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The Companion Equations and the Moyal-Nahm Equations

Linda Margaret Baker

A thesis presented for the degree
of Doctor of Philosophy

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Abstract

The Companion Equations and the Moyal-Nahm Equations
Ph.D. thesis submitted by Linda Baker, August 2000.

The first part of this thesis is concerned with the companion equations. These are equations of motion for the companion Lagrangian which is proposed to be the Lagrangian for a field theory associated with strings and branes, similar to the Klein-Gordon field description for particles. The form of this Lagrangian can be related to the Hamilton-Jacobi formalism for strings and branes. Some solutions to the companion equations are found and their integrability is discussed.

There is an equivalence between the equations of motion for different companion Lagrangians when some constraints are applied. Under these constraints, the companion equations for a Lagrangian without a square root are equivalent to the companion equations for a Lagrangian with a square root but in one dimension less.

The appearance of Universal Field Equations, generalised Bateman equations, in the companion equations leads to the study of an iterative procedure for Lagrangians which are homogeneous of weight one in the first derivatives in the fields the theory describes. The Universal Field equations appear after several iterations.

Also, it is shown how Lagrangians for a large family of field theories are a divergence or vanish on the space of solutions of the equations of motion. Such theories could be called 'pseudo-topological'.

The second part of this thesis is concerned with finding solutions to the Moyal-Nahm equations in four and eight dimensions. These equations are the Nahm equations, which give a set of solutions to self-dual Yang-Mills, but with the commutators replaced with Moyal brackets. Solutions are found in terms of generalised Wigner functions. Also, matrix representations of the algebra generated by the equivalent Nahm equations in eight dimensions are obtained. Solutions to the Nahm equations in eight dimensions are also given.

Declaration

This thesis is the result of research carried out by the author between October 1997 and August 2000 in the Department of Mathematical Sciences at the University of Durham. No part of this thesis has been submitted for a degree at this or any other university.

Chapter 1, section 2.2, sections 4.1, 4.2 and sections 6.1 to 6.3 all contain necessary background material and no claim of originality is made. The remaining work is believed to be original, unless otherwise stated.

Chapters 2-5 are based on joint work with my supervisor, Professor David Fairlie, and this can be found in [1] and [2], the first of which is published. Chapter 6 is based on joint work with my supervisor which has been published in [3]. The proof in Appendix A is my own work. It is based on [4] which is, as yet, unpublished.

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Finally, thanks to Geoff - just for being there.

Contents

Chapter 1

Introduction

This thesis contains work on two topics. The first is a proposal for a field theory associated with strings and branes with equations motion which have been named the companion equations. The second part is concerned with finding solutions to a set of equations known as the Moyal-Nahm equations. This introduction gives a brief review of some of the background behind the work and introduces some other useful topics that will be needed later.

1.1 Particles, Strings and Branes

A lot of the current research in theoretical high energy physics is based on the study of extended objects called strings and branes. In the first part of this thesis, we develop a field theory which can be associated with these strings and branes. To do this, we first need to explain what strings and branes are. These objects are reviewed in [5][6].

1.1.1 Particles

It is easiest to first consider point particles. Consider a relativistic particle in d-dimensional space-time. It is a zero-dimensional object which traces out a one-dimensional trajectory in space-time, a world-line. This world-line can be parameterised by one parameter, τ say. The motion of the particle can be described by d functions $X^\mu(\tau)$ where $\mu = 0, 1, \ldots, d-1$.

The action for a particle is given by

$$S = \int \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\right)^2} \mathrm{d}\tau. \tag{1.1}$$

Summation over indices is assumed here and throughout the thesis, unless specified otherwise. The equations of motion can be found by minimising this action with respect to X^μ.

1.1.2 Strings

A string is a one-dimensional object which traces out a two-dimensional world-sheet in d-dimensional space-time. This world-sheet can be parametrised by two coordinates (σ, τ). The motion of the string is described by d functions of these coordinates $X^\mu(\sigma, \tau)$, where $\mu = 0, 1, \ldots, d-1$. Strings can be either open or closed. As their names suggest, closed strings form a loop and open strings have two ends.

A natural extension of the particle action is to take the string action to be

$$\begin{aligned} S &= \int \sqrt{\left(\frac{\partial X^\mu}{\partial \sigma}\right)^2 \left(\frac{\partial X^\nu}{\partial \tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \tau}\right)^2} \mathrm{d}\sigma \mathrm{d}\tau \\ &= \int \sqrt{\det \left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|} \mathrm{d}\sigma \mathrm{d}\tau \qquad \sigma^i = (\sigma, \tau), \quad i = 1, 2. \end{aligned} \tag{1.2}$$

This is the Nambu-Goto action for a string. The sign under the square root changes according to whether the theory is for Euclidean or Minkowski space-time. The action given above is the Euclidean version. When dealing with Minkowski space-time the action is

$$S = -\int \sqrt{-\det \left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|} \mathrm{d}\sigma \mathrm{d}\tau \qquad \sigma^i = (\sigma, \tau), \quad i = 1, 2. \tag{1.3}$$

However, there is also another action which is classically equivalent to the Nambu-Goto action. This is the Schild Lagrangian [7],

$$\mathcal{L} = \left(\frac{\partial X^\mu}{\partial \sigma}\right)^2 \left(\frac{\partial X^\nu}{\partial \tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \tau}\right)^2, \tag{1.4}$$

which is the square of the Nambu-Goto Lagrangian. The equations of motion for the Schild Lagrangian imply that the Lagrangian is a constant. If it is a non-zero constant then the equations of motion are classically equivalent to the equations of motion from Nambu-Goto Lagrangian. If the constant is zero then we get a new set of solutions, the null strings. The Nambu-Goto Lagrangian cannot admit solutions where the Lagrangian is equal to zero.

There is an analogue for this in the particle case. Consider the two Lagrangians

$$\mathcal{L}_1 = \sqrt{\left(\frac{\partial X^\mu}{\partial\tau}\right)^2}, \qquad \mathcal{L}_2 = \left(\frac{\partial X^\mu}{\partial\tau}\right)^2. \tag{1.5}$$

The equations of motion for the second Lagrangian imply that the Lagrangian is a constant. If it is a non-zero constant then the equations of motion imply the equations of motion for the first Lagrangian. However, the first Lagrangian does not permit solutions where the Lagrangian is zero. It only allows time-like and space-like solutions. The second Lagrangian allows time-like, space-like and null solutions since the Lagrangian can be zero.

The main problem with the Schild Lagrangian is that it is not reparametrisation invariant, in contrast with the Nambu-Goto Lagrangian which is. However, it has been used in the literature by Eguchi [8], to quantise the string, and by Nambu [9], to find a generalisation of Hamiltonian dynamics for strings.

1.1.3 Branes

A p-brane is a p-dimensional object which traces out a $(p{+}1)$-dimensional world-volume in d-dimensional space-time. This world-volume is parameterised by $p{+}1$ coordinates σ^i, where $i = 0, 1, \ldots, p$. The p-brane is described by d functions $X^\mu(\sigma^i)$ where $\mu = 0, 1, \ldots, d-1$. A 0-brane is a point particle and a 1-brane is a string.

Dirac-Born-Infeld Action

The action for a p-brane is

$$S = -T_p \int \mathrm{d}^{p+1}\sigma\; e^{-\Phi}\sqrt{-\det|G_{ij} + B_{ij} + 2\pi\alpha\prime F_{ij}|}. \tag{1.6}$$

This is the Dirac-Born-Infeld action [10][11]. G_{ij} is the induced metric on the brane given by

$$G_{ij} = \frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X^\nu}{\partial\sigma^j}G_{\mu\nu}, \tag{1.7}$$

where $G_{\mu\nu}$ is the space-time metric. B_{ij} is the pullback of the antisymmetric tensor $B_{\mu\nu}$ on the brane.

$$B_{ij} = \frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X^\nu}{\partial \sigma^j}B_{\mu\nu}. \tag{1.8}$$

F_{ij} is the antisymmetric field strength tensor for the $U(1)$ gauge field $A^i(\sigma^j)$ living on the brane. $e^{-\Phi}$ gives the dilaton dependence and T_p is the tension of the p-brane.

Often, the choice is made to split the induced metric G_{ij} into two parts by picking what is know as the static gauge. The world-volume coordinates are chosen to be equal to the first $p+1$ target space coordinates. The remaining target space coordinates are the transverse coordinates and are labelled y^m, say. So we have made the choice

$$X^i = \sigma^i, \qquad i = 0, 1, \dots p \tag{1.9}$$

$$X^m = y^m, \qquad m = p+1, \dots d-1. \tag{1.10}$$

The induced metric can now be written as

$$G_{ij} = \eta_{ij} + \frac{\partial y^m}{\partial \sigma^i}\frac{\partial y_m}{\partial \sigma^j}. \tag{1.11}$$

Although this choice helps with some calculations, it is harder to see some of the properties of the action. In this work, such a choice will not be made and as a result it is easier to see some of the more global properties of the theory and its equations of motion.

This action arose from the Born-Infeld action [10] which was first proposed as a non-linear theory for electrodynamics. Born-Infeld theory allows finite energy solutions. A pure Born-Infeld action takes the form [12]

$$S = \int \mathrm{d}^d x \sqrt{\det|\eta_{\mu\nu} + F_{\mu\nu}|}. \tag{1.12}$$

$\eta_{\mu\nu}$ is the space-time metric and $F_{\mu\nu}$ is the electromagnetic field strength. The classical action for a string in d dimensions is the same as the d-dimensional Born-Infeld action [13].

The original idea for branes came from [14]. Dai *et.al.* considered p-dimensional membrane type objects which had Dirichlet boundary conditions in some directions. These were named D-branes. A D-brane is an extended object that open strings can end on [15]. This was not an entirely new idea since Dirichlet boundary conditions had been considered for strings previously [16]. The Dirac-Born-Infeld action was found to give the required classical equations of motion for these D-branes [17].

The action (1.6), given above, is for arbitrary fields in arbitrary space-time. For much of the work in the following chapters we will be considering the case where there is flat space-time and no antisymmetric part to the action. For this choice, the action is of the form

$$S = \int \mathrm{d}^{p+1}\sigma \sqrt{\det \left| \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X_\mu}{\partial \sigma^j} \right|}. \tag{1.13}$$

This is the higher dimensional analogue of the Nambu-Goto action for strings (1.2). This Lagrangian can either be written as the square root of a determinant (as above) or as the square root of a sum of squares of Jacobians,

$$\mathcal{L} = \sqrt{\frac{1}{(p+1)!} \left(\frac{\partial(X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_{p+1}})}{\partial(\ \sigma^1\ ,\ \sigma^2\ , \ldots\ ,\ \sigma^{p+1}\ )} \right)^2}. \tag{1.14}$$

There are always more target space or dependent variables, X^μ, than there are base space or independent coordinates, σ^i. One motivation behind the structure of the companion Lagrangian, which appears in later chapters, is that it is of the same structure as the Dirac-Born-Infeld Lagrangian but the number of dependent coordinates, ϕ^i, is less than the number of independent coordinates, x^μ. Such a Lagrangian can still be written as the square root of a determinant or the sum of squares of Jacobians,

$$\mathcal{L} = \sqrt{\det \left| \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x_\mu} \right|} = \sqrt{\frac{1}{(p+1)!} \left(\frac{\partial(\ \phi^1,\ \phi^2, \ldots\ , \phi^{p+1}\ )}{\partial(x^{\mu_1}, x^{\mu_2}, \ldots, x^{\mu_{p+1}})} \right)^2}. \tag{1.15}$$

It should also be noted that if the antisymmetric F_{ij} terms are put back into the Dirac-Born-Infeld Lagrangian, then it can still be written as the square root of the sum of squares [18]. For example, if p=3,

$$\begin{aligned}\mathcal{L} &= \sqrt{\det \left| \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X_\mu}{\partial \sigma^j} + F_{ij} \right|} \\ &= \left[\frac{1}{4!} \left(\epsilon_{ijkl} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \frac{\partial X^\rho}{\partial \sigma^k} \frac{\partial X^\eta}{\partial \sigma^l} \right)^2 + \frac{1}{8} \left(\epsilon_{ijkl} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} F_{kl} \right)^2 \right. \\ &\qquad \left. + \left(\frac{1}{8} \epsilon_{ijkl} F_{ij} F_{kl} \right)^2 \right]^{\frac{1}{2}} .\end{aligned} \tag{1.16}$$

1.2 Field Theory and the Klein-Gordon Equation

Fundamental theories of matter need to be consistent with both relativity and quantum mechanics. Field theories make this possible. The need for field theories can be seen by considering a single particle relativistic wave equation, such as the Klein-Gordon equation [19]. This describes a particle with no spin, a scalar particle.

It arises from the energy-momentum equation for a relativistic particle,

$$E^2 - \mathbf{p}^2 = m^2. \tag{1.17}$$

E is the energy, $\mathbf{p}$ is 3-momentum and m is the mass of the particle. The convention $c = 1$, $\hbar = 1$ is assumed. Using the correspondence principle to make the substitution $E \to i\frac{\partial}{\partial t}$, $\mathbf{p} \to -i\nabla$ and letting these operators act on the one component wavefunction $\phi(x)$ we find that (1.17) becomes

$$(\Box + m^2)\phi = 0. \tag{1.18}$$

This is the Klein-Gordon equation.

Unfortunately, there are problems with interpreting this as a single particle wave equation. These include the existence of negative energy solutions, a probability amplitude which is sometimes negative (a probability, by definition, cannot be negative) and the violation of causality. These problems initially led to the Klein-Gordon equation being abandoned.

However, by interpreting the Klein-Gordon equation as a field equation these problems are solved. Such a theory allows the number of particles in the theory to be non-constant. It allows pair creation and the existence of multi-particle states and virtual particles. This in turn removes the problems of negative energy and causality violation.

In the chapters that follow, the Klein-Gordon equation is a field equation. When proposing a field theory for strings and branes the initial idea was to generalise the Klein-Gordon equation, which is for particles, to a theory for higher dimensional objects.

1.2.1 Lagrangian Field Theory

Much of this thesis is concerned with the Lagrangians of various field theories. In classical mechanics, one of the fundamental quantities is the action, S, which is the time integral of the Lagrangian L of a dynamical system [20]. It is also the integral of the Lagrangian density, $\mathcal{L}$, over space-time. $\mathcal{L}$ is a function of the field $\phi(x^\mu)$ and its first derivatives $\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$.

$$S = \int L \, \mathrm{d}t = \int \mathcal{L}(\phi, \partial_\mu \phi) \, \mathrm{d}^4 x. \tag{1.19}$$

From now on, the Lagrangian density $\mathcal{L}$ will be referred to as the Lagrangian.

To find the equations of motion for a theory we use the principle of least action which basically says that as a system evolves between two times, t_1 and t_2, the action S is extremised. It is usually a minimum. This condition can be imposed by setting $\delta S = 0$. Therefore,

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\} \mathrm{d}^4 x \\ &= \int_{t_1}^{t_2} \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \phi \right) \right\} \mathrm{d}^4 x = 0. \end{aligned} \tag{1.20}$$

The last term can be written as a surface term. Since the initial and final field configurations are fixed then $\delta \phi = 0$ at $t = t_1, t_2$. Therefore this term vanishes. Since the remainder must vanish for arbitrary $\delta \phi$ then we find

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{1.21}$$

This is the Euler-Lagrange equation of motion for a field ϕ. This is easily extended for a field theory with n fields ϕ^i, $i = 1, 2, \ldots, n$, with Lagrangian $\mathcal{L}(\phi^i, \partial_\mu \phi^i)$. In this case, there are n equations of motion written as

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) = 0 \qquad i = 1, 2, \ldots, n. \tag{1.22}$$

These equations of motion will be used extensively when finding the equations of motion for the companion Lagrangian which depends on the derivatives of n fields ϕ^i.

Lagrangian for the Klein-Gordon Equation

Consider the Lagrangian for a field $\phi(x^\mu)$,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{1.23}$$

Putting this into the Euler-Lagrange equations of motion (1.21) gives

$$(\Box + m^2) \phi = 0. \tag{1.24}$$

This is the Klein-Gordon equation. Therefore (1.23) is the Lagrangian for the Klein-Gordon equation. It is this Lagrangian we will be generalising when we propose the companion Lagrangian for strings and branes.

Quantisation

When quantised, the Klein-Gordon equation is a quantum field theory which allows many particle solutions. The number of particles is a quantum variable. The first part of this thesis looks at developing a field theory for strings which is similar to the Klein-Gordon field theory for particles. This is not an entirely new idea. Morris [21, 22] tried to develop a field theory for strings and used the quantisation of string theory as his main motivation. The idea was to find a field theory where it was not necessary to specify the number strings. The number of strings was a quantum number and neither the strings nor the world-sheet appeared explicitly in the formulation. In this respect the theory was analogous to the theory for the Klein-Gordon equation. However, one of the main problems with this idea, and a similar idea of Hosotani [23], was that their formulations in the particle case did not resemble the Klein-Gordon case. Instead of a theory with one field ϕ, there were many fields.

A later idea of Hosotani and Nakayama [24] was also partially motivated by the search for a quantum string theory. Their idea was to use the classical Hamilton-Jacobi equation for strings in order the find a quantum field theory for strings and p-branes. The Hamilton-Jacobi equation can be viewed as the classical limit for a quantum theory. The Hamilton-Jacobi equations for strings and branes will be used as a further motivation for the companion Lagrangian for strings and branes.

It should be noted that all these ideas for field theories for strings, and the theory involving companion Lagrangian to be proposed in this thesis, are different from string field theory [25][26]. In string field theory, the field is a functional $\Psi[X^{\mu}(\sigma), p_{+}, \tau]$ which depends on the curve traced out by the string, $X^{\mu}(\sigma)$, and the string length, p_{+}.

1.3 Bateman Equation

When the equations of motion for the companion Lagrangian are studied, they often take the form of what is known as the Bateman equation, or equations related to the Bateman equation. This section looks at what this equation is, what its properties are and how it can be generalised.

The Bateman equation is

$$\left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial x^2}-2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}+\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial y^2}=0, \tag{1.25}$$

for a field $\phi(x,y)$ in two dimensions. It first appeared in [27] where Bateman discussed hydrodynamics. Using the notation

$$\phi_x=\frac{\partial\phi}{\partial x},\quad \phi_y=\frac{\partial\phi}{\partial y},\quad \phi_{xx}=\frac{\partial^2\phi}{\partial x^2},\quad \phi_{xy}=\frac{\partial^2\phi}{\partial x\partial y},\quad \phi_{yy}=\frac{\partial^2\phi}{\partial y^2}, \tag{1.26}$$

it can also be written as a determinant

$$\det\begin{vmatrix}0 & \phi_x & \phi_y\\ \phi_x & \phi_{xx} & \phi_{xy}\\ \phi_y & \phi_{xy} & \phi_{yy}\end{vmatrix}=0. \tag{1.27}$$

The Bateman equation has many important properties. It is not only invariant under Euclidean (Lorentz) coordinate transformations but is also invariant under general linear transformations of the group $GL(2,R)$. Also, if ϕ is a solution to the Bateman equation then so is any function, $f(\phi)$ say, of ϕ. This means the equation is covariant, a property which will be desirable in our field theory.

The general solution to the Bateman equation is the solution for ϕ of the following equation,

$$xF(\phi)+yG(\phi)=c, \tag{1.28}$$

where F and G are arbitrary functions of ϕ. c can be any constant, including zero.

The Bateman equation is equivalent to the Monge nonlinear wave equation

$$\frac{\partial u}{\partial x}=u\frac{\partial u}{\partial y},\qquad \text{where}\qquad u\doteq\frac{\phi_x}{\phi_y}. \tag{1.29}$$

This is a first order differential equation. From the this equation is easy to show that the Bateman equation possesses an infinite number of conservation laws since (1.29) implies that

$$\frac{\partial u^n}{\partial x}=\frac{\partial}{\partial y}\left(\frac{n}{n+1}u^{n+1}\right). \tag{1.30}$$

This property leads to the fact that the Bateman equation is completely integrable.

The general solution to the Monge equation is given by solutions to the equation

$$u=W(y+ux), \tag{1.31}$$

where W is an arbitrary function. The general solution to the Bateman equation can be derived from this since

$$W^{-1}(u) = y + ux. \tag{1.32}$$

Then, put $u = V(\phi)$, where V is some function of ϕ. This is allowed since

$$u = \frac{u_x}{u_y} = \frac{V_x}{V_y} = \frac{\phi_x}{\phi_y}, \tag{1.33}$$

which is consistent with (1.29). Therefore,

$$\begin{aligned} W^{-1}(V(\phi)) &= y + V(\phi)x \\ 1 &= \frac{1}{W^{-1}(V(\phi))}y + \frac{V(\phi)}{W^{-1}(V(\phi))}x. \end{aligned} \tag{1.34}$$

This is equivalent to $xF(\phi) + yG(\phi) = c$ where F and G are arbitrary functions, as required.

The Bateman equation can also be derived from the three dimensional Laplace wave equation when this is subject to the constraint that the gradient of ϕ is a null vector.

$$\phi_{xx} + \phi_{yy} \pm \phi_{zz} = 0, \tag{1.35}$$

$$\phi_x^2 + \phi_y^2 \pm \phi_z^2 = 0. \tag{1.36}$$

To show this, simply eliminate the ϕ_z and ϕ_{zz} from the above. It is an extension of this property which leads to an equivalence theorem between the equations of motion for the companion Lagrangians, with and without square roots, in different dimensions. It should be noted that the left hand side of (1.36) is the Lagrangian for equations of motion which take the form of (1.35).

Finally, any Lagrangian which is homogeneous of weight one in the derivatives $\phi_\mu = \frac{\partial\phi}{\partial x^\mu}$, $\mu=1,2$, has the Bateman equation as its equation of motion. If a Lagrangian, $\mathcal{L}$, is a homogeneous function of weight m in the derivatives ϕ_μ then it satisfies

$$\phi_\mu \frac{\partial \mathcal{L}}{\partial \phi_\mu} = m\mathcal{L}. \tag{1.37}$$

This equation will be made use of later.

1.3.1 Generalising the Bateman Equation

Work has been done to find other field theories which have fully covariant solutions, just like the Bateman equation does. This has been achieved by generalising the

Bateman equation to more fields and more dimensions resulting in what are known as the Universal Field Equations [28][29].

There are two ways to do this. One is just to increase the number of dimensions. Generalising the determinantal structure of the Bateman equation we can construct an equation for a field $\phi(x^\mu)$ in d dimensions,

$$\det \begin{vmatrix} 0 & \phi_1 & \phi_2 & \cdots & \phi_d \\ \phi_1 & \phi_{11} & \phi_{12} & \cdots & \phi_{1d} \\ \phi_2 & \phi_{12} & \phi_{22} & \cdots & \phi_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_d & \phi_{1d} & \phi_{2d} & \cdots & \phi_{dd} \end{vmatrix} = 0. \tag{1.38}$$

If ϕ is a solution to this equation then so is any function of ϕ, so the equation is covariant, as required. The notation used is

$$\phi_\mu = \frac{\partial \phi}{\partial x^\mu}, \qquad \phi_{\mu\nu} = \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu}. \tag{1.39}$$

Such notation will also be used elsewhere in the thesis.

The other way is to increase the number of fields (and the number of dimensions). In particular, we shall be considering the case of n fields in $n+1$ dimensions. For two fields in three dimensions the generalisation in determinantal form is

$$\det \begin{vmatrix} 0 & 0 & \phi_x & \phi_y & \phi_z \\ 0 & 0 & \psi_x & \psi_y & \psi_z \\ \phi_x & \psi_x & \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_y & \psi_y & \phi_{xy} & \phi_{yy} & \phi_{yz} \\ \phi_z & \psi_z & \phi_{xz} & \phi_{yz} & \phi_{zz} \end{vmatrix} = 0, \tag{1.40}$$

where the fields are ϕ and ψ and the space-time coordinates are $(x,\ y,\ z)$. In general, for n fields, ϕ^i, in $n+1$ dimensions, $\{x_\mu\}$, the Universal Field Equations can be written as

$$J_\mu J_\nu \phi^i_{\mu\nu} = 0 \qquad i = 1, \ldots, n, \tag{1.41}$$

where $J_\mu = \epsilon_{\mu\nu_1\nu_2\ldots\nu_n}\phi^1_{\nu_1}\phi^2_{\nu_2}\cdots\phi^n_{\nu_n}$. J_μ is a Jacobian and could also be written as $\frac{\partial(\phi^1,\phi^2,\ldots,\phi^n)}{\partial(x^{\nu_1},x^{\nu_2},\ldots,x^{\nu_d})}$. Later on, the notation for a Jacobian for n fields $\phi^i(x^\mu)$ in d dimensions will be

$$J_{\mu_1\mu_2\ldots\mu_{d-n}} = \epsilon_{\mu_1\mu_2\ldots\mu_{d-n}\nu_1\nu_2\ldots\nu_n}\phi^1_{\nu_1}\phi^2_{\nu_2}\cdots\phi^n_{\nu_n}. \tag{1.42}$$

The Bateman equation and its generalisations, the Universal Field Equations, appear in the equations of motion for the companion Lagrangians we will be considering. They are also involved in the iterative procedure to be studied in a later chapter. In this procedure, each iteration involves multiplying by some function and then applying the Euler operator $\mathcal{E}$ which is the operator which gives the equations of motion.

