Some classical integrable systems with topological solitons

Winn, Elizabeth Amanda

Some Classical Integrable Systems with Topological Solitons

Amanda Elizabeth Winn

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematical Sciences
University of Durham,
England

July 1999

Abstract

Some Classical Integrable Systems with Topological Solitons

Amanda Elizabeth Winn

This thesis is concerned with some low dimensional non-linear systems of partial differential equations and their solutions. The systems are all in the classical domain and aside from a version of one model in Appendix D, are continuous. To begin with we examine the field equations of motion derived from Hamiltonian and Lagrangian densities, respectively defining the $(1+1)$-dimensional Hyperbolic Heisenberg and Hyperbolic sigma models, where the metric on the target manifold is indefinite.

The models are integrable in the sense that a suitable Lax pair exists, and admit solitonic solutions classifiable by an integer winding number. Such solutions are explicitly derived in both the static and time dependent cases where physical space X is the circle S^1. The existence of travelling wave solutions of topological type is discussed for each model with $X = S^1$ and $X = \mathbb{R}$; explicit solutions are derived for the $X = S^1$ case and it is shown for both the Heisenberg and sigma models, that no such travelling wave solutions exist if X is the real line. Nevertheless, time dependent solutions (not of travelling wave type) are possible in each case for $X = \mathbb{R}$, some examples of which are derived explicitly.

A further integrable system; the Hyperbolic 'Pivotal' model is proposed as a special case of a more general model on Hermitian symmetric spaces. Of particular interest is the fact that the Pivotal model interpolates between the previous two models. To begin with the integrability of the model is established via a Lax representation. Solutions analogous to some of those of the previous models are then derived and the interpolative limits examined with respect to the Heisenberg and sigma models. Conserved currents for the model are also briefly discussed.

Finally, some conclusions and further possibilities are noted including a brief examination of a discrete version of the sigma model where the target manifold is positive definite. A *Bogomol'nyi* bound is shown to exist for the systems energy in terms of a well defined winding number.

Preface

This thesis derives from work done by the author between October 1995 and July 1999 under the supervision of Professor R. S. Ward, in the Department of Mathematical Sciences, at the University of Durham. No part of it has been submitted previously for any degree at any university.

No claim on the originality of the reviews of Chapter 1, sections (2.1), (2.2), (3.1), (3.2) and (4.1) is made, however, it is believed that the remainder of the material is original. Parts of Chapters 2 and 3 were done in collaboration with Richard Ward and appear in a joint paper [1] published in *J. Phys. A: Math. Gen.*. Some of the remainder of these two chapters is to be published as part of the proceedings of *NEEDS '98* in a forthcoming issue of *J. Math. Phys.* [2]. The results of Chapter 4 have recently been submitted for publication.

Acknowledgements

The work in this thesis was supported by the EPSRC to whom I am extremely grateful. Without the help and support of my family, friends, academic and non-academic staff both at Durham and elsewhere, however, this thesis would no doubt still be a figment of my overactive imagination. I would therefore like to express my gratitude to at least a few of these people by name here. And to those who remain un-named due to my forgetfulness or reasons of space, a thousand thanks for everything.

My supervisor, Richard Ward, for whom I have the greatest respect and admiration, set me on the right track from the outset. Furthermore, and to my suprise, with his seemingly inexhaustable patience and wisdom, he managed to keep me from wandering off that track. For this and so much more I am truly grateful. Many thanks also to Wojtek Zakrzewski, John Bolton, Edward Corrigan, the office staff and all other members of staff at Durham who have given their advice and support over the last four years.

Life in Durham was made infinitely more fun by the many friends I made there and amongst these, in particular I owe thanks to Suzanne and Moncef for their many parties and other diversions and indeed, for their friendship; also to Andrew Pocklington, Medina Ablikim, Filipe Bonjour, David Berman, Jens Gladikowski and all the members of the maths football team. Outside the maths department I would especially like to thank Rick Coles who has done so much for me over the last few years with his immense generosity of spirit and who is without a doubt, the kindest, most honourable person I have ever met, and could ever hope to meet. Thanks also to Hazel Tucker for being there with me; to Georgina Fletcher, Tuncer Demir and all the other members of the household; to my good friend Flo for her kind support and reminders that life exists outside academia, and to Sally, Beverley and Con. For endless hours of football fun thanks to the various Sunday morning footballers, in particular to John Campbell for being the truly nice guy that he is. Many thanks also to the organisers and participants of the NEEDS '97, '98 and '99 conferences for making them so much fun that I just didn't want to come home.

I would like finally, to thank my family; my brothers Tony and Simon who have always been there for me and my uncle Alan for being like a father to me but most importantly, a very special thanks to my mother Lilian, whose faith in me has never wavered and whose daughter I am immensely proud to be.

Contents

Chapter 1

Introduction

The solution of non-linear partial differential equations (PDE's) and the determination of whether a particular equation can be solved at all relied, until relatively recently, more on trial and error and some luck than anything else. Whilst well established algorithms and consistently applicable strategies for the solution of their linear counterparts had been known for some time, none were available for non-linear PDE's until the mid to late 1960's when major breakthroughs were made. Since it's inception at that time, what has come to be known as the 'Theory of Solitons' has evolved at a breathtaking pace; various methods of solution and analysis of solvability applicable to 'soliton equations' have emerged whose implementation, whilst being far from trivial, has yielded an abundance of interesting features and solutions of a large number of equations. Furthermore, the increased interest generated by the expansion of the theory has had many positive ramifications; various connections have been made between a wide range of physical phenomena and some soliton equations, and soliton theory as a whole has flourished into a strikingly rich and diverse area of mathematical physics.

From a mathematical perspective the theories and constructs employed and developed to produce solitonic solutions of known equations and indeed new equations, range from Gardner, Greene, Kruskal and Miura's first formulation of the Inverse Scattering Transform (IST) [3] for solving the initial value problem for the Korteweg-de Vries (KdV) equation, to use of the ring of Gödel (hyperbolic) quaternions by Lambert and Piette in their derivation [4] of solutions of σ-models on non-compact manifolds. From a physical point of view, one can find applications of soliton theory extending from, on the smallest scale, elementary particle theory to cosmology on the largest, with hydrodynamics, non-linear optics and biophysics in between.

With such a wealth of theory and applications established together with enormous scope for further possible expansion, little or no encouragement is needed for further exploration into the world of solitons and in this thesis we examine certain non-linear PDE's with solitonic features from which, it is hoped, some contribution may be made to the theory of solitons.

As a starting point and to establish some foundations on which to build, in the following section the notion of a soliton is discussed, initially in a general sense and subsequently with more specific definitions as applicable to the non-linear systems considered later. Two well known integrable models are then described with particular emphasis placed on how the relevant solitonic features occur in these systems.

1.1 Solitons

The term *soliton* was first used [5] in 1965 by Zabusky and Kruskal in reference to the remarkable particle-like behaviour manifested by non-linear wave solutions of the KdV equation; when two or more of these waves are brought together, following a strong interaction, they continue on an almost undisturbed path retaining their shape after collision thus acting in a particle-like manner. To be a little more precise, use of the word soliton initially and subsequently pertained to localized lumps of energy, retaining constant velocity and profile on collision and which are stable to perturbations. More recently however, the term has been employed to cover all manner of localized configurations and it is in this looser sense that we apply it. So to avoid any ambiguity later, when referring to solitons throughout this text we use the following definition:

> *A soliton is a spatially localized, non-singular solution of a classical non-linear partial differential equation with constant velocity and profile.*

Effectively two (almost) distinct classes of soliton exist: topological solitons and those arising from what are known as *integrable* equations. The quintessential difference being that topological solitons owe their stability to the global topology of the system whereas 'integrable' solitons are stable due to local details of the evolution equation. The latter are comparatively rare since the balance between dispersive and non-linear sharpening terms, which gives the solitons their stability, is rather delicate and apt to collapse under perturbations of the evolution equation.

The majority of known integrable systems, scarce enough as it is, are restricted to $(1+1)$ and $(2+0)$ dimensions where solitons admitted by $(2+0)$-dimensional systems are the static solutions of the system in $(2+1)$ dimensions. For example, what serve as instantons of the (integrable) Euclidean version of the $(1+1)$-dimensional $O(3)$ model are the static solutions of the model in $(2+1)$ dimensions. The higher dimensional model is not, however, integrable. Some higher dimensional integrable equations are known though, for example, the Davey Stewartson equation [6], its vector counterpart - Ishimori's equation [7], and the Kadomtsev-Petviashvili (KP) equation [8] (a two dimensional extension of the KdV equation). However, whilst many of the $(1+1)$-dimensional equations such as the well known sine-Gordon equation and the $O(3)$ model in $(1+1)$ dimensions are Lorentz invariant, this is not the case for the higher dimensional integrable equations noted above. In fact, apart from a semi-relativistic chiral system in $(2+1)$ dimensions [9], and the self-dual Yang Mills (SDYM) equations in four dimensions [10, 11], there appear to be no known higher dimensional Lorentz invariant integrable systems. Lorentz invariance aside, there are actually no known $(3+1)$-dimensional *integrable* equations other than the SDYM equations (which are conjectured [12] and partly verified [13, 14] as a sort of 'master equation'), and to find a non-linear integrable evolution equation in $(3+1)$ dimensions is, in the words of Ablowitz and Clarkson [15], *"the most important open problem in soliton theory"*.

So integrable equations, residing mainly in low dimensions, tend for the most part, to be of more interest for their mathematical properties such as the possibility of constructing infinitely many conserved quantities and, if the Inverse Scattering Transform[1] can be applied, the initial value problem may, at least in theory, be solved and a large number of exact, non-trivial solutions explicitly derived.

Topological solitons on the other hand, may be found in any number of dimensions their stability, due to the topology of the configuration space, being more robust hence, lending themselves more readily to physical interpretation. Such solitons are classifiable by an integer winding number or topological charge, conserved under time evolution and soliton collision and this may be interpreted as one of the conserved quantities of particle physics. As an example, in the $(3+1)$-dimensional Skyrme model [17, 18] soliton solutions are thought of as baryons (in particular protons and neutrons) where the topological charge is identified with the conserved baryon number.

In contrast to integrable models and aside from the topological charge, systems admitting topological solitons tend to have only a small number of conserved quan-

[1]The IST is a kind of non-linear analogue of the application of the Fourier Transform in solving linear PDE's.

tities corresponding to the global symmetries of the system for example, energy, momentum and angular momentum. It is also more difficult (in the absence of the IST and related theories) to find non-trivial, and in particular, exact solutions and those which can be derived tend to be restricted to the static case. Numerical methods are, however, becoming increasingly helpful in the study of the dynamics of such solitons.

1.1.1 Integrability

Despite their relative rarity, non-linear partial differential equations which are 'integrable' arise as a recurrent feature in the study of non-linear phenomena and to give meaning to the notion of integrability we first recall the situation with respect to ordinary differential equations: the classic Liouville definition for integrability applies, namely; for Hamiltonian systems with n degrees of freedom there should exist $(n-1)$ independent functions g_i on phase space which are in involution with the Hamiltonian and with each other and these functions must exist globally. A continuous system of ODE's should therefore have an infinite number of such conserved quantities to be integrable.

For partial differential equations the situation is less straightforward, indeed there does not appear to be a universally accepted definition of integrability for such equations. There are, however, various ways in which a system might be thought of as integrable and we list some of these as follows:

(i) In close analogy to the Liouville definition one may require that a Hamiltonian system admits an infinite number of conserved quantities and integrals of the motion. A *completely integrable* system as defined by Faddeev and Zakharov [19, 20] is one for which the conserved quantities are in involution and the action angle variables with which the Hamiltonian may be expressed can be found.

(ii) For many integrable equations the IST can be applied to solve the initial value problem via the scattering data.

(iii) As an indicator of integrability one may require that the system satisfy the Painlevé test (c.f. [21, 22]) which, loosely speaking says that its solutions should be meromorphic functions of the complexified independent variables.

(iv) Or, lastly, one might require that the equations of motion for the system can be written in the form of a compatibility condition for a suitable overdetermined system of linear equations.

In fact (c.f. [15]), the IST **(ii)**, is thought to encompass many of the other definitions proposed for integrability. In what follows we consider a system to be integrable in the sense of **(iv)** above. In particular, given the overdetermined system of linear matrix equations

$$\frac{\partial F}{\partial x} = U(x,t,\lambda)F \tag{1.1a}$$

$$\frac{\partial F}{\partial t} = V(x,t,\lambda)F \tag{1.1b}$$

where U, V are 2×2 matrices, F a matrix valued function and λ the 'spectral parameter'. Then the compatibility condition for (1.1) is obtained by differentiating (1.1a) with respect to t, (1.1b) with respect to x and subtracting so that

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0. \tag{1.2}$$

For the relevant non-linear system to be integrable, (1.2) must be equivalent to the equation(s) of motion for a suitably chosen U and V. Equation (1.2) is known as the zero curvature representation or condition and must hold for all complex constants λ. And the equations (1.1) are known as a Lax pair.

1.1.2 Conventions, Basic Concepts and Topology

As has been noted, topological solitons owe their stability to global aspects of the system, i.e. its topology. The stability relies in particular, on the (dis)connectedness of the configuration space and in order to explain this more fully and to establish the basic framework for our discussion we begin by specifying the conventions in use throughout. Some of the necessary topological and physical concepts are then defined with specific emphasis placed on those of direct relevance to the models considered in subsequent chapters.

The conventions in use unless otherwise stated are as follows: $t \in \mathbb{R}$ denotes time and $x \in X$ is the space variable so that $X \times \mathbb{R}$ is space-time and a space-time event is given by $x^\mu = (x^0, x^1) = (t, x)$. $\eta^{\mu\nu} = \text{diag}(1, -1)$ is the (inverse) space-time metric and we use Einstein's index summation convention throughout. The tensor ϵ_{abc} is completely anti-symmetric such that $\epsilon_{abc} = -\epsilon_{acb}$ and is identically zero if any two indices are the same (similarly for $\epsilon_{\mu\nu}$).

Given that it is the topology of the system which is of interest, the following definitions and ideas will be useful: first, a topological space χ is **connected** if it is *not* the union of two (or more) non-empty disjoint open sets. In other words if $\chi = \chi_1 \cup \chi_2$ (χ_1, χ_2 open) and $\chi_1 \cup \chi_2 = \emptyset$ then χ is **disconnected**.

Now let χ and X be topological spaces and let $\Phi(\chi, X)$ be the set of continuous maps from χ to X. If f and g are two maps in Φ then f is **homotopic to** g if the image $f(\chi)$ may be continuously deformed to $g(\chi)$ in X. The set Φ is therefore divided by homotopy into equivalence classes or 'homotopy sectors'. Of particular relevance here is the choice that X be the circle S^1. In this case, the homotopy sectors are the classes of loops on X. More precisely, if X is a topological space and $I = [0,1]$; a continuous map $f : I \longrightarrow X$ is a **path** with initial point x_0 and endpoint x_1 if $f(0) = x_0$ and $f(1) = x_1$. If then $f(0) = f(1) = x_0$ the path is instead called a **loop** and x_0 its **basepoint**. Now letting f and $g : I \longrightarrow X$ be loops with the basepoint x_0; f and g are homotopic ($f \sim g$) if a continuous map $\phi : I \times I \longrightarrow X$ exists such that

$$\begin{aligned} \phi(s,0) &= f(s), \qquad \phi(s,1) = g(s) \qquad \forall s \in I \\ \phi(0,t) &= \phi(1,t) = x_0 \qquad \forall t \in I. \end{aligned}$$

The equivalence class of loops is denoted by $[f]$ and called the **homotopy class**.

The (first) **fundamental group** for the topological space X is defined as the set of homotopy classes of loops based at $x_0 \in X$ and is denoted by $\pi_1(X, x_0)$. In particular, if $X = S^1$ one has the fundamental group $\pi_1(S^1)$ and this is isomorphic to the group of integers $\mathbb{Z}$ (where the basepoint may be left out since S^1 is arcwise connected). Of considerable relevance is the following: let f and $g : S^1 \longrightarrow S^1$ be maps defined in such a way that $f(0) = g(0) = 1 \in S^1$ and denote the **degree** of the map f by $\deg(f) = N \in \mathbb{Z}$, corresponding to the number of times the (target) circle is traversed under the map. Then $\deg(f) = \deg(g)$ if and only if f is homotopic to g. In addition, for *any* $N \in \mathbb{Z}$ there exists a map $f : S^1 \longrightarrow S^1$ such that $\deg(f) = N$. What all this means is that the fundamental group $\pi_1(S^1)$ is comprised of classes of loops, each of which is classified by a particular integer N, and by constancy of the winding number [24] a configuration in one homotopy sector cannot be continuously deformed into one in another; so $\pi_1(S^1)$ is a disjoint union of homotopy classes and is therefore disconnected. The degree of the map, i.e. the integer N, is referred to as the **winding number** or **topological charge**.

So how is all of this somewhat abstract topology related to soliton theory? In order to make the connection, some slightly more physical concepts must enter the arena which are presented below, and from these we shall see that the stability of topological solitons for the models of subsequent chapters is dependent on the above topological ideas.

(i) As noted, X is physical space and $X \times \mathbb{R}$ space-time.

(ii) A **configuration** Ψ is a map from physical space to some target manifold M; $\Psi : X \longrightarrow M$.

(iii) A **field** $\tilde{\Psi}$ is a continuous time sequence of configurations, i.e. a map from space-time to the target manifold $\tilde{\Psi} : X \times \mathbb{R} \longrightarrow M$. So that the time evolution of a field is thought of as a smooth progression through a family of configurations.

(iv) **Configuration space** $\mathcal{C}$ is then defined as the space of configurations so that each configuration is a point in $\mathcal{C}$ and a trajectory in $\mathcal{C}$ is a solution of the field equations. $\mathcal{C}$ is infinite dimensional.

It is the topology, and in particular, the connectedness of the configuration space $\mathcal{C}$ which is the crucial factor for the stability of topological solitons and it will be helpful to specify some spaces in order to make this clear. This also provides an appropriate opportunity to introduce those spaces pertaining to the Hyperbolic Heisenberg, Hyperbolic Sigma and Hyperbolic Pivotal models which are the main subject of this thesis: the field is a three vector denoted by $\vec{\psi}(t,x) = (\psi^1, \psi^2, \psi^3)$. Physical space X is such that either $X = S^1$, in which case $\vec{\psi}$ is periodic in x, or $X = \mathbb{R}$ and the boundary condition $\vec{\psi}(t,\infty) = \vec{\psi}(t,-\infty)$ is imposed. The target manifold M is taken to be the cylinder $S^1 \times \mathbb{R}$ which may be thought of as the hyperboloid of one sheet H^2 in $\mathbb{R}^{2+1}$ (see figure (1.1)). Then $\vec{\psi}$ satisfies

$$\eta_{ab}\psi^a\psi^b = 1 \tag{1.3}$$

where $\eta_{ab} = \mathrm{diag}(1, 1, -1)$. The metric on the hyperboloid (1.3) is taken to be that induced by the metric η_{ab}, then M is a symmetric space[2] with group structure

[2] A symmetric space may be defined as follows: given a manifold $M = G/H$ where G is a connected Lie group and H is a closed subgroup of G; if for each point $p \in M$, $\exists$ a smooth isometry τ_p of M of order two which preserves geodesics through p and has p as an isolated fixed point then M is a symmetric space.

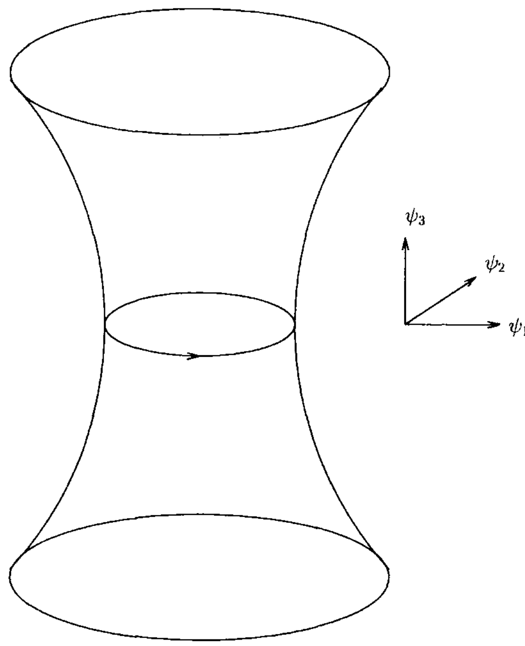


Figure 1.1: The hyperboloid $H^2 : (\psi^1)^2 + (\psi^2)^2 - (\psi^3)^2 = 1$

$SO(2,1)/SO(1,1)$, and is sometimes referred to as two dimensional de Sitter space-time. Having specified physical space X in two different ways we consider the cases separately:

(i) $\underline{X = S^1}$: Here $\vec{\psi}$ is periodic in x and configuration space $\mathcal{C}$ is the space of smooth maps from S^1 to H^2 and for non-trivial maps, the circle should be wrapped around H^2. Effectively one then has a map from S^1 to S^1 with $\pi_1(H^2) \cong \pi_1(S^1) = \mathbb{Z}$ and the topological charge is the degree of the map. Recalling the definition of $\mathcal{C}$ as the space of configurations and a solution of the field equations as a trajectory in $\mathcal{C}$; if such a solution resides in a particular homotopy class of configuration space, it cannot be continuously deformed into one in another sector by constancy of the winding number. Such a solution is therefore confined to its homotopy sector and is, in this sense, stable.

(ii) $\underline{X = \mathbb{R}}$: In this case $\mathcal{C}$ is the space of smooth maps $\vec{\psi}$ from $\mathbb{R}$ to H^2. The boundary condition $\vec{\psi}(t,\infty) = \vec{\psi}(t,-\infty)$ is imposed, i.e. $\lim_{x\to\pm\infty} \vec{\psi}(x) = p_0 \in H^2$ and $\vec{\psi}$ reaches the limit sufficiently fast that integrals converge. (We note that in some cases one may specify that a configuration have finite energy, in which case, the 'energy density' denoted by ε, which is a function of $\vec{\psi}$ and its first derivatives, must vanish on the boundary of space (i.e. at $x = \pm\infty$), otherwise the integral of ε over X will diverge.) Thus $\vec{\psi}$ may be regarded as a map from the one point compactification $\mathbb{R} \cup \{\infty\} \cong S^1$ to S^1. And again we have a map from the circle to the circle classified by an integer winding number - for each homotopy class the real line is, in effect, wrapped around the hyperboloid with winding number N where the point p_0 corresponds to the map $\vec{\psi}$ at $x = \pm\infty$. Each configuration is therefore trapped in one of the homotopy classes of the fundamental group $\pi_1(S^1) = \mathbb{Z}$. (If finite (non zero) energy is required one must also have $\nabla\vec{\psi} \neq 0$ in some region of X space ensuring the presence of a lump or lumps of energy.)

The fundamental group $\pi_1(S^1)$ is the **first** fundamental group and the idea of homotopy groups may be generalized; the n**th homotopy group**, denoted by $\pi_n(X, x_0)$ for a topological space X and n-loops based at x_0 is called a **higher** homotopy group if $n \geq 2$ (where an n-loop at x_0 is a map from n copies of the unit cube to X).

Topological 'defects' owing their stability to non-trivial $\pi_n(X, x_0)$ are called **textures** and those arising in the systems we shall describe with the above considerations fall into this category. Skyrmions and $\mathbb{C}P^1$ lumps are further examples of textures and they may also be found in condensed matter theory, for example, in the 'superfluid' ^{3}He-A [23]. In addition to textures, a further class of topological defects called **monopoles** arises from similar considerations where here the stability is due to the non-triviality of $\pi_{n-1}(V_0)$, V_0 being the vacuum manifold, i.e. where the potential energy reaches its absolute minimum. However, since these are not relevant for the models under examination, we simply make note of their existence and refer to various texts in which they are discussed in detail, see for example [16, 23, 25, 26].

1.2 Combining Integrability and Topological Solitons

Integrability and topological solitons have been discussed in the previous section as unrelated concepts, excepting the fact that they both pertain to the solvability or solutions of non-linear PDE's. However, one might naturally consider the question; do systems exist which exhibit both features? In fact, there are such systems although, as one can imagine, since integrable systems are rare enough objects as it is, models which are both integrable *and* admit topological solitons are extremely scarce. Various known *elliptic* systems of PDE's exhibit both properties, however, in these systems there is no time dependence so that only static solutions exist. The Euclidean version of the $O(3)$ model in $(1+1)$ dimensions is one example(c.f. [16]) where the instantons of the model are classifiable by an integer winding number and the model is characterized by an infinite number of conserved quantities [28, 29]. An example of a higher dimensional integrable model with instanton solutions is the self-dual Yang Mills system which is briefly examined later in the section. Prior to this however, to put some of the ideas described into action, we discuss what is virtually the only (at least well-)known example of a *time* dependent system which is both integrable and admits topological solitons.

1.2.1 The Sine-Gordon Model

Having initially appeared in 1870 in differential geometry [32] where it described surfaces of constant negative Gaussian curvature, the sine-Gordon equation[3] has, over the ensuing years, found many and various physical applications. For example, the propagation of a dislocation in a crystal, with its periodicity represented by $\sin\psi$, was shown in 1939 [33] to be governed by the sine-Gordon equation and in 1962 it was tentatively considered [34] as a model of an elementary particle and shown in 1975 [35] to be an equivalent form of the Thirring model. This remarkably diverse equation has therefore been studied extensively and various other applications found in addition to those noted, however, let us restrict ourselves here to reviewing the model in the context of the ideas discussed earlier in this section.

The sine-Gordon model is formulated as follows: taking physical space X to be the real line and the target manifold M to be the circle S^1; consisting of a single scalar field $\phi(t,x)$ in $(1+1)$ dimensions, the model is governed by the Lagrangian density

$$\mathcal{L}(t,x) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{m^2}{\beta^2}\left[\cos(\beta\phi) - 1\right] \tag{1.4}$$

where m and β are positive parameters. Physically the model describes a relativistic field theory in two space dimensions and m and β play the role of the mass and coupling constant respectively. The equation of motion is obtained via Hamilton's

[3]The name 'sine-Gordon' came about much later as a lightly controversial pun on Klein-Gordon, attributed to Finkelstein and Rubenstein [16] although the latter passed on parentage to M. Kruskal in 1970.

variational principle by extremising the action functional

$$S[\phi] = \int dt \int dx\ \mathcal{L}.$$

Variation of S yields

$$\delta S[\phi] = \int dt \int dx \left\{ -\frac{m^2}{\beta} \sin\ \beta\phi - \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} \right\} \delta\phi$$

and for ϕ to be a local extremum of S requires

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \frac{m^2}{\beta} \sin\ \beta\phi = 0. \tag{1.5}$$

The Hamiltonian (or conserved energy integral) for the system is then

$$H = \int_X \mathcal{H}\, dx = \int_X \left\{ \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{m^2}{\beta^2}\left[1 - \cos\ \beta\phi\right] \right\} dx \tag{1.6}$$

where $\mathcal{H} = \pi(x)\dot{\phi}(x) - \mathcal{L}$ with the canonically conjugate momentum variable $\pi(x) = \frac{\partial \mathcal{L}}{\partial \phi}$. We note also that with the Poisson bracket structure defined by $\{\phi(x), \phi(y)\} = \{\pi(x), \pi(y)\} = 0, \{\pi(x), \phi(y)\} = \delta(x - y)$, and with the Hamiltonian (1.6), equation (1.5) may be written in Hamiltonian form $\frac{\partial \phi}{\partial t} = \{H, \phi\}$; $\frac{\partial \pi}{\partial t} = \{H, \pi\}$.

Non-singular finite energy solutions are the objects of interest here and for both static and time dependent solutions, for finite energy, cos $\beta\phi$ must tend to a constant at each end of space (and $\nabla\phi$ to zero). Further, from (1.6), this constant must be a zero of the potential $\mathcal{V}(\phi) = \frac{m^2}{\beta^2}\left[1 - \cos\ \beta\phi\right]$, and these are given by $\phi = \frac{2\pi n}{\beta}\ :\ n \in \mathbb{Z}$. If cos $\beta\phi$ tends to the *same*[4] constant as $x \longrightarrow \pm\infty$, finite energy configurations are maps from the one point compactification $\mathbb{R} \cup \{\infty\} \cong S^1$ to the target space S^1 and as noted previously, these fall into disjoint homotopy classes since $\pi_1(S^1) = \mathbb{Z}$. Configuration space is the union of these disjoint classes where each class is labeled by its topological charge or winding number. For the sine-Gordon model this is defined specifically as

$$N = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} dx$$

[4]This need not be the case; cos $\beta\phi$ may tend to different values at each end of space and solitons (or kinks as they are called in $(1+1)$ dimension) resulting from such a choice of boundary conditions are classified as monopoles.

where $N \in \mathbb{Z}$. So solitons and indeed all finite energy solutions of the sine-Gordon system are topological in nature.

That the model is completely integrable is well established, see for example, [36, 37] so rather than going into unnecessary detail and in keeping with the definition of integrability as applied to the models which are the subject of this thesis, we simply note that the equation of motion (1.5) is representable as the Zero Curvature condition (1.2) with $U(x,t,\lambda), V(x,t,\lambda)$ given (c.f. [63]) by

$$\begin{aligned} U &= \frac{\beta}{4i}\pi\sigma_3 + \frac{k_0}{i}\sin\frac{\beta\phi}{2}\sigma_1 + \frac{k_1}{i}\cos\frac{\beta\phi}{2}\sigma_2 \\ V &= \frac{\beta}{4i}\frac{\partial\phi}{\partial x}\sigma_3 + \frac{k_1}{i}\sin\frac{\beta\phi}{2}\sigma_1 + \frac{k_0}{i}\cos\frac{\beta\phi}{2}\sigma_2, \end{aligned}$$

i.e. the sine-Gordon equation may be written as the compatibility condition for an overdetermined linear system. Here the $\sigma_i\,:\,i=1,2,3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.7}$$

$$\text{and} \quad k_0 = \frac{m}{4}\left(\lambda + \frac{1}{\lambda}\right), \quad k_1 = \frac{m}{4}\left(\lambda - \frac{1}{\lambda}\right)$$

with spectral parameter λ.

We are essentially considering a system whose dynamics is governed by the Lorentz invariant Lagrangian density

$$\mathcal{L}(t,x) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \mathcal{V}(\phi)$$

where the potential $\mathcal{V}(\phi)$ has a minimum value of zero for certain ϕ and is positive semi-definite. Applying the variational principle results in the wave equation

$$\Box\phi = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial\mathcal{V}}{\partial\phi},$$

the non-linear terms depending on the choice of $\mathcal{V}(\phi)$ which in the sine-Gordon case is $\frac{m^2}{\beta}\left[1-\cos\beta\phi\right]$. For static solutions one then has

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\mathcal{V}}{\partial\phi}$$

which, on multiplying by $\frac{\partial\phi}{\partial x}$ and integrating results in

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 = \mathcal{V}(\phi)$$

or

$$\frac{d\phi}{dx} = \pm[2\mathcal{V}(\phi)]^{\frac{1}{2}}.$$

A further integration yields

$$\pm\int\frac{d\phi}{\sqrt{2\mathcal{V}(\phi)}} = x - a$$

with a constant. For the sine-Gordon equation this translates as

$$\pm\frac{\sqrt{\beta}}{2m}\int\frac{d\phi}{\sin\frac{\beta\phi}{2}} = x - a.$$

The left hand side is easily integrated resulting in the solutions

$$\phi(x) = \pm\frac{2}{\beta}\tan^{-1}\left\{\exp[m\sqrt{\beta}(x-a)]\right\}. \tag{1.8}$$

In the positive case the solution has winding number $N = 1$ where (1.8) is then called the soliton of the system and goes from 0 to 2π (or 2π to 4π etc.), whereas in the negative case the winding number is $N = -1$ and (1.8) is called the antisoliton. The antisoliton is related to the soliton by the symmetry $\phi \longleftrightarrow -\phi$ under which the Lagrangian (1.4) is invariant.

Since the system is relativistically invariant, moving solitons can immediately be generated by Lorentz boosting (1.8); replacing $(x - a)$ by $\frac{(x-a-vt)}{\sqrt{1-v^2}}$ one has

$$\phi(t,x) = \pm\frac{2}{\beta}\tan^{-1}\left\{\exp\left[\frac{m\sqrt{\beta}}{\sqrt{1-v^2}}(x-a-vt)\right]\right\}$$

which recovers (1.8) if $v = 0$. As well as the soliton and antisoliton the sine-Gordon model admits a third type of solitonic solution called the doublet or breather which is thought of as a 'bound' solution of a soliton-antisoliton pair. The soliton and antisoliton oscillate with respect to one another with a given period; however, such solutions will not be examined here.

This completes our discussion of the sine-Gordon system as an illustrative example of a $(1+1)$-dimensional time dependent system which is both integrable and admits topological solitons. Prior to laying out a synopsis of the contents of the thesis proper we conclude the introductory material with a brief description of a higher dimensional integrable system which may well turn out to be at the foundation of all integrable systems.

1.2.2 The Self-Dual Yang Mills System

The gauge invariant self-dual Yang Mills (SDYM) system is possibly the most important known integrable system; it has been conjectured by Ward [12] that

> *"many (and perhaps all?) of the ordinary and partial differential equations that are regarded as being integrable or solvable may be obtained from the self-dual gauge field equations (or its generalization) by reduction."*

Whilst this remains as yet unproven much progress has been made to substantiate the claim. For example, in $(2+1)$ dimensions a $SU(N)$ chiral field equation with torsion has been shown (c.f. [9, 38]) to be a reduction of SDYM, as have the Bogomol'nyi equations relevant in the study of magnetic monopoles [39, 40, 41]. The Toda Molecule equation (a $(2+1)$-dimensional version of the $(1+1)$-dimensional Toda lattice) is obtained by reduction and it has been shown [15] that the Toda Molecule reduces asymptotically to the KP equation which can itself be asymptotically reduced to the Davey Stewartson (DS) equation, indicating that it may be possible to obtain the KP and DS equations by exact reductions of SDYM. In $(1+1)$ dimensions the KdV and non-linear Schrödinger equations arise as reductions [14], as do the sine-Gordon equation of the previous section and the Ernst equations. There are also various examples of ordinary differential equations which have been shown to be reductions of SDYM. And with more and more equations being shown to be obtainable from the SDYM equations the conjecture looks increasingly likely to be true however, as there are so many possibilities a complete analysis of all reductions poses an immensely challenging problem.

