i)})^{(j)} = R^{(i+j)}$. Consider

$$\alpha\colon R^{(1)} \to P$$
$$x \mapsto (\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})x.$$

We shall demonstrate that α is a $\mathbb{Z}_p$-isomorphism of modules, and that it respects the actions of σ and τ.

α is clearly a homomorphism. For the first part, it will therefore suffice to show that $(\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})/(\zeta - 1)$ is a unit in R, since then we will have $(\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})R = (\zeta - 1)R = P$. But

$$\begin{aligned}\frac{\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i}}{\zeta - 1} &= \frac{\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i} - \sum_{i=0}^{q-1} \theta^{-i}}{\zeta - 1} \\ &= \sum_{i=0}^{q-1} \theta^{-i}\frac{\zeta^{r^i} - 1}{\zeta - 1} \\ &= \sum_{i=0}^{q-1} (\theta^{-i} \sum_{j=0}^{r^i - 1} \zeta^j).\end{aligned}$$

Now, this is a unit in $\mathbb{Z}_p[\zeta]$ if and only if its image is a unit in $\mathbb{Z}_p[\zeta]/(1 - \zeta)$. So, reducing our expression modulo $1 - \zeta$, we obtain

$$\sum_{i=0}^{q-1} (\theta^{-i} \sum_{j=0}^{r^i - 1} \zeta^j) \equiv \sum_{i=0}^{q-1} r^{-i} r^i = q \pmod{1 - \zeta}$$

which is invertible as $q \mid p-1$.

For the action of σ, take $x = a\zeta^j \in R^{(1)}$, $a \in \mathbb{Z}_p^{(1)}$. We need to show that $\alpha(x^\sigma) = \alpha(x)^\sigma$. Now $x^\sigma = a\zeta^{j+1}$, thus

$$\alpha(x^\sigma) = (\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})a\zeta^{j+1} = a\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i+j+1}.$$

Now

$$\alpha(x) = (\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})a\zeta^{j} = a\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i+j},$$

so

$$\alpha(x)^\sigma = a\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i+j+1} = \alpha(x^\sigma)$$

as required.

For the action of τ, again take $x = a\zeta^j$, $a \in \mathbb{Z}_p^{(1)}$. Then $x^\tau = a\theta\zeta^{rj}$, thus

$$\alpha(x^\tau) = (\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})a\theta\zeta^{rj} = a\sum_{i=0}^{q-1} \theta^{1-i}\zeta^{r^i+rj}.$$

Now

$$\alpha(x) = (\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i})a\zeta^{j} = a\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^i+j},$$

so

$$\begin{aligned}
\alpha(x)^\tau &= a\sum_{i=0}^{q-1} \theta^{-i}\zeta^{r^{i+1}+rj} \\
&= a\sum_{i=1}^{q} \theta^{1-i}\zeta^{r^i+rj} \\
&= a\sum_{i=0}^{q-1} \theta^{1-i}\zeta^{r^i+rj} = \alpha(x^\tau)
\end{aligned}$$

as required. □

$\mathcal{K}_0(\mathbb{Q}_pG_{q,p})$ is isomorphic to the $\mathbb{Q}_p$-character ring of $G_{q,p}$, with the isomorphism sending the class of a $\mathbb{Q}_pG_{q,p}$-module to its character. Thus it is clear that the image of $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$ in $\mathcal{K}_0(\mathbb{Q}_pG_{q,p})$ is generated by χ_i, $i = 0, \ldots, q-1$, and χ, that is, $q+1$ distinct characters.

Lemma 2.4.4. *$\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$ has rank $2q-1$ and is freely generated by*

$$[P^i] + [\mathbb{Z}_p^{(i)}] - [V_i], \quad i = 0, \ldots, q-1$$

and

$$[P^i] - [P^{i+1}], \quad i = 1, \ldots, q-1.$$

Proof. $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$ has rank $3q$, and its image in $\mathcal{K}_0(\mathbb{Q}_pG_{q,p})$ has rank $q+1$, therefore $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$ has rank $3q-(q+1) = 2q-1$. By comparing characters we see that $[P^i]+[\mathbb{Z}_p^{(i)}]-[V_i]$, $i = 0, \ldots, q-1$ lie in $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$, as do $[P^i] - [P^{i+1}]$, $i = 1, \ldots, q-1$.

By including $[P]$, and $[\mathbb{Z}_p^{(i)}]$, $i = 0, \ldots, q-1$ we can expand this set to get a base for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$—to verify this, we observe that each of the usual generators is an integer combination of these elements: we have $[P^2] = [P] - ([P] - [P^2])$, and thus $[P^{i+1}] = [P^i] - ([P^i] - [P^{i+1}])$; $[R] = [P^q]$; and $[V_i] = [P^i] + [\mathbb{Z}_p^{(i)}] - ([P^i] + [\mathbb{Z}_p^{(i)}] - [V_i])$.

Therefore the $2q-1$ elements listed are linearly independent and hence freely generate a subgroup of finite index in $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$. However, since we can extend this set of $2q-1$ elements to a base for the whole of $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}))$, the subgroup they generate must be of index 1. □

We have an exact sequence, the Heller sequence

$$\mathcal{K}_1^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p})) \xrightarrow{[\otimes\mathbb{Q}]} \mathcal{K}_1(\mathbb{Q}_p G_{q,p})$$
$$\xrightarrow{\partial} \mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q}) \xrightarrow{\delta} \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p})).$$

Since $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ is free, this sequence splits at $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$, that is,

$$\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q}) \cong \mathrm{Im}(\mathcal{K}_1(\mathbb{Q}_p G_{q,p})) \oplus \tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p})).$$

In order to identify a base for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$, our approach will be as follows. We will identify a generating set for each of $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ and $\mathrm{Im}(\mathcal{K}_1(\mathbb{Q}_p G_{q,p}))$. We will then identify the image of our generating set for $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ under some splitting map, and adding our generating set for $\mathrm{Im}(\mathcal{K}_1(\mathbb{Q}_p G_{q,p}))$ will give us a generating set for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$. By computing the image under $\psi_p^{\mathcal{C}}$ of each element of this generating set we may identify the kernel of $\psi_p^{\mathcal{C}}$.

$\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ is easy:

Lemma 2.4.5. *Our desired subgroup of $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ which is isomorphic to $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}))$ is generated by the triples*

$$[P^{i+1}, 1, P^i], \quad i = 1, \ldots, q-1,$$

and

$$[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}], \quad i = 0, \ldots, q-1.$$

Proof. We observe that $[P^{i+1}, 1, P^i]$ is a preimage of $[P^i] - [P^{i+1}]$; similarly $[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$ is a preimage of $[R^{(i)}] + [\mathbb{Z}_p^{(i)}] - [V_i]$, and by lemma 2.4.3, $[R^{(i)}] = [P^i]$. □

Note that our choice of elements of $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ is not unique—our choice here is motivated by convenience since this set of elements simplifies the following calculations. In particular we could have avoided mixing P^i's and $R^{(i)}$'s, for example by using $[V_i, ((\sum_{j=0}^{q-1} \theta^{-j}\zeta^{r^j})^i \circ \pi, \epsilon^{(i)}), P^i \oplus \mathbb{Z}_p^{(i)}]$ in place of $[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$; again, this choice is based upon convenience. To simplify the calculations of $\psi_p^{\mathcal{C}}$, we observe the following two lemmas:

Lemma 2.4.6. *The sequence*

$$0 \to P^{i+1} \to P^i \to \mathbb{F}_p^{(i)} \to 0$$

is exact.

Proof.

$$0 \to P \to R \to \mathbb{F}_p \to 0$$

is clearly exact, and $\otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)}$ is flat. Therefore

$$0 \to P^{(i)} \to R^{(i)} \to \mathbb{F}_p^{(i)} \to 0$$

is exact, and the result follows from lemma 2.4.3. □

Lemma 2.4.7.

$$\begin{array}{ccc} V_i & \xrightarrow{\epsilon^{(i)}} & \mathbb{Z}_p^{(i)} \\ {\scriptstyle \pi^{(i)}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ R^{(i)} & \xrightarrow[\pi_2]{} & \mathbb{F}_p^{(i)} \end{array}$$

is a cartesian square, where π_1 is the projection

$$\pi_1 \colon \mathbb{Z}_p^{(i)} \to \mathbb{Z}_p^{(i)}/p\mathbb{Z}_p^{(i)} \cong \mathbb{F}_p^{(i)}$$

$$x \mapsto x \pmod p$$

and π_2 is the projection

$$\pi_2 \colon R^{(i)} \to R^{(i)}/P^{(i)} \cong \mathbb{F}_p^{(i)}$$

$$x\zeta^j \mapsto x \pmod p.$$

Proof. Since $\otimes_{\mathbb{Z}_p}\mathbb{Z}_p^{(i)}$ is flat, it is sufficient to show that

$$\begin{array}{ccc} V_0 & \xrightarrow{\epsilon} & \mathbb{Z}_p \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ R & \xrightarrow[\pi_2]{} & \mathbb{F}_p \end{array}$$

is a cartesian square. This will follow from considering whether

$$0 \to V_0 \xrightarrow{(\epsilon,\pi)} \mathbb{Z}_p \oplus R \xrightarrow{\alpha} \mathbb{F}_p \to 0$$

is exact, where $\alpha = (-\pi_1, \pi_2)$, that is,

$$\alpha(-a_0, \sum_{i=1}^{p-1} a_i\zeta^i) = \sum_{i=0}^{p-1} a_i \pmod p.$$

Clearly (ϵ, π) is injective, α is surjective and $\alpha \circ (\epsilon, \pi) = 0$. It therefore only remains to show that $\ker\alpha \subseteq \operatorname{Im}(\epsilon, \pi)$. But if $\alpha(a_0, \sum_{i=1}^{p-1} a_i\zeta^i) = 0$ then $a_0 \equiv \sum_{i=1}^{p-1} a_i \pmod p$, that is,

$$a_0 = \sum_{i=1}^{p-1} a_i + pc \text{ for some } c \in \mathbb{Z}_p.$$

Consider $\sum_{i=0}^{p-1} b_i\sigma^i \in V_0$, where $b_0 = c$ and $b_i = a_i + c$, $i = 1, \ldots, p-1$. Then

$$(\epsilon, \pi)(\sum_{i=0}^{p-1} b_i\sigma^i) = (c + \sum_{i=1}^{p-1}(a_i + c), c + \sum_{i=1}^{p-1}(a_i + c)\zeta^i) = (a_0, \sum_{i=1}^{p-1} a_i\zeta^i)$$

as required. □

We now turn our attention to $\mathrm{Im}(\mathcal{K}_1(\mathbb{Q}_pG_{q,p}))$. It will help to think of $\mathcal{K}_1(\mathbb{Q}_pG_{q,p})$ as the isomorphic group $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^\times)$. We begin by identifying a generating set for the representation ring $\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}$.

Definition 2.4.8. We define the following representations of $G_{q,p}$. For $i = 0, 1, \ldots, q-1$, we define

$$\phi_i \colon G_{q,p} \to \mathbb{Z}_p^\times$$
$$\sigma \mapsto 1$$
$$\tau \mapsto \theta^i,$$

and for $i = 1, 2, \ldots, q-1$, we define

$$\rho_i \colon \sigma \mapsto \begin{pmatrix} \zeta^i & 0 & \cdots & 0 \\ 0 & \zeta^{ir} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^{ir^{q-1}} \end{pmatrix},$$

$$\tau \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 2.4.9. *Let ρ_i have character χ_i. Then*

$$\chi_i \colon \tau^k \sigma^j \mapsto 0 \quad \textit{if } q \nmid k,$$
$$\sigma^j \mapsto \sum_{k=0}^{q-1} (\zeta^{ij})^{r^k}$$

Also, $\mathrm{Gal}(R/\mathbb{Q}_p)$ *acts transitively on the set* $\{\chi_i \colon i = 1, \dots, q-1\}$.

Proof. Clearly if $q \nmid k$

$$\chi_i(\tau^k \sigma^j) = \mathrm{Tr}(\rho_i(\tau^k \sigma^j)) = 0,$$

and

$$\chi_i(\sigma^j) = \mathrm{Tr}(\rho_i(\sigma^j)) = \sum_{k=0}^{q-1} \zeta^{ir^k j} = \sum_{k=0}^{q-1} (\zeta^{ij})^{r^k}.$$

For $p \nmid ij$, ζ^{ij} is a primitive pth root of 1. $\mathrm{Gal}(R/\mathbb{Q}_p)$ acts transitively on the set $\{\zeta^k : k = 1, 2, \dots, p-1\}$. Suppose then that $\omega \in \mathrm{Gal}(R/\mathbb{Q}_p)$ is such that $(\zeta^i)^\omega = \zeta^{i'}$. Then

$$\chi_i^\omega \colon \sigma^j \mapsto \chi_i(\sigma^j)^\omega = (\sum_{k=0}^{q-1} (\zeta^{ij})^{r^k})^\omega = \sum_{k=0}^{q-1} (\zeta^{i'j})^{r^k},$$

and

$$\chi_i^\omega \colon \sigma^j \tau^k \mapsto \chi_i(\sigma^j \tau^k)^\omega = 0 \text{ if } q \nmid k,$$

therefore $\chi_i^\omega = \chi_{i'}$ and as required $\mathrm{Gal}(R/\mathbb{Q}_p)$ acts transitively on the set $\{\chi_i \colon i = 1, \dots, q-1\}$. □

We know by [3], corollary 47.14 that each of the irreducible matrix representations of $G_{q,p}$ is either one of the ϕ_i, $i = 0, 1, \dots, q-1$, or one of the ρ_i, $i = 1, \dots, q-1$. Hence each of the irreducible characters of $G_{q,p}$ is either one of the ϕ_i, $i = 0, 1, \dots, q-1$, or one of the χ_i, $i = 1, \dots, q-1$. We now determine when χ_i and $\chi_{i'}$ (and hence ρ_i and $\rho_{i'}$) are equivalent.

Lemma 2.4.10. *χ_i and $\chi_{i'}$ are equivalent if and only if $i^{-1}i' \in <r>$.*

Proof. χ_i and $\chi_{i'}$ are equivalent if and only if there exists $g \in G_{q,p}$ such that $\chi_i(\tau^k\sigma^j) = \chi_{i'}(g^{-1}\tau^k\sigma^j g)$ for all j and k. Suppose $g = \sigma^l\tau^m$. Then $g^{-1} = \tau^{-m}\sigma^{-l}$ and $g^{-1}\tau^k\sigma^j g = \tau^k\sigma^{-lr^{k+m}+(j+l)r^m}$. Therefore

$$\chi_i(\tau^k\sigma^j) = \chi_{i'}(g^{-1}\tau^k\sigma^j g) = 0 \text{ whenever } k \neq 0.$$

Supposing $k = 0$, $g^{-1}\sigma^j g = \sigma^{jr^m}$, therefore χ_i and $\chi_{i'}$ are equivalent if and only if there exists an integer m such that

$$\begin{aligned}
& \chi_i(\sigma) = \chi_{i'}(\sigma^{r^m}) \\
\Leftrightarrow\quad & \sum_{n=0}^{q-1}(\zeta^i)^{r^n} = \sum_{n=0}^{q-1}(\zeta^{i'})^{r^{m+n}} = \sum_{n=m}^{q+m-1}(\zeta^{i'})^{r^n} = \sum_{n=0}^{q-1}(\zeta^{i'})^{r^n} \\
\Leftrightarrow\quad & \zeta^i = \zeta^{i'r^n} \qquad \text{for some } n = 0, 1, \ldots, q-1 \\
\Leftrightarrow\quad & i \equiv i'r^n \pmod p \qquad \text{for some } n = 0, 1, \ldots, q-1 \\
\Leftrightarrow\quad & i^{-1}i' \in <r>
\end{aligned}$$

as required, the last implication holding since $1 \leq i, i' \leq p-1$ and hence are both invertible (mod p). □

2.4.11. Remark. By the above lemma, we see that for each χ_i there are exactly $|<r>| = q$ of the $\chi_{i'}$ in the same equivalence class. The ϕ_i are clearly all distinct; alternatively we note that we have $(p-1)/q$ distinct absolutely irreducible q-dimensional representations of $G_{q,p}$, we know that all the others are 1 dimensional, and the number of distinct 1 dimensional characters must be

$$|G_{q,p}| - \frac{p-1}{q} \times q^2 = qp - q(p-1) = q.$$

Thus we have a set of generators for $R_{G_{q,p},\overline{\mathbb{Q}}_p}$, namely ϕ_i $(i = 0, 1, \ldots, q-1)$, and $(p-1)/q$ of the χ_i, one chosen from each equivalence class. Note that it will not matter which character we choose from each equivalence class, since we know that $\mathrm{Gal}(K/\mathbb{Q}_p)$ acts transitively on $\{\chi_i\}$. However we shall for convenience assume χ_1 is chosen as the representative of its class.

We may now characterise elements of $\mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$.

Lemma 2.4.12. *Let $K' = \mathbb{Q}_p[\sum_{k=0}^{q-1} \zeta^{r^k}]$. Then K' is a subfield of K of index q. Suppose $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$. Then α is completely defined by the values $\mu_i = \alpha(\phi_i) \in \mathbb{Q}_p$, $i = 0, 1, \ldots, q-1$, and $\lambda = \alpha(\chi_1) \in K'$. Conversely, any such collection of μ_i and λ define such an α.*

Proof. Firstly, $\mathrm{Gal}(K/\mathbb{Q}_p)$ is isomorphic to C_{p-1}. We choose a generator ω' for $\mathrm{Gal}(K/\mathbb{Q}_p)$ as follows. Let r' be a generator of $\mathbb{F}_p^{\times} \cong C_{p-1}$, chosen so that $r'^{(p-1)/q} = r$. Then ω' is chosen so that $\omega'(\zeta) = \zeta^{r'}$. We also define ω so that $\omega(\zeta) = \zeta^r$.

Since $K/\mathbb{Q}_p$ is a Galois extension, we will show that $K' = K^{<\omega>}$, and thus $|K : K'| = |<\omega>| = q$. This amounts to determining when ω'^i fixes K', that is, when ω'^i fixes $\sum_{k=0}^{q-1} \zeta^{r^k}$. Now $\omega'^i(\sum_{k=0}^{q-1} \zeta^{r^k}) = \sum_{k=0}^{q-1} \zeta^{r^k r'^i}$ which equals $\sum_{k=0}^{q-1} \zeta^{r^k}$ if and only if $\zeta^{r^0} = \zeta = \zeta^{r^k r'^i}$ for some k and i, that is, if and only if $r'^i \in <r>$. Therefore ω'^i fixes K' precisely when $\omega'^i \in <\omega>$.

Suppose $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(R_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$, $\mu_i = \alpha(\phi_i)$, $i = 0, 1, \ldots, q-1$, and $\alpha(\chi_1) = \lambda$. We begin by demonstrating that $\lambda \in K'$.

Clearly χ_i is fixed by Ω_K. Now, $\chi_i^{\omega'^j}(\sigma) = \sum_{k=0}^{q-1} \zeta^{ir^k r'^j}$, so $\chi_i^{\omega'^j} = \chi_{ir'^j}$. Hence $\chi_i^{\omega'^j}$ is equivalent to χ_i if and only if $\omega'^i \in <\omega>$ and hence the largest subgroup of $\Omega_{\mathbb{Q}_p}$ which fixes χ_i is $\Omega_{K'}$. Therefore $\lambda \in K'$. It is necessary to consider only the image of χ_1, since $\mathrm{Gal}(K/\mathbb{Q}_p)$ acts transitively on the χ_i's.

Since ϕ_i is fixed by $\Omega_{\mathbb{Q}_p}$, μ_i is fixed by $\Omega_{\mathbb{Q}_p}$, that is, $\mu_i \in \overline{\mathbb{Q}}_p^{\Omega_{\mathbb{Q}_p}} = \mathbb{Q}_p$. Also, since for any given i the orbit of ϕ_i is just ϕ_i itself, there is no relation between the μ_i's. Thus α is completely determined by the values $\mu_i = \alpha(\phi_i) \in \mathbb{Q}_p$, $i = 0, 1, \ldots, q-1$, and $\lambda = \alpha(\chi_1) \in K'$.