For one field ϕ, the required function is already known to be the Lagrangian $\mathcal{L}$ [28]. In this case, the iterative procedure is

$$\mathcal{E}\mathcal{L}, \qquad \mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}, \qquad \mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}, \qquad \ldots \tag{1.43}$$

where the Euler operator is

$$\mathcal{E} = -\frac{\partial}{\partial\phi} + \partial_\mu \frac{\partial}{\partial\phi_\mu} - \partial_\mu\partial_\nu \frac{\partial}{\partial\phi_{\mu\nu}} + \cdots . \tag{1.44}$$

For a theory in d dimensions, after $d-1$ iterations we obtain the Universal Field equation for one field in d dimensions (1.38). Part of this thesis is concerned with generalising this procedure to more than one field with the aim of obtaining the Universal Field equations after a finite number of iterations.

1.4 Topological Field Theories

We will also be looking at a property of the companion Lagrangian which extends to other field theories. This property is that for a large family of field theories, the Lagrangian of the theory vanishes or is a divergence on the space of solutions of its equations of motion. A large set of examples will be given. The fact that we obtain a divergence, leads to describing these theories as 'pseudo-topological'. This is because, for a fully topological theory the Lagrangian is a divergence or zero without having to put any constraints on it. An example of such a topological theory is gravity in two dimensions [30]. In this case the Lagrangian can just be picked to be zero. In our examples of free fields, we need to put in the constraint that the equations of motion are satisfied before the Lagrangian is zero or a divergence. This is where the 'pseudo' part of the name comes from.

1.5 Yang-Mills Fields

The final part of the thesis is concerned with finding solutions to the Moyal-Nahm equations in four and eight dimensions. The Moyal-Nahm equations are the Nahm equations, but with Moyal brackets instead of commutators. Solutions to the Nahm equations give a set of solutions to Yang-Mills theory. In the next few sections we briefly review Yang-Mills fields, Nahm equations, Moyal brackets and give some motivation for studying such topics in more than four dimensions.

Non-abelian gauge theories, *i.e.* theories with a higher symmetry than $SO(2)$ or $U(1)$, can be described by Yang-Mills theory. Work on this was originally done by Yang and Mills in at attempt to treat isospin as a local symmetry [31]. Although this was the wrong thing to do, Yang-Mills theory did successfully describe the $SU(2)$ symmetry of the weak interaction and the $SU(3)$ symmetry of the strong interaction of quarks [32]. It works for other symmetry groups, such as $U(N)$. The easiest symmetry group to consider is $SU(2)$.

The Lagrangian for pure Yang-Mills is

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{1.45}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. A_μ is the gauge field and $F_{\mu\nu}$ is the gauge field strength. The equation of motion for this theory is

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \tag{1.46}$$

D_μ is the covariant derivative.

One way to obtain solutions to pure Yang-Mills is by solving the Nahm equations [33]. Any solution of the Nahm equations is automatically a solution of the full Yang-Mills equation of motion. Again, although Nahm equations can be found for any gauge group, $SU(2)$ is the easiest to consider. In four space-time dimensions, the Nahm equations are

$$\begin{aligned}\frac{\partial A^1}{\partial t} &= [A^2, A^3],\\ \frac{\partial A^2}{\partial t} &= [A^3, A^1],\\ \frac{\partial A^3}{\partial t} &= [A^1, A^2].\end{aligned} \tag{1.47}$$

The gauge choice $A^0 = 0$ has been made.

1.6 M-Theory and M(atrix) Theory

M-Theory

M-Theory is possibly the best candidate we have at present for a 'Theory of Everything' [34]. It appears to have two definitions. One is that it is the eleven-dimensional theory which is the strong coupling limit of Type IIA superstring theory and has eleven dimensional supergravity as its low energy limit. However, its definition is often broadened to be the eleven dimensional quantum theory which has the five different superstring theories as various limits [5][35]. Very little is known about M-Theory and there is obviously much that still needs to be understood. It is also unclear what the M in M-Theory stands for. In the past it has been taken to stand for membrane, matrix, mother, mystery, and magical.

M(atrix) Theory

A big step towards understanding M-Theory was made by Banks, Fischler, Shenker and Susskind when they proposed M(atrix) Theory [36]. Their conjecture was that M-Theory in the infinite momentum frame is equivalent to matrix supersymmetric quantum mechanics for N D0-branes in the $N \to \infty$ limit. It follows from taking 9+1 dimensional $U(N)$ super Yang-Mills theory and dimensionally reducing it to 0+1 dimensions. The infinite momentum frame, in simplest terms, is when the total momentum of the system is very large [37]. The action for this $U(N)$ super-Yang-Mills quantum mechanics is

$$S = \frac{1}{2g} \int \mathrm{Tr} \left(\dot{X}^{\mu} \dot{X}_{\mu} + 2\theta^T \dot{\theta} - \frac{1}{2}[X^{\mu}, X^{\nu}]^2 - 2\theta^T \gamma_{\mu} [\theta, X^{\mu}] \right) \mathrm{d}\tau. \tag{1.48}$$

The X^{μ} ($\mu = 1, \ldots, 9$) are nine $N \times N$ matrices and θ represents the 16 fermionic superpartners. Derivatives with respect to τ are denoted by $\dot{X}^{\mu}$ or $\dot{\theta}$. g is the coupling constant.

Matrix String Theory

A similar approach was later used to construct matrix string theory, to give a two dimensional $\mathcal{N} = 8$ supersymmetric $U(N)$ Yang-Mills theory, rather than a one dimensional theory [38]. The description is now for D1-branes, or strings, instead of D0-branes which are particles. The action for such a theory is

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{Tr} \left((D_i X^{\mu})^2 + \theta^T \not{D} \theta + g_s^2 F_{ij}^2 - \frac{1}{g_s^2}[X^{\mu}, X^{\nu}]^2 + \frac{1}{g_s} \theta^T \gamma_{\mu} [X^{\mu}, \theta] \right) \mathrm{d}\sigma \mathrm{d}\tau. \tag{1.49}$$

The X^{μ} ($\mu = 1, \ldots, 8$) are eight scalar fields and the θ are the eight fermionic fields. They are all $N \times N$ hermitian matrices. (σ, τ) are the world-sheet coordinates. g_s is the string coupling constant. Again, to obtain a description of M-theory, we need to take the $N \to \infty$ limit. This theory gives a new way to approach M-Theory and investigate the string and brane states which occur and their interactions [39]. It has been used to study the high energy scattering processes in M-Theory [40].

It is matrix string theory which provides some of the motivation for studying the Nahm equations in eight dimensions. Matrix string theory involves Yang-Mills theory for eight fields X^μ, for which a set of solutions for Yang-Mills can be found from the Nahm equations in eight dimensions. Therefore, by studying the Nahm equations in eight dimensions we can find a set of solutions for Yang-Mills.

1.7 The Moyal Bracket

The Moyal Bracket for two functions $f(x,p)$ and $g(x,p)$ on two-dimensional phase space (x,p) is defined as [41]

$$\{f,g\}_{MB} = \frac{1}{2i}(f \star g - g \star f). \tag{1.50}$$

The star, $\star$, denotes the star product which is defined as

$$\star = \exp\left[i\lambda\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial x}\right)\right]. \tag{1.51}$$

The Moyal bracket is a one parameter deformation of the Poisson bracket, where λ is the deformation parameter. The Poisson bracket is written as

$$\{f,g\}_{PB} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{1.52}$$

In the limit $\lambda \to 0$, the Moyal Bracket is just the Poisson bracket.

The Moyal brackets can also be associated with commutators [42]. If the deformation parameter is set to be $\lambda = \frac{2\pi}{N}$, where N is an odd integer, then the Moyal bracket of two functions $\{X^\mu, X^\nu\}_{MB}$ reproduces the commutators of $N \times N$ matrices, A^μ. These are $SU(N)$ matrices and the matrix components of A^μ are the fourier modes of the functions X^μ. As $N \to \infty$ then $\lambda \to 0$, so in the large N limit the Moyal Bracket is the Poisson bracket. Therefore the Poisson bracket can be identified with the commutator of $SU(\infty)$ matrices. The link between Moyal Brackets, Poisson brackets and commutators will become more apparent in the examples that follow.

Quantum Mechanics

Moyal Brackets were first used to write down a formulation of Quantum Mechanics [43]. The way to incorporate the Moyal brackets is to replace all ordinary multiplication with the star product. Therefore, all commutators in the usual formulation of quantum mechanics are now Moyal Brackets. The deformation parameter is set to be $\hbar$. Finding the classical limit of the theory is then both easy and natural. It simply amounts to taking the limit $\hbar \to 0$. In this limit, all the star products are reduced to ordinary multiplication again.

Non-commutative Geometry

Recent work of Seiberg and Witten [44] has led to a large number of papers on the subject of non-commutative geometry and the use of the star product and the Moyal bracket. The Seiberg and Witten paper showed that there was an equivalence between ordinary Yang-Mills and non-commutative Yang-Mills for open strings in a constant non-zero B-field. This work resulted in many papers being written where ordinary multiplication was replaced with the star product in order to make the theory non-commutative. In such papers it is space-time which is non-commutative.

For example, in [45] a non-commutative scalar theory for field ϕ in 2+1 dimensions is studied. Consider the theory where the energy is given by

$$E = \frac{1}{g^2}\int \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi)\right) \mathrm{d}^2x. \tag{1.53}$$

$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{3}\lambda\phi^3 + \ldots$ is the potential term. To turn this ordinary scalar field theory into a non-commutative one, then the spatial coordinates become non-commutative such that

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \tag{1.54}$$

where $\theta_{\mu\nu}$ (μ,ν=1,2) are the components of a totally antisymmetric matrix. Let $\theta_{12} = \theta$. This θ is then the deformation parameter in the star product. We now put star products in place of ordinary multiplication to give

$$E = \frac{1}{g^2}\int \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{3}\lambda\phi\star\phi\star\phi + \ldots\right) \mathrm{d}^2x. \tag{1.55}$$

Note that we do not need star products for the quadratic terms because when the integral over the whole space is taken the following property holds:

$$\int f(x_1,x_2)\star g(x_1,x_2)\ \mathrm{d}^2x = \int f(x_1,x_2)g(x_1,x_2)\ \mathrm{d}^2x. \tag{1.56}$$

Therefore, we only need put in the star product for cubic terms and above. If we were not considering the integral then the star product would need to be put in everywhere. The potential would be $V(\phi) = \frac{1}{2}m^2\phi \star \phi + \frac{1}{3}\lambda\phi \star \phi \star \phi + \ldots$, for example. Similar procedures have been used to study other non-commutative field theories.

M(atrix) theory

Moyal brackets can also be used in M(atrix) theory [46]. They can be used to give a new interpretation to the $N \to \infty$ limit which needs to be taken in order to recover M-Theory from M(atrix) theory. By setting the deformation parameter to be $\lambda = \frac{2\pi}{N}$ and replacing all multiplication by star products we have an action containing Moyal Brackets. Taking the large N limit all the Moyal Brackets become Poisson brackets. Therefore, a new way of looking at the $N \to \infty$ limit of M(atrix) theory is to have a theory with Poisson brackets, not commutators.

Nahm Equations

A similar procedure is used to turn the Nahm equations (1.47) into the Moyal-Nahm equations. As with the other cases, the ordinary multiplication is replaced with star brackets. This results in the right hand side of the Moyal-Nahm equations being Moyal brackets, as shown below.

$$\begin{aligned}
\frac{\partial A^1}{\partial t} &= \{A^2, A^3\}_{MB}, \\
\frac{\partial A^2}{\partial t} &= \{A^3, A^1\}_{MB}, \\
\frac{\partial A^3}{\partial t} &= \{A^1, A^2\}_{MB}.
\end{aligned} \tag{1.57}$$

These are the Moyal-Nahm equations in four dimensions. The main motivation for studying these is that we live in four large dimensions and the Nahm equations in four dimensions themselves have already been studied in great detail.

However, this thesis is also concerned with finding solutions to the Moyal-Nahm equations in eight dimensions. The main motivation for this arises from the appearance of Yang-Mills in matrix string theory. In this theory, Yang-Mills field theory for eight fields X^μ appears in the action. If solutions to the Nahm equations can be found then these are automatically solutions to the equations from Yang-Mills. Therefore, it makes sense to study the Nahm equations and the Moyal-Nahm equations in eight dimensions. The solutions which are found may have some bearing on matrix string theory and so ultimately on M-Theory. Another reason for considering the Moyal-Nahm equations, rather than just the Nahm equations, is the use of Moyal brackets in a possible interpretation of the large N limit as described above.

1.8 Layout of Thesis

This chapter has been a brief introduction into some of the background material needed for the main part of this thesis. Some of these topics will be expanded later on.

In Chapter 2, we introduce the companion Lagrangian and give some motivation as to why we want to study it. Chapter 3 looks at the equations of motion for this Lagrangian, the companion equations, and discusses their integrability. In Chapter 4 we extend an iterative procedure to Lagrangians for more than one field, such as the companion Lagrangian. Chapter 5 deals with a property of a large set of Lagrangians, not just the companion Lagrangian. This property is that many field theory Lagrangians are zero or a divergence on the space of solutions of the equations of motion. In Chapter 6 we study the Moyal-Nahm equations, explaining what they are and solving them in four and eight dimensions. In Chapter 7 we give the final conclusions to all this work, giving suggestions for further research.

Chapter 2

The Companion Lagrangian

The next few chapters involve a Lagrangian called the companion Lagrangian which has equations of motion known as the companion equations. It is a Lagrangian for a field theory associated with strings and branes. These chapters are based on work in [1][2].

This chapter discusses the motivation behind looking at such a theory and the problems encountered by similar theories in the past. Equivalence theorems between different companion Lagrangians are stated. Finally, we look at the covariance of the theory, the inclusion of a background metric and the possible ways of including electromagnetism in the theory.

2.1 The Big Idea

In quantum mechanics we come across the concept of particle-wave duality. On the quantum level, particles take on wave-like characteristics such as electrons going through slits exhibiting interference effects and waves take on particle-like characteristics, for example, electromagnetic waves being made up of photons. A classical point particle has the Lagrangian

$$\mathcal{L} = \sqrt{\left(\frac{\partial X^\mu}{\partial \tau}\right)^2}, \tag{2.1}$$

but when we go over to quantum mechanics it can be described by a Klein-Gordon field, $\phi(x^\mu)$ which has Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^\mu}\right)^2. \tag{2.2}$$

This is an example of particle-wave duality. A point particle traces out a one-dimensional world-line in space-time which can be parameterised by just one parameter, τ. It should be noted that there is one parameter τ and one field ϕ. The big question is: Is there an alternative description for strings and branes so that they too have a field theory description similar to the Klein-Gordon?

Strings can be described by the Nambu-Goto Lagrangian

$$\mathcal{L} = \sqrt{\left(\frac{\partial X^\mu}{\partial \sigma}\right)^2 \left(\frac{\partial X^\nu}{\partial \tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial \sigma}\frac{\partial X_\mu}{\partial \tau}\right)^2}. \tag{2.3}$$

Strings trace out a two-dimensional world-sheet which is parameterised by two world-sheet coordinates (σ, τ). The conjecture is that they can also be described by a theory with two fields and a Lagrangian which is some power of

$$\mathcal{L} = \left(\frac{\partial \phi}{\partial x^\mu}\right)^2 \left(\frac{\partial \psi}{\partial x^\nu}\right)^2 - \left(\frac{\partial \phi}{\partial x^\mu}\frac{\partial \psi}{\partial x_\mu}\right)^2. \tag{2.4}$$

$\phi(x^\mu)$ and $\psi(x^\mu)$ are the two fields and they depend on the space time coordinates x^μ $(\mu = 1, \dots, d)$. This idea can also be extended to branes which, in simplest form in the absence of a $U(1)$ field, are described by the Lagrangian

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|}. \tag{2.5}$$

A p-brane sweeps out a $p+1$ dimensional world-volume which is parameterised by the $p+1$ world-volume coordinates σ^i $(i = 0, \dots, p)$. The field theory conjectured to be associated with branes has a Lagrangian which is some power of

$$\mathcal{L} = \det\left|\frac{\partial \phi^i}{\partial x^\mu}\frac{\partial \phi^j}{\partial x_\mu}\right|. \tag{2.6}$$

ϕ^i are $p+1$ fields. In every case the number of fields is equal to the number of world-volume coordinates. This is analogous to the particle case where there is one (Klein-Gordon) field and a one dimensional world-line. The new field Lagrangians, (2.4) and (2.6), will be referred to as *companion Lagrangians* and their equations of motion will be the *companion equations.*

Similar ideas have appeared before in the literature. Hosotani [23] considered the case of a string theory in four dimensions and showed this was mathematically equivalent to a scalar field theory with two fields. The equations of motion in both theories are the same. Strings have a world-sheet with coordinates (σ, τ). He introduced two new parameters (S, T) so that (σ, τ, S, T) covered a four-dimensional domain in space-time. The Nambu-Goto Lagrangian for string theory (2.3) was shown to be the same, up to a determinantal factor, as the Lagrangian for a scalar field theory where $S(x^\mu)$ and $T(x^\mu)$ are the two scalar fields and the Lagrangian is

$$\mathcal{L} = \sqrt{\left(\frac{\partial S}{\partial x^\mu}\right)^2 \left(\frac{\partial T}{\partial x^\nu}\right)^2 - \left(\frac{\partial S}{\partial x^\mu}\frac{\partial T}{\partial x_\mu}\right)^2}. \tag{2.7}$$

Morris [21][22] later discussed a very similar idea but extended the number of space-time dimensions. His starting point was to consider the string world-sheet, not as the function of two world-sheet variables, $X^\mu(\sigma,\tau)$, but as the intersection of $d-2$ hypersurfaces, $f_k(x) = 0$. (This is similar to the way a curve can be thought of as the intersection of two planes.) A Lagrangian for a string in d dimensions was related to a Lagrangian for $d-2$ fields. This amounted to an interchange of independent and dependent variables, where the independent variables were complemented with $d-2$ extra variables. He showed the equations of motion involving these fields are mathematically equivalent at a classical level to the equations of motion from the Nambu-Goto Lagrangian. The Lagrangians for the two theories are the same up to a determinantal factor. This was also extended to show the equivalence of a theory of p-branes with Lagrangian (2.5) to a theory with $d-p-1$ fields.

But these field theories differ from the companion Lagrangian idea. In the companion Lagrangian case, the number of fields is equal to the dimension of the world-volume, $p+1$. In the theories of Hosotani and Morris the number of fields is essentially the complement of this, *i.e.* $d-p-1$. However, one of the initial motivations for this work was that the field theory for the strings and branes should be analogous to the particle/Klein-Gordon case where there is one Klein-Gordon field, irrespective of the number of space-time dimensions. This is not the case for the ideas of Hosotani and Morris where a particle in d space-time dimensions would be described by a theory with $d-1$ fields. This would clearly not look like Klein-Gordon theory. Morris was aware that the particle case should look like Klein-Gordon and thought maybe some form of gauge-fixing would solve this problem, but the companion Lagrangian seems a simpler way of achieving this.

More recent work of Hosotani and Nakayama [24] has been done on field theories for strings with two fields (not $d-2$) which is much the same as the companion Lagrangian idea. It is based on the Hamilton-Jacobi equations for strings and branes.

2.2 Hamilton-Jacobi Equations

The work on Hamilton-Jacobi equations for theories with more than one independent variable began with Carathéodory [47] and Velte [48]. Their ideas were developed by Nambu [9, 49], Rund [50], Kastrup [51][52] and Rinke [53] to find Hamilton-Jacobi equations for strings. Rinke [53] was the first to give a derivation for the Hamilton-Jacobi equation for strings. However, for Kastrup and Rinke the motivation was to try to relate string theory to a Maxwell field, not to find a generalisation to the Klein-Gordon field. Hosotani and Nakayama [24] used these Hamilton-Jacobi equations to construct their field theory associated with strings and branes. Similar ideas have also been discussed in [54].

This section is based on work by Nambu [9] which is easier to follow than some of the other papers on this subject. A Hamilton-Jacobi type formalism for strings which can be extended to branes is given. Equations analogous to the Hamiltonian equations and Hamilton-Jacobi equation for a point particle can be obtained for strings. This will give further motivation for the form of the companion Lagrangian.

2.2.1 Point Particles

For a point particle we have the one-form relation

$$dS = \sum_i p_i dx^i - H dt, \qquad \text{where} \qquad H = H(p_i, x^i), \ \ S = S(x^i, t). \tag{2.8}$$

from which we can obtain the usual Hamiltonian equations

$$\frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial x^i}, \qquad \frac{\partial x^i}{\partial t} = \frac{\partial H}{\partial p_i}. \tag{2.9}$$

H is the Hamiltonian for the system and S is the action. The Hamilton-Jacobi equation for a point particle, with mass m, is

$$\left(\frac{\partial S}{\partial x^\mu}\right)^2 = m^2. \tag{2.10}$$

It can be viewed as the classical limit of a quantum field theory.

2.2.2 Hamilton-Jacobi Equation for Strings

This idea is now extended to strings. We start by writing a two-form analogous to the one-form above (2.8).

$$dS_1 \wedge dT_1 + dS_2 \wedge dT_2 = \sum_{i>j} p_{ij}\, dx^i \wedge dx^j - H d\sigma \wedge d\tau. \tag{2.11}$$

$$\text{where} \quad S_m = S_m(x^i,\sigma,\tau), \;\; T_m = T_m(x^i,\sigma,\tau), \qquad m=1,2.$$
$$H = H(p_{ij}, x^i). \tag{2.12}$$

This results in the following

$$\begin{aligned} p_{ij} &= \sum_{m=1,2} \frac{\partial(S_m,T_m)}{\partial(\,x^i\,,\,x^j\,)}, & -H &= \sum_{m=1,2} \frac{\partial(S_m,T_m)}{\partial(\,\sigma\,,\,\tau\,)}, \\ 0 &= \sum_{m=1,2} \frac{\partial(S_m,T_m)}{\partial(\,\sigma\,,\,x^j\,)}, & 0 &= \sum_{m=1,2} \frac{\partial(S_m,T_m)}{\partial(\,x^i\,,\,\tau\,)}. \end{aligned} \tag{2.13}$$

The last two expressions are essentially constraints and their left hand sides are zero because there are no cross terms between dx^j and $d\sigma$ or $d\tau$ in the two-form. A suitable ansatz would be to set S_1 and T_1 to both be functions of σ and τ only, and set S_2 and T_2 to be functions of the x^i only. *i.e.* $S_1(\sigma,\tau), T_1(\sigma,\tau), S_2(x^i), T_2(x^i)$. This ansatz will suffice since it satisfies the two constraints to leave the equations

$$p_{ij} = \frac{\partial(S_2,\,T_2)}{\partial(x^i,\,x^j)}, \qquad -H = \frac{\partial(S_1,\,T_1)}{\partial(\,\sigma\,,\,\tau\,)}. \tag{2.14}$$

From (2.11), by taking the exterior derivative we can see that

$$0 = \sum_{i>j} dp_{ij} \wedge dx^i \wedge dx^j - \left(\sum_{i>j} \frac{\partial H}{\partial p_{ij}} dp_{ij} + \sum_i \frac{\partial H}{\partial x^i} dx^i \right) \wedge d\sigma \wedge d\tau, \tag{2.15}$$

and by equating the coefficients of dp_{ij} and dx^i we obtain

$$\frac{\partial(x^i,\,x^j)}{\partial(\,\sigma,\,\tau\,)} = \frac{\partial H}{\partial p_{ij}}, \qquad \sum_j \frac{\partial(p_{ij},\,x^j)}{\partial(\,\sigma\,,\,\tau\,)} = -\frac{\partial H}{\partial x^i}. \tag{2.16}$$

These equations are the analogues of the Hamiltonian equations (2.9). Substituting the string Hamiltonian equations (2.16) back into the two form (2.11) we have

$$\sum_m dS_m \wedge dT_m = \left(\sum_{i>j} p_{ij} \frac{\partial H}{\partial p_{ij}} - H \right) d\sigma \wedge d\tau = L\, d\sigma \wedge d\tau. \tag{2.17}$$

This defines the Lagrangian L in terms of the Hamiltonian H.

These equations (2.16) also imply

$$\frac{\partial H}{\partial \sigma} = 0, \qquad \frac{\partial H}{\partial \tau} = 0, \tag{2.18}$$

so H is a constant of the motion and does not depend upon the evolution parameters (σ,τ).