In addition to its as yet unquantified importance in the general theory of integrable equations, the SDYM system is an example of a higher dimensional integrable model (of elliptic type) with topological solitons . And following a description of the model and an example of the reduction process, the integrability and presence of instanton solutions are briefly discussed.

The Yang Mills **curvature** or **gauge** field may be defined as

$$F_{ab} = \partial_a A_b - \partial_b A_a - [A_a, A_b] \tag{1.9}$$

where $x_a : a = 1,2,3,4$ are coordinates in physical space $X = \mathbb{R}^4$ equipped with some metric. In particular, instanton solutions occur in Euclidean space where the signature of the metric is $(+,+,+,+)$. The A_a are the Yang Mills potentials taking values in some Lie algebra ϱ; if we take the associated Lie group $\mathcal{G}$ to be $SU(2)$ then the A_a are elements of the gauge group $SU(2)$ in its 2×2 representation. In terms of the standard null coordinates

$$\tau = x_1 + ix_2 \qquad \sigma = x_3 + ix_4$$
$$\bar{\tau} = x_1 - ix_2 \qquad \bar{\sigma} = x_3 - ix_4$$

where the metric is $ds^2 = d\tau d\bar{\tau} + d\sigma d\bar{\sigma}$, the corresponding gauge potentials are

$$A_\tau = A_1 + iA_2 \qquad A_\sigma = A_3 + iA_4$$
$$A_{\bar{\tau}} = A_1 - iA_2 \qquad A_{\bar{\sigma}} = A_3 - iA_4.$$

The self-dual Yang Mills equations are then given by

$$F_{\tau\sigma} = 0 \tag{1.10a}$$
$$F_{\bar{\tau}\bar{\sigma}} = 0 \tag{1.10b}$$
$$F_{\tau\bar{\tau}} + F_{\sigma\bar{\sigma}} = 0 \tag{1.10c}$$

which are equivalent to the self-duality condition $\tilde{F}_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd} = \pm F_{ab}$ and are invariant under the gauge transformation

$$A_a \longrightarrow UA_aU^{-1} - (\partial_a U)U^{-1}, \qquad F_{ab} \longrightarrow U^{-1}F_{ab}U.$$

The SDYM equations may be defined on Euclidean $\mathbb{R}^4$ as above, where instanton solutions occur, however, to introduce time into reductions, a mixed metric signature must be imposed and only with the signature $(+,+,-,-)$ do the SDYM equations admit real solutions. (In $(3+1)$ dimensions, i.e. with signature $(+,+,+,-)$ no real solutions are possible).

Whilst the reduction process is not of direct relevance to the contents of this thesis, a potentially interesting project for the future may be to consider whether or not the Pivotal model of Chapter 4 is a reduction of the SDYM equations. With this in mind, the following demonstrates how the procedure works for an integrable model with topological solitons, and an appropriate choice of model to briefly illustrate the reduction process is a version of the sine-Gordon model of the previous section. The equation

$$\nabla\phi = \sin\phi \tag{1.11}$$

(where $\nabla\phi = \phi_{xx} + \phi_{tt}$) is obtained from the SDYM equations (1.10) in the following ways [15, 42]: let the A_a depend only on τ and $\bar{\tau}$ and still take their values in the Lie algebra $su(2)$ generated by the Pauli matrices (1.7) which satisfy $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. $A_{\bar{\tau}}$ can be set to zero by using an appropriate gauge freedom. Referring forward to (1.14) this results in the Lax pair

$$\partial_\tau\Psi = (A_\tau + \lambda A_{\bar{\sigma}})\Psi, \qquad \lambda\partial_{\bar{\tau}}\Psi = -A_\sigma\Psi. \tag{1.12}$$

One can then choose

$$A_\tau = c\sigma_3, \qquad A_\sigma = (a+b)\sigma_2, \qquad A_{\bar{\sigma}} = \sigma_1$$

with a, b, c functions of τ and $\bar{\tau}$ and substitution into (1.12) yields the three first order equations

$$\partial_\tau a = -2ibc \tag{1.13a}$$
$$\partial_\tau b = 2iac \tag{1.13b}$$
$$\partial_{\bar{\tau}} c = 2ib. \tag{1.13c}$$

These have the solution $a^2 + b^2 =$ constant $= 1$. So that with the choice $a = \cos\phi$, $b = \sin\phi$, c is given by $c = \frac{i}{2}\partial_\tau\phi$ and from equation (1.13c) one gets a sine-Gordon equation (1.11) with $\nabla\phi = \frac{1}{4}\partial_\tau\partial_{\bar{\tau}}\phi = \frac{1}{4}(\phi_{x_2x_2} + \phi_{x_1x_1})$.

Alternatively, instead of $su(2)$ suppose the A_a lie in the Lie algebra $sl(2,\mathbb{C})$. Now let A_a depend only on x and t so that $\tau = \bar{\tau} = x_1 = t$ and $\sigma = \bar{\sigma} = x_3 = x$. Then (1.10a) and (1.10b) are satisfied so long as ϕ satisfies the sine-Gordon equation

(1.11) by choosing

$$A_\tau = \frac{1}{4}\begin{pmatrix} -\phi_x & -2\sin\frac{\phi}{2} \\ 2\sin\frac{\phi}{2} & \phi_x \end{pmatrix}, \qquad A_\sigma = \frac{1}{4}\begin{pmatrix} \phi_t & -2\cos\frac{\phi}{2} \\ -2\cos\frac{\phi}{2} & -\phi_t \end{pmatrix}$$
$$A_{\bar\tau} = \frac{1}{4}\begin{pmatrix} -\phi_x & 2\sin\frac{\phi}{2} \\ -2\sin\frac{\phi}{2} & \phi_x \end{pmatrix}, \qquad A_{\bar\sigma} = \frac{1}{4}\begin{pmatrix} \phi_t & 2\cos\frac{\phi}{2} \\ 2\cos\frac{\phi}{2} & -\phi_t \end{pmatrix}.$$

And equation (1.10c) is satisfied identically.

In general there are two reductive procedures by which an integrable system might be obtained from the SDYM equations (c.f. [13, 43])

(i) By factoring out a subgroup of the Poincaré group reduce the number of independent variables. This is know as **dimensional reduction**.

(ii) By imposing algebraic constraints on the gauge potential A_a (in a way that is consistent with the equations) the number of dependent variables may be reduced; **algebraic reduction**.

For the sine-Gordon equation above a combination of the two was utilized as is often the case in order to obtain a particular integrable system.

Returning to the SDYM equations themselves (1.10); in the context of integrability, consider the overdetermined linear system

$$\begin{aligned}(\partial_\tau + \lambda\partial_{\bar\sigma})\Psi &= (A_\tau + \lambda A_{\bar\sigma})\Psi \\ (\partial_\sigma - \lambda\partial_{\bar\tau})\Psi &= (A_\sigma - \lambda A_{\bar\tau})\Psi\end{aligned} \tag{1.14}$$

where λ is the complex valued spectral parameter. The SDYM equations may be written as the compatibility condition, expressed as the following polynomial in λ, for the system (1.14):

$$(\partial_\tau + \lambda\partial_{\bar\sigma})(A_\sigma - \lambda A_{\bar\tau}) - (\partial_\sigma - \lambda\partial_{\bar\tau})(A_\tau + \lambda A_{\bar\sigma}) = \left[A_\tau + \lambda A_{\bar\sigma}, A_\sigma - \lambda A_{\bar\tau}\right].$$

Equations (1.10) are then produced by equating powers of λ and are therefore integrable in the sense noted earlier. In fact they are regarded as completely integrable as a consequence of what is known as the 'twistor' construction [10, 44, 45, 46], relating solutions of the equations to certain holomorphic vector bundles.

With regard to the existence of topological solitons for the system; loosely defined, instantons are localised finite action solutions of the classical Euclidean field equations of a theory (c.f. [16]) where 'Euclidean', in 4 dimensions, refers to replacement of the Minkowskian indefinite metric by the definite Euclidean one so that the Euclidean theory is invariant under $O(4)$ transformations rather than the Lorentz transformations. The Euclidean system can be thought of as an analytical continuation of the Minkowskian one. The finiteness of energy requirement of solitons is replaced by requiring finite Euclidean action; at the classical level the Euclidean action has the same structure as the energy of a static field configuration in one higher dimension. The Euclidean action for the Yang Mills system is given by

$$\mathcal{S} = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}[F_{ab}F_{ab}] \tag{1.15}$$

where 'Tr' denotes trace, and extremising (1.15) through the variational principle results in the corresponding field equations

$$D_aF_{ab} \equiv \partial_a F_{ab} + [A_a, F_{ab}] = 0. \tag{1.16}$$

Using the identity

$$-\int d^4x \, \mathrm{Tr}[(F_{ab} \pm \tilde{F}_{ab})^2] \geq 0,$$

and since $\mathrm{Tr}[F_{ab}F_{ab}] = \mathrm{Tr}[\tilde{F}_{ab}\tilde{F}_{ab}]$ one has

$$\begin{aligned} -\int d^4x \mathrm{Tr}[F_{ab}F_{ab}] &\geq \pm \int d^4x \mathrm{Tr}[\tilde{F}_{ab}F_{ab}] \\ \text{or} \qquad \mathcal{S} &\geq \frac{8\pi^2}{g^2}|N| \\ \text{where} \qquad N &= -\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}[\tilde{F}_{ab}F_{ab}] \end{aligned} \tag{1.17}$$

and N is the winding number. This denotes the number of times the group space $SU(2)$ is wrapped around the target space which is taken to be S^3. In fact, (c.f. [16, 26]) from finite action considerations and since $\pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}$, configuration space is the disjoint union of finite action configurations classifiable by the integer N. In a given homotopy class, fields extremising the action (1.15) are solutions of (1.16) in that class. The absolute minimum value of (1.15) is attained in a given homotopy sector as

$$S = \frac{8\pi^2}{g^2}|N|$$

and from (1.17) this occurs when $\tilde{F}_{ab} = \pm F_{ab}$. The action (1.15) is therefore extremised by the self and antiself-dual configurations. The absolute minima may not, of course, be the only extrema of the action so that whilst solutions of the self-dual equations (1.10) are always solutions of the Yang Mills system (1.16), the converse is not necessarily the case. The Yang Mills equations themselves however, are not integrable, (admitting chaotic solutions) whereas the self-dual system is.

In the derivation of instanton solutions of (1.10), various methods have been used; originally the one-instanton solution was produced by Belavin *et al* [47] and subsequently one and many-instanton solutions obtained by a technique evolved through the work of a collection of authors c.f. [48, 49, 50, 51] and also [52]. Since such methods are not, however, utilized in our study of the Hyperbolic Heisenberg, sigma and Pivotal models we refrain from examining them here.

This concludes the introductory matter. The thesis proper begins in Chapter 2, where we begin by introducing the $SU(2)$ Heisenberg Ferromagnet system. Following this short review a hyperbolic version of the model is examined; static and simple time dependent solutions of winding type are explicitly derived and the existence of travelling wave solutions of the model discussed. A further family of time dependent winding solutions for $X = \mathbb{R}$ is also derived.

Following a brief description of the compact $O(3)$ Sigma model, Chapter 3 is concerned with a hyperbolic or non-compact version of the model. After formulating the non-compact model we derive time dependent solutions and again discuss the existence of travelling wave solutions. The self-dual HSM equations are then examined and solutions discussed.

In Chapter 4 a new (Pivotal) model is proposed which interpolates between the Heisenberg and sigma models, arising as a particular case of a class of integrable models introduced by Oh [53]. The integrability of the system is established and some topological solutions derived. The reduction of the Lax representation and the solutions with respect to the Heisenberg and sigma models are examined and conserved quantities for the Pivotal system briefly discussed.

Chapter 5 contains a synopsis of the results and discusses further research possibilities, in particular for the Pivotal model. This includes reference to the positive-definite continuous and discrete versions of the sigma model discussed in Appendices C and D.

Chapter 2

The Hyperbolic Heisenberg Model (HHM)

2.1 Introduction

Occurring in solid state physics, the continuous Heisenberg Ferromagnet arises as a special case of the Landau-Lifshitz (LL) model of a continuous anisotropic ferromagnet. The set up for these models is such that the fields are three-vectors; $\vec{S}(x) = (S_1, S_2, S_3)$, taking values on the unit sphere S^2 so that

$$\vec{S}(x)^2 = \sum_{a=1}^{3} S_a(x)^2 = 1.$$

Typical boundary conditions satisfied by $\vec{S}$ are

(i) periodic such that $\vec{S}(x+2L) = \vec{S}(x)$ or

(ii) 'rapidly decreasing' in which case, $\vec{S}(x) \longrightarrow \vec{S}_0$ as $|x| \longrightarrow \infty$ and the limiting values are approached sufficiently fast (for instance, $\vec{S}$ is infinitely differentiable and together with all its derivatives, decays faster than any power of $|x|^{-1}$ as $|x| \longrightarrow \infty$). Without loss of generality due to $O(3)$ invariance, $\vec{S}_0$ can be fixed as $\vec{S}_0 = (0,0,1)$.

The Hamiltonian (LL) equation of motion is given by

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times \frac{\partial^2 \vec{S}}{\partial x^2} + \vec{S} \times J\vec{S} \tag{2.1}$$

where $J = \text{diag}(J_1, J_2, J_3) \ : J_1 \leq J_2 \leq J_3$. When $J_1 = J_2 = J_3 = 0$ the resultant equation is known as the continuous isotropic Heisenberg ferromagnet model (HM)[1] describing the classical spin $\vec{S}$ distributed along the line. For the HM model, the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left(\frac{\partial \vec{S}}{\partial x}\right)^2,$$

together with the Poisson structure

$$\{S_a(x), S_b(y)\} = -\epsilon_{abc} S_c(x)\delta(x-y), \quad a, b = 1, 2, 3$$

give rise to the equation of motion

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times \frac{\partial^2 \vec{S}}{\partial x^2}. \tag{2.2}$$

And this may be written in the Hamiltonian form

$$\frac{\partial \vec{S}}{\partial t} = \{H, \vec{S}\},$$

where $H = \int \mathcal{H} dx$ with limits appropriate to the boundary conditions. It is a non-compact version of this model which is the main concern of this chapter. We first, however, make note of some features of the compact model and various aspects of both compact and non-compact versions discussed in the literature.

Some of the most interesting developments to emerge from the theory of solitons are the connections which have come to light between systems which previously appeared unrelated. The various reductions of the self-dual Yang Mills equations (1.10) are a prime example; in this case, low-dimensional systems are related to one another via the higher dimensional SDYM system and are all, in some sense, different elements of the same system. On a more 'planar' level, in the sense that no change of dimension occurs; the application of gauge theoretical methods has been extremely fruitful in relating one system to another. In particular, where the linear spectral problem for a non-linear system may be written in the form of a U, V pair,

[1] Makhankov [20] is careful to point out that whilst both the Landau-Lifshitz (2.1) and HM (2.2) equations are often referred to as continuous Heisenberg models, the *true* Heisenberg model is a quantum system written in terms of spin operators whereas both (2.1) and (2.2) describe classical systems and the correspondence between the classical and quantum cases is not completely clear. When referring to the Heisenberg model we mean the model whose dynamics is described by (2.2)

a formulation in the language of gauge field theory is possible; gauge transforming the linear system leads to a further linear system for which a different non-linear equation (or set of equations) is the compatibility condition. Furthermore, solutions can be generated in the frame of a given system or equation by the use of solutions of another.

In essence, the gauge transformation comprises a field coordinate transformation by which the non-linear field equations, the conserved quantities and solutions of two systems may be transformed into one another. Such models are 'gauge equivalent' and the $SU(2)$ Heisenberg model (2.2) and the non-linear Schrödinger equation (2.5) below, constitute a well documented and illustrative example of such an equivalence.

The connection between these two models was first discussed by Lakshmanan [54] and Corones [55] and their complete gauge equivalence established by Zakharov and Takhtajan [56]. The following provides a synopsis of how this equivalence is manifested (c.f. [57, 58]). Consider the two pairs of linear equations

$$\frac{\partial F_i}{\partial x} = U_i F_i, \qquad \frac{\partial F_i}{\partial t} = V_i F_i \tag{2.3}$$

(with no sum implied) where $i = 1$ stands for the non-linear Schrödinger equation, $i = 2$ for the HM model. The equations (2.3) for $i = 1, 2$, may be obtained from one another by a gauge transformation, independent of the spectral parameter λ, of the form

$$U_1 = gU_2g^{-1} + g_x g^{-1}; \qquad V_1 = gV_2g^{-1} + g_t g^{-1}. \tag{2.4}$$

Here $g \in SU(2)$ and $g_x \equiv \frac{\partial g}{\partial x}$, $g_t \equiv \frac{\partial g}{\partial t}$.

The non-linear Schrödinger equation (NLS) (with a cubic non-linearity and taking $X = \mathbb{R}$) is given by

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2\psi = 0 \tag{2.5}$$

and is the compatibility condition for (2.3) for $i = 1$ and with

$$U_1 = i\lambda\Lambda + P; \qquad V_1 = Q + 2\lambda P + 2i\lambda^2\Lambda \tag{2.6}$$

where $\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & \psi \\ -\psi* & 0 \end{pmatrix}$ and $Q = -i\Lambda P_x + i\Lambda P^2$. Defining $g(x,t)$ as a solution of (2.3) for $\lambda = 0$ results in

$$\begin{aligned} g_x &= Pg \\ g_t &= Qg. \end{aligned} \tag{2.7}$$

Further, with the gauge transformation (2.4) and defining

$$S = g^{-1}\Lambda g$$

one has

$$\begin{aligned} U_2 &= g^{-1}U_1 g - g^{-1}g_x \\ &= i\lambda g^{-1}\Lambda g + g^{-1}Pg - g^{-1}g_x \\ &= i\lambda S \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} V_2 &= g^{-1}V_1 g - g^{-1}g_t \\ &= 2i\lambda^2 g^{-1}\Lambda\, g + 2\lambda g^{-1}Pg \\ &= 2i\lambda^2 S + \lambda SS_x \end{aligned} \tag{2.9}$$

using in (2.9) the fact that $\Lambda P\Lambda = -P$. The linear system with U_2, V_2 as above then has as its compatibility condition the equation

$$\frac{\partial S}{\partial t} = \frac{1}{2i}\Big[S, \frac{\partial^2 S}{\partial x^2}\Big] \tag{2.10}$$

where with the Pauli matrices σ_a, $S = S^a\sigma_a \in SU(2)$; and this equation (2.10) is equivalent to the HM equation (2.2) with the condition $\vec{S}^2 = 1$. Hence there is gauge equivalence between the linear systems (2.3) which have the NLS (2.5) and the HM (2.2) equations as their compatibility conditions for $i = 1, 2$ respectively.

With regard to the gauge transformation of solutions we simply note (c.f. [56, 57]) that if $\psi(x,t)$ is a solution of (2.5) with $\psi \longrightarrow 0$ as $t \longrightarrow \infty$ then the solution of (2.7) with $g \longrightarrow 1$ as $x \longrightarrow \infty$ has $S \longrightarrow \sigma_3$ as $x \longrightarrow \infty$ and is such that $S = g^{-1}\sigma_3\, g$ is a solution of (2.10).

This whole process has been generalized for the HM and NLS systems for higher compact symmetry groups (c.f.[58, 59, 60]), non-compact groups [61], and to the most general formulation in terms of the algebraic theory of gauge fields [62].

The gauge equivalence described above provides as an added bonus, proof that the HM model is a reduction of the SDYM equations; the NLS equation was shown by Mason and Sparling [14] to be a reduction in 1989 and the connection to the HM model is discussed by Mason and Woodhouse [43] for the gauge groups $SU(2)$ and $SU(1,1)$, i.e. in a compact and a non-compact case.

With the U, V pair (2.8, 2.9) the compact $SU(2)$ HM model is integrable in $(1+1)$ dimensions by our definition in Chapter 1 and has in fact, been shown to be solvable by the Inverse Scattering Transform and completely integrable [63]. The compact $(1+1)$-dimensional model does not however, admit topological solitons as a result of the fact that $\pi_1(S^2) = 0$, i.e. maps from a circle to a sphere are trivial; any circle wrapped around the sphere shrinks to a point. In $(2+1)$ dimensions there is a non-trivial topological charge [64] since the map is from S^2 to S^2 however, this model does not appear to be integrable.

Having discussed a compact version of the Heisenberg model we next consider the model where the field space is non-compact. In 1982 a $SU(1,1)$ variant of the $SU(2)$ HM model was proposed by Kundu [65], and the models gauge equivalence with a non-linear Schrödinger equation of repulsive type established. With the two-sheeted hyperboloid parametrized by the polar angles $(\theta,\ \phi)$ and the spin vector taken as $\vec{S} = (\sinh\theta\,\cos\phi, \sinh\theta\,\sin\phi, \cosh\theta)$, both singular and regular soliton solutions were obtained and mapped through a 'Miura' transformation to solutions of the given NLS equation. Via this gauge equivalence, reference has been made [65, 67] to a connection between Kundu's model and the Yang and Yang system[2] [80]. So this non-compact Heisenberg spin chain, besides its pure mathematical interest, may approximate some physical model in the classical and continuous limit. It may also be linked with a model for explaining the metal-dielectic transition in some organic salts [65, 69].

Whilst Kundu's model and the one we shall discuss in this chapter are both associated with non-compact manifolds, these manifolds differ in their group structure;

[2]Yang and Yang in 1966 [80] established the connection between the thermodynamic behaviour of the spin $\frac{1}{2}$ quantum Heisenberg chain with anisotropy and the Bose gas with repulsive interaction, and further that as a field theory this is equivalent also to the quantum NLSE of repulsive type; in relation to the classical reduction of this, Kundu and Pashaev [67] and Nakamura and Sasada [68] have discussed various gauge equivalences of certain cases of the Landau-Lifshitz equation with some non-linear Schrödinger type equations.

Kundu's $SU(1,1)/U(1)$ model has the field taking values on the two-sheeted hyperboloid as noted, whilst in our case, with the symmetric space $SO(2,1)/SO(1,1)$ (described in chapter 1), it takes its values on the hyperboloid of one sheet. Certain features of the Heisenberg model in this latter case have been discussed recently in the literature; for example, in connection to gravity, c.f. [70, 71], where it is claimed that the model is equivalent to a gauge fixing condition for a low-dimensional gravity model known as the **Jackiw-Teitelboim** model (or 'lineal gravity'). It is shown that the model is gauge equivalent to a 'resonance NLSE' related to non-ideal Bose gas, and to a two component reaction-diffusion system. What is called a **magnetic dissipaton** solution is given where the properties of the solution are velocity dependent. In particular, a topological soliton on the hyperboloid is related to a black hole; it is claimed that the existence of black holes and an event horizon are related to the topologically non-trivial nature of solutions of the Heisenberg model on the hyperboloid of one sheet. The model and its gauge partners are further connected to non-linear optics and the speculation made that experimental realization of resonance collisions of solitons might be an interesting tool in optical communications systems.

So, whilst as Kundu says [65], non-linear models with non-compact phase spaces may have previously been considered physically uninteresting, such models have more recently been connected to some interesting physical phenomena as indicated above. In the following sections, the existence of various types of solution for one such model is discussed.

2.2 Formulation of HHM

The object of attention for the remainder of this chapter is the continuous Hyperbolic Heisenberg model in (1+1) dimensions, which shall henceforth be abbreviated to HHM. Here the field takes its values on the non-compact manifold $M = H^2$ described in the previous chapter. The model arises from the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\psi_x^a \psi_x^b \eta_{ab} \tag{2.11}$$

(where the subscript x denotes differentiation with respect to x), with Poisson brackets

$$\{\psi^a(x), \psi^b(y)\} = -\delta(x-y)\epsilon^{abc}\psi^d(x)\eta_{cd}. \tag{2.12}$$

Note that if η_{ab} is replaced by the Euclidean metric δ_{ab}, the analogous $M = S^2$ system is recovered. Using

$$\frac{d}{dt}\psi^a(y) = \{H, \psi^a(y)\}$$

where H is the integral over X of (2.11), and the relation

$$\left\{\frac{\partial}{\partial x}\psi^a(x), \psi^b(y)\right\} = -\epsilon^{abc}\eta_{cd}\,\psi^d(y)\delta'(x-y)$$

results in the following expression

$$\frac{d}{dt}\psi^a(y) = \int \left[\eta_{bc}\frac{\partial\psi^c}{\partial x}\,\epsilon^{abd}\eta_{de}\,\psi^e(y)\,\delta'(x-y)\right] dx.$$

Integration by parts then yields the equations of motion for the system which are given by

$$\frac{\partial}{\partial t}\psi^a(x) = \eta^{ab}\epsilon_{bcd}\psi^c\,\psi^d_{xx}. \tag{2.13}$$

The Hamiltonian for the system

$$H = \frac{1}{2}\int_X \mathcal{H}\,dx \tag{2.14}$$

is non-positive definite due to the indefinite metric on the field space.

The aim in this chapter is to seek out, or indeed in some instances eliminate the possibility of, solutions of this model which are classifiable by an integer winding number N. However, since only *integrable* systems with such solutions are of interest here, that the HHM is integrable must first be established and this may be seen as follows: define a 2×2 matrix $S \in SL(2,\mathbb{R})$ by

$$S = \begin{bmatrix} \psi_1 & \psi_2+\psi_3 \\ \psi_2-\psi_3 & -\psi_1 \end{bmatrix}. \tag{2.15}$$

Then the consistency condition for the linear system

$$\begin{aligned}\partial_x F &= UF \\ \partial_t F &= VF\end{aligned}$$

is exactly the equation of motion (2.13) where

$$U = \lambda S; \qquad V = -\lambda(2\lambda S + \lambda S_x S)$$

and U, V satisfy the zero-curvature condition

$$U_t - V_x + [U, V] = 0. \tag{2.16}$$

Hence, the HHM system is integrable in the required sense.

In what follows two different parametrizations of the hyperboloid H^2 are used to investigate the existence of 'winding' solutions for the HHM and the models discussed in later chapters. The first of these is in terms of the polar angles θ, ϕ where $\theta \in (-\infty, \infty)$, $\phi \in [0, 2\pi]$ and we take

$$\begin{aligned}\psi^1 &= \cosh\theta \cos\phi \\ \psi^2 &= \cosh\theta \sin\phi \\ \psi^3 &= \sinh\theta.\end{aligned} \tag{2.17}$$

The second parametrization is in terms of a stereographic projection (c.f. [4]) and takes the form

$$\begin{aligned}\psi^1 &= \frac{(1 - u^2 + v^2)}{(1 + u^2 - v^2)} \\ \psi^2 &= \frac{2u}{(1 + u^2 - v^2)} \\ \psi^3 &= \frac{2v}{(1 + u^2 - v^2)}\end{aligned} \tag{2.18}$$

The two parametrizations are related in the following way; let $f^2 = u^2 - v^2$, then equating (2.17) with (2.18) one finds that

$$f^2 = \frac{1 - \cosh\theta \cos\phi}{1 + \cosh\theta \cos\phi}$$

and hence,

$$u(t,x) = \frac{\cosh\theta\,\sin\phi}{1+\cosh\theta\,\cos\phi} \quad v(t,x) = \frac{\sinh\theta}{1+\cosh\theta\,\cos\phi}.$$

2.3 Some Static and Simple Time Dependent Winding Solutions for HHM

Taking the first of the parametrizations (2.18) of the hyperboloid given above, i.e. in terms of 'polar angles' θ, ϕ, the following shows how some simple winding solutions can be derived for the HHM; in particular, where physical space X is the circle S^1. We note first, however, that the Hamiltonian of the system with this parametrization is given by

$$H = \frac{1}{2}\int_X \left[\cosh^2\theta\,(\phi_x^2) - \theta_x^2\right]\,dx \tag{2.19}$$

which, as expected, may not be positive definite if θ depends on x.

Equation (2.13) is equivalent to the following equations of motion in terms of $\theta(t,x)$, $\phi(t,x)$:

$$\theta_t = 2\theta_x\phi_x\sinh\theta + \phi_{xx}\cosh\theta \tag{2.20a}$$

$$\phi_t = \theta_{xx}\mathrm{sech}\theta + \phi_x^2\sinh\theta \tag{2.20b}$$

and we begin by considering first static winding solutions of the model. Taking θ and ϕ as functions of x only, (2.20a) yields

$$-2\theta_x\,\tanh\theta = \frac{\phi_{xx}}{\phi_x}.$$

Integrating both sides with respect to x we have

$$\phi_x = B\,\mathrm{sech}^2\theta \tag{2.21}$$

with B constant, and on substitution into (2.20b) this results in the equation

$$\theta_{xx} = -B^2\,\tanh\theta\,\mathrm{sech}^2\theta.$$

Multiplying both sides by θ_x and integrating again with respect to x;

$$\theta_x = \sqrt{B^2 \operatorname{sech}^2 \theta + A} \tag{2.22}$$

with A constant. For a solution of winding type it is necessary that $\theta(x)$ be real and bounded and specifically, non-monotonic. For this to be the case θ_x must have at least one real zero and letting $y = \sinh\theta$ in (2.22) and squaring both sides,

$$y_x^2 = B^2 + A(1 + y^2) \qquad (= Y(y) \quad \text{say.})$$

Requiring then that $Y(y_0) = 0$ for some $y_0 \in \mathbb{R}$ one then has

$$\begin{aligned} -\left(\tfrac{B^2}{A} + 1\right) &= y_0^2 \\ \Longrightarrow \quad \left(\tfrac{B^2}{A} + 1\right) &< 0 \end{aligned}$$

i.e. $-B^2 \leq A < 0$. Writing $A = -N^2$ where $0 < N \leq B$,

$$N(x - x_0) = \int \frac{dy}{\sqrt{(\frac{B^2}{N^2} - 1) - y^2}}$$

so that after integrating one has the solution

$$\sinh\theta = \left(\sqrt{\frac{B^2}{N^2} - 1}\ \sin N(x - x_0)\right) \tag{2.23}$$

where x_0 is constant. This is periodic with period 2π, provided N is an integer. An explicit expression for $\phi(x)$ can be obtained by integrating the smooth function (2.21). However, to check that the solution is of winding type it is sufficient to compute

$$\Delta\phi = \int_0^{2\pi} B \operatorname{sech}^2 \theta \, dx. \tag{2.24}$$

Referring forward to section 2.4 and the analogous travelling wave solution (2.34) where it is shown that for $N = 1$, $\Delta\phi = 2\pi$, we find in this case that $\Delta\phi = 2\pi N$ and so have a space-periodic static solution with winding number N. Furthermore, with (2.21) and (2.22) in (2.19) the Hamiltonian is given explicitly by $H = \pi N^2$ (since $A = -N^2$), hence the solution is a static topological soliton with finite energy.

Noting that if $B = N$ in (2.23) so that $\theta = 0$ and $\phi = Nx$ one has a solution which winds N times around the 'waist' of the hyperboloid and a simple time dependent generalization of this static solution can be found as follows: Taylor expanding around the initial data $\phi = Nx$ and $\theta = c$ (constant) > 0, at $t = 0$,

$$\begin{aligned}\theta(x,t) &\approx \theta(x,0) + t\theta_t(x,0) + \frac{t^2}{2}\theta_{tt}(x,0) + O(t^3)\\ \phi(x,t) &\approx \phi(x,0) + t\phi_t(x,0) + \frac{t^2}{2}\phi_{tt}(x,0) + O(t^3).\end{aligned}$$

Using the equations of motion (2.20) yields

$$\theta(x,0) = c\ ; \quad \theta_t(x,0)\Big|_{\substack{\phi=Nx\\ \theta=c}} = 0\ ; \quad \theta_{tt}(x,0)\Big|_{\substack{\phi=Nx\\ \theta=c}} = 0.$$

Similarly for $\phi(x,t)$;

$$\phi(x,0) = Nx\ ; \quad \phi_t(x,0)\Big|_{\substack{\phi=Nx\\ \theta=c}} = N^2\sinh c\ ; \quad \phi_{tt}(x,0)\Big|_{\substack{\phi=Nx\\ \theta=c}} = 0$$

so that to second order,

$$\theta(x,t) \equiv c \quad \forall\, x,\, t \tag{2.25a}$$

and

$$\phi(x,t) = Nx + tN^2\sinh(c). \tag{2.25b}$$

In fact, this is an exact solution (to all orders) which can be seen via substitution into the equations of motion (2.20). Hence, both static and simple time-dependent solitonic winding solutions exist for the HHM where periodic boundary conditions are imposed.

2.4 Travelling Waves for HHM

In the quest for winding solutions the next most natural step is to look for those of travelling wave type where it is customary to define a travelling wave such that, with the characteristic variable $\xi = x - vt$,

$$\theta(t,x) = f(\xi) \tag{2.26a}$$

$$\phi(t,x) = g(\xi) + ct \tag{2.26b}$$

$$\mathcal{H}(t,x) = \mathcal{H}(\xi) \tag{2.26c}$$

where v corresponding to the speed, and c are constants. Then solutions will have a constant profile, as will the corresponding density $\mathcal{H}$. One might relax (2.26a, 2.26b) so that only $\mathcal{H}$ is required to depend solely on ξ, see for example [70], however this is likely to produce, some additional (for example, exponential) t dependence in at least one of θ, ϕ resulting in unstable or non-uniform behaviour. We therefore impose that θ and ϕ depend on ξ where ϕ may have the additional term ct. Of course, this term may be absent and what occurs in this event is discussed presently.