For the converse, suppose $\mu_i \in \mathbb{Q}_p$, $i = 0, 1, \ldots, q-1$, and $\lambda \in K'$. We define a homomorphism $\alpha \in \mathrm{Hom}(\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$ as follows: $\alpha(\phi_i) = \mu_i$, $i = 0, 1, \ldots, q-1$, and $\alpha(\chi_i) = \lambda^{\omega_i}$, $i = 0, 1, \ldots, q-1$, where $\omega_i \in \mathrm{Gal}(K/\mathbb{Q}_p)$ is chosen so that $\chi_1^{\omega_i} = \chi_i$. Note that ω_i must exist since $\mathrm{Gal}(K/\mathbb{Q}_p)$ acts transitively on the χ_i's. We extend linearly to $\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}$. To demonstrate that $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$ we observe that μ_i and ϕ_i are fixed by $\Omega_{\mathbb{Q}_p}$, λ and χ_i are fixed by Ω_K and if $\omega' \in \mathrm{Gal}(K/\mathbb{Q}_p)$ then $\alpha(\chi_i^{\omega'}) = \alpha(\chi_i)^{\omega'}$. □

Lemma 2.4.13. *The image of $\mathcal{K}_1(\mathbb{Q}_p G_{q,p})$ in $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ is freely generated by the triples $[\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$, and $[R, \times\lambda', R]$, where λ' is a generator of P', the unique maximal ideal in $R' = \mathbb{Z}_p[\sum_{k=0}^{q-1} \zeta^{r^k}]$.*

Proof. Suppose $\alpha \in \mathrm{Hom}_{\Omega_{\mathbb{Q}_p}}(\mathrm{R}_{G_{q,p},\overline{\mathbb{Q}}_p}, \overline{\mathbb{Q}}_p^{\times})$, with $\mu_i = \alpha(\phi_i) \in \mathbb{Q}_p$ for $i = 0, 1, \ldots, q-1$, and $\alpha(\chi_1) = \lambda \in K'$. Then its image in $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ is

$$\sum_{i=0}^{q-1} [\mathbb{Z}_p^{(i)}, \times\mu_i, \mathbb{Z}_p^{(i)}] + [R, \times\lambda, R].$$

$[\mathbb{Z}_p^{(i)}, \times\mu_i, \mathbb{Z}_p^{(i)}] = [\mathbb{Z}_p^{(i)}, \times\mu_i u, \mathbb{Z}_p^{(i)}]$ where $u \in \mathbb{Z}_p^{\times}$. Similarly, $[R, \times\lambda, R] = [R, \times\lambda u, R]$ where $u \in R'^{\times}$. Thus we see that $\mathrm{Im}(\mathcal{K}_1(\mathbb{Q}_p G_{q,p}))$ is freely generated by triples $[\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$, and $[R, \times\lambda', R]$ as required. □

Thus combining lemmas 2.4.5 and 2.4.13, we have

Theorem 2.4.14. *$\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ is freely generated by the $3q$ triples*

$$[\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}], \qquad i = 0, 1, \ldots, q-1,$$

$$[R, \times\lambda', R],$$

$$[P^{i+1}, 1, P^i], \qquad i = 1, 2, \ldots, q-1,$$

and

$$[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}], \quad i = 0, 1, \ldots, q-1.$$

In order to compute $\psi_p^{\mathcal{C}}$, we must determine exactly what triples (Γ', χ, Δ) lie in $\mathcal{C}$.

Lemma 2.4.15. *The following table is an exhaustive list of those triples in $\mathcal{C}$, together with the value of $\tilde{e}_\chi$:*

Γ'	χ	Δ	e_χ
$\{1\}$	ϕ_0'	$\{1\}$	*1*
C_p	χ_0'	C_p	*1*
C_q	$\phi_0\colon \tau \mapsto 1$	C_q	*1*
C_q	$\phi_i\colon \tau \mapsto \theta^i$, $i = 1, \ldots, q-1$	$\{1\}$	$\sum_{j=0}^{q-1} \theta^{-ij}\tau^j$
$G_{q,p}$	$\chi_0\colon \tau \mapsto 1, \sigma \mapsto 1$	$G_{q,p}$	*1*
$G_{q,p}$	$\chi_i\colon \tau \mapsto \theta^i, \sigma \mapsto 1$, $i = 1, \ldots, q-1$	C_p	$\sum_{j=0}^{q-1} \theta^{-ij}\tau^j$

Proof. The distinct subgroups of $G_{q,p}$ up to conjugacy are $\{1\}$ the trivial subgroup, C_p, C_q and $G_{q,p}$ itself. We have listed all the characters of $\{1\}$ and C_q; for C_p and $G_{q,p}$, recall that we also need $p \nmid |\Gamma' : \Delta|$. For χ any of the characters listed in the table, since $\chi\colon \Gamma' \to \mathbb{Z}_p^\times$, we have $\tilde{e}_\chi = e_\chi$. □

It now only remains to calculate $v_p(|T^\Delta \tilde{e}_\chi|)$ for each (Γ', χ, Δ) listed above and $[T]$ running through the $3q$ generators of $\mathcal{K}_0^\oplus(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ listed in theorem 2.4.14. To aid this process, we introduce the following two lemmas:

Lemma 2.4.16. *Suppose* $[T] \in \mathcal{K}_0^\oplus(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$.

For (C_q, ϕ_0, C_q),

$$|T^{C_q}| = \Big|T \cdot \frac{1}{q}\sum_{j=0}^{q-1} \tau^j\Big|,$$

that is, we may substitute $\tilde{e}_{\phi_0} = \frac{1}{q}\sum_{j=0}^{q-1}\tau^j$ *and* $\Delta = \{1\}$ *for the old definition of* $\tilde{e}_{\phi_0}$ *and* $\Delta = C_q$ *without altering the value of* $|T^\Delta \tilde{e}_{\phi_0}|$.

For $(G_{q,p}, \chi_0, G_{q,p})$,

$$|T^{G_{q,p}}| = \Big|T^{C_p} \cdot \frac{1}{q}\sum_{j=0}^{q-1} \tau^j\Big|,$$

that is, we may substitute $\tilde{e}_{\chi_0} = \frac{1}{q}\sum_{j=0}^{q-1}\tau^j$ *and* $\Delta = C_p$ *for the old definition of* $\tilde{e}_{\chi_0}$ *and* $\Delta = G_{q,p}$ *without altering the value of* $|T^\Delta \tilde{e}_{\chi_0}|$.

Proof. Since q is invertible in $\mathbb{Z}_p$, fixing by C_q is the same as hitting by the idempotent $\frac{1}{q}\sum_{j=0}^{q-1}\tau^j$. □

Definition 2.4.17. Whenever we use this alternative version of $\tilde{e}_{\phi_0}$ (respectively $\tilde{e}_{\chi_0}$) and Δ, we will refer to the corresponding triples in $\mathcal{C}$ as $(C_q, \phi_0, \{1\})$ (respectively $(G_{q,p}, \chi_0, C_p)$).

Lemma 2.4.18. *Suppose* $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$. *Then*

$$|T\tilde{e}_{\phi'_0}| = \prod_{i=0}^{q-1} |T\tilde{e}_{\phi_i}|$$

and

$$|T^{C_p}\tilde{e}_{\chi'_0}| = \prod_{i=0}^{q-1} |T^{C_p}\tilde{e}_{\chi_i}|.$$

Proof. Note that

$$\prod_{i=0}^{q-1} |T\tilde{e}_{\phi_i}| = |\bigoplus_{i=0}^{q-1} T\tilde{e}_{\phi_i}| = |T\sum_{i=0}^{q-1} \tilde{e}_{\phi_i}|$$

and

$$\prod_{i=0}^{q-1} |T^{C_p}\tilde{e}_{\chi_i}| = |\bigoplus_{i=0}^{q-1} T^{C_p}\tilde{e}_{\chi_i}| = |T^{C_p}\sum_{i=0}^{q-1} \tilde{e}_{\chi_i}|.$$

The lemma now follows from observing that

$$\sum_{i=0}^{q-1} \tilde{e}_{\chi_i} = \sum_{i=0}^{q-1} \frac{1}{q} \sum_{j=0}^{q-1} \theta^{-ij}\tau^j = \frac{1}{q}\sum_{j=0}^{q-1} \tau^j \sum_{i=0}^{q-1} (\theta^{-j})^i = \frac{1}{q}\tau^0 \sum_{i=0}^{q-1} 1 = 1$$

since θ^{-j} is a primitive qth root of 1 for $j = 1, 2, \ldots, q-1$. □

2.4.19. Remark. These last two lemmas allow us to calculate ψ_p^C by considering only $(C_q, \phi_i, \{1\})$ and $(G_{q,p}, \chi_i, C_p)$, $i = 0, 1, \ldots, q-1$.

We now turn to the actual calculations of $v_p|T^{\Delta}\tilde{e}_{\chi}|$.

- $[T] = [\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, with $(C_q, \phi_j, \{1\})$.

  Let $x \in \mathbb{Z}_p^{(i)}$. Then

  $$x\tilde{e}_{\phi_j} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk} x^{\tau^k} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk} x \theta^{ik} = x\frac{1}{q}\sum_{k=0}^{q-1} \theta^{(i-j)k} = x\delta_{ij}$$

  where δ_{ij} is the Kronecker δ. Therefore in $K_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q})$, we have $[T]\tilde{e}_{\phi_j} = \delta_{ij}[\mathbb{Z}_p, \times p, \mathbb{Z}_p]$ and hence $v_p|T\tilde{e}_{\phi_j}| = \delta_{ij}$.

- $[T] = [\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, with $(G_{q,p}, \chi_j, C_p)$.

  Since σ acts trivially on $\mathbb{Z}_p^{(i)}$, we again have $v_p|T\tilde{e}_{\chi_j}| = \delta_{ij}$.

- $[T] = [R, \times\lambda', R]$, with $(C_q, \phi_j, \{1\})$.

  Note that

$$\begin{aligned}[R, \times\lambda', R] &= [P^i, 1, R] + [R, \times\lambda', R] + [R, 1, P^i] \\ &= [P^i, \times\lambda', P^i] \\ &= [R^{(i)}, \times\lambda', R^{(i)}].\end{aligned}$$

  Therefore we shall consider instead $[T] = [R^{(j)}, \times\lambda', R^{(j)}]$. Consider $x = \zeta^i \in R^{(j)}$. (We recall that in $R^{(j)}$, $\zeta^\tau = \theta^j\zeta^\tau$.) Then

$$x\tilde{e}_{\phi_j} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk}(\zeta^i)^{\tau^k} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk}\theta^{jk}\zeta^{ir^k} = \frac{1}{q}\sum_{k=0}^{q-1} \zeta^{ir^k}.$$

  Therefore $R^{(j)}\tilde{e}_{\phi_j}$ is isomorphic as a $\mathbb{Z}_p$-module to $R' = \mathbb{Z}_p[\sum_{k=0}^{q-1}\zeta^{r^k}]$. Since the multiplying factor λ' was chosen to be the generator of $P'R$, we have that $[T\tilde{e}_{\phi_j}] = [R', \times\lambda', R'] = [R', \times\lambda', P'] + [P', 1, R']$. We know that $[R', \times\lambda', P'] = 0$ by lemma 1.1.6, and $R'/P' \cong \mathbb{F}_p$. Therefore $v_p|T\tilde{e}_{\phi_j}| = 1$ for each j.

- $[T] = [R, \times\lambda', R]$, with $(G_{q,p}, \chi_j, C_p)$.

  Since $R^{C_p} = \{0\}$, $v_p|T^{C_p}\tilde{e}_{\chi_j}| = 0$ for each j.

- $[T] = [P^{i+1}, 1, P^i]$, with $(C_q, \phi_j, \{1\})$.

  Since $0 \to P^{i+1} \to P^i \to \mathbb{F}_p^{(i)} \to 0$ is exact, and hitting by $\tilde{e}_{\phi_j}$ is exact, $v_p|T\tilde{e}_{\phi_j}| = v_p|\mathbb{F}_p^{(i)}\tilde{e}_{\phi_j}|$.

Let $x \in \mathbb{F}_p^{(i)}$. Then

$$x\tilde{e}_{\phi_j} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk} x^{\tau^k} = \frac{1}{q}\sum_{k=0}^{q-1} \theta^{-jk} x \theta^{ik} = x\frac{1}{q}\sum_{k=0}^{q-1} \theta^{(i-j)k} = x\delta_{ij},$$

therefore $v_p|\mathbb{F}_p^{(i)}\tilde{e}_{\phi_j}| = \delta_{ij}$.

- $[T] = [P^{i+1}, 1, P^i]$, with $(G_{q,p}, \chi_j, C_p)$.

  Since $(P^i)^{C_p} = \{0\}$, $v_p|T^{C_p}\tilde{e}_{\chi_j}| = 0$ for each j.

- $[T] = [V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$, with $(C_q, \phi_j, \{1\})$.

  From the cartesian square

$$\begin{array}{ccc} V_i & \xrightarrow{\epsilon^{(i)}} & \mathbb{Z}_p^{(i)} \\ {\scriptstyle \pi^{(i)}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ R^{(i)} & \xrightarrow[\pi_2]{} & \mathbb{F}_p^{(i)} \end{array}$$

  we get the commutative square

$$\begin{array}{ccc} V_i\tilde{e}_{\phi_j} & \longrightarrow & \mathbb{Z}_p^{(i)}\tilde{e}_{\phi_j} \\ \downarrow & & \downarrow \\ R^{(i)}\tilde{e}_{\phi_j} & \longrightarrow & \mathbb{F}_p^{(i)}\tilde{e}_{\phi_j} \end{array}$$

  This is in fact also a cartesian square, since hitting by $\tilde{e}_{\phi_j}$ is exact. Thus $v_p|T\tilde{e}_{\phi_j}| = v_p|\mathbb{F}_p^{(i)}\tilde{e}_{\phi_j}| = \delta_{ij}$.

- $[T] = [V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$, with $(G_{q,p}, \chi_j, C_p)$.

  Since V_i is a free $\mathbb{Z}_pC_p$-module, $H^1(C_p, V_i)$ is trivial, so the square

$$\begin{array}{ccc} V_i^{C_p} & \longrightarrow & (\mathbb{Z}_p^{(i)})^{C_p} \\ \downarrow & & \downarrow \\ (R^{(i)})^{C_p} & \longrightarrow & (\mathbb{F}_p^{(i)})^{C_p} \end{array}$$

is cartesian. Hence we get the cartesian square

$$\begin{array}{ccc} V_i^{C_p}\tilde{e}_{\chi_j} & \longrightarrow & (\mathbb{Z}_p^{(i)})^{C_p}\tilde{e}_{\chi_j} \\ \downarrow & & \downarrow \\ (R^{(i)})^{C_p}\tilde{e}_{\chi_j} & \longrightarrow & (\mathbb{F}_p^{(i)})^{C_p}\tilde{e}_{\chi_j} \end{array}$$

thus $v_p|T^{C_p}\tilde{e}_{\chi_j}| = v_p|(\mathbb{F}_p^{(i)})^{C_p}\tilde{e}_{\chi_j}| = \delta_{ij}$.

Thus we have established

Theorem 2.4.20. *$\psi_p^{\mathcal{C}}$ is completely determined by the following table of values of $v_p|T^{\Delta}\tilde{e}_{\chi}|$:*

Triples \\ (Γ', χ, Δ)	$(C_q, \phi_j, \{1\})$	$(G_{q,p}, \chi_j, C_p)$
$[\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$	δ_{ij}	δ_{ij}
$[R, \times\lambda', R]$	1	0
$[P^{i+1}, 1, P^i]$, $i = 1, 2, \ldots, q-1$	δ_{ij}	0
$[V_i, (\pi^{(i)}, \epsilon^{(i)}), P^i \oplus \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$	δ_{ij}	δ_{ij}

This tabular form is ideal for computing the kernel of $\psi_p^{\mathcal{C}}$. Since these $3q$ triples freely generate $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$, any element $[T]$ contained in $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$ is a $\mathbb{Z}_p$-linear sum of these generators. If $[T]$ lies in the kernel of $\psi_p^{\mathcal{C}}$ then $v_p(|T^{\Delta}\tilde{e}_{\chi}|) = 0$ for each $(\Gamma', \chi, \Delta) \in \mathcal{C}$. Thus to compute the kernel of $\psi_p^{\mathcal{C}}$ we need to row reduce the table of theorem 2.4.20. By our careful choice of generators this is easy. We clearly have:

Theorem 2.4.21. *The kernel of $\psi_p^{\mathcal{C}}$ is of rank q and is freely generated by triples*

$$[V_i, (\pi^{(i)}, \epsilon^{(i)}), P^i \oplus \mathbb{Z}_p^{(i)}] - [\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}],$$

where $i = 0, 1, \ldots, q-1$.

Chapter 3

The Coinvariance Factorizability Theories

WE WILL define in this chapter a second large family of factorizability theories, namely the coinvariance factorizability theories. This family is closely related to the invariance factorizability theories of the previous chapter. In fact, we will show that in the weak cyclic case the two theories give identical values, and in the strong cyclic case that they are equivalent but not identical.

The Coinvariance Factorizability Theories are locally defined theories based upon the co-fixing functor $\cdot_\Delta$ for various subgroups Δ of Γ instead of the fixing functor $\cdot^\Delta$, again hit by the idempotents $\tilde{e}_\chi$ corresponding to characters of subgroups of Γ. The reason behind the name "co-fixing" functor is that, whereas M^Δ is the largest submodule of M upon which Δ acts trivially, M_Δ is the largest quotient module of M upon which Δ acts trivially.

3.1 Coinvariance factorizability theories

In this section we define the coinvariance factorizability theories, and examine the connection between the invariance and coinvariance factorizability theories.

Just as the invariance factorizabilities could be considered as factoring through a homomorphism between relative groups induced from the functor $\mathcal{F}_{(\Gamma',\chi,\Delta)}$ where $\mathcal{F}_{(\Gamma',\chi,\Delta)}(M) = M^{\Delta}\tilde{e}_{\chi}$, we will show that the coinvariance factorizabilities can be considered as factoring through a homomorphism induced this time from the functor $\mathcal{F}'_{(\Gamma',\chi,\Delta)}$ where $\mathcal{F}'_{(\Gamma',\chi,\Delta)}(M) = M_{\Delta}\tilde{e}_{\chi}$. The modules $\mathcal{F}(M)$ and $\mathcal{F}'(M)$ are related by a four term exact sequence

$$\hat{H}^{-1}(\Delta, M)\tilde{e}_{\chi} \hookrightarrow M_{\Delta}\tilde{e}_{\chi} \xrightarrow{\mathrm{Tr}_M} M^{\Delta}\tilde{e}_{\chi} \twoheadrightarrow \hat{H}^{0}(\Delta, M)\tilde{e}_{\chi}$$

where Tr_M is the trace map; this will be used to compare the invariance and coinvariance factorizability theories.

Definition 3.1.1. Let $S \subseteq \mathrm{uRep}_p$. The *$S$-coinvariance factorizability theory* $\psi_{S,p}$ is a local factorizability theory

$$\psi_{S,p}\colon \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \mathrm{Map}(S, \mathbb{Z})$$
$$[M, f, N] \mapsto\ : (\Gamma', \chi, \Delta) \mapsto v_p(\left|M_{\Delta}\tilde{e}_{\chi}, f, N_{\Delta}\tilde{e}_{\chi}\right|).$$

If we choose a collection $\underline{S} = \{S_p : S_p \subseteq \mathrm{uRep}_p\}$ of local theories, then we may define a global theory $\psi_{\underline{S}}$, a locally defined factorizability theory with local theories $\psi_{S_p,p}$.

We again have

Proposition 3.1.2. *If $S' \subseteq S \subseteq \mathrm{uRep}_p$ then $\psi_{S,p} \succcurlyeq \psi_{S',p}$.*

Proof. C.f. proposition 2.1.8. □

3.1.3. Remark. Since $\mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{M}$, we have that $\psi_{\mathcal{C}_0,p} \preccurlyeq \psi_{\mathcal{C},p} \preccurlyeq \psi_{\mathcal{M},p}$ and hence $\psi_{\underline{\mathcal{C}_0}} \preccurlyeq \psi_{\underline{\mathcal{C}}} \preccurlyeq \psi_{\underline{\mathcal{M}}}$.

3.1.4. Remark. Following on from our discussion on the invariant factorizability theories (see remark 2.1.5), we observe that the coinvariance factorizability theories may be thought of as a homomorphism induced from a functor, composed with the generalised index and the p-adic valuation; this time the functor is

$$\mathcal{F}'_{(\Gamma',\chi,\Delta)} \colon \mathrm{mod}(\mathbb{Z}_p\Gamma) \to \mathrm{mod}(\mathbb{Z}_p)$$
$$M \mapsto M_\Delta \tilde{e}_\chi$$

and the corresponding induced functor on $\mathrm{mod}(\mathbb{Q}_p\Gamma)$.

Lemma 3.1.5. *For any $S \subseteq \mathrm{uRep}_p$, ψ_p^S and $\psi_{S,p}$ agree on the image of $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$ in the relative group $\mathcal{K}_0^\oplus(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$.*

Proof. For $[\beta\times, \mathbb{Q}_p\Gamma] \in \mathcal{K}_1(\mathbb{Q}_p\Gamma)$, and a triple (Γ', χ, Δ) in $S \subseteq \mathrm{uRep}_p$, we have

$$(\psi_p^S \circ \tilde{\partial}_p^\oplus)([\beta\times, \mathbb{Q}_p\Gamma]) \colon (\Gamma', \chi, \Delta) \mapsto v_p \det{}_{\mathbb{Q}_p}([\beta\times, \mathbb{Q}_p\Gamma_\Delta \tilde{e}_\chi]);$$

c.f. corollary 2.1.7, but using our new $\mathcal{F}'$ defined above in place of $\mathcal{F}$. Note that the order of χ, being coprime to p, is necessarily invertible in $\mathbb{Q}_p$. Therefore $\mathbb{Q}_p\Gamma^\Delta\tilde{e}_\chi = \mathbb{Q}_p\Gamma e_\Delta\tilde{e}_\chi = \mathbb{Q}_p\Gamma_\Delta\tilde{e}_\chi$, and ψ_p^S agrees with $\psi_{S,p}$ on the image of $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$. □

In preparation for our examination of the four term exact sequence

$$\hat{H}^{-1}(\Delta, M) \hookrightarrow M_\Delta \xrightarrow{\mathrm{Tr}_M} M^\Delta \twoheadrightarrow \hat{H}^0(\Delta, M)$$

we present the following lemma.