Now we consider the Schild string. Remember this has Lagrangian

$$\mathcal{L} = \frac{1}{4}(\{x_\mu, x_\nu\})^2 \qquad \text{where} \qquad \{x_\mu, x_\nu\} = \frac{\partial(x_\mu, x_\nu)}{\partial(\sigma, \tau)}, \tag{2.19}$$

and equations of motion

$$\{x^\mu, \{x_\mu, x_\nu\}\} = 0. \tag{2.20}$$

Choose the Hamiltonian to be

$$H = \frac{1}{2}\sum_{\mu>\nu} p_{\mu\nu}^2. \tag{2.21}$$

From (2.17) and (2.18) we have $\mathcal{L} = H =$ constant. By putting this Hamiltonian into the Hamiltonian equations (2.16) we obtain

$$\frac{\partial(x_\mu, x_\nu)}{\partial(\sigma, \tau)} = p_{\mu\nu}, \quad \sum_\nu \frac{\partial(p_{\mu\nu}, x^\nu)}{\partial(\sigma, \tau)} = 0, \tag{2.22}$$

which leads to

$$\{x^\mu, \{x_\mu, x_\nu\}\} = 0, \tag{2.23}$$

which is the equation of motion for the Schild string. (The Lagrangian is also that for the Schild string.) Using the ansatz for S_m, T_m that we had before (2.14) then

$$p_{\mu\nu} = \frac{\partial(S_2, T_2)}{\partial(x^\mu, x^\nu)}, \tag{2.24}$$

so using (2.21) we have

$$\left(\frac{\partial S_2}{\partial x^\mu}\right)^2 \left(\frac{\partial T_2}{\partial x^\nu}\right)^2 - \left(\frac{\partial S_2}{\partial x^\mu}\frac{\partial T_2}{\partial x_\mu}\right)^2 = \text{constant}. \tag{2.25}$$

This is the Hamilton-Jacobi equation for strings.

Hosotani and Nakayama based their analysis on the Nambu-Goto action rather than the Schild action for strings but some of their findings work for both cases. The equation of motion for both is

$$\frac{\partial(p_{\mu\nu}, x^\nu)}{\partial(\sigma, \tau)} = 0, \tag{2.26}$$

where $p_{\mu\nu}$ is the conjugate momentum in each case.

Now, consider a family of solutions to the equations of motion for a theory in d space-time dimensions. $d-2$ parameters λ_a specify these solutions

$x^\mu = x^\mu(\sigma, \tau, \lambda_1, \dots \lambda_{d-2})$. This defines a mapping from $\{\sigma, \tau, \lambda_a\}$ to $\{x^\mu\}$. If this mapping is one-to-one then $p_{\mu\nu}(\sigma, \tau, \lambda_a)$ can be treated as a local field $p_{\mu\nu}(x^\rho)$. This means the equations of motion can be rewritten as

$$\frac{\partial(p_{\mu\nu}, x^\nu)}{\partial(\ \sigma\ ,\ \tau\ )} = \frac{\partial p_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\rho}{\partial \sigma}\frac{\partial x^\nu}{\partial \tau} - \frac{\partial p_{\mu\nu}}{\partial x^\rho}\frac{\partial x^\rho}{\partial \tau}\frac{\partial x^\nu}{\partial \sigma} = p^{\rho\nu}\partial_\rho p_{\mu\nu} = 0. \tag{2.27}$$

But, using the fact $p_{\mu\nu}$ is antisymmetric with respect to indices μ and ν then

$$\begin{aligned} p^{\rho\nu}\partial_\rho p_{\mu\nu} &= \frac{1}{2}p^{\rho\nu}\partial_\rho p_{\mu\nu} - \frac{1}{2}p^{\rho\nu}\partial_\nu p_{\mu\rho} + \frac{1}{2}p^{\rho\nu}\partial_\mu p_{\nu\rho} - \frac{1}{2}p^{\rho\nu}\partial_\mu p_{\nu\rho} \\ &= \frac{1}{2}p^{\rho\nu}\partial_{[\rho}p_{\mu\nu]} - \frac{1}{4}\partial_\mu(p^{\rho\nu}p_{\nu\rho}) \\ &= -\frac{1}{4}\partial_\mu(p^{\rho\nu}p_{\nu\rho}) = 0. \end{aligned} \tag{2.28}$$

so $p^{\rho\nu}p_{\rho\nu}$ =constant. From the ansatz (2.14) for S_m and T_m earlier, using the definition for $p_{\mu\nu}$, then

$$\frac{1}{2}p^{\mu\nu}p_{\mu\nu} = \left(\frac{\partial S_2}{\partial x^\mu}\right)^2\left(\frac{\partial T_2}{\partial x^\nu}\right)^2 - \left(\frac{\partial S_2}{\partial x^\mu}\frac{\partial T_2}{\partial x_\mu}\right)^2 = \text{constant}, \tag{2.29}$$

which is the Hamilton-Jacobi equation for strings. For the Nambu-Goto string the constant is related to the way the theory is normalised. It should be noted that the constant may be zero for the Schild string but not for the Nambu-Goto string.

In general, if S_2 and T_2 satisfy the Hamilton-Jacobi equation (2.29) then $p_{\mu\nu}$ as given in (2.14) satisfies the equation of motion for the string (2.26).

The question still remains: what are S_2 and T_2? For the point particle $S(x)$ is the action at point x, but as yet the meaning of S_2 and T_2 has yet to be worked out.

2.2.3 Hamilton-Jacobi Equation for Branes

The Hamilton-Jacobi equation for p-branes can be found in a similar way. The conjugate momentum tensor is

$$p_{\mu_1\mu_2\dots\mu_{p+1}} = \frac{\partial(\ S_1\ ,\ S_2\ ,\dots,\ S_{p+1}\ )}{\partial(x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_{p+1}})}. \tag{2.30}$$

Note that here the local fields S_i are analogous to the S_2 and T_2 from the string case. The Hamilton-Jacobi equation is

$$\left(\frac{\partial(\ S_1\ ,\ S_2\ ,\dots,\ S_{p+1}\ )}{\partial(x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_{p+1}})}\right)^2 = \text{constant}. \tag{2.31}$$

2.2.4 Obtaining the Companion Lagrangians

We observe that the Hamilton-Jacobi equation for a massless point particle

$$\left(\frac{\partial S}{\partial x^\mu}\right)^2 = 0 \tag{2.32}$$

takes the same form as the Klein-Gordon Lagrangian

$$\mathcal{L} = \left(\frac{\partial \phi}{\partial x^\mu}\right)^2. \tag{2.33}$$

By analogy, for the string case we take the field Lagrangian to be of the same form as the Hamilton-Jacobi equation for strings

$$\left(\frac{\partial S_2}{\partial x^\mu}\right)^2 \left(\frac{\partial T_2}{\partial x^\nu}\right)^2 - \left(\frac{\partial S_2}{\partial x^\mu}\frac{\partial T_2}{\partial x_\mu}\right)^2 = 0, \tag{2.34}$$

$$\Rightarrow \qquad \mathcal{L} = \left(\frac{\partial \phi}{\partial x^\mu}\right)^2 \left(\frac{\partial \psi}{\partial x^\nu}\right)^2 - \left(\frac{\partial \phi}{\partial x^\mu}\frac{\partial \psi}{\partial x_\mu}\right)^2. \tag{2.35}$$

Similarly for the p-brane, the companion Lagrangian is the same form as the Hamilton-Jacobi equation for a p-brane (2.31) and is therefore

$$\mathcal{L} = \frac{1}{(p+1)!}\left(\frac{\partial(\phi^1, \phi^2, \dots, \phi^{p+1})}{\partial(x^{\mu_1}, x^{\mu_2}, \dots, x^{\mu_{p+1}})}\right)^2 = \det\left|\frac{\partial \phi^i}{\partial x^\mu}\frac{\partial \phi^j}{\partial x_\mu}\right|. \tag{2.36}$$

However, it may prove a good idea to take the square root of this as the Lagrangian. This would look similar to the Nambu-Goto action for strings or the Born-Infeld action for branes.

2.3 Equivalence Theorems

2.3.1 Equivalent Lagrangians

In some cases it may not matter if the companion Lagrangian has a square root or not, since if the Lagrangian is a non-zero constant the equations of motion for both Lagrangians are the same.

For the non-square root companion Lagrangian, $\mathcal{L}$, the equations of motion are

$$\frac{\partial^2 \mathcal{L}}{\partial \phi^i_\mu \partial \phi^j_\nu}\phi^j_{\mu\nu} = 0. \tag{2.37}$$

For the square root Lagrangian $\sqrt{\mathcal{L}}$, with the same number of fields and dimensions as the non-square root Lagrangian above, the equations of motion are

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\sqrt{\mathcal{L}}}{\partial \phi^i_\mu}\right) = \frac{1}{2\sqrt{\mathcal{L}}}\frac{\partial^2 \mathcal{L}}{\partial \phi^i_\mu \partial \phi^j_\nu}\phi^j_{\mu\nu} - \frac{1}{4\mathcal{L}^{3/2}}\frac{\partial \mathcal{L}}{\partial \phi^i_\mu}\frac{\partial \mathcal{L}}{\partial x^\mu} = 0. \tag{2.38}$$

If the Lagrangian is a non-zero constant, $\mathcal{L} = c$ say, then

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = 0. \tag{2.39}$$

Therefore, the second term in the equations of motion for the square root case vanishes leaving the first term which is the equivalent to the equations of motion for the non-square root case. Therefore, the two Lagrangians give the same equations of motion if the Lagrangian is a non-zero constant.

2.3.2 Equivalence Theorem for Companion Lagrangians in Different Dimensions

There is a another way to relate the theories of companion Lagrangians with and without square roots. The main difference here is that the number of dimensions in each theory is not the same.

Theorem: The equations of motion for a companion Lagrangian for n fields without a square root, subject to some constraints, are equivalent to the equations of motion for a companion Lagrangian for n fields with a square root in one less space-time dimension.

Klein-Gordon equation

This began with a simple observation regarding the Klein-Gordon equation. Consider the massless Klein-Gordon Lagrangian in d space-time dimensions,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x^\mu}\right)^2 \qquad \text{where} \qquad \mu = 1, \dots, d. \tag{2.40}$$

Impose the condition $\mathcal{L} = 0$. Using this condition to eliminate partial derivatives with respect to one coordinate, x^d say, then we find the equations of motion for the Klein-Gordon Lagrangian are now the same as the equations of motion for the square root of the Klein-Gordon Lagrangian in one less dimension.

$$\mathcal{L}' = \sqrt{\frac{1}{2}\left(\frac{\partial \phi}{\partial x^\alpha}\right)^2} \qquad \text{where} \qquad \alpha = 1, \dots, d-1. \tag{2.41}$$

This can be seen as follows:

Using the constraint $\mathcal{L} = 0$ we find

$$\phi_d = \sqrt{-\phi_\alpha \phi_\alpha}, \qquad \text{so} \qquad \phi_{dd} = -\frac{\phi_{\alpha\beta}\phi_\alpha\phi_\beta}{\phi_\gamma\phi_\gamma}. \tag{2.42}$$

Therefore, the Klein-Gordon equation can be written as

$$\phi_{\mu\mu} = \phi_{\alpha\alpha} + \phi_{dd} = \phi_{\alpha\alpha} - \frac{\phi_{\alpha\beta}\phi_\alpha\phi_\beta}{\phi_\gamma\phi_\gamma} = \frac{\phi_{\alpha\alpha}\phi_\beta\phi_\beta - \phi_{\alpha\beta}\phi_\alpha\phi_\beta}{\phi_\gamma\phi_\gamma} = 0. \tag{2.43}$$

The numerator of which is the same as the equations of motion for Lagrangian (2.41). In all of the above then $\alpha, \beta, \gamma = 1, 2, \ldots, (d-1)$.

This property also extends to more general Lagrangians of the form

$$\mathcal{L} = \frac{1}{2}\phi_d^2 + \frac{1}{2}F(\phi_\alpha) \qquad \text{where} \qquad \alpha = 1, \ldots, d-1, \tag{2.44}$$

where $F(\phi_\alpha)$ is an arbitrary function of the ϕ_α. As before, by imposing the condition $\mathcal{L} = 0$, we can eliminate derivatives with respect to x^d since

$$\phi_d = \sqrt{-F}, \qquad \text{so} \qquad \phi_{dd} = -\frac{\phi_{\alpha\beta}F_\alpha F_\beta}{4F}, \tag{2.45}$$

$$\text{where} \qquad F_\alpha = \frac{\partial F}{\partial \phi_\alpha}, \qquad F_{\alpha\beta} = \frac{\partial^2 F}{\partial \phi_\alpha \partial \phi_\beta}.$$

So the equation of motion for (2.44) is

$$\phi_{dd} + \frac{1}{2}\phi_{\alpha\beta}F_{\alpha\beta} = \frac{(2F_{\alpha\beta}F - F_\alpha F_\beta)\phi_{\alpha\beta}}{4F} = 0. \tag{2.46}$$

The numerator of this is the equation of motion for the Lagrangian

$$\mathcal{L}' = \sqrt{F(\phi_\alpha)} \qquad \alpha = 1, \ldots d-1. \tag{2.47}$$

Again, a theory for a field in d dimensions has been reduced to a theory in $d-1$ dimensions by setting the Lagrangian of the first theory equal to zero.

The question to ask now is, does any of this generalise to the companion equations for strings and branes? The answer is yes!

Strings

A similar thing can be shown for the companion Lagrangians for strings which have two fields $\phi(x^\mu)$ and $\psi(x^\mu)$. However, this time, as well as imposing the condition $\mathcal{L} = 0$, we also need the constraints

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x^\mu}\right)} \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} = 0, \qquad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \psi}{\partial x^\mu}\right)} \frac{\partial^2 \psi}{\partial x^\mu \partial x^\nu} = 0. \tag{2.48}$$

With these constraints then the equations of motion for the string companion Lagrangian without a square root are the same as the equations of motion for the string companion Lagrangian with a square root but in one less dimension.

Branes

This also extends to p-branes. When the equations of motion for the companion Lagrangian for $p+1$ fields ϕ^i without a square root are subject to the constraints

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi^i}{\partial x^\mu}\right)} \frac{\partial^2 \phi^i}{\partial x^\mu \partial x^\nu} = 0, \qquad \text{no summation over } i, \tag{2.49}$$

then they are the same as the equations of motion for the companion Lagrangian with a square root in one less space-time dimension.

It is easy to prove this equivalence in the Klein-Gordon case and this has already been shown. Computer calculations using the package MAPLE can also be used to check results for low numbers of dimensions and fields. However, a general proof has been found which proves this equivalence for any number of fields, n, and any number of fields, d, where $d > n$ [4]. This proof is given in Appendix A.

It should be noted that for $d = n + 1$ the equations of motion, when subject to the constraints, vanish identically. This is because when the number of fields is the same as the number of dimensions in the square root case, the Lagrangian is a divergence so therefore the equations of motion will be identically zero.

This equivalence theorem has some similarity with 't Hooft's Holographic Principle [55], which has also been studied by Susskind [56]. The principle says that a three-dimensional object can be described on a two-dimensional surface, just like a hologram. Therefore, a theory can be dimensionally reduced by one dimension, the same as in the companion Lagrangian equivalence theorem.

As yet, we do not have a full interpretation of what the constraints mean. The constraints we have used may turn out to be too strong, since there are a lot of them. We may need fewer constraints to obtain the same equivalence and the constraints that have been used here would just be a special class of a more general set of constraints. Also, the constraints we have used are not invariant under simple transformations such as

$$\phi \to \frac{1}{\sqrt{2}}(\phi + \psi), \qquad \psi \to \frac{1}{\sqrt{2}}(\phi - \psi). \tag{2.50}$$

This is something to look for in more general constraints, particularly as the Lagrangian is invariant under such a transformation. Finding another set of constraints may make the interpretation of the constraints and the equivalence theorem easier.

Overall, it seems likely that the theorem and constraints have some importance since it is non-trivial that the two Lagrangians give the same equations of motion when the constraints are applied.

2.4 Covariance

It is unclear at present whether it is best to take the companion Lagrangian with or without the square root. One argument for taking the square root case is that of general covariance. For the companion Lagrangian with n fields in d dimensions, under the field redefinition

$$\phi^i \to \Phi^i(\phi^1, \phi^2, \dots, \phi^n), \tag{2.51}$$

the Lagrangian transforms as

$$\mathcal{L} \mapsto J\mathcal{L}, \tag{2.52}$$

and so is multiplied by a factor J, which is the Jacobian of the transformation *i.e.*

$$J = \frac{\partial(\Phi^1, \Phi^2, \dots, \Phi^n)}{\partial(\phi^1, \phi^2, \dots, \phi^n)}. \tag{2.53}$$

However, the equations of motion are unchanged under this transformation since

$$\begin{aligned}
\frac{\partial(J\mathcal{L})}{\partial\phi^i} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial(J\mathcal{L})}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\right) &= \frac{\partial J}{\partial\phi^i}\mathcal{L} - \frac{\partial}{\partial x^\mu}\left(J\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\right) \\
&= \frac{\partial J}{\partial\phi^i}\mathcal{L} - \frac{\partial J}{\partial\phi^j}\frac{\partial\phi^j}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)} - J\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\right) \\
&= -J\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\right) = 0.
\end{aligned} \tag{2.54}$$

In the middle line, the first two terms cancel with each other because the Lagrangian is homogeneous of weight one in the $\frac{\partial\phi^i}{\partial x^\mu}$ and depends on these derivatives in such a way that

$$\frac{\partial\phi^j}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)} = \delta_i^j\mathcal{L}. \tag{2.55}$$

This just leaves the original equation of motion for ϕ^i. Therefore, the square root Lagrangian is generally covariant which means any function of a solution to the equations of motion is also a solution to the equations of motion. The Lagrangian without the square root does not possess this property of general covariance. The Lagrangian acquires a factor of J^2 under the field redefinition (2.51) and the equations of motion change under this transformation.

2.5 Including a Background Metric

Companion Lagrangians and Born-Infeld type Lagrangians can be written in terms of a quadratic form of Jacobians. If we include a background metric, $g_{\mu\nu}$, then this property still holds.

A companion Lagrangian with a background metric is

$$\mathcal{L} = \sqrt{\det \left| g_{\mu\nu} \frac{\partial \phi^i}{\partial x_\mu} \frac{\partial \phi^j}{\partial x_\nu} \right|}. \tag{2.56}$$

For the string case, where we have two fields (ϕ, ψ), this can be re-written in terms of Jacobians to give

$$\mathcal{L} = \sqrt{\frac{1}{4}(g_{\mu\nu}g_{\rho\kappa} - g_{\mu\rho}g_{\nu\kappa}) \left(\frac{\partial(\phi, \psi)}{\partial(x_\mu, x_\kappa)} \right) \left(\frac{\partial(\phi, \psi)}{\partial(x_\nu, x_\rho)} \right)}. \tag{2.57}$$

This is very similar to the Nambu-Goto string. If we put a background metric into the Lagrangian then it becomes

$$\mathcal{L} = \sqrt{\det \left| g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X^\nu}{\partial \sigma^j} \right|}, \tag{2.58}$$

which can be written in terms of Jacobians as follows

$$\mathcal{L} = \sqrt{\frac{1}{4}(g_{\mu\nu}g_{\rho\kappa} - g_{\mu\rho}g_{\nu\kappa}) \left(\frac{\partial(X^\mu, X^\kappa)}{\partial(\sigma, \tau)} \right) \left(\frac{\partial(X^\nu, X^\rho)}{\partial(\sigma, \tau)} \right)}. \tag{2.59}$$

A general companion Lagrangian, for n fields in d dimensions, with a background metric can be written in terms of Jacobians

$$\mathcal{L} = \sqrt{\frac{1}{n!((d-n)!)^2} \epsilon_{\mu_1\mu_2\ldots\mu_d} \epsilon_{\nu_1\nu_2\ldots\nu_d} g_{\mu_1\nu_1} g_{\mu_2\nu_2} \cdots g_{\mu_n\nu_n} J_{\mu_{n+1}\mu_{n+2}\cdots\mu_d} J_{\nu_{n+1}\nu_{n+2}\cdots\nu_d}}, \tag{2.60}$$

where $J_{\mu_{n+1}\mu_{n+2}\cdots\mu_d}$ is the usual Jacobian for companion Lagrangians as defined in (1.42). Note that it is still of quadratic form inside the square root. This is also the form for Born-Infeld type Lagrangians with a background metric but where the Jacobians $J_{\mu_{n+1}\mu_{n+2}\cdots\mu_d}$ are now the Jacobians for Born-Infeld type theories.

2.6 Electromagnetic Interactions

The question arises as to how to incorporate electromagnetic interactions into our theory with companion Lagrangians. As yet, the following ideas are somewhat speculative but do point towards ways in which this could be done.

In Born-Infeld theory, electromagnetism is incorporated by adding an antisymmetric term, F_{ij}, to the induced metric g_{ij} so the Lagrangian is now

$$\mathcal{L} = \sqrt{\det |g_{ij} + F_{ij}|} = \sqrt{\det \left| \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X_\mu}{\partial \sigma^j} + F_{ij} \right|}, \tag{2.61}$$

$$\text{where} \qquad F_{ij} = \frac{\partial A^j}{\partial \sigma^i} - \frac{\partial A^i}{\partial \sigma^j}.$$

If we want to copy this structure for the companion Lagrangian, then one way of doing this is to assume that the gauge field depends only on x^μ through the fields $\phi^i(x^\mu)$. The gauge fields would be written as $\mathcal{A}^i(\phi^j)$ and the companion Lagrangian with electromagnetism would be

$$\mathcal{L} = \sqrt{\det \left| \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x_\mu} + \frac{\partial \mathcal{A}^j}{\partial \phi^i} - \frac{\partial \mathcal{A}^i}{\partial \phi^j} \right|}. \tag{2.62}$$

This possibility is gauge invariant.

Another possibility is to consider the conserved currents for the theory and to couple the electromagnetic fields to these. This ensures gauge invariance. An example of a conserved current is

$$J^{ij}_\mu = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi^i}{\partial x_\mu} \right)} \phi^j, \qquad i \neq j. \tag{2.63}$$

It is easy to see that this is conserved since,

$$\partial_\mu J^{ij}_\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi^i}{\partial x^\mu} \right)} \phi^j \right) = \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi^i}{\partial x^\mu} \right)} \right)}_{e.o.m} \phi^j + \underbrace{\frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \phi^i}{\partial x^\mu} \right)} \frac{\partial \phi^j}{\partial x^\mu}}_{=0} = 0. \tag{2.64}$$

The first term is zero because it is the equation of motion. The second term is zero because this is a condition due to the fact the Lagrangian is a function of Jacobians and $i \neq j$, as seen from (2.55).

Also, the currents $J^{ii}_\mu - J^{jj}_\mu$ (no summation over indices i and j) are conserved since

$$\partial_\mu (J^{ii}_\mu - J^{jj}_\mu) = 0$$

$$\Rightarrow \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\right)}_{e.o.m.}\phi^i - \underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^j}{\partial x^\mu}\right)}\right)}_{e.o.m.}\phi^j + \underbrace{\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^i}{\partial x^\mu}\right)}\frac{\partial\phi^i}{\partial x^\mu}}_{=\mathcal{L}} - \underbrace{\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^j}{\partial x^\mu}\right)}\frac{\partial\phi^j}{\partial x^\mu}}_{=\mathcal{L}} = 0. \tag{2.65}$$

In the above, the μ index is summed over but the i and j indices are not. These currents, J^{ij}_μ, carry two indices so they naturally couple to a two index gauge field, A^{ij} say, which transforms under the group $SO(n)$. The contribution to the Lagrangian for the gauge field coupling to the fields ϕ^i would be

$$\sum_{i,j} A^{ij}_\mu J^{ij}_\mu. \tag{2.66}$$

This is gauge invariant up to a divergence.

A third suggestion would be to consider the Kalb-Ramond string interaction term for an antisymmetric B-field

$$B_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\nu}{\partial\tau}. \tag{2.67}$$

For the companion Lagrangian, the analogous term would be of the form

$$B_{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}. \tag{2.68}$$

A final way of including $U(1)$ gauge fields in companion Lagrangians was suggested in [57]. Instead of having the usual brane Lagrangian (2.61), we consider

$$\mathcal{L} = \sqrt{\det|g_{ij}+F_{ij}|} = \sqrt{\det\left|\frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X_\mu}{\partial\sigma^j} + \left(\frac{\partial p}{\partial\sigma^i}\frac{\partial q}{\partial\sigma^j} - \frac{\partial p}{\partial\sigma^j}\frac{\partial q}{\partial\sigma^i}\right)\right|}, \tag{2.69}$$

where F_{ij} is now the Lagrange Bracket,

$$F_{ij} = \frac{\partial p}{\partial\sigma^i}\frac{\partial q}{\partial\sigma^j} - \frac{\partial p}{\partial\sigma^j}\frac{\partial q}{\partial\sigma^i}. \tag{2.70}$$

This is still a $U(1)$ theory.