The equations of motion (2.20) are reformulated with f, g as follows: with

$$-v\frac{df}{d\xi} = 2\frac{df}{d\xi}\frac{dg}{d\xi}\sinh f + \frac{d^2g}{d\xi^2}\cosh f, \tag{2.27}$$

from (2.20a), substitution of the relationship

$$\frac{d}{d\xi}(\cosh^2 f\,\frac{dg}{d\xi}) = 2\frac{df}{d\xi}\frac{dg}{d\xi}\cosh f \sinh f + \frac{d^2g}{d\xi^2}\cosh^2 f \tag{2.28}$$

results in the equation

$$\frac{d}{d\xi}(-v\sinh f) = \frac{d}{d\xi}(\cosh^2 f\,\frac{dg}{d\xi}).$$

Integration then yields

$$\frac{dg}{d\xi} = (k - v\sinh f)\ \mathrm{sech}^{\,2} f \tag{2.29}$$

where k is constant. From the equation of motion for ϕ, (2.20b), using (2.29) and with some algebra one finds the following equation for f:

$$\frac{d^2f}{d\xi^2} = (v^2 - k^2)\tanh f\ \mathrm{sech}^2 f + kv\ \mathrm{sech}\, f\,(1 - 2\ \mathrm{sech}^2 f) + c\cosh f.$$

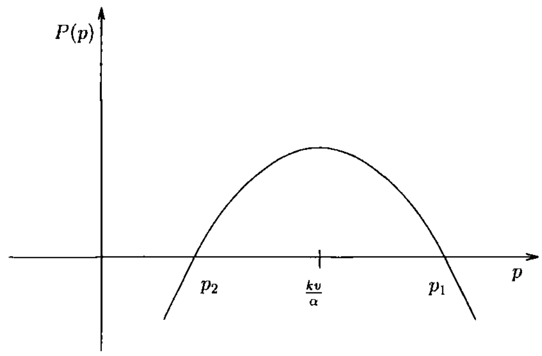


Figure 2.1: General profile of $P(p)$ in the case $c = 0$.

This has first integral

$$\left(\frac{df}{d\xi}\right)^2 = \frac{(v^2-k^2)\sinh^2 f - 2kv\,\sinh f + 2c\,\sinh f\,\cosh^2 f + 2Q\cosh^2 f}{\cosh^2 f}$$

with Q constant and may be simplified by letting $p(\xi) = \sinh f$ resulting in

$$\frac{p'^2}{2} = cp^3 + \frac{p^2}{2}\left(v^2 - k^2 + 2Q\right) + p\left(c - kv\right) + Q \, = P(p), \tag{2.30}$$

with the 4 real parameters k, v, c, Q and where 'prime' denotes differentiation with respect to ξ.

Solutions in terms of elliptic functions can be found for $p(\xi)$ by rearranging and integrating (2.30) and these will be discussed presently; however, for the moment, we restrict ourselves to examining a less complicated case by taking the constant $c = 0$ in (2.26b). In (2.30) this results in a quadratic in p with the three parameters k, v, Q:

$$\frac{p'^2}{2} = \frac{\alpha}{2}p^2 - kvp + Q = P(p) \tag{2.31}$$

where $\alpha = v^2 - k^2 + 2Q$. For the solution to be of winding type then requires $\alpha < 0$ and $Q > \frac{k^2v^2}{2\alpha}$ so that $P(p)$ is greater than zero and bounded between the two zeros p_1, p_2 say, i.e. $P(p)$ takes the general form of Figure (2.1) so that p has a maximum and a minimum and the solution oscillates between p_1 and p_2. Reorganizing the constants for simplicity one then has

$$\sqrt{2}(\xi - \xi_0) = \int \frac{dp}{\sqrt{A - B(p-p_0)^2}} \tag{2.32}$$

where $A - Bp_0^2 = Q$, $-2B = \alpha$, $2Bp_0 = -kv$, $(A, B > 0, \xi_0, p_0$ constants) i.e.,

$$\begin{aligned} B &= \frac{k^2 - v^2 - 2Q}{2} \\ p_o &= \frac{kv}{v^2 - k^2 + 2Q} \\ A &= Q - \frac{k^2 v^2}{2(v^2 - k^2 + 2Q)}. \end{aligned}$$

From (2.32) the solution is then

$$p(\xi) = p_0 + \sqrt{\frac{A}{B}} \sin(\sqrt{2B}\xi) \tag{2.33}$$

where for this to be of winding type put

$$N = \sqrt{2B} = \pm\sqrt{k^2 - v^2 - 2Q} \quad \in \mathbb{Z}.$$

By scaling x, α can be set to -1 so that Q can now be given in terms of v and k, namely, $Q = \frac{1}{2}(k^2 - v^2 - 1)$. N is then equal to $\pm$ unity and we have the one-wind solution. In terms of the remaining parameters v and k (and putting $\xi_0 = 0$ for simplicity) this is given by

$$p(\xi) = -kv + \sqrt{(v^2 + 1)(k^2 - 1)} \sin(\xi) \tag{2.34}$$

with period 2π. Note that this is a travelling wave version of the static solution (2.23).

For a solution to be of winding type requires also that the integral of (2.29) over (in this case) $X = S^1$ be a finite integer multiple of 2π and with the substitution $p = \sinh f$ in (2.29) one has

$$\frac{dg}{d\xi} = \frac{k - vp}{1 + p^2}. \tag{2.35}$$

In the case of solution (2.34) where there are two parameters, $k \geq 1$ and v, solitons with unit winding number exist as follows:

(i) if $v = 0$, then $\Delta\phi = 2\pi$, $\forall\, k \geq 1$ which can be seen by following a similar analysis as that given for case **(iii)** below.

(ii) if $k = 1$, then $\Delta\phi = 2\pi$, $\forall\, v$ (where this winding solution has $p \equiv -v$).

(iii) Retaining the value of $\alpha = -1$ and taking now $v \neq 0$ and $k > 1$, (2.35) is of the following form:

$$\frac{dg}{d\xi} = \frac{k(v^2+1) - v\sqrt{(v^2+1)(k^2-1)}\,\sin(\xi)}{1 + \left[-kv + \sqrt{(v^2+1)(k^2-1)}\,\sin(\xi)\right]^2}. \tag{2.36}$$

This is a rational function with the denominator quadratic in $\sin\xi$ which can therefore be easily factorized to

$$\frac{dg}{d\xi} = \frac{k(v^2+1) - v\gamma\sin\xi}{\left[\gamma\sin\xi - (kv+i)\right]\left[\gamma\sin\xi - (kv-i)\right]}$$

where $\gamma = \sqrt{(k^2+1)(v^2-1)}$. This may then be decomposed using partial fractions to give

$$\frac{dg}{d\xi} = \frac{1}{2}\left[\frac{(v+ik)}{(kv+i) - \gamma\sin\xi} + \frac{(v-ik)}{(kv-i) - \gamma\sin\xi}\right].$$

And using standard integral formulae yields the solution

$$g(\xi) = \mathrm{Re}\left[-\frac{i}{2}\left[\ln\left\{\frac{\tan\frac{\xi}{2} - \frac{(i\gamma+ik-v)}{(1+ikv)}}{\tan\frac{\xi}{2} - \frac{(i\gamma-ik+v)}{(1+ikv)}}\right\} - \ln\left\{\frac{\tan\frac{\xi}{2} + \frac{i\gamma+ik+v}{(1-ikv)}}{\tan\frac{\xi}{2} + \frac{i\gamma-ik-v}{(1-ikv)}}\right\}\right] + \Lambda\right] \tag{2.37}$$

where Λ is a constant of integration. Using the formula $\ln\frac{A}{B} = \ln A - \ln B$ this can be written as

$$\begin{aligned} g(\xi) = \mathrm{Re}\Bigg[&-\frac{i}{2}\ln\left\{\tan\frac{\xi}{2} + \left(\frac{-i\gamma - ik + v}{1+ikv}\right)\right\} \\ &+ \frac{i}{2}\ln\left\{\tan\frac{\xi}{2} + \left(\frac{-i\gamma + ik - v}{1+ikv}\right)\right\} \\ &+ \frac{i}{2}\ln\left\{\tan\frac{\xi}{2} + \left(\frac{i\gamma + ik + v}{1-ikv}\right)\right\} \\ &- \frac{i}{2}\ln\left\{\tan\frac{\xi}{2} + \left(\frac{i\gamma - ik - v}{1-ikv}\right)\right\} + \Lambda\Bigg]. \end{aligned}$$

Further, letting $a = -i\gamma - ik + v$, $b = 1 + ikv$, $C = -i\gamma + ik - v$ and given that the third term is the complex conjugate of the first, and the fourth the complex conjugate of the second, we then have

$$g(\xi) = \mathrm{Re}\left[-i\left\{\ln\left(\tan\frac{\xi}{2}+\frac{a}{b}\right) - \ln\left(\tan\frac{\xi}{2}+\frac{C}{b}\right)\right\}+\Lambda\right]$$

i.e.

$$g(\xi) = \mathrm{Re}\left[-i\ln\left\{\frac{\tan\frac{\xi}{2}+\frac{a}{b}}{\tan\frac{\xi}{2}+\frac{C}{b}}\right\}+\Lambda\right]. \tag{2.38}$$

It is not immediately clear whether this solution is of winding type or not; however, Appendix A demonstrates that (2.38) has the form

$$g(\xi) = \mathrm{Re}\left[-i\ln\left\{\frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}}\right\}\right] \tag{2.39}$$

where $\Omega = \frac{v-ik-i\gamma}{1+ikv}\left(=\frac{a}{b}\right)$. And this *is* of winding type as is shown by analysing the behaviour of the function

$$\Xi(\xi) = \frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}}$$

as x goes from $-\pi$ to π as follows: note first that Ξ is a continuous function of x which is never zero (for Ξ to be zero requires $x \in \mathbb{C}$), and that as $x \longrightarrow \pm\pi$, $\Xi \longrightarrow -1$, so that Ξ takes the form of a loop beginning and ending at -1. If a branch cut is then taken from the origin of Ξ space along the negative real axis, it remains to show that Ξ wraps once around the origin so that

$$\Delta\phi = \int_{-\pi}^{\pi}\frac{dg}{d\xi}\,dx \;=\; \pm 2\pi.$$

For this to be the case, Ξ must pass through the positive real axis once and only once, i.e. $\Xi = \overline{\Xi}$ must have a *unique*, real solution x. Since

$$\begin{aligned}
\frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}} &= \frac{\tan\frac{\xi}{2}+\overline{\Omega}}{-\tan\frac{\xi}{2}-\overline{\Omega}^{-1}},\\
\text{which} \quad \Longrightarrow \quad \tan\frac{\xi}{2} &= \frac{\overline{\Omega}\Omega^{-1}-\Omega\overline{\Omega}^{-1}}{\Omega+\overline{\Omega}^{-1}-\Omega^{-1}-\overline{\Omega}},
\end{aligned}$$

and $\Omega \in \mathbb{C}$, i.e. $\Omega = r + is$ for $r = \frac{v-k^2v-\gamma kv}{1+k^2v^2}$ and $s = -\frac{(k+\gamma+kv^2)}{1+k^2v^2}$, one has

$$\tan\frac{\xi}{2} = \frac{-4ir}{2i(r^2+s^2+1)} = \frac{-2r}{(r^2+s^2+1)} \quad \in \mathbb{R}$$

so that

$$\xi = 2\tan^{-1}\left[\frac{-2(v-k^2v-\gamma kv)(1+k^2v^2)}{(v-k^2v-\gamma kv)^2+(k+\gamma+kv^2)^2+(1+k^2v^2)^2}\right]$$

giving a real and unique solution x to $\Xi = \overline{\Xi}$ for each real k and v. Hence, $\triangle\phi = \pm 2\pi$. The solution (2.39) is therefore a travelling wave of winding type (with $N = 1$) and hence, by Appendix A, so is (2.38). The Hamiltonian is given by the conserved positive quantity $H = \pi(v^2+1)$ so the solution given by (2.34) and (2.39) is a moving soliton with unit winding number.

Thus, with the constant $c = 0$ in (2.30), there exist travelling wave topological soliton solutions with unit winding number (which may be generalized for $N > 1$) for $v \geq 0$ and $k \geq 1$.

It has been shown thus far, that as well as static and simple time-dependent winding solutions, a family of topologically classifiable travelling waves exists for this model, where $X = S^1$. It would be pleasing to find such solutions for $X = \mathbb{R}$ also and in an attempt to find a more general solution from which a solution with $X = \mathbb{R}$ may result, we take $c \neq 0$, and in particular $c = 1$ (by scaling x and transforming $p \mapsto -p$ if necessary). With the substitutions $\alpha = \frac{1}{6}(k^2 - v^2 - 2Q)$, $\triangle = \alpha^2 - \frac{1}{3}(1-kv)$, equation (2.30) is then transformed to

$$P(p) = \frac{p'^2}{2} = p^3 - 3p^2\alpha + 3(\alpha^2 - \triangle)p + Q, \tag{2.40}$$

retaining the 3 real parameters k, v, Q.

For a solution to wind in $p = \sinh f$, $P(p)$ must take the form shown in Figure (2.2), i.e. $P(p)$ should have three real zeros so that there exists a positive bounded region in which the solution can oscillate in an appropriate sense with regard to the space X.

How the limits of this behaviour are manifested for the spaces $X = S^1$ and $X = \mathbb{R}$ will be examined presently. However, it must first be determined whether or not

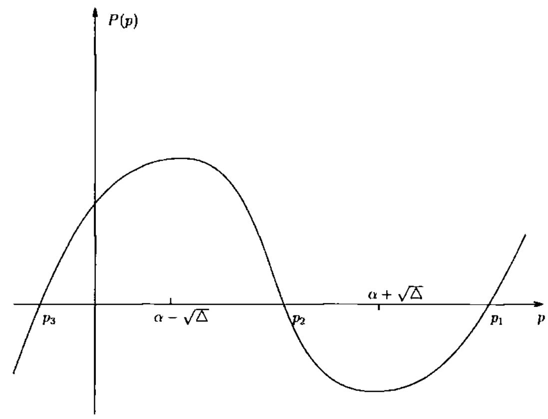


Figure 2.2: Requisite form of $P(p)$ for solutions winding in p if $c = 1$

(2.40) can in fact take the form shown. To this end we note that

$$P'(p) = 3p^2 - 6p\alpha + 3(\alpha^2 - \triangle)$$

where 'prime' denotes differentiation with respect to p, and setting this equal to zero implies $p = \alpha \pm \sqrt{\triangle}$. Further, $P''(p) = 6(p - \alpha)$, so that on substitution of the critical values $p = \alpha \pm \sqrt{\triangle}$ into $P''(p)$, one has

$$\begin{aligned} P''(\alpha + \sqrt{\triangle}) &= 6\sqrt{\triangle} \\ P''(\alpha - \sqrt{\triangle}) &= -6\sqrt{\triangle}. \end{aligned}$$

Then one of the points $p = \alpha \pm \sqrt{\triangle}$ is a minimum of $P(p)$ whilst the other is a maximum, depending on the sign of $\sqrt{\triangle}$. For $P(p)$ to take the required form we need only then impose the restrictions that $\triangle > 0$, i.e. $\sqrt{3}\alpha > \pm\sqrt{1 - kv} \Longrightarrow 1 > kv$ and $\alpha^3 + 2\triangle\sqrt{\triangle} - 3\alpha\triangle + Q > 0$.

Having established that the cubic (2.30) (with $c = 1$) can take the form shown in Figure (2.2), i.e. that three real zeros $p_1 > p_2 > p_3$ say, exist if the above specifications are satisfied, the solution is then given implicitly by

$$\xi - \xi_0 = \pm \int_{p_3}^{p} \frac{dy}{\{2(y - p_1)(y - p_2)(y - p_3)\}^{\frac{1}{2}}} \tag{2.41}$$

where the $\pm$ is according to $p' >< 0$. Following Abramowitz and Stegun's notation for elliptic functions [74], we then have

$$\begin{aligned} u &= (\xi - \xi_0)\left\{\frac{p_1 - p_3}{2}\right\}^{\frac{1}{2}} \\ &= \frac{\{p_1 - p_3\}^{\frac{1}{2}}}{2}\int_{p_3}^{p} \frac{dy}{\{(y-p_1)(y-p_2)(y-p_3)\}^{\frac{1}{2}}} \end{aligned}$$

where $\operatorname{sn} u = \sin\phi = \operatorname{sn}\left[(\xi-\xi_0)\left\{\frac{p_1-p_3}{2}\right\}^{\frac{1}{2}}\middle| m\right]$, with $\phi = \operatorname{am} u$, the amplitude, and where the parameter $m = \frac{p_2 - p_3}{p_1 - p_3}$. And the solution results as follows:

$$\begin{aligned} \sin^2\phi &= \frac{p(\xi) - p_3}{p_2 - p_3} \\ &= \operatorname{sn}^2\left[(\xi-\xi_0)\left\{\frac{p_1-p_3}{2}\right\}^{\frac{1}{2}}\middle| m\right] \end{aligned}$$

so that

$$p(\xi) = p_3 + (p_2 - p_3)\operatorname{sn}^2\left[(\xi-\xi_0)\left\{\frac{p_1-p_3}{2}\right\}^{\frac{1}{2}}\middle| m\right] \tag{2.42}$$

or, using the identity $\operatorname{sn}^2(u|m) + \operatorname{cn}^2(u|m) = 1$ this has the form

$$p(\xi) = p_2 - (p_2 - p_3)\operatorname{cn}^2\left[(\xi-\xi_0)\left\{\frac{p_1-p_3}{2}\right\}^{\frac{1}{2}}\middle| m\right]. \tag{2.43}$$

We observe here that this is also a travelling wave solution of the KdV equation (c.f. [75]) and given values of p_1, p_2, p_3, the shape of the cnoidal wave can be obtained either from tables of Jacobian elliptic functions or by direct computation. Further, from (2.43) one can see that $p = p_2$ describes one peak of the wave and $p = p_3$ the trough (since $0 \le \operatorname{cn}^2 \le 1$ and $p_3 < p_2$) so $\frac{1}{2}(p_2 - p_3)$ could be regarded as the amplitude of the wave. The wavelength can also be determined as $2K(m)\left\{\frac{2}{p_1-p_3}\right\}^{\frac{1}{2}}$ recalling that the period of $\operatorname{cn}^2(u|m)$ is $2K(m)$ where

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac{1}{2}}}$$

and the shape of the wave is governed by the parameter m.

Having thus far derived a travelling wave solution (2.43) in the special case $c = 1$,

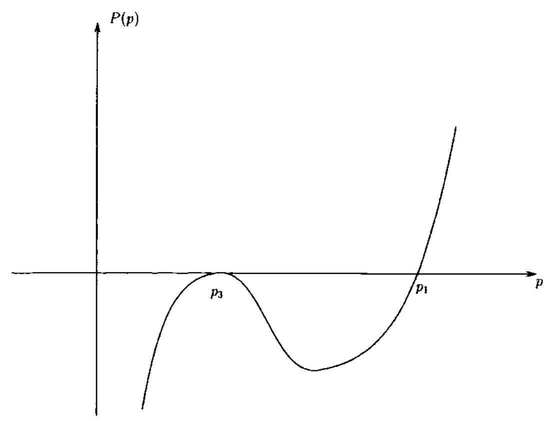


Figure 2.3: $P(p)$ in the limit $m = 0$

we have yet to determine whether or not the solution is actually of winding type for our model. For this to be the case, as usual, the integral of $\frac{dq}{d\xi}$ (2.35) must be a finite integer multiple of 2π. However, it is no simple matter (and may not be possible at all) to integrate (2.35) if $p(\xi)$ is given by the solution (2.43). So rather than attempt this let us investigate what happens in the limiting cases, i.e. where the parameter $m = 0$ or 1.

When $m = 0$, $p_2 = p_3$ so that $P(p)$ takes the form of Figure (2.3). The solution is then $p = p_3 = \sinh f$ and in this case, $\frac{dq}{d\xi} = \frac{k-vp_3}{(1+p_3^2)} =$ constant so that

$$\phi = ax + t$$

where $a = \frac{k-vp_3}{(1+p_3^2)}$. And if $a = N \in \mathbb{Z}$, this is simply the winding solution $\phi = Nx + t$, $\sinh\theta = p_3$, constant for $X = S^1$, a version of (2.25b).

From a graphical point of view one can see that a further limiting case occurs when $P(p)$ takes the form shown in Figure (2.4), i.e. $P(p)$ has one repeated and one single zero. This corresponds to a travelling wave on $X = \mathbb{R}$ where the solution has a minimum at $p = p_3$ (since $P'(p_3) > 0$) and reaches $p = p_1$ as $\xi \longrightarrow \pm\infty$. Referring back to the solution (2.43) and considering the limit where the parameter $m = \frac{p_2-p_3}{p_1-p_3} = 1$, in this limit p_2 must equal p_1 which is exactly the situation shown in Figure (2.4). And since $\text{cn}\ (u|m) = \text{sech}\ (u)$ for $m = 1$ the solution (2.43) reduces in the limit to

$$p(\xi) = p_1 - (p_1 - p_3)\ \text{sech}^2\left[(\xi - \xi_0)\left\{\frac{p_1 - p_3}{2}\right\}^{\frac{1}{2}}\right] \tag{2.44}$$

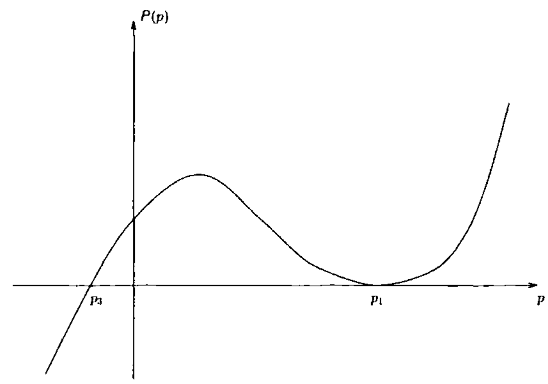


Figure 2.4: $P(p)$ in the limit $m = 1$

for $m = 1$. Explicit expressions can be found for p_1, p_3 as follows: to give the correct picture (i.e. Figure (2.4)) requires

$$\begin{aligned} P(p) &= p^3 - 3\alpha p^2 + 3(\alpha^2 - \triangle)p + Q \\ &= (p - p_1)^2(p - p_3). \end{aligned} \tag{2.45}$$

so that on equating coefficients, $p_1 = \alpha \pm \sqrt{\triangle}, p_3 = \alpha \mp 2\sqrt{\triangle}$ and $Q = -p_1^2 p_3$ (where the signs are ordered).

To check that (2.44) is indeed a solution of (2.40), replace $p_1 = \alpha + \sqrt{\triangle}$, $p_3 = \alpha - 2\sqrt{\triangle}$ in (2.44). Then substituting in the resultant expression

$$p(\xi) = (\alpha + \sqrt{\triangle}) - 3\sqrt{\triangle}\ \mathrm{sech}^{\,2}\left[(\xi - \xi_0)\left\{\frac{3\sqrt{\triangle}}{2}\right\}^{\frac{1}{2}}\right] \tag{2.46}$$

(where $\sqrt{\triangle} > 0$) one can see that this solution satisfies (2.40) if and only if Q satisfies the following equation:

$$Q = -\alpha^3 + 3\alpha\triangle + 2\triangle\sqrt{\triangle} \tag{2.47}$$

which is the same as saying $Q = -p_1^2 p_3$. Presuming that such a $Q \in \mathbb{R}$ exists we then have a solution for $p(\xi) = \sinh f$ which is a travelling wave satisfying $p(x = -\infty) = p(x = \infty)$, i.e. where $X = \mathbb{R}$ and the solution winds in $f(\xi) = \theta(t, x)$. This looks hopeful; however, it does not necessarily mean that $\vec{\psi}(t, x)$ is of winding type on H^2. For this to be the case it must also be checked that with $X = \mathbb{R}$, for the one-wind solution, $\triangle\phi = \pm 2\pi$. To determine whether or not this is the case put (for simplicity) $\sqrt{\triangle} = \frac{2}{3}$, which gives the one-wind solution in f so that

$$p(\xi) = \sinh f = \alpha + \frac{2}{3} - 2 \operatorname{sech}^2 X$$

where $X = \xi - \xi_0$. From (2.35) one then has

$$\frac{dg}{d\xi} = \frac{(k - v\gamma) + 2v \operatorname{sech}^2 X}{1 + \left[\gamma - 2 \operatorname{sech}^2 X\right]^2}$$

where $\gamma = \alpha + \frac{2}{3}$, which may be rewritten as

$$\frac{dg}{d\xi} = \frac{(k - v\gamma) + 2v \operatorname{sech}^2 X}{4\left[\operatorname{sech}^2 X - \frac{(\gamma+i)}{2}\right]\left[\operatorname{sech}^2 X - \frac{(\gamma-i)}{2}\right]}.$$

And splitting into partial fractions produces

$$\begin{aligned} \frac{dg}{d\xi} = \frac{k - v\gamma}{(\gamma^2 - 1)} &+ \Omega\left[\frac{1}{\lambda - \cosh X} + \frac{1}{\lambda + \cosh X}\right] \\ &+ \overline{\Omega}\left[\frac{1}{\overline{\lambda} - \cosh X} + \frac{1}{\overline{\lambda} + \cosh X}\right] \end{aligned} \tag{2.48}$$

where $\Omega = \frac{(v-ik)\sqrt{\gamma+i}}{2\sqrt{2}(\gamma+i)^2}$ and $\lambda = \left[\frac{2}{\gamma+i}\right]^{\frac{1}{2}}$. Integrated over $X = \mathbb{R}$, (2.48) diverges and this solution is therefore *not* of winding type.

This rather unsatisfactory divergent behaviour also manifests itself in the corresponding Hamiltonian density which can be seen as follows: with $\theta = f$, $\phi = g + ct$ in (2.19),

$$\mathcal{H} = g'^2 \cosh^2 f - f'^2$$

which, with $p = \sinh f$ is then given by

$$\mathcal{H} = \frac{(1 + p^2)^2 g'^2 - p'^2}{(1 + p^2)}.$$

And since $g'^2 = \frac{(k-vp)^2}{(1+p^2)^2}$, this translates to

$$\mathcal{H} = \frac{(k - vp)^2 - p'^2}{(1 + p^2)}. \tag{2.49}$$

As an explicit example, consider the simple case $p_1 = 0$ in (2.45). This implies

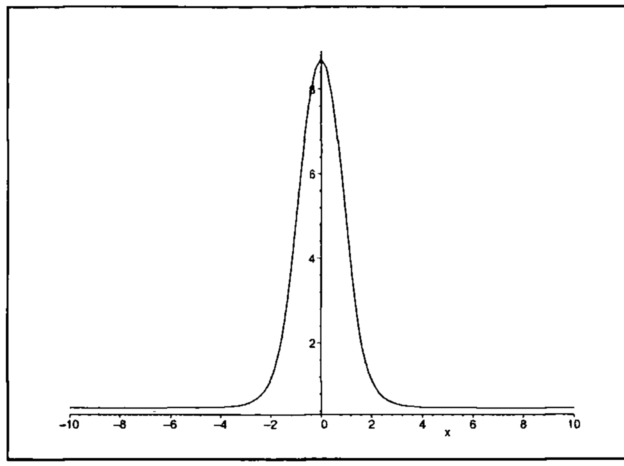


Figure 2.5: Hamiltonian Density for the one-wind solution when $c = 1$ in the limit $m = 1$ with $p_0 = 0$

$\alpha = \sqrt{\triangle}$, so that

$$\begin{aligned} P(p) &= p^3 - 3\sqrt{\triangle}p^2 + Q \\ &= p(p - p_3). \end{aligned}$$

Comparing coefficients one has $Q = 0$ and $p_3 = 3\sqrt{\triangle}$ so that in the one-wind case (with $\sqrt{\triangle} = -\frac{2}{3}$), $p_3 = -2$. (Note that (2.47) is satisfied by these values so there does exist such a Q). Further, since $\alpha = \frac{1}{6}(k^2 - v^2 + 2Q)$, $\triangle = \alpha^2 - \frac{1}{3}(1 - kv)$, one has $(v^2 - k^2) = 4$ and $k = \frac{1}{v}$ so that $v^2 = 2 + \sqrt{5}$. Substitution of the solution (2.46) with these values into the Hamiltonian density (2.49) results in

$$\mathcal{H} = \frac{\left[1 + 2(2+\sqrt{5})\mathrm{sech}^2(X)\right]^2 - 16(\sqrt{2+\sqrt{5}})\mathrm{sech}^4(X)\tanh^2(X)}{(\sqrt{2+\sqrt{5}})(1 + 4\,\mathrm{sech}^4(X))},$$

the profile of which can be seen in Figure (2.5). Note that as $X = \xi - \xi_0 \longrightarrow \pm\infty$, $\mathcal{H}$ does not tend to zero so that the integral over X is divergent and hence the Hamiltonian is infinite. Since the shape is that of a rather attractive uniform lump, however, some renormalization procedure might be utilized to remove this infinity. Whether this would be fruitful or not remains an open question which will not be pursued here.

2.5 Topological Travelling Waves when $X = \mathbb{R}$

Having established that travelling wave solutions of winding type exist for $X = S^1$ (in the special case $c = 0$); when c is non-zero, (in particular $c = 1$), we have been unable to derive any travelling waves with finite winding number when X is the real line. Solutions of *winding* type do exist when $X = \mathbb{R}$; Pashaev and Lee's solution (c.f. [70]), which we noted earlier in this chapter is of winding type with $X = \mathbb{R}$ within specific ranges of values of the speed, however, their solution is not a travelling wave in the strictest sense described earlier - with non-zero v, their solution exhibits behaviour exponential in t in addition to the required ξ dependence. A further winding solution with $X = \mathbb{R}$ is shown to exist presently, however, again this is not of travelling wave type. In fact, no such travelling wave solutions, as defined[3], exist for HHM and this section is devoted to proving this fact.

In order to show that there are no travelling waves of winding type when $X = \mathbb{R}$ for HHM, the necessary conditions for such a solution to exist must first be stated. These are as follows:

(i) the solution $p(\xi)$ should be real and bounded so $P(p) = p'^2 \geq 0$.

(ii) $P(p)$ should have two real zeros; one repeated and one simple (which we denote by p_0 and p_1 respectively), where the solution $p(\xi)$ has a minimum at $p = p_1$ and $p(x) \longrightarrow p_0$ as $x \longrightarrow \pm\infty$.

Recalling further that

$$\triangle\phi = \int_X \frac{dg}{d\xi}\, dx = \int_{-\infty}^{\infty} \frac{k - vp}{1 + p^2}\, dx$$

we must also then demand that

(iii) $p(\xi) \longrightarrow \frac{k}{v}$ as $x \longrightarrow \pm\infty$, where, $\frac{k}{v}$ is the zero of $\frac{dg}{d\xi}$, so that $p_0 = \frac{k}{v}$ (from **(ii)** above), and

(iv) $p'(\xi) \longrightarrow M$, constant as $x \longrightarrow \pm\infty$ and in fact, on $X = \mathbb{R}$, **(iii)** and **(i)** above $\Longrightarrow M = 0$, so that $P(p = \frac{k}{v}) = 0$.

[3]Pashaev, in private communications, considers the definition for a travelling wave here as too restrictive however, we shall stick with this definition for travelling waves so that no 'noisy' extra t dependence can occur.

To show that these conditions cannot be satisfied for the HHM we need only, to begin with, use **(iv)**, in equation (2.30): note first that with $p_0 = \frac{k}{v}$, the double zero, $P(p)$ must have the form

$$P(p) = \left(p - \frac{k}{v}\right)^2 (\alpha p - \beta) \tag{2.50}$$

for some α, $\beta \in \mathbb{R}$. Then substiting $p = \frac{k}{v}$ into (2.30) results in

$$(k^2 + v^2)(2Qv + 2ck - k^2 v) = 0 \tag{2.51}$$

presenting the following possibilities:

$$\begin{array}{lrcl} \text{(a)} & v & = & \pm ik \\ \text{(b)} & v & = & c \; = \; 0 \\ \text{(c)} & v & = & k \; = \; 0 \\ \text{(d)} & k & = & Q \; = \; 0 \\ \text{(e)} & Q & = & \frac{k}{2v}(kv - 2c) \\ \text{(f)} & c & = & \frac{v}{2k}(k^2 - 2Q). \end{array}$$

The first three can be immediately discounted since v should be real and non-zero. In case (d), $k = 0$ means that the double zero $p_0 = \frac{k}{v}$ must be zero so that with the additional condition $Q = 0$ one has (from (2.50))

$$P(p) = p^2(\alpha p - \beta) = cp^3 + \frac{v^2}{2}p^2 + cp$$

for α, $\beta \in \mathbb{R}$. For this to be true requires $c = \alpha = 0$ leaving

$$P(p) = \frac{v^2}{2}p^2$$

which evidently has only one (double) zero, contradicting condition **(ii)** above. Cases (e) and (f) remain; substitution of (e) into (2.30) results in

$$P(p) = cp^3 + \frac{p^2}{2}(v^2 - 2c\frac{k}{v}) + p(c - kv) + \frac{k}{2v}(kv - 2c). \tag{2.52}$$

And equating this with (2.50) and comparing coefficients results in the following set of equations:

$$\begin{aligned}\alpha &= c && (2.53a)\\ \beta &= \frac{v}{2k}(2c-kv) && (2.53b)\\ \beta+\frac{2k}{v}\alpha &= \frac{1}{2}\left(2\frac{ck}{v}-v^2\right) && (2.53c)\\ 2\beta+\alpha\frac{k}{v} &= \frac{v}{k}(c-kv). && (2.53d)\end{aligned}$$

Substitution of (2.53a) into the other three equations then yields

$$-\frac{1}{2v}(v^3+2kc)=\frac{1}{2kv}(v^2c-kv^3-ck^2)=\frac{v}{2k}(2c-kv).$$

The only solution to this is again that, $v=\pm ik$.

There is one possibility left to consider, namely, case (f); on substitution of $c=\frac{v}{2k}(k^2-2Q)$ into (2.30) so that

$$P(p)=\frac{v}{2k}(k^2-2Q)p^3+\frac{p^2}{2}(v^2-k^2+2Q)-\frac{pv}{2k}(2Q+k^2)+Q$$

and following the same procedure as in case (e), a similar set of equations to (2.53) results. The solution of these is $B=-\frac{v^2Q}{k^2}$ leading to

$$\frac{v}{2k}(k^2-2Q)=\frac{v}{4k^3}(k^4-k^2(v^2+2Q)+2v^2Q)=\frac{v^3}{2k^3}(2Q-k^2).$$

Equating each of these expressions with the others results always in the equation

$$k^4+k^2(v^2-2Q)-2Qv^2=0$$

so that either again, $v=\pm ik$, or $k^2=2Q$. Substitution of the latter into (2.30) then gives us that

$$P(p)=\frac{1}{2}\left(p-\frac{k}{v}\right)^2$$

which again does not satisfy stipulation **(ii)**, i.e. that two repeated and one simple zero must exist.