Lemma 3.1.6. *For $M, N \in \mathrm{mod}(\mathbb{Z}_pG)$, G any finite group, suppose a homomorphism $f\colon M \to N$ induces an isomorphism $(f\otimes 1)\colon M\otimes\mathbb{Q} \xrightarrow{\sim} N\otimes\mathbb{Q}$. Then we have in $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ that*

$$[M, f\otimes 1, N] = [\ker(f), 0, \mathrm{coker}(f)].$$

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\ker f & \hookrightarrow & M & \xrightarrow{f} & N & \twoheadrightarrow & \mathrm{coker}\, f \\
\downarrow & & \downarrow f & & \downarrow \| & & \downarrow \\
0 & \hookrightarrow & N & \xrightarrow{=} & N & \twoheadrightarrow & 0
\end{array}$$

hence from the corresponding exact sequence relation in $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ we have

$$[M, f\otimes 1, N] = [N, 1, N] + [\ker f, 0, 0] - [\mathrm{coker}\, f, 0, 0] = [\ker f, 0, \mathrm{coker}\, f]$$

as required. □

Definition 3.1.7. For a finite group G and $M \in \mathrm{mod}(\mathbb{Z}_pG)$, we define the trace map tr_M:

$$tr_M \colon M_G \to M^G$$

$$m \mapsto \sum_{g \in G} mg.$$

Lemma 3.1.8. *In $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q})$, for a triple $[M, f, N]$, and a triple $(\Gamma', \chi, \Delta) \in S$, the following equality holds:*

$$[M_\Delta \tilde{e}_\chi, f_\Delta, N_\Delta \tilde{e}_\chi] + [N_\Delta \tilde{e}_\chi, tr_N \otimes 1, N^\Delta \tilde{e}_\chi]$$
$$= [M_\Delta \tilde{e}_\chi, tr_M \otimes 1, M^\Delta \tilde{e}_\chi] + [M^\Delta \tilde{e}_\chi, f^\Delta, N^\Delta \tilde{e}_\chi]$$

where f_Δ, f^Δ are the maps induced from f, and tr_M, tr_N are the trace maps restricted to $M_\Delta \tilde{e}_\chi$ and $N_\Delta \tilde{e}_\chi$ respectively.

Proof. We have a commutative diagram

$$\begin{array}{ccc} M_\Delta \tilde{e}_\chi \otimes \mathbb{Q} & \xrightarrow{f_\Delta} & N_\Delta \tilde{e}_\chi \otimes \mathbb{Q} \\ {\scriptstyle tr_M \otimes 1} \downarrow & & \downarrow {\scriptstyle tr_N \otimes 1} \\ M^\Delta \tilde{e}_\chi \otimes \mathbb{Q} & \xrightarrow[f^\Delta]{} & N^\Delta \tilde{e}_\chi \otimes \mathbb{Q} \end{array}$$

Hence

$\text{LHS} = [M_\Delta \tilde{e}_\chi, (tr_N \otimes 1) \circ f_\Delta, N^\Delta \tilde{e}_\chi]$	by composition rule
$= [M_\Delta \tilde{e}_\chi, f^\Delta \circ (tr_M \otimes 1), N^\Delta \tilde{e}_\chi]$	by commutivity of the diagram
$= \text{RHS}$	by composition rule

concluding the proof. □

In view of lemma 3.1.6, the four-term exact sequence of $\mathbb{Z}_p\Gamma'$-modules

$$\hat{H}^{-1}(\Delta, M) \hookrightarrow M_\Delta \xrightarrow{tr_M} M^\Delta \twoheadrightarrow \hat{H}^0(\Delta, M),$$

where $\hat{H}^i(\Delta, M)$, $i = -1, 0$ are the usual Tate cohomology groups, motivates the following definition.

Definition 3.1.9. Let $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$. We define the *Herbrandt quotient* to be

$$h_{(\Gamma',\chi,\Delta)}(M) = \frac{|\hat{H}^0(\Delta, M)\tilde{e}_\chi|}{|\hat{H}^{-1}(\Delta, M)\tilde{e}_\chi|}.$$

Lemma 3.1.10. *Let* $(\Gamma', \chi, \Delta) \in S$, *for* $S \subseteq \mathrm{uRep}_p$. *Then*

$$\psi_p^S([M, f, N])((\Gamma', \chi, \Delta)) + v_p(h_{(\Gamma',\chi,\Delta)}(M)) \\ = v_p(h_{(\Gamma',\chi,\Delta)}(N)) + \psi_{S,p}([M, f, N])((\Gamma', \chi, \Delta)).$$

Proof. We have the four-term exact sequence

$$\hat{H}^{-1}(\Delta, M) \hookrightarrow M_\Delta \xrightarrow{tr_M} M^\Delta \twoheadrightarrow \hat{H}^0(\Delta, M),$$

giving the exact sequence

$$\hat{H}^{-1}(\Delta, M)\tilde{e}_\chi \hookrightarrow M_\Delta\tilde{e}_\chi \to M^\Delta\tilde{e}_\chi \twoheadrightarrow \hat{H}^0(\Delta, M)\tilde{e}_\chi.$$

Therefore by lemma 3.1.6, in $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_p), \otimes\mathbb{Q})$

$$[M_\Delta\tilde{e}_\chi, tr_M, M^\Delta\tilde{e}_\chi] = [\hat{H}^{-1}(\Delta, M)\tilde{e}_\chi, 0, \hat{H}^0(\Delta, M)\tilde{e}_\chi].$$

Now,

$$\psi_p^S([M, f, N])((\Gamma', \chi, \Delta)) = v_p(|M^\Delta\tilde{e}_\chi, f^\Delta, N^\Delta\tilde{e}_\chi|)$$

and

$$\psi_{S,p}([M, f, N])((\Gamma', \chi, \Delta)) = v_p(|M_\Delta\tilde{e}_\chi, f_\Delta, N_\Delta\tilde{e}_\chi|).$$

The lemma follows by applying $v_p(|\cdot|)$ to the equation of lemma 3.1.8. □

Any $\mathbb{Z}_p\Gamma$-module M is a $\mathbb{Z}_p\Gamma'$-module by restriction. We then know that $\hat{H}^i(\Delta, M)$ is a $\mathbb{Z}_p\Gamma'/\Delta$-module, or a $\mathbb{Z}_p\Gamma'$ module on which Δ acts trivially; see for example [15], XI.9. We may therefore view the Tate cohomology groups as functors

$$\hat{H}^i(\Delta, \cdot)\colon \operatorname{mod}(\mathbb{Z}_p\Gamma') \to \operatorname{mod}(\mathbb{Z}_p\Gamma')$$

which induce functors

$$\hat{H}^i(\Delta, \cdot)\colon \Phi(\operatorname{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q}) \to \Phi(\operatorname{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$$

and hence, composed with restriction, homomorphisms

$$\hat{H}^i(\Delta, \cdot)\colon \mathcal{K}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q}) \to \mathcal{K}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q}).$$

Via this approach, an alternative description of this lemma is

Lemma 3.1.11. *Let* $(\Gamma', \chi, \Delta) \in S$ *and* $T \in \Phi(\operatorname{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$ *(and therefore* $[T] \in \mathcal{K}_0^{\oplus}(\operatorname{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$*). Then*

$$(\psi_p^S - \psi_{S,p})([T])\colon (\Gamma', \chi, \Delta) \mapsto v_p|[\hat{H}^0(\Delta, T)\tilde{e}_\chi]| - v_p|[\hat{H}^{-1}(\Delta, T)\tilde{e}_\chi]|.$$

Proof. Let $T = (M, f, N)$. By lemma 3.1.10,

$$(\psi_p^S - \psi_{S,p})([T])\colon (\Gamma', \chi, \Delta) \mapsto v_p(h_{(\Gamma',\chi,\Delta)}(N)) - v_p(h_{(\Gamma',\chi,\Delta)}(M)).$$

But

$$v_p(h_{(\Gamma',\chi,\Delta)}(N)) - v_p(h_{(\Gamma',\chi,\Delta)}(M)) = v_p\left|\frac{\hat{H}^0(\Delta, N)\tilde{e}_\chi}{\hat{H}^{-1}(\Delta, N)\tilde{e}_\chi}\right| - v_p\left|\frac{\hat{H}^0(\Delta, M)\tilde{e}_\chi}{\hat{H}^{-1}(\Delta, M)\tilde{e}_\chi}\right|$$

$$= v_p|\hat{H}^0(\Delta, N)\tilde{e}_\chi| - v_p|\hat{H}^{-1}(\Delta, N)\tilde{e}_\chi| - v_p|\hat{H}^0(\Delta, M)\tilde{e}_\chi| + v_p|\hat{H}^{-1}(\Delta, M)\tilde{e}_\chi|$$

and

$$v_p|\hat{H}^i(\Delta, N)\tilde{e}_\chi| - v_p|\hat{H}^i(\Delta, M)\tilde{e}_\chi| = v_p|\hat{H}^i(\Delta, [T])\tilde{e}_\chi|$$

for $i = 0, -1$. The result then follows. □

3.2 Equivalence of the weak cyclic cases

In this section we show that $\psi_p^{\mathcal{C}_0}$ and $\psi_{\mathcal{C}_0,p}$ are not only equivalent factorizability theories, but yield identical values. Hence any corresponding pair of weaker theories must also not only be equivalent but in fact yield identical values. This result will rely upon work on the Herbrandt quotient $h_{(\Gamma',\chi,\Delta)}$. By lemma 3.1.10 we see that if $h_{(\Gamma',\chi,\Delta)}(M) = h_{(\Gamma',\chi,\Delta)}(N)$ for all $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$ then $\psi_p^{\mathcal{C}_0}([M, f, N]) = \psi_{\mathcal{C}_0,p}([M, f, N])$. It is this approach we will take.

We begin by presenting three well-known results.

Lemma 3.2.1. *1. Suppose A is a $\mathbb{Z}_p\Delta$-module, where Δ is a cyclic group. Then A has periodic Tate cohomology with period 2, that is, $\hat{H}^i(\Delta, A) \cong \hat{H}^{i+2}(\Delta, A)$.*

2. Suppose

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of $\mathbb{Z}_p\Delta$-modules, and Δ is a cyclic group. Then the Tate cohomology long exact sequence

$$\cdots \xrightarrow{\beta_{i+1}} \hat{H}^{i+1}(\Delta, C) \xrightarrow{\gamma_{i+1}} \hat{H}^i(\Delta, A) \xrightarrow{\alpha_i} \hat{H}^i(\Delta, B) \xrightarrow{\beta_i} \hat{H}^i(\Delta, C) \xrightarrow{\gamma_i} \cdots$$

is periodic, with period 6.

3. Suppose

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of $\mathbb{Z}_p\Gamma'$*-modules, where* $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$ *(and hence* Γ' *and* Δ *are cyclic groups). Then the Tate cohomology long exact sequence*

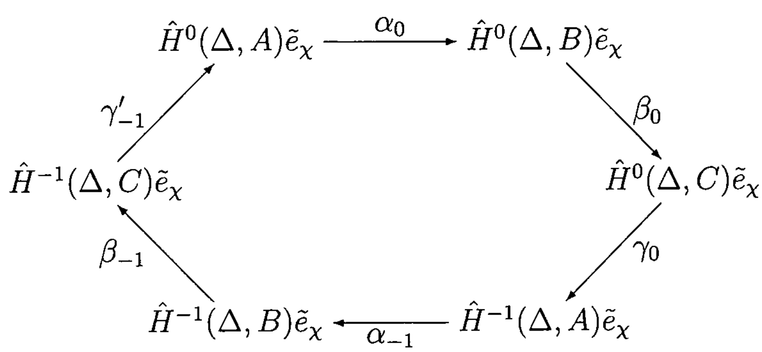
$$\xrightarrow{\beta_{i+1}} \hat{H}^{i+1}(\Delta, C)\tilde{e}_\chi \xrightarrow{\gamma_{i+1}} \hat{H}^i(\Delta, A)\tilde{e}_\chi \xrightarrow{\alpha_i} \hat{H}^i(\Delta, B)\tilde{e}_\chi \xrightarrow{\beta_i} \hat{H}^i(\Delta, C)\tilde{e}_\chi \xrightarrow{\gamma_i} \cdots$$

is again periodic, with period 6.

Lemma 3.2.2. *Suppose*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of $\mathbb{Z}_p\Gamma$*-modules, and* $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$*. Then the hexagon*



is exact, where α_i, β_i, $i = -1, 0$ *and* γ_0 *are the usual maps from the Tate cohomology long exact sequence, and* γ'_{-1} *is the usual* γ_{-1} *composed with the isomorphism from* $\hat{H}^{-2}(\Delta, A)\tilde{e}_\chi$ *to* $\hat{H}^0(\Delta, A)\tilde{e}_\chi$*.*

Proof. Clear from long exact sequence from Tate cohomology in the previous lemma. □

Lemma 3.2.3. *Suppose*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of $\mathbb{Z}_p\Gamma$*-modules, and* $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$*. Then*

$$h_{(\Gamma',\chi,\Delta)}(B) = h_{(\Gamma',\chi,\Delta)}(A) \cdot h_{(\Gamma',\chi,\Delta)}(C).$$

Proof. Construct the hexagon of lemma 3.2.2. Let

$$a_0 = |\mathrm{Im}(\alpha_0)|, \qquad b_0 = |\mathrm{Im}(\beta_0)|, \qquad c_0 = |\mathrm{Im}(\gamma_0)|,$$

$$a_{-1} = |\mathrm{Im}(\alpha_{-1})|, \qquad b_{-1} = |\mathrm{Im}(\beta_{-1})|, \qquad c_{-1} = |\mathrm{Im}(\gamma_{-1})|.$$

Then

$$|\hat{H}^0(\Delta, A)\tilde{e}_\chi| = c_{-1}a_0, \quad |\hat{H}^0(\Delta, B)\tilde{e}_\chi| = a_0 b_0, \quad |\hat{H}^0(\Delta, C)\tilde{e}_\chi| = b_0 c_0,$$

$$|\hat{H}^{-1}(\Delta, A)\tilde{e}_\chi| = c_0 a_{-1}, \quad |\hat{H}^{-1}(\Delta, B)\tilde{e}_\chi| = a_{-1}b_{-1}, \quad |\hat{H}^{-1}(\Delta, C)\tilde{e}_\chi| = b_{-1}c_{-1}.$$

Hence

$$|\hat{H}^0(\Delta, A)\tilde{e}_\chi||\hat{H}^{-1}(\Delta, B)\tilde{e}_\chi||\hat{H}^0(\Delta, C)\tilde{e}_\chi|$$
$$= |\hat{H}^{-1}(\Delta, A)\tilde{e}_\chi||\hat{H}^0(\Delta, B)\tilde{e}_\chi||\hat{H}^{-1}(\Delta, C)\tilde{e}_\chi|.$$

The result follows by rearranging this equation. □

Lemma 3.2.4. *If A is a finite* $\mathbb{Z}_p\Gamma$*-module then* $h_{(\Gamma',\chi,\Delta)}(A) = 1$.

Proof. We have short exact sequences

$$0 \to A^\Delta \to A \to (1-\sigma)A \to 0$$

and

$$0 \to (1-\sigma)A \to A \to A_\Delta \to 0,$$

where $\Delta = <\sigma>$. Hence we have short exact sequences

$$0 \to A^\Delta \tilde{e}_\chi \to A\tilde{e}_\chi \to (1-\sigma)A\tilde{e}_\chi \to 0$$

and

$$0 \to (1-\sigma)A\tilde{e}_\chi \to A\tilde{e}_\chi \to A_\Delta \tilde{e}_\chi \to 0.$$

Hence

$$|A_\Delta \tilde{e}_\chi| = |A^\Delta \tilde{e}_\chi| = \frac{|A\tilde{e}_\chi|}{|(1-\sigma)A\tilde{e}_\chi|}.$$

We have the four-term exact sequence

$$\hat{H}^{-1}(\Delta, A)\tilde{e}_\chi \hookrightarrow A_\Delta \tilde{e}_\chi \xrightarrow{tr_A} A^\Delta \tilde{e}_\chi \twoheadrightarrow \hat{H}^0(\Delta, A)\tilde{e}_\chi.$$

Therefore

$$h_{(\Gamma',\chi,\Delta)}(A) = \frac{|\hat{H}^0(\Delta, A)\tilde{e}_\chi|}{|\hat{H}^{-1}(\Delta, A)\tilde{e}_\chi|} = \frac{|A_\Delta \tilde{e}_\chi|}{|A^\Delta \tilde{e}_\chi|},$$

that is, $h_{(\Gamma',\chi,\Delta)}(A) = 1$. □

Lemma 3.2.5. *$h_{(\Gamma',\chi,\Delta)}(A)$ depends only on $A \otimes \mathbb{Q}$, that is, if $A \otimes \mathbb{Q} \cong B \otimes \mathbb{Q}$ then $h_{(\Gamma',\chi,\Delta)}(A) = h_{(\Gamma',\chi,\Delta)}(B)$.*

Proof. Denote the image of A in $A \otimes \mathbb{Q}$ by $\overline{A}$. Then we have a short exact sequence

$$0 \to T(A) \to A \to \overline{A} \to 0,$$

where $T(A)$ is the torsion part of A. But since A is finitely generated, $T(A)$ is a finite $\mathbb{Z}_p\Gamma$-module. Hence $h_{(\Gamma',\chi,\Delta)}(T(A)) = 1$, and

$$h_{(\Gamma',\chi,\Delta)}(A) = h_{(\Gamma',\chi,\Delta)}(T(A))h_{(\Gamma',\chi,\Delta)}(\overline{A}) = h_{(\Gamma',\chi,\Delta)}(\overline{A}).$$

Now, since $\overline{A}$ and $\overline{B}$ are $\mathbb{Z}_p\Gamma$-lattices which span the same $\mathbb{Q}_p\Gamma$-vector space, they are isomorphic as modules and hence $h_{(\Gamma',\chi,\Delta)}(\overline{B}) = h_{(\Gamma',\chi,\Delta)}(\overline{A})$. Therefore $h_{(\Gamma',\chi,\Delta)}(A) = h_{(\Gamma',\chi,\Delta)}(B)$ as required. □

Therefore we have

Theorem 3.2.6. *$\psi_p^{\mathcal{C}_0}([T]) = \psi_{\mathcal{C}_0,p}([T])$ for all $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$.*

Proof. Let $[T] = [M, f, N] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$. Then by lemma 3.2.5, we have $h_{(\Gamma',\chi,\Delta)}(M) = h_{(\Gamma',\chi,\Delta)}(N)$. Therefore by lemma 3.1.10,

$$(\psi_p^{\mathcal{C}_0}([T]))((\Gamma', \chi, \Delta)) = (\psi_{\mathcal{C}_0,p}([T]))((\Gamma', \chi, \Delta))$$

for all $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$. □

3.3 Differences in the strong cyclic cases

The proof that $\psi_p^{\mathcal{C}_0}$ and $\psi_{\mathcal{C}_0,p}$ give identical values does not, unfortunately, generalise to $\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$. This is because the Tate cohomology groups $\hat{H}^i(\Delta, A)$, with Γ' acting, are not necessarily periodic of period 2 if Γ' is not cyclic.

In fact, $\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$ do not yield identical values but are equivalent. In this section we will demonstrate that the invariance and coinvariance factorizability theories do not in general yield identical values by investigating the metacyclic group $G_{q,p}$ which we first looked at in section 2.4. As in that section, we will produce a table completely determining the values of $\psi_{\mathcal{C},p}$. We will then be in a position to demonstrate that, at least for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$,

$\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$ are equivalent. In the next section we will prove this equivalence in general, but this specific case will help motivate some of the steps of the general proof.

Our approach in this section will be similar to section 2.4. We will keep the notation of that section. We will use the same generating set for $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$ that we obtained in theorem 2.4.14. The proofs of the lemmas corresponding to lemmas 2.4.16 and 2.4.18 go through with only minimal changes, so we will not reproduce them here. We draw from these results the same conclusion as we did in section 2.4—that to completely determine the values taken by $\psi_{\mathcal{C},p}$ we need only consider the elements $(C_q, \phi_j, \{1\})$ and $(G_{q,p}, \chi_j, C_p)$ of $\mathcal{C}$, $j = 0, 1, \dots, q-1$. We have already established that $\psi_p^{\mathcal{C}_0}$ and $\psi_{\mathcal{C}_0,p}$ take the same values, and therefore $v_p|T^{\Delta}\tilde{e}_\chi| = v_p|T_{\Delta}\tilde{e}_\chi|$ for all $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$, that is, for $(\Gamma', \chi, \Delta) = (C_q, \phi_j, \{1\})$ (alternatively, observe that $T^{\{1\}} = T_{\{1\}} = T$.) Furthermore, by lemma 3.1.5, we know that $\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$ agree on those generators of $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$ arising from the image of $\mathcal{K}_1(\mathbb{Q}_p\Gamma)$. Therefore in order to determine the values taken by $\psi_{\mathcal{C},p}$ all that remains is to calculate $v_p|T_{C_q}\tilde{e}_{\chi_j}|$, $j = 0, 1, \dots, q-1$ for each of the $2q-1$ specified generators of $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$ arising from $\tilde{\mathcal{K}}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma))$.