The equivalent companion Lagrangian would be

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu} + \left(\frac{\partial\phi^i}{\partial p}\frac{\partial\phi^j}{\partial q} - \frac{\partial\phi^j}{\partial p}\frac{\partial\phi^i}{\partial q}\right)\right|}, \tag{2.71}$$

where the antisymmetric term, F_{ij}, is now

$$F_{ij} = \frac{\partial \phi^i}{\partial p}\frac{\partial \phi^j}{\partial q} - \frac{\partial \phi^j}{\partial p}\frac{\partial \phi^i}{\partial q}. \tag{2.72}$$

This is the Poisson bracket. This term is the commutator term in an $SU(\infty)$ theory. This is because in the $N \to \infty$ limit, the $SU(N)$ algebra corresponds to the Poisson Bracket algebra [58]. It should be noted that the Poisson bracket and Lagrange bracket are inverses of each other. This Lagrangian is covariant.

Although it is not yet clear which of these ideas is the best, it has been shown that there are possible covariant or gauge invariant ways of introducing gauge fields. The concept of a field description of strings and branes via the companion Lagrangian and equations is strengthened by the fact that gauge fields can be added to the theory.

2.7 Summary

In this chapter, the structure of the companion Lagrangian, a Lagrangian for a field theory for strings and branes, has been given. The Lagrangian can be written in terms of Jacobians which always appear in quadratic form. This form is maintained even when a background metric is added.

The main motivation behind studying it is to formulate a field theory which gives equations of motion similar to the Klein-Gordon equation but for strings and branes rather than particles. The number of fields should always be equal to the number of world-volume coordinates. It is further motivated by the Hamilton-Jacobi equations for strings and branes.

It is not clear whether the Lagrangian should be taken with or without a square root. However, if the Lagrangian is a non-zero constant, this does not matter since the equations motion for Lagrangians with and without square roots are the same in this case. There is also an equivalence theorem which states the equations of motion for the companion Lagrangian without a square root for n fields in d dimensions, when subjected to some constraints, are equivalent to the equations of motion for a companion Lagrangian with a square root with n fields but $d-1$ dimensions. From the point of view of wanting a covariant theory, the square root companion Lagrangian is the best choice.

Chapter 3

Companion Equations and Integrability

The aim of this chapter is to discuss the equations of motion for the companion Lagrangian. First, we look at the equations of motion for the Born-Infeld type Lagrangians *i.e* those for the relativistic particle, Nambu-Goto string and p-branes. Then we consider the companion equations. These are sums of Bateman equations or Universal Field Equations. The integrability of some of these equations is discussed, mainly for the case where there is one more dimension than there are fields. Finally, a proof is given to show that all Lagrangians with two fields in three dimensions, which are homogeneous functions of weight one in the Jacobians, have the same equations of motion.

3.1 Equations of Motion of Born-Infeld Type

We begin by describing the equations of motion for Born-Infeld type Lagrangians and show how they can be written in a compact form. For a classical point particle with Lagrangian

$$\mathcal{L} = \sqrt{\left(\frac{\partial X^{\mu}}{\partial \tau}\right)^2}, \tag{3.1}$$

the equations of motion can be written as

$$\frac{\partial^2 X^{\mu}}{\partial \tau^2}\frac{\partial X^{\nu}}{\partial \tau} - \frac{\partial^2 X^{\nu}}{\partial \tau^2}\frac{\partial X^{\mu}}{\partial \tau} = 0. \tag{3.2}$$

For d dimensions it is easy to verify that there are $d-1$ independent equations of motion in this case. This can be seen if we rewrite them as

$$\frac{\frac{\partial^2 X^1}{\partial\tau^2}}{\frac{\partial X^1}{\partial\tau}} = \frac{\frac{\partial^2 X^2}{\partial\tau^2}}{\frac{\partial X^2}{\partial\tau}} = \cdots = \frac{\frac{\partial^2 X^d}{\partial\tau^2}}{\frac{\partial X^d}{\partial\tau}}, \tag{3.3}$$

assuming $\frac{\partial X^\mu}{\partial\tau} \neq 0$ for $\mu = 1, 2, \ldots d$. In the case of the point particle, the number of equations of motion depends on the number of space-time dimensions, d.

Now we consider the Nambu-Goto string with Lagrangian

$$\mathcal{L} = \sqrt{\left(\frac{\partial X^\mu}{\partial\sigma}\right)^2 \left(\frac{\partial X^\nu}{\partial\tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau}\right)^2}. \tag{3.4}$$

In three dimensions, $d = 3$, there is only one equation of motion. This can be written as

$$\begin{pmatrix} J_1 & J_2 & J_3 \end{pmatrix} \begin{pmatrix} X^1_{\sigma\sigma} & X^1_{\sigma\tau} & X^1_{\tau\tau} \\ X^2_{\sigma\sigma} & X^2_{\sigma\tau} & X^2_{\tau\tau} \\ X^3_{\sigma\sigma} & X^3_{\sigma\tau} & X^3_{\tau\tau} \end{pmatrix} \begin{pmatrix} (X^1_\tau)^2 + (X^2_\tau)^2 + (X^3_\tau)^2 \\ -2(X^1_\sigma X^1_\tau + X^2_\sigma X^2_\tau + X^3_\sigma X^3_\tau) \\ (X^1_\sigma)^2 + (X^2_\sigma)^2 + (X^3_\sigma)^2 \end{pmatrix} = 0, \tag{3.5}$$

$$\text{where} \qquad X^\mu_i = \frac{\partial X^\mu}{\partial\sigma^i}, \qquad X^\mu_{ij} = \frac{\partial^2 X^\mu}{\partial\sigma^i\partial\sigma^j}, \qquad \sigma^i = (\sigma, \tau), \tag{3.6}$$

$$\text{and} \qquad J_\rho = \epsilon_{\rho\mu\nu} X^\mu_\sigma X^\nu_\tau = \frac{1}{2}\epsilon_{\rho\mu\nu} \begin{vmatrix} X^\mu_\sigma & X^\nu_\sigma \\ X^\mu_\tau & X^\nu_\tau \end{vmatrix}. \tag{3.7}$$

Therefore, the equation of motion (3.5) could be written as

$$J_\nu X^\nu_{ij} (L^{-1})_{ij} = 0, \tag{3.8}$$

where L is a matrix which has components $[L]_{ij} = \frac{\partial X^\mu}{\partial\sigma^i}\frac{\partial X_\mu}{\partial\sigma^j}$. L^{-1} is the inverse of this matrix. For a string in d dimensions, the equations of motion can be written in a similar form,

$$J_{\mu\nu_2\ldots\nu_{d-2}} J_{\nu_1\nu_2\ldots\nu_{d-2}} X^{\nu_1}_{ij} (L^{-1})_{ij} = 0, \tag{3.9}$$

where the Jacobians are

$$J_{\nu_1\nu_2\ldots\nu_{d-2}} = \epsilon_{\nu_1\nu_2\ldots\nu_d} X^{\nu_{d-1}}_\sigma X^{\nu_d}_\tau = \frac{1}{2}\epsilon_{\nu_1\nu_2\ldots\nu_d} \begin{vmatrix} X^{\nu_{d-1}}_\sigma & X^{\nu_d}_\sigma \\ X^{\nu_{d-1}}_\tau & X^{\nu_d}_\tau \end{vmatrix}. \tag{3.10}$$

Only $d-2$ of these equations of motion (3.9) are independent.

This can be extended to branes. For a p-brane with the Born-Infeld Lagrangian

$$\mathcal{L} = \sqrt{\det \left| \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X_\mu}{\partial \sigma^j} \right|}, \tag{3.11}$$

the equations of motion can be written as

$$J_{\mu\nu_2\nu_3 \dots \nu_{d-p-1}} J_{\nu_1\nu_2 \dots \nu_{d-p-1}} X^{\nu_1}_{ij} (L^{-1})_{ij} = 0, \tag{3.12}$$

where

$$J_{\nu_1\nu_2\dots\nu_{d-p-1}} = \epsilon_{\nu_1\nu_2\dots\nu_d} \frac{\partial X^{\nu_{d-p}}}{\partial \sigma^1} \frac{\partial X^{\nu_{d-p+1}}}{\partial \sigma^2} \cdots \frac{\partial X^{\nu_d}}{\partial \sigma^{p+1}}. \tag{3.13}$$

Again, L is a matrix with components $[L]_{ij} = \frac{\partial X^\mu}{\partial \sigma^i} \frac{\partial X_\mu}{\partial \sigma^j}$. In general, an object (particle, string or brane) which sweeps out an $(p+1)$-dimensional world volume in d-dimensional space-time has only $d-p-1$ independent equations of motion. In the case $d=p+1$, the Lagrangian is a divergence, so all the equations of motion vanish.

3.2 Companion Equations

We now turn our attention to the companion equations which are the equations of motion for the companion Lagrangians. We will be considering the Lagrangians with a square root. Firstly, we discuss the case of one field $\phi(x^\mu)$ with Lagrangian

$$\mathcal{L} = \sqrt{\left(\frac{\partial \phi}{\partial x^\mu} \right)^2}. \tag{3.14}$$

For two space-time dimensions, $d=2$, the companion equation is

$$\left(\frac{\partial \phi}{\partial x_1} \right)^2 \frac{\partial^2 \phi}{\partial x_2^2} + \left(\frac{\partial \phi}{\partial x_2} \right)^2 \frac{\partial^2 \phi}{\partial x_1^2} - 2 \left(\frac{\partial \phi}{\partial x_1} \right) \left(\frac{\partial \phi}{\partial x_2} \right) \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0. \tag{3.15}$$

This is the Bateman equation. For $d=3$ the equation of motion is

$$\begin{aligned} \phi_1^2\phi_{22} + \phi_2^2\phi_{11} - 2\phi_1\phi_2\phi_{12} + \phi_1^2\phi_{33} + \phi_3^2\phi_{11} - 2\phi_1\phi_3\phi_{13} \\ + \phi_2^2\phi_{33} + \phi_3^2\phi_{22} - 2\phi_2\phi_3\phi_{23} = 0, \end{aligned} \tag{3.16}$$

where

$$\phi_i = \frac{\partial \phi}{\partial x_i}, \qquad \phi_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \tag{3.17}$$

This is the sum of three Bateman equations. This pattern generalises as we increase the number of dimensions. For d space-time dimensions the companion equation is the sum of $\binom{d}{2}$ Bateman equations. There is always only one equation of motion, irrespective of the number of space-time dimensions. This is due to the fact there is only one field, ϕ.

We now consider the companion equations for strings which come from the companion Lagrangian

$$\mathcal{L} = \sqrt{\left[(\phi_\mu)^2 (\psi_\nu)^2 - (\phi_\mu \psi_\mu)^2\right]} = \sqrt{\frac{1}{2} (\phi_\mu \psi_\nu - \phi_\nu \psi_\mu)^2}. \tag{3.18}$$

Note that the Lagrangian depends on the derivatives ϕ_μ and ψ_ν only through the Jacobians which are of the form $(\phi_\mu \psi_\nu - \phi_\nu \psi_\mu)$. For three space-time dimensions, d=3, then the equations of motion are of the form

$$\det \begin{vmatrix} 0 & 0 & \phi_1 & \phi_2 & \phi_3 \\ 0 & 0 & \psi_1 & \psi_2 & \psi_3 \\ \phi_1 & \psi_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_2 & \psi_2 & \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_3 & \psi_3 & \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix} = 0. \tag{3.19}$$

There are two equations of motion. The second has the same structure as (3.19) but with ϕ and ψ interchanged. This is a Universal Field Equation, a generalisation of the Bateman equation.

For d=4, the equations of motion are the the sum of four Universal Field Equations like (3.19),

$$\begin{vmatrix} 0 & 0 & \phi_1 & \phi_2 & \phi_3 \\ 0 & 0 & \psi_1 & \psi_2 & \psi_3 \\ \phi_1 & \psi_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_2 & \psi_2 & \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_3 & \psi_3 & \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & \phi_1 & \phi_2 & \phi_4 \\ 0 & 0 & \psi_1 & \psi_2 & \psi_4 \\ \phi_1 & \psi_1 & \phi_{11} & \phi_{12} & \phi_{14} \\ \phi_2 & \psi_2 & \phi_{12} & \phi_{22} & \phi_{24} \\ \phi_4 & \psi_4 & \phi_{14} & \phi_{24} & \phi_{44} \end{vmatrix}$$
$$+ \begin{vmatrix} 0 & 0 & \phi_1 & \phi_3 & \phi_4 \\ 0 & 0 & \psi_1 & \psi_3 & \psi_4 \\ \phi_1 & \psi_1 & \phi_{11} & \phi_{13} & \phi_{14} \\ \phi_3 & \psi_3 & \phi_{13} & \phi_{33} & \phi_{34} \\ \phi_4 & \psi_4 & \phi_{14} & \phi_{34} & \phi_{44} \end{vmatrix} + \begin{vmatrix} 0 & 0 & \phi_2 & \phi_3 & \phi_4 \\ 0 & 0 & \psi_2 & \psi_3 & \psi_4 \\ \phi_2 & \psi_2 & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_3 & \psi_3 & \phi_{23} & \phi_{33} & \phi_{34} \\ \phi_4 & \psi_4 & \phi_{24} & \phi_{34} & \phi_{44} \end{vmatrix} = 0. \tag{3.20}$$

The other equation of motion is the same except ϕ and ψ are interchanged. For general d, the equations of motion are the sum of $\binom{d}{3}$ Universal Field Equations.

For a p-brane, the companion Lagrangian can again be written in terms of Jacobians,

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|} = \sqrt{\frac{1}{(p+1)!}\left(\frac{\partial(\ \phi^1,\ \phi^2,\ldots,\ \phi^{p+1})}{\partial(x^{\mu_1},x^{\mu_2},\ldots,x^{\mu_{p+1}})}\right)^2}. \tag{3.21}$$

In the case d=p+2, the equation of motion are just Universal Field Equations again, of the form

$$\det\begin{vmatrix} 0 & 0 & \cdots & 0 & \phi^1_1 & \phi^1_2 & \cdots & \phi^1_{p+2} \\ 0 & 0 & \cdots & 0 & \phi^2_1 & \phi^2_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \phi^{p+1}_1 & \cdots & \cdots & \phi^{p+1}_{p+2} \\ \phi^1_1 & \phi^2_1 & \cdots & \phi^{p+1}_1 & \phi^1_{11} & \phi^1_{12} & \cdots & \phi^1_{1,p+2} \\ \phi^1_2 & \phi^2_2 & \cdots & \phi^{p+1}_2 & \phi^1_{12} & \phi^1_{22} & \cdots & \phi^1_{2,p+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^1_{p+2} & \phi^2_{p+2} & \cdots & \phi^{p+1}_{p+2} & \phi^1_{1,p+2} & \cdots & \cdots & \phi^1_{p+2,p+2} \end{vmatrix} = 0. \tag{3.22}$$

For higher space-time dimensions, the equations of motion are sums of $\binom{d}{p+2}$ Universal Field Equations (3.22). There are always p+1 equations of motion, the same as the number of fields. The fact that the companion equations are all made up of Universal Field Equations is related to the fact that the companion Lagrangians are all square roots of squares of Jacobians. In fact, the equations of motion can alternatively be written using the Jacobians. They take the form,

$$J_{\mu\mu_2\mu_3\cdots\mu_{d-p-1}}J_{\nu\mu_2\mu_3\cdots\mu_{d-p-1}}\phi^i_{\mu\nu} = 0 \qquad i = 1,2,\ldots,p+1, \tag{3.23}$$

where the Jacobian is defined as

$$J_{\mu_1\mu_2\mu_3\cdots\mu_{d-p-1}} = \epsilon_{\mu_1\mu_2\cdots\mu_{d-p-1}\nu_1\nu_2\cdots\nu_{p+1}}\phi^1_{\nu_1}\phi^2_{\nu_2}\cdots\phi^{p+1}_{\nu_{p+1}}. \tag{3.24}$$

The calculation to show these are the equations of motion of the companion Lagrangian is in Appendix B.

3.3 Integrability

As mentioned before, the companion equation for the companion Lagrangian with a square root for one field in two dimensions (3.14) is the Bateman equation (3.15). This equation is fully integrable. The equation has the general solution

$$F(\phi)x_1 + G(\phi)x_2 = c = \text{constant}, \tag{3.25}$$

where F and G are arbitrary functions. It is covariant since if ϕ is a solution then so is any function of ϕ. The Bateman equation is equivalent to the Monge non-linear wave equation

$$\frac{\partial u}{\partial x_1} = u\frac{\partial u}{\partial x_2}, \qquad \text{where} \qquad u = \frac{\frac{\partial \phi}{\partial x_1}}{\frac{\partial \phi}{\partial x_2}} = \frac{\phi_1}{\phi_2}. \tag{3.26}$$

This is a first order equation for u. The companion Lagrangian can be written so that it contains u,

$$\mathcal{L} = \phi_2\sqrt{(1+u^2)}. \tag{3.27}$$

The equation of motion is

$$\frac{\partial}{\partial x_2}\frac{1}{\sqrt{1+u^2}} + \frac{\partial}{\partial x_1}\frac{u}{\sqrt{1+u^2}} = 0, \tag{3.28}$$

which is equivalent to the Monge equation (3.26). In fact, replacing $\sqrt{1+u^2}$ with any differentiable function $f(u)$ will give the same equation of motion.

This was known already, but what happens in the next case up when we have two fields in three dimensions? The equations of motion are known to be covariant in this case (see section 2.4) and so we ought to be able to express them in terms of two ratios of Jacobians

$$\begin{aligned} u &= \frac{\phi_1\psi_2 - \phi_2\psi_1}{\phi_2\psi_3 - \phi_3\psi_2}, \\ v &= \frac{\phi_3\psi_1 - \phi_1\psi_3}{\phi_2\psi_3 - \phi_3\psi_2}. \end{aligned} \tag{3.29}$$

These are analogous to the u defined in (3.26) for the one field case. The Lagrangian can be written as

$$\mathcal{L} = (\phi_2\psi_3 - \phi_3\psi_2)\sqrt{(1+u^2+v^2)}, \tag{3.30}$$

and the equations of motion are

$$\begin{aligned} &v\left[\frac{\partial}{\partial x_1}\frac{v}{\sqrt{1+u^2+v^2}} - \frac{\partial}{\partial x_2}\frac{1}{\sqrt{1+u^2+v^2}}\right] \\ &+ u\left[\frac{\partial}{\partial x_1}\frac{u}{\sqrt{1+u^2+v^2}} - \frac{\partial}{\partial x_3}\frac{1}{\sqrt{1+u^2+v^2}}\right] = 0, \end{aligned}$$

$$\begin{aligned} &\left[\frac{\partial}{\partial x_2}\frac{1}{\sqrt{1+u^2+v^2}} - \frac{\partial}{\partial x_1}\frac{v}{\sqrt{1+u^2+v^2}}\right] \\ &+ u\left[\frac{\partial}{\partial x_2}\frac{u}{\sqrt{1+u^2+v^2}} - \frac{\partial}{\partial x_3}\frac{v}{\sqrt{1+u^2+v^2}}\right] = 0. \end{aligned} \tag{3.31}$$

As before, $\sqrt{1+u^2+v^2}$ can be replaced by an arbitrary function $f(u,v)$. For any such function, we can write the equations of motion down as

$$\begin{aligned}\frac{\partial u}{\partial x_1}+v\frac{\partial u}{\partial x_2}+u\frac{\partial u}{\partial x_3}&=0,\\ \frac{\partial v}{\partial x_1}+v\frac{\partial v}{\partial x_2}+u\frac{\partial v}{\partial x_3}&=0.\end{aligned} \tag{3.32}$$

These are equivalent to (3.31), independent of the function $f(u,v)$, and look like generalisations of the Monge equation.

They have an implicit solution for u and v which can be found by solving the equations

$$u=F(x_3-ux_1,x_2-vx_1), \qquad v=G(x_3-ux_1,x_2-vx_1), \tag{3.33}$$

where F and G are arbitrary functions of two variables. By setting $u=U(\phi,\psi)$ and $v=V(\phi,\psi)$, where U and V are also arbitrary functions of two variables, and then solving the equations (3.33) for ϕ and ψ then we have a general solution to the equations of motion. This is similar to they way the general solution to the Monge equation leads to the general solution for the Bateman equation. This shows that these equations of motion are integrable.

This procedure is easily generalised for n fields in n+1 dimensions. As yet, a generalisation for n fields in d dimensions has not been found, but it is hoped that this does exist and that the equations of motion are integrable. However, for the cases where $d>n+1$, the equations of motion are sums of Bateman equations of Universal Field Equations. A large class of solutions to these equations of motion can be found by solving each Bateman equation or Universal Field Equation separately. For example, for the one field case the equation of motion is the sum of $\binom{d}{2}$ Bateman equations. Solving the equation below for ϕ,

$$x^\mu F_\mu(\phi)=c, \tag{3.34}$$

where F_μ are d arbitrary functions of ϕ, and c is a constant, gives solutions to all the individual Bateman equations and therefore to the whole equation of motion. This works the same for more fields. By finding solutions which satisfy each Universal Field Equation by itself, the equations of motion are satisfied.

3.4 Lagrangians with the Same Equations of Motion

In the previous section we said that all Lagrangians for two fields in three dimensions which are homogeneous of weight one in the Jacobians have the same equations of motion *i.e.* Lagrangians of the form $\mathcal{L} = (\phi_2\psi_3 - \phi_3\psi_2)f(u, v)$ where f is an arbitrary function and u, v are the ratios of Jacobians, given in (3.29). This can be shown either by calculating the equations of motion for such a Lagrangian directly, which can be a bit messy, or as follows. Take Lagrangian $\mathcal{L} = \mathcal{L}(J_1, J_2, J_3)$ where J_μ are the Jacobians defined as $J_\mu = \epsilon_{\mu\nu\rho}\phi_\nu\psi_\rho$. Since the Lagrangian is a homogeneous function of weight one in the Jacobians then the following are true:

$$\phi_\mu\frac{\partial\mathcal{L}}{\partial\phi_\mu} = \mathcal{L}, \qquad \psi_\mu\frac{\partial\mathcal{L}}{\partial\phi_\mu} = 0, \tag{3.35}$$

$$\phi_\mu\frac{\partial\mathcal{L}}{\partial\psi_\mu} = 0, \qquad \psi_\mu\frac{\partial\mathcal{L}}{\partial\psi_\mu} = \mathcal{L}. \tag{3.36}$$

The first and the last equation arise because the Lagrangian is a homogeneous function of weight one in ϕ_μ and also in ψ_ν. The other two equations arise because the Lagrangian is a function of Jacobians. The equations of motion are

$$\begin{aligned}\phi_{\mu\nu}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} + \psi_{\mu\nu}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu} &= 0,\\ \psi_{\mu\nu}\frac{\partial^2\mathcal{L}}{\partial\psi_\mu\partial\psi_\nu} + \phi_{\mu\nu}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu} &= 0.\end{aligned} \tag{3.37}$$

Differentiating the constraints (3.35) with respect to ϕ_ν, we obtain

$$\phi_\mu\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} = 0, \qquad \psi_\mu\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} = 0. \tag{3.38}$$

This gives six equations, five of which are independent. Solving these, it is possible to write all $\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}$ in terms of $\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\phi_1}$, say. Similarly, differentiating the constraints (3.35) with respect to ψ_ν and constraints (3.36) with respect to ϕ_ν we find

$$\begin{aligned}\phi_\mu\left(\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu} + \frac{\partial^2\mathcal{L}}{\partial\psi_\mu\partial\phi_\nu}\right) &= 0,\\ \psi_\mu\left(\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu} + \frac{\partial^2\mathcal{L}}{\partial\psi_\mu\partial\phi_\nu}\right) &= 0.\end{aligned} \tag{3.39}$$

Again, there are six equations for which five of the $(\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu} + \frac{\partial^2\mathcal{L}}{\partial\psi_\mu\partial\phi_\nu})$ can be found in terms of the sixth, $\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\psi_1}$, say. Differentiating (3.36) with respect to ψ_ν means we can also write all $\frac{\partial^2\mathcal{L}}{\partial\psi_\mu\partial\psi_\nu}$ in terms of $\frac{\partial^2\mathcal{L}}{\partial\psi_1\partial\psi_1}$. Putting these into the equations of motion gives

$$\begin{aligned}
&\frac{1}{J_1^2}\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\phi_1}J_\mu J_\nu\phi_{\mu\nu}+\frac{1}{J_1^2}\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\psi_1}J_\mu J_\nu\psi_{\mu\nu}=0,\\
&\frac{1}{J_1^2}\frac{\partial^2\mathcal{L}}{\partial\psi_1\partial\psi_1}J_\mu J_\nu\psi_{\mu\nu}+\frac{1}{J_1^2}\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\psi_1}J_\mu J_\nu\phi_{\mu\nu}=0,
\end{aligned}\tag{3.40}$$

which are equivalent to

$$\begin{aligned}
J_\mu J_\nu\phi_{\mu\nu}&=0,\\
J_\mu J_\nu\psi_{\mu\nu}&=0.
\end{aligned}\tag{3.41}$$

These equations (3.41) are the Universal Field equations which were written in determinantal form in (3.19). This should generalise to n fields in n+1 dimensions. The proof follows a similar pattern to give the result that, for a given n, all such Lagrangians which are homogeneous functions of weight one in the Jacobians have same equations of motions. These equations are the Universal Field equations (3.22).