All possibilities have been exhausted for solutions of (2.51) to satisfy the required conditions **(i)-(iv)** for a travelling wave of winding type as defined, and where

$X = \mathbb{R}$. And we surmise that for the Hyperbolic Heisenberg model, whilst they do exist for $X = S^1$, such travelling wave solutions cannot exist when physical space X is the real line.

2.6 Solutions from a Second Parametrization

Up to now solutions of the equations of motion (2.20a) and (2.20b) have been considered with the parametrization (2.17) of the hyperboloid H^2 in terms of θ and ϕ. Turning now to the second parametrization (2.18) in terms of u and v, the equations of motion under the stereographic projection take the form

$$u_t = v_{xx} + 2\alpha v(u_x^2 + v_x^2) - 4\alpha u u_x v_x \tag{2.54}$$

$$v_t = u_{xx} - 2\alpha u(u_x^2 + v_x^2) + 4\alpha v u_x v_x \tag{2.55}$$

where $\alpha = (1 + u^2 - v^2)^{-1}$. From these equations a simple time dependent solution on $X = \mathbb{R}$ can be found by looking for solutions in which α is a function only of x, i.e. letting $u^2 - v^2 = \big[f(x)\big]^2$. Defining the function $g = vu_x - uv_x$ so that $g_x = vu_{xx} - uv_{xx}$, noting that $2ff_t = 0 = 2(uu_t - vv_t)$, i.e. $uu_t = vv_t$, and multiplying (2.54), (2.55) by u and v respectively, one has

$$\begin{aligned}
\frac{\partial g}{\partial x} &= vu_{xx} - uv_{xx} \\
&= \frac{4uv(u_x^2 + v_x^2) - 4u_x v_x(v^2 + u^2)}{1 + u^2 - v^2} \\
&= 4(vu_x - uv_x)\frac{(u^2 - v^2)^{\frac{1}{2}}(uu_x - vv_x)}{(u^2 - v^2)^{\frac{1}{2}}(1 + u^2 - v^2)}.
\end{aligned}$$

And since $f_x = \frac{(uu_x - vv_x)}{(u^2 - v^2)^{\frac{1}{2}}}$, this is equivalent to

$$\frac{\partial g}{\partial x} = \frac{4gf_x f}{(1 + f^2)},$$

i.e. $g = vu_x - uv_x$ satisfies the equation

$$\frac{\partial g}{\partial x} = 2g\,\frac{\partial}{\partial x}\big[\log(1 + f^2)\big]. \tag{2.56}$$

Taking the simplest solution of this, namely $g \equiv 0$ then $uv_x = vu_x$ and u, v must have the form

$$u(t,x) = f(x)\cosh(h(t)) \tag{2.57}$$
$$v(t,x) = f(x)\sinh(h(t)). \tag{2.58}$$

Substitution of these into (2.54), (2.55) results in

$$fh_t = f_{xx} - \frac{2ff_x^2}{1+f^2},$$

i.e. $\frac{dh}{dt} = m$ constant, since $f = f(x)$, so that f satisfies the following equation:

$$\frac{d^2f}{dx^2} = \frac{2f}{(1+f^2)}\left(\frac{df}{dx}\right)^2 + mf. \tag{2.59}$$

Observe that this equation exhibits the singularities $f = \pm i$, and following the methodology of Ince [76] to ensure that any general solution be uniform; the absence of *movable* critical points, i.e. that any branch points and essential singularities are fixed, must be established. In order that this be the case, the equation should be of the form

$$\frac{d^2\omega}{dz^2} = L(z,\omega)\left(\frac{d\omega}{dz}\right)^2 + M(z,\omega)\left(\frac{d\omega}{dz}\right) + N(z,\omega)$$

where L, M, N are rational functions of ω with coefficients analytic in z. $L(z,\omega)$ should have only simple poles and any poles of M and N must be included amongst those of L and hence be simple also, and clearly these conditions are satisfied by equation (2.59) (where ω corresponds to f here and z to x). Making the transformation $W = \frac{f+i}{f-i}$ the equation (2.59) then has the form

$$\frac{d^2W}{dz^2} = \frac{1}{W}\left(\frac{dW}{dz}\right)^2 - mW^2 + m$$

which has as its first integral

$$\left(\frac{dW}{dz}\right)^2 = -mW^3 + QW^2 - mW$$

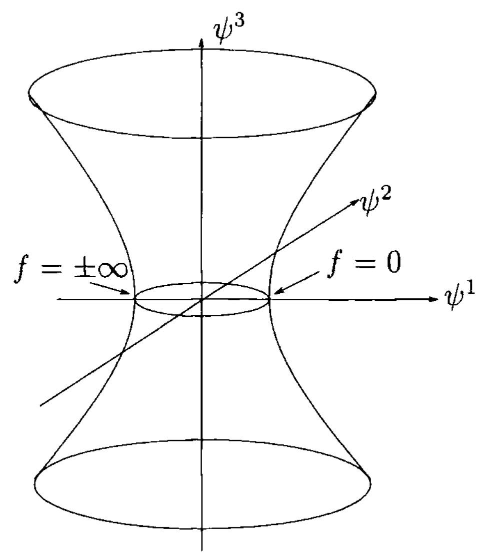


Figure 2.6: The solution f wraps around the waist of the hyperboloid at $t = t_0$ passing through $\vec{\psi} = (\pm 1, 0, 0)$.

where Q is an arbitrary constant. Transforming back to f now results in the equation

$$P(f) = \left(\frac{df}{dx}\right)^2 = C(1+f^2)^2 - m(1+f^2) \tag{2.60}$$

where $C = \frac{1}{4}(2m - Q)$. For winding solutions certain restrictions must be placed on the constants C and m which can be seen as follows: with u, v as given in (2.57),

$$\vec{\psi} = \frac{1}{1+f^2}(1-f^2, 2f\cosh h, 2f\sinh h), \tag{2.61}$$

and without loss of generality, t_0 may be chosen such that $h(t_0) = 0$; there is such a t_0 since $h(t) = -mt + k$. Then at $t = t_0$

$$\vec{\psi} = \frac{1}{1+f^2}(1-f^2, 2f, 0).$$

Now to wind, f must pass through $f = 0$ and $f = \pm\infty$ so that it passes through the points $(\pm 1, 0, 0)$ as shown in Figure (2.6). Hence, we may put

(i) on S^1,where $x \in (-\pi, \pi)$, with $N = 1$ for simplicity,

$$\begin{aligned} f(0) &= 0 \\ \text{and} \quad f(\pm\pi) &= \pm\infty \end{aligned} \tag{2.62}$$

(ii) and on $\mathbb{R}$ where $x \in (-\infty, \infty)$ again taking $N = 1$,

$$\begin{aligned} f(\pm\infty) &= 0 \\ \text{and} \quad f(0) &= \pm\infty. \end{aligned} \tag{2.63}$$

In case **(i)** with $X = S^1$, f takes the form shown in Figure (2.7(a)). Then the inverse of f must look like Figure (2.7(b)) and since f^{-1} is the same as $\int \frac{df}{\sqrt{P(f)}}$, $\frac{1}{\sqrt{P(f)}}$ must have the shape of Figure (2.7(c)). Hence, $P(f)$ takes the form of Figure (2.7(d)). It can be seen from (2.60) that for this to be the case, C must be ≥ 0. And since here no real zeros can exist for $P(f)$ it must also be the case that $\frac{m}{C} < 1$ (for $C \neq 0$).

In case **(ii)** for $X = \mathbb{R}$, f takes the form of Figure (2.8(a)) and since f^{-1} has almost the same profile, $\int \frac{df}{\sqrt{P(f)}}$ must also have a similar form. This being so, $\frac{1}{P(f)}$ then looks like Figure (2.8(b)) so that $P(f)$ takes the form of Figure (2.8(c)). Therefore $P(f)$ must have one real, repeated zero and be greater than zero elsewhere.

Since $C \geq 0$ in both cases above, let us take $C > 0$ and scale x such that $C = 1$. This then imposes the condition in the S^1 case that $m < 1$, whilst $m = 1$ satisfies the $X = \mathbb{R}$ case. So we take $0 \leq m \leq 1$ (noting that for $-1 \leq m < 0$ similar solutions are obtained). Rearranging (2.60), we then have the integral

$$\int \frac{df}{[(1+f^2)(f^2 + (1-m))]^{\frac{1}{2}}} = (x - x_0)$$

x_0 an arbitrary constant. And the solution of (2.60) is given by

$$f(x) = (1-m)^{\frac{1}{2}} \operatorname{sc}(x|m) \tag{2.64}$$

in the notation of [74], where $\operatorname{sc}(x|m) = \frac{\operatorname{sn}(x|m)}{\operatorname{cn}(x|m)}$. The limits of (2.64) with respect to the elliptic modulus (here, m), are given by the following: for $m = 0$, $\operatorname{cn}(x|0) = \cos x$ and $\operatorname{sn}(x|0) = \sin x$ so that

$$f(x) = \tan x \qquad \text{for } m = 0. \tag{2.65}$$

For the limit $m = 1$ to obtain f of the required form a shift in x is necessary: to this end note that $\operatorname{sc}(K - u|m) = \frac{1}{\sqrt{m_1}} \operatorname{cs}(u|m)$ where $m_1 = 1 - m$, $\operatorname{cs}(u|m) = \frac{\operatorname{cn}(u|m)}{\operatorname{sn}(u|m)}$

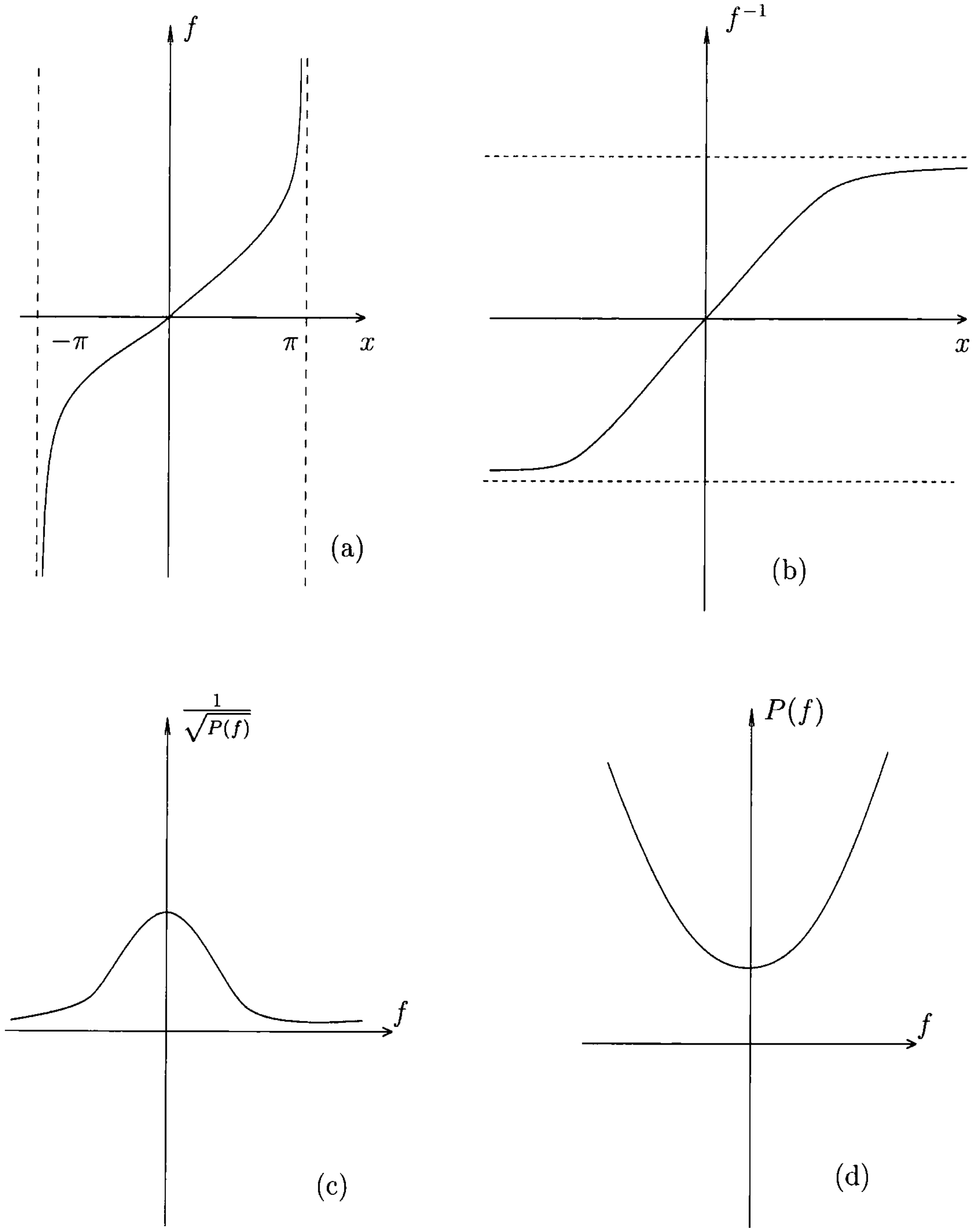


Figure 2.7: For $X = S^1$: from f in (a) one then has f^{-1} in (b) hence, through (c), the profile of $P(f)$ can be deduced, i.e. (d)

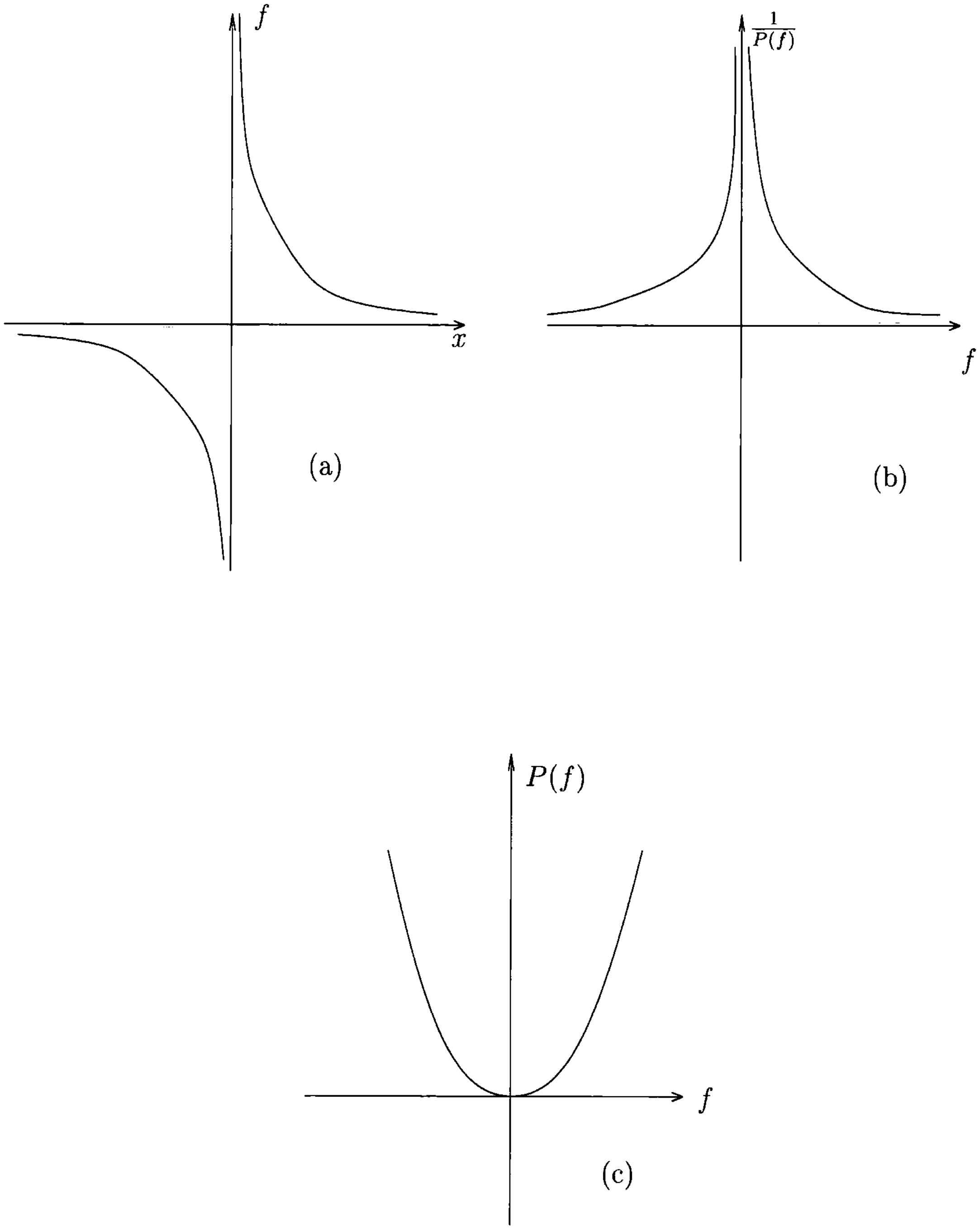


Figure 2.8: For $X = \mathbb{R}$ from f in (a) the required shape of $P(f)$ for winding solutions can be deduced through (b).

and

$$K(m) = K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - m\sin^2\theta)^{\frac{1}{2}}} \qquad \in \mathbb{R}. \tag{2.66}$$

And when $m = 1$, cs $(u|m) = \operatorname{cosech} u$. With this shift in x, the coefficient in (2.64) reduces to $m_1^{-\frac{1}{2}}\sqrt{1-m} = 1$, and

$$f(x) = \operatorname{cosech} x \qquad \text{for } m = 1. \tag{2.67}$$

Although, in effect, x has been shifted by an infinite amount since $K(m)$ '$=$' ∞ when $m = 1$, this solution is consistent with the equations of motion and further, since $K(m)$ essentially defines the period of the elliptic functions, the solution has infinite period which is what one would expect since what we have here is the one-wind solution on the real line.

To summarize, in the case where $f = f(x)$, $g \equiv 0$ there exists a family of winding solutions parametrized by $m \in [0, 1]$ with $h(t) = -mt$ and where $f(x)$ is specified in (2.64) (or a shift thereof), (2.65) or (2.67). For (2.64), (2.65) the solution has finite period whilst for (2.67) it lives on the real line $X = \mathbb{R}$. This third solution in terms of $\vec{\psi}$ is

$$\vec{\psi} = (1 - 2 \operatorname{sech}^2 x, 2 \operatorname{sech}^2 x \sinh x \cosh t, -2 \operatorname{sech}^2 x \sinh x \sinh t) \tag{2.68}$$

and is a 'stationary' 1-soliton with winding number $N = 1$. The solution (2.68) is the hyperbolic version of the 'stationary' 1-soliton solution of the S^2 Heisenberg Model (c.f. [63]) which, after analytic continuation to fit on the hyperboloid H^2 and with full time dependence, takes the form

$$\begin{aligned}
\psi^1(t,x) &= 1 - 2\Lambda^2\mu \\
\psi^2(t,x) &= 2\mu\left(-\Lambda^2 \cosh\Omega \sinh\Theta + \Lambda v \sinh\Omega \cosh\Theta\right) \\
\psi^3(t,x) &= 2\mu\left(-\Lambda v \cosh\Omega \cosh\Theta + \Lambda^2 \sinh\Omega \sinh\Theta\right)
\end{aligned} \tag{2.69}$$

where $\mu = (\Lambda^2 - v^2)^{-1} \operatorname{sech}^2\Theta$,

$$\begin{aligned}
\Omega &= \gamma_0 + \frac{vx}{2} - \frac{(\Lambda^2 + v^2)}{4}t, \\
\Theta &= \frac{\Lambda}{2}(x - vt - x_0)
\end{aligned}$$

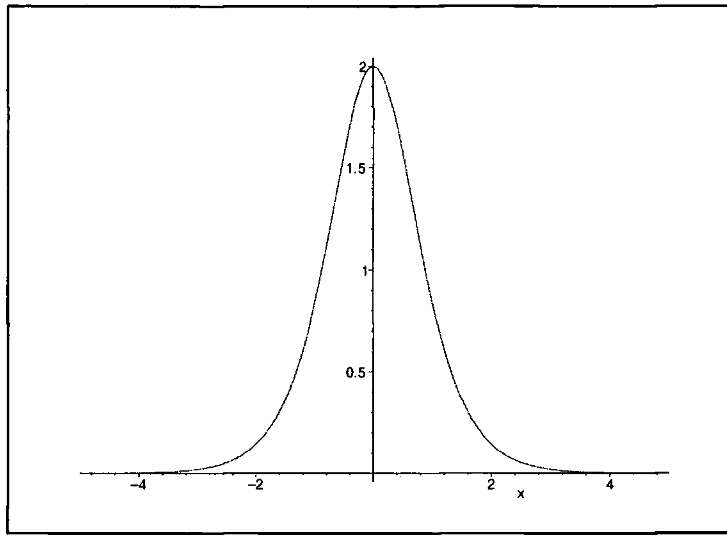


Figure 2.9: Energy density for solution (2.69) with $v = 0$

and $\Lambda \neq \pm v \in \mathbb{R}$. The soliton is specified by the four real parameters: velocity v, the initial phase γ_0, the initial center of inertia coordinate x_0 and the amplitude $\mathcal{A}$ of the coefficient $\psi^1(t,x)$ given by

$$\mathcal{A} = \frac{\Lambda^2 + v^2}{\Lambda^2 - v^2}.$$

The solution (2.69) is, in fact, equivalent to that of Pashaev and Lee [70], mentioned earlier in the chapter. Note that for $v = 0$, $\vec{\psi} = (\psi^1, \psi^2, \psi^3) = (1, 0, 0)$ as $x \longrightarrow \pm\infty$ and at $x = 0$, $\vec{\psi} = (-1, 0, 0)$ so that the solution wraps once around the 'waist' of the hyperboloid and it is this particular case which is related to (2.67) above where $m = \frac{\Lambda^2}{4}$. The energy density of this stationary one wind solution is given by $\epsilon = 2\mathrm{sech}^2 x$ as shown in figure (2.9).

2.7 Concluding Remarks

This completes our discussion of winding solutions for the HHM. A pair U, V has been specified which satisfies the zero curvature condition (1.2), with the equations of motion for HHM as the consistency condition for a linear system for the given U, V, thereby indicating that the model is integrable. With the two different parametrizations of the hyperboloid given in Chapter 1, we have been able to show the existence of, and explicitly derive, both static and simple time dependent solutions of winding type for the model for both $X = S^1$ and $X = \mathbb{R}$. It has also been shown that whilst they exist for the space $X = S^1$, no travelling waves of winding type are possible for HHM when X is the real line $\mathbb{R}$.

Chapter 3

The Hyperbolic Sigma Model (HSM)

3.1 Introduction

The sine-Gordon equation of Chapter 1 is, as noted, an example of a (1 + 1)-dimensional non-linear system whose dynamics is governed by the Lorentz invariant Lagrangian density

$$\mathcal{L}(t,x) = \frac{1}{2}\Big(\frac{\partial \phi}{\partial t}\Big)^2 - \frac{1}{2}\Big(\frac{\partial \phi}{\partial x}\Big)^2 - \mathcal{V}(\phi) \tag{3.1}$$

where $\phi(t,x)$ is a single scalar field and non-linear terms depend on the choice of the potential $\mathcal{V}(\phi)$. If $\mathcal{V}(\phi)$ is set to zero, application of the variational principle results in the *linear* wave equation

$$\Box\phi = \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{3.2}$$

with the general solution, in terms of characteristic variables $x \pm t$ and arbitrary functions f and g,

$$\phi(t,x) = f(x-t) + g(x+t).$$

The introduction of non-linearities via the choice of potential $\mathcal{V}$ may be thought of in an alternative way; taking as a starting point the Lagrangian

$$\mathcal{L}(t,x) = \frac{1}{2}\partial_\mu \vec{\phi}\, \partial^\mu \vec{\phi} \tag{3.3}$$

(corresponding to (3.1) with $\mathcal{V} \equiv 0$), non-linearities may occur in the resultant equations of motion via the imposition of constraints on the fields[1]. What is known variously as the non-linear $O(n)$ model, the *chiral* model and the $O(n)$ sigma model[2] arises in just such a way. By imposing the constraint

$$\vec{\phi}.\vec{\phi} = 1 \tag{3.4}$$

where $\vec{\phi}(t,x) = \phi_i(t,x) : i = 1, \ldots, n$ is a set of n coupled scalar fields (replacing the single scalar field ϕ in (3.1)), the fields are constrained to take values on an n-dimensional sphere and the Lagrangian (3.3) is invariant under $O(n)$ rotation. For example, the equations of motion for the $O(3)$ sigma model are obtained via the variational principle by extremizing the action

$$S[\vec{\phi}] = \int dx \int dt \Big[\frac{1}{2}\partial_\mu \vec{\phi}\,.\,\partial^\mu \vec{\phi} + \lambda(t,x)\big(\vec{\phi}.\vec{\phi} - 1\big)\Big]. \tag{3.5}$$

The constraint (3.4) is imposed through the Lagrange multiplier $\lambda(t,x)$, and the resultant field equations are

$$\Box\vec{\phi} - \big(\vec{\phi}.\Box\vec{\phi}\big)\vec{\phi} = 0. \tag{3.6}$$

Pohlmeyer, [28] in 1976, produced a set of 'reduced' $O(n)$ models where the reductions involved the imposition of additional constraints. And much interest has been generated by the fact that these models possess dual symmetry and are generalizations of the sine-Gordon theory (see Pohlmeyer, and also [58, 65, 77]). The integrability of both the reduced models and the original systems is well established for $(1+1)$-dimensional space-time and for general n, c.f. [27, 28, 77], and in the $n = 3$ case in particular, *static* topological solitons have been derived explicitly for both the $(1+1)$ and $(2+1)$-dimensional models (c.f.[16]). However, whilst the $O(3)$

[1] In the sine-Gordon case both the potential $\mathcal{V}$ and field constraints are constituents of the model.

[2] The name 'sigma model' is often used generically to describe a whole class of relativistic systems (c.f. [20] for the sigma model condition), however, the term as used throughout this text refers to a particular class of relativistic systems associated with Lagrangians of the type (3.3).

model in $(1+1)$ dimensions is integrable (as noted earlier), the $(2+1)$-dimensional model is not.

Moreover, the $O(n)$ chiral fields represent (c.f. [27]), in some sense, the same object as the U, V systems which play an important role in integrability theory, and thereby admit a strong resemblance to the four dimensional self-dual Yang Mills system discussed earlier. The chiral equations are also, of course, reductions of the SDYM system ([9, 38, 15]). However, having previously reviewed gauge equivalence and the SDYM reduction process for specific examples, we refrain from applying these ideas directly here.

In light of the fact that the main concern of this chapter is a non-compact version of the above model, we note first the following results which have emerged in the literature in relation to various non-compact sigma models. In 1982, Kundu [65, 66] extended Pohlmeyer's reduced compact case and its sine-Gordon equivalence to the analogous $SU(2,1)/U(1,1)$ sigma model (i.e. on the two-sheeted hyperboloid), demonstrating its gauge equivalence to a sinh-Gordon equation. Also in 1982, Mazur [83, 84] showed that the Ernst equations for the stationary Einstein-Maxwell fields are connected with a non-linear sigma model on the space $SU(2,1)/S(U(2) \times U(1))$ and similarly that the vacuum Ernst equations can be interpreted as a sigma model on $SU(1,1)/U(1)$.

For Euclidean sigma models on non-compact Grassmanian manifolds (and their supersymmetric extensions) a method of constructing explicit solutions is presented by Antoine and Piette [86]; the "holomorphic" (DZS) method of Din, Zakrzewski and Sasaki [87] for compact manifolds is generalized to those of non-compact type. Lambert and Piette [4] showed further that the DZS method can be extended to Minkowskian sigma models on both compact and non-compact manifolds and in particular, the hyperboloid of one sheet.

In the context of string theory, DeVega and Sanchez [78] adapt Pohlmeyer's reduced $O(n)$ sigma model to fit onto $SO(2,1)/SO(1,1)$, i.e. de-Sitter space-time, and show the complete integrability of string propagation and further that the string equations in two dimensions are equivalent to the Liouville equation. Their model, following the formulation of Pohlmeyer [28], has the additional string constraint

$$\psi_\xi^a \psi_\xi^b \eta_{ab} = \psi_\eta^a \psi_\eta^b \eta_{ab} = 1$$

and is a special case of the model under scrutiny in this chapter.

Many diverse and complex elements of pure mathematics are utilized in the above works, including hyperbolic complex numbers, harmonic maps and Riemannian geometry and indeed, in the higher dimensional cases of Lambert and Piette [4], and Lambert and Rembielinski [85] (as noted in Chapter 1), Gödel quaternions are implemented to study sigma models on a four-dimensional hyperboloid. So, as well as the physical applications of non-compact non-linear sigma models, for example, string theory and relativity, such systems are of aesthetic interest since analysis of the models and their solutions involves some attractive mathematics, which is in some instances, of a highly non-trivial nature.

This chapter is concerned with the existence of solutions of winding type for the HSM - the *Hyperbolic non-linear Sigma Model* where the field takes its values on the hyperboloid of one sheet. The following section is devoted to formulation of the model and a U, V pair given to establish its integrability in the sense described in Chapter 1.

3.2 Formulation of HSM

The generalized field equations of motion for the model on H^2 are obtained as follows: the Lagrangian density for the HSM is given by

$$\mathcal{L} = \frac{1}{2}\psi^a_\mu \, \psi^b_\nu \, \eta_{ab}\eta^{\mu\nu}, \tag{3.7}$$

i.e. (3.3) with the metric η_{ab} replacing δ_{ab} in the compact case, and by imposing the constraint (1.3) through the arbitrary parameter λ one has the action

$$S[\vec{\psi}] = \int dt \int dx \Big[\frac{1}{2}\psi^a_\mu \, \psi^b_\nu \, \eta^{\mu\nu} \, \eta_{ab} + \lambda(\psi^a \, \psi^b \, \eta_{ab} - 1)\Big].$$

Applying the Euler-Lagrange variational principle by varying with respect to λ then results in the constraint (1.3). And varying with respect to ψ^a produces

$$\eta^{\mu\nu}\psi^a_{\mu\nu} - 2\lambda\psi^a = 0$$

from which $2\lambda = (\psi^a_{\mu\nu} \, \psi^b) \, \eta^{\mu\nu} \, \eta_{ab}$. Now

$$(\psi^a_{\mu\nu} \, \psi^b)\eta^{\mu\nu} \, \eta_{ab} = \big[(\psi^a_\mu \, \psi^b)_\nu - \psi^a_\mu \, \psi^b_\nu\big]\eta^{\mu\nu} \, \eta_{ab}$$

and since by the constraint (1.3)

$$\left(\psi_\mu^a \psi^b\right)_\nu \eta^{\mu\nu} \eta_{ab} = 0$$

one then has

$$2\lambda = -\psi_\mu^a \psi_\nu^b \eta^{\mu\nu} \eta_{ab},$$

resulting in the following equation of motion:

$$\left[\psi_{\mu\nu}^a + \psi^a(\psi_\mu^b \psi_\nu^c)\eta_{bc}\right]\eta^{\mu\nu} = 0. \tag{3.8}$$

As with the previous chapter,the objective here is to investigate solutions of winding type, however, it must first be established that the model is integrable: with the change of variables $\xi = \frac{1}{2}(t+x)$, $\eta = \frac{1}{2}(t-x)$ so that (3.8) becomes $\psi_{\eta\xi}^a + \psi^a(\psi_\eta^b \psi_\xi^c)\eta_{bc} = 0$, one can simply adapt the U, V pair given in [77] for the $\mathbb{C}P^N$ case to fit the model on H^2. With (1.3), the pair

$$\begin{aligned} U &= \frac{1}{4}(\lambda - 1)\,[S, \partial_\xi S] \\ V &= \frac{1}{4}\frac{(1-\lambda)}{\lambda}\,[S, \partial_\eta S] \end{aligned}$$

satisfy the zero curvature condition

$$\partial_\eta U - \partial_\xi V + [U, V] = 0.$$

Where here $S = \begin{bmatrix} \psi^1 & \psi^2 + \psi^3 \\ \psi^2 - \psi^3 & -\psi^1 \end{bmatrix} \in SL(2, \mathbb{R})$ so that the consistency condition for the linear system

$$\begin{aligned} \partial_\xi F &= UF \\ \partial_\eta F &= VF \end{aligned}$$

is exactly the equation of motion (3.8) in terms of ξ, η. Hence the system is integrable. The change of variables above is not strictly necessary since the zero curvature condition etc. still hold for the original variables x, t substituted into the U, V pair however, the form is more complicated making the actual calculations much more so. Hence this change of variables has been implemented here for the sake of simplicity only.

The HSM under the stereographic parametrisation (2.18) was formulated and examined by Lambert and Piette [4] and will, for completeness, be discussed briefly later in the chapter. For the most part however, we shall work with the θ, ϕ parametrisation (2.17) of the hyperboloid. The Lagrangian density (3.7) of the system in terms of θ, ϕ is

$$\mathcal{L} = \frac{1}{2}\left(\cosh^2\theta(\phi_t^2 - \phi_x^2) - (\theta_t^2 - \theta_x^2)\right) \tag{3.9}$$

and the energy density,

$$\varepsilon = T^{00} = \sum_{a=1,2} \frac{\delta\mathcal{L}}{\delta(\partial_0 A_a)}(\partial_0 A_a) - \mathcal{L}\eta^{00}$$

(where here, $A_{1,2} = \theta, \phi$), is, in this case, given by

$$\varepsilon = \cosh^2\theta\,(\phi_t^2 + \phi_x^2) - (\theta_t^2 + \theta_x^2). \tag{3.10}$$

The 'energy' is then

$$E = \int_X \varepsilon\, dx$$

which again, one can see, is not positive definite due to the indefinite metric on the field space.

The equations of motion in terms of θ, ϕ may be derived either from the Lagrangian (3.9) using the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A)}\right) - \frac{\partial\mathcal{L}}{\partial A} = 0$$

where $A = \theta$, ϕ, or by direct substitution of the parametrisation (2.17) into (3.8). Either way, the resultant equations are

$$\theta_{tt} - \theta_{xx} = -\cosh\theta\,\sinh\theta\,(\phi_t^2 - \phi_x^2) \tag{3.11a}$$

$$(\phi_t\cosh^2\theta)_t = (\phi_x\cosh^2\theta)_x \tag{3.11b}$$

and we can now begin investigating the existence of topological solitons for the HSM.