- $[T] = [V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \dots, q-1$.

Consider the cartesian square

$$\begin{array}{ccc} V_i & \xrightarrow{\epsilon^{(i)}} & \mathbb{Z}_p^{(i)} \\ {\scriptstyle \pi^{(i)}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ R^{(i)} & \xrightarrow[\pi_2]{} & \mathbb{F}_p^{(i)} \end{array}$$

Since V_i is a free $\mathbb{Z}_p C_p$-module, $H_1(C_p, V_i)$ is trivial, so the square

$$\begin{array}{ccc} (V_i)_{C_p} & \longrightarrow & (\mathbb{Z}_p^{(i)})_{C_p} \\ \downarrow & & \downarrow \\ (R^{(i)})_{C_p} & \longrightarrow & (\mathbb{F}_p^{(i)})_{C_p} \end{array}$$

is cartesian. Hence we get the cartesian square

$$\begin{array}{ccc} (V_i)_{C_p}\tilde{e}_{\chi_j} & \longrightarrow & (\mathbb{Z}_p^{(i)})_{C_p}\tilde{e}_{\chi_j} \\ \downarrow & & \downarrow \\ (R^{(i)})_{C_p}\tilde{e}_{\chi_j} & \longrightarrow & (\mathbb{F}_p^{(i)})_{C_p}\tilde{e}_{\chi_j} \end{array}$$

thus $v_p|T_{C_p}\tilde{e}_{\chi_j}| = v_p|(\mathbb{F}_p^{(i)})_{C_p}\tilde{e}_{\chi_j}| = \delta_{ij}$.

- $[T] = [P^{i+1}, 1, P^i]$, $i = 1, 2, \ldots, q-1$.

  $P^i_{C_p} = \mathbb{F}_p^{(i)}$, so $[T_{C_p}] = [\mathbb{F}_p^{(i)}, 0, \mathbb{F}_p^{(i+1)}]$. Therefore $v_p|T_{C_p}\tilde{e}_{\chi_i}| = \delta_{ij} - \delta_{i+1,j}$, $i = 1, 2, \ldots, q-2$, and equals $\delta_{ij} - \delta_{i+1-q,j}$ when $i = q-1$.

Therefore we find

Theorem 3.3.1. *Restricting to $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$, $\psi_{C,p}$ is completely determined by the following table of values of $v_p|T_\Delta \tilde{e}_\chi|$:*

Triples \\ (Γ', χ, Δ)	$(C_q, \phi_j, \{1\})$	$(G_{q,p}, \chi_j, C_p)$
$[\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$	δ_{ij}	δ_{ij}
$[R, \times\lambda', R]$	1	0
$[P^{i+1}, 1, P^i]$, $i = 1, 2, \ldots, q-2$	δ_{ij}	$\delta_{ij} - \delta_{i+1,j}$
$[P^{i+1}, 1, P^i]$, $i = q-1$	δ_{ij}	$\delta_{ij} - \delta_{i+1-q,j}$
$[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}]$, $i = 0, 1, \ldots, q-1$	δ_{ij}	δ_{ij}

Theorem 3.3.2.

$$\psi_p^C \neq \psi_{C,p}$$

Proof. The tables of theorems 2.4.20 and 3.3.1 are different. In particular, $v_p|T_{C_p}\tilde{e}_{\chi_i}| = \delta_{i,j} \neq v_p|T^{C_p}\tilde{e}_{\chi_i}|$ for $[T] = [P^{i+1}, 1, P^i]$, $i = 1, 2, \ldots, q-1$. □

However, if we compute the kernel of $\psi_{C,p}$ by row reducing the table, we find

Theorem 3.3.3. *Restricting to* $\mathcal{K}_0^{\oplus}(\text{lat}(\mathbb{Z}_p G_{q,p}), \otimes\mathbb{Q})$*, the kernel of* $\psi_{C,p}$ *is of rank* q *and is freely generated by triples*

$$[V_i, (\pi^{(i)}, \epsilon^{(i)}), R^{(i)} \oplus \mathbb{Z}_p^{(i)}] - [\mathbb{Z}_p^{(i)}, \times p, \mathbb{Z}_p^{(i)}],$$

where $i = 0, 1, \ldots, q-1$.

Hence,

Corollary 3.3.4. *Restricting to* $\mathcal{K}_0^{\oplus}(\mathrm{lat}(\mathbb{Z}_pG_{q,p}), \otimes\mathbb{Q})$,

$$\psi_{\mathcal{C},p} \sim \psi_p^{\mathcal{C}}.$$

3.4 Equivalence of the strong cyclic cases

In this section we will prove that $\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$ are equivalent factorizability theories. Our approach will be to prove that for each $y \in \mathcal{C}$ and each $T \in \Phi(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$, we can write

$$(\psi_p^{\mathcal{C}}([T]))(y) = \sum_{x \in \mathcal{C}} m_x(\psi_{\mathcal{C},p}([T]))(x)$$

and

$$(\psi_{\mathcal{C},p}([T]))(y) = \sum_{x \in \mathcal{C}} n_x(\psi_p^{\mathcal{C}}([T]))(x)$$

where m_x, n_x are integers independent of $[T]$, since then if $\psi_p^{\mathcal{C}}([T])(x) = 0$ for all $x \in \mathcal{C}$ then $\psi_{\mathcal{C},p}([T])(y) = 0$, and vice-versa.

We recall that by theorem 3.2.6, for $(\Gamma', \chi, \Delta) \in \mathcal{C}_0$,

$$(\psi_p^{\mathcal{C}}([T]))((\Gamma', \chi, \Delta)) = (\psi_{\mathcal{C},p}([T]))((\Gamma', \chi, \Delta)).$$

In general we know by lemma 3.1.11 that

$$(\psi_p^{\mathcal{C}} - \psi_{\mathcal{C},p})([T]) \colon (\Gamma', \chi, \Delta) \mapsto v_p|[\hat{H}^0(\Delta, T)\tilde{e}_\chi] - [\hat{H}^{-1}(\Delta, T)\tilde{e}_\chi]|.$$

Our first goal, therefore, is to explicitly calculate $\hat{H}^0(\Delta, T)$ and $\hat{H}^{-1}(\Delta, T)$ together with the action of Γ' upon these groups.

We begin with some notation.

Notation 3.4.1. Suppose $(\Gamma', \chi, \Delta) \in \mathcal{C}$. Then $\Gamma' = C_{p^k} \rtimes G$ with the action of G on C_{p^k} given by some homomorphism $\alpha \colon G \to \mathrm{Aut}(C_{p^k})$. Throughout this section, the following definitions hold: C denotes the group C_{p^k}, σ denotes a chosen, fixed generator of C. $\phi = \chi|_G$, $D = \ker(\phi) \subseteq G$. $G_0 = \ker(\alpha) \subseteq G$, $q = |G/G_0|$, and τ denotes a chosen, fixed G_0-coset representative such that τG_0 generates G/G_0 (which is necessarily cyclic). Note that τG_0 has order q in G/G_0, so τ^q, although not necessarily equal to the identity in G, certainly lies in G_0.

3.4.2. Remarks. Note that

$$\chi \colon C \rtimes G \to \overline{\mathbb{Q}}_p^\times$$
$$(\sigma^i, g) \mapsto \phi(g),$$

so $\Delta = \ker(\chi) = C \rtimes D$.

Note that G/G_0 is necessarily cyclic and of order q dividing both $|G|$ and $|\mathrm{Aut}(C)| = p^{k-1}(p-1)$ if $k > 0$, or equals 1 if $k = 0$. Since $p \nmid |G|$, necessarily $q \mid p-1$.

Note that if $(\Gamma', \chi, \Delta) \in \mathcal{C}$, then $(G, \phi, D) \in \mathcal{C}$ also.

Note that the action of G on C may be thought of as factoring through the quotient group $G/G_0 = <\tau G_0>$.

It is this last remark which prompts us to extend the notation of 2.4.1.

Notation 3.4.3. Let θ be a primitive qth root of 1 in $\mathbb{Z}_p$. Let r be a qth root of 1 in $(\mathbb{Z}_p/p^k\mathbb{Z}_p)^\times$ with $\theta \equiv r \pmod{p^k}$.

We may view $\mathbb{Z}_p$ as a $\mathbb{Z}_p\Gamma'$-module in several ways: we define $\mathbb{Z}_p^{(i)}$ to be a copy of $\mathbb{Z}_p$ where the $C \rtimes G$ action is that $(\sigma^j, \tau^k g_0)$ acts as multiplication by θ^{ik}, where $g_0 \in G_0$. That is, the Γ' action may be thought of as factoring through G/G_0, with τG_0 acting as multiplication by θ^i.

For any $\mathbb{Z}_p\Gamma'$-module M we again define $M^{(i)} = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)}$. Define $\zeta = \zeta_{p^k}$ a primitive p^kth root of 1. R denotes the ring $\mathbb{Z}_p[\zeta]$, and P denotes the unique maximal ideal $(1-\zeta)R$. ϵ denotes the usual augmentation map from $\mathbb{Z}_pC$ to $\mathbb{Z}_p$. π denotes the usual projection map from $\mathbb{Z}_pC$ to R, where $\pi(\sigma) = \zeta$.

We introduce here a technical lemma:

Lemma 3.4.4. *The homomorphism*

$$(1-\sigma)\mathbb{Z}_pC \xrightarrow{\pi} R$$

is injective.

Proof. Suppose $a \in (1-\sigma)\mathbb{Z}_pC$ lies in the kernel of this map. Then, writing

$$a = \sum_{i=1}^{p^k-1} a_i(1-\sigma^i),$$

we have

$$\pi(a) = \sum_{i=1}^{p^k-1} a_i(1-\zeta^i) = \sum_{i=1}^{p^k-1} a_i - \sum_{i=1}^{p^k-1} a_i\zeta^i = 0.$$

For this to be so, we need $a_1 = a_2 = \cdots = a_{p^k-1}$. But then

$$\pi(a) = (p^k-1)a_1 - a_1\sum_{i=1}^{p^k-1} \zeta^i = p^k a_1.$$

So $a_1 = 0$, and hence $a = 0$. □

3.4.5. Remarks. $\Gamma'/C \cong G$, and thus we may view both $[T^C]$ and $[T_C]$ as lying in $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$. We can think of both $\psi_p^C([T])((\Gamma', \chi, \Delta))$ and $\psi_{C,p}([T])((\Gamma', \chi, \Delta))$ as being calculated in $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ since we know that $T^{\Delta} = (T^C)^D$ and $T_{\Delta} = (T_C)_D$.

In fact, we can do more than this. Since $p \nmid |G|$, the order of D is invertible in $\mathbb{Z}_p$. Therefore for any $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$, $[T^D] = [T_D] = [Te_D]$ where e_D is the idempotent $\frac{1}{|D|}\sum_{d\in D} d \in \mathbb{Z}_pG$.

This leads us to consider another interpretation of $\tilde{e}_\chi$. Call

$$\tilde{e}_\chi = \sum_{\chi' \in \mathrm{orb}(\chi)} e_{\chi'}, \text{ where } e_\chi = \frac{1}{|\Gamma'/\Delta|} \sum_{\gamma \in \Gamma'/\Delta} \gamma^{-1}\chi(\gamma)$$

the "old" interpretation of $\tilde{e}_\chi$, and call

$$\tilde{e}_\chi = \sum_{\chi' \in \mathrm{orb}(\chi)} e_{\chi'}, \text{ where } e_\chi = \frac{1}{|G|} \sum_{\gamma \in G} \gamma^{-1}\chi(\gamma)$$

the "new" interpretation of $\tilde{e}_\chi$. Then the "old" interpretation of $\tilde{e}_\chi$ lies in $\mathbb{Z}_p\Gamma'/\Delta$ and the "new" interpretation of $\tilde{e}_\chi$ lies in $\mathbb{Z}_pG$. However,

$$\begin{aligned} e_De_\chi \text{ (``old'' interpretation)} &= \frac{1}{|\Gamma'/\Delta|}\frac{1}{|D|} \sum_{\gamma \in \Gamma'/\Delta} \sum_{d \in D} d\gamma^{-1}\chi(\gamma) \\ &= \frac{1}{|\Gamma'/C|} \sum_{\gamma \in \Gamma'/C} \gamma^{-1}\chi(\gamma) \\ &= e_\chi \text{ (``new'' interpretation)}, \end{aligned}$$

and thus for $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$

$$\begin{aligned} T^D\tilde{e}_\chi \text{ (``old'' interpretation)} &= Te_D\tilde{e}_\chi \text{ (``old'' interpretation)} \\ &= T\tilde{e}_\chi \text{ (``new'' interpretation)} \\ &= Te_D\tilde{e}_\chi \text{ (``new'' interpretation)}. \end{aligned}$$

Therefore we can use the "new" interpretation in place of the "old" interpretation without needing to make any other changes to our formulae.

Lemma 3.4.6. *For* $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$,

$$\psi_p^{\mathcal{C}}([T])((G,\phi,D)) = \psi_{\mathcal{C},p}([T])((G,\phi,D)).$$

Proof. When $(\Gamma', \chi, \Delta) = (G, \chi, D)$, $T^C = T_C = T$ and therefore

$$\psi_p^{\mathcal{C}}([T])((G,\phi,D)) = \psi_{\mathcal{C},p}([T])((G,\phi,D)) = v_p|T\tilde{e}_\phi|.$$

□

Since $\Delta = C \rtimes D$, and $p \nmid |D|$, $\hat{H}^i(\Delta, T) = \hat{H}^i(C,T)e_D$ and hence $\hat{H}^i(\Delta,T)\tilde{e}_\chi = \hat{H}^i(C,T)\tilde{e}_\chi$. Therefore we may restrict our attention to calculating $\hat{H}^i(C,T)$. In order to compute the Tate cohomology groups, we shall construct a complete, projective resolution of $\mathbb{Z}_pC$, exact as a sequence of $\mathbb{Z}_p\Gamma'$-modules.

Proposition 3.4.7. *We have a $\mathbb{Z}_pC$-free complete resolution, exact as a sequence of $\mathbb{Z}_p\Gamma'$-modules, which is periodic with period $2q$*

$$\cdots \xrightarrow{\nu_2} \mathbb{Z}_p^{(1)}C \xrightarrow{\mu_1} \mathbb{Z}_p^{(1)}C \xrightarrow{\nu_1} \mathbb{Z}_p^{(0)}C \xrightarrow{\mu_0} \mathbb{Z}_p^{(0)}C \xrightarrow{\nu_0} \mathbb{Z}_p^{(-1)}C \xrightarrow{\mu_{-1}} \cdots$$

$$\mu_0 : \mathbb{Z}_p^{(0)}C \xrightarrow{\epsilon} \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^{(0)}C$$

where $\mu_i\colon \mathbb{Z}_p^{(i)}C \to \mathbb{Z}_p^{(i)}C$ is multiplication by $\sum_{j=1}^{p^k-1}\sigma^j$ and $\nu_i\colon \mathbb{Z}_p^{(i)}C \to \mathbb{Z}_p^{(i-1)}C$ is multiplication by $\sum_{j=1}^{q-1}\theta^{-j}\sigma^{r^j}$.

Proof. To prove this proposition, we must show that the sequence above is exact as a sequence of $\mathbb{Z}_pC$-modules, and that it respects the action of τ. The periodicity of $2q$ is clear.

For exactness, first note that

$$\sum_{i=0}^{p^k-1}\sigma^i\sum_{j=0}^{q-1}\theta^{-j}\sigma^{r^j}=\sum_{i=0}^{p^k-1}\sigma^i\sum_{j=0}^{q-1}\theta^{-j}=0.$$

In fact, it is well known that the resolution

$$\cdots\xrightarrow{\sum\sigma^i}\mathbb{Z}_pC\xrightarrow{1-\sigma}\mathbb{Z}_pC\xrightarrow{\sum\sigma^i}\mathbb{Z}_pC\xrightarrow{1-\sigma}\cdots$$

is exact. Therefore to establish the exactness, we will show that

$$\operatorname{Im}(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})=\operatorname{Im}(1-\sigma)=(1-\sigma)\mathbb{Z}_pC$$

and

$$\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})=\ker(1-\sigma).$$

We deal with $\operatorname{Im}(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})=\operatorname{Im}(1-\sigma)$ first. Since $(1-\sigma)\mathbb{Z}_pC$ is mapped injectively into R by π by lemma 3.4.4, it will suffice to show that $(\sum_{i=0}^{q-1}\theta^{-i}\zeta^{r^i})/(\zeta-1)$ is a unit in R. But

$$\frac{\sum_{i=0}^{q-1}\theta^{-i}\zeta^{r^i}}{\zeta-1}=\frac{\sum_{i=0}^{q-1}\theta^{-i}\zeta^{r^i}-\sum_{i=0}^{q-1}\theta^{-i}}{\zeta-1}=\sum_{i=0}^{q-1}\theta^{-i}\frac{\zeta^{r^i}-1}{\zeta-1}=\sum_{i=0}^{q-1}(\theta^{-i}\sum_{j=0}^{r^i-1}\zeta^j).$$

Now, this is a unit in $\mathbb{Z}_p[\zeta]$ if and only if its image is a unit in $\mathbb{Z}_p[\zeta]/(1-\zeta)$. So, reducing our expression modulo $1-\zeta$, we obtain

$$\sum_{i=0}^{q-1}(\theta^{-i}\sum_{j=0}^{r^i-1}\zeta^j)\equiv\sum_{i=0}^{q-1}r^{-i}r^i=q\pmod{1-\zeta}$$

which is invertible.

To show $\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i}) = \ker(1-\sigma)$ we proceed as follows. Clearly both $\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})$ and $\ker(1-\sigma)$ are submodules of $\mathbb{Z}_pC$, and we have just established that $\mathrm{Im}(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i}) = \mathrm{Im}(1-\sigma)$, so they are both torsion free and of the same rank. Furthermore,

$$\ker(1-\sigma) = \mathrm{Im}(\sum_{i=0}^{p^k-1}\sigma^i) \subseteq \ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i}).$$

Now, $\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})/\ker(1-\sigma)$ is a torsion module, and

$$\frac{\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})}{\ker(1-\sigma)} \subseteq \frac{\mathbb{Z}_pC}{\ker(1-\sigma)} \cong \mathrm{Im}(\sum_{i=0}^{p^k-1}\sigma^i) \subseteq \mathbb{Z}_pC.$$

Therefore since $\mathbb{Z}_pC$ is torsion free, $\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i})/\ker(1-\sigma)$ is the trivial module and $\ker(\sum_{i=0}^{q-1}\theta^{-i}\sigma^{r^i}) = \ker(1-\sigma)$. Thus we have established exactness.

For the Γ'-action, we perform the following checks. Firstly we show for μ_i that $(\cdot)\tau\cdot\sum_{j=0}^{p^k-1}\sigma^j = (\cdot)\sum_{j=0}^{p^k-1}\sigma^j\cdot\tau$. Consider σ^l in $\mathbb{Z}_p^{(i)}C$.

$$\sigma^l\cdot\sum_{j=0}^{p^k-1}\sigma^j = \sum_{j=0}^{p^k-1}\sigma^j, \qquad \sigma^l\cdot\tau = \theta^i\sigma^{rl},$$

$$\theta^i\sigma^{rl}\cdot\sum_{j=0}^{p^k-1}\sigma^j = \theta^i\sum_{j=0}^{p^k-1}\sigma^j = (\sum_{j=0}^{p^k-1}\sigma^j)\cdot\tau.$$

Similarly, we show for ν_i that $(\cdot)\tau\cdot\sum_{j=0}^{q-1}\theta^j\sigma^{r^j} = (\cdot)\sum_{j=0}^{q-1}\theta^j\sigma^{r^j}\cdot\tau$. Consider again σ^l in $\mathbb{Z}_p^{(i)}C$.

$$\sigma^l\cdot\sum_{j=0}^{q-1}\theta^{-j}\sigma^{r^j} = \sum_{j=0}^{q-1}\theta^{-j}\sigma^{r^j+l}, \qquad \sigma^l\cdot\tau = \theta^i\sigma^{rl},$$

$$\theta^i\sigma^{rl}\cdot\sum_{j=0}^{q-1}\theta^{-j}\sigma^{r^j} = \theta^{i-1}\sum_{j=0}^{q-1}\theta^{-(j-1)}\sigma^{r(r^{j-1}+l)} = (\sum_{j=0}^{q-1}\theta^{-j}\sigma^{r^j+l})\cdot\tau,$$

thus verifying the Γ' action. □

Definition 3.4.8. For triples $T \in \Phi(\mathrm{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$, consider the maps

$$1 \otimes \mu_i \colon T^{(i)} \to T^{(i)}$$

and

$$1 \otimes \nu_i \colon T^{(i)} \to T^{(i-1)}.$$

We define $W_i = \ker(1 \otimes \mu_i)$ and $X_i = \mathrm{Im}(1 \otimes \mu_i)$, and $Y_i = \ker(1 \otimes \nu_i)$ and $Z_{i-1} = \mathrm{Im}(1 \otimes \nu_i)$.