3.5 Summary

For the usual particle, Nambu-Goto string and brane with a Born-Infeld type Lagrangian, then a p-dimensional object in d-dimensional space-time has $d-p-1$ independent equations of motion which can be written in the general form (3.12). The number of equations depends on both the number of world-sheet coordinates and space-time coordinates. For the companion Lagrangian with n fields in d dimensions there are n independent equations of motion which can be written in the general form of sums of (3.22). The number of equations only depends on the number of fields, not the number of dimensions unlike the Born-Infeld type cases. However, the structure of both types of equation of motion are similar.

The companion equations are sums of $\binom{d}{n+1}$ Bateman equations or Universal Field equations. If d=n, the equations of motion are automatically zero. If d=n+1, the equations of motion are each just one Bateman or Universal Field equation. In fact, if d=n+1, then all Lagrangians which are homogeneous of weight one in the Jacobians have the same equations of motion. These equations are the Universal Field Equations. Such cases appear to be integrable.

Chapter 4

An Iterative Procedure

The structure of the companion Lagrangians and the appearance of Bateman and Universal Field Equations in the companion equations led to investigations into the extension of an iterative procedure known for Lagrangians depending on one field to Lagrangians depending on more than one field.

In this chapter, we explain how the iterative procedure works for Lagrangians which are homogeneous functions of weight one in the first derivatives of a field ϕ. The penultimate iteration always gives a Universal Field Equation. We then explain how such a procedure can be extended to Lagrangians which are homogeneous functions of weight one in the first derivatives of several fields, such as the companion Lagrangian. We concentrate mainly on the case of two fields in three dimensions, giving explicit examples of how the procedure works, and how the Universal Field Equations appear.

4.1 Universal Field Equations

Until now, generalisations of the Bateman equation have involved increasing the number of fields. We now consider generalisations where the number of dimensions are increased, without changing the number of fields. The resulting field equation is also known as a Universal Field Equation [28][29][59]. This generalised Bateman equation in d dimensions for a field $\phi(x^\mu)$ can be written as

$$\det\begin{vmatrix} 0 & \phi_1 & \phi_2 & \dots & \phi_d \\ \phi_1 & \phi_{11} & \phi_{12} & \dots & \phi_{1d} \\ \phi_2 & \phi_{12} & \phi_{22} & \dots & \phi_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_d & \phi_{1d} & \phi_{2d} & \dots & \phi_{dd} \end{vmatrix} = 0. \tag{4.1}$$

This equation possesses general covariance like the Bateman equation does. It is also integrable just like the Bateman equation is. This was proved in [60] by linearising the Universal Field Equation using the Legendre transform. A large class of solutions to (4.1) can be written as solutions to the equation

$$x^\mu F_\mu(\phi) = c, \tag{4.2}$$

where F_μ (μ=1,2, ... , d) are arbitrary functions of ϕ. c is a constant. This has the same form as the general solution to the Bateman Equation (1.28).

4.2 Iterative Lagrangians

These Universal Field Equations can be obtained by using an iterative procedure involving Lagrangians and equations of motion. Equations of motion for a theory involving one field ϕ can be found using the Euler operator

$$\mathcal{E} = -\frac{\partial}{\partial\phi} + \partial_\mu\frac{\partial}{\partial\phi_\mu} - \partial_\mu\partial_\nu\frac{\partial}{\partial\phi_{\mu\nu}} + \cdots. \tag{4.3}$$

This summation in the operator can be continued forever, but for the present discussion the expansion can be terminated after the third term because we will only be dealing with first and second derivatives. In general, the equation of motion for Lagrangian $\mathcal{L}$ would be written $\mathcal{E}\mathcal{L}$=0.

Now consider Lagrangians, $\mathcal{L}$, which only involve first derivatives, ϕ_μ, and are homogeneous functions of weight one in these derivatives. Since the Lagrangian does not depend explicitly on the field ϕ then $\mathcal{E}\mathcal{L}$ is a divergence. For such cases, it can be shown that $\mathcal{E}^2\mathcal{L} = 0$.

The iterative procedure is as follows

$$\begin{gathered} \mathcal{E}\mathcal{L}, \\ \mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}, \\ \mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}, \\ \dots \end{gathered} \tag{4.4}$$

It should be noted that applying operator $\mathcal{E}$ reduces the function to one which is weight zero in the derivatives of ϕ, in the sense that

$$\sum_{\mu} \phi_\mu \frac{\partial(\mathcal{E}\mathcal{L})}{\partial \phi_\mu} + \sum_{\nu \geq \mu} \phi_{\mu\nu} \frac{\partial(\mathcal{E}\mathcal{L})}{\partial \phi_{\mu\nu}} = 0. \tag{4.5}$$

By multiplying it by $\mathcal{L}$ we return to a function which is weight one in the derivatives of ϕ, just like the original Lagrangian. After each iteration then only first and second derivatives are left. All third derivatives cancel with each other. For a Lagrangian in d dimensions, this iterative procedure terminates after d iterations. At this point everything vanishes identically. After $d-1$ iterations then we obtain the Universal Field Equation for a field in d dimensions (4.1). This is all independent of the Lagrangian we started with.

For any Lagrangian, $\mathcal{L} = \mathcal{L}(\phi_\mu)$ which is homogeneous of weight one in the ϕ_μ, after $d-1$ iterations we obtain the Universal Field Equation. Even if the original Lagrangian possesses no symmetry we can reach an equation which is invariant under the group $GL(d)$, despite the equation being highly non-linear. It is this which makes the Universal Field Equations universal, because they can be found from an infinite number of starting Lagrangians. There is a proof for this in [29].

The Universal Field Equations can be obtained from an infinite number of starting Lagrangians. Since these Lagrangians only depend on first derivatives of the field, and not on the field ϕ itself, this means the equation of motion is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_\mu}\right) = 0. \tag{4.6}$$

Since this is a divergence and there are an infinite number of Lagrangians $\mathcal{L}$, then there are an infinite number of conservation laws. There is one for each possible Lagrangian. This is one of the properties which first led to the idea these Universal Field Equations might be integrable.

4.3 Extension to Higher Dimensions

The main aim of this work was to try to find an iterative procedure for Lagrangians which involve more than one field. Most of this discussion will be for theories with two fields, ϕ and ψ, where the Lagrangian is homogeneous of weight one in the first order derivatives ϕ_μ and ψ_μ.

This process is not as simple as the iterative procedure $\mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}\mathcal{E}\mathcal{L}$ for one field where we just multiply by the Lagrangian $\mathcal{L}$ before re-applying the Euler operator $\mathcal{E}$.

From now on, $\mathcal{E}_\phi$ denotes the Euler operator with respect to field ϕ and $\mathcal{E}_\psi$ denotes the Euler operator with respect to field ψ. If we just carry out the process $\mathcal{E}_\phi \mathcal{L} \mathcal{E}_\phi \mathcal{L}$ then we now obtain third order derivatives and higher, unlike in the one field case where we only had first and second derivatives. As a result, there appears to be no simplification like in the one field case. As the iteration continues, the resulting expression becomes increasingly complicated and involves higher and higher orders of derivatives. Also, $\mathcal{E}_\phi \mathcal{L}$ gives a function of weight zero in the derivatives of ϕ and weight one in derivatives of ψ. Multiplying by $\mathcal{L}$ gives $\mathcal{L}\mathcal{E}_\phi \mathcal{L}$. This is a function of weight one in derivatives of ϕ but weight two in derivatives of ψ, which is not the same as the Lagrangian which is weight one in both derivatives of ϕ and ψ. This is also different to the one field case where multiplying by the Lagrangian always returned the object to a function of weight one in derivatives of ϕ_μ, which is the same as the starting Lagrangian.

To overcome these problems, instead of multiplying by $\mathcal{L}$ each time, we multiply by a function f which depends on the Lagrangian in some way. f should depend on ϕ_μ and ψ_μ only, as $\mathcal{L}$ does. Also, f should be a homogeneous function of weight one in ϕ_μ and weight zero in ψ_μ. This means that $f\mathcal{E}_\phi \mathcal{L}$ is weight one in both ϕ_μ and ψ_μ, the same as the Lagrangian. Finally, f should be chosen so that there are no third order derivatives when the Euler operator is applied for the second time, $\mathcal{E}_\phi f \mathcal{E}_\phi \mathcal{L}$.

To find $f(\phi_\mu, \psi_\mu)$ we need to find the conditions on f required to ensure that there are no terms involving third derivatives. Applying the first Euler operator, $\mathcal{E}_\phi$, gives

$$\mathcal{E}_\phi \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_\mu} \right) = \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} \phi_{\mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \psi_\nu} \psi_{\mu\nu}. \tag{4.7}$$

Multiply this by an as yet unknown function f, which depends only on ϕ_μ and ψ_μ. Then apply the Euler operator, $\mathcal{E}_\phi$, again.

$$\begin{aligned} \mathcal{E}_\phi f \mathcal{E}_\phi \mathcal{L} = \partial_\rho \Bigg(& \frac{\partial f}{\partial \phi_\rho} \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} \phi_{\mu\nu} + f \frac{\partial^3 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu \partial \phi_\rho} \phi_{\mu\nu} \\ & + \frac{\partial f}{\partial \phi_\rho} \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \psi_\nu} \psi_{\mu\nu} + f \frac{\partial^3 \mathcal{L}}{\partial \phi_\rho \partial \phi_\mu \partial \psi_\nu} \psi_{\mu\nu} \Bigg) - \partial_\mu \partial_\nu \left(f \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} \right). \end{aligned} \tag{4.8}$$

Looking at the terms involving third order derivatives only then we find

$$\begin{aligned}&\frac{\partial f}{\partial \phi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}\phi_{\mu\nu\rho}+f\frac{\partial^3\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu\partial\phi_\rho}\phi_{\mu\nu\rho}+\frac{\partial f}{\partial \phi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu}\psi_{\mu\nu\rho}\\&+f\frac{\partial^3\mathcal{L}}{\partial\phi_\rho\partial\phi_\mu\partial\psi_\nu}\psi_{\mu\nu\rho}-\frac{\partial f}{\partial \phi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}\phi_{\mu\nu\rho}-\frac{\partial f}{\partial \psi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}\psi_{\mu\nu\rho}\\&-f\frac{\partial^3\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu\partial\phi_\rho}\phi_{\mu\nu\rho}-f\frac{\partial^3\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu\partial\psi_\rho}\psi_{\mu\nu\rho}.\end{aligned}\tag{4.9}$$

So, for all this to vanish we require

$$\frac{\partial f}{\partial \phi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu}\psi_{\mu\nu\rho}-\frac{\partial f}{\partial \psi_\rho}\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}\psi_{\mu\nu\rho}=0.\tag{4.10}$$

In general, this will be satisfied if the function f satisfies the following condition:

$$\frac{\frac{\partial f}{\partial\phi_\mu}}{\frac{\partial f}{\partial\psi_\mu}}=\frac{\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\phi_1}}{\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\psi_1}}\qquad\text{for all }\mu.\tag{4.11}$$

There is no summation over indices in this condition (4.11).

If we apply the Euler operator $\mathcal{E}_\psi$ to $f\mathcal{E}_\phi\mathcal{L}$ then we obtain the same condition (4.10).

Similarly, if we consider multiplying $\mathcal{E}_\psi\mathcal{L}$ by some function $g(\phi_\mu,\psi_\nu)$ then g needs to satisfy

$$\frac{\frac{\partial g}{\partial\phi_\mu}}{\frac{\partial g}{\partial\psi_\mu}}=\frac{\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\psi_1}}{\frac{\partial^2\mathcal{L}}{\partial\psi_1\partial\psi_1}}\qquad\text{for all }\mu,\tag{4.12}$$

so that $\mathcal{E}_\psi g\mathcal{E}_\psi\mathcal{L}$ and $\mathcal{E}_\phi g\mathcal{E}_\psi\mathcal{L}$ only involve first and second derivatives.

4.4 Specific Examples

We now consider some specific Lagrangians, find the functions f which satisfy the conditions (4.11) and discuss the iterations. We will concentrate on two fields in three dimensions. In each case, f must satisfy the conditions

$$\frac{\frac{\partial f}{\partial\phi_1}}{\frac{\partial f}{\partial\psi_1}}=\frac{\frac{\partial f}{\partial\phi_2}}{\frac{\partial f}{\partial\psi_2}}=\frac{\frac{\partial f}{\partial\phi_3}}{\frac{\partial f}{\partial\psi_3}}=\frac{\frac{\partial^2\mathcal{L}}{\partial\phi_1\partial\phi_1}}{\frac{\partial^2\mathcal{L}}{\partial\psi_1\partial\phi_1}}.\tag{4.13}$$

First we look at the companion Lagrangian with a square root

$$\mathcal{L}=\sqrt{J_1^2+J_2^2+J_3^2},\tag{4.14}$$

where the J_i are the usual Jacobians. The required function f is

$$f = \frac{\mathcal{L}}{\sqrt{\psi_1^2 + \psi_2^2 + \psi_3^2}}. \tag{4.15}$$

Note that any function of this f will satisfy the condition (4.13). However, it makes sense to choose f to be weight one in ϕ_μ and weight zero in ψ_μ so that $f\mathcal{E}_\phi\mathcal{L}$ is a function of weight one in derivatives of ϕ and in derivatives of ψ_μ, as stated earlier.

The iterative sequence is then

$$\mathcal{E}_\phi f \mathcal{E}_\phi \mathcal{L} = \frac{1}{(\psi_1^2 + \psi_2^2 + \psi_3^2)^{3/2}} \det \begin{vmatrix} 0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_1 & \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_2 & \psi_{12} & \psi_{22} & \psi_{23} \\ \psi_3 & \psi_{13} & \psi_{23} & \psi_{33} \end{vmatrix}. \tag{4.16}$$

After two iterations we have some factor multiplied by the Universal Field Equation for field ψ in three dimensions. The expression is completely independent of ϕ and its derivatives. Similarly, if we make the valid choice for g

$$g = \frac{\mathcal{L}}{\sqrt{\phi_1^2 + \phi_2^2 + \phi_3^2}}, \tag{4.17}$$

then

$$\mathcal{E}_\psi g \mathcal{E}_\psi \mathcal{L} = \frac{1}{(\phi_1^2 + \phi_2^2 + \phi_3^2)^{3/2}} \det \begin{vmatrix} 0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_2 & \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_3 & \phi_{13} & \phi_{23} & \phi_{33} \end{vmatrix}. \tag{4.18}$$

This is some factor multiplied by the Universal Field Equation for field ϕ and is completely independent of ψ.

The following list of Lagrangians for two fields in three dimensions all behave similarly. Suitable functions f have been found for them all.

$$\begin{aligned} \mathcal{L} &= \sqrt{a_{ij}J_iJ_j}, & f &= \frac{\mathcal{L}}{\sqrt{2\epsilon_{i_1i_2i_3}\epsilon_{j_1j_2j_3}a_{i_2j_2}a_{i_3j_3}\psi_{i_1}\psi_{j_1}}}, \\ \mathcal{L} &= \frac{a_{ij}J_iJ_j}{c_kJ_k}, & f &= \frac{c_kJ_k}{\sqrt{\epsilon_{i_1i_2i_3}\epsilon_{j_1j_2j_3}a_{i_2j_2}c_{i_3}c_{j_3}\psi_{i_1}\psi_{j_1}}}, \\ \mathcal{L} &= \mathcal{L}(b_jJ_j, c_kJ_k), & f &= \frac{\mathcal{L}}{\epsilon_{ijk}b_ic_j\psi_k}. \end{aligned} \tag{4.19}$$

The a_{ij}, b_k, c_k are all constants and summation over indices is assumed. All indices run from 1 to 3. The first of these examples is the companion Lagrangian (4.14) with a background metric. Setting $a_{11}, a_{22}, a_{33} = 1$ and all other $a_{ij} = 0$ we just have the normal companion Lagrangian. In all the cases above, the iterative sequence $\mathcal{E}_\phi f \mathcal{E}_\phi \mathcal{L}$ is always of the same form as (4.16). The determinant part, the Universal Field Equation, always appears and is multiplied by some factor. The factor depends on the starting Lagrangian. The whole expression always only involves derivatives of ψ and so is completely independent of derivatives of ϕ.

In some ways it is not surprising that the second iteration of these Lagrangians is the same, since the equations of motion of all these Lagrangians are equivalent. What is surprising is that the iteration only depends on the first and second derivatives of ψ and has no dependence on derivatives of ϕ at all. Similarly, $\mathcal{E}_\psi g \mathcal{E}_\psi \mathcal{L}$ only depends on derivatives of ϕ. The fields seem to completely decouple. Another important point is that the second iteration always involves the generalised Bateman equation (the Universal Field Equation). This is analogous to the one field case where after $d-1$ iterations we obtain the Universal Field Equation.

This can be generalised to higher dimensional cases where the number of fields is one less than the number of dimensions. Functions f can be found for Lagrangians of the same form as (4.19) but with the Jacobians redefined for $d-1$ fields in d dimensions. The f's have a similar structure to those in (4.19). It is hoped that this can be extended to n fields in d dimensions.

4.5 Summary

The iterative procedure for Lagrangians which are homogeneous functions of weight one in derivatives of one field, ϕ_μ, where the $(d-1)$th iteration is the Universal Field Equation, has been generalised to more than one field.

Rather than multiplying each time by the Lagrangian, it is necessary to multiply by a function f which depends on the Lagrangian. For two fields in three dimensions, several examples of Lagrangians which are homogeneous functions of weight one in ϕ_μ and ψ_μ have been given. In these cases we have found a suitable function f and shown that the second iteration always gives a Universal Field Equation.

As yet, the list of examples does not cover all Lagrangians which are weight one in the first derivatives of the fields. However, the list does involve a large class of such Lagrangians, including the companion Lagrangian.

Extension to $d-1$ fields in d dimensions is possible and functions, f, can be found for a similar list of Lagrangians to those found for two fields in three dimensions. Extension to n fields in d dimensions still remains to be done. It is likely that iterations for these will result in Universal Field Equations appearing somewhere.

Chapter 5

A Special Property of a Family of Field Theories

In this chapter, we discuss a property of a family of field Lagrangians, not just companion Lagrangians. This property is that for these field theories, the Lagrangian vanishes or is a divergence on the space of solutions of the equations of motion. The list of examples is given below. Basic background on most of the examples can be found in [19][32].

5.1 Klein-Gordon Field

The Klein-Gordon equation is a field equation for a scalar field $\phi(x^\mu)$. Its Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{5.1}$$

where m is the mass. The equation of motion is

$$(\Box + m^2)\phi = 0. \tag{5.2}$$

This is the Klein-Gordon equation. To show the property that the Lagrangian is a divergence on the space of solutions then rewrite the Lagrangian (5.1) using partial integration as follows.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu(\phi\partial^\mu\phi) - \frac{1}{2}\phi\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\underbrace{\partial_\mu(\phi\partial^\mu\phi)}_{\text{divergence}} - \frac{1}{2}\phi(\underbrace{\Box\phi + m^2\phi}_{\text{e.o.m.}}).\end{aligned} \tag{5.3}$$

When the equation of motion (e.o.m.) is satisfied the second term is zero, so the Lagrangian is a divergence.

5.2 The Dirac Equation

The Dirac equation is a first order equation for fermions with spin 1/2. The Lagrangian is

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi, \tag{5.4}$$

where $\psi(x^\mu)$ is a spinor and $\bar{\psi} = \psi^\dagger\gamma^0$ is its adjoint. The equations of motion are

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \text{and} \qquad \bar{\psi}(i\gamma^\mu\overleftarrow{\partial_\mu} + m) = 0. \tag{5.5}$$

The first is the Dirac equation and the second is its Hermitian conjugate. It is very easy to see that the Lagrangian for Dirac field vanishes on the space of the solutions of the equations of motion since the Lagrangian is just

$$\mathcal{L} = \bar{\psi}\underbrace{(i\gamma^\mu\partial_\mu\psi - m\psi)}_{\text{e.o.m.}}. \tag{5.6}$$

5.3 Maxwell Theory

Maxwell Theory describes electromagnetism. It is a $U(1)$ gauge theory. The Lagrangian for a free electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5.7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and A_μ is the gauge field. The equation of motion is

$$\partial_\mu F^{\mu\nu} = 0. \tag{5.8}$$

Rewriting the Lagrangian (5.7) using the antisymmetry of $F_{\mu\nu}$ and partial integration we find

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} \\ &= -\frac{1}{2}\partial_\mu A_\nu F^{\mu\nu} \\ &= -\frac{1}{2}\partial_\mu(A_\nu F^{\mu\nu}) + \frac{1}{2}A_\nu\partial_\mu F^{\mu\nu}.\end{aligned} \tag{5.9}$$

The last term is zero when the equations of motion are satisfied so the Lagrangian is a divergence on the space of solutions of the equations of motion.

5.4 Self-Dual Gauge Fields

In general, the result does not hold for non-abelian gauge theories. However, in the special case of self-dual Yang-Mills the Lagrangian can be written as a divergence. The non-abelian field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{5.10}$$

The self-duality condition is

$$F_{\mu\nu} = {}^*F_{\mu\nu}, \qquad \text{where} \qquad {}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{5.11}$$

so the self-dual Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{8}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}). \tag{5.12}$$

This can be rewritten as a total derivative,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu[\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma)]. \tag{5.13}$$

It is the total derivative of the Chern-Simons term which can be studied as a Lagrangian in its own right [61]. Therefore, self-dual Yang-Mills has a Lagrangian which is a divergence. In the previous examples the Lagrangian is only a divergence when the equations of motion are satisfied. Here, it may at first seem like the equations of motion have not been considered. However, if the self-duality condition is satisfied then so are the equations of motion for Yang-Mills, so the condition the equations of motion are satisfied was taken account of in (5.12). For more on self-duality, see Chapter 6 on the Moyal-Nahm equations.

5.5 Gravity

The Lagrangian for gravity is

$$\mathcal{L} = \sqrt{-g}R, \tag{5.14}$$

where g is the determinant of the space-time metric and R is the Ricci scalar. If there is no matter in the theory, the equation of motion is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0, \tag{5.15}$$

where $R^{\mu\nu}$ is the Ricci tensor. With matter, the right hand side of this would be the stress-energy tensor $T^{\mu\nu}$ but since we have empty space-time this implies $R^{\mu\nu} = 0$ and therefore $R = 0$. Therefore, the Lagrangian for pure gravity vanishes on the space of solutions of the equations of motion.

5.6 Strings and Branes

The Lagrangian for strings and p-branes is

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|}. \tag{5.16}$$

$X^\mu(\sigma^i)$ ($\mu = 0, 1, \ldots, d-1$) are the d target space coordinates. σ^i ($i = 0, 1, \ldots, p$) are the $p+1$ world-volume coordinates. For string theory, set $p = 1$. The equations of motion are

$$\frac{\partial}{\partial \sigma^i}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial X^\mu}{\partial \sigma^i}\right)}\right) = 0. \tag{5.17}$$

It can be shown, via the theorem of false cofactors, that

$$\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial X^\mu}{\partial \sigma^j}\right)} = \delta_i^j \mathcal{L}. \tag{5.18}$$

This is shown as follows [18]:

Let $\mathcal{L} = \sqrt{L}$, so

$$L = \det\left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|. \tag{5.19}$$

Also, let $\hat{L}_{ij}$ be the cofactor, or signed minor, of the $i-j$th component in matrix of which L is the determinant. Therefore,

$$\begin{aligned}
\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial X^\mu}{\partial \sigma^j}\right)} = \frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial \sqrt{L}}{\partial\left(\frac{\partial X^\mu}{\partial \sigma^j}\right)} &= \frac{1}{2}\frac{\partial X^\mu}{\partial \sigma^i}\frac{1}{\sqrt{L}}\frac{\partial L}{\partial\left(\frac{\partial X^\mu}{\partial \sigma^j}\right)} \\
&= \frac{1}{2}\frac{1}{\sqrt{L}}\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^k}(\hat{L}_{kj} + \hat{L}_{jk}) \\
&= \frac{1}{\sqrt{L}}\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^k}\hat{L}_{kj} \\
&= \frac{1}{\sqrt{L}} L\,\delta_{ij} \\
&= \delta_{ij}\mathcal{L}.
\end{aligned} \tag{5.20}$$

In the third line we can see that if $i = j$ then we just get the determinant, L. But, if $i \neq j$ then this is same as finding the determinant of a matrix with two rows which are the same. The determinant of such a matrix is always zero. This is the theorem of false cofactors. It should be noted that the case $i = j$ shows that the Lagrangian $\mathcal{L}$ is homogeneous of degree one in the $\frac{\partial X^\mu}{\partial \sigma^i}$ for each value of i.