3.3 Some Simple Time Dependent Winding Solutions

For the static problem, the equations (and hence their solutions) are the same as those of the static Heisenberg case previously discussed. We therefore begin by looking for winding solutions with some time dependence. Taking as a first possibility $\phi = Nx$ with $N \in \mathbb{Z}$ and where physical space $X = S^1$, has $\phi_t = 0$, $\phi_x = N$ which, in (3.11b) leaves

$$(N \cosh^2 \theta)_x = 0.$$

And it is clear from this that θ can be a function of t only. With $\theta = \theta(t)$ in (3.11a), θ then satisfies the first integral

$$\theta_t^2 = N^2 \sinh^2 \theta + c \tag{3.12}$$

where c is an arbitrary constant. This is integrable in terms of elliptic functions, the exact form of the solution depending on the constants c and N^2. For example, let $c = \rho^2 > 0$ and $0 \leq N^2 \leq \rho^2$ then with the substitution $\tanh \theta = v$ one has

$$\frac{\rho}{(\rho^2 - N^2)^{\frac{1}{2}}} \int \frac{dv}{\left[(1 - v^2)(\frac{\rho^2}{\rho^2 - N^2} - v^2)\right]^{\frac{1}{2}}} = \rho(t - t_0)$$

where t_0 is constant. And with the parameter $m = 1 - \frac{N^2}{\rho^2} \leq 1$ this yields the solution

$$\theta(t) = \tanh^{-1} [\, \mathrm{sn}\ (\rho(t - t_0)|m)] \,. \tag{3.13}$$

Since $\tanh^{-1} u \longrightarrow \infty$ as $u \longrightarrow 1$, clearly $\theta(t) \longrightarrow \infty$ as $\mathrm{sn}\,(\rho(t - t_0)|m) \longrightarrow 1$, i.e. $\theta(t)$ reaches infinity in a finite time. This behaviour may be examined in more detail for the field $\vec{\psi}$ as follows: letting $t_0 = 0$ so that $\theta(t) = \tanh^{-1}[\mathrm{sn}\ (\rho t)]$ then $\theta(t) \longrightarrow \infty$ for some finite value of t, say $\tilde{t}$. Now putting

$$\mathrm{sn}\ (\rho t|m) = \mathrm{sn}(\rho\tilde{t} + \rho(t - \tilde{t})|m),$$

with $u = \rho\tilde{t}$, $v = \rho(t - \tilde{t})$ in the identity

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\,\mathrm{cn}v\,\mathrm{dn}v + \mathrm{cn}u\,\mathrm{sn}v\,\mathrm{dn}u}{1 - m\ \mathrm{sn}^2u\,\mathrm{sn}^2v},$$

and replacing $\rho(t-\tilde{t})$ by ϵ suitably small, one then has

$$\mathrm{sn}(\rho t) \approx \mathrm{cn}\epsilon\,\mathrm{dn}\epsilon$$

since when $\mathrm{sn}\,u = 1$, $\mathrm{cn}\,u = 0$ (and omitting the parameter m for simplicity). Furthermore, to second order, $\mathrm{cn}u \approx 1 - \frac{u^2}{2} + O(u^3)$ and $\mathrm{dn}u \approx 1 - \frac{mu^2}{2} + O(u^3)$ so that

$$\begin{aligned}\mathrm{sn}(\rho t) &\approx 1 - \frac{\epsilon^2}{2}(m+1)\\ &= 1 - (c - \frac{N^2}{2})(t-\tilde{t})^2\end{aligned}$$

(which tends to 1 as $t \longrightarrow \tilde{t}$). Substituting $\phi = Nx$ and the above into the field coordinates $\vec{\psi} = (\psi^1, \psi^2, \psi^3)$, and using the identities $\tanh^{-1} x = \ln(\frac{1+x}{1-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, one has

$$\begin{aligned}\psi^1 &= \frac{\cos Nx}{A(t-\tilde{t})[2 - A^2(t-\tilde{t})^2]^{\frac{1}{2}}}\\ \psi^2 &= \frac{\sin Nx}{A(t-\tilde{t})[2 - A^2(t-\tilde{t})^2]^{\frac{1}{2}}}\\ \psi^3 &= \frac{1 - A^2(t-\tilde{t})^2}{A(t-\tilde{t})[2 - A^2(t-\tilde{t})^2]^{\frac{1}{2}}}\end{aligned}$$

where $A^2 = c - \frac{N^2}{2}$, each of which clearly blows up in a finite time, i.e. as $t \longrightarrow \tilde{t}$. One therefore has a loop located at $\theta(t)$ which goes from $\theta = -\infty$ to $\theta = +\infty$ in a finite time, as $t \longrightarrow \tilde{t}$ which can be seen pictorially in Figure (3.1).

As was noted earlier, the 'energy' density for the HSM is, in general, non positive definite and with $\phi = Nx$, this is given by

$$E = \int_0^{2\pi} N^2\cosh^2\theta - (N^2\sinh^2\theta + \rho^2)\,dx, \tag{3.14}$$

i.e., $E = 2\pi\,[A(\theta) - B(\theta)]$ where $A(\theta) = N^2\cosh^2\theta$ and $B(\theta) = N^2\sinh^2\theta + \rho^2$. It can be shown that for the solution (3.13), the two positive functions A and B reach

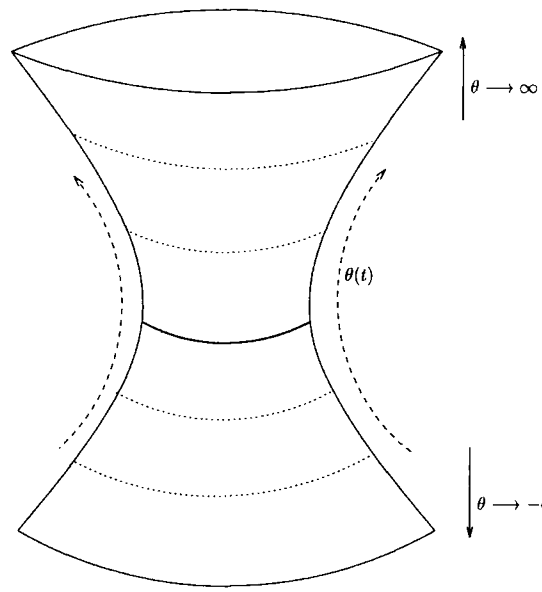


Figure 3.1: The loop is located at $\theta(t)$ travelling from $\theta = -\infty$ to $\theta = +\infty$ in a finite time, i.e. as $t \longrightarrow \tilde{t}$

infinity in a finite time (both at the same rate). However, the 'energy' density for the solution $\phi = Nx$ is a finite constant, i.e. $\varepsilon = N^2 - \rho^2$ so that $E = 2\pi(N^2 - \rho^2)$ (negative since $N^2 \leq \rho^2$). One therefore has a conserved negative quantity, although how this may be interpreted physically is not clear.

A simpler version of solution (3.13) occurs in the limit $m = 0$ where sn $(v|0) = \sin v$ (requiring $\rho^2 = N^2$) so that $\theta(t) = \tanh^{-1}\left[\sin N(t - t_0)\right]$ and as t goes from $-\pi$ to π the solution θ goes from $-\infty$ to ∞. Solutions with $c > 0$ all exhibit this kind of behaviour. In the opposite limit where $m = 1$, the winding number N is required to be zero so that the solution is trivial. And in the special case $c = 0$, the solution

$$\theta(t) = 2\tanh^{-1}(ke^t)$$

includes the static case $\theta \equiv 0$ and tends asymptotically to $\theta = 0$ as $t \longrightarrow -\infty$.

We thus have a family of t-dependent solutions wrapped around the hyperboloid with $\phi = Nx$, $\theta = \theta(t)$ and where there is manifest instability since $\theta \longrightarrow \infty$ in a finite time. Further, the solutions admit a finite, conserved, negative quantity associated in some sense, with the energy of the system.

3.4 Travelling Waves for HSM

In investigating solutions for HSM the next step is to follow the approach of the previous chapter, where the existence of travelling wave winding solutions was examined using the characteristic variable $\xi = x - vt$ with

$$\begin{aligned} \theta(t,x) &= f(\xi) \\ \phi(t,x) &= g(\xi) + ct. \end{aligned}$$

The equations of motion are then

$$f''(v^2-1) = -\left[g'^2(v^2-1) - 2vcg' + c^2\right]\cosh f\,\sinh f \tag{3.15}$$
$$g''(v^2-1) = -2\left[f'g'(v^2-1) - cvf'\right]\tanh f \tag{3.16}$$

where 'prime' refers to differentiation with respect to ξ.

As a first case let $c = 0$, so that

$$f'' = -g'^2\cosh f\,\sinh f \tag{3.17}$$
$$g'' = -2f'g'\tanh f. \tag{3.18}$$

Noting that

$$\begin{aligned} \frac{d}{d\xi}(g'\cosh^2 f) &= 2f'g'\cosh f\,\sinh f + g''\cosh^2 f \\ &= 0 \end{aligned}$$

by (3.18), one then has the following first integrals for g and f:

$$g' = B\,\mathrm{sech}^2 f \tag{3.19}$$
$$f' = \sqrt{(B^2-N^2) - B^2\tanh^2 f} \tag{3.20}$$

where B and N are constants. The form of these equations suggests that their solution be analogous to the static solution (2.23) of both the HHM and this sigma model. Indeed this *is* simply a travelling wave version of that solution taking the form

$$\sinh f = \sqrt{\frac{B^2}{N^2} - 1}\,\sin\left[N(x - vt - x_0)\right] \tag{3.21}$$

And again, provided $N \in \mathbb{Z}$, $\sinh f$ is 2π periodic. In fact, since this model is Lorentz invariant one might simply have boosted the solution (2.23) - the same does not apply in the non-relativistic HHM case.

To find an explicit expression for $g(\xi)$ the same procedure as was followed with the HHM version of this solution is applied. In this case one finds (for $N = 1$)

$$\begin{aligned} g(\xi) &= \frac{B(B^2-1)}{\sqrt{(B^2-1)^2+1}}\mathcal{R}e\left\{ i\ln\left[\frac{\tan\frac{\xi}{2} - i\left((B^2-1) - \sqrt{(B^2-1)^2+1}\right)}{\tan\frac{\xi}{2} - i\left((B^2-1) + \sqrt{(B^2-1)^2+1}\right)}\right]\right. \\ &\quad \left. + i\ln A\right\}. \end{aligned} \tag{3.22}$$

And using the methodology of Appendix A, it can be shown that $g(\xi)$ has the form

$$g(\xi) = \frac{B(B^2-1)}{\sqrt{(B^2-1)^2+1}}\mathcal{R}e\left\{ i\ln\left(\frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}}\right)\right\} \tag{3.23}$$

where $\Omega = -i\left(B^2 - 1 - \sqrt{(B^2-1)^2+1}\right)$. It has been demonstrated in the previous chapter, by analyzing the function $\Xi = \frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}}$, that $\triangle\phi = \pm 2\pi$ (for $N = 1$) and the same applies here, specifically in the one-wind case for $B = \sqrt{2}$. This may all be generalized for $|N| > 1$ hence solution (3.21) is a travelling wave with winding number N and period 2π.

Extending the search for travelling waves we may look for a more general solution of (3.15), (3.16) by taking $c \neq 0$. Using the relationship (2.28), one then arrives at the following equation:

$$\frac{d}{d\xi}(g'\cosh^2 f) = \frac{df}{d\xi}\frac{2cv}{(v^2-1)}\sinh f\,\cosh f$$

and after integration equations for g' and f' are produced:

$$\begin{aligned} g' &= \frac{cv}{(v^2-1)} - \left[\frac{cv}{2(v^2-1)} - R\right]\operatorname{sech}^2 f \\ f' &= \pm\frac{\left[2\left(c^2 + 2Q(v^2-1)^2\right) + 4c^2\sinh^2 f + \left(4R(v^2-1)^2 - cv\right)^2\operatorname{sech}^2 f\right]^{\frac{1}{2}}}{2(v^2-1)} \end{aligned}$$

where R and Q are constants of integration. Letting $p = \sinh f$ to simplify things,

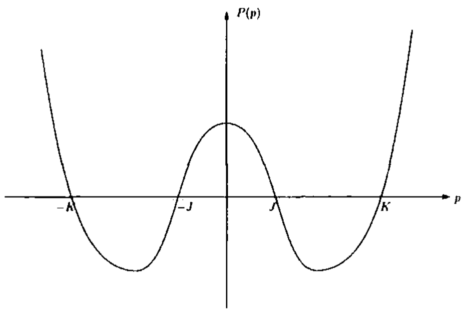


Figure 3.2: $P(p)$ with 4 real zeros, positive and bounded between $p = \pm J$

these equations transform to

$$\begin{aligned} p'^2 &= \frac{c^2}{(v^2-1)^2}\Big[p^4 + p^2\frac{(6c^2+4Q(v^2-1)^2)}{4c^2} \\ &\quad + \frac{2c^2+4Q(v^2-1)^2+\left(4R(v^2-1)^2-cv\right)^2}{4c^2}\Big] \end{aligned} \tag{3.24}$$

$$g' = \frac{cv+2R(v^2-1)+2cvp^2}{2(v^2-1)(1+p^2)}. \tag{3.25}$$

Requiring that p be periodic with respect to physical space X, one can see from the form of the quartic (3.24) in the absence of a cubic term, that the only possibility for periodic solutions is that

$$P(p) = \frac{(v^2-1)^2}{c^2}p'^2 = (J^2-p^2)(K^2-p^2) \tag{3.26}$$

for some $J, K \in \mathbb{R}$, as depicted in Figure (3.2) (or limits thereof which shall be discussed presently), so that $P(p)$ is positive and bounded between the two zeros $p = \pm J$. A solution $p(\xi)$ would then have a local minimum at $-J$ (since $P'(-J) < 0$) and a local maximum at J (since $P'(J) > 0$) and oscillate between $-J$ and J with a finite period.

Before any investigation of solutions for equations (3.24) and (3.25), it must first be ascertained whether or not $P(p)$ can take the form (3.26) and for proof that this is possible we refer to Appendix B; whilst no specific J, K are found, it is established that they do exist so that from (3.26) one has

$$\frac{cK(\xi-\xi_0)}{(v^2-1)} = K\int\frac{dp}{\sqrt{(J^2-p^2)(K^2-p^2)}}.$$

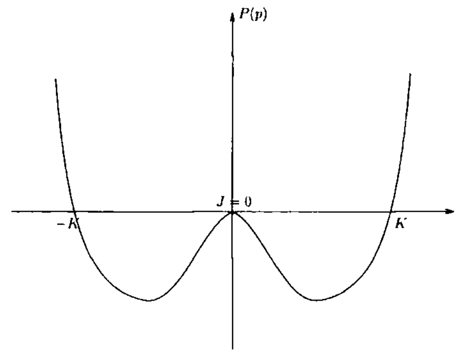


Figure 3.3: $P(p)$ with double zero $J = 0$.

This can be solved in terms of Jacobian elliptic functions to give

$$p(\xi) = J \text{ cd } \left[\frac{cK(\xi - \xi_0)}{(v^2 - 1)}|m\right]$$

where $m = \frac{J^2}{K^2}$ and $\text{cd}u = \frac{\text{cn } u}{\text{dn } u}$. If the argument is now changed so that $\frac{cK(\xi-\xi_0)}{(v^2-1)} \mapsto \frac{cK(\xi-\xi_0)}{(v^2-1)} - \mathcal{K}$ (with the quarter period $\mathcal{K}$) then

$$\text{cd } \left[\frac{cK(\xi - \xi_0)}{(v^2 - 1)} - \mathcal{K}|m\right] \mapsto \text{ sn } \left[\frac{cK(\xi - \xi_0)}{(v^2 - 1)}|m\right]$$

so that

$$p(\xi) = J \text{ sn } \left[\frac{cK(\xi - \xi_0)}{(v^2 - 1)}|m\right]. \tag{3.27}$$

Substitution of (3.27) into (3.25) then gives us an expression for $\frac{dg}{d\xi}$ however, the integration of this is somewhat problematic so rather than attempt this, to determine whether or not the solution is of winding type, the limiting cases of the elliptic solution may be considered as follows: for $m = 0$ one must have $J = 0$ so that the solution (3.27) is identically zero, corresponding to figure (3.3) for $P(p)$. In this case, $\frac{dg}{d\xi} = N$, constant, so that if $N \in \mathbb{Z}$ the HHM solution $\theta = 0$, $\phi = Nx + ct$ is recovered which is of winding type for $X = S^1$. Note that for $c = N$ the soliton travels at the speed of light around the waist of the hyperboloid.

For $m = 1$, J must equal K, renamed p_0, then

$$p(\xi) = p_0 \tanh\left[\frac{cp_0(\xi - \xi_0)}{(v^2 - 1)}\right]. \tag{3.28}$$

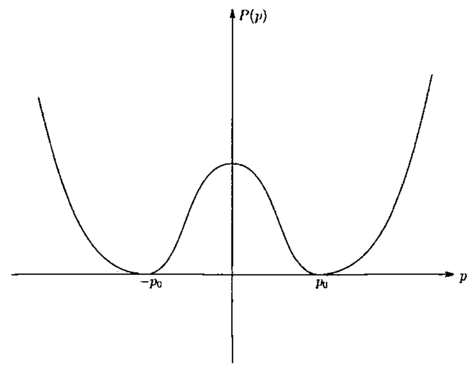


Figure 3.4: $P(p)$ with two double zeros at $\pm p_0$.

Looking for solutions in the $X = \mathbb{R}$ case, diagramaticaly the form of $P(p)$ is represented in figure (3.4). Since $\tanh u$ goes from -1 to $+1$ as x goes from $-\infty$ to $+\infty$; at $x = -\infty, p = -p_0$ and the solution travels to p_0 at $x = +\infty$. And having effectively run out of x space, the soliton cannot return to it's starting point so that there is no periodicity. In fact, this shows that there are no travelling wave solutions of winding type for HSM on the real line for $c \neq 0$ since no other bounded region is possible for $P(p)$, i.e. one cannot have a double zero together with two real and distinct zeros due to the absence of a cubic term in (3.24).

As previously noted, for a solution to be topological requires that $\Delta\phi = 2\pi N$. Now given the form of $P(p)$ shown in figure (3.2), of which (3.3) and (3.4) above are limiting cases, if the solution (3.27) is of winding type, since the solution has traveled effectively half a period one might reasonably expect in the limiting picture (figure 3.4), that here $\Delta\phi = \pi$. However, as we shall see this is not the case and whilst this does not constitute a proof, it is at least a solid indication that the general solution (3.27) may not be of winding type. The following shows that $\Delta\phi \neq \pi$ in the limit $m = 1$: for $p(\xi)$ given by (3.28) one has

$$\frac{dg}{d\xi} = \frac{cv + 2R(v^2-1) + 2cvp_0^2\tanh^2 X}{2(v^2-1)(1+p_0^2\tanh^2 X)}$$

where $X = \pm\frac{cp_0(\xi-\xi_0)}{(v^2-1)}$. Splitting into partial fractions results in

$$\frac{dg}{d\xi} = \frac{cv}{(v^2-1)} + \frac{2R(v^2-1) - cv}{2(v^2-1)[1+p_0^2\tanh^2 X]}, \tag{3.29}$$

and the integral of the above over $X = \mathbb{R}$ clearly diverges with v, c non-zero so that

$\triangle\phi \neq \pi$. Therefore with $c \neq 0$, it appears that no winding solutions of travelling wave type exist for HSM.

3.5 The Self-dual HSM

The idea of self-duality, with its origins in the SDYM system discussed earlier, applies to theories in which certain terms of the action and their couplings are not free but are constrained by relations between one another [88]. At the point of self-duality, the equations of motion are reduced to first order differential equations which promise to be easier to analyse than those of higher order and can, at least in some cases, be solved explicitly. Fields satisfying a set of self-dual equations tend to give vacuum solutions, i.e. the energy-momentum tensor will vanish for such solutions. Furthermore, solutions of the self-dual equations of a theory are automatically solutions of the general equations of motion for the model whilst the converse is not necessarily the case.

In this section an example of a family of exact topological solitons for HSM arising from the following self-dual equations

$$\phi_x = \theta_t \mathrm{sech}\theta \tag{3.30a}$$
$$\phi_t = \theta_x \mathrm{sech}\theta, \tag{3.30b}$$

is derived[3]. Moreover, it is shown that winding solutions admitted by these self-dual equations exist for a finite time only. Prior to this however; that (3.30) imply the equations of motion (3.11) may be seen as follows; for (3.11b) multiply (3.30) by $\cosh^2\theta$ and differentiate the new (3.30a) and (3.30b) respectively with respect to x and t, from which (3.11b) is recovered. For (3.11a) differentiate (3.30a) with respect to t and (3.30b) with respect to x, so that

$$\begin{aligned}\phi_{xt} &= -\theta_t^2 \text{ sech } \theta \tanh\theta + \theta_{tt}\text{sech } \theta \\ \phi_{tx} &= -\theta_x^2\text{sech } \theta \tanh\theta + \theta_{xx}\text{sech } \theta.\end{aligned}$$

Subtracting, one then has

$$\theta_{tt} - \theta_{xx} = (\theta_t^2 - \theta_x^2)\tanh\theta$$

[3]The term *self-dual* is applicable with respect to the HSM since in $(1+1)$ dimensions this sigma model is equivalent to the $SL(2,\mathbb{R})$ Chiral equation, which is itself self-dual.

but

$$\phi_x \cosh\theta = \theta_t \qquad \Longrightarrow \qquad \phi_x^2 \cosh^2\theta = \theta_t^2$$

and

$$\phi_t \cosh\theta = \theta_x \qquad \Longrightarrow \qquad \phi_t^2 \cosh^2\theta = \theta_x^2.$$

Hence,

$$\theta_{tt} - \theta_{xx} = -(\phi_t^2 - \phi_x^2)\sinh\theta\,\cosh\theta,$$

i.e. (3.11a), and the equations of motion (3.11) are implied by the self-dual equations (3.30).

Solutions arise from equations (3.30) as follows; it is easily shown that ϕ and $\mu = 2\tan^{-1}\exp\theta$ satisfy $\phi_x = \mu_t$, $\phi_t = \mu_x$ and are therefore conjugate solutions of the (1+1) dimensional wave equation so that their general solution is

$$\begin{aligned} \phi &= f(x+t) + g(x-t) \\ \mu &= f(x+t) - g(x-t) \end{aligned}$$

where f and g are arbitrary functions. Since the boundary conditions for winding solutions on $X = \mathbb{R}$ (referring forward to (3.32), (3.33)) require that f and g essentially have a kink-like profile, we take the following choice as an example for f and g;

$$\begin{aligned} f(x+t) &= f(\xi_+) = \tfrac{\pi}{2}\tanh(\xi_+) + \tfrac{\pi}{4} \\ g(x-t) &= f(\xi_-) = \tfrac{\pi}{2}\tanh(\xi_-) - \tfrac{\pi}{4}. \end{aligned}$$

This leads to

$$\phi = \frac{\pi\sinh\,2x}{\cosh\,2x + \cosh\,2t} \tag{3.31a}$$

$$\mu = \frac{\pi}{2} + \frac{\pi\,\sinh\,2t}{\cosh\,2x + \cosh\,2t} \tag{3.31b}$$

and since we need $0 < \mu = 2\tan^{-1}\exp\theta < \pi$, this imposes that

$$\left|\frac{\sinh 2t}{\cosh 2x\,\cosh 2t}\right| < \frac{1}{2}.$$

As $\cosh y \geq 1 \, \forall \, y$, one then has

$$\left|\frac{\sinh 2t}{\cosh 2x + \cosh 2t}\right| \leq \left|\frac{\sinh 2t}{1 + \cosh 2t}\right|$$

so that

$$-\frac{1}{2} < \frac{\sinh 2t}{1 + \cosh 2t} < \frac{1}{2}.$$

Then with the use of hyperbolic trigonometric identities one finds

$$-\cosh t < 2 \sinh t < \cosh t.$$

And on changing cosh and sinh to their exponential form, it can be shown that (3.31) represents a smooth solution only for

$$\mid t \mid < \frac{1}{2} \log 3.$$

(At $\mid t \mid = \frac{1}{2} \log 3$ the solution $\vec{\psi}$ diverges.) The solution (3.31) is of winding type where $X = \mathbb{R}$ with unit winding number since, as $x \longrightarrow -\infty$, $\phi \longrightarrow -\pi$ and as $x \longrightarrow +\infty$, $\phi \longrightarrow \pi$, and further, $\mu \longrightarrow \frac{\pi}{2}$ as $x \longrightarrow \pm\infty$.

As is the case in the above example, it turns out, in fact, that all winding solutions derived from the self-dual equations for $X = \mathbb{R}$ exist only for a finite time. This may be seen as follows; by definition for such a solution to exist, the boundary conditions

$$\begin{array}{rcccccc} & \theta & \longrightarrow & \theta_0 & \text{as} & x & \longrightarrow & \pm\infty \\ \Longrightarrow & \mu & \longrightarrow & \mu_0 & \text{as} & x & \longrightarrow & \pm\infty \end{array} \tag{3.32}$$

and

$$\begin{array}{ccccccc} \phi & \longrightarrow & \phi_0 & \text{as} & x & \longrightarrow & -\infty \\ \phi & \longrightarrow & \phi_0 - 2\pi N & \text{as} & x & \longrightarrow & +\infty \end{array} \tag{3.33}$$

must hold for all t. Also f and g should be smooth functions and one must have $\mu_x \longrightarrow 0$ as $x \longrightarrow \pm\infty$. Recall that $\mu = \tan^{-1}(e^{\theta})$ so that $0 < \mu < \pi \, \forall \, t$ and

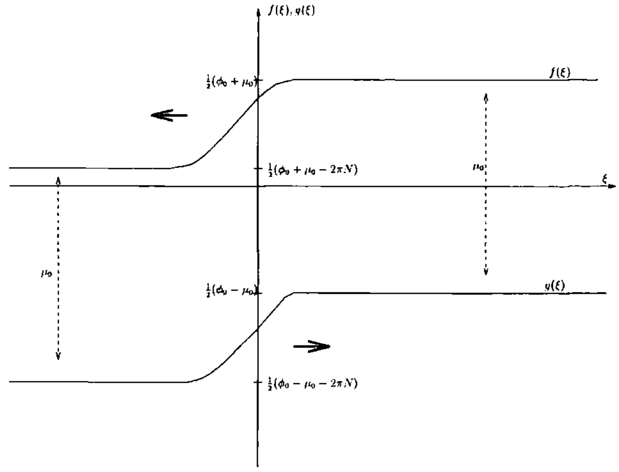


Figure 3.5: The two kinks $f(\xi), g(\xi)$ at $t = 0$.

therefore $0 < \mu_0 < \pi$ also. Now,

$$\begin{aligned} \mu &= 2\tan^{-1}\exp\theta &= f(x+t) - g(x-t) \\ & \phi &= f(x+t) + g(x-t) \end{aligned}$$

and with the boundary conditions above one has

$$\begin{aligned} f(\infty + t) &= \tfrac{(\phi_0+\mu_0)}{2} \\ g(\infty - t) &= \tfrac{(\phi_0-\mu_0)}{2} \\ f(-\infty + t) &= \tfrac{(\phi_0-2\pi N+\mu_0)}{2} \\ g(-\infty - t) &= \tfrac{(\phi_0-2\pi N-\mu_0)}{2}. \end{aligned}$$

So that as x goes from $-\infty$ to ∞, f goes from $\frac{1}{2}(\phi_0 - 2\pi N - \mu_0)$ to $\frac{1}{2}(\phi_0 + \mu_0)$ and g goes from $\frac{1}{2}(\phi_0 - 2\pi N - \mu_0)$ to $\frac{1}{2}(\phi_0 - \mu_0)$. If, without loss of generality, the kinks f and g start off at $t = 0$ such that f and g lie in the same relative position along the ξ axis, i.e. the g kink is simply the f kink shifted downwards as shown in Figure (3.5); at this point in time, the largest distance between f and g (i.e. μ at $t = 0$ for any given x) is $\frac{1}{2}(\phi_0+\mu_0) - \frac{1}{2}(\phi_0-\mu_0) = \frac{1}{2}(\phi_0 - 2\pi N + \mu_0) - \frac{1}{2}(\phi_0 - 2\pi N - \mu_0) = \mu_0$. As t increases, the f kink moves to the left and the g kink moves to the right so that eventually, for some t,

$$\mu = f - g = \frac{1}{2}(\phi_0 + \mu_0) - \frac{1}{2}(\phi_0 - 2\pi N - \mu_0) = \mu_0 + \pi N.$$

And this does not satisfy $0 < \mu_0 < \pi$ for all non-zero $N \in \mathbb{Z}$. Similarly, as t increases in the negative direction,

$$\mu = f - g = \frac{1}{2}(\phi_0 - 2\pi N + \mu_0) - \frac{1}{2}(\phi_0 - \mu_0) = \mu_0 - \pi N$$

which again does not satisfy $0 < \mu_0 < \pi$ for $N \neq 0$. So for some x, and as $\mid t \mid$ gets large, μ does not satisfy the required inequality hence, solutions of this type can exist only for a finite time.

Similar solutions have been discussed by De Vega and Sanchez [78] (1993), in their work on the exact integrability of strings in D-dimensional de Sitter space-time; the string is wound around the de Sitter space and all strings in their model exhibit similar behaviour, i.e. the string solutions are defined only for a finite time. The string constraints on the world sheet are (in particular in $D = 2$) that the energy-momentum tensor must vanish, corresponding to 'vacuum type' solutions. In two dimensions the conformal invariance on the world sheet is reflected in the fact that strings wound around and evolving with the de Sitter universe depend on two arbitrary functions. The evolution period $\tau \in [0, \pi]$, is where half of the string evolution $(0 < \tau < \frac{\pi}{2})$ corresponds to the expansion time of the de Sitter universe and $(\frac{\pi}{2} < \tau < \pi)$ corresponds to the contraction phase. If the string winds $N \in \mathbb{Z}$ times around de Sitter space the evolution period is reduced to $\triangle\tau = \frac{\pi}{2N}$, i.e. expansion/contraction is N times faster. In the context of cosmological backgrounds, with the cosmic time u, for $u \longrightarrow \pm\infty$, the dependence on τ is logarithmic and in these regions the proper length of the string stretches infinitely, i.e. the conformal factor $e^{\alpha(\sigma,\tau)}$ blows up and the string is therefore unstable. DeVega and Sanchez note also that as well as for strings falling into space-time singularities, the same unstable behaviour is found in inflationary backgrounds. A further solution describes the trajectory of a massless particle which is a geodesic in 2-dimensional de Sitter space-time travelling from what corresponds to our $\theta = -\infty$ to $\theta = +\infty$. This particle goes over half the de Sitter circle which corresponds to our ϕ moving through an angle of π.

Prior to DeVega and Sanchez's work on the HSM, Lambert and Piette [4] in 1988 also examined the self-dual non-linear HSM. A stereographic projection (2.18) of the hyperboloid was formulated in terms of the set Ω of hyperbolic complex numbers defined by

$$\Omega = \{z = t + \epsilon x | (t, x) \in \mathbb{R}^2, \epsilon^2 = 1\}.$$

And with the Grassmanian manifold ΩP^1 (isomorphic to $H(2,1)$ which is the hyperboloid of one sheet $SO(2,1)/SO(1,1)$ in $\mathbb{R}^3$), vacuum solutions were constructed from the self-dual equations in terms of the projection operator P. Further, with $P = \frac{hh^\#}{h^\# h}$ where $h = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$, $\mid \alpha \mid^2 \neq -1$; from the (non self-dual) equation for α

$$\Box\alpha = \frac{2\tilde{\alpha}\partial_-\alpha\,\partial_+\alpha}{1+\mid\alpha\mid^2} \tag{3.34}$$

where $\partial_\pm = (\partial_t \pm \epsilon\partial_x) \Longrightarrow \Box = \partial_+\partial_- = \partial_-\partial_+$, both vacuum and non-vacuum solutions travelling at the speed of light with both positive definite and non-positive definite finite energies were derived. In terms of the variables u, v of (2.18), where here $\alpha = u + \epsilon v$, $\tilde{\alpha} = u - \epsilon v$, equation (3.34) translates to

$$\begin{aligned} u_{xx} - u_{tt} &= 2u(u_x^2 + v_x^2 - u_t^2 - v_t^2)\mu^{-1} - 4v(u_x v_x - u_t v_t)\mu^{-1} \\ v_{xx} - v_{tt} &= -2v(u_x^2 + v_x^2 - u_t^2 - v_t^2)\mu^{-1} + 4u(u_x v_x - u_t v_t)\mu^{-1} \end{aligned}$$

where $\mu = (1 + u^2 - v^2)$. And we note simply here that the solution

$$\begin{aligned} u &= \tan\frac{x}{2} \\ v &= 0 \end{aligned}$$

corresponds to a one-wind static (non-vacuum) solution for $X = S^1$ with positive definite energy density equal to $\frac{1}{2}$ (which is obviously also valid for the HHM).

3.6 Concluding remarks

This concludes our discussion of the HSM where, in this chapter, time-dependent and travelling wave solutions for $X = S^1$, classifiable by a winding number N, have been shown to exist (in the latter case specifically where the constant $c = 0$). When $c \neq 0$ there are no travelling wave solutions for $X = \mathbb{R}$ and evidence has been given to indicate that no travelling waves of winding type exist in the general case. Further, from the self-dual equations it has been demonstrated that time dependent winding solutions exist when X is the real line and that such solutions are defined only on a finite time interval.