Thus W_i, X_i, Y_i and Z_i are all contained in $T^{(i)}$, and we have the following exact sequences:

$$W_i \hookrightarrow T^{(i)} \twoheadrightarrow X_i \subseteq T^{(i)}$$

and

$$Y_i \hookrightarrow T^{(i)} \twoheadrightarrow Z_{i-1} \subseteq T^{(i-1)}.$$

Lemma 3.4.9. *$W_i^{(j)} = W_{i+j}$, $X_i^{(j)} = X_{i+j}$, $Y_i^{(j)} = Y_{i+j}$ and $Z_i^{(j)} = Z_{i+j}$.*

Proof. If we tensor the short exact sequence

$$W_i \hookrightarrow T^{(i)} \twoheadrightarrow X_i \subseteq T^{(i)}$$

over $\mathbb{Z}_p$ by $\mathbb{Z}_p^{(j)}$, we get the new sequence

$$W_i^{(j)} \hookrightarrow T^{(i+j)} \twoheadrightarrow X_i^{(j)} \subseteq T^{(i+j)}.$$

The result follows by comparing this sequence with the sequence

$$W_{i+j} \hookrightarrow T^{(i+j)} \twoheadrightarrow X_{i+j} \subseteq T^{(i+j)},$$

and performing the same manipulations for the sequence

$$Y_i \hookrightarrow T^{(i)} \twoheadrightarrow Z_{i-1} \subseteq T^{(i-1)}.$$

□

Lemma 3.4.10.

$$Y_0 = T^C.$$

Proof. As we observed in the proof of proposition 3.4.7,

$$\nu_0 \colon \mathbb{Z}_p C \to \mathbb{Z}_p^{(-1)} C$$

and

$$(1-\sigma) \colon \mathbb{Z}_p C \to \mathbb{Z}_p C$$

have the same kernel. Therefore

$$1 \otimes \nu_0 \colon T \to T^{(-1)}$$

and

$$(1-\sigma) \colon T \to T$$

also have the same kernel. But from this second map, the kernel is seen to be T^C. □

We are now able to work out the Tate cohomology triples $\hat{H}^i(C,T)$ as triples of $\mathbb{Z}_p G$-modules.

Proposition 3.4.11. *For any integer $i \in \mathbb{Z}$, and $T \in \Phi(\mathrm{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$,*

$$\hat{H}^{-2i}(C,T) = Y_i/X_i,$$

$$\hat{H}^{-2i-1}(C,T) = W_i/Z_i.$$

Proof. Let F denote the complete resolution of proposition 3.4.7. We can compute the Tate cohomology via the homology of the complex $T \otimes_{\mathbb{Z}_p C} F$.

$$\hat{H}^{-2i-1}(C,T) = \hat{H}_{2i}(C,T) = \ker(1 \otimes \mu_i)/\operatorname{Im}(1 \otimes \nu_{i+1}) = W_i/Z_i,$$

and

$$\hat{H}^{-2i}(C,T) = \hat{H}_{2i-1}(C,T) = \ker(1 \otimes \nu_i)/\operatorname{Im}(1 \otimes \mu_i) = Y_i/X_i,$$

As required. For a description of the Tate homology groups $\hat{H}_i(C,T)$, and a proof that $\hat{H}^i(C,T) = \hat{H}_{-i-1}(C,T)$, see [2], page 135 and following. □

3.4.12. Remark. An alternative approach avoiding the Tate homology groups would be to observe that the Tate cohomology groups are periodic with period $2q$, as are the W_i, X_i, Y_i and Z_i. Therefore taking $n \in \mathbb{Z}$ large enough that $2i + 2nq - 1 > 0$,

$$\hat{H}^{-2i}(C,T) = \hat{H}^{-2i-2nq}(C,T) = H_{2i+2nq-1}(C,T) = Y_i/X_i,$$

and

$$\hat{H}^{-2i-1}(C,T) = \hat{H}^{-2i-2nq-1}(C,T) = H_{2i+2nq}(C,T) = W_i/Z_i.$$

3.4.13. Remark. Before we continue, it is helpful to recall that if $(\Gamma', \chi, \Delta) \in \mathcal{C}$, then $(G, \phi, D) \in \mathcal{C}$. Just as we restrict $[T]$ to $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$ when dealing with (Γ', χ, Δ), we restrict $[T]$ to $\mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p G), \otimes\mathbb{Q})$ when dealing with (G, ϕ, D). Thus we are allowed to work with $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p G), \otimes\mathbb{Q})$ as well as $[T^C]$ and $[T_C]$.

Proposition 3.4.14. *Let* $T \in \Phi(\mathrm{mod}(\mathbb{Z}_p\Gamma'), \otimes\mathbb{Q})$. *In* $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ *we have*

$$[\hat{H}^0(C,T)] - [\hat{H}^{-1}(C,T)] = [T^C] - [T] + [T^{(1)}] - [(T^C)^{(1)}].$$

Proof. In $\Phi(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ we have the following short exact sequences:

$$0 \to X_0 \to Y_0 \to \hat{H}^0(C,T) \to 0,$$

$$0 \to Z_0 \to W_0 \to \hat{H}^{-1}(C,T) \to 0,$$

$$0 \to W_0 \to T \to X_0 \to 0$$

$$0 \to Y_0^{(1)} \to T \to Z_0 \to 0.$$

Using the relations in $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$ due to these exact sequences, we get

$$\begin{aligned}
[\hat{H}^0(C,T)] - [\hat{H}^{-1}(C,T)] &= [Y_0] - [X_0] - [W_0] + [Z_0] \\
&= [Y_0] - [T] + [Z_0] \\
&= [Y_0] - [T] + [T^{(1)}] - [Y_0^{(1)}] \\
&= [T^C] - [T] + [T^{(1)}] - [(T^C)^{(1)}]
\end{aligned}$$

as required. □

Corollary 3.4.15. *In* $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$,

$$[T_C] = [(T^C)^{(1)}] + [T] - [T^{(1)}]$$

and

$$[T^C] = [T_C^{(-1)}] + [T] - [T^{(-1)}].$$

Proof. Recall that $[\hat{H}^0(C,T)] - [\hat{H}^{-1}(C,T)] = [T^C] - [T_C]$. The corollary follows from rearranging the equation of proposition 3.4.14. □

Thus if we can show that for $T \in \Phi(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$, $T^{(i)}\tilde{e}_\chi \cong T\tilde{e}_{\chi'}$ as $\mathbb{Z}_p$-modules, for some other character χ' of order not divisible by p, we're done. We shall in fact do more than this, explicitly identifying χ'.

Definitions 3.4.16. For ϕ a 1-dimensional character of G we define $\phi_{(i)}$ to be a 1 dimensional character of G as follows. We write $g \in G$ as $\tau^j g_0$, where $g_0 \in G_0$. Then

$$\phi_{(i)}(\tau^j g_0) = \theta^{-ij}\phi(\tau^j g_0).$$

Note that, since $\theta \in \mathbb{Z}_p$, $\mathbb{Z}_p[\phi] = \mathbb{Z}_p[\phi_{(i)}]$ for each $i \in \mathbb{Z}$, and if we take $\omega \in \mathrm{Gal}(\mathbb{Q}_p[\phi]/\mathbb{Q}_p)$,

$$\phi \sim \phi^\omega \qquad \Leftrightarrow \qquad \phi_{(i)} \sim \phi^\omega_{(i)}.$$

Therefore

$$\tilde{e}_\phi = \sum_{\phi^\omega \in \mathrm{orb}(\phi)} e_{\phi^\omega}$$

and

$$\tilde{e}_{\phi_{(i)}} = \sum_{\phi^\omega \in \mathrm{orb}(\phi)} e_{\phi^\omega_{(i)}},$$

the summation over the *same* choice of elements of the Galois group.

For χ a 1-dimensional character of Γ', we define $\chi_{(i)}$ to be a 1-dimensional character of Γ' as follows. If $\phi = \chi|_G$,

$$\chi_{(i)}(\sigma^k, g) = \phi_{(i)}(g).$$

We now calculate $T^{(i)}\tilde{e}_\phi$.

Lemma 3.4.17. *For $T \in \Phi(\mathrm{mod}(\mathbb{Z}_pG), \otimes\mathbb{Q})$,*

$$T^{(i)}\tilde{e}_\phi \cong T\tilde{e}_{\phi_{(i)}}$$

as $\mathbb{Z}_p$-modules.

Proof.

$$\begin{aligned}
T^{(i)}\tilde{e}_\phi &= T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)} \sum_{\phi^\omega \in \mathrm{orb}(\phi)} e_{\phi^\omega} \\
&= T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)} \frac{1}{|G|} \sum_{\phi^\omega \in \mathrm{orb}(\phi)} \sum_{g_0 \in G_0} \sum_{k=0}^{q-1} g_0^{-1}\tau^{-j}\phi^\omega(\tau^j g_0) \\
&= T \frac{1}{|G|} \sum_{\phi^\omega \in \mathrm{orb}(\phi)} \sum_{g_0 \in G_0} \sum_{k=0}^{q-1} g_0^{-1}\tau^{-j}\theta^{-ij}\phi^\omega(\tau^j g_0) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)} \\
&= T \frac{1}{|G|} \sum_{\phi^\omega \in \mathrm{orb}(\phi)} \sum_{g \in G} g^{-1}\phi^\omega_{(i)}(g) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)} \\
&= T \sum_{\phi^\omega \in \mathrm{orb}(\phi)} e_{\phi^\omega_{(i)}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)} \\
&= T\tilde{e}_{\phi_{(i)}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^{(i)}
\end{aligned}$$

and thus as $\mathbb{Z}_p$-modules $T^{(i)}\tilde{e}_\phi \cong T\tilde{e}_{\phi_{(i)}}$. □

Thus we have

Proposition 3.4.18. *For $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$,*

$$\psi_{\mathcal{C},p}([T])((\Gamma', \chi, \Delta)) = \psi_p^{\mathcal{C}}([T])((\Gamma', \chi_{(1)}, \ker\chi_{(1)})) + \psi_p^{\mathcal{C}}([T])((G, \phi, D)) - \psi_p^{\mathcal{C}}([T])((G, \phi_{(1)}, \ker\phi_{(1)}))$$

and

$$\psi_p^{\mathcal{C}}([T])((\Gamma',\chi,\Delta)) = \psi_{\mathcal{C},p}([T])((\Gamma',\chi_{(-1)},\ker\chi_{(-1)})) + \psi_{\mathcal{C},p}([T])((G,\phi,D)) - \psi_{\mathcal{C},p}([T])((G,\phi_{(-1)},\ker\phi_{(-1)})).$$

Proof. For the first equation, in $\mathcal{K}_0(\mathrm{mod}(\mathbb{Z}_pG),\otimes\mathbb{Q})$, we have

$$[T_C] = [(T^C)^{(1)}] + [T] - [T^{(1)}],$$

by corollary 3.4.15, and therefore

$$v_p|T_\Delta\tilde{e}_\chi| = v_p|(T^C)^{(1)}\tilde{e}_\chi| + v_p|T\tilde{e}_\chi| - v_p|T^{(1)}\tilde{e}_\chi|.$$

Now, $|(T^C)^{(1)}\tilde{e}_\chi| = |T^C\tilde{e}_{\chi_{(1)}}|$ by lemma 3.4.17. Similarly $|T\tilde{e}_\chi| = |T^D\tilde{e}_\phi|$ and $|T^{(1)}\tilde{e}_\chi| = |T^D\tilde{e}_{\phi_{(1)}}|$.

The second equation follows from corollary 3.4.15

$$[T^C] = [T_C^{(-1)}] + [T] - [T^{(-1)}]$$

in a similar way. □

Hence

Theorem 3.4.19.

$$\psi_p^{\mathcal{C}} \sim \psi_{\mathcal{C},p}.$$

Proof. Suppose $[T]\in\ker(\psi_p^{\mathcal{C}})$. Then $\psi_p^{\mathcal{C}}([T])((\Gamma',\chi,\Delta)) = 0$ for all (Γ',χ,Δ) in $\mathcal{C}$, and thus by proposition 3.4.18, $\psi_{\mathcal{C},p}([T])((\Gamma',\chi,\Delta)) = 0$. Conversely, suppose $[T]\in\ker(\psi_{\mathcal{C},p})$. Then $\psi_{\mathcal{C},p}([T])((\Gamma',\chi,\Delta)) = 0$ for all (Γ',χ,Δ) in $\mathcal{C}$, and thus by proposition 3.4.18, $\psi_p^{\mathcal{C}}([T])((\Gamma',\chi,\Delta)) = 0$. Thus $\psi_p^{\mathcal{C}}$ and $\psi_{\mathcal{C},p}$ have the same kernel and are therefore equivalent factorizability theories. □

3.5 Stronger cases

We have demonstrated that the weak cyclic invariance and coinvariance factorizability theories were identical, and the strong cyclic invariance and coinvariance factorizability theories were equivalent but nonidentical. We turn our attention now to stronger theories and some of the barriers to showing them to be equivalent. In the strong cyclic case, we allowed $\Gamma' = C \rtimes G$ where C is a cyclic p-group and $p \nmid |G|$. Our approach here will be to consider weakening the assumptions on Γ'. Throughout this section we will continue to use the notation developed in the previous section for use with the strong cyclic case.

Conjecture 3.5.1. Recall that, if $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$, then the order of χ is prime to p. As a possible first step, therefore, we could consider triples where $\Gamma' = C \rtimes (G \times P)$, where P is some other p-group whose action on C is trivial. Since the order of χ is prime to p, necessarily $P \subseteq \Delta$. In fact, we have the case that $\Delta = C \rtimes (D \times P)$. For $[T] \in \mathcal{K}_0^{\oplus}(\mathrm{mod}(\mathbb{Z}_p\Gamma), \otimes\mathbb{Q})$ we have that $T^{\Delta} = (T^C)^{D\times P} = ((T^C)^P)^D$, and similarly $T_{\Delta} = ((T_C)_P)_D$. The only difficulty lies in computing the Tate cohomology groups $\hat{H}^i(C \rtimes P, T)$, however I conjecture that these problems are not insurmountable, and the invariance and coinvariance factorizability theories are still equivalent in this case.

Conjecture 3.5.2. A second possibility would be to take $\Gamma' = P \rtimes G$, where

P is any abelian p-group. The idea here would be to express

$$P = P_1 \times P_2 \times \cdots \times P_m$$

as a product of cyclic groups. Using

$$T^{\Delta} = ((T^{P_m})^{P_1 \times P_2 \times \cdots \times P_{m-1}})^D,$$

my aim would be to express T^{P_m} in terms of triples of the form $T_{P'}$ for some other p-groups P'. Thus by applying an inductive argument, I conjecture that the invariance and coinvariance factorizability theories are still equivalent in this case.

3.5.3. Remark. In the stronger case where $\Gamma' = P \rtimes G$ for an arbitrary p-group P, the problem is going to be in computing the Tate cohomology groups, since the current approach relies on the well known complex when P is cyclic.

Chapter 4

Monomial Mackey Γ-functors

THE FACTORIZABILITY theories discussed so far in this thesis are all based upon collections of closely related functors, either $\cdot^{\Delta}\tilde{e}_{\chi}$ or $\cdot_{\Delta}\tilde{e}_{\chi}$ for some collection of triples (Γ', χ, Δ). We were able to connect these two theories via the Tate cohomology groups, and thus prove the equivalence of a number of cases.

In this chapter we turn our attention to a different kind of relation between the theories—this time seeking relations between the values of $\cdot^{\Delta_i}\tilde{e}_{\chi_i}$ for various triples $(\Gamma'_i, \chi_i, \Delta_i)$, and thus find relations within the values taken by $\psi_p^S([T])((\Gamma'_i, \chi_i, \Delta_i))$ which are independent of the triple T in question. Our approach will be to define monomial Mackey Γ-functors, $\mathcal{A}((\Gamma', \chi, \Delta))$, of which both $M^{\Delta}\tilde{e}_{\chi}$ and $M_{\Delta}\tilde{e}_{\chi}$ are examples for any $M \in \mathrm{mod}(\mathbb{Z}_p\Gamma)$ (as, in fact, are $\hat{H}^i(\Delta, M)\tilde{e}_{\chi}$). Our definition will be related to the definition of G-functors of [20]. However we will expand this definition to cover triples (Γ', χ, Δ) rather than subgroups of Γ alone. We go on to demonstrate that a direct sum relation of the form

$$\bigoplus_i \mathbb{Z}_p[\Gamma/\Delta_i]\tilde{e}_{\chi_i} = \bigoplus_j \mathbb{Z}_p[\Gamma/\Delta'_j]\tilde{e}_{\phi_j}$$

for triples $(\Gamma'_i, \chi_i, \Delta_i), (\Gamma''_j, \phi_j, \Delta'_j) \in S$ yields a direct sum relation of the form

$$\bigoplus_i \mathcal{A}((\Gamma'_i, \chi_i, \Delta_i)) = \bigoplus_j \mathcal{A}((\Gamma''_j, \phi_j, \Delta'_j)).$$

We conclude this chapter by considering a number of examples of monomial Mackey Γ-functors, paying particular attention to the classgroup.

4.1 Monomial Mackey Γ-functors

As we said in the introduction, the factorizability theories discussed in this thesis are all based on a collection of closely related functors. In this section we will provide an axiomatic definition of a *monomial Mackey* Γ*-functor*, or M.M. Γ-functor for short. We will draw on the functors used in the factorizability theories as examples of M.M. Γ-functors. We will go on to look at two general construction tools for M.M. Γ-functors—essentially we will define the concept of a "quotient" M.M. Γ-functor by a "subfunctor", and a mechanism for extending the base-ring. We will go on to use these tools to look at some further examples of M.M. Γ-functors.

This axiomatic approach was inspired by Yoshida's paper, *On G-functors (II): Hecke operators and G-functors*, [20]. The definition of a Γ-functor over a ring R found in [20] and elsewhere is a functor $\mathcal{A}$ from the category of subgroups of Γ to $\mathrm{mod}(R)$ together with homomorphisms in $\mathrm{mod}(R)$

corresponding to restriction, induction and conjugation of subgroups of Γ. Following the work on factorizability theories, it seemed useful to expand the definition to be a functor from the category of triples (Γ', χ, Δ) (with Γ' a subgroup of Γ) to mod(R), or from a suitable subcategory of the category of triples (Γ', χ, Δ).

As a working definition, we will consider a M.M. Γ-functor $\mathcal{A} = (\mathcal{A}, \tau, \rho, \sigma)$ over a ring R to be a functor $\mathcal{A}$ from a collection of triples (Γ', χ, Δ) to mod(R), together with maps τ, ρ, σ between the modules $\mathcal{A}(\Gamma', \chi, \Delta)$ corresponding to induction, restriction and conjugation respectively. The maps τ, ρ and σ must obey certain axioms. In order to make this definition precise we must state the axioms, but first we must define what we mean by induction, restriction and conjugation for triples (Γ', χ, Δ).

Throughout the following definitions, we take S to be a collection of triples (Γ', χ, Δ) with Γ' a subgroup of Γ. We will impose conditions on the triples $(\Gamma', \chi, \Delta) \in S$ later.

Definition 4.1.1. For (H, χ, H_0) and (K, ϕ, K_0) triples in S, we say that $(H, \chi, H_0) \leq (K, \phi, K_0)$ whenever $H \subseteq K$ and $\chi = \phi|_H$. We say that $(H, \chi, H_0) < (K, \phi, K_0)$ whenever $H \subsetneq K$ and $\chi = \phi|_H$; or equivalently, if $(H, \chi, H_0) \leq (K, \phi, K_0)$ but $H \neq K$. Similar definitions exist for $\geq$ and $>$. Note that if $(H, \chi, H_0) \leq (K, \phi, K_0)$ and $(H, \chi, H_0) \geq (K, \phi, K_0)$ then necessarily $(H, \chi, H_0) = (K, \phi, K_0)$ in the usual sense.

Definition 4.1.2. Let $\gamma \in \Gamma$. We define the conjugate of a triple (H, χ, H_0)

in S to be $(H, \chi, H_0)^\gamma = (H^\gamma, \chi_{(\gamma)}, H_0^\gamma)$ where $\chi_{(\gamma)}(h^\gamma) = \chi(h)$ for all $h \in H$. Note that $((H, \chi, H_0)^\gamma)^{\gamma'} = (H, \chi, H_0)^{(\gamma\gamma')}$.

Definition 4.1.3. For $(H, \chi, H_0), (K, \phi, K_0) \leq (L, \theta, L_0)$, define L to be the largest subgroup of $H \cap K$ upon which χ and ϕ agree, and define L_0 to be the kernel of $\chi|_L$. We define the intersection of the triples (H, χ, H_0), (K, ϕ, K_0) to be

$$(H, \chi, H_0) \cap (K, \phi, K_0) = (L, \chi|_L, L_0).$$

Note that this definition is symmetric since necessarily $\chi|_L = \phi|_L$. Also note that necessarily $L_0 = H_0 \cap K_0$, since χ and ϕ necessarily agree on the intersection of their kernels.

Definitions 4.1.4. Suppose $\mathcal{A}\colon S \to \mathrm{mod}(R)$ is a functor, and $x, y \in S$ with $x \leq y$. Then τ will denote a homomorphism

$$\tau_x^y\colon \mathcal{A}(x) \to \mathcal{A}(y).$$

Where the choice of x is clear, we will denote the image of $\alpha \in \mathcal{A}(x)$ by α^y. We may also denote τ_x^y by τ^y or even τ, where this will not cause confusion.