This result and partial integration can be used to give

$$\begin{aligned}\mathcal{L} &= \frac{1}{p+1}\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial \mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial \sigma^i})} \\ &= \frac{1}{p+1}\frac{\partial}{\partial \sigma^i}\left(X^\mu \frac{\partial \mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial \sigma^i})}\right) - \frac{1}{p+1}X^\mu \frac{\partial}{\partial \sigma^i}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial \sigma^i})}\right).\end{aligned} \tag{5.21}$$

Therefore, it can now be seen that when the equations of motion are satisfied the Lagrangian is a divergence. It should be noted that this works for any power of the Lagrangian (5.16). For

$$\mathcal{L} = \left(\det\left|\frac{\partial X^\mu}{\partial \sigma^i}\frac{\partial X_\mu}{\partial \sigma^j}\right|\right)^{N/2}, \tag{5.22}$$

where N is some number, then the Lagrangian can be rewritten as

$$\mathcal{L} = \frac{1}{N(p+1)}\frac{\partial}{\partial \sigma^i}\left(X^\mu \frac{\partial \mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial \sigma^i})}\right) - \frac{1}{N(p+1)}X^\mu \frac{\partial}{\partial \sigma^i}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial X^\mu}{\partial \sigma^i})}\right), \tag{5.23}$$

so is again a divergence on the space of solutions for any power $\frac{N}{2}$. The important values of N are $N = 1$ which is the case given and $N = 2$, the Schild string (when $p = 1$).

5.7 Companion Equations

This property is also true for the companion field theory for strings and branes which was described earlier. The Lagrangian is

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial \phi^i}{\partial x^\mu}\frac{\partial \phi^j}{\partial x_\mu}\right|}, \tag{5.24}$$

where ϕ^i $(i = 1, 2, \ldots, n)$ are the fields and x^μ $(\mu = 1, 2, \ldots, d)$ are the spacetime coordinates. The equations of motion are

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\right) = 0. \tag{5.25}$$

The Lagrangian is homogeneous of weight one in the first partial derivatives of $\phi^1 = \frac{\partial \phi^1}{\partial x^\mu}$. This means

$$\mathcal{L} = \partial_\mu \phi^1 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^1)}. \tag{5.26}$$

By rewriting this using partial integration, the Lagrangian is

$$\mathcal{L} = \partial_\mu \left(\phi^1 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^1)} \right) - \phi^1 \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^1)} \right). \tag{5.27}$$

So, the companion Lagrangians are also a divergence on the space of solutions of the equations of motion.

As in the string/brane example, this works the same for any power of the Lagrangian (5.24). In particular, the property holds for the companion Lagrangian either with or without a square root.

5.8 Supersymmetric Lagrangians: Chiral Superfields

The property also extends to some supersymmetric Lagrangians. Firstly, we consider a chiral superfield. Chiral superfields obey the condition $\bar{D}_{\dot{\alpha}}\Phi = 0$. A general chiral superfield in superspace $(x, \theta, \bar{\theta})$ has the form [62]

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= A(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box A(x) \\ &\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x).\end{aligned} \tag{5.28}$$

Consider a Lagrangian involving only chiral superfields as below,

$$\begin{aligned}\mathcal{L} &= \Phi_i^\dagger\Phi_i|_{\theta\theta\bar{\theta}\bar{\theta}} + \frac{1}{2}m_{ij}(\Phi_i\Phi_j|_{\theta\theta} + \Phi_i^\dagger\Phi_j^\dagger|_{\bar{\theta}\bar{\theta}}) \qquad (5.29)\\ &= i\partial_m\bar{\psi}_i\bar{\sigma}^m\psi_i + A_i^*\Box A_i + F_i^*F_i + m_{ij}(A_iF_j - \frac{1}{2}\psi_i\psi_j + A_i^*F_j^* - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j),\end{aligned}$$

where m_{ij} is symmetric with respect to indices i and j. The equations of motion for this Lagrangian are as follows:

$$\begin{aligned}&F_i^* + m_{ij}A_j = 0, \qquad \Box A_i^* + m_{ij}F_j = 0, \qquad i\partial_m\bar{\psi}_i\bar{\sigma}^m - m_{ij}\psi_j = 0,\\ &F_i + m_{ij}A_j^* = 0, \qquad \Box A_i + m_{ij}F_j^* = 0, \qquad i\bar{\sigma}^m\partial_m\psi_i + m_{ij}\bar{\psi}_j = 0.\end{aligned} \tag{5.30}$$

Rewriting the Lagrangian using partial integration, we find

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(i\partial_m\bar{\psi}_i\bar{\sigma}^m - m_{ij}\psi_j)\psi_i - \frac{1}{2}\bar{\psi}_i(i\bar{\sigma}^m\partial_m\psi_i + m_{ij}\bar{\psi}_j)\\ &\quad + A_i^*(\Box A_i + m_{ij}F_j^*) + (F_i^* + m_{ij}A_j)F_i + \frac{i}{2}\partial_m(\bar{\psi}_i\bar{\sigma}^m\psi_i).\end{aligned} \tag{5.31}$$

The first four terms will vanish when the equations of motion are satisfied and the last term is a divergence. A more general supersymmetric Lagrangian for chiral fields would involve the addition of the terms $\frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k$, $\lambda_i\Phi_i$ and their hermitian conjugates. However, if these are added then the Lagrangian is no longer a divergence when the equations of motion are satisfied.

5.9 Supersymmetric Lagrangians: Vector Superfields

Secondly, we consider vector supersymmetric Lagrangians. Vector superfields obey the condition $V = V^\dagger$. A general vector superfield takes the form [62]

$$\begin{aligned} V = {} & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x) - iN(x)] \\ & - \theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}[\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^m\partial_m\chi(x)] - i\bar{\theta}\bar{\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^m\partial_m\bar{\chi}(x)] \\ & + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) + \frac{1}{2}\Box C(x)]. \end{aligned} \tag{5.32}$$

For a vector field, the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{4}(W^\alpha W_\alpha|_{\theta\theta} + \bar{W}^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}|_{\bar{\theta}\bar{\theta}}) + m^2V^2, \tag{5.33}$$

where $W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V$ and $\bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V$. For the massless case, we can use the Wess-Zumino gauge in which the component fields C, χ, M and N are all zero. The Lagrangian is then just

$$\mathcal{L} = \frac{1}{2}D^2 - \frac{1}{4}v^{mn}v_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda}, \tag{5.34}$$

where $v_{mn} = \partial_m v_n - \partial_n v_m$. The equations of motion are

$$D = 0, \qquad -i\sigma^m\partial_m\bar{\lambda} = 0, \qquad i\partial_m\lambda\sigma^m = 0, \qquad \partial^m v_{mn} = 0. \tag{5.35}$$

Rewriting the Lagrangian, we find

$$\mathcal{L} = \frac{1}{2}D^2 + \frac{1}{2}v^n\partial^m v_{mn} - i\lambda\sigma^m\partial_m\bar{\lambda} - \frac{1}{2}\partial^m(v^n v_{mn}). \tag{5.36}$$

So, for the massless vector superfield, when the equations of motion are satisfied the Lagrangian is a divergence. For the massive case we cannot use the Wess-Zumino gauge. However, the action can be rewritten as [63]

$$S[V] = \frac{1}{8}\int VD^\alpha\bar{D}^2D_\alpha V \,\mathrm{d}^4x\mathrm{d}^4\theta + m^2\int V^2 \,\mathrm{d}^4x\mathrm{d}^4\theta, \tag{5.37}$$

and the equations of motion are

$$\frac{1}{8}D^\alpha\bar{D}^2D_\alpha V + m^2V = 0. \tag{5.38}$$

From this it can be see that the Lagrangian is a divergence when the equations of motion are satisfied.

5.10 Discussion

The Lagrangians of the following field theories have been shown to be a divergence on the space of solutions of the equations of motion of that theory: Klein-Gordon, Dirac, Maxwell, Self-Dual Yang-Mills, gravity, string theory, brane theory, companion field theory, supersymmetric chiral and vector superfields. The theories can be massive or massless.

This property suggests there is a 'pseudo-topological' nature to the Lagrangians. Lagrangians which are divergences with no other constraints are fully topological, such as gravity in two dimensions [30]. Here the Lagrangian is only a divergence when the equations of motion are satisfied. However, it should be noted that there are many important properties of fully topological theories which do not appear in these 'pseudo-topological' theories.

Kastrup [52] considered what solutions of equations of motion implied $\mathcal{L} = 0$ for various theories including field theories. He had also noticed that $\mathcal{L} = 0$ for all solutions to the Dirac equation and pure gravity, but did not notice that for some other field theories $\mathcal{L}$ is a divergence. His interpretation was that solutions with $\mathcal{L} = 0$ were bifurcations or phase transitions of the theory

This is best seen in statistical mechanics where $\mathcal{L} = F$ with F defined as the density of the free energy. $\mathcal{L} = 0$ marks the transition between ordered and unordered phases in the theory. In the examples we have given, when the Lagrangian is zero or a divergence we have a non-interacting theory but as soon as other terms are added this condition is lost and we have interactions. In Yang-Mills theory we the Lagrangian is a divergence only for self-dual gauge fields. $\mathcal{L} = 0$ seems to mark special solutions to a theory (*e.g.* solutions with no interactions, no matter, self-duality).

It should be noted that each theory can be written as a Lagrangian which vanishes on the space of solutions, since all Lagrangians which are equivalent up to a divergence give the same equation of motion.

Chapter 6

The Moyal-Nahm Equations

6.1 Introduction

The main aim of this chapter is to find solutions to the Moyal-Nahm equations in four and eight dimensions. The Nahm equations give solutions for a particular set of self-dual Yang-Mills fields. When the commutators are replaced by Moyal brackets these equations become the Moyal-Nahm equations.

Firstly, we discuss self-duality and the Nahm equations in four dimensions. This is then extended to higher dimensions. We look at why you would want to this and how to go about it. In particular, we focus on Nahm equations in eight dimensions.

Next, we look at Moyal brackets and star products. These objects are defined and some of their properties are given. Motivation is given as to why you might want to consider them. Wigner functions, a type of phase space distribution function, are also discussed since the solutions will be in terms of generalised Wigner functions.

Finally, we try to solve the four dimensional Moyal-Nahm equations and the eight dimensional Nahm and Moyal-Nahm equations. We use an ansatz based on generalised Wigner functions and sets of matrices which obey the algebra generated by the Nahm equations. Finding sets of such matrices was an important part of this work.

This chapter is based on work in [3].

6.2 Nahm Equations

6.2.1 Self-Dual Gauge Fields in Four Dimensions

Non-abelian gauge fields can be described by Yang-Mills field theory. For the moment, consider gauge fields in four dimensional Euclidean space-time with coordinates x^μ, $(\mu = 0, \ldots, 3)$. If the theory only involves the gauge fields themselves, $A^\mu(x^\nu)$ say, this is pure Yang-Mills. The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{6.1}$$

The Lagrangian for pure Yang-Mills is

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{6.2}$$

The equation of motion is

$$D_\mu F^{\mu\nu} = 0, \tag{6.3}$$

where D_μ is the covariant derivative. The self-duality condition in four dimensions is

$$F_{\mu\nu} = {}^*F_{\mu\nu}, \qquad \text{where} \qquad {}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{6.4}$$

i.e. the field strength is equal to the dual field strength ${}^*F_{\mu\nu}$. This results in the following three equations:

$$F_{01} = F_{23}, \qquad F_{02} = F_{31}, \qquad F_{03} = F_{12}. \tag{6.5}$$

If the self-duality equation (6.4) is satisfied then the equations of motion for the Yang-Mills theory (6.3) are automatically satisfied via the Bianchi identity,

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0. \tag{6.6}$$

This is important, since it means all solutions of the self-dual equations are solutions of the full Yang-Mills theory. It is known that the Yang-Mills equations are not completely solvable, however the self-dual Yang-Mills equations are, in general, solvable [64]. Instanton and BPS[1] monopole solutions both satisfy the self-dual Yang-Mills equations.

[1]where BPS stands for Bogomol'nyi-Prasad-Somerfield.

6.2.2 Nahm Equations

Instantons are solutions of the self-dual Yang-Mills equations where the action is finite. All instanton solutions can be generated by the ADHM construction of Atiyah, Drinfield, Hitchin and Manin [65]. This construction reduces the problem to a set of non-linear algebraic equations. Nahm generalised the ADHM construction to monopole solutions which have finite energy and are invariant under shifts in Euclidean time but do not have finite action. This generalisation resulted in what are know as the Nahm equations [33].

The Nahm equations can also be constructed in the following way. Consider the self-dual Yang-Mills equations (6.4) where the gauge fields A^μ depend on only one space time coordinate, $x^0 = t$ say. Also, fix the gauge so that $A^0 = 0$. This is the most convenient gauge to use and makes life easier later. The self-dual equations are now

$$\begin{aligned}\frac{\partial A^1}{\partial t} &= [A^2, A^3],\\ \frac{\partial A^2}{\partial t} &= [A^3, A^1],\\ \frac{\partial A^3}{\partial t} &= [A^1, A^2].\end{aligned} \tag{6.7}$$

These are the Nahm equations.

Given that $A^0 = 0$ and the other gauge fields A^j $(j = 1, 2, 3)$ only depend on t, the Yang-Mills Lagrangian is now [66]

$$\mathcal{L} = \mathrm{Tr}\left(\left(\frac{\partial A^j}{\partial t}\right)^2 + \frac{1}{2}[A^j, A^k][A^j, A^k]\right). \tag{6.8}$$

The equations of motion are

$$\frac{\partial^2 A^j}{\partial t^2} + [A^k, [A^k, A^j]] = 0, \tag{6.9}$$

and the energy is

$$E = \mathrm{Tr}\left(\left(\frac{\partial A^j}{\partial t}\right)^2 - \frac{1}{2}[A^j, A^k][A^j, A^k]\right). \tag{6.10}$$

It is easy to see that solutions of the Nahm equations (6.7) satisfy the equations of motion, simply by differentiating.

The Lagrangian (6.8) can also be written as

$$\mathcal{L} = \mathrm{Tr}\Bigg(\left(\frac{\partial A^1}{\partial t} - [A^2, A^3]\right)^2 + \left(\frac{\partial A^2}{\partial t} - [A^3, A^1]\right)^2 + \left(\frac{\partial A^3}{\partial t} - [A^1, A^2]\right)^2 + 2\frac{\partial}{\partial t}(A^1A^2A^3 - A^1A^3A^2)\Bigg). \tag{6.11}$$

This is just sums of squares of the Nahm equations plus a divergence. Since squares of real objects are always positive then the Lagrangian must be greater or equal to the divergence. This is the Bogomol'nyi bound [67]. Therefore, when the Nahm equations are satisfied the Bogomol'nyi bound is satisfied. The Nahm equations are just Bogomol'nyi equations. Also, by squaring the Nahm equations it can be seen that the energy (6.10) for solutions to the Nahm equations is zero.

6.2.3 Self-Duality in Higher Dimensions

As well as considering gauge fields in four dimensions, a lot of work has been done in extending such theories to higher dimensions [64][68][69][70]. The aim is usually to consider the theory in a higher dimension, where there may be new and interesting physics, and then dimensionally reduce the theory via compactification to one in (preferably) four large dimensions. Recently, there has been an interest in Yang-Mills in higher dimensions because of the appearance of Yang-Mills actions in M(atrix) Theory [36][38]. M(atrix) theory is based on the conjecture that M-theory can be described by the $N \to \infty$ limit of supersymmetric quantum mechanics.

Corrigan et al. [68] were particularly interested in finding analogues of the self-dual Yang-Mills equation in higher dimensions. They wanted to find a linear relationship for the field strength, solutions to which automatically satisfied the Yang-Mills equations in $D > 4$ dimensions via the Bianchi identity.

$$\frac{1}{2}T_{\mu\nu\rho\sigma}F_{\rho\sigma} = \lambda F_{\mu\nu}. \tag{6.12}$$

$T_{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor and λ is a constant. Therefore, this linear relationship implies the Yang-Mills equations are satisfied. For $D = 4$ the choice for $T_{\mu\nu\rho\sigma}$ is essentially unique and is

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = \lambda F_{\mu\nu}. \tag{6.13}$$

$\lambda = 1$ gives the usual self-dual equations. $\lambda = -1$ will give the anti-self-dual equations. All other values of λ require $F_{\mu\nu} = 0$. The most interesting, and closest analogy to the $D = 4$ case is that for $D = 8$. One possible linear relationship is

$$F_{0a} = \frac{1}{2} c_{abc} F_{bc}, \tag{6.14}$$

where c_{abc} are the octonionic structure constants (see section 6.2.5). This is in complete analogy with the self-dual equations in four dimensions, which may be written as

$$F_{0a} = \frac{1}{2} \epsilon_{abc} F_{bc}, \tag{6.15}$$

where ϵ_{abc} are the quaternionic structure constants. Equations (6.14) are the self-dual equations in eight dimensions that we will be considering. They can be written as

$$\begin{aligned}
&F_{10} + F_{27} + F_{63} + F_{54} = 0, && F_{20} + F_{71} + F_{53} + F_{46} = 0, \\
&F_{30} + F_{16} + F_{25} + F_{47} = 0, && F_{40} + F_{15} + F_{62} + F_{73} = 0, \\
&F_{50} + F_{41} + F_{32} + F_{67} = 0, && F_{60} + F_{31} + F_{24} + F_{75} = 0, \\
&F_{70} + F_{12} + F_{34} + F_{56} = 0.
\end{aligned} \tag{6.16}$$

6.2.4 Nahm Equations in Eight Dimensions

The Nahm equations in eight dimensions can be derived in much the same way as in four dimensions. Again, the gauge field A^μ is assumed to only depend on one space-time coordinate, $x^0 = t$, and the gauge choice $A^0 = 0$ is made. Putting these constraints into the the self-dual equations (6.16) results in the following set of equations:

$$\begin{aligned}
\frac{\partial A^1}{\partial t} - [A^2, A^7] - [A^6, A^3] - [A^5, A^4] &= 0, \\
\frac{\partial A^2}{\partial t} - [A^7, A^1] - [A^5, A^3] - [A^4, A^6] &= 0, \\
\frac{\partial A^3}{\partial t} - [A^1, A^6] - [A^2, A^5] - [A^4, A^7] &= 0, \\
\frac{\partial A^4}{\partial t} - [A^1, A^5] - [A^6, A^2] - [A^7, A^3] &= 0, \\
\frac{\partial A^5}{\partial t} - [A^4, A^1] - [A^3, A^2] - [A^6, A^7] &= 0, \\
\frac{\partial A^6}{\partial t} - [A^3, A^1] - [A^2, A^4] - [A^7, A^5] &= 0, \\
\frac{\partial A^7}{\partial t} - [A^1, A^2] - [A^3, A^4] - [A^5, A^6] &= 0.
\end{aligned} \tag{6.17}$$

In terms of the octonionic structure constants they are

$$\frac{\partial A^i}{\partial t} = \frac{1}{2} c_{ijk} [A^j,\ A^k]. \tag{6.18}$$

These are the Nahm equations in eight dimensional Euclidean space-time.

6.2.5 Octonions

The octonions $\mathcal{O}$ are one of the four division algebras [69]. The other three are the real, complex and quaternionic numbers $(\mathcal{R}, \mathcal{C}, \mathcal{H})$. The octonions are non-associative and non-commutative, so they do not have a matrix representation. However, they are alternative and so for any $x, y \in \mathcal{O}$, so

$$x(y^2) - (xy)y = 0 \qquad \text{and} \qquad (x^2)y - x(xy) = 0. \tag{6.19}$$

The basis for the octonions is $\{1, e_a\}$ where $a = 1, \ldots, 7$. Any octonion q can be written as $q = q_0 + q_a e_a$ where all q_μ are real. In this work, only the imaginary octonions (e_a, $a = 1, \ldots, 7$) will be considered. The octonions obey the multiplication rule

$$e_i e_j = -\delta_{ij} + c_{ijk} e_k, \tag{6.20}$$

where c_{ijk} are the octonionic structure constants. The structure constants will be taken to be

$$c_{127} = c_{631} = c_{541} = c_{532} = c_{246} = c_{347} = c_{567} = 1. \tag{6.21}$$

These are totally antisymmetric. All other c_{ijk} are zero. There are many examples of the octonions appearing in physics, especially when the groups $SO(8)$, $SO(7)$ and G_2 are discussed [68][69][71][72].

6.3 Moyal Brackets and Star Products

6.3.1 The Star Product

The star product of two functions $f(x,p)$ and $g(x,p)$ which are functions of a 2-dimensional phase space (x,p) can be written in several ways [41].

$$f \star g = \exp\left[i\lambda \left(\frac{\partial}{\partial x}\frac{\partial}{\partial p'} - \frac{\partial}{\partial p}\frac{\partial}{\partial x'} \right) \right] f(x,p) g(x',p')|_{x=x',p=p'}, \tag{6.22}$$

or

$$f \star g = \sum_{s=0}^{\infty} \frac{(i\lambda)^s}{s!} \sum_{t=0}^{s} (-1)^t \binom{s}{t} [\partial_x^{s-t} \partial_p^t f][\partial_x^t \partial_p^{s-t} g], \tag{6.23}$$

or

$$f \star g = \frac{1}{4\pi^2\lambda^2} \int e^{\frac{i}{\lambda} \det \begin{vmatrix} 1 & 1 & 1 \\ x & x' & x'' \\ p & p' & p'' \end{vmatrix}} f(x', p') g(x'', p'')\, \mathrm{d}x' \mathrm{d}x'' \mathrm{d}p' \mathrm{d}p''. \tag{6.24}$$

λ is a parameter. The last definition can be checked against the others by using the Fourier transforms of f and g. The star product can easily be generalised for functions on a $2N$-dimensional phase space (x_j, p_j) as follows:

$$f \star g = f \exp\left[i\lambda \left(\frac{\overleftarrow{\partial}}{\partial x_j} \frac{\overrightarrow{\partial}}{\partial p_j} - \frac{\overleftarrow{\partial}}{\partial p_j} \frac{\overrightarrow{\partial}}{\partial x_j} \right) \right] g, \tag{6.25}$$

or

$$f \star g = \sum_{j=1}^{N} \sum_{s=0}^{\infty} \frac{(i\lambda)^s}{s!} \sum_{t=0}^{s} (-1)^t \binom{s}{t} [\partial_{x_j}^{s-t} \partial_{p_j}^t f][\partial_{x_j}^t \partial_{p_j}^{s-t} g]. \tag{6.26}$$

The star product is associative, so for any three functions f, g, h then

$$f \star (g \star h) = (f \star g) \star h, \tag{6.27}$$

but it is non-commutative, so in general

$$f \star g \neq g \star f. \tag{6.28}$$

The star product can be expanded as a power series in the parameter λ.

$$f \star g = fg + i\lambda \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) + O(\lambda^2). \tag{6.29}$$

It should be noted that the first term in the expansion is ordinary multiplication of f and g. Therefore, in the limit $\lambda \to 0$ the star product tends to ordinary multiplication. The second term in the expansion is the Poisson bracket which will be discussed later.

Properties

As well as being associative and non-commutative, the star product has the property

$$\int f \star g \, \mathrm{d}p\mathrm{d}x = \int fg \, \mathrm{d}p\mathrm{d}x. \tag{6.30}$$

This is because all the terms in the star product are divergences except the first term fg. Assuming the functions f and g and their derivatives vanish at infinity these divergences are just integrated out. This property is most easily seen from definition (6.24) of the star product.

A useful result due to Ian Strachan [41] which will be used in later calculations is that

$$e^{py} f(x) \star e^{py'} g(x) = e^{p(y+y')} f(x+y') g(x-y). \tag{6.31}$$

6.3.2 Moyal Brackets

The Moyal Bracket was first introduced by Moyal over 50 years ago [43]. It is the imaginary part of the star product. The Moyal Bracket of two functions f and g is therefore

$$\{f,g\}_{MB} = \frac{1}{2i}(f \star g - g \star f). \tag{6.32}$$

The real part of the star product is known as Baker's cosine bracket (named after George Baker, not Linda Baker),

$$((f,g)) = \frac{1}{2}(f \star g + g \star f). \tag{6.33}$$

The Moyal Bracket is a one parameter deformation of the Poisson Bracket which is given by

$$\{f,g\}_{PB} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}. \tag{6.34}$$

The deformation parameter is the λ in the star product. It is a Lie algebra, like the Poisson bracket is, and so satisfies the Jacobi Identity just like matrix commutators and Poisson Brackets do.

$$\{\{f,g\}_{MB},h\}_{MB} + \{\{g,h\}_{MB},f\}_{MB} + \{\{h,f\}_{MB},g\}_{MB} = 0. \tag{6.35}$$

Arveson showed that the Moyal bracket is the only function of iterated Poisson brackets which can satisfy the Jacobi identity [73]. The Moyal Bracket is the unique one parameter associative deformation of the Poisson bracket and in the limit $\lambda \to 0$ then $\frac{1}{\lambda}\{f,g\}_{MB}$ becomes the Poisson bracket.