Chapter 4

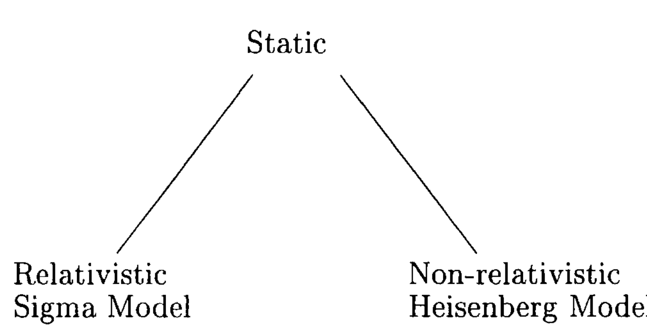
A Pivotal Model

4.1 Introduction

That Heisenberg and sigma models of the types discussed so far may be associated with similar physical phenomena, (in particular, (an)isotropic ferromagnets), should not be entirely unexpected; thinking of such models in the static case, the expression for the energy of both systems is essentially given by

$$E = \int \left(\partial_x \vec{\psi}\right)^2 dx.$$

And, as noted earlier, the HHM and HSM models have the same static equations and therefore the same static solutions. Introducing some time dependence into the static model may result in two types of dynamical system: (i) those with non-relativistic evolution; i.e. where the system is invariant under the Galilean transformation, or (ii) systems with relativistically invariant motion, i.e. where there is Lorentz invariance. The Heisenberg model is a simple example of (i) and the sigma model an example of (ii). So stemming from the same static energy two separate time



dependent systems are obtained. (A similar situation occurs with linear systems in quantum mechanics; the relativistic Klein-Gordon and non-relativistic Schrödinger equations arise from the same static quantum mechanical system.) The relativistic and non-relativistic models are not derived from one another, rather, one might regard them as cousins coming from the same static picture.

Nevertheless, some form of interpolation between the *time* dependent cousins would certainly be desirable. In fact, such an interpolation (at least for the Heisenberg and sigma models) is possible via a third non-linear, time dependent integrable system and this chapter is concerned with just such a model. This 'Pivotal' system *includes* both the HHM and HSM, and (at least some of) its properties and solutions reduce to those of the two simpler models.

The Pivotal model arises as a variation on an integrable extension [53], on Hermitian symmetric target spaces, of the non-linear $O(3)$ sigma model (3.6). The model proposed in [53] is formulated as follows: if the cross product of $\vec{Q}$ is taken with the equation

$$\vec{Q} \times \vec{Q}_{xx} - \vec{Q} \times \vec{Q}_{tt} = 0 \tag{4.1}$$

then the $O(3)$ sigma equation of motion (3.6) results. And from this observation the equation

$$\vec{Q} \times \vec{Q}_{tt} - \vec{Q} \times \vec{Q}_{xx} = \gamma^0 \vec{Q}_t + \gamma^1 \vec{Q}_x \tag{4.2}$$

is proposed where $\gamma^\mu : \mu = 0, 1$ are constants. This is motivated by the further observation that if $\gamma^1 = 0$ and in the absence of the second order t derivative on the left hand side, the integrable S^2 Heisenberg equation (2.2) is recovered.

Equation (4.2) is then generalized for Hermitian symmetric (HSS) target spaces via the covariantised symplectic structure properties of such spaces, resulting in the equation of motion

$$\partial_\mu [Q, \partial^\mu Q] + \gamma^\mu \partial_\mu Q = 0 \tag{4.3}$$

(where $Q = Q^a t^a$ such that the infinitesimal generators t^a are the basis for the tangent space $T_e(G/H)$, the target space having the group structure G/H, H a closed subgroup of G). The integrability of (4.3) is established via a zero curvature representation as follows: let

$$A_\mu = a\big[Q, \partial_\mu Q\big] + b\epsilon_\mu{}^\rho\big[Q, \partial_\rho Q\big] + c_\mu Q \tag{4.4}$$

where

$$a = \frac{2}{\lambda^2 - 1}, \qquad b = \frac{2\lambda}{\lambda^2 - 1} \tag{4.5}$$

and

$$c_\mu = \frac{4\lambda^2}{(\lambda^2-1)^2}\gamma_\mu + \frac{2\lambda(\lambda^2+1)}{(\lambda^2-1)^2}\epsilon_\mu{}^\nu\gamma_\nu. \tag{4.6}$$

Then using the identity (c.f. [89])

$$\Big[Q, \big[Q, \partial_\mu Q\big]\Big] = -\partial_\mu Q, \tag{4.7}$$

A_μ satisfies the equation

$$\epsilon^{\mu\nu}\Big(\partial_\mu A_\nu - \partial_\nu A_\mu + \big[A_\mu, A_\nu\big]\Big) = 0 \tag{4.8}$$

if and only if Q satisfies the equation of motion (4.3), i.e. (4.3) is integrable with respect to a zero curvature representation. Further, an explicit expression for non-local conservation laws may be found which will be briefly discussed in a later section.

It is from equation (4.2) that the Pivotal model emerges and the following sections are devoted to its formulation, solutions and properties, some of which are derived from the above.

4.2 Formulation and Integrability

Replacing the parameters γ^μ with a single parameter ω, where, to accommodate the reductive models, $0 \leq \omega \leq 1$, then equation (4.2) may be modified to take the form

$$(1-\omega)\vec{\psi}_t = \vec{\psi} \times \vec{\psi}_{xx} - \omega\vec{\psi} \times \vec{\psi}_{tt} \tag{4.9}$$

(with $\vec{\psi}$ replacing $\vec{Q}$). The S^2 Heisenberg and sigma models can be recovered immediately from this equation[1] by taking, respectively, $\omega = 0$ and $\omega = 1$. Hence, the above contains both a relativistic (i.e. Lorentz invariant) and a non-relativistic, Galilean invariant system. It is interesting to note also that, with respect to the classification of equations, in the limits $\omega = 0$ and $\omega = 1$ respectively a parabolic and a hyperbolic equation result. In theory, if one has solutions for an equation of one of these two classes, one would not expect to be able to produce solutions for an equation of the other class from the first set of solutions; the space of solutions for a given equation is specified by the space of initial data which is not the same in the parabolic and hyperbolic cases due to the differing orders in t dependence. However, as we shall see, whilst it is probable that in general one cannot simply interpolate between solutions of the two limiting models via the parameter ω, it is possible in some special cases.

Since the two models discussed previously have had the non-compact H^2 as their target manifold, the Pivotal model will be examined in the same context since it is from this choice of manifold that topological solitons arise; with the indefinite metric one has the field equations of motion

$$(1-\omega)\psi^a_t = \epsilon^a{}_{bc}\left(\psi^b\psi^c_{xx} - \omega\psi^b\psi^c_{tt}\right), \tag{4.10}$$

i.e. the Hyperbolic Pivotal Model equation. (For ease of reference this will be abbreviated to HPM.) With equation (4.10) a continuous interpolation via the single parameter ω is facilitated between the HHM ($\omega = 0$) and HSM ($\omega = 1$) models.

The integrability of the HPM may be established by a suitable modification of the linear spectral problem of [53] as follows: using the identity (4.7) and with the pair

$$U = a\left[S, \partial_x S\right] - b\omega\left[S, \partial_t S\right] + c_1 S \tag{4.11a}$$

$$V = a\left[S, \partial_t S\right] - b\left[S, \partial_x S\right] + c_0 S \tag{4.11b}$$

where, as usual for the hyperbolic version,

$$S = \begin{bmatrix} \psi_1 & \psi_2 + \psi_3 \\ \psi_2 - \psi_3 & -\psi_1 \end{bmatrix} \quad \in \ SL(2, \mathbb{R});$$

[1] It will be evident in the non-compact hyperbolic case, that with the θ, ϕ parametrisation of the hyperboloid, the $\omega = 1$ reduction explicitly yields the HSM equations (3.11).

if

$$a = \frac{2\omega}{\lambda^2 - \omega}, \qquad b = \frac{2\lambda}{\lambda^2 - \omega}$$

(i.e. replacing the spectral parameter λ with $\frac{\lambda}{\sqrt{\omega}}$ in (4.5)), and

$$c_0 = b^2(1-\omega) = \frac{4\lambda^2(1-\omega)}{(\lambda^2-\omega)^2}, \tag{4.12}$$

$$c_1 = b(\omega-1)(1+a) = \frac{2\lambda(\omega-1)(\lambda^2+\omega)}{(\lambda^2-\omega)^2}, \tag{4.13}$$

then (4.11) satisfy the equation

$$\partial_t U - \partial_x V + \big[V, U\big] = 0 \tag{4.14}$$

if and only if ψ^a satisfies the HPM equation (4.10). In other words, equation (4.10) may be written in the form of the compatibility condition for an overdetermined linear system[2] and hence, is integrable.

Furthermore, the integrability of both the HHM and HSM models may be verified via the U, V pair (4.11); in the limit $\omega = 0$ one has $a = 0$, $b = \frac{2}{\lambda}$, $c_0 = \frac{4}{\lambda^2}$ and $c_1 = -\frac{2}{\lambda}$ so that

$$U = -\frac{2}{\lambda} S \tag{4.15a}$$

$$V = -\frac{2}{\lambda}\big[S, \partial_x S\big] + \frac{4}{\lambda^2} S. \tag{4.15b}$$

And substitution of these into the zero curvature condition (4.14) results in the equation

$$\frac{2}{\lambda}\Big(\partial_x\big[S, \partial_x S\big] - \partial_t S\Big) = 0.$$

This is equivalent to the HHM equation of motion (2.13) hence, in the limit $\omega = 0$ the U, V pair (4.11) reduces to one for the HHM equation[3].

Conversely, in the $\omega = 1$ limit, one has $a = \frac{2}{\lambda^2 - 1}$, $b = \frac{2\lambda}{\lambda^2 - 1}$ and $c_0 = c_1 = 0$. And

[2]Equation (4.14) is the compatibility condition for the linear system obtained by replacing U, V with $-U, -V$ in (1.1).

[3]This reduced U, V pair differs slightly from that given in Chapter 1 for HHM which is permissible (c.f. [20]) since, due to the relatively arbitrary nature of the matrix F in the linear system (1.1), U, V pairs are not unique.

the pair (4.11) reduce to

$$U = a\big[S, \partial_x S\big] - b\big[S, \partial_t S\big] \tag{4.16a}$$
$$V = a\big[S, \partial_t S\big] - b\big[S, \partial_x S\big]. \tag{4.16b}$$

In this case, it is the HSM equation (3.11), equivalent to

$$\big[S, \partial_x^2 S\big] - \big[S, \partial_t^2 S\big] = 0$$

which may be written in the form of the compatibility condition for a linear system. Hence, not only is the HPM equation (4.10) itself integrable, the integrability of both the HHM an HSM models may be established via this third model.

4.3 Winding Solutions for the Pivotal Model

In keeping with the previous chapters on the Heisenberg and sigma models, where topological solitons were the main object of study, the existence of the same type of solution will be examined for the Pivotal model. And following the same format as for the previous models; if equations (4.10) are parametrized in terms of the 'polar angles' θ, ϕ (c.f. (2.17)) one finds

$$\phi_t(1-\omega) = [\phi_x^2 - \omega\phi_t^2]\sinh\theta + [\theta_{xx} - \omega\theta_{tt}]\,\mathrm{sech}\,\theta \tag{4.17a}$$
$$\theta_t(1-\omega) = [\phi_{xx} - \omega\phi_{tt}]\cosh\theta + 2[\theta_x\phi_x - \omega\theta_t\phi_t]\sinh\theta. \tag{4.17b}$$

Recalling that static solutions are the same for both the HHM and HSM, we may expect that this will also apply to the HPM which is indeed the case; in the absence of any time derivatives equations (4.17) simply reduce to (2.20) of Chapter 2 and hence their static solutions are the same.

As a simple t dependent example, consider the solution $\phi = Nx + tN^2\sinh c$ where $\theta = c$, constant and $N \in \mathbb{Z}$, i.e. solution (2.25) of the HHM model where $X = S^1$. Into the pivotal equations (4.17) this yields

$$\omega(1 - N^2\sinh^2 c) = 0$$

so either $\omega = 0$ (the HHM case) or

$$\sinh\theta = \sinh c = \pm\frac{1}{N}.$$

In the latter case one then has

$$\phi = N(x \pm t)$$

which is also admitted by the HSM equations (3.11), and corresponds to a moving version of the static solution $\phi = Nx$, $\theta = 0$, c.f. Section 3.4 where the observation was made that in this sigma model case the solution moves with the speed of light. Solutions for all three models therefore result from this first example although there is no explicit interpolation via the parameter ω; the ω dependence (excepting the specific $\omega = 0$ case corresponding to the HHM) disappears on substitution of the Heisenberg solution into the Pivotal model equations.

4.3.1 Travelling Waves for HPM

As usual, the next natural step in the search for winding solutions is to examine the equations for travelling wave solutions. With the anzatz

$$\begin{aligned}\theta(t,x) &\mapsto f(x-vt) = f(\xi)\\ \phi(t,x) &\mapsto g(x-vt) + \Omega t = g(\xi) + \Omega t\end{aligned}$$

where Ω is constant, the equations of motion are then

$$\begin{aligned}(\Omega - vg')(1-\omega) &= [g'^2(1-\omega v^2) + 2v\omega\Omega g' - \omega\Omega^2]\sinh f\\ &\quad + f''(1-\omega v^2)\mathrm{sech}\ f \qquad (4.18)\\ -vf'(1-\omega) &= g''(1-\omega v^2)\cosh f + 2f'[v\omega\Omega\\ &\quad + g'(1-\omega v^2)]\sinh f \qquad (4.19)\end{aligned}$$

where prime denotes differentiation with respect to ξ. Letting $\chi = g'(1-\omega v^2)\cosh f$ so that

$$\chi' = f'g'(1-\omega v^2)\sinh f + g''(1-\omega v^2)\cosh f;$$

then using (4.19) for $g'' \cosh f\,(1-\omega v^2)$ and $g' = \frac{\chi}{(1-\omega v^2)}\mathrm{sech} f$ from the definition of χ, the following equation results for χ:

$$\chi' + \chi f' \tanh f = -f'\big[2v\omega\Omega \sinh f + v(1-\omega)\big].$$

This is linear in χ and can therefore be solved using elementary methods to give

$$\chi = -v\omega\Omega \cosh f + \frac{v\omega\Omega}{2}\ \mathrm{sech} f - v(1-\omega)\tanh f + k\ \mathrm{sech} f \tag{4.20}$$

where k is a constant of integration. With this expression one can now formulate a first order equation for g' such that

$$(1-\omega v^2)g' = -v\omega\Omega + \big(\frac{v\omega\Omega}{2} + k\big)\ \mathrm{sech}^2 f - v(1-\omega)\tanh f\ \ \mathrm{sech} f \tag{4.21}$$

which may then be substituted into (4.18) to give

$$\begin{aligned} f''(1-\omega v^2)^2 = {} & \frac{\omega\Omega^2}{2}\sinh 2f + \Omega(1-\omega)\cosh f \\ & + \Big[(1-\omega)^2 v^2 - \big(\frac{v\omega\Omega}{2} + k\big)^2\Big]\mathrm{sech}^2 f \tanh f \\ & + v(1-\omega)\big(\frac{v\omega\Omega}{2} + k\big)\ \mathrm{sech} f - 2v(1-\omega)\big[\frac{v\omega\Omega}{2} + k\big]\ \mathrm{sech}^3 f. \end{aligned} \tag{4.22}$$

And on integration this yields the following first order equation for f:

$$\begin{aligned} 2(1-\omega v^2)^2 f'^2 \cosh^2 f = {} & \omega\Omega^2\cosh^2 f + 2\omega\Omega^2 \cosh^2 f \sinh^2 f \\ & + 4\Omega(1-\omega)\sinh f\ \cosh^2 f - 4v(1-\omega)\big[\frac{v\omega\Omega}{2} + k\big]\sinh f \\ & + 4q\cosh^2 f - 2(1-\omega)^2 v^2 + 2\big[\frac{v\omega\Omega}{2} + k\big]^2 \end{aligned} \tag{4.23}$$

where q is a constant of integration. In the HHM limit ($\omega = 0$) the constant q here is equivalent to $\frac{\alpha}{2} = \frac{v^2 - k^2 + 2Q}{2}$ of the travelling wave equations for the HHM and similarly, k of the Pivotal Model corresponds to $-k$ of the HSM. To make things more manageable put $p(\xi) = \sinh f$ so that

$$\begin{aligned} P(p) = 2(1-\omega v^2)^2 p'^2 &= 2\omega\Omega^2 p^4 + 4\Omega(1-\omega)p^3 + (3\omega\Omega^2 + 4q)p^2 \\ &\quad + 4p(1-\omega)\big[\Omega - v\big(\frac{v\omega\Omega}{2} + k\big)\big] \\ &\quad + \omega\Omega^2 + 4q - 2(1-\omega)^2 v^2 + 2\big(\frac{v\omega\Omega}{2} + k\big)^2 \end{aligned} \tag{4.24}$$

and

$$(1-\omega v^2)g' = \frac{-2vp\big[\omega\Omega p + (1-\omega)\big] + 2k - v\omega\Omega}{2(1+p^2)}. \tag{4.25}$$

Note that all of the preceding equations are consistent with the corresponding equation for the HHM and HSM in the respective limits $\omega = 0$, $\omega = 1$. Further, to exclude any zeros in the denominators and in keeping with the interpolation between the two limiting cases one must impose the condition $0 < 1 - \omega v^2 \Longrightarrow \omega v^2 < 1$, i.e. $v^2 < \omega^{-1}$ which complies with the limiting models in the sense that, for the σ-model ($\omega = 1$) which is a Lorentz invariant system, the speed v is bounded by the speed of light (unity throughout) and in the non-relativistic HHM ($\omega = 0$), the speed is unbounded.

Consider first the special case where $\Omega = 0$, and recall that for the HHM and HSM travelling wave solutions of winding type with $X = S^1$ were shown to exist in this case (where Ω here corresponds to the constant c in the limiting models). An analogous solution can be found for the Pivotal Model as follows: $\Omega = 0$ leaves

$$P_{\Omega=0}(p) = 2(1-\omega v^2)^2 p'^2 = 4qp^2 - v(1-\omega)kp + 4q - 2\big[(1-\omega)^2 v^2 - k^2\big] \tag{4.26}$$

and

$$(1-\omega v^2)g' = \frac{vp(\omega - 1) + k}{1 + p^2} \tag{4.27}$$

so that from (4.26) one has

$$\frac{1}{\sqrt{2}(1-\omega v^2)} \int d\xi = \int \frac{dp}{\big[P_{\Omega=0}(p)\big]^{\frac{1}{2}}}.$$

For a winding solution corresponding to (2.33) of HHM and (3.21) of HSM we then require that

$$P_{\Omega=0}(p) = A - B(p-p_0)^2$$

where $A, B > 0$ and p_0 are constants. Comparison with (4.26) yields $p_0 = \frac{kv(1-\omega)}{2q}$, $B = -2q$ and $A = 2q - v^2(1-\omega)^2 + k^2 - \frac{2k^2v^2(1-\omega)}{q}$. With these parameters one then has

$$\frac{(\xi-\xi_0)}{(1-\omega v^2)} = \int \frac{dp}{\left[A - B(p-p_0)^2\right]^{\frac{1}{2}}}$$

where ξ_0 is constant, admitting the solution

$$p(\xi) = \sinh f = \sqrt{\frac{A}{B}} \sin\left[\frac{\sqrt{B}(\xi-\xi_0)}{(1-\omega v^2)}\right] + p_0. \tag{4.28}$$

And this exhibits the required periodicity for a solution of winding type where $X = S^1$. To check that the solution is fully topological; on substitution into the expression (4.27) one has

$$(1-\omega v^2)\frac{dg}{d\xi} = \frac{v(\omega-1)\left(\sqrt{\frac{A}{B}}\sin\alpha + p_0\right) + k}{1 + \left(\sqrt{\frac{A}{B}}\sin\alpha + p_0\right)^2} \tag{4.29}$$

where $\alpha = \frac{\sqrt{B}(\xi-\xi_0)}{(1-\omega v^2)}$. And again following the analysis for the analogous solution of the HHM (2.33); decomposition of the RHS of (4.29) into partial fractions results in

$$\begin{aligned} g(\xi) = {} & \frac{(v(\omega-1)-ik)}{2\sqrt{A}} \int \frac{d\xi}{\sin\xi + \sqrt{\frac{B}{A}}(p_0 - i)} \\ & + \frac{(v(\omega-1)-ik)}{2\sqrt{A}} \int \frac{d\xi}{\sin\xi + \sqrt{\frac{B}{A}}(p_0 + i)} \end{aligned} \tag{4.30}$$

where $q = -\frac{1}{2}(1-\omega v^2)^2, \xi_0 = 0$, corresponds to the one-wind solution. After integration this yields

$$g(\xi) = \mathcal{R}e\left\{-i\ln\left[\frac{\tan\frac{\xi}{2} + \frac{a}{c}}{\tan\frac{\xi}{2} + \frac{b}{c}}\right] + \Lambda\right\} \tag{4.31}$$

where $a = \sqrt{A} - k - iv(\omega-1)$, $b = \sqrt{A} + k + iv(\omega-1)$, $c = \sqrt{B}(p_0 - i)$ and $\Lambda = i\ln\Theta$,

Θ a complex constant. As in both the HHM and HSM, following the methodology of Appendix A, it can be shown that (4.31) is equivalent to

$$g(\xi) = \mathcal{R}e\left\{ -i \ln \left[\frac{\tan\frac{\xi}{2} + \Phi}{-\tan\frac{\xi}{2} - \Phi^{-1}} \right] \right\} \tag{4.32}$$

where $\Phi = \frac{a}{c}$. And this solution *is* of winding type as was demonstrated in section 4 of Chapter 2. So that we again have a travelling wave solution with unit winding number in the $\Omega = 0$ case, and where physical space $X = S^1$. This solution is consistent with the limiting models for $\omega = 0$ (HHM) and $\omega = 1$ (HSM) where in the latter, (4.28) is related to a Lorentz boost of the static solution (2.23).

Taking $\Omega \neq 0$, it may be fruitful to follow an analogous path as that taken for the S^2 model of [53], since the resultant solution exhibits at least some topology. Having previously defined the function $\chi = g'(1-\omega v^2)\cosh f$, this may now be used to replace various terms in equation (4.19) to give

$$\begin{aligned} f'' = & \frac{(1-\omega)\Omega\cosh f}{(1-\omega v^2)} + \frac{\omega\Omega^2\sinh 2f}{2(1-\omega v^2)} - \frac{\chi^2\tanh f}{(1-\omega v^2)^2} \\ & - \frac{2v\omega\Omega\chi\sinh f}{(1-\omega v^2)^2} - \frac{v(1-\omega)\chi}{(1-\omega v^2)^2}. \end{aligned} \tag{4.33}$$

A further expression (4.20) was found for χ, and differentiation of this yields

$$-\frac{\chi'}{f'} = \chi\tanh f + v(1-\omega) + 2v\omega\Omega\sinh f,$$

so that (4.33) can be rewritten as

$$f'' = \frac{(1-\omega)\Omega\cosh f}{(1-\omega v^2)} + \frac{\omega\Omega^2\sinh 2f}{2(1-\omega v^2)} + \frac{\chi'\chi}{f'(1-\omega v^2)^2}.$$

Multiplying through by f' one can then produce the first integral

$$f'^2 - \frac{2\Omega(1-\omega)\sinh f}{(1-\omega v^2)} - \frac{\omega\Omega^2\cosh 2f}{2(1-\omega v^2)} - \frac{\chi^2}{(1-\omega v^2)^2} = E$$

(E constant). And replacing χ, i.e. (4.20), in the above and putting $2k = v\omega\Omega$, $(1-\omega) = -\omega\Omega$, and $E = \frac{\omega\Omega^2}{2(1-\omega v^2)}$ one has

$$f'^2 = \frac{(1-\omega)^2(\sinh f - 1)^2\left[\sinh^2 f + (1-\omega v^2)\right]}{\omega(1-\omega v^2)^2\cosh^2 f}.$$

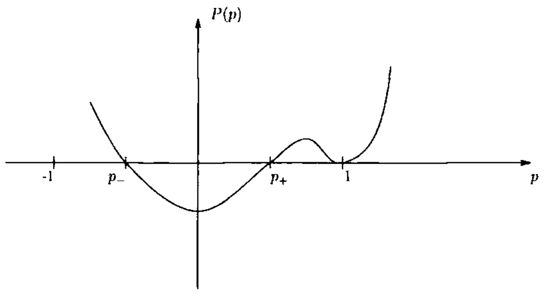


Figure 4.1: With a repeated zero at $p = 1$ and distinct real zeros at $p_\pm = \pm\sqrt{\omega v^2 - 1}$; $P(p)$ has the correct form for a travelling wave with $X = \mathbb{R}$.

Now letting $p = \sinh f$ the following equation results:

$$P(p) = \frac{\omega(\omega v^2 - 1)^2}{(1-\omega)^2} p'^2 = (1-p)^2\big[p^2 - (\omega v^2 - 1)\big] \tag{4.34}$$

from which

$$\pm\frac{(1-\omega)}{\sqrt{\omega}(\omega v^2 - 1)}(\xi - \xi_0) = \int \frac{dp}{(1-p)\sqrt{p^2 - (\omega v^2 - 1)}}. \tag{4.35}$$

The form of $P(p)$ (4.34) is shown in Figure (4.1) giving the perfect picture for a travelling wave of winding type for $X = \mathbb{R}$; there is a positive bounded region in which the solution moves and it exhibits a minimum at $p = \sqrt{\omega v^2 - 1}$, attaining $p = 1$ as $\xi \longrightarrow \pm\infty$. Note that for this picture the condition $\omega v^2 > 1$ must be imposed which does not satisfy the constraints on the speed v for the HHM and HSM models. More specifically; in the $\omega = 0$ Heisenberg case the condition becomes nonsense, and for the sigma model any solution would have to travel faster than the speed of light. It is also the case that for the Pivotal model itself with $\omega v^2 > 1$, the soliton should remain in perpetual motion, i.e. never be static.

On integrating (4.35), one finds

$$\alpha(\xi - \xi_0) = \ln\left[\frac{2\left[(2-\omega v^2) - (1-p) + \sqrt{2-\omega v^2}\sqrt{p^2 - (\omega v^2 - 1)}\right]}{(1-p)}\right] \tag{4.36}$$

where $\alpha = \frac{\sqrt{2-\omega v^2}(1-\omega)}{\sqrt{\omega}(1-\omega v^2)}$ and requiring $2 > \omega v^2$. For this to be a valid solution within the constraints, the argument of the logarithm must be positive which is the case for p in the bounded region (c.f. Figure (4.1)); $(1-p) > 0$ for $2 > \omega v^2 > 1$ and the numerator of the argument is positive within the same range. A solution for $p(\xi)$

can be found from (4.36), namely

$$p(\xi) = \frac{\exp\left[\pm 2\alpha(\xi-\xi_0)\right] + 4(\omega v^2 - 1)\exp\left[\pm\alpha(\xi-\xi_0)\right] + 4(\omega v^2 - 1)}{\exp\left[\pm 2\alpha(\xi-\xi_0)\right] + 4\exp\left[\pm\alpha(\xi-\xi_0)\right] + 4(\omega v^2 - 1)} \tag{4.37}$$

which is topological since as $\xi \longrightarrow \pm\infty$, $p \longrightarrow 1$[4]. So thus far, the combination of the two models in the Pivotal system does appear to yield an attractive solution although this does not reduce in any meaningful sense to the HHM and HSM.

As usual, of course, in order to check that this solution is *fully* topological, the behaviour of $\frac{dg}{d\xi}$ must be examined and this, it turns out, is at least a little more promising than in the previous models. Here, the parameters have been organized such that $\Omega = -\frac{(1-\omega)}{\omega}$ and $2k = v\omega\Omega$ so that (4.25) is now

$$(1-\omega v^2)\frac{dg}{d\xi} = \frac{v(1-\omega)p\left[p-1\right]}{(1+p^2)}. \tag{4.38}$$

And at the extremities of X space, i.e. $\pm\infty$; $p \longrightarrow 1$ so that $\frac{dg}{d\xi} \longrightarrow 0$. Since this occurs at an exponential rate, the integral over $X = \mathbb{R}$ of $\frac{dg}{d\xi}$ does not diverge and is therefore finite.

Of course, only if this definite integral (i.e. $\Delta\phi$) is equal to $\pm 2\pi$ can one say that the solution is fully topological. And actually doing the integration has proved somewhat problematic and inconclusive; the expression for $\frac{dg}{d\xi}$ with (4.37) is

$$(\omega v^2 - 1)\frac{dg}{d\xi} = \frac{4Ev(1-\omega)(2-\omega v^2)\left[E^2 + 4E(\omega v^2 - 1) + 4(\omega v^2 - 1)\right]}{\left[E^2 + 4E(\omega v^2 - 1) + 4(\omega v^2 - 1)\right]^2 + \left[E^2 + 4E + 4(\omega v^2 - 1)\right]^2} \tag{4.39}$$

where $E = \exp\left[\pm\alpha(\xi - \xi_0)\right]$, which is certainly not easily integrated. Taking $\omega = \frac{1}{2}$ and $v^2 = 3 \Longrightarrow \alpha = 1$ simplifies things slightly so that

$$\frac{dg}{d\xi} = \frac{\sqrt{3}}{2}\frac{E\left[E^2 + 2E + 2\right]}{\left[E^2 + 2E + 2\right]^2 + \left[E^2 + 4E + 2\right]^2}$$

although this again, does not lend itself to any obvious integration techniques. With the use of Maple^{TM}, definite integration over $X = \mathbb{R}$ produces $\Delta\phi = \pm\frac{\pi}{3}$ which,

[4]Note that with the definition of the parameter α above, the solution for the sigma model is $p = \beta$, constant anyway so the restrictions on the speed v become superfluous since p is static.

if correct, means that the soliton travels through only a portion of the required $\pm 2\pi$. With this rather odd result then, it is unclear whether or not the solution is of winding type, however, erring on the side of optimism, the possibility that the solution winds with $X = \mathbb{R}$ still remains.

In an attempt to shed more light on the situation let us consider next a more general case where

$$\frac{\Omega\sqrt{\omega}(\xi - \xi_0)}{(1 - \omega v^2)} = \int \frac{dp}{[P(p)]^{\frac{1}{2}}} \tag{4.40}$$

and $P(p)$ is now

$$\begin{aligned} P(p) = p^4 + \frac{2}{\omega\Omega}(1-\omega)p^3 + \Big(\frac{3}{2} + \frac{2q}{\omega\Omega^2}\Big)p^2 \\ + \frac{2p(1-\omega)}{\omega\Omega^2}\Big(\Omega - v\big[\frac{v\omega\Omega}{2} + k\big]\Big) \\ + \frac{\omega\Omega^2 + 4q - 2(1-\omega)^2v^2 + 2\big[\frac{v\omega\Omega}{2} + k\big]^2}{2\omega\Omega^2}. \end{aligned} \tag{4.41}$$

Bearing in mind the previous solution let us look for simple cases corresponding to special polynomials $P(p)$ where, for the $X = \mathbb{R}$ case, $P(p)$ should have one double zero and two distinct real and simple zeros. (A triple zero or two double zeros fail to give the correct picture for $X = \mathbb{R}$ as was seen in the sigma model case.) Let then $P(p)$ take the form

$$P(p) = p^4 + bp^3 + cp^2 \tag{4.42}$$

so that the double zero is at $p = 0$ for simplicity. From (4.41) one then has

$$2(1-\omega)\left(\Omega - v\Big(\frac{v\omega\Omega}{2} + k\Big)\right) = 0 \tag{4.43a}$$

$$\omega\Omega^2 = 4q - 2(1-\omega)^2v^2 + 2\Big(\frac{v\omega\Omega}{2} + k\Big)^2 = 0 \tag{4.43b}$$

which may be solved for k and q to give

$$k = \frac{\Omega}{v} - \frac{v\omega\Omega}{2} \tag{4.44}$$

$$q = \frac{2(1-\omega)^2v^4 - 2\Omega^2 - \omega\Omega^2v^2}{4v^2}. \tag{4.45}$$

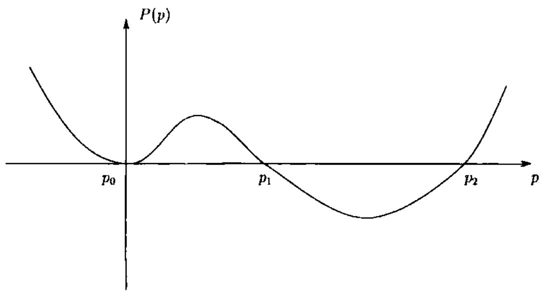


Figure 4.2: With a repeated zero at $p_0 = 0$ and distinct real zeros at p_1, p_2 then $P(p)$ has the correct form for a travelling wave with $X = \mathbb{R}$.

Hence,

$$b = \frac{2}{\omega\Omega}(1-w), \qquad c = \frac{\omega\Omega^2 v^2 + (1-w)^2 v^4 - \Omega^2}{\omega\Omega^2 v^2} \tag{4.46}$$

and without loss of generality one may then choose Ω to be less than zero so that $b < 0$. That two real and distinct roots exist requires the further condition that $b^2 - 4c > 0$, i.e.,

$$\frac{4(1-\omega v^2)}{\omega^2\Omega^2 v^2}\left(\Omega^2\omega + (1-\omega)^2 v^2\right) > 0$$

which is the case if $(1-\omega v^2) > 0$ and this is consistent with HHM and HSM. Further, for solutions on $X = \mathbb{R}$ both of the single zeros of $P(p)$ must have the same sign, otherwise no bounded region in which a solution may move can occur (c.f. Figure (4.2)). The requirement is then $-b > \sqrt{b^2 - 4c}$ (with $-b > 0$), i.e. $0 > -4c$ or $c > 0$, imposing the condition $\frac{(1-\omega)^2 v^4}{(1-\omega v^2)} > \Omega^2$. Note however, that when $\omega = 1$ this is not satisfied and further that $\omega = 0$ produces infinities in both b and c, so again any solution resulting from this formulation will not reduce to the HHM and HSM models.