In a similar way, ρ will denote a homomorphism

$$\rho_x^y\colon \mathcal{A}(y) \to \mathcal{A}(x).$$

Where the choice of y is clear, we will denote the image of $\beta \in \mathcal{A}(y)$ by β_x. We may also denote ρ_x^y by ρ_x or even ρ, where this will not cause confusion.

Suppose also that $\gamma \in \Gamma$. σ will denote a homomorphism

$$\sigma_x^\gamma\colon \mathcal{A}(x) \to \mathcal{A}(x^\gamma).$$

Where the choice of x is clear, we will denote the image of $\delta \in \mathcal{A}(x)$ by δ^{γ}. We may also denote σ_x^{γ} by σ^{γ} or even σ, where this will not cause confusion.

Definition 4.1.5. Throughout what follows, S will denote a set of triples (Γ', χ, Δ), closed under conjugation and intersection of triples, with the order of χ invertible in the ring R; that is, if $x, y \in S$ and $\gamma \in \Gamma$, then $x \cap y \in S$ and $x^{\gamma} \in S$.

4.1.6. Remark. Note that one possible way of ensuring that the order of χ invertible in the ring R is to take $R = \mathbb{Z}_p$ and requiring the order of χ to be coprime to p. This will automatically ensure that, if $x = (H, \chi, H_0)$ and $y = (K, \phi, K_0)$ with the orders of χ and ϕ both coprime to p, then the character involved in both x^{γ} and $x \cap y$ will be of order coprime to p. For example, the sets $\mathcal{C}_0$, $\mathcal{C}$ and $\mathcal{M}$ of chapter 2 satisfy this condition.

We now state our axioms.

Definitions 4.1.7. In the axioms that follow, x, y, z are triples in S with $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, and $\gamma, \gamma' \in \Gamma$.

(MF.1) If $\alpha \in \mathcal{A}(x)$ and $x \leq y \leq z$, then

$$\alpha^x = \alpha \qquad \text{and} \qquad (\alpha^y)^z = \alpha^z.$$

(MF.2) If $\beta \in \mathcal{A}(z)$ and $x \leq y \leq z$, then

$$\beta_z = \beta \qquad \text{and} \qquad (\beta_y)_x = \beta_x.$$

(MF.3) If $\alpha \in \mathcal{A}(x)$ and $h \in H$, then

$$\alpha^h = \alpha \qquad \text{and} \qquad (\alpha^{\gamma})^{\gamma'} = \alpha^{\gamma\gamma'}.$$

(MF.4) If $x \le y$, $\alpha \in \mathcal{A}(x)$ and $\beta \in \mathcal{A}(y)$, then

$$(\alpha^y)^\gamma = (\alpha^\gamma)^{y^\gamma} \qquad \text{and} \qquad (\beta_x)^\gamma = (\beta^\gamma)_{x^\gamma}.$$

(MF.5) *(Mackey axiom)* If $x, y \le z$ and $\alpha \in \mathcal{A}(x)$, then

$$\alpha^z{}_y = \sum_{\gamma \in T} \alpha^\gamma{}_{x^\gamma \cap y}{}^y,$$

where T is a complete set of double coset representatives $H_0 \backslash L_0 / K_0$.

(MF.C) *(cohomologicality axiom)* If $x \le y$ and $\beta \in \mathcal{A}(y)$, then

$$(\beta_x)^y = |K_0 : H_0| \cdot \beta.$$

Definition 4.1.8. A monomial Mackey Γ-functor (called an M.M. Γ-functor for short), over a ring R, $\mathcal{A} = (\mathcal{A}, \tau, \rho, \sigma)$ is a functor

$$\mathcal{A}\colon S \to \operatorname{mod}(R)$$

from some collection of triples $S = \{(H, \chi, H_0)\}$, together with maps τ, ρ, σ as defined above, which satisfies axioms MF.1 to MF.5. If in addition it satisfies axiom MF.C, we call it a cohomological monomial Mackey Γ-functor (or C.M.M. Γ-functor for short), over R.

4.1.9. Remark. Note that the definition of G-functor given in [20] coincides with this definition when S is taken to be $S = \{(H, \chi_0, H)\}$, where H ranges over all subgroups of Γ and χ_0 is the trivial character on H.

We are now in a position to introduce a number of tools which will simplify our attempts to demonstrate examples of (C.)M.M. Γ-functors. Note that we will use this notation from now on—for (C.)M.M. Γ-functor over R, read "M.M. Γ-functor over R, respectively C.M.M. Γ-functor over R".

Lemma 4.1.10. *Let $\mathcal{A} = (\mathcal{A}, \tau, \rho, \sigma)$ be a (C.)M.M. Γ-functor over R, and suppose for each $x \in S$ there exist submodules $\mathcal{A}'(x) \subseteq \mathcal{A}(x)$. Suppose also that, for each $x, y \in S$ with $x \leq y$, and $\gamma \in \Gamma$, the following three statements are true:*

$$\tau_x^y(\mathcal{A}'(x)) \subseteq \mathcal{A}(y),$$

$$\rho_x^y(\mathcal{A}'(y)) \subseteq \mathcal{A}(x),$$

$$\sigma_x^\gamma(\mathcal{A}'(x)) \subseteq \mathcal{A}(x^\gamma).$$

Then $\mathcal{A}' = (\mathcal{A}', \tau, \rho, \sigma)$ is also a (C.)M.M. Γ-functor.

Proof. To prove this lemma we need to check the axioms. But due to the containment requirements, they are clearly satisfied. □

Lemma 4.1.11. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are as in the previous lemma. Then $(\mathcal{A}'', \tau'', \rho'', \sigma'')$ is a (C.)M.M. Γ-functor, where, for $x, y \in S$ with $x \leq y$, $\mathcal{A}''(x) = \mathcal{A}(x)/\mathcal{A}'(x)$ (quotient of modules), and*

$$\tau''^y_x \colon \mathcal{A}(x)/\mathcal{A}'(x) \to \mathcal{A}(y)/\mathcal{A}'(y)$$

is induced from τ (similarly ρ'' and σ'').

Proof. We begin by showing that τ'' is well defined. Suppose for $x \leq y$ and $\alpha, \alpha' \in \mathcal{A}(x)$ that $\alpha + \mathcal{A}'(x) = \alpha' + \mathcal{A}'(x)$. Then $\alpha - \alpha' = \beta \in \mathcal{A}'(x)$. But then

$$\begin{aligned}\tau''(\alpha' + \mathcal{A}'(x)) &= \tau(\alpha') + \mathcal{A}'(y)\\ &= \tau(\alpha) + \tau(\beta) + \mathcal{A}'(y)\\ &= \tau(\alpha) + \mathcal{A}'(y)\\ &= \tau''(\alpha + \mathcal{A}'(x))\end{aligned}$$

so τ'' is well defined. Similarly ρ'', σ'' are well defined. The lemma is then clear as the axioms follow from $\mathcal{A}$. □

Lemma 4.1.12. *Suppose that $\mathcal{A} = (\mathcal{A}, \tau, \rho, \sigma)$ is a (C.)M.M. Γ-functor over a ring R. Suppose also that R' is an extension of R in the sense that the ring R' may be viewed as a left R-module. Then*

$$\mathcal{A} \otimes_R R' = (\mathcal{A} \otimes_R R', \tau \otimes 1, \rho \otimes 1, \sigma \otimes 1),$$

defined below, is a (C.)M.M. Γ-functor over R'. Here,

$$\mathcal{A} \otimes_R R' \colon x \mapsto \mathcal{A}(x) \otimes_R R'$$

and the homomorphisms $\tau \otimes 1$, $\rho \otimes 1$ and $\sigma \otimes 1$ are induced from τ, ρ and σ respectively.

In particular, if $R = \mathbb{Z}$ then R' can be any ring.

Proof. The result is clear—the axioms are still satisfied after the modules concerned are tensored up with R'. □

Definition 4.1.13. A morphism between M.M. Γ-functors over R is a family of R-homomorphisms $\theta(x) \colon \mathcal{A}(x) \to \mathcal{B}(x)$, for each $x \in S$, which commute with τ, ρ and σ.

We denote the category of M.M. Γ-functors over R by $\mathcal{M}_R(S,\Gamma)$. The full subcategory of C.M.M. Γ-functors over R we denote by $\mathcal{M}^c_R(S,\Gamma)$.

4.2 Equivalence to the Hecke category

In this section we will define the Hecke category $\mathcal{H}_{R\Gamma,S}$ of the group Γ, and demonstrate that the category of cohomological monomial Γ-functors $\mathcal{M}^c_R(S,\Gamma)$ is equivalent to the category of additive functors from the Hecke category to mod(R). In the next chapter we will use this to find direct sum relations between the modules $\mathcal{A}(H,\chi,H_0)$ for any $\mathcal{A} \in \mathcal{M}^c_R(S,\Gamma)$. Our definition of the Hecke category closely follows that of Yoshida [20] in its approach, with modifications to incorporate unramified monomial modules in addition to permutation modules. The objects of our category will be the monomial R-modules $R[\Gamma/H_0]\tilde{e}_\chi$ for each $(H,\chi,H_0) \in S$. The morphisms will be the $R\Gamma$-homomorphisms between these modules.

We begin by giving an alternative description of the homomorphisms between the modules $R[\Gamma/H_0]\tilde{e}_\chi$ and $R[\Gamma/K_0]\tilde{e}_\phi$.

Lemma 4.2.1. *Let (H,χ,H_0) and (K,ϕ,K_0) be triples in S. Then we have isomorphisms*

$$\Phi\colon \tilde{e}_\chi R[H_0\backslash\Gamma/K_0]\tilde{e}_\phi \to \mathrm{Hom}_{R\Gamma}(R[\Gamma/H_0]\tilde{e}_\chi, R[\Gamma/K_0]\tilde{e}_\phi)$$

and

$$\Phi'\colon \tilde{e}_\chi R[H_0\backslash\Gamma/K_0]\tilde{e}_\phi \to \mathrm{Hom}_{R\Gamma}(R[K_0\backslash\Gamma]\tilde{e}_\phi, R[H_0\backslash\Gamma]\tilde{e}_\chi)$$

where

$$\Phi(\tilde{e}_\chi H_0 x K_0 \tilde{e}_\phi)\colon \gamma H_0 \tilde{e}_\chi \mapsto \sum_{u \in H_0/(H_0 \cap K_0^{x^{-1}})} \gamma u x K_0 \tilde{e}_\phi = \sum_{u' \in H_0 x K_0/K_0} \gamma u' K_0 \tilde{e}_\phi$$

and

$$\Phi'(\tilde{e}_\chi H_0 x K_0 \tilde{e}_\phi)\colon K_0 \gamma \tilde{e}_\phi \mapsto \sum_{v \in H_0^x \cap K_0 \backslash K_0} H_0 x v \gamma \tilde{e}_\chi = \sum_{v' \in H_0 \backslash H_0 x K_0} H_0 v' \gamma \tilde{e}_\chi.$$

Note that u, u', v and v' are coset representatives.

Proof. From [20], lemma 3.1 we have isomorphisms

$$\Phi\colon R[H_0\backslash\Gamma/K_0] \to \mathrm{Hom}_{R\Gamma}(R[\Gamma/H_0], R[\Gamma/K_0])$$

$$\Phi'\colon R[H_0\backslash\Gamma/K_0] \to \mathrm{Hom}_{R\Gamma}(R[K_0\backslash\Gamma], R[H_0\backslash\Gamma]).$$

Our result follows from hitting these isomorphisms with the idempotents $\tilde{e}_\chi$ and $\tilde{e}_\phi$. This preserves the isomorphisms. □

The isomorphism of lemma 4.2.1 will provide a useful description of the homomorphisms between the modules $R[\Gamma/H_0]\tilde{e}_\chi$ and $R[\Gamma/K_0]\tilde{e}_\phi$. From this isomorphism, and composition of homomorphisms

$$\alpha \cdot \beta\colon R[\Gamma/H_0]\tilde{e}_\chi \xrightarrow{\alpha} R[\Gamma/K_0]\tilde{e}_\phi \xrightarrow{\beta} R[\Gamma/L_0]\tilde{e}_\theta$$

we have an R-bilinear map

$$\tilde{e}_\chi R[H_0\backslash\Gamma/K_0]\tilde{e}_\phi \times \tilde{e}_\phi R[K_0\backslash\Gamma/L_0]\tilde{e}_\theta \to \tilde{e}_\chi R[H_0\backslash\Gamma/L_0]\tilde{e}_\theta$$

$$(\alpha, \beta) \mapsto \alpha \cdot \beta.$$

Our next step will be to find a better description of $\alpha \cdot \beta \in \tilde{e}_\chi R[H_0\backslash\Gamma/L_0]\tilde{e}_\theta$.

Lemma 4.2.2.

$$(H_0xK_0)\cdot(K_0yL_0) = \sum_{z\in H_0\backslash\Gamma/L_0} m(x,y;z)(H_0zL_0)$$

where $m(x,y;z) = |(H_0xK_0 \cap zL_0y^{-1}K_0)/K_0|$.

Proof. Direct verification—or see [20], lemma 3.2. □

Definition 4.2.3. The Hecke category $\mathcal{H}_{R\Gamma,S}$ is the category whose objects are the monomial modules $R[\Gamma/H_0]\tilde{e}_\chi$ for each $(H,\chi,H_0)\in S$, where the morphisms from $R[\Gamma/H_0]\tilde{e}_\chi$ to $R[\Gamma/K_0]\tilde{e}_\phi$ are given by the set $\tilde{e}_\chi R[H_0\backslash\Gamma/K_0]\tilde{e}_\phi$.

As we said in the introduction to this section, we introduced the Hecke category with the aim of showing the equivalence between C.M.M. Γ-functors over R and the Hecke category $\mathcal{H}_{R\Gamma,S}$. Towards this end, we will define some morphisms of the Hecke category. These will have properties similar to the homomorphisms τ, ρ and σ which will parallel the axioms of the C.M.M. Γ-functors.

Definition 4.2.4. Suppose that $x=(H,\chi,H_0)$, $y=(K,\phi,K_0)$ are triples in S, and suppose that $x\le y$ and $\gamma\in\Gamma$. We define the following morphisms of $\mathcal{H}_{R\Gamma,S}$:

$$t_x^y = \tilde{e}_\chi(H_0 1 K_0)\tilde{e}_\phi\colon R[\Gamma/H_0]\tilde{e}_\chi \to R[\Gamma/K_0]\tilde{e}_\phi$$

$$r_x^y = \tilde{e}_\phi(K_0 1 H_0)\tilde{e}_\chi\colon R[\Gamma/K_0]\tilde{e}_\phi \to R[\Gamma/H_0]\tilde{e}_\chi$$

$$s_x^\gamma = \tilde{e}_\chi(H_0\gamma H_0^\gamma)\tilde{e}_{\chi^\gamma}\colon R[\Gamma/H_0]\tilde{e}_\chi \to R[\Gamma/H_0^\gamma]\tilde{e}_{\chi^\gamma}$$

$$i_x = \tilde{e}_\chi(H_0 1 H_0)\tilde{e}_\chi\colon R[\Gamma/H_0]\tilde{e}_\chi \to R[\Gamma/H_0]\tilde{e}_\chi.$$

Where it is unambiguous and aids the clarity of the notation, we may write t^y for t^y_x, r_x for r^y_x, s^γ for s^γ_x, and i for i_x.

These morphisms are sufficient to define each of the morphisms of $\mathcal{H}_{R\Gamma,S}$, since we have

Lemma 4.2.5. *Suppose* $x = (H, \chi, H_0)$ *and* $y = (K, \phi, K_0)$ *are elements of* S*,* $\gamma \in \Gamma$ *and* $z = (L, \theta, L_0) = x^\gamma \cap y$*. Then the following equality of morphisms of* $\mathcal{H}_{R\Gamma,S}$ *holds:*

$$\tilde{e}_\chi(H_0\gamma K_0)\tilde{e}_\phi = s^\gamma_x \cdot r^{x^\gamma}_z \cdot t^y_z.$$

Proof. Direct verification, using the formula of lemma 4.2.2.

$$\begin{aligned} s^\gamma_x \cdot r^{x^\gamma}_z &= \tilde{e}_\chi H_0\gamma H^\gamma_0 \tilde{e}_{\chi^\gamma} \cdot \tilde{e}_{\chi^\gamma} H^\gamma_0 1 L_0 \tilde{e}_\theta \\ &= \sum_{w \in H_0\backslash\Gamma/L_0} m(\gamma, 1; w)\tilde{e}_\chi(H_0 w L_0)\tilde{e}_\theta \end{aligned}$$

where

$$m(\gamma, 1; w) = |(H_0\gamma H^\gamma_0 \cap w L_0 1 H^\gamma_0)/H^\gamma_0|.$$

But

$$H_0\gamma H^\gamma_0 = H_0\gamma\gamma^{-1}H_0\gamma = H_0\gamma = \gamma H^\gamma_0$$

and

$$w L_0 1 H^\gamma_0 = w(H^\gamma_0 \cap K_0)1H^\gamma_0 = wH^\gamma_0$$

and therefore

$$\begin{aligned} m(\gamma, 1; w) &= |(\gamma H^\gamma_0 \cap wH^\gamma_0)/H^\gamma_0| \\ &= \begin{cases} 1 & \text{if } w \in \gamma H^\gamma_0 = H_0\gamma \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$s_x^\gamma \cdot r_z^{x^\gamma} = \tilde{e}_\chi(H_0\gamma L_0)\tilde{e}_\theta.$$

Therefore

$$\begin{aligned} s_x^\gamma \cdot r_z^{x^\gamma} \cdot t_z^y &= \tilde{e}_\chi(H_0\gamma L_0)\tilde{e}_\theta \cdot \tilde{e}_\theta(L_0 1 K_0)\tilde{e}_\phi \\ &= \sum_{v\in H_0\backslash\Gamma/K_0} m(\gamma,1;v)\tilde{e}_\chi(H_0 v K_0)\tilde{e}_\phi \end{aligned}$$

where

$$\begin{aligned} m(\gamma,1;v) &= |(H_0\gamma L_0 \cap vK_0 1 L_0)/L_0| \\ &= |(\gamma H_0^\gamma L_0 \cap vK_0)/L_0| \\ &= |(H_0^\gamma \cap \gamma^{-1}vK_0)/(H_0^\gamma \cap K_0)| \\ &= \begin{cases} 1 & \text{if } \gamma \in vK_0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and hence

$$s_x^\gamma \cdot r_z^{x^\gamma} \cdot t_z^y = \tilde{e}_\chi(H_0\gamma K_0)\tilde{e}_\phi$$

as required. □

Lemma 4.2.6. *In the equations that follow, $x, y, z \in S$ with $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, and $\gamma, \gamma' \in \Gamma$.*

(H.1) *If $x \le y \le z$, then*

$$t_x^x = i_x \qquad \text{and} \qquad t_x^y t_y^z = t_x^z.$$

(H.2) *If $x \le y \le z$, then*

$$r_x^x = i_x \qquad \text{and} \qquad r_y^z r_x^y = r_x^z.$$

(H.3) *If $h \in H$, then*

$$s_x^h = i_x \qquad \textit{and} \qquad s_x^\gamma s_{x^\gamma}^{\gamma'} = s_x^{\gamma\gamma'}.$$

(H.4) *If $x \leq y$, then*

$$t_x^y s_y^\gamma = s_x^\gamma t_{x^\gamma}^{y^\gamma} \qquad \textit{and} \qquad r_x^y s_x^\gamma = s_y^\gamma r_{y^\gamma}^{x^\gamma}.$$

(H.5) (Mackey decomposition) *If $x, y \leq z$, then*

$$t_x^z r_y^z = \sum_{\gamma \in T} s_x^\gamma r_{x^\gamma \cap y}^{x^\gamma} t_{x^\gamma \cap y}^y$$

where T is a complete set of double coset representatives $H_0 \backslash L_0 / K_0$.

(H.C) (cohomologicality) *If $x \leq y$, then*

$$r_x^y t_x^y = |K_0 : H_0| i.$$

Proof. These are all proved by direct verification, using the formula of lemma 4.2.2. $\square$

We are now in a position to begin to prove the main result of this chapter, namely, that $\mathcal{M}_R^c(S, \Gamma)$ is equivalent to the category of R-additive functors from $\mathcal{H}_{R\Gamma,S}$ to $\mathrm{mod}(R)$. Our approach will be as follows. For any cohomological monomial Γ-functor over R, $\mathcal{A} \in \mathcal{M}_R^c(S, \Gamma)$, we define a map $A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R)$. We go on to show that this A is an R-additive functor. To show this, we must show that, for suitable morphisms of the Hecke category,

$$A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(\tilde{e}_\phi H_0 \gamma' L_0 \tilde{e}_\theta) = A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot \tilde{e}_\phi H_0 \gamma' L_0 \tilde{e}_\theta).$$

Since

$$\tilde{e}_\chi(H_0\gamma K_0)\tilde{e}_\phi = s_x^\gamma \cdot r_z^{x^\gamma} \cdot t_z^y,$$

we need only consider certain combinations of morphisms of the type r_y^x, t_y^x and s^γ.

At this point we will have shown that $\mathcal{M}_R^c(S,\Gamma)$ is equivalent to some subcategory of the category of R-additive functors from $\mathcal{H}_{R\Gamma,S}$ to $\mathrm{mod}(R)$. To complete the proof, we will identify a map in the opposite direction, and show that they are inverses of one another.