6.3.3 Wigner Functions

One set of objects commonly used with Moyal Brackets are Wigner functions. These are phase space distribution functions and were invented by Wigner and Szilard [74]. Wigner used them as a kind of probability distribution function constructed from wave functions when he was studying quantum corrections in statistical mechanics. Since then, Wigner functions have been used in dynamical systems (especially collision theory), quantum optics, quantum chemistry and M-theory [75][76][77][46]. The time independent Wigner function on phase space (x,p) is [78]

$$f(x,p) = \frac{1}{2\pi}\int \psi^*(x - \frac{\hbar}{2}y)e^{-iyp}\psi(x + \frac{\hbar}{2}y)\,\mathrm{d}y. \tag{6.36}$$

ψ is an eigenfunction of the Schrödinger equation, $H\psi = E\psi$. As it stands, the Wigner function is not a probability distribution function since it can sometimes be negative. However, integrating it over one of the phase space coordinates, either x or p, results in an object which is always positive and can be considered to be probability distribution function. It is non-local, an important property for some of its uses. The Wigner function can be generalised [41]. For example, one could consider

$$f_{ab}(x,p) = \frac{1}{2\pi}\int \psi_a^*(x - \frac{\hbar}{2}y)e^{-iyp}\psi_b(x + \frac{\hbar}{2}y)\,\mathrm{d}y, \tag{6.37}$$

where the wavefunctions ψ_a are orthogonal,

$$\int \psi_a^*(x)\psi_b(x)\,\mathrm{d}x = \delta_{ab}. \tag{6.38}$$

It will be a generalised Wigner function which is used to solve the Moyal-Nahm equations.

6.3.4 Uses and Motivation

Quantisation

The original context for the use of Moyal Brackets was in a formulation of Quantum Mechanics. This is known as the Weyl-Wigner-Moyal formalism which uses Wigner distribution functions. Moyal wrote an evolution equation for these phase space distribution functions $f(x,p,t)$,

$$\frac{\partial f(x,p,t)}{\partial t} = \{H, f\}_{MB}. \tag{6.39}$$

H is the Hamiltonian. As is well known, when writing classical objects as quantum mechanical operators the ordering matters since the operators do not commute. In the Weyl-Wigner-Moyal formalism, Weyl ordering is used as the way of choosing the order the operators are written down. Baker showed that the Moyal evolution equation and Wigner distribution functions imply quantum mechanics [79].

Essentially, the Moyal quantisation process involves replacing all multiplication with star products and using Wigner distribution functions instead of the usual wavefunctions. Wherever there is usually a commutator there is now a Moyal bracket. The deformation parameter is $\hbar$. This is quite a natural way to quantise since in the classical limit ($\hbar \to 0$) the Moyal bracket reduces to the Poisson bracket as expected.

Bayen et al. stated that the Moyal bracket is the only deformation of the Poisson bracket which can be used like this [80], while Arveson showed that the Moyal Bracket is the only such object which can be used in the phase-space formulation of Quantum Mechanics [73].

M-Theory

Moyal brackets can not only be used in quantisation but they can also appear in association with M-Theory [46][81]. M-Theory is the 11-dimensional theory which has SUGRA as its low energy effective description. It also reduces to the five string theories in various limits. Banks et al. have constructed a M(atrix) Theory which is a matrix model which describes this theory when the large N limit is taken [36]. N is the size of the matrices. A typical action [38], in this case for matrix string theory, is

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{Tr} \left((D_i X^\mu)^2 + \theta^T \not{D} \theta + g_s^2 F_{ij}^2 - \frac{1}{g_s^2}[X^\mu, X^\nu]^2 + \frac{1}{g_s}\theta^T \gamma_\mu [X^\mu, \theta] \right) \mathrm{d}\sigma \mathrm{d}\tau. \tag{6.40}$$

This is the action for $\mathcal{N} = 8$ supersymmetric $U(N)$ Yang-Mills theory. The X^μ are $N \times N$ Hermitian matrices and are the scalar fields. The θ_L^α, θ_R^α are eight fermionic fields. The σ and τ are the world-sheet coordinates. To recover M-Theory we need to take the large N limit. The Moyal brackets give a way of approaching this limit. If we rewrite the theory by replacing all multiplication by star products, then the commutators are replaced by Moyal brackets, fermionic terms involve the cosine bracket and the matrices X^μ are now functions over a phase space, (α, β)

say. The deformation parameter is $\lambda = \frac{2\pi}{N}$. The action is now

$$S_{MB} = \frac{1}{2\pi\alpha'}\int\Bigg((D_i X^\mu)^2 + ((\theta^T, \not{D}\theta)) + g_s^2 \mathrm{Tr} F_{ij}^2 - \left(\frac{1}{\lambda g_s}\{X^\mu, X^\nu\}_{MB}\right)^2 + \frac{1}{g_s}((\theta^T\gamma_\mu, \frac{1}{\lambda}\{X^\mu, \theta\}_{MB}))\Bigg) \mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}\alpha\mathrm{d}\beta. \tag{6.41}$$

We now need to take the large N limit. As the value N (taken to be an odd integer else this does not work) is increased the Moyal bracket becomes an infinite sum of copies of the commutator $[X^\mu, X^\nu]$. In the large N limit this is the Poisson bracket. So, when we take the $N \to \infty$ limit of the Moyal action then we get an action involving Poisson brackets,

$$S_{PB} = \frac{1}{2\pi\alpha'}\int\Bigg((D_i X^\mu)^2 + \theta^T\not{D}\theta + g_s^2 \mathrm{Tr} F_{ij}^2 - \frac{1}{g_s^2}\{X^\mu, X^\nu\}_{PB}^2 + \frac{1}{g_s}\theta^T\gamma_\mu\{X^\mu, \theta\}_{PB}\Bigg) \mathrm{d}\sigma\mathrm{d}\tau\mathrm{d}\alpha\mathrm{d}\beta. \tag{6.42}$$

Moyal brackets just give a different way of considering the large N limit.

String Theory and Non-commutative Geometry

One of the most recent uses of the star product was in the work of Seiberg and Witten [44] (and all the spin-off papers from this work) which showed the equivalence between ordinary gauge fields and non-commutative ones. They considered non-commutative geometry with coordinates x^i which have a non-zero commutator given by

$$[x^i, x^j] = i\theta^{ij}. \tag{6.43}$$

In this case, the deformation parameter of the star product is taken to be the antisymmetric matrix θ which has components θ^{ij}. The star product is

$$\star = e^{\frac{i}{2}\theta^{ij}\partial_i\partial_j}. \tag{6.44}$$

They showed that ordinary Yang-Mills and non-commutative Yang-Mills are equivalent for open strings in a constant, non-zero B-field and that it is possible to go from one theory to the other simply via a change of variables.

6.4 4D Moyal-Nahm Equations and Solutions

The aim of this section is to find solutions to the Moyal-Nahm equations in four dimensional Euclidean space-time.

The Moyal-Nahm equations are simply the usual Nahm equations (6.7) where the commutators have been replaced by Moyal brackets and the matrices $A^k(t)$ are replaced by functions $X^k(t,x,p)$. This X^k (k=0,1,2,3) is a field in four dimensions which depends upon only one coordinate, in this case t, and phase space (x,p). The Moyal Nahm equations in four dimensions are

$$\begin{aligned}\frac{\partial X^1}{\partial t} &= \{X^2, X^3\}_{MB},\\ \frac{\partial X^2}{\partial t} &= \{X^3, X^1\}_{MB},\\ \frac{\partial X^3}{\partial t} &= \{X^1, X^2\}_{MB}.\end{aligned} \tag{6.45}$$

To solve this set of equations we use the ansatz

$$X^i = \frac{1}{i}\int_{-\infty}^{\infty} \psi_j^\dagger(x-y,t)\epsilon^{ijk}\psi_k(x+y,t)e^{2\pi i p y/\lambda}\,\mathrm{d}y, \tag{6.46}$$

which takes the form of a generalised Wigner function. The ϵ^{ijk} is the usual totally antisymmetric ϵ symbol (with convention $\epsilon^{123} = +1$). The $\psi(x,t)$ are three component wavefunctions. These wavefunctions were chosen to be of the form

$$\psi(x,t) = \begin{pmatrix}\psi_1(x,t)\\ \psi_2(x,t)\\ \psi_3(x,t)\end{pmatrix} = \begin{pmatrix} f_1(t)\phi_1(x)\\ f_2(t)\phi_2(x)\\ f_3(t)\phi_3(x)\end{pmatrix}, \tag{6.47}$$

where the $\phi_i(x)$ are orthonormal wavefunctions. The star product of X^j and X^k is calculated as follows:

$$\begin{aligned}X^j \star X^k = -\iint & \psi_i^\dagger(x-y,t)\epsilon^{jil}\psi_l(x+y,t)e^{2\pi i p y/\lambda}\star\\ & \psi_m^\dagger(x-y',t)\epsilon^{kmn}\psi_n(x+y',t)e^{2\pi i p y'/\lambda}\,\mathrm{d}y\mathrm{d}y'.\end{aligned}$$

Using (6.31)

$$\begin{aligned}= -\iint & \psi_i^\dagger(x-y+y',t)\epsilon^{jil}\psi_l(x+y+y',t)e^{2\pi i p y/\lambda}\\ & \psi_m^\dagger(x-y'-y,t)\epsilon^{kmn}\psi_n(x+y'-y,t)e^{2\pi i p y'/\lambda}\,\mathrm{d}y\mathrm{d}y',\end{aligned}$$

and via a change of variables

$$= -\frac{1}{2}\int \psi^{\dagger}_{m}(x-y,t)\epsilon^{kmn}Z_{nj}(t)\epsilon^{jil}\psi_{l}(x+y,t)e^{2\pi ipy/\lambda}\,\mathrm{d}y \qquad (6.48)$$
$$= -\frac{1}{2}\int \phi^{\dagger}_{m}(x-y)f^{\dagger}_{ms}(t)\epsilon^{ksn}Z_{nj}(t)\epsilon^{jir}f_{rl}(t)\phi_{l}(x+y)e^{2\pi ipy/\lambda}\,\mathrm{d}y,$$

where orthogonality of the wavefunctions $\phi_k(x)$ is assumed to be of the form

$$\int_{-\infty}^{\infty}\phi^{\dagger}_{j}(x)\phi_{k}(x)\,\mathrm{d}x = \delta_{jk}, \qquad (6.49)$$

and

$$Z(t) = ff^{\dagger} \qquad \text{where} \qquad f = \begin{pmatrix} f_1(t) & 0 & 0 \\ 0 & f_2(t) & 0 \\ 0 & 0 & f_3(t) \end{pmatrix}. \qquad (6.50)$$

The partial derivative $\frac{\partial X^i}{\partial t}$, a much simpler calculation, can be written as

$$\frac{\partial X^i}{\partial t} = \frac{1}{i}\int \phi^{\dagger}_{j}(x-y)\frac{\partial}{\partial t}(f^{\dagger}(t)\epsilon^{ijk}f(t))\phi_k(x+y)e^{2\pi ipy/\lambda}\mathrm{d}y. \qquad (6.51)$$

By putting these into the Moyal-Nahm equations we obtain three matrix equations of the form

$$\frac{1}{i}\frac{\partial}{\partial t}(f^{\dagger}(t)\epsilon^{1}f(t)) = \frac{-1}{4i}(f^{\dagger}(t)\epsilon^{3}Z(t)\epsilon^{2}f(t) - f^{\dagger}(t)\epsilon^{2}Z(t)\epsilon^{3}f(t)), \qquad (6.52)$$

where ϵ^i is a 3×3 matrix with jk^{th} entry ϵ^{ijk}. Equating the entries in the matrices gives differential equations of the form

$$\begin{aligned} \frac{\partial}{\partial t}(f_2^* f_3) &= -\frac{1}{4}|f_1|^2(f_2^* f_3), \\ \frac{\partial}{\partial t}(f_3^* f_2) &= -\frac{1}{4}|f_1|^2(f_3^* f_2), \end{aligned} \qquad (6.53)$$

and cyclic combinations of these. These result in the following set of three differential equations:

$$\begin{aligned} \frac{\partial}{\partial t}(|f_2|^2|f_3|^2) &= -\frac{1}{2}|f_1|^2|f_2|^2|f_3|^2, \\ \frac{\partial}{\partial t}(|f_3|^2|f_1|^2) &= -\frac{1}{2}|f_1|^2|f_2|^2|f_3|^2, \\ \frac{\partial}{\partial t}(|f_1|^2|f_2|^2) &= -\frac{1}{2}|f_1|^2|f_2|^2|f_3|^2. \end{aligned} \qquad (6.54)$$

Note that for each of the above, the right hand side of the equations is always the same. It is these equations which need to be solved to find the solutions to the Moyal-Nahm equations.

6.4.1 Simplest Solution

The simplest solution is to set all the f_i equal to each other. This gives the solution

$$|f_1|^2 = |f_2|^2 = |f_3|^2 = \frac{4}{t+K}, \tag{6.55}$$

so that

$$f_1(t) = f_2(t) = f_3(t) = \frac{2}{\sqrt{t+K}}, \tag{6.56}$$

where K is an arbitrary constant. Since each component of the field X^i is dependent on $|f_i|^2$ then the t dependence in this example is a simple pole.

6.4.2 Another Simple Solution

By setting two of the f_i equal to each other then a solution in terms of the hyperbolic functions can be found.

$$\begin{aligned} |f_1|^2 = |f_2|^2 &= 4q\coth(qt+K), \\ |f_3|^2 &= 8q\,\mathrm{csch}(2qt+2K), \end{aligned} \tag{6.57}$$

so that

$$\begin{aligned} f_1(t) = f_2(t) &= 2\sqrt{q\coth(qt+K)}, \\ f_3(t) &= 2\sqrt{2q\,\mathrm{csch}(2qt+2K)}, \end{aligned} \tag{6.58}$$

where K and q are both real constants.

6.4.3 General Solution

However, ideally we want a general solution to these equations. In this case the solutions are written in terms of elliptic functions sn, cn and dn. The most general solution was found to be

$$\begin{aligned} |f_1|^2 &= 4qk^2\frac{\mathrm{sn}(qt+c)\mathrm{cn}(qt+c)}{\mathrm{dn}(qt+c)}, \\ |f_2|^2 &= -4q\frac{\mathrm{cn}(qt+c)\mathrm{dn}(qt+c)}{\mathrm{sn}(qt+c)}, \\ |f_3|^2 &= 4q\frac{\mathrm{dn}(qt+c)\mathrm{sn}(qt+c)}{\mathrm{cn}(qt+c)}. \end{aligned} \tag{6.59}$$

q, c and k are all constants but may have to be carefully chosen in order to ensure that all the $|f_i|^2$ are positive. k depends on the elliptic functions.

6.5 8D Nahm Equations and Solutions

The work is now extended from four dimensions to eight dimensions. The Nahm equations in eight dimensions are

$$\frac{\partial A_i}{\partial \tau} = \frac{1}{2} c_{ijk}[A_j,\ A_k], \tag{6.60}$$

where the c_{ijk} are the structure constants which define the multiplication of the imaginary octonions. The equations are written out in full in (6.17). These equations are self-dual Yang-Mills equations in 8-dimensional Euclidean space where the gauge fields depend on only one coordinate , $x^0 = \tau$ (the zeroth coordinate), and the gauge is fixed so that $A^0 = 0$.

We now attempt to find solutions to the 8D Nahm equations. In particular, we are looking for matrix solutions. To do this we must first find sets of matrices which satisfy the algebra generated by the Nahm equations, *i.e.* sets of matrices, B_i $(i = 1, \ldots, 7)$, which satisfy

$$mB_i = \frac{1}{2} c_{ijk}[B_j, B_k], \tag{6.61}$$

where m is some number.

6.5.1 Solution 1

It is known that a solution to the 8D Nahm equations is

$$A_i = -\frac{1}{6\tau} e_i, \tag{6.62}$$

where the e_i form the basis of the imaginary octonions. Since the octonions are non-associative, there are no matrix representations of the octonion algebra. It would therefore be reasonable to question whether a matrix solution to the 8D Nahm equations exists at all.

However, the octonionic structure constants can be used to find a matrix solution of the Nahm equations. Seven matrices, $B_i (i = 1, \ldots, 7)$, were constructed where the j-kth component of the ith matrix is the octonionic structure constant c_{ijk}. *i.e.*

$$[B_i]_{jk} = c_{ijk}. \tag{6.63}$$

These matrices, B_i, are written out in full in Appendix D. The matrices satisfy the equations

$$3B_i = \frac{1}{2} c_{ijk}[B_j,\ B_k], \tag{6.64}$$

which are basically the Nahm equations without the partial derivative. The possible τ dependence of solutions based on these matrices was explored. In this case, the most general solution to the Nahm equations to be found was one involving a simple pole.

$$A_i = -\frac{1}{3\tau}B_i. \tag{6.65}$$

6.5.2 Solution 2

Solutions with a more general τ dependence can be found using a different set of matrices. Consider a set of matrices which are a direct sum of representations of the $SU(2)$ algebra. These matrices are not reducible. The example we use is below, although obviously there are other possible constructions.

$$\begin{aligned}
B_1 &= -i\begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, &\quad B_2 &= -i\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & b\sigma_2 & 0 \\ 0 & 0 & ic\sigma_3 \end{pmatrix}, \\
B_3 &= -i\begin{pmatrix} a\sigma_3 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & ic\sigma_2 \end{pmatrix}, &\quad B_4 &= -i\begin{pmatrix} ia\sigma_2 & 0 & 0 \\ 0 & b\sigma_3 & 0 \\ 0 & 0 & \sigma_2 \end{pmatrix}, \\
B_5 &= -i\begin{pmatrix} a\sigma_2 & 0 & 0 \\ 0 & -ib\sigma_3 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix}, &\quad B_6 &= -i\begin{pmatrix} ia\sigma_3 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & c\sigma_2 \end{pmatrix}, \\
B_7 &= -i\begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & ib\sigma_2 & 0 \\ 0 & 0 & c\sigma_3 \end{pmatrix}.
\end{aligned} \tag{6.66}$$

Each matrix is a direct sum of three sigma matrices. The sigma matrices are the usual 2×2 matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.67}$$

The set involves three arbitrary parameters $(a,\, b,\, c)$. None of these parameters can be set to zero if the solution is to stay irreducible. These seven matrices satisfy the equations (6.61), just like the last set of matrices, but this time $m = 2$. The aim is now to find solutions to the Nahm equations with non trivial τ dependence (setting $a = 1$, $b = 1$, $c = 1$). The most obvious way of doing this is to multiply each matrix by a function of τ, $f_i(\tau)$. So

$$A_i = f_i(\tau)B_i \qquad i = 1, \ldots, 7. \tag{6.68}$$

However, this ansatz is too restrictive and the only solution to be found is

$$A_i = -\frac{1}{2\tau}B_i \qquad i = 1, \ldots, 7, \tag{6.69}$$

as before. However, a more general ansatz gives a different result. This time multiply each matrix B_i by a diagonal 6×6 matrix C_i given by

$$C_i = \begin{pmatrix} f_i(\tau)\mathbb{1}_2 & 0 & 0 \\ 0 & g_i(\tau)\mathbb{1}_2 & 0 \\ 0 & 0 & h_i(\tau)\mathbb{1}_2 \end{pmatrix}. \tag{6.70}$$

This amounts to multiplying each σ matrix entry in each matrix B_i by a different τ dependent function. It is easiest to consider each row of σ matrices separately.

First, we shall look at the top row of sigma matrices and put in the τ dependence by multiplying each matrix by a function $f_i(\tau)$. Putting these 2×2 matrices into the Nahm equations gives the following set of differential equations,

$$\begin{aligned} &\frac{\partial f_2}{\partial \tau} = 2f_1f_7 + 2f_5f_3 - 2f_4f_6, \\ &\frac{\partial f_1}{\partial \tau} = 2f_2f_7, \qquad \frac{\partial f_3}{\partial \tau} = 2f_2f_5, \qquad \frac{\partial f_4}{\partial \tau} = 2f_2f_6, \\ &\frac{\partial f_5}{\partial \tau} = 2f_2f_3, \qquad \frac{\partial f_6}{\partial \tau} = 2f_2f_4, \qquad \frac{\partial f_7}{\partial \tau} = 2f_2f_1, \end{aligned} \tag{6.71}$$

and the following constraints,

$$f_7f_3 = f_1f_5, \qquad f_6f_7 = f_1f_4, \qquad f_3f_4 = f_5f_6. \tag{6.72}$$

Note that all of the differential equations involve f_2, but none of the constraints do.

These can be solved in terms of elliptic functions. It was found that

$$\begin{aligned} f_6 = K_1f_3 = K_1M_1f_1 &= \frac{1}{2}K_1M_1Q_1\mathrm{sn}(q_1\tau + d_1), \\ f_4 = K_1f_5 = K_1M_1f_7 &= \frac{-i}{2}K_1M_1Q_1\mathrm{cn}(q_1\tau + d_1), \\ f_2 &= \frac{i}{2}q_1\mathrm{dn}(q_1\tau + d_1), \end{aligned} \tag{6.73}$$

where cn, sn, dn are elliptic functions and K_1, M_1, q_1, Q_1, d_1 are all constants. The elliptic functions are related to each other by a parameter k_1 as follows

$$\begin{aligned} \mathrm{sn}^2(x) + \mathrm{cn}^2(x) &= 1, \\ \mathrm{dn}^2(x) + k_1^2 \mathrm{sn}^2(x) &= 1, \end{aligned} \tag{6.74}$$

where $k_1 = \frac{Q_1}{q_1}\sqrt{1 + M_1^2(1 - K_1^2)}$.

Similar results are obtained when the second and third rows of sigma matrices in the B_i are considered and the functions g_i and h_i have a similar form to the f_i. Putting all of this together, the following set of matrices satisfy Nahm's equations in eight dimensions.

$$\begin{aligned}
A_1 &= -i\begin{pmatrix} f_1\sigma_3 & 0 & 0 \\ 0 & g_1\sigma_3 & 0 \\ 0 & 0 & h_1\sigma_3 \end{pmatrix}, \qquad A_2 = -i\begin{pmatrix} f_2\sigma_1 & 0 & 0 \\ 0 & g_2\sigma_2 & 0 \\ 0 & 0 & ih_2\sigma_3 \end{pmatrix}, \\
A_3 &= -i\begin{pmatrix} f_3\sigma_3 & 0 & 0 \\ 0 & g_3\sigma_2 & 0 \\ 0 & 0 & ih_3\sigma_2 \end{pmatrix}, \qquad A_4 = -i\begin{pmatrix} if_4\sigma_2 & 0 & 0 \\ 0 & g_4\sigma_3 & 0 \\ 0 & 0 & h_4\sigma_2 \end{pmatrix}, \\
A_5 &= -i\begin{pmatrix} f_5\sigma_2 & 0 & 0 \\ 0 & -ig_5\sigma_3 & 0 \\ 0 & 0 & h_5\sigma_1 \end{pmatrix}, \qquad A_6 = -i\begin{pmatrix} if_6\sigma_3 & 0 & 0 \\ 0 & g_6\sigma_1 & 0 \\ 0 & 0 & h_6\sigma_2 \end{pmatrix}, \\
A_7 &= -i\begin{pmatrix} f_7\sigma_2 & 0 & 0 \\ 0 & ig_7\sigma_2 & 0 \\ 0 & 0 & h_7\sigma_3 \end{pmatrix},
\end{aligned} \tag{6.75}$$

where f_i are given in (6.73) the other τ dependent functions are given by

$$\begin{aligned} g_5 = K_2 g_4 = K_2 M_2 g_1 &= \frac{1}{2} K_2 M_2 Q_2 \mathrm{sn}(q_2\tau + d_2), \\ g_7 = K_2 g_2 = K_2 M_2 g_3 &= \frac{-i}{2} K_2 M_2 Q_2 \mathrm{cn}(q_2\tau + d_2), \\ g_6 &= \frac{i}{2} q_2 \mathrm{dn}(q_2\tau + d_2), \end{aligned} \tag{6.76}$$

$$\begin{aligned} h_2 = K_3 h_7 = K_3 M_3 h_1 &= \frac{1}{2} K_3 M_3 Q_3 \mathrm{sn}(q_3\tau + d_3), \\ h_3 = K_3 h_6 = K_3 M_3 h_4 &= \frac{-i}{2} K_3 M_3 Q_3 \mathrm{cn}(q_3\tau + d_3), \\ h_5 &= \frac{i}{2} q_3 \mathrm{dn}(q_3\tau + d_3). \end{aligned} \tag{6.77}$$

The matrices used here are not the only solutions which can constructed using direct sums of representations of $SU(2)$. The sigma matrices in each row can be replaced by representations of $SU(2)$ of any dimension. Also, the matrix B_i does not have to be a direct sum of three objects, it could be a sum of any number of two or more objects. For example, B_i which are 4×4 matrices can be found by omitting the last two rows and columns of the B_i used above.