With the above considerations $P(p)$ then takes the form of Figure (4.2) and the right hand side of equation (4.40) may be integrated to give the following:

$$\pm\frac{\Omega\sqrt{\omega}(\xi - \xi_0)}{(1-\omega v^2)} = -\frac{1}{\sqrt{c}}\ln\left[\frac{2\sqrt{c}\sqrt{p^2 + bp + c} + bp + 2c}{p}\right]. \tag{4.47}$$

It may be shown that the right hand side is real and non-singular in the required region, i.e. where $0 < p < p_1 = \frac{-b\pm\sqrt{b^2-4c}}{2}$, and the solution

$$p = \frac{4\Big[(1-\omega)^2v^4 - \bar{\Omega}^2(1-\omega v^2)\Big]\exp\big[\alpha X\big]}{\omega^2\bar{\Omega}^2v^2\exp\big[2\alpha X\big] + 4v^2\omega\bar{\Omega}(1-\omega)\exp\big[\alpha X\big] + 4(1-\omega v^2)\big[v^2(1-\omega)+\omega\bar{\Omega}^2\big]} \tag{4.48}$$

results, where $\alpha X = \pm\frac{\sqrt{c\omega}\bar{\Omega}}{1-\omega v^2}(\xi - \xi_0)$ and $\bar{\Omega} = -\Omega > 0$. As $\xi \longrightarrow \pm\infty$, $p \longrightarrow 0$ as required for a topological soliton with $X = \mathbb{R}$, so this looks hopeful with respect to the topology. As usual, of course, a full verification requires that the behaviour of $\frac{dg}{d\xi}$ (4.25), be examined and simply by substituting (4.44) for k in (4.25) one finds

$$(1-\omega v^2)\frac{dg}{d\xi} = -\frac{\big(pv^2(1-w)+\bar{\Omega}\big)}{v(1+p^2)} + v\omega\bar{\Omega}.$$

It is clear that at $p = 0$ (i.e. when $\xi \longrightarrow \pm\infty$), $\frac{dg}{d\xi} = -\frac{\bar{\Omega}}{v}$ which is non zero unless $\bar{\Omega} = 0$. Hence, the integral of $\frac{dg}{d\xi}$ over $X = \mathbb{R}$ will diverge. And from this it may be deduced that the solution is sadly, not of winding type.

Whilst this does not seem a positive result with respect to the topological nature of (4.37) it does not discount the possibility that (4.37) is a winding solution. What may be surmised however, is that even if (4.37) is a travelling wave of winding type for $X = \mathbb{R}$, $P(p)$ may not simply be shifted by adding a constant to p. In conclusion then, although travelling waves which wind around the hyperboloid for $X = \mathbb{R}$ do not exist for the HHM and HSM models, there still remains the possibility that such a solution does exist for the Pivotal model.

4.3.2 Solutions from the Stereographic Coordinates

Having discussed the existence of travelling waves for the Pivotal model we next consider solutions of a different type, utilizing some of the methods applied to one of the previous models. In the Heisenberg case where the equations of motion were constructed in terms of the stereographic parametrisation of the hyperboloid (2.18), (c.f. section 2.6), a family of time dependent solutions with $X = \mathbb{R}$ resulted. And since this approach was fruitful for the HHM, it seems expedient to follow the same path for the Pivotal model; the HPM equations in terms of the variables u, v are then

$$\begin{aligned} u_{xx} - \omega u_{tt} &= (1-\omega)v_t \\ &+ \frac{2u\left[(u_x^2+v_x^2) - \omega(u_t^2+v_t^2)\right] - 4v(u_x v_x - \omega u_t v_t)}{(1+u^2-v^2)} \qquad (4.49a) \\ v_{xx} - \omega v_{tt} &= (1-\omega)u_t \\ &- \frac{2v\left[(u_x^2+v_x^2) - \omega(u_t^2+v_t^2)\right] + 4u(u_x v_x - \omega u_t v_t)}{(1+u^2-v^2)} \qquad (4.49b) \end{aligned}$$

which reduce to both the HHM and HSM equations of this type in the relevant limits and, of course, to the static equations for all three models. In the Heisenberg case the relevant family of winding solutions arose from the constraint $u^2 - v^2 = f(x)^2$, a function of x only, from which u and v then took the form

$$\begin{aligned} u(t,x) &= f(x)\cosh(mt) \qquad (4.50a) \\ v(t,x) &= f(x)\sinh(mt) \qquad (4.50b) \end{aligned}$$

with m constant. And one might envisage that similar solutions occur in the HPM case. This is indeed the case, which can be seen as follows: substitution of (4.50) into equations (4.49) results in a second order equation for $f(x)$;

$$f'' = fm(1-\omega) + \frac{fm^2\omega(1-f^2)}{(1+f^2)} + \frac{2ff'^2}{(1+f^2)}$$

(where 'prime' (') denotes differentiation with respect to x). Implementation of Ince's methods [76] for the solution of O.D.E.'s, as before, with the substitution $F = \frac{(f+i)}{(f-i)}$ (since $f = \pm i$ are the (simple) poles) one then has

$$\frac{d^2F}{dz^2} = \frac{1}{F}\left(\frac{dF}{dz}\right)^2 - \frac{m^2\omega}{4F} + \frac{m^2\omega}{4}F^3 - \frac{m(1-\omega)}{2}F^2 + \frac{m(1-\omega)}{2}$$

which has as its first integral

$$\left(\frac{dF}{dz}\right)^2 = \frac{m^2\omega}{4}F^4 - m(1-\omega)F + \frac{m^2\omega}{4} + kF^2$$

where k is constant. Replacing now F with $\frac{(f+i)}{(f-i)}$ produces the first order equation for f

$$f'^2 = \frac{1}{4}\Big(\frac{m^2\omega}{2} - k\Big)\big(1+f^2\big)^2 + \frac{m(1-\omega)}{2}\big(f^2+1\big)\big(f^2-1\big) - \frac{m^2\omega}{4}\big(f^2-1\big)^2. \tag{4.51}$$

If the constant k is now carefully chosen as

$$k = -\frac{m^2\omega}{2} - 2m(1-\omega)$$

one has

$$\begin{aligned} f'^2 &= f^2\Big[m^2\omega + m(1-\omega) + m(1-\omega)\Big]f^2 \\ \Longrightarrow \quad (x-x_0) &= \int \frac{df}{f\left[(m^2\omega + m(1-\omega)) + m(1-\omega)f^2\right]^{\frac{1}{2}}} \end{aligned}$$

with x_0 constant. And the right hand side may be integrated in terms of elementary functions to give

$$x - x_0 = \frac{1}{\sqrt{m^2\omega + m(1-\omega)}}\tanh^{-1}\left[\Big(\frac{m^2\omega + m(1-\omega)}{(f^2+1)m(1-\omega) + m^2\omega}\Big)^{\frac{1}{2}}\right].$$

Using the hyperbolic trigonometric identities and since $\tanh^{-1}(\frac{1}{x}) = \coth^{-1}(x)$, one has the solution

$$f(x) = \pm\Big(\frac{m\omega}{(1-\omega)} + 1\Big)^{\frac{1}{2}}\operatorname{cosech}\left[\sqrt{m^2\omega + m(1-\omega)}(x-x_0)\right], \tag{4.52}$$

which is analogous to solution (2.67) of the HHM. In fact in the limiting case $\omega = 0$, (2.67) is recovered exactly. Furthermore, with $f(x)$ as given above (i.e. (4.50) and (4.52)) in

$$\vec{\psi} = \Big(\frac{1-f^2}{1+f^2}, \frac{2u}{1+f^2}, \frac{2v}{1+f^2}\Big) \tag{4.53}$$

one has

$$\begin{aligned}
\psi^1 &= \frac{(1-\omega)\sinh^2\left[\Lambda(x-x_0)\right]-(m\omega+(1-\omega))}{(1-\omega)\sinh^2\left[\Lambda(x-x_0)\right]+(m\omega+(1-\omega))}\\
\psi^2 &= \frac{2\Gamma\sinh\left[\Lambda(x-x_0)\right]\cosh(mt)}{(1-\omega)\sinh^2\left[\Lambda(x-x_0)\right]+(m\omega+(1-\omega))}\\
\psi^3 &= \frac{2\Gamma\sinh\left[\Lambda(x-x_0)\right]\sinh(mt)}{(1-\omega)\sinh^2\left[\Lambda(x-x_0)\right]+(m\omega+(1-\omega))}.
\end{aligned}$$

where $\Lambda = \sqrt{m^2\omega + m(1-w)}$ and $\Gamma = \sqrt{m\omega + (1-\omega)^2}$ and m is taken to be greater than zero. Without loss of generality (as in the HHM case), for some t_0 the above is such that as $x \longrightarrow \pm\infty$, $\vec{\psi} \longrightarrow (1,0,0)$ and at $x = x_0$, $\vec{\psi} = (-1,0,0)$, and hence, winds once around the hyperboloid.

The solution is therefore attractive in two senses; first in that, if $X = \mathbb{R}$, it is of winding type for the Pivotal model, and second because it reduces exactly to one of the same type in the HHM ($\omega = 0$) limit. In the HSM ($\omega = 1$) limit however, the reduction does not carry through since here, $f(x)$ diverges. Nevertheless, it may be possible to produce something of interest for the sigma model by considering a more general case. Not withstanding this minor setback let us therefore recall that in the Heisenberg case the 'cosech' solution occurred as a limiting case of a solution found in terms of elliptic functions (i.e. (2.64)). One might speculate the same to be true of the solution in the Pivotal model case, and if this were so, that it might be possible to find at least some interpolation between all three models via such a solution.

Consider then the anzatz

$$f(x) = A\,\mathrm{sc}\left[B(x-x_0)|M\right] \tag{4.54}$$

where A, B, x_0 and M are constants (M being the modulus of the elliptic function). It is possible to find expressions for these parameters directly from this solution and (4.51) and the resultant solutions reduction in the Heisenberg case is straightforward. However, this method is not amenable to the Pivotal model itself, in the sense that it is not obvious how the parameters behave when the elliptic function reduces to its limits. We rather follow a different path and transform the solution as it stands by a shift in x, as was done in the original Heisenberg case: shifting x by the half period $K : x \longrightarrow K - x$ one has

$$A\,\mathrm{sc}[K - x|M] = \frac{A}{\sqrt{1-M}}\,\mathrm{cs}[Bx|M] \tag{4.55}$$

where $\mathrm{cs}u = \frac{\mathrm{cn}u}{\mathrm{sn}u}$. If this is now put into equation (4.51) with $\frac{d}{du}[\mathrm{cs}u] = -\mathrm{ns}u\,\mathrm{ds}u$, $\mathrm{ns}u = \frac{1}{\mathrm{sn}u}$, $\mathrm{ds}u = \frac{\mathrm{dn}u}{\mathrm{sn}u}$ and $\mathrm{dn}^2u = (1 - M\,\mathrm{sn}^2u)$; the following equation results

$$\begin{aligned}\frac{A^2B^2}{1-M}\mathrm{ns}^2(Bx)\,\mathrm{ds}^2(Bx) &= \frac{1}{8}\frac{m^2\omega + 2l}{(1-M)^2}\Big[A^2\,\mathrm{cs}^2(Bx) + (1-M)\Big]^2\\ &+ \frac{m(1-w)}{2(1-M)^2}\Big[A^4\,\mathrm{cs}^4(Bx) - (1-M)^2\Big]\\ &- \frac{m^2\omega}{4(1-M)^2}\Big[A^2\,\mathrm{cs}^2(Bx) - (1-M)\Big]^2\end{aligned} \tag{4.56}$$

where the constant k has been replaced by $-l < 0$. By the elliptic function identities and equating coefficients a set of equations for A^2, B^2 and M, in terms of m, ω and l may then be found, namely;

$$\begin{aligned}2B^2(1-M) &= A^2\Big[\frac{l}{2} + m(1-\omega) - \frac{m^2\omega}{4}\Big],\\ B^2M(1-M) &= \frac{A^2}{4}\big[m^2\omega + 2l + 4m(1-\omega)\big] - \frac{(1-M)}{4}\big[3m^2\omega + 2l\big],\\ 0 &= \frac{m^2\omega + 2l}{4}\big[A^2 - (1-M)\big]^2 + m(1-\omega)\big[A^4 - (1-M)^2\big]\\ &- \frac{m^2\omega}{2}\big[A^2 + (1-M)\big]^2.\end{aligned}$$

And these can be solved to give

$$B^2 = \frac{1}{8}\Big[2l + 3m^2\omega \pm 2m\sqrt{2\big[2\omega l + m^2\omega^2 + 2(1-\omega)^2\big]}\Big], \tag{4.57a}$$

$$A^2 = \frac{3m^2\omega + 2l \mp 2m\sqrt{2\big[2\omega l + m^2\omega^2 + 2(1-\omega)^2\big]}}{2l - m^2\omega + 4m(1-w)}, \tag{4.57b}$$

$$M = \frac{\pm 4m\sqrt{2\big[2\omega l + m^2\omega^2 + 2(1-\omega)^2\big]}}{2l + 3m^2\omega \pm 2m\sqrt{2\big[2\omega l + m^2\omega^2 + 2(1-\omega)^2\big]}}. \tag{4.57c}$$

where the signs are ordered throughout. Taking $m > 0$ and the positive square root in B^2, i.e. the same in M and the negative root in A^2; M is then greater than or equal to zero (as is B^2). The only constraints therefore result from A^2 and the fact that M should be less than or equal to one so that

$$\begin{aligned} 3m^2\omega + 2l &\geq 2m\sqrt{2\left[2\omega l + m^2\omega^2 + 2(1-\omega)^2\right]} && (4.58a)\\ 2l - m^2\omega &> -4m(1-\omega). && (4.58b)\end{aligned}$$

From (4.58a) one has the inequality

$$4l^2 - 4m^2\omega l + m^4\omega^2 - 16m^2(1-\omega)^2 \geq 0$$

or $(2l - \Lambda_+)(2l - \Lambda_-) \geq 0$ where $\Lambda_\pm = m^2\omega \pm 4m(1-\omega)$ hence both (4.58a) and (4.58b) are satisfied if and only if

$$2l - m^2\omega > 4m(1-\omega) \tag{4.59}$$

for positive m. It is now possible to find an explicit expression for the coefficient $\frac{A}{\sqrt{1-M)}}$ ($= \mu$ say), of (4.55) in terms of w, l and m:

$$\mu = \sqrt{\frac{4m\left[2\left(2l\omega + m^2\omega^2 + 2(1-\omega)^2\right)\right]^{\frac{1}{2}}}{2l - m^2\omega + 4m(1-\omega)}} \tag{4.60}$$

and hence the solution

$$f(x) = \mu\,\mathrm{cs}(Bx|M) \tag{4.61}$$

is well defined with the given parameters (4.57) and constraint (4.59). Further, with (4.61) and u, v as in (4.50); $\vec{\psi} = \frac{1}{1+f^2}(1-f^2, 2u, 2v)$ is a topological (t dependent) stationary soliton by the considerations of Section 2.6 for periodic X, where here the period is $2K$, (c.f. (2.66)).

Considering now the limits of (4.61) with respect to the parameter M; from the original solution $f(x) = A\,\mathrm{sc}(Bx|M)$, and with A, B and M as above, for $M = 0$ it is necessary that m also be zero. Then $A = \pm 1$, $B = \pm\frac{\sqrt{l}}{2}$ so that with $l = 4$ the static solution $f(x) = \tan x$ is recovered exactly which is, of course, a solution for all three models. In the opposite limit, for $M = 1$ the requirement is $2l - m^2\omega = \pm 4m(1-\omega)$ and taking the positive case results in $\mu = \sqrt{\frac{m\omega}{(1-\omega}}$, and $B = \sqrt{m^2\omega + m(1-\omega)}$, recovering exactly (4.52) which winds once around the hyperboloid for $X = \mathbb{R}$ as was previously shown.

It remains now to examine how (4.61) behaves in the limiting models. To begin with, in both cases it is easily shown that $M = 0$ requires again $m = 0$ and the static solution $f(x) = \tan x$ is produced. For the general case and the $M = 1$ limit we consider the limiting models separately as follows:

(i) If $\omega = 0$ (i.e. for HHM), one has $B = \pm\frac{\sqrt{l+2m}}{2}$, $A = \sqrt{\frac{2l-4m}{2l+4m}}$, $M = \frac{8m}{2l+4m}$ and $\mu = \sqrt{M}$. Hence (4.61) is valid for the Heisenberg model if $l \geq 2m$ and is in exact correspondence with the solution (2.64) (in its 'sc' form and hence its 'cs' counterpart) if $l = 2(2-m)$. Taking the limit $M = 1$ requires $l = 2m$ and this reduction corresponds identically with solution (2.67) if $l = 2$ and where $m = 1$.

(ii) In the $w = 1$ (i.e. HSM) case, the situation is almost, but not quite, so satisfactory. Here one has

$$\begin{aligned} B^2 &= \frac{1}{8}\left(2l + 3m^2 + 2m\sqrt{2(2l+m^2)}\right), \\ A^2 &= \frac{3m^2 + 2l - 2m\sqrt{2(2l+m^2)}}{2l - m^2}, \\ M &= \frac{4m\sqrt{2(2l+m^2)}}{2l + 3m^2 + 2m\sqrt{2(2l+m^2)}} \end{aligned}$$

resulting in $\mu = \sqrt{\frac{4m\left[2(2l+m^2)\right]^{\frac{1}{2}}}{2l-m^2}}$. If $2l > m^2$ the solution (4.61) is perfectly valid for the sigma model and is topological in its elliptic form (and of course, in the static ($M = m = 0$) case). However, taking the limit $M = 1$ requires $2l = m^2$ so that $f(x)$ diverges. That this occurs may seem obvious and was indeed, to be expected. Nevertheless, it is interesting to note that as M gets *close* to unity the solution does actually come close to a 'cosech'. For example, taking $l = 2.1$ and $m = 2$ (so that $2l$ is close to m^2) results in $M \approx 0.999962$ and $\mu \approx 161.99$ (with $B \approx 4$) so that such a solution is approximated as the limit is approached.

To briefly summarize what has been shown; with the anzatz (4.54) an explicit family of stationary solitons has been found which is topological in its elliptic form and its trigonometric (static) and hyperbolic trig. (t dependent) reductions for the Pivotal model, where respectively, $X = S^1$ and $X = \mathbb{R}$. The solutions reduce exactly, in all their forms in the Heisenberg model case, and in their elliptic and trigonometric forms for the sigma model. However, in the HSM reduction, for the $X = \mathbb{R}$ solution,

$f(x)$ diverges in the exact hyperbolic trig. limit. An approximation for this latter case does, however, appear possible.

This concludes the discussion of topological solutions for the Pivotal model on the hyperboloid. Of course there are many and varied families of solutions which are non-topological in nature and these may be of interest in their own right, however it is those of winding type which are of concern here and the existence of such solutions has been shown explicitly and their various reductions examined.

4.4 Conserved Quantities

It has been shown [53], that there exist an infinite number of non-local conservation laws for equation (4.3) by using the technique applied to the principle chiral model [90, 91]. Utilized for (4.3) this is briefly stated as follows: taking the Laurent expansion in λ

$$F(x;\lambda) = \sum_{n=0}^{\infty} \frac{F_n}{\lambda^n}$$

(where $F_0 = 1$), in the linear problem

$$(\partial_\mu + A_\mu)F(x;\lambda) = 0,$$

and defining the conserved currents as

$$J_\mu^{(n)} = \epsilon_{\mu\nu}\partial^\nu F_n(x)$$

where these satisfy the current conservation $\partial_\mu J^\mu = 0$; the $F_n's$ are defined recursively with the A_μ of the spectral problem (4.4). So that to lowest order $(\frac{1}{\lambda})$

$$\partial_\mu F_1 + \epsilon_{\mu\nu}(j^\nu + 2\gamma^\nu Q) = 0$$

(with $j^\mu = 2\epsilon^{\mu\nu}[Q, \partial_\nu Q]$) and the current conservation is just the equation of motion (4.3). From this, the higher order currents are obtained, so that for $n = 2$

$$\partial_\mu F_2 + (j_\mu + 4\gamma_\mu Q) + \epsilon_{\mu\nu}(j^\nu + 2\gamma^\nu Q)F_1 = 0$$

and to higher order

$$
\begin{aligned}
0 = \partial_\mu F_{p+3} + \epsilon_{\mu\nu} \sum_{m=0}^{\frac{p}{2}+1} (j^\nu + 2(m+1)\gamma^\nu Q) F_{p-2-2m} \\
+ \sum_{m=0}^{\frac{p+1}{2}} (j_\mu + 4(m+1)\gamma_\mu Q) F_{p+1-2m} + 2\epsilon_{\mu\nu}\gamma^\nu Q \sum_{m=0}^{\frac{p}{2}} (m+1) F_{p-2m}
\end{aligned}
$$

for $p \geq 0$.

These currents apply to the model on general Hermitian symmetric spaces, and since the Pivotal model is in some sense a special case of this general model, satisfying the zero curvature condition, they should also be valid for the HPM. In fact, the first order current is a *local* conservation law since the equation of motion is a vector divergence equation; in the Pivotal model case the equation of motion (4.9) may be rewritten in the form

$$\partial_t\big[(1-\omega)\vec{\psi} + \omega\vec{\psi}\times\vec{\psi}_t\big] = \partial_x(\vec{\psi}\times\vec{\psi}_x)$$

resulting in the vector conserved quantity

$$\int_X \big[(1-\omega)\vec{\psi} + \omega\vec{\psi}\times\vec{\psi}_t\big]dx. \tag{4.62}$$

Following the formulation noted above; expanding to second order in $\frac{1}{\lambda}$, the second order current for the Pivotal model is given by

$$
\begin{aligned}
0 = (\partial_0 - \partial_1)F_2 + 2\omega\big([S,\partial_0 S] - [S,\partial_1 S]\big) + 4(1-\omega)S \\
- 2F_1\big([S,\partial_1 S] - \omega[S,\partial_0 S] - (1-\omega)S\big).
\end{aligned} \tag{4.63}
$$

Whilst the first order current is a local quantity, those of higher order (at least for the general model and hence, no doubt for the Pivotal model also), are all non-local. The above expressions for first and second order currents, reduce also to conserved quantities for the HHM and HSM simply by taking the limits of the parameter ω as usual.

As a simple example of a first order current with respect to the solutions of the model; for the static solution (in the stereographic parametrisation where $u = \tan\frac{x}{2}, v = 0$), $\vec{\psi} = (\cos x, \sin x, 0)$ and where $X = S^1$, the integral (4.62) is clearly zero. Taking the solution $f(x) = \pm\big(\frac{m\omega}{(1-w)} + 1\big)^{\frac{1}{2}}\text{cosech}\big[\sqrt{m^2\omega + m(1-\omega)}(x - x_0)\big]$, i.e. (4.52) as a further example; in this case the integral diverges. This is to be expected since it is pointed out [53] that for the general equation (4.3) in the rapidly decreasing case (i.e. $\lim_{|x|\longrightarrow\infty} Q(x) = Q_0 \neq 0$) the integrals in general diverge, becoming finite only after subtracting their values in the ground state.

4.5 Concluding Remarks

This chapter has been concerned with a novel non-linear system which is both integrable and admits topological solitons. Derived from an extension of the non-linear sigma model (c.f. [53]), the Pivotal model has been formulated with the hyperboloid of one sheet as its target manifold and contains both the Heisenberg and sigma models discussed in the previous chapters. It therefore combines and interpolates between, a Lorentz invariant and a non Lorentz invariant system, both of which stem from the same static picture.

The integrability of the HPM has been established via a zero curvature representation and the existence of various types of topological soliton examined. Where physical space X is the circle S^1, a family of travelling waves were found which carry through all three models by variation of the parameter ω. For the $X = \mathbb{R}$ case a set of travelling waves has been derived exhibiting most of the requirements for a topological soliton and whilst it has not been shown conclusively that this solution is fully topological, the possibility remains. So whilst they do not exist for the HHM and HSM models, it is possible that a family of topological travelling waves with $X = \mathbb{R}$ may exist for the Pivotal model. As one would expect, no reduction to the limiting models is possible in this family of solutions.

In addition to the travelling waves, a time dependent (stationary) set of solutions was shown to exist; first in the $X = \mathbb{R}$ case where the solutions reduce only in the Heisenberg limit and then in a more general elliptic form. This latter elliptic solution is valid and winds for all three models (with X periodic) and reduces to a static solution for $X = S^1$ (again valid for all the models). Reduction to the hyperbolic trigonometric limit, i.e. in the $X = \mathbb{R}$ case, is only possible for the Heisenberg and Pivotal models and is in exact correspondence with the first set of solutions found. For the sigma model the solutions in this limit diverge although an approximation does appear possible; a complete investigation into the solutions behaviour in this limit is, however, left for the future. Such elliptic solutions of the Pivotal model are particularly pleasing from an aesthetic point of view since the reductions via the elliptic modulus M to the hyperbolic and trigonometric function are in some sense reminiscent of the Pivotal reduction to the Heisenberg and sigma models via the pivotal parameter ω.

Finally, the existence of conserved quantities with respect to the model has been discussed, albeit briefly, with reference to the general model on Hermitian symmetric spaces [53], and much remains to be done in the analysis of these and other possible conservation laws.

Chapter 5

Conclusion

5.1 Synopsis

The main aim of this thesis has been to investigate the existence of topological solitons for three integrable (1+1)-dimensional systems of classical non-linear partial differential equations; the hyperbolic Heisenberg (HHM), sigma (HSM) and Pivotal (HPM) models. The HHM and HSM are known models, various aspects of which are represented in the literature however, the HPM is a new system and is of particular interest since it is a combination of the other two models, thereby incorporating both the non-relativistic (HHM) and relativistic (HSM) models in a single system. Furthermore, the form the Pivotal model equations take provides an interpolation between the two constituent models, and therefore at least some of their solutions and properties, via a single scalar parameter.

The integrability has been demonstrated for each model via a zero curvature representation and the zero curvature representation for the HPM shown to reduce to such a representation for each of the other two models. That all three models are indeed integrable is an added bonus to the fact that, as has been shown and is summarized below, they admit solitons of winding type, since systems with both properties are comparatively rare.

With respect to the existence of topological solitons for the models, the choice of target manifold M, together with the boundary conditions imposed on physical space X is crucial. Here we have chosen M to be the hyperboloid of one sheet and taken X to be either periodic or the real line $\mathbb{R}$. Each model has been examined separately with varying results:

(i) The Heisenberg case was discussed first and some simple static and t dependent solutions of winding type derived. A family of travelling waves of winding type for $X = S^1$ was also shown to exist and a proof given that for $X = \mathbb{R}$ no such solutions are possible. In addition, a set of elliptic t dependent, stationary solitons was found which reduced in the limits of the elliptic modulus to solutions for both $X = S^1$ and $X = \mathbb{R}$. In the former case the solution is static and hence, a valid solution for all three models.

(ii) The sigma model was examined next and again, some simple t dependent winding solutions found. Travelling waves of winding type were shown to exist (analogous to those of the HHM) for $X = S^1$ and it was demonstrated that they cannot occur if $X = \mathbb{R}$. The sigma model possesses the property of self-duality and t dependent solutions for the $X = \mathbb{R}$ case were derived from the self-dual equations. It was further shown that such solutions are defined only for a finite time.

(iii) Examination of the Pivotal model produced as usual, a simple family of t dependent solutions and topological travelling waves for $X = S^1$, where taking the limits of the parameter ω, the corresponding solutions of the previous models resulted. Investigation into the existence of travelling waves of winding type on $X = \mathbb{R}$ proved at least a little more hopeful for the HPM than for the other two models; it was found that a family of such solutions *may* exist for the Pivotal system where these solitons are in perpetual motion. As expected, they do not reduce to analogous solutions for the limiting models. An investigation into a more general solution of this type proved unproductive and it seems unlikely that a generalization is possible. In addition to the above, a set of elliptic t dependent stationary solitons which are themselves topological, was derived. Taking the limits of the elliptic parameter yielded topological solitons in both the $X = S^1$ and $X = \mathbb{R}$ cases for HPM, and the elliptic solution and its limits reduced exactly to the analogous Heisenberg solutions. Both the elliptic and $X = S^1$ static solution were valid in the sigma model limit but in the $X = \mathbb{R}$ case only an approximation appeared possible; in the exact limit a divergence occurs.

Conserved quantities for the Pivotal model have been briefly discussed and explicit expressions given for the currents up to second order where the first order current is a local quantity and the higher order currents, are non-local. It seems probable that an expression for currents with $n \geq 3$ may be obtained from the marriage of the lower order explicit HPM currents and the definition of those of higher order for the general model on Hermitian Symmetric spaces.

5.2 Further Possibilities

The discovery of any new model, particularly one of integrable type, opens up all manner of potential avenues of research. And what has been discussed with respect to the Pivotal model here, i.e. its integrability via a zero curvature representation and the existence of topological solitons, provides only a glimpse of the fruits further investigation of the model might yield. This section is therefore devoted to listing some of these possibilities although the list is by no means exhaustive.

5.2.1 Integrability

As noted above, it has been shown that the Pivotal model is integrable via a zero curvature representation, however; alluding to the various definitions given in Chapter 1, this is not the only way in which a model may be shown to be integrable. Various other definitions and tests may be applied or performed; not only in order to check for integrability, but which may also produce solutions of different types to those examined here. In particular, the Inverse Scattering Transform, which encompasses many of these, might be applied to the model. Systems which are solvable via this method are known to possess a number of remarkable properties amongst which, the admittance of a Lax or zero curvature representation and the the existence of solitons and an infinite number of conserved quantities are but a few. Such models, solvable by the IST, are *completely integrable Hamiltonian systems* where the IST is interpreted (c.f. [15]) as a non-linear transformation from physical variables to an infinite set of action-angle variables. To check whether or not the Pivotal system possesses such a Hamiltonian structure may prove an interesting exercise. A system integrable via the IST should also produce N-soliton solutions resulting from Hirota's method [92]. And some form of Bäcklund transformation ought to be possible [93], relating the solution of an equation either to another solution of the equation, or to a solution of another equation. Furthermore, the possession of the 'Painlevé property' may be tested for such models.

In Chapter 1 the conjecture that many if not all integrable systems may be obtained as reductions of the Self-dual Yang Mills equations was noted, and an example of the process of reduction given. Since both the Heisenberg and sigma models occur from such reductions one might envisage that the Pivotal model may also be produced in a similar way. And indeed that this might occur via some combination of the reductions for the two constituent models.

Related also to the integrability of a model is the idea of gauge transformation of a model and its solutions, an example of which was given in Chapter 1. What kind of model the Pivotal system might be associated with via this formalism is unclear, however, it may prove interesting to find out. The conclusion is that there are many possibilities for further investigation with respect to the integrability of the HPM and its solutions.

5.2.2 $(2+1)$-dimensional Extensions of the Pivotal Model

The models discussed in this thesis are all (1+1)-dimensional with respect to physical space-time and whilst integrable models in $(2+1)$-dimensions (some of which admit topological solitons), are scarce, they do exist. In addition, there are of course, many non-integrable systems in higher dimensions with solitons classifiable by an integer winding number. And it turns out that it is in fact, possible to find extensions of the Pivotal model in $(2+1)$-dimensions. The Ishimori equation [7] (or vector Davey-Stewartson equation [6]) and an equation formulated by Myrzakulov [95]; the M-XX equation, which are both $(2+1)$-dimensional integrable models with topological solitons, may be reformed into Pivotal type equations as follows: the Ishimori equation is given by

$$\vec{S}_t + \vec{S} \times \vec{S}_{xx} + \sigma^2 \vec{S} \times \vec{S}_{yy} + \phi_x \vec{S}_y + \phi_y \vec{S}_x = 0$$
$$\phi_{xx} - \sigma^2 \phi_{yy} + 2\sigma^2 \vec{S}(\vec{S}_x \times \vec{S}_y) = 0$$

where $|\vec{S}|^2 = 1$, $\sigma^2 = \pm 1$ and $\phi(x, y, t)$ is a scalar function generated by the topological charge. And letting $\partial_y = \phi = 0$ results in the Heisenberg model in $(1+1)$-dimensions. One can reformulate the above equation by introducing the Pivotal parameter ω and a further scalar parameter a in the following way:

$$(\omega - 1)\vec{S}_t + a(\vec{S} \times \vec{S}_{xx}) + \omega\sigma^2(\vec{S} \times \vec{S}_{yy}) + (\omega - 1)(a - 1)\left[\phi_x \vec{S}_y + \phi_y \vec{S}_x\right] = 0$$
$$a\phi_{xx} - \omega\sigma^2 \phi_{yy} + 2(\omega - 1)(a - 1)\sigma^2 \vec{S}(\vec{S}_x \times \vec{S}_y) = 0.$$

And letting $y \longrightarrow t$, $\phi = 0$, $\sigma^2 = -1$ and $a = 1$ results in the $(1+1)$-dimensional Pivotal equation (4.9).

Similarly, the Myrzakulov MXX equation

$$\vec{S}_t - \vec{S} \times \left[(b+1)\vec{S}_{xx} - b\vec{S}_{yy}\right] - (b+1)u_x\vec{S}_x - bu_y\vec{S}_y = 0$$
$$u_{xy} - \alpha\kappa\vec{S}(\vec{S}_x \times \vec{S}_y) = 0$$

with b, α, κ constant and u a scalar function, reduces to the Heisenberg model if $b = u = \kappa = 0$. And again, by introducing the Pivotal parameter ω and the scalar a in an appropriate way, a Pivotal type equation in (2+1)-dimensions results. Namely,

$$(\omega - 1)\vec{S}_t + \vec{S} \times \left[a(b+1)\vec{S}_{xx} - \omega b\vec{S}_{yy}\right] + (b+1)(a-1)u_x\vec{S}_x + b(1-\omega)u_y\vec{S}_y = 0$$
$$u_{xy} = \alpha\kappa\vec{S}(\vec{S}_x \times \vec{S}_y).$$

The $(1+1)$-dimensional Pivotal equation (4.9) is then produced by letting $y \longrightarrow t$, $u = \kappa = 0$, $b = 1$ and $a = \frac{1}{2}$. Since both of the above $(2+1)$-dimensional Pivotal equations reduce to the $(1+1)$-dimensional equation (4.9), they further reduce to *both* the Heisenberg and sigma model equations.

The Ishimori equation above admits a zero curvature representation however, there appears to be no way to adapt this to fit to the altered Pivotal version, and whilst no attempt has yet been made to adapt the MXX Lax representation to the Pivotal case, it seems unlikely that this will be possible; as noted, integrable equations in more than $(1+1)$-dimensions are rare and whilst it would certainly be pleasing if the above $(2+1)$-dimensional Pivotal equations were integrable, it would not be entirely unexpected if this were not the case. What is more likely however, is that topological solitons exist for the models although this also, has yet to be investigated.

5.2.3 Positive Definite and Discrete Versions

In each of the systems discussed in the main body of this thesis, the metric on the target space M is indefinite:

$$\begin{aligned} ds^2 &= \eta_{ab}\, d\psi^a\, d\psi^b \\ &= -d\theta^2 + \cosh^2\theta\, d\phi^2. \end{aligned}$$

For the HHM and HSM models this gives rise to the indefinite energy densities of the systems. One can replace this by an analogous positive-definite metric on the cylinder, namely

$$ds^2 = d\theta^2 + \cosh^2\theta\, d\phi^2. \tag{5.1}$$

And both the Heisenberg and sigma models remain integrable, since they are obtained simply by making the replacement $\phi \mapsto i\phi$ (and in the HHM case, also $t \mapsto it$). In Appendix C the sigma model in this positive-definite case is discussed, where the model is reformed and a family of topological solitons found, analogous to those of Section 3.3. Further, an expression for the winding number is given and a '*Bogomol'nyi*' bound for the energy shown to exist, dependent on the winding number N.