Definition 4.2.7. Let $\mathcal{A} = (\mathcal{A}, \rho, \tau, \sigma)$ be a C.M.M. Γ-functor over R, and $x = (H, \chi, H_0)$ and $y = (K, \phi, K_0)$. We define a map $A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R)$ by

$$A(R[\Gamma/H_0]\tilde{e}_\chi) = \mathcal{A}(x)$$

and for $\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi \in \tilde{e}_\chi R[H_0\backslash\Gamma/K_0]\tilde{e}_\phi$,

$$A(\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi) = \sigma_x^\gamma \rho_z^{x^\gamma} \tau_z^y \colon \mathcal{A}(x) \to \mathcal{A}(y)$$

where $z = x^\gamma \cap y$.

Lemma 4.2.8. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $\gamma, \gamma' \in \Gamma$. Then*

$$A(\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi \cdot s_y^{\gamma'}) = A(\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi) \cdot A(s_y^{\gamma'}).$$

Proof. By lemma 4.2.2,

$$\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi \cdot s_y^{\gamma'} = \tilde{e}_\chi H_0\gamma\gamma' K_0^{\gamma'}\tilde{e}_{\phi^{\gamma'}},$$

and thus

$$A(\tilde{e}_\chi H_0\gamma K_0\tilde{e}_\phi \cdot s_y^{\gamma'}) = \sigma_x^{\gamma\gamma'} \rho_w^{x^{\gamma\gamma'}} \tau_w^{y^{\gamma'}}$$

where $w = x^{\gamma\gamma'} \cap y^{\gamma'}$. Let $z = x^{\gamma} \cap y$, and hence $w = z^{\gamma'}$. Therefore, by the axioms for C.M.M. Γ-functors,

$$\begin{aligned} A(\tilde{e}_{\chi}H_0\gamma K_0\tilde{e}_{\phi}) \cdot A(s_y^{\gamma'}) &= \sigma_x^{\gamma}\rho_z^{x^{\gamma}}\tau_z^{y} \cdot \sigma_y^{\gamma'}\rho_{y^{\gamma'}}^{y^{\gamma'}}\tau_{y^{\gamma'}}^{y^{\gamma'}} \\ &= \sigma_x^{\gamma}\rho_z^{x^{\gamma}}\tau_z^{y}\sigma_y^{\gamma'} \\ &= \sigma_x^{\gamma}\rho_z^{x^{\gamma}}\sigma_z^{\gamma'}\tau_{z^{\gamma'}}^{y^{\gamma'}} \\ &= \sigma_x^{\gamma}\sigma_{x^{\gamma}}^{\gamma'}\rho_{z^{\gamma'}}^{x^{\gamma\gamma'}}\tau_w^{y^{\gamma'}} \\ &= \sigma_x^{\gamma\gamma'}\rho_w^{x^{\gamma\gamma'}}\tau_w^{y^{\gamma'}} \end{aligned}$$

as required. □

Lemma 4.2.9. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, with $z \geq y$, and $\gamma \in \Gamma$. Then*

$$A(\tilde{e}_{\chi}H_0\gamma K_0\tilde{e}_{\phi} \cdot t_y^z) = A(\tilde{e}_{\chi}H_0\gamma K_0\tilde{e}_{\phi}) \cdot A(t_y^z).$$

Proof. Let $v = (V, \theta', V_0) = x^{\gamma} \cap y$ and $w = (W, \theta'', W_0) = x^{\gamma} \cap z$. By lemma 4.2.2,

$$\begin{aligned} \tilde{e}_{\chi}H_0\gamma K_0\tilde{e}_{\phi} \cdot t_y^z &= m(\gamma, 1; \gamma)\tilde{e}_{\chi}H_0\gamma L_0\tilde{e}_{\theta} \\ &= |(H_0\gamma K_0 \cap \gamma L_0 K_0)/K_0|\tilde{e}_{\chi}H_0\gamma L_0\tilde{e}_{\theta} \\ &= |(H_0\gamma K_0 \cap \gamma L_0)/K_0|\tilde{e}_{\chi}H_0\gamma L_0\tilde{e}_{\theta} \\ &= |W_0 : V_0|\tilde{e}_{\chi}H_0\gamma L_0\tilde{e}_{\theta} \end{aligned}$$

and therefore

$$A(\tilde{e}_{\chi}H_0\gamma K_0\tilde{e}_{\phi} \cdot t_y^z) = |W_0 : V_0|\sigma_x^{\gamma}\rho_w^{x^{\gamma}}\tau_w^{z}.$$

Now,

$$A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(t_y^z) = \sigma_x^\gamma \rho_v^{x^\gamma} \tau_v^y \cdot \tau_y^z$$
$$= \sigma_x^\gamma \rho_w^{x^\gamma} \rho_v^w \tau_v^w \tau_w^z$$
$$= |W_0 : V_0| \sigma_x^\gamma \rho_w^{x^\gamma} \tau_w^z$$

as required. □

Lemma 4.2.10. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $\gamma, \gamma' \in \Gamma$. Then*

$$A(s_{x^{\gamma'^{-1}}}^{\gamma'} \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) = A(s_{x^{\gamma'^{-1}}}^{\gamma'}) \cdot A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi).$$

Proof. Let $z = x^\gamma \cap y$. By lemma 4.2.2,

$$s_{x^{\gamma'^{-1}}}^{\gamma'} \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi = \tilde{e}_{\chi^{\gamma'^{-1}}} H_0^{\gamma'^{-1}} \gamma' \gamma K_0 \tilde{e}_\phi$$

and therefore

$$A(s_{x^{\gamma'^{-1}}}^{\gamma'} \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) = \sigma_{x^{\gamma'^{-1}}}^{\gamma' \gamma} \rho_z^{x^\gamma} \tau_z^y.$$

Also,

$$A(s_{x^{\gamma'^{-1}}}^{\gamma'}) \cdot A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) = \sigma_{x^{\gamma'^{-1}}}^{\gamma'} \cdot \sigma_x^\gamma \rho_z^{x^\gamma} \tau_z^y$$
$$= \sigma_{x^{\gamma'^{-1}}}^{\gamma' \gamma} \rho_z^{x^\gamma} \tau_z^y$$

as required. □

Lemma 4.2.11. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, with $z \geq x$, and $\gamma \in \Gamma$. Then*

$$A(r_x^z \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) = A(r_x^z) \cdot A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi).$$

Proof. Let $v = (V, \phi', V_0) = x^\gamma \cap y$ and $w = (W, \phi'', W_0) = z^\gamma \cap y$. By lemma 4.2.2,

$$\begin{aligned} r_x^z \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi &= m(1, \gamma; \gamma) \tilde{e}_\theta L_0 \gamma K_0 \tilde{e}_\phi \\ &= |(L_0 1 H_0 \cap \gamma H_0 \gamma^{-1} K_0)/H_0| \tilde{e}_\theta H_0 \gamma L_0 \tilde{e}_\phi \\ &= |(L_0 \cap H_0^\gamma K_0)/H_0| \tilde{e}_\theta H_0 \gamma L_0 \tilde{e}_\phi \\ &= |W_0 : V_0| \tilde{e}_\theta H_0 \gamma L_0 \tilde{e}_\phi \end{aligned}$$

and therefore

$$A(r_x^z \cdot \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) = |W_0 : V_0| \sigma_z^\gamma \rho_w^{z^\gamma} \tau_w^y.$$

Now, using the cohomologicality axiom,

$$\begin{aligned} A(r_x^z) \cdot A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) &= \rho_x^z \cdot \sigma_x^\gamma \rho_v^{x^\gamma} \tau_v^y \\ &= \sigma_z^\gamma \rho_{x^\gamma}^{z^\gamma} \rho_v^{x^\gamma} \tau_v^w \tau_w^y \\ &= \sigma_z^\gamma \rho_{x^\gamma}^{z^\gamma} \rho_v^{x^\gamma} \tau_v^w \tau_w^y \\ &= \sigma_z^\gamma \rho_w^{z^\gamma} \rho_v^w \tau_v^w \tau_w^y \\ &= |W_0 : V_0| \sigma_z^\gamma \rho_w^{z^\gamma} \tau_w^y \end{aligned}$$

as required. □

Lemma 4.2.12. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, with $x, y \leq z$, and $\gamma \in \Gamma$. Then*

$$A(t_x^z \cdot r_y^z) = A(t_x^z) \cdot A(r_y^z).$$

Proof. We use the Mackey axiom (MF.5) of C.M.M. Γ-functors, and the

Mackey decomposition formula (H.5) of lemma 4.2.6:

$$t_x^z \cdot r_y^z = \sum_{\gamma \in H_0 \backslash L_0 / K_0} s_x^\gamma r_{x^\gamma \cap y}^{x^\gamma} t_{x^\gamma \cap y}^y$$

and therefore

$$A(t_x^z \cdot r_y^z) = \sum_{\gamma \in H_0 \backslash L_0 / K_0} \sigma_x^\gamma \rho_{x^\gamma \cap y}^{x^\gamma} \tau_{x^\gamma \cap y}^y.$$

However,

$$A(t_x^z) \cdot A(r_y^z) = \tau_x^z \cdot \rho_y^z = \sum_{\gamma \in H_0 \backslash L_0 / K_0} \sigma_x^\gamma \rho_{x^\gamma \cap y}^{x^\gamma} \tau_{x^\gamma \cap y}^y$$

as required. □

Lemma 4.2.13. *Let* $x = (H, \chi, H_0)$*,* $y = (K, \phi, K_0)$ *and* $z = (L, \theta, L_0)$*, with* $y \geq z$*, and* $\gamma, \gamma' \in \Gamma$*. Then*

$$A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot r_z^y) = A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(r_z^y).$$

Proof. Let $v = x^\gamma \cap y$. Then

$$\tilde{e}_\chi (H_0 \gamma K_0) \tilde{e}_\phi = s_x^\gamma r_v^{x^\gamma} t_v^y,$$

so

$$\begin{aligned} A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot r_z^y) &= A(s_x^\gamma r_v^{x^\gamma} t_v^y r_z^y) \\ &= A(s_x^\gamma) \cdot A(r_v^{x^\gamma}) \cdot A(t_v^y) \cdot A(r_z^y) \end{aligned}$$

by lemmas 4.2.10, 4.2.11, and 4.2.12,

$$= A(s_x^\gamma) \cdot A(r_v^{x^\gamma} t_v^y) \cdot A(r_z^y)$$

by lemma 4.2.11,

$$= A(s_x^{\gamma} r_v^{x^{\gamma}} t_v^y) \cdot A(r_z^y)$$

by lemma 4.2.10, as required. □

Lemma 4.2.14. *Let $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $z = (L, \theta, L_0)$, then*

$$A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot \tilde{e}_\phi K_0 \gamma' L_0 \tilde{e}_\theta) = A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(\tilde{e}_\phi K_0 \gamma' L_0 \tilde{e}_\theta).$$

Proof. Let $w = y^{\gamma'} \cap z$. Then

$$\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot \tilde{e}_\phi K_0 \gamma' L_0 \tilde{e}_\theta = \tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot s_y^{\gamma'} \cdot r_w^{y^{\gamma'}} \cdot t_w^z$$

and therefore

$$A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi \cdot \tilde{e}_\phi K_0 \gamma' L_0 \tilde{e}_\theta) = A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(s_y^{\gamma'}) \cdot A(r_w^{y^{\gamma'}}) \cdot A(t_w^z)$$

by lemmas 4.2.9, 4.2.13 and 4.2.8,

$$= A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(s_y^{\gamma'} \cdot r_w^{y^{\gamma'}} \cdot t_w^z)$$

by lemmas 4.2.13 and 4.2.9,

$$= A(\tilde{e}_\chi H_0 \gamma K_0 \tilde{e}_\phi) \cdot A(\tilde{e}_\phi K_0 \gamma' L_0 \tilde{e}_\theta)$$

as required. □

Proposition 4.2.15. *Let $\mathcal{A} = (\mathcal{A}, \rho, \tau, \sigma)$ be a C.M.M. Γ-functor over R. Then A of definition 4.2.7 is an R-additive functor $A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R)$.*

Proof. This is clear: all we need to check is that A respects composition of the morphisms, and by lemma 4.2.14 this is clear. □

Proposition 4.2.16. *Let A be a R-additive functor*

$$A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R).$$

For $x = (H, \chi, H_0)$, $y = (K, \phi, K_0)$ and $\gamma \in \Gamma$, define

$$\mathcal{A}(x) = A(R[\Gamma/H_0]\tilde{e}_\chi),$$

$$\tau_x^y = A(t_x^y),$$

$$\rho_x^y = A(r_x^y),$$

$$\sigma_x^\gamma = A(s_x^\gamma).$$

Then $(\mathcal{A}, \tau, \rho, \sigma)$ is a C.M.M. Γ-functor over R.

Proof. By lemma 4.2.6, $(\mathcal{A}, \tau, \rho, \sigma)$ satisfies the axioms for C.M.M. Γ-functors over R. □

We are now in a position to prove the main theorem.

Theorem 4.2.17. *The category of C.M.M. Γ-functors over R, $\mathcal{M}_R^c(S, \Gamma)$, is equivalent to the category of R-additive functors from $\mathcal{H}_{R\Gamma,S}$ to $\mathrm{mod}(R)$.*

Proof. By proposition 4.2.15 we know that for each C.M.M. Γ-functor over R, $\mathcal{A}$, we have an R-additive functor $A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R)$. By proposition 4.2.16 we know that for each R-additive functor $A\colon \mathcal{H}_{R\Gamma,S} \to \mathrm{mod}(R)$, we have C.M.M. Γ-functor over R, $\mathcal{A}$. Clearly these two processes are inverses of one another. We need to show that this approach takes morphisms between C.M.M. Γ-functors to morphisms between R-additive functors $\mathcal{H}_{R\Gamma,S} \to \text{mod}(R)$, and vice-versa. But this is clear from the definition of a morphism between C.M.M. Γ-functors over R. □

This theorem is important in its own right. However, of greater importance to us is the following corollary.

Corollary 4.2.18. *Suppose $\mathcal{A}$ is a C.M.M. Γ-functors over R, and we have triples $(\Gamma'_i, \chi_i, \Delta_i) \in S$ for $i = 1, 2, \ldots, n$ and $i = n+1, n+2, \ldots, n+m$. If*

$$\bigoplus_{i=1}^{n} R[\Gamma/\Delta_i]\tilde{e}_{\chi_i} \cong \bigoplus_{i=n+1}^{n+m} R[\Gamma/\Delta_i]\tilde{e}_{\chi_i}$$

then

$$\bigoplus_{i=1}^{n} \mathcal{A}((\Gamma'_i, \chi_i, \Delta_i)) \cong \bigoplus_{i=n+1}^{n+m} \mathcal{A}((\Gamma'_i, \chi_i, \Delta_i)).$$

In the next chapter we will find all relations in the Hecke category of this form.

4.3 Examples of C.M.M. Γ-functors

In this section we give two examples of C.M.M. Γ-functors. We also mention without proof a number of other examples.

Proposition 4.3.1. *Let $M \in \text{mod}(\mathbb{Z}_p\Gamma)$. Then*

$$\mathcal{A} \colon (\Gamma', \chi, \Delta) \mapsto M^{\Delta}\tilde{e}_{\chi}$$

with ρ, τ, σ the usual restriction, induction and conjugation maps, is a C.M.M. Γ-functor over $\mathbb{Z}_p$.

Proof. This result can be proved directly, by showing that ρ, τ and σ obey the necessary axioms. Alternatively, this can be proved via proposition 4.2.16: we have a $\mathbb{Z}_p$-additive functor

$$A\colon \mathcal{H}_{\mathbb{Z}_p\Gamma,S} \to \mathrm{mod}(\mathbb{Z}_p)$$
$$\mathbb{Z}_p[\Gamma/\Delta]\tilde{e}_\chi \mapsto M^\Delta \tilde{e}_\chi$$

and hence the corresponding $\mathcal{A}$ is a C.M.M. Γ-functor over $\mathbb{Z}_p$, with ρ, τ, σ the usual restriction, induction and conjugation maps. □

4.3.2. Remark. In a similar way, it can be shown that

$$\mathcal{A}\colon (\Gamma', \chi, \Delta) \mapsto \hat{H}^i(\Delta, M)\tilde{e}_\chi$$

with ρ, τ, σ the usual restriction, induction and conjugation maps, is a C.M.M. Γ-functor over $\mathbb{Z}_p$, as is

$$\mathcal{A}\colon (\Gamma', \chi, \Delta) \mapsto M_\Delta \tilde{e}_\chi$$

with ρ, τ, σ the usual restriction, induction and conjugation maps.

Let F be an algebraic number field, a Galois extension of $\mathbb{Q}$ with Galois group $\Gamma = \mathrm{Gal}(F/\mathbb{Q})$. For our second example, we will show that

$$\mathcal{A}\colon (\Gamma', \chi, \Delta) \mapsto \mathrm{Cl}(F^\Delta)_p \tilde{e}_\chi$$

with ρ, τ, σ defined below, is a C.M.M. Γ-functor over $\mathbb{Z}_p$.

Definition 4.3.3. In what follows, for F an algebraic number field we denote the ring of algebraic integers in F by $\mathcal{O}_F$. $\mathcal{I}(\mathcal{O}_F)$ denotes the ideals of $\mathcal{O}_F$, and $\mathcal{P}(\mathcal{O}_F)$ denotes the principle ideals of $\mathcal{O}_F$. $\Gamma = \mathrm{Gal}(F/\mathbb{Q})$, and Γ acts on F in the obvious way. Thus $\mathrm{Cl}(F^\Delta) = \mathcal{I}(\mathcal{O}_{F^\Delta})/\mathcal{P}(\mathcal{O}_{F^\Delta})$ and $\mathrm{Cl}(F^\Delta)_p = \mathcal{I}(\mathcal{O}_{F^\Delta})_p/\mathcal{P}(\mathcal{O}_{F^\Delta})_p$.

Lemma 4.3.4.

$$\mathcal{B}\colon (\Gamma', \chi, \Delta) \mapsto \mathcal{I}(\mathcal{O}_{F^\Delta})_p \tilde{e}_\chi$$

is a C.M.M. Γ-functor over $\mathbb{Z}$, where (for $\Delta \subseteq \Delta'$), ρ, τ, σ are induced from the norm map $N\colon F^\Delta \to F^{\Delta'}$, inclusion $F^{\Delta'} \hookrightarrow F^\Delta$ and conjugation by elements of the galois group, respectively.

Lemma 4.3.5. *The inclusion*

$$\mathcal{P}(\mathcal{O}_{F^\Delta})_p \tilde{e}_\chi \subseteq \mathcal{I}(\mathcal{O}_{F^\Delta})_p \tilde{e}_\chi$$

satisfies the requirements of lemma 4.1.10.

Thus we have

Proposition 4.3.6. *Let F be an algebraic number field, a Galois extension of $\mathbb{Q}$ with Galois group $\Gamma = \mathrm{Gal}(F/\mathbb{Q})$. Then by lemma 4.1.11,*

$$\mathcal{A}\colon (\Gamma', \chi, \Delta) \mapsto \mathrm{Cl}(F^\Delta)_p \tilde{e}_\chi$$

with ρ, τ, σ defined below, is a C.M.M. Γ-functor over $\mathbb{Z}_p$.

An interesting question which may be considered is, how $\mathrm{Cl}(F^\Delta)_p\tilde{e}_\chi$ and $\mathrm{Cl}(F)_p^\Delta \tilde{e}_\chi$ relate to one another, bearing in mind that they are both C.M.M. Γ-functors over $\mathbb{Z}_p$. We do not deal with this question here.

Chapter 5

Relations in C.M.M. Γ-functors

IN THE previous chapter we demonstrated that, for a C.M.M. Γ-functor over $\mathbb{Z}_p$, $\mathcal{A}$, a direct sum relation of the form

$$\bigoplus_i \mathbb{Z}_p[\Gamma/\Delta_i]\tilde{e}_{\chi_i} = \bigoplus_j \mathbb{Z}_p[\Gamma/\Delta'_j]\tilde{e}_{\phi_j}$$

for triples $(\Gamma'_i, \chi_i, \Delta_i), (\Gamma''_j, \phi_j, \Delta'_j) \in \mathrm{uRep}_p$ yielded a direct sum relation of the form

$$\bigoplus_i \mathcal{A}((\Gamma'_i, \chi_i, \Delta_i)) = \bigoplus_j \mathcal{A}((\Gamma''_j, \phi_j, \Delta'_j)).$$

In the first section of this chapter we discuss how we can go about finding such relations. We go on to actually find all such relations in the case $\Gamma = G_{q,p}$ in the second section.

5.1 General considerations

In this section we consider a general approach to finding the direct sum relations of the form

$$\bigoplus_i \mathbb{Z}_p[\Gamma/\Delta_i]\tilde{e}_{\chi_i} = \bigoplus_j \mathbb{Z}_p[\Gamma/\Delta'_j]\tilde{e}_{\phi_j}$$

for triples $(\Gamma'_i, \chi_i, \Delta_i), (\Gamma''_j, \phi_j, \Delta'_j) \in \mathrm{uRep}_p$. Our approach is to look at the direct sum Grothendieck group of monomial modules of this form, $\mathrm{PP}'(\mathbb{Z}_p\Gamma)$, which is contained within $\mathrm{PP}(\mathbb{Z}_p\Gamma)$. We introduce species $s_{H,c}$ (where H is a p-hypoelementary subgroup of Γ) because these species separate elements of $\mathrm{PP}(\mathbb{Z}_p\Gamma)$—that is, two permutation projective modules M and N are isomorphic if and only if $s_{H,c}(M) = s_{H,c}(N)$ for every species. In fact, we use a subset of the set of all species which is still big enough to separate permutation projective modules. Therefore by looking at the values of $s_{H,c}(M)$ as M ranges over a generating set for $\mathrm{PP}'(\mathbb{Z}_p\Gamma)$ we can determine when the direct sums of two different sets of generators are in fact isomorphic, that is, we have a relation in $\mathrm{PP}'(\mathbb{Z}_p\Gamma)$.

We begin by establishing some notation.

Notation 5.1.1. Throughout what follows, μ will be a finite group, large enough to include all the roots of unity taken by the characters χ for (Γ', χ, Δ) lying in uRep_p with $\mathbb{Q}_p[\mu]$ a Galois extension of $\mathbb{Q}_p$. For example, we could take all the kth roots of unity, where $|\Gamma| = p^r k$, and $p \nmid k$.

We denote the ring $\mathbb{Z}_p[\mu]$ by R. Our choice of μ ensures that R is of finite degree over $\mathbb{Z}_p$.

5.1.2. Remark. Inclusion is an injective map

$$i\colon \mathrm{PP}(\mathbb{Z}_p\Gamma) \hookrightarrow \mathrm{PP}(R\Gamma).$$

To see this, note that inclusion map $\mathrm{PP}(\mathbb{Z}_p\Gamma) \to \mathrm{PP}(R\Gamma)$ composed with the restriction $\mathrm{PP}(R\Gamma) \to \mathrm{PP}(\mathbb{Z}_p\Gamma)$ is the same as multiplication by the degree of R over $\mathbb{Z}_p$ (which is finite), and $\mathrm{PP}(\mathbb{Z}_p\Gamma)$ is torsion free.

We are interested in finding direct sum relations between modules of the form $\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi$ for triples $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$. These modules are clearly permutation projective modules, since

$$\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi \oplus \mathbb{Z}_p\Gamma/\Delta(1-\tilde{e}_\chi) = \mathbb{Z}_p\Gamma/\Delta.$$

Therefore we define

Definition 5.1.3. We denote the subgroup of $\mathrm{PP}(\mathbb{Z}_p\Gamma)$ generated by the classes $[\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi]$ for $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$ by $\mathrm{PP}'(\mathbb{Z}_p\Gamma)$. We denote its image in $\mathrm{PP}(R\Gamma)$ by $\mathrm{PP}'(R\Gamma)$.

We present now a series of results taken from [5] which will allow us to tell when two elements of $\mathrm{PP}(R\Gamma)$ are distinct.

Definition 5.1.4. Let $H \subseteq \Gamma$ be a p-hypoelementary group, that is, $H = P \rtimes C$ for a p-group P, and C a cyclic group of order coprime to p. Let $C = \langle c \rangle$. We define the *species* $s_{H,c}$ to be a map from the permutation projective $R\Gamma$-modules as follows. For a permutation projective $R\Gamma$-module

M, we define M' to be the sum of the indecomposable summands of $M\downarrow_H^\Gamma$ with vertex P. (The vertex of a module is the maximal p-group which acts trivially. Note that necessarily the indecomposable summands of $M\downarrow_H^\Gamma$ have vertex a subgroup of P.) $s_{H,c}(M)$ is then computed by considering the value at c of the Brauer character afforded by M'. For further details, see [5], pages 880 and following. $s_{H,c}$ induces a homomorphism (also denoted $s_{H,c}$)

$$s_{H,c}\colon \mathrm{PP}(R\Gamma) \to R.$$

We define $s'_{H,c}$ to be a map from the permutation projective $\mathbb{Z}_p\Gamma$-modules where $s'_{H,c} = s_{H,c} \circ i$. $s'_{H,c}$ induces a homomorphism (also denoted $s'_{H,c}$)

$$s'_{H,c}\colon \mathrm{PP}(\mathbb{Z}_p\Gamma) \to R.$$

These species separate the elements of $\mathrm{PP}(R\Gamma)$ in the sense that $M \cong N$ if and only if $s_{H,c}(M) = s_{H,c}(N)$ for all species $s_{H,c}$. However we can say more than this. We will define a minimal set of species $\mathcal{S}$ which together still separate elements of $\mathrm{PP}(R\Gamma)$. See [5], remark 81.23.

Definition 5.1.5. We define $\mathcal{S} = \{s_{H,c}\}$ to be a collection of species as follows. Let P range over a full set of nonconjugate p-subgroups of Γ. Denote the normaliser of P in Γ by $N_\Gamma(P)$. For each P, let $C = <c>$ where c ranges over a full set of nonconjugate elements of order coprime to p of $N_\Gamma(P)/P$. For each P and c above, we include $s_{H,c}$ in $\mathcal{S}$ where $H = P \rtimes C$.

Thus in effect we take P to range over a full set of nonconjugate p-subgroups of Γ and for each P take one H from each conjugacy class of subgroups $P \rtimes C \subseteq \Gamma$.

We define $n = |\mathcal{S}|$.

We will label the distinct species $s_{H,c}$ in $\mathcal{S}$ as s_i, $i = 1, \ldots, n$.

Theorem 5.1.6.

$$s\colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{PP}(R\Gamma) \to (\mathbb{Q} \otimes_{\mathbb{Z}} R)^n$$

$$[M] \mapsto (s_i(M))$$

is an isomorphism.

Proof. See [5], theorem 81.24 and corollary 81.26. To provide an outline of the proof, we establish idempotents e_i in the ring $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{PP}(R\Gamma)$ such that $s_j(e_i) = \delta_{ij}$ and $\sum_{i=1}^{n} e_i = 1$. □

We can now determine when two elements of $\mathrm{PP}(R\Gamma)$ are equal, and hence identify the relations in $\mathrm{PP}'(\mathbb{Z}_p\Gamma)$. Our approach will be as follows. We will calculate $s'_{H,c}(\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi)$ for each $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$ with $[\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi] \in \mathrm{PP}'(\mathbb{Z}_p\Gamma)$ distinct and each $s_{H,c} \in \mathcal{S}$. It is then simple linear algebra to work out those linear combinations of elements $[\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi] \in \mathrm{PP}'(\mathbb{Z}_p\Gamma)$ which map to zero; these linear combinations provide all the relations.

We conclude this section with some preliminary calculations towards computing $s'_{H,c}(\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi)$.

Lemma 5.1.7. *Consider a triple $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$, and let R_χ denote a copy of R where Γ' acts via χ. Let $s_{H,c} \in \mathcal{S}$ with $H = P \rtimes C$ and $C = <c>$. Then*

$$s_{H,c}(M) = \sum_{\substack{\gamma \in \Gamma' \backslash \Gamma / H: \\ H \subseteq \Gamma'^{\gamma}}} \chi(c^{\gamma^{-1}}).$$

Proof. Firstly,

$$M\downarrow^{\Gamma}_{H} = R_{\chi}\uparrow^{\Gamma}_{\Gamma'}\downarrow^{\Gamma}_{H} = \bigoplus_{\gamma\in\Gamma'\backslash\Gamma/H} (R_{\chi})^{\gamma}\downarrow^{\Gamma'^{\gamma}}_{\Gamma'^{\gamma}\cap H}\uparrow^{H}_{\Gamma'^{\gamma}\cap H}$$

using the Mackey formula. The sum of the indecomposable summands of $M\downarrow^{\Gamma}_{H}$ with vertex equal to P is

$$M' = \bigoplus_{\substack{\gamma\in\Gamma'\backslash\Gamma/H:\\ P\subseteq\Gamma'^{\gamma}\cap H}} (R_{\chi})^{\gamma}\downarrow^{\Gamma'^{\gamma}}_{\Gamma'^{\gamma}\cap H}\uparrow^{H}_{\Gamma'^{\gamma}\cap H}.$$

$s_{H,c}(M)$ equals the trace of c acting on M'. Now, the trace of c acting on $(R_{\chi})^{\gamma}\downarrow^{\Gamma'^{\gamma}}_{\Gamma'^{\gamma}\cap H}\uparrow^{H}_{\Gamma'^{\gamma}\cap H}$ is zero whenever $\Gamma'^{\gamma}\cap H \neq H$. Therefore $s_{H,c}(M)$ equals the trace of c acting on M'' where

$$M'' = \bigoplus_{\substack{\gamma\in\Gamma'\backslash\Gamma/H:\\ P\subseteq\Gamma'^{\gamma}\cap H,\\ \Gamma'^{\gamma}\cap H=H}} (R_{\chi})^{\gamma}\downarrow^{\Gamma'^{\gamma}}_{H} = \bigoplus_{\substack{\gamma\in\Gamma'\backslash\Gamma/H:\\ H\subseteq\Gamma'^{\gamma}}} \left(R_{\chi}\downarrow^{\Gamma'}_{H^{\gamma^{-1}}}\right)^{\gamma}.$$

For $c \in H$, therefore, we consider how c acts on the RH-module $\left(R_{\chi}\downarrow^{\Gamma'}_{H^{\gamma^{-1}}}\right)^{\gamma}$. For $r \in R_{\chi}$ denote the corresponding element of $\left(R_{\chi}\downarrow^{\Gamma'}_{H^{\gamma^{-1}}}\right)^{\gamma}$ by $(r)^{\gamma}$. Then

$$(r)^{\gamma}\cdot c = (r\cdot c^{\gamma^{-1}})^{\gamma} = (r\chi(c^{\gamma^{-1}}))^{\gamma}$$

and the value of the Brauer character at c is seen to be $\chi(c^{\gamma^{-1}})$. Therefore

$$s_{H,c}(M) = \sum_{\substack{\gamma\in\Gamma'\backslash\Gamma/H:\\ H\subseteq\Gamma'^{\gamma}}} \chi(c^{\gamma^{-1}})$$

as required. □

Lemma 5.1.8. *Let $s_{H,c} \in \mathcal{S}$ with $H = P \rtimes C$ and $C = <c>$. Consider a triple $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$, and let $M = \mathbb{Z}_p\Gamma/\Delta\tilde{e}_{\chi}$. Then*

$$s'_{H,c}(M) = \sum_{\substack{\gamma\in\Gamma'\backslash\Gamma/H:\\ H\subseteq\Gamma'^{\gamma}}} \tilde{\chi}(c^{\gamma^{-1}}).$$

Proof. For (Γ', χ, Δ), let R_χ denote a copy of R where Γ' acts via χ. Then

$$\begin{aligned} s_{H,c}(M) &= \sum_{\chi' \in \mathrm{orb}(\chi)} s_{H,c}(R_{\chi'}) \\ &= \sum_{\chi' \in \mathrm{orb}(\chi)} \sum_{\substack{\gamma \in \Gamma' \backslash \Gamma / H: \\ H \subseteq \Gamma'^\gamma}} \chi'(c^{\gamma^{-1}}) \\ &= \sum_{\substack{\gamma \in \Gamma' \backslash \Gamma / H: \\ H \subseteq \Gamma'^\gamma}} \tilde{\chi}(c^{\gamma^{-1}}) \end{aligned}$$

as required. □

5.2 Example—the group $G_{q,p}$

To conclude this chapter, we will find all relations of the desired form for $\Gamma = G_{q,p}$, the metacyclic group we considered in section 2.4. We will consequently prove the relations used in lemma 2.4.18 to simplify the calculations in section 2.4. In fact, we will prove that these are (up to linear combination) the only relations for $G_{q,p}$.

We use the notation developed in section 2.4.

Lemma 5.2.1. *For $\Gamma = G_{q,p}$, the triples contained in* uRep$_p$ *are precisely* $(\{1\}, \phi'_0, \{1\})$, (C_q, ϕ_0, C_q), $(C_q, \phi_i, \{1\})$ *for* $i = 1, \ldots, q-1$, *where* $\phi_i \colon \tau \to \theta^i$, (C_p, χ'_0, C_p), $(G_{q,p}, \chi_0, G_{q,p})$, *and* $(G_{q,p}, \chi_i, C_p)$ *for* $i = 1, \ldots, q-1$, *where* $\chi_i \colon \tau \to \theta^i$.

Proof. See lemma 2.4.15. □

5.2.2. Remark. Note that each of the characters involved maps into $\mathbb{Z}_p$, and therefore we can take $R = \mathbb{Z}_p$ and $s'_{H,c} = s_{H,c}$.

Lemma 5.2.3. *For* $\Gamma = G_{q,p}$,

$$\mathcal{S} = \{s_{\{1\},1}, s_{C_q,\tau}, s_{C_p,1}, s_{G_{q,p},\tau}\}.$$

Proof. Recalling the notation of definition 5.1.5, we see that either $P = \{1\}$ or $P = C_p$, and in either case, $C = \{1\}$ or $C = C_q$. The lemma follows. □

We now calculate $s_{H,c}(\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi)$ for each $s_{H,c} \in \mathcal{S}$ and each (Γ', χ, Δ) in uRep_p. We will work the values out in two stages, using the formula of lemma 5.1.7.

Lemma 5.2.4. *The entries in the following table are those double coset representatives* $\gamma \in \Gamma' \backslash \Gamma / H$ *such that* $H \subseteq \Gamma'^\gamma$, *as* $s_{H,c}$ *ranges over* $\mathcal{S}$ *and* (Γ', χ, Δ) *ranges over* uRep_p.

	$s_{\{1\},1}$	$s_{C_q,\tau}$	$s_{C_p,1}$	$s_{G_{q,p},\tau}$
$(\{1\}, \phi'_0, \{1\})$	Γ	$\emptyset$	$\emptyset$	$\emptyset$
(C_q, ϕ_0, C_q)	C_p	$\{1\}$	$\emptyset$	$\emptyset$
$(C_q, \phi_i, \{1\})$	C_p	$\{1\}$	$\emptyset$	$\emptyset$
(C_p, χ'_0, C_p)	C_q	$\emptyset$	C_q	$\emptyset$
$(G_{q,p}, \chi_0, G_{q,p})$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$(G_{q,p}, \chi_i, C_p)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Proof. Instead of the requirement that $H \subseteq \Gamma'^\gamma$, we will use the equivalent requirement that $H^{\gamma^{-1}} \subseteq \Gamma'$.

The empty sets arise since $H^{\gamma^{-1}} \not\subseteq \Gamma'$ if $|H| > |\Gamma'|$.

For the bottom two rows, we observe that $\Gamma' = \Gamma$ and so necessarily $H^{\gamma^{-1}} \subseteq \Gamma'$, and also $\Gamma' \backslash \Gamma = \{1\}$.

For the first column, we observe that $H = \{1\}$, and therefore $H^{\gamma^{-1}} \subseteq \Gamma'$ for any γ, and $\Gamma' \backslash \Gamma / H = \Gamma' \backslash \Gamma$.

This leaves three entries requiring further calculations.

$$C_q \backslash G_{q,p} = \{C_q, C_q\sigma, \ldots, C_q\sigma^{p-1}\}.$$

Now, $C_q\sigma^k\tau = C_q\tau\sigma^{rk}$, so $C_q\sigma^k C_q = C_q\sigma^{r^i k} C_q$ for all $i \in \mathbb{Z}$. Therefore

$$C_q \backslash G_{q,p} / C_q = \{\sigma^k : k \in \mathbb{F}_p^\times / <r>\},$$

that is, $(p-1)/q$ different double coset representatives. Now, $\sigma\tau\sigma^{-1} = \tau\sigma^{r-1}$, and thus

$$\sigma^k \tau^l \sigma^{-k} = \tau\sigma^{(r^l-1)k}.$$

Therefore $C_q^{\sigma^k} = C_q$ if and only if $p \mid (r^l - 1)k$ for every l, that is, if and only if $p \mid (r-1)k$, which only happens if $C_q\sigma^k C_q = C_q 1 C_q$.

Finally, $C_p \backslash G_{q,p} / C_p = C_q$, and $C_p^{\tau^k} = C_p$ for any $k \in \mathbb{Z}$. □

Lemma 5.2.5. *The following table contains the values of* $s_{H,c}(\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi)$, *as* $s_{H,c}$ *ranges over* $\mathcal{S}$ *and* (Γ', χ, Δ) *ranges over* uRep_p.

	$s_{\{1\},1}$	$s_{C_q,\tau}$	$s_{C_p,1}$	$s_{G_{q,p},\tau}$
$(\{1\},\phi'_0,\{1\})$	pq	0	0	0
(C_q,ϕ_0,C_q)	p	1	0	0
$(C_q,\phi_i,\{1\})$	p	θ^i	0	0
(C_p,χ'_0,C_p)	q	0	q	0
$(G_{q,p},\chi_0,G_{q,p})$	1	1	1	1
$(G_{q,p},\chi_i,C_p)$	1	θ^i	1	θ^i

Proof. Repeated use of the formula of lemma 5.1.7. The calculations are simplified for four of the six rows since the character involved is the trivial character, and therefore the value in the table is equal to the number of double coset representatives in the corresponding entry of the previous table. For the remaining two rows, two of the columns have $c = 1$ and so in each case $\chi(c^{\gamma^{-1}}) = \chi(1) = 1$, and in the other two columns there is at most one double coset representative to consider. □

We are now in a position to identify all relations in $\mathrm{PP}'(\mathbb{Z}_pG_{q,p})$.

Theorem 5.2.6. *All direct sum relations occuring between modules of the form $\mathbb{Z}_p\Gamma/\Delta\tilde{e}_\chi$, where $(\Gamma',\chi,\Delta) \in \mathrm{uRep}_p$, are integer combinations of the relations*

$$\mathbb{Z}_pG_{q,p} \cong \mathbb{Z}_pC_p \oplus \bigoplus_{i=1}^{q-1} \mathbb{Z}_pG_{q,p}\tilde{e}_{\phi_i}$$

and

$$\mathbb{Z}_pC_q \cong \mathbb{Z}_p \oplus \bigoplus_{i=1}^{q-1} \mathbb{Z}_pC_q\tilde{e}_{\chi_i}.$$

Proof. Direct sum relations between modules of the form $\mathbb{Z}_p G_{q,p}/\Delta\tilde{e}_\chi$, where $(\Gamma', \chi, \Delta) \in \mathrm{uRep}_p$ are in one to one correspondence with linear relations in $\mathrm{PP}'(\mathbb{Z}_p G_{q,p})$. Suppose for some $u, v, w_i, x, y, z_i \in \mathbb{Z}$ for $i = 1, \ldots, q-1$ we have a relation in $\mathrm{PP}'(\mathbb{Z}_p G_{q,p})$:

$$u[\mathbb{Z}_p G_{q,p}/\{1\}\tilde{e}_{\phi_0'}] + v[\mathbb{Z}_p G_{q,p}/C_q\tilde{e}_{\phi_0}] + \sum_{i=1}^{q-1} w_i[\mathbb{Z}_p G_{q,p}/\{1\}\tilde{e}_{\phi_i}]$$
$$+ x[\mathbb{Z}_p G_{q,p}/C_p\tilde{e}_{\chi_0'}] + y[\mathbb{Z}_p G_{q,p}/G_{q,p}\tilde{e}_{\chi_0}] + \sum_{i=1}^{q-1} z_i[\mathbb{Z}_p G_{q,p}/C_p\tilde{e}_{\chi_i}] = 0,$$

that is, a relation

$$u[\mathbb{Z}_p G_{q,p}] + v[\mathbb{Z}_p C_p] + \sum_{i=1}^{q-1} w_i[\mathbb{Z}_p G_{q,p}\tilde{e}_{\phi_i}] + x[\mathbb{Z}_p C_q] + y[\mathbb{Z}_p] + \sum_{i=1}^{q-1} z_i[\mathbb{Z}_p C_q\tilde{e}_{\chi_i}] = 0.$$

Denote the left hand side of this relation by LHS. We know that, for each $s_{H,c} \in \mathcal{S}$, $s_{H,c}(LHS) = 0$. We now use this to obtain relations between the u, v, w_i, x, y, z_i.

$$s_{G_{q,p},\tau}(LHS) = y + \sum_i z_i\theta^i = 0$$

which implies that $y = z_i$ for each $i = 1, \ldots, q-1$.

$$s_{C_q,\tau}(LHS) = v + \sum_i w_i\theta^i + y + \sum_i z_i\theta^i = 0$$

which implies that $v = w_i$ for each $i = 1, \ldots, q-1$.

$$s_{C_p,1}(LHS) = qx + y + \sum_i z_i = 0$$

which implies that $x = -y$.

$$s_{\{1\},1}(LHS) = pqu + pv + p\sum_i w_i + qx + y + \sum_i z_i = 0$$

which implies that $u = -v$. Thus all relations are of the form

$$u[\mathbb{Z}_p G_{q,p}] - u[\mathbb{Z}_p C_p] - u \sum_{i=1}^{q-1} [\mathbb{Z}_p G_{q,p} \tilde{e}_{\phi_i}] + x[\mathbb{Z}_p C_q] - x[\mathbb{Z}_p] - x \sum_{i=1}^{q-1} [\mathbb{Z}_p C_q \tilde{e}_{\chi_i}] = 0$$

as required. □

5.2.7. Remark. These two relations are precisely the relations used in lemma 2.4.18 to simplify the calculations.

Appendix A

Index of Definitions

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