As in the Heisenberg case, with this metric on the target manifold the Pivotal model also requires $\phi \mapsto i\phi$ and $t \mapsto it$ and the resultant equation in the 'polar angle' parametrization is given by

$$\begin{aligned}(1-\omega)\phi &= \left[\phi_x^2 + \omega\phi_t^2\right]\sinh\theta - \left[\theta_{xx} + \omega\theta_{tt}\right]\,\mathrm{sech}\;\theta \\ (1-\omega)\theta &= \left[\phi_{xx} + \omega\phi_{tt}\right]\cosh\theta + 2\left[\theta_x\phi_x + \omega\theta_t\phi_t\right]\sinh\theta.\end{aligned}$$

The reduction to the sigma model is not the same as that of Appendix C due to the fact that $t \mapsto it$ here, and the integrability and solutions of this positive-definite Pivotal equation have yet to be investigated. However, one might expect both the integrability to survive and that topological solitons exist since both properties appear to be retained for the sigma and Heisenberg models.

A simple discrete version of the positive-definite sigma model is proposed and briefly examined in Appendix D, where the winding number is defined in terms of the discretized ϕ and a Bogomol'nyi bound again found for the energy. However, this investigation has yet to be extended to the positive definite Heisenberg and Pivotal systems, or indeed the indefinite cases. Certainly a discrete (and completely integrable) version of the S^2 Heisenberg model exists [63] which one would hope may be adapted to the indefinite HPM however, whether such an integrable lattice Pivotal model is possible remains to be seen.

5.3 Final Remarks

There are obviously many other possibilities for further investigation of the Pivotal system although one is liable to be kept busy for quite some time with those noted above. Two important points which have thus far been omitted are the symmetries of the system and its applications. The former is, of course important but also a curiosity since the Heisenberg terms in the Pivotal equation are Galileo invariant whilst the sigma model terms are Lorentz invariant. So what may result from an investigation of this is an intriguing question.

With regard to applications of the system, as has been noted, the hyperbolic Heisenberg and sigma models have been connected with gravity and strings respectively, so what applications may be associated with the H^2 Pivotal model also poses an interesting question. A further point worthy of note is the previously mentioned fact that the S^2 Heisenberg model is related to anisotropic ferromagnets and the analogous sigma model to those of isotropic type; consultation with some condensed matter physicists may prove beneficial in determining whether or not any materials exist exhibiting both properties. With these open (and intriguing) questions the thesis proper is concluded.

Appendix A

Supplement to Section 2.4

In Section 2.4 it was shown that with $c = 0$ in equation (2.30),

$$g(\xi) = \mathrm{Re}\left[-i\ln\left\{\frac{\tan\frac{\xi}{2}+\frac{a}{b}}{\tan\frac{\xi}{2}+\frac{C}{b}}\right\}+\Lambda\right] \tag{A.1}$$

where $a = -i\gamma - ik + v$, $b = 1 + ikv$, $C = -i\gamma + ik - v$, $\gamma = \sqrt{(k^2+1)(v^2-1)}$ and $\alpha = v^2 - k^2 + 2Q = -1$, is a travelling wave solution of the HHM, and further that

$$g(\xi) = \mathrm{Re}\left[-i\ln\left\{\frac{\tan\frac{\xi}{2}+\Omega}{-\tan\frac{\xi}{2}-\Omega^{-1}}\right\}\right] \tag{A.2}$$

(where $\Omega = \frac{a}{b}$) is a travelling wave of *winding* type. What follows is a demonstration that (A.1) may take the form of (A.2) by specifying the constant $\Lambda = i\ln\Theta$ (Θ complex) in terms of the parameters v and k, thereby proving that (A.1) is also a winding solution. Letting $T = \tan\frac{\xi}{2}$ for simplicity the requirement is that

$$\mathrm{Re}\left[-i\ln\left\{\frac{T+\frac{a}{b}}{T+\frac{C}{b}}\right\}+\Lambda\right] = \mathrm{Re}\left[-i\ln\left\{\frac{T+\frac{a}{b}}{-T-\frac{b}{a}}\right\}\right]$$

and using $\ln\frac{A}{B} = \ln A - \ln B$, one has

$$\begin{aligned}&\mathrm{Re}\left[-i\ln\left(\frac{bT+a}{b}\right)+i\ln\left(\frac{bT+C}{b}\right)+i\ln\Theta\right]\\ =\ &\mathrm{Re}\left[-i\ln\left(\frac{bT+a}{b}\right)+i\ln\left(\frac{-aT-b}{a}\right)\right]\end{aligned}$$

or

$$\begin{aligned}
&\mathrm{Re}\left[-i\ln(bT+a)+i\ln b+i\ln(bT+C)-i\ln b+i\ln\Theta\right]\\
&= \mathrm{Re}\left[-i\ln(bT+a)+i\ln b+i\ln(-aT-b)-i\ln a\right]
\end{aligned}$$

so that

$$\begin{aligned}
\mathrm{Re}\left[i\ln(bT+C)+i\ln\Theta\right] &= \mathrm{Re}\left[i\ln b+i\ln(-aT-b)-i\ln a\right]\\
\Longrightarrow \mathrm{Re}\left[i\ln(\Theta(bT+C))\right] &= \mathrm{Re}\left[i\ln\left(\frac{b}{a}(-aT-b)\right)\right]. \qquad \text{(A.3)}
\end{aligned}$$

Recall now that if $x = r\,e^{i\theta}$, $y = s\,e^{i\phi}$ for some $r, s, \theta, \phi \in \mathbb{R}$ then

$$\begin{aligned}
\mathrm{Re}\left[-i\ln x\right] &= \mathrm{Re}\left[-i\ln y\right]\\
\Longrightarrow \quad \mathrm{Re}\left[-i(\ln r+i\theta)\right] &= \mathrm{Re}\left[-i(\ln s+i\phi)\right]\\
\Longrightarrow \quad \theta &= \phi.
\end{aligned}$$

And from (A.3)

$$\mathrm{Re}\left(-i\ln\Theta\right) = \mathrm{Re}\left[-i\ln\left(\frac{-bT-\frac{b^2}{a}}{bT+C}\right)\right].$$

Hence, defining $\Theta = r_1 e^{i\theta_1}$, $\frac{-bT-\frac{b^2}{a}}{bT+C} = r_2 e^{i\theta_2}$,

$$\theta_1 = \theta_2 = \tan^{-1}\left[\frac{\mathrm{Im}}{\mathrm{Re}}\left\{\frac{-bT-\frac{b^2}{a}}{bT+C}\right\}\right].$$

The dependence on $T = \tan\frac{\xi}{2}$ can be eliminated by noting that

$$\begin{aligned}
\frac{-bT-\frac{b^2}{a}}{bT+C} &= \frac{-(bT+C)+C-\frac{b^2}{a}}{(bT+C)}\\
&= -1+\frac{C-\frac{b^2}{a}}{bT+C}
\end{aligned}$$

so that to eliminate T, the second term on the right hand side must be identically zero. This then requires $C - \frac{b^2}{a} = 0$ which is true if and only if

$$\begin{aligned} aC &= b^2 \\ \iff \quad (-i\gamma + ik - v)(-i\gamma - ik + v) &= (1 + ikv)^2 \\ \iff \quad -\gamma^2 + k^2 - v^2 + 2ikv &= (1 + 2ikv - k^2v^2). \end{aligned}$$

And since $\gamma^2 = (v^2+1)(k^2-1)$ the equality holds. One then has $\theta_1 = \theta_2 = \tan^{-1}(-1)$, i.e. $\theta_1 = \theta_2 = -\frac{\pi}{4}$. Hence, the solution (A.1) does indeed have the form (A.2) which is a travelling wave solution of winding type.

Appendix B

$P(p)$ of HSM has the Correct Form for Topological Solitons

In order to investigate topological travelling waves for the HSM in the case where $c \neq 0$, it must be established that there exist $J, K \in \mathbb{R}$ such that the equation (3.24), i.e.

$$p'^2 = \frac{c^2}{(v^2-1)^2}\left[p^4 + p^2\frac{(6c^2+4Q(v^2-1)^2)}{4c^2} + \frac{2c^2+4Q(v^2-1)^2+\left[4R(v^2-1)^2-cv\right]^2}{4c^2}\right] \tag{B.1}$$

may take the form (3.26), i.e.

$$P(p) = \frac{(v^2-1)^2}{c^2}p'^2 = (J^2-p^2)(K^2-p^2). \tag{B.2}$$

That this is the case can be seen as follows: comparing coefficients of (B.2) with those of (B.1) results in the following two equations for J and K in terms of the constants Q, R, v and c;

$$-(J^2+K^2) = \frac{3c^2+2Q(v^2-1)^2}{2c^2} < 0 \tag{B.3a}$$

$$J^2K^2 = \frac{2c^2+4Q(v^2-1)^2+\left[4R(v^2-1)^2-cv\right]^2}{4c^2} > 0. \tag{B.3b}$$

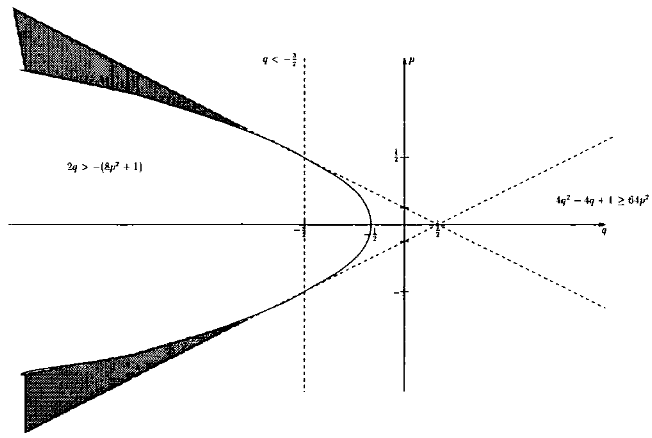


Figure B.1: Shaded regions depict the ranges of p, q for which solutions can exist.

These equations may be simplified by letting $q = Q(v^2 - 1)^2$ and $r = R(v^2 - 1)^2$, $\forall\, v$ so that

$$-(J^2 + K^2) = \frac{3c^2 + 2q}{2c^2} < 0 \tag{B.4}$$

$$J^2K^2 = \frac{2c^2 + 4q + (4r - cv)^2}{4c^2} > 0 \tag{B.5}$$

from which the conditions $q < -\frac{3}{2}c^2$, and $2c^2 + 4q + (4r - cv)^2 > 0$ result. Further, from (B.4, B.5), one can obtain an equation for K^2 namely

$$K^4 + K^2\frac{(3c^2 + 2q)}{2c^2} + \frac{2c^2 + 4q + (4r - cv)^2}{4c^2} = 0. \tag{B.6}$$

Then coupled with the conditions noted above, and letting $4\rho = 4r - cv$ and without loss of generality, $c = 1$; and since we require $K^2 > 0$, for J, K to exist satisfying (B.3), q and ρ must satisfy the following inequalties:

$$\begin{aligned} q &< -\frac{3}{2} \\ 2q &> -(8\rho^2 + 1) \\ 4q(q-1) &\geq 16\rho^2 - 1. \end{aligned} \tag{B.7}$$

Plotting ρ against q (Figure (B.1)) one can see that solutions exist for (B.7) if ρ and q take values in the shaded regions. So that for such $q = Q(v^2 - 1)^2$, $r = R(v^2 - 1)^2$ and $4\rho = 4r - cv$, there exist J and $K \in \mathbb{R}$ satisfying the equations (B.3) and hence, (B.1) may take the form of (B.2).

Appendix C

A Positive-Definite Version of the Sigma Model

With the positive definite metric given in Section 5.2.3, the sigma model may be formulated in the following way where we are still thinking of ϕ as being a periodic coordinate (and looking for solutions which wind in ϕ): in terms of the θ, ϕ parametrization (2.17), the Lagrangian density in this case is given by

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\left[(\cosh^2\theta)\phi_\mu\phi_\nu + \theta_\mu\theta_\nu\right] \tag{C.1}$$

from which one obtains the equations of motion

$$\theta_{tt} - \theta_{xx} = \cosh\theta\,\sinh\theta(\phi_t^2 - \phi_x^2), \tag{C.2}$$

$$(\phi_t\cosh^2\theta)_t = (\phi_x\cosh^2\theta)_x. \tag{C.3}$$

The winding number N, i.e. the number of times physical space X is wrapped around the cylindrical manifold M, is given here by

$$N = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\phi}{\partial x}\,dx \tag{C.4}$$

which is independent of θ. The most general static winding solution is now $\theta \equiv 0$, $\phi = Nx$. This is what one would expect; in the positive definite case, the string will try to minimize its length, and $\theta = 0$ is where the cylinder is narrowest (with respect to the metric (5.1)). If one looks for more general solutions having $\phi = Nx$, then θ, as in the case of the indefinite metric, must be a function of t only, with

$$\tanh\theta = \sqrt{m}\,\mathrm{sn}\,(\rho t|m), \tag{C.5}$$

where ρ and m are constants with $\rho > |N|$ and $m = 1 - N^2/\rho^2$. In other words, the string oscillates between the values $\theta_\pm = \pm\tanh^{-1}\sqrt{m}$.

Regarding solutions of travelling wave type; it appears that no solution analogous to (3.21) is possible for this positive definite version. However, in the analogous Heisenberg model (where in addition to $\phi \mapsto i\phi$ one also has $t \mapsto it$) such a solution does exist so that the same may be true with $t \mapsto it$ here; further investigation is obviously necessary, as it is with topological travelling waves for the $X = \mathbb{R}$ case.

The energy for this system is given by

$$E = \int \varepsilon\, dx$$

where ε is the energy density. So that in the θ, ϕ parametrization one has

$$E = \int_0^{2\pi} (\partial_t\theta)^2 + (\partial_x\theta)^2 + \left[(\partial_t\phi)^2 + (\partial_x\phi)^2\right]\cosh^2\theta\, dx. \tag{C.6}$$

Note that

$$E \geq \int_0^{2\pi} \sigma^2\, dx \tag{C.7}$$

where $\sigma = \partial_x\phi$ is a function of x and t, and so has a minimum for some values of x and t. The above inequality is saturated if and only if $\theta = 0$ and $\partial_t\phi = 0$ in which case, as noted above, the strings length will be minimized. All of this would indicate that some form of 'Bogomol'nyi' type argument can be formulated to give a lower bound for the energy, where the Bogomol'nyi bound is dependent on the winding number N. This is indeed the case which we now show: observing that the following inequality is true for any $k \in \mathbb{R}$;

$$\int_0^{2\pi} (\sigma - k)^2\, dx \geq 0$$

$$\Longrightarrow \int_0^{2\pi} (\sigma^2 - 2k\sigma + k^2)\, dx \geq 0,$$

and taking into account (C.7), the above implies that

$$E - 2k \int_0^{2\pi} \partial_x \phi \, dx + k^2 \int_0^{2\pi} dx \ \geq 0.$$

The first integral in this expression is equal to $-4kN\pi$ (N being the winding number (C.4)) so that

$$E - 4kN\pi + 2\pi k^2 \ \geq 0 \tag{C.8}$$

which is quadratic in k. For (C.8) to be true for all $k \in \mathbb{R}$ the discriminant $b^2 - 4ac$ must be ≤ 0. This implies

$$\begin{array}{rcl} & 16N^2\pi^2 - 8E\pi & \leq \ 0 \\ \Longleftrightarrow & 2N^2\pi & \leq \ E \end{array}$$

giving a lower ('Bogomol'nyi') bound for the energy E which is strictly positive and depends on the winding number N, as required.

For this positive-definite sigma model then, it has been shown that for $\phi = Nx$, both static and t dependent topological solitons and a Bogomol'nyi bound for the energy exist. The model is yet to be examined with $t \mapsto it$, which is the requirement for the analogous Heisenberg and Pivotal cases.

Appendix D

A Spatially Discrete Sigma Model

In this section the discussion of the sigma model with a positive definite metric is continued in the form of a simple discrete version of this system which may be formulated as follows: whilst time t remains continuous such that $t \in \mathbb{R}$, the space X now takes n discrete values resulting in a lattice system. Configuration space is the space of maps from the lattice L^1 into the Hyperboloid where $\phi(t,x)$ is replaced by $\phi_j(t)$, with $j \in \mathbb{Z}$ and $\phi_{j+n}(t) = \phi_j(t)$, and similarly for θ. Labeling the discrete n values of the space X by the subscript $i \in \{1 \ldots n\}$, and letting $x_i = \frac{2\pi i}{n}$ with equal lattice spacing $h = \frac{2\pi}{n}$ for simplicity, the winding number N may be defined as

$$N = \frac{1}{2\pi} \sum_{i=1}^{n} A_i. \tag{D.1}$$

where A_i is defined in terms of ϕ_{i+1} and ϕ_i in the following way:

Case 1 : If $0 \leq \phi_{i+1} - \phi_i < \pi$ then $A_i = \phi_{i+1} - \phi_i$,

Case 2 : If $\pi < \phi_{i+1} - \phi_i < 2\pi$ then $A_i = \phi_{i+1} - \phi_i - 2\pi$,

Case 3 : If $0 \leq \phi_i - \phi_{i+1} < \pi$ then $A_i = -(\phi_i - \phi_{i+1})$, i.e. $A_i = \phi_{i+1} - \phi_i$,

Case 4 : If $\pi \leq \phi_i - \phi_{i+1} < 2\pi$ then $A_i = \phi_{i+1} - \phi_i + 2\pi$.

Note that in analogy to (C.4), the definition (D.1) is independent of θ and further, that for the above to be valid one must have $N \in \mathbb{Z}$. To prove that this is indeed the case; if

$$N = \frac{1}{2\pi} \sum_{j=1}^{n} A_j$$

then

$$2\pi i N = i\sum_{j=1}^{n} A_j$$

where here $i = \sqrt{-1}$. Since it is the case that $\exp(2\pi i p) = 1 \,\forall p \in \mathbb{Z}$; requiring $N \in \mathbb{Z}$ imposes that

$$\exp(2\pi i N) = 1.$$

And this is the case if and only if

$$\exp(\sum_{j=1}^{n} iA_j) = 1$$

The latter is true since for all cases in the definition of A_j;

$$\exp(iA_j) = \exp(i(\phi_{j+1} - \phi_j))$$

and

$$\begin{aligned}\exp(i\sum_{j=1}^{n} A_j) &= \exp(i\sum_{j=1}^{n}(\phi_{j+1} - \phi_j)) \\ &= \exp(i(\phi_{n+1} - \phi_1)) \\ &= e^{i0} = 1.\end{aligned}$$

Hence, $N \in \mathbb{Z}$ as required. The sign of A_j is ambiguous if ϕ_{j+1} and ϕ_j are antipodal points so the fields must be restricted to those of 'continuous' type, defined such that for each j, $\phi_{j+1} - \phi_j \neq \pm\pi$.

The next step is to devise an expression for the energy of a 'continuous' static field, such that the Bogomol'nyi bound of the continuous system remains valid and has the correct continuum limit and where physical space X remains discrete. Since the field is static for the moment, we need only find an expression for the potential energy so letting

$$E = \frac{1}{h}\sum_{j=1}^{n}[\theta_{j+1}(t) - \theta_j(t)]^2 + (A_j)^2\cosh^2\theta_j(t) \tag{D.2}$$

one has

$$E \geq \frac{n}{2\pi} \sum_j A_j^2. \tag{D.3}$$

There *is* a Bogomol'nyi bound which can be seen by utilizing a similar inequality as in the continuum case:

$$\sum_j [A_j - s]^2 \geq 0 \quad \forall s \in \mathbb{R},$$

or

$$\sum_j A_j^2 - 2s \sum_j A_j + ns^2 \geq 0.$$

Using E, i.e. (D.3) and N (D.1) then gives

$$\frac{2\pi}{n} E - 4s\pi N + ns^2 \geq 0$$

which is quadratic in s. And for this inequality to hold $\forall\, s \in \mathbb{R}$, the discriminant, must be ≥ 0, i.e.

$$\begin{array}{rcl} 16\pi^2 N^2 - 8\pi E & \leq & 0 \\ \Longrightarrow \qquad 2\pi N^2 & \leq & E. \end{array}$$

This provides a lower bound for the static energy with the correct continuum limit as required.

Having discretized the potential energy (D.2), in order to formulate a Lagrangian for the system an expression for the kinetic energy must also be obtained where in this case, the fact that time t is continuous in the lattice system must be taken into account so that the time derivative remains unchanged. With this in mind let the kinetic energy for the system be defined as

$$E_K = h \sum_j [\partial_t \theta_j(t)]^2 + [\partial_t \phi_j(t)]^2 \cosh^2 \theta_j(t) \tag{D.4}$$

where $j \in \{1, \ldots, N\}$. The Lagrangian, given by $L = E_K - E_P$, is then

$$\begin{aligned} L = \sum_j L_j = h \sum_j (\partial_t \theta_j)^2 + (\partial_t \phi_j)^2 \cosh^2 \theta_j \\ - h \sum_j \left[\frac{\theta_{j+1} - \theta_j}{h}\right]^2 + \left[\frac{A_j}{h}\right]^2 \cosh^2 \theta_j \end{aligned} \tag{D.5}$$

and from this, the following equations of motion may be derived: for $\phi_j(t)$;

$$\partial_t^2 \phi_j(t) = -2 \tanh \theta_j \left[(\partial_t \phi_j)(\partial_t \theta_j) + \left(\frac{\theta_{j+1} - \theta_j}{h}\right) \frac{A_j}{h}\right] + \left[\frac{A_j - A_{j-1}}{h}\right] \tag{D.6}$$

whilst for $\theta_j(t)$

$$\partial_t^2 \theta_j = \left[\frac{(\theta_{j+1} - \theta_j)}{h} - \frac{\theta_j - \theta_{j-1})}{h}\right] - \left(\frac{A_j}{h}\right)^2 \cosh \theta_j \sinh \theta_j. \tag{D.7}$$

If in analogy to the continuous case of the previous section where solutions were found in the case $\phi = Nx$, putting here, $\phi_j = Nx_j$, having defined $x_j = \frac{2\pi j}{n}$, and using the fact that any time derivative of ϕ_j vanishes, the x derivative can be discretized such that

$$\begin{aligned} \tfrac{A_j}{h} &= N\left(\tfrac{x_{j+1} - x_j}{h}\right) \\ &= N \end{aligned}$$

so that it's square is N^2. Substituting into (D.6) results in

$$N \tanh \theta_j \left(\frac{\theta_{j+1} - \theta_j}{h}\right) = 0,$$

and for non-trivial θ_j this requires $\frac{\theta_{j+1} - \theta_j}{h} = 0$ so that, as in the continuum case, the discrete θ variable does not depend on x_j. The equation of motion for θ_j (D.7) reduces to

$$\partial_t^2 \theta_j = -N^2 \cosh \theta_j \sinh \theta_j \tag{D.8}$$

which one would expect admits solutions for θ_j, independent of x_j, analogous to those those of the continuum case.

Note that in the above case where $\phi_j = Nx_j$ there is a restriction on the winding number which may be seen as follows:

$$\phi_j = Nx_j = \frac{2\pi jN}{n}$$

so that

$$\begin{aligned} \phi_{j+1} - \phi_j &= N(x_{j+1} - x_j) \\ &= \tfrac{2\pi N}{n}. \end{aligned}$$

And since $|\phi_{j+1} - \phi_j| < \pi$, the above implies that

$$\| \frac{2\pi N}{n} \| < \pi$$

i.e. $|N| < \frac{n}{2}$.

With this, the brief discussion of the discrete sigma model is concluded. Of course there remains much to be considered for this system, in particular its integrability and further topological solutions; what has been noted here are simply a few basic properties of the model and further discussion is left for the future.

Bibliography

[1] Ward RS and Winn AE 1998, *J. Phys. A: Math. Gen.* **31** L261

[2] Winn AE 1998, *Preprint: math.AP/9811135*,
to be published in Proc. NEEDS '98

[3] Gardner CS, Greene JM, Kruskal MD and Miura RM 1967,
Phys. Rev. Lett., **19** 1095

[4] Lambert D and Piette B 1988, *Class. Quantum Grav.* **5** 307

[5] Zabusky NJ and Kruskal MD 1965, *Phys. Rev. Lett.*, **15** 240

[6] Davey A and Stewartson K 1974, *Proc. Roy. Soc. London A*, **338** 101

[7] Ishimori Y 1984, *Prog. Theor. Phys.*, **72** 33

[8] Kadomtsev BB and Petviashvili VI 1970, *Sov. Phys. Dokl.*, **15** 539

[9] Ward RS 1988, *Nonlinearity*, **1** 671

[10] Ward RS 1977, *Phys. Lett.*, **61A** 81

[11] Belavin AA and Zakharov VE 1978 *Phys. Lett.*, **73B** 53

[12] Ward RS 1985, *Phil. Trans. R. Soc. Lond. A*, **315** 451

[13] Ward RS 1987, in *Field Theory, Quantum Gravity and Strings II*,
eds. De Vega HJ and Sanchez N, Springer Lecture Notes in Physics, 280

[14] Mason LJ and Sparling GAJ 1989, *Phys. Lett.*, **137A** 29

[15] Ablowitz MJ and Clarkson PA 1991, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press,
LMS Lecture Note Series, 149

[16] Rajaraman R 1989, *Solitons and Instantons*, North-Holland, Elsevier Science Publishers BV

[17] Skyrme THR 1961, *Proc. Roy. Soc. A*, **262** 237

[18] Skyrme THR 1962, *Nucl. Phys.*, **31** 556

[19] Faddeev LD and Zakharov VE 1971, *Funct. Analy. Appl.*, **5** 280

[20] Makhankov VG 1990, *Soliton Phenomenology*, MIA (Soviet Series), Kluwer Academic Publishers

[21] Ablowitz MJ and Ramani A, Segur H 1978 *Lett. Nuovo Cim.*, **23** 333, 1980 *J. Math. Phys.*, **21** 715

[22] Hastings SP and McLeod JB 1980, *Arch. Rat. Mech. Anal.*, **73** 31

[23] Nakahara M 1996, *Geometry, Topology and Physics*, IOP Graduate Student Series in Physics

[24] Chinn WG and Steenrod NE 1966, *First Concepts of Topology*, New Mathematical Library, MAA

[25] Speight JM 1995, *PhD. Thesis: On the Dynamics of Topological Solitons*, University of Durham

[26] Coleman S 1995, *Aspects of Symmetry*, Selected Erice Lectures, Cambridge University Press

[27] Zakharov VE and Mikhailov AV 1978, *Sov. Phys. JETP* **47**(6) 1017

[28] Pohlmeyer K 1976, *Commun. Math. Phys.* **46** 207

[29] Luscher M and Pohlmeyer K 1978, *Nucl. Phys.*, **B137** 46

[30] Luscher M 1978, *Nucl. Phys.*, **B135** 1

[31] Polyakov AM 1977, *Phys. Lett.*, **72B** 224

[32] Ennerper A 1870, *Nachr. Königl. Ges. Wiss. Göttingen* 4 93

[33] Frenkel J and Kontorova T 1939, *Fiz. Zhurnal*, **1** 137

[34] Perring JK and Skyrme THR 1962, *Nucl. Phys.*, **31** 550

[35] Coleman S 1975, *Phys. Rev. D*, **11** 2088

[36] Ablowitz MJ, Kaup DJ, Newell AC and Segur H 1973, *Phys. Rev. Lett.*, **30** 1262

[37] Novikov SP, Manakov SV, Pitaevskii LP and Zakharov VE 1984, *Theory of Solitons: The Scattering Method*, Contemporary Soviet Mathematics 276, NY Consultants Bureau

[38] Ward RS 1988, *J. Math. Phys.*, **29** 386

[39] Atiyah MF and Hitchen NJ 1985, *Phys. Lett.*, **107A** 21

[40] Atiyah MF and Hitchen NJ 1985, *Phil. Trans. R. Soc. Lond. A*, **315** 459

[41] Hitchen NJ 1987, *Séminaire de Mathématiques Supérieures, Montreal*, **105**

[42] Chakravarty S and Ablowitz MJ 1991, *Painlevé Transcendents, their Asymptotic and Physical Applications*, Proceeding, SC. Adèle, Québec, 1990. Eds. Levi D and Winternitz P, Plenum New York

[43] Mason LJ and Woodhouse NMJ 1996, *Integrability, Self-Duality and Twistor Theory*, LMS Monographs, New Series, Clarendon Press, Oxford

[44] Atiyah MF and Ward RS 1977, *Commun. Math. Phys.*, **55** 117

[45] Corrigan EF, Fairlie DB, Yates RG and Goddard P 1978, *Commun. Math. Phys.*,**58** 223

[46] Ward RS 1981, *Commun. Math. Phys.*, **80** 563

[47] Belavin AA, Polyakov AM, Schwartz AS and Tyupkin Yu S 1975, *Phys. Lett.*, **B59** 85

[48] t'Hooft G 1976, *Phys. Rev. Lett.*, **37** 8

[49] Wilczec F 1977, in *Quark Confinement and Field Theory*, eds. Stump D and Weingarten D, John Wiley and Sons, New York

[50] Corrigan EF and Fairlie DB 1977, *Phys. Lett.* **67B** 69

[51] Jackiw R, Nohl C and Rebbi C 1977 *Phys. Rev.*, **D15** 1642 1978 in *Particles and Fields* 199, eds. Boal D and Kamal AN, Plenum Publishing Co., New York

[52] Witten E 1977, *Phys. Rev. Lett.*, **38** 121

[53] Oh P 1998, *J. Phys. A: Math. Gen.* **31** L325

[54] Lakshmanan M 1977, *Phys. Lett.*, **61A** 1 53

[55] Corones J 1977, *Phys. Rev.*, **16B** 1763

[56] Zakharov VE and Takhtajan L 1979, *Teor. Mat. Fiz.*, **38** 27

[57] Lund F 1984, *J. Math. Phys.*, **25** (2) 256

[58] Honerkamp J 1981, *J. Math. Phys.* **22** (2) 277

[59] Orfandis SJ 1980, *Phys. Rev.*, **21D** 1513

[60] Orfandis SJ 1980, *Phys. Lett.*, **75A** 304

[61] Makhankov VG and Pashaev OK 1982, *Teor. Mat. Fiz*, **53** 55, 1983 *Phys. Lett.*, **95A** 95

[62] Fordy AP and Kulish PP 1983, *Commun. Math. Phys.*, **89** 427

[63] Faddeev LD and Takhtajan LA 1987, *Hamiltonian Methods in the Theory of Soltions*, Springer-Verlag, Berlin

[64] Belavin AA and Polyakov AM 1975, *JETP Lett.*, Vol **22**, 10, 245

[65] Kundu A 1982, *Lett. Math. Phys.* **6** 479

[66] Kundu A 1986, *J. Phys. A: Math. Gen.* **19** 1303

[67] Kundu A and Pashaev OK 1983, *J. Phys. C: Solid State Phys.*, **16** L585

[68] Nakamura K and Sasada T 1982, *J. Phys. C: Solid State Phys.*, **15** L915

[69] Ovchinnikov A, Ukrainski I and Kvintsel G 1972, *Usp. Fiz. Nauk (USSR)* **108** 81

[70] Pashaev OK and Lee J-H 1998, *Resonance NLS Solitons as Black Holes in Madelung Fluid*

[71] Martina L, Pashaev OK and Soliani G 1997, *Class. Quantum Grav.* **14** 3179. 1998, *Phys. Rev. D*: **58** 084025

[72] Lee J-H and Pashaev OK 1997, *Abelian Gauge Theory and Integrable σ-Models*, MIAS 97-3

[73] Pashaev OK 1997, *Nucl. Phys. B (Proc. Suppl.)* **57** 338,

[74] Abramowitz M and Stegun IA 1964, *Handbook of Mathematical Functions*, Washington: National Bureau of Standards

[75] Drazin PF and Johnson RS 1989, *Soltions: an introduction* Cambridge University Press

[76] Ince EL 1927, *Ordinary Differential Equations*, Longmans, Green and Co. Ltd.

[77] Eichenherr H and Honerkamp J 1981, *J. Math. Phys.* **22** (2) 374

[78] de Vega HJ and Sanchez N 1993, *Phys. Rev. D* **47** (8) 3394

[79] Ward RS 1997, *Integrable Systems and Twistors*, Lectures Series; Integrable Systems Workshop, Oxford

[80] Yang CN and Yang CP 1966, *Phys. Rev.* **150** 321, 327

[81] Nesterenko VV 1982, *Dubna Preprints* JINR-E2-82-745,

[82] Gradshteyn IS, Ryzhik IM 1980, *Tables of Integrals, Series and Products*, Academic Press, Inc., JINR-E2-82-76

[83] Mazur PO 1983, *Acta Physica Polonica* **B**14 219

[84] Mazur PO 1984, *Physics Letters* **100A** 341

[85] Lambert D, Rembielinski J 1988, *J. Phys. A: Math. Gen.* **21** 2677

[86] Antoine J-P, Piette B 1987, *J. Math. Phys.* **28** (11) 2753, *J. Math. Phys.* **29** (7) 1687

[87] Zakrzewski WJ, 1984, *J. Geom. Phys.* **1**, 39

[88] Gladikowski J 1997, *PhD. Thesis: Solitons in Low Dimensional Sigma Models*, University of Durham

[89] Oh P and Park Q-H 1996, *Phys. Lett. B*, **383** 333

[90] de Vega HJ 1979, *Phys. Lett.* **87B** 233

[91] Brezin E, Itzykson C, Zinn-Justin J and Zuber J-B, 1979, *Phys. Lett.* **82B** 442

[92] Hirota R. 1971, *Phys. Rev. Lett.* **27**, 1192

[93] Bäcklund AV 1876, *Math. Ann.* **17**, 285

[94] Ablowitz MJ and Segur 1981, *"Solitons and the Inverse Scattering Transform"* SIAM, Philadelphia.

[95] Myrzakulov R 1987, *Pre-print*, (Alma-Ata:HEPI)