6.6 8D Moyal-Nahm Equations and Solutions

The Moyal-Nahm equations in eight dimensions are:

$$\begin{aligned}
\frac{\partial X^1}{\partial t} &= \{X^2, X^7\}_{MB} + \{X^6, X^3\}_{MB} + \{X^5, X^4\}_{MB}, \\
\frac{\partial X^2}{\partial t} &= \{X^7, X^1\}_{MB} + \{X^5, X^3\}_{MB} + \{X^4, X^6\}_{MB}, \\
\frac{\partial X^3}{\partial t} &= \{X^1, X^6\}_{MB} + \{X^2, X^5\}_{MB} + \{X^4, X^7\}_{MB}, \\
\frac{\partial X^4}{\partial t} &= \{X^1, X^5\}_{MB} + \{X^6, X^2\}_{MB} + \{X^7, X^3\}_{MB}, \\
\frac{\partial X^5}{\partial t} &= \{X^4, X^1\}_{MB} + \{X^3, X^2\}_{MB} + \{X^6, X^7\}_{MB}, \\
\frac{\partial X^6}{\partial t} &= \{X^3, X^1\}_{MB} + \{X^2, X^4\}_{MB} + \{X^7, X^5\}_{MB}, \\
\frac{\partial X^7}{\partial t} &= \{X^1, X^2\}_{MB} + \{X^3, X^4\}_{MB} + \{X^5, X^6\}_{MB}.
\end{aligned} \tag{6.78}$$

Again, they were obtained from the Nahm equations by replacing the commutators with Moyal Brackets and the matrices A_i with functions X^i. An ansatz based on the generalised Wigner function, similar to the one used for the 4D Moyal-Nahm equations, is used. It is

$$X_i = i\int_{-\infty}^{\infty} \psi^\dagger(x-y,\tau)B_i\psi(x+y,\tau)e^{2\pi i p y/\lambda}\mathrm{d}y, \tag{6.79}$$

where $\psi(x,t)$ are six component wave functions of the form

$$\psi_j = f_j(\tau)\phi_j(x), \quad j \text{ not summed}, \tag{6.80}$$

and B_i are 6×6 matrices. Again, the $\phi_j(x)$ are orthonormal.

6.6.1 Sigma Solution

In the first solution, the matrices, B_i, are the direct sums of sigma matrices as given in (6.66) but with parameters $a, b, c = 1$. There is no loss of generality when doing this since these parameters can be considered to be contained in the τ dependent functions $f_i(\tau)$. As before, this ansatz was put into the Moyal-Nahm equations. The differential equations for the functions $f_i(t)$ were found to of the form

$$\begin{aligned}\frac{\partial|f_1|^2}{\partial\tau} &= -|f_1|^2|f_2|^2, & \frac{\partial(f_1^*f_2)}{\partial\tau} &= -\frac{1}{2}(|f_1|^2+|f_2|^2)f_1^*f_2,\\ \frac{\partial|f_2|^2}{\partial\tau} &= -|f_1|^2|f_2|^2, & \frac{\partial(f_1f_2^*)}{\partial\tau} &= -\frac{1}{2}(|f_1|^2+|f_2|^2)f_1f_2^*.\end{aligned} \tag{6.81}$$

The differential equations are similar for f_3, f_4 and f_5, f_6.

They can be solved to give the following solution.

$$\begin{aligned}&\{f_1,f_2,f_3,f_4,f_5,f_6\} = \\ &\left\{\frac{2\sqrt{K_1}e^{i\theta_1}}{\sqrt{1-e^{-K_1\tau}}},\frac{2\sqrt{K_1}e^{i\theta_2}}{\sqrt{e^{K_1\tau}-1}},\frac{2\sqrt{K_2}e^{i\theta_3}}{\sqrt{1-e^{-K_2\tau}}},\frac{2\sqrt{K_2}e^{i\theta_4}}{\sqrt{e^{K_2\tau}-1}},\frac{2\sqrt{K_3}e^{i\theta_5}}{\sqrt{1-e^{-K_3\tau}}},\frac{2\sqrt{K_3}e^{i\theta_6}}{\sqrt{e^{K_3\tau}-1}}\right\}.\end{aligned} \tag{6.82}$$

All K_i and θ_j are real constants. This solution can be generalised to solutions with B_i matrices which are direct sums of any number of, but at least two, sigma matrices.

6.6.2 Epsilon Solution

Another solution can be found using the three dimensional representation of $SU(2)$, which involves the completely antisymmetric matrices ϵ_{ijk}. This time, the matrices B_i are the direct sums of two epsilon matrices. For our example, we used

$$\begin{aligned}&B_1=-\begin{pmatrix}\epsilon_3 & 0\\ 0 & \epsilon_3\end{pmatrix},\ B_2=-\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix},\ B_3=-\begin{pmatrix}\epsilon_3 & 0\\ 0 & \epsilon_2\end{pmatrix},\ B_4=-\begin{pmatrix}i\epsilon_2 & 0\\ 0 & \epsilon_3\end{pmatrix},\\ &B_5=-\begin{pmatrix}\epsilon_2 & 0\\ 0 & i\epsilon_3\end{pmatrix},\ B_6=-\begin{pmatrix}i\epsilon_3 & 0\\ 0 & \epsilon_1\end{pmatrix},\ B_7=-\begin{pmatrix}\epsilon_2 & 0\\ 0 & i\epsilon_2\end{pmatrix},\end{aligned} \tag{6.83}$$

where the jk^{th} entry of the matrix ϵ_i is given by the totally antisymmetric tensor ϵ_{ijk}. *i.e.*

$$\epsilon_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\qquad \epsilon_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\qquad \epsilon_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}. \tag{6.84}$$

Putting these B_i matrices into the ansatz we now solve the differential equations for the functions $f_i(\tau)$ obtained from the Moyal-Nahm equations. These are of the same form as the ones obtained from the 4D Moyal-Nahm equations when solved using ϵ matrices (6.53). The τ dependent functions $f_i(\tau)$ are therefore same form as the solution (6.59). The solutions are

$$\begin{aligned}
|f_1|^2 &= 4qk^2\frac{\mathrm{sn}(q\tau+c)\mathrm{cn}(q\tau+c)}{\mathrm{dn}(q\tau+c)}, & |f_4|^2 &= 4QK^2\frac{\mathrm{sn}(Q\tau+b)\mathrm{cn}(Q\tau+b)}{\mathrm{dn}(Q\tau+b)},\\
|f_2|^2 &= -4q\frac{\mathrm{cn}(q\tau+c)\mathrm{dn}(q\tau+c)}{\mathrm{sn}(q\tau+c)}, & |f_5|^2 &= -4Q\frac{\mathrm{cn}(Q\tau+b)\mathrm{dn}(Q\tau+b)}{\mathrm{sn}(Q\tau+b)},\\
|f_3|^2 &= 4q\frac{\mathrm{dn}(q\tau+c)\mathrm{sn}(q\tau+c)}{\mathrm{cn}(q\tau+c)}, & |f_6|^2 &= 4Q\frac{\mathrm{dn}(Q\tau+b)\mathrm{sn}(Q\tau+b)}{\mathrm{cn}(Q\tau+b)}.
\end{aligned} \tag{6.85}$$

q, Q, c, b, k, K are all constants. k, K are the parameters which depend on the elliptic functions. These solutions can be extended for direct sums of more than two ϵ_i matrices. For a direct sum of n ϵ_i matrices, the B_i will be $3n \times 3n$ matrices and $\psi(x,\tau)$ will be $3n$ component wavefunctions.

It would also be possible to find solutions using other direct sums of representations of $SU(2)$ for the B_i in the ansatz.

6.6.3 Octonion Solution

The same method can be used to find solutions based on the seven matrices constructed using the octonionic structure constants. The same ansatz (6.79) is used as before but this time the matrices B_i are (6.63) and the $\psi(x,\tau)$ are seven component wavefunctions. Putting the ansatz into the Moyal Nahm equations gives differential equations for seven functions f_i. However, the only solution to be found for these equations was

$$f_i = -\frac{1}{\sqrt{\tau}}, \tag{6.86}$$

so the whole function X^i has a simple pole solution. The only way of obtaining other solutions is if several of the f_i are set to zero.

6.7 Summary

In this chapter we have discussed the Moyal-Nahm equations and their solutions. These equations come from the Nahm equations, the solutions to which form a set of solutions to self-dual Yang-Mills. The concept of self-duality has mostly been studied in four dimensions but has been extended to higher dimensions.

The Nahm equations in eight dimensions have been given. The motivation for studying eight dimensions is the existence of eight dimensional Yang-Mills in matrix string theory and the fact that eight dimensional self-duality closely resembles self-duality in four dimensions. This last part is most clearly seen when the self-duality condition is written in terms of quaternionic structure constants in four dimensions and octonionic structure constants in eight dimensions.

To obtain the Moyal-Nahm equations we simply replaced the matrices in the Nahm equations with functions and all multiplication with star products. The commutators in the Nahm equations became Moyal Brackets. The Moyal Bracket is a one parameter deformation of the Poisson bracket and was first introduced in the context of writing down a formulation for quantum mechanics.

Solutions to the Moyal-Nahm equations in four and eight dimensions were found using an ansatz based on the generalised Wigner function. Such Wigner functions often appear in theories involving Moyal Brackets. Solutions to the eight dimensional Nahm equations were also found.

During the construction of solutions, sets of matrices which satisfy the algebra created by the 8D Nahm equations when the partial derivatives are removed were obtained. One of these sets of matrices was constructed using the octonionic structure constants. The octonions seem to feature strongly in the eight dimensional case.

Chapter 7

Conclusion

The question asked at the beginning of this work was is there a field theory associated with strings and branes analogous to the Klein-Gordon theory for particles? The conjecture of a field theory with the companion Lagrangian and its equations of motion seems to be a good candidate for such a theory. Although it may require some alterations, in principle this proposal is a good one and deserves further investigation.

The companion Lagrangian is a better idea than the early proposals of Hosotani and those of Morris since it reduces to the Klein-Gordon Lagrangian in the particle case which these other ideas do not. In the particle case they have many fields, not one. The companion Lagrangian has the same number of fields as the number of world-sheet coordinates for the object it is describing. Like the later work of Hosotani and Nakayama, the companion Lagrangian is further motivated by the Hamilton-Jacobi formalism for strings and branes.

One of the remaining questions is whether to take the Lagrangian with or without a square root. While the non-square root case is a direct analogue of the Klein-Gordon Lagrangian, the square root case has many things in its favour. It possesses general covariance, the equations of motion have either been shown to be integrable or show signs they will be, and the Lagrangian is a direct continuation of the Dirac-Born-Infeld Lagrangian but for more base space coordinates than target space coordinates. However, recently it has been shown that the equations of motion for the companion Lagrangians with and without square roots are the same if the Lagrangian is set to be a non-zero constant.

These two types of Lagrangians can also be linked together by an equivalence theorem which states that the equations of motion for a companion Lagrangian without a square root when subjected to some constraints are the same as the equations of motion for a companion Lagrangian with a square root in one less dimension but with the same number of fields. A proof has been given for this. However, these constraints have not been fully understood so further work needs to be done, either to understand these constraints or to find other ones which lead to the same equivalence. The constraints that have been found may turn out to be sufficient but not necessary. They could be a special case of some more general constraints. It would also be interesting to find out if any other types of Lagrangians have a similar equivalence theorem. The proof in the appendix depends on the use of an epsilon identity which could be useful in other calculations and proofs.

Both the Born-Infeld Lagrangian and companion Lagrangian can be written as the square root of Jacobians in quadratic form. This persists even when a background metric is added. The equations of motion for these theories have a similar structure and both involve Jacobians. However, the number of independent equations of motion differs in each case. For the Born-Infeld case the number of equations depends on the number of target space coordinates and base space coordinates. For the companion equations this number depends only on the number of fields, not the number of dimensions. The companion equations are sums of Bateman equations or Universal Field equations. This makes the theory integrable or at least makes it easy to find a large class of solutions. More work could be done in this area to find more general solutions for theories with any number of fields or dimensions.

The inclusion of electromagnetism in the theory was investigated briefly and four possible ways of incorporating a gauge field were given. It is not clear which proposal is the right one, although the proposals which maintain covariance are the strongest candidates since this is a desirable property in our theory.

The fact that the companion Lagrangian with a square root is a homogeneous function of weight one in the Jacobians made it possible to extend the iterative procedure for Lagrangians with one field to Lagrangians with two or more fields. At present this has only really been done for two fields in three dimensions. The most interesting aspect of this was that after two iterations the expression only depends on one of the fields in the form of a generalised Bateman equation. It is completely independent of the other field. The extension to $d-1$ fields in d dimensions is relatively straightforward. However, further work could be done to investigate more d=3, n=2 cases. Also, extension to more fields in higher dimensions should be done so that more than two iterations can be considered.

One of the properties of the companion Lagrangian is that it is a divergence on the space of solutions of its equations of motion. In fact, this is also true for other theories. Lagrangians for a large family of field theories are a divergence or vanish on the space of solutions of their equations of motion. This property means such theories could be called 'pseudo-topological' because a Lagrangian which is a divergence without any additional constraints is fully topological. The full meaning and implication of this property is, as yet, unknown. Therefore, further investigation is required.

Overall, an important and interesting question concerning a field theory associated with strings and branes has been discussed and solutions to its equations of motion have been found. Although the ideas may require some modification, they give a good basis for further investigation. The study of the companion Lagrangian and its equations of motion has also led to other observations which are relevant to other Lagrangians too. This includes the iterative procedure work which covers many Lagrangians which are homogeneous functions of weight one in the Jacobians, not just those Lagrangians with the structure of the companion Lagrangian. It also includes the property that all free field theory Lagrangians are a divergence on the space of solutions of the equations of motion, of which the companion Lagrangian is just one example. There is scope for a lot more research into this subject.

The last part of this thesis was a search for solutions to the Moyal-Nahm equations. It was found that solutions to the Moyal-Nahm equations do exist.

Solutions to the 8D Nahm equations and 4D and 8D Moyal-Nahm equations have been found, although the list given is by no means exhaustive. The solutions are constructed from generalised Wigner functions. The dependence on the coordinate t or τ is often based on a simple pole or elliptic functions.

The solutions in four dimensions are useful because this is the number of dimensions we like to think we live in. The solutions in eight dimensions may turn out to be useful in the context of M(atrix) theory which has a Yang-Mills action which involves eight bosonic fields, X^{μ}.

As well as the solutions, sets of matrices which satisfy the algebra generated by the eight dimensional Nahm equations have been found. One set is based on the octonionic structure constants. The other set is based on representations of $SU(2)$. The matrices are a direct sum of any number of representations of $SU(2)$.

Any representation can be used. These sets of matrices may be useful elsewhere, possibly in other areas which are related to the octonions. There may be other matrices which obey the algebra.

With the increase in interest in non-commutative gauge fields and replacing multiplication with star products in many theories, these results could turn out to be useful in the future. This work shows that by putting star products into self-dual Yang-Mills, solutions can be found and a way to construct an ansatz for such theories has been given. A similar ansatz, involving Wigner functions, could be used to find solutions to other theories involving Moyal Brackets.

Further research could be done to find more solutions to the Moyal-Nahm equations in four, eight, and maybe other dimensions, as well as finding solutions to other theories containing star products and Moyal brackets.

Appendix A

Proof for the Equivalence Theorem

The proof in this appendix is based on [4].

A.1 Theorem

The equations of motion for the companion Lagrangian, without a square root, with n fields, ϕ^i, in d space-time dimensions (x^μ),

$$\mathcal{L} = \det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|, \qquad \mu = 1, 2, \ldots, d, \tag{A.1}$$

subject to the constraints

$$\frac{\partial\mathcal{L}}{\partial\phi^i_\mu}\phi^i_{\mu\nu} = 0 \qquad i \text{ not summed, } \mu \text{ is summed,} \tag{A.2}$$

and the Lagrangian vanishing, are the same as the equations of motion for the companion Lagrangian with a square root with n fields but $d-1$ space-time dimensions, *i.e.* in one dimension less,

$$\mathcal{L}' = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|}, \qquad \mu = 1, 2, \ldots, d-1. \tag{A.3}$$

A.2 Conventions and Notation

The n fields are ϕ^i where $i = 1, \ldots, n$. They depend on the d space-time coordinates x^μ, where $\mu = 1, \ldots, d$. Partial derivatives are denoted by

$$\frac{\partial\phi^i}{\partial x^\mu} = \phi^i_\mu, \qquad \frac{\partial^2\phi^i}{\partial x^\mu\partial x^\nu} = \phi^i_{\mu\nu}. \tag{A.4}$$

There is summation over indices unless otherwise stated.

Totally antisymmetric tensors $\epsilon_{\nu_1\nu_2\ldots\nu_d}$ are used throughout the proof with the convention $\epsilon_{12\ldots d} = +1$. When indices have an arrow above them then they represent several indices. They can be thought of as vectors with several components. $\vec{\mu}, \vec{\nu}, \vec{\rho}, \vec{\sigma}$ each have $(n-1)$ components. For example, $\vec{\mu}$ denotes $\{\mu_2, \mu_3, \ldots, \mu_n\}$. $\vec{\tau}, \vec{\kappa}$ each have $(d-n)$ components. For example, $\vec{\kappa}$ denotes $\{\kappa_1, \kappa_2, \ldots, \kappa_{d-n}\}$. $\vec{\kappa}'$ denotes $\{\kappa_2, \kappa_3, \ldots, \kappa_{d-n}\}$ and $\vec{\kappa}''$ denotes $\{\kappa_3, \ldots, \kappa_{d-n}\}$.
For the product of $(n-1)$ fields we use the notation $\Phi_{\vec{\nu}} = \phi^2_{\nu_2}\phi^3_{\nu_3}\cdots\phi^n_{\nu_n}$.
Also, $r = d - n$.

A.3 Useful Epsilon Identity

A useful identity for the antisymmetric epsilon tensors, which will be used throughout the proof, is

$$\begin{aligned}\epsilon_{\mu\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\rho_2\ldots\rho_d} = \epsilon_{\rho_1\nu_2\nu_3\ldots\nu_d}\epsilon_{\mu\rho_2\ldots\rho_d} + \epsilon_{\rho_2\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\mu\rho_3\ldots\rho_d} + \cdots \\ \cdots + \epsilon_{\rho_d\nu_2\nu_3\ldots\nu_d}\epsilon_{\rho_1\rho_2\ldots\rho_{d-1}\mu}.\end{aligned} \tag{A.5}$$

It amounts to swapping the index μ from the first epsilon with each index from the second epsilon. For a more involved explanation of this identity, see Appendix C.

A.4 Equations of motion

Consider the Lagrangian for n fields ϕ^i in d space-time dimensions x^μ which does not involve a square root,

$$\mathcal{L} = \det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|. \tag{A.6}$$

The equations of motion for this Lagrangian are

$$\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\phi^j_{\mu\nu} = 0. \tag{A.7}$$

These determinantal Lagrangians can be written as the sum of squares of Jacobians. The Jacobians will be denoted as

$$J_{\vec{\kappa}} = J_{\kappa_1\kappa_2\ldots\kappa_{d-n}} = \epsilon_{\kappa_1\kappa_2\ldots\kappa_{d-n}\nu_1\nu_2\ldots\nu_n}\phi^1_{\nu_1}\phi^2_{\nu_2}\cdots\phi^n_{\nu_n}. \tag{A.8}$$

For the square root case the Lagrangian is

$$\mathcal{L}' = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|} = \sqrt{\frac{1}{(d-n)!}J_{\vec{\kappa}}J_{\vec{\kappa}}}. \tag{A.9}$$

The equations of motion for this can be written as

$$J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\phi^i_{\mu\nu} = 0. \tag{A.10}$$

The calculation for obtaining these equations of motion is given in Appendix B. For n fields in $d-1$ dimensions, the equations of motion for the companion Lagrangian with a square root can be written as

$$J_{d\alpha\vec{\kappa}''}J_{d\beta\vec{\kappa}''}\phi^i_{\alpha\beta} = 0. \tag{A.11}$$

This is the expression we will be looking for.

A.5 The Constraints

The equations of motion for the non-square root case will be subject to the following constraints.

$$\frac{\partial\mathcal{L}}{\partial\phi^i_\mu}\phi^i_{\mu\nu} = 0. \tag{A.12}$$

There is no summation over the index i, but there is over index μ. Also, we set $\mathcal{L} = 0$.

The idea is to reduce the number of dimensions from d to $d-1$. The constraints (A.12) can be used to eliminate all second derivatives of the fields which involve a partial derivative with respect to x^d, the dth dimension. *i.e.* From the constraints

$$\phi^i_{d\beta} = -\frac{\frac{\partial\mathcal{L}}{\partial\phi^i_\alpha}}{\frac{\partial\mathcal{L}}{\partial\phi^i_d}}\phi^i_{\alpha\beta}, \qquad \phi^i_{dd} = \frac{\frac{\partial\mathcal{L}}{\partial\phi^i_\alpha}\frac{\partial\mathcal{L}}{\partial\phi^i_\beta}}{\left(\frac{\partial\mathcal{L}}{\partial\phi^i_d}\right)^2}\phi^i_{\alpha\beta}. \tag{A.13}$$

Again, there is no summation over i but there is over $\alpha, \beta = 1, 2, \ldots (d-1)$. Putting these constraints into the equations of motion (A.7) we have

$$\sum_{j=1}^{n}\frac{1}{\left(\frac{\partial\mathcal{L}}{\partial\phi^j_d}\right)^2}\left[\left(\frac{\partial\mathcal{L}}{\partial\phi^j_d}\right)^2\frac{\partial^2\mathcal{L}}{\partial\phi^i_\alpha\partial\phi^j_\beta} - \frac{\partial\mathcal{L}}{\partial\phi^j_d}\frac{\partial\mathcal{L}}{\partial\phi^j_\beta}\frac{\partial^2\mathcal{L}}{\partial\phi^i_\alpha\partial\phi^j_d} - \frac{\partial\mathcal{L}}{\partial\phi^j_d}\frac{\partial\mathcal{L}}{\partial\phi^j_\alpha}\frac{\partial^2\mathcal{L}}{\partial\phi^i_\beta\partial\phi^j_d} + \frac{\partial\mathcal{L}}{\partial\phi^j_\alpha}\frac{\partial\mathcal{L}}{\partial\phi^j_\beta}\frac{\partial^2\mathcal{L}}{\partial\phi^i_d\partial\phi^j_d}\right]\phi^j_{\alpha\beta} = 0. \tag{A.14}$$

The equations of motion no longer involve any second derivatives with respect to x^d. Note that the indices $\alpha, \beta = 1, 2, \ldots, (d-1)$ throughout this proof.

A.6 The Proof

For the moment we shall consider the equation of motion with respect to field $\phi^1 = \phi$ and we are only looking at the component which involves the terms $\phi_{\alpha\beta}$. The other components will work in the same way. Writing the Lagrangian without a square root in terms of Jacobians then we have

$$\begin{aligned}\mathcal{L} &= \frac{1}{(d-n)!} J_{\vec{\kappa}} J_{\vec{\kappa}} \\ &= \frac{1}{(d-n)!} \epsilon_{\nu\vec{\kappa}\vec{\nu}} \epsilon_{\rho\vec{\kappa}\vec{\rho}} \phi_\nu \phi_\rho \Phi_{\vec{\nu}} \Phi_{\vec{\rho}} \\ &= \frac{1}{(d-n)!} \phi_\nu \phi_\rho B_{\nu\rho} \qquad \text{where} \qquad B_{\nu\rho} = \epsilon_{\nu\vec{\kappa}\vec{\nu}} \epsilon_{\rho\vec{\kappa}\vec{\rho}} \Phi_{\vec{\nu}} \Phi_{\vec{\rho}},\end{aligned} \tag{A.15}$$

using the notation defined in section A.2. The first and second partial derivatives of the Lagrangian can therefore be written as

$$\frac{\partial \mathcal{L}}{\partial \phi_\mu} = \frac{2}{(d-n)!} \phi_\nu B_{\mu\nu}, \qquad \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} = \frac{2}{(d-n)!} B_{\mu\nu}. \tag{A.16}$$

The numerator of the coefficient of $\phi_{\alpha\beta}$ in (A.14) becomes

$$[B_{\mu d}(B_{\nu d} B_{\alpha\beta} - B_{\nu\beta} B_{\alpha d}) + B_{\mu\alpha}(B_{\nu\beta} B_{dd} - B_{\nu d} B_{\beta d})]\phi_\mu \phi_\nu. \tag{A.17}$$

Now,

$$\begin{aligned}B_{\nu d} B_{\alpha\beta} - B_{\nu\beta} B_{\alpha d} &= \left[\epsilon_{\nu\vec{\kappa}\vec{\mu}} \epsilon_{d\vec{\kappa}\vec{\nu}} \epsilon_{\alpha\vec{\tau}\vec{\rho}} \epsilon_{\beta\vec{\tau}\vec{\sigma}} - \epsilon_{\nu\vec{\kappa}\vec{\mu}} \epsilon_{\beta\vec{\kappa}\vec{\nu}} \epsilon_{\alpha\vec{\tau}\vec{\rho}} \epsilon_{d\vec{\tau}\vec{\sigma}}\right] \Phi_{\vec{\mu}} \Phi_{\vec{\nu}} \Phi_{\vec{\rho}} \Phi_{\vec{\sigma}} \\ &= \epsilon_{\nu\vec{\kappa}\vec{\mu}} \epsilon_{\alpha\vec{\tau}\vec{\rho}} [\epsilon_{d\vec{\kappa}\vec{\nu}} \epsilon_{\beta\vec{\tau}\vec{\sigma}} - \epsilon_{\beta\vec{\kappa}\vec{\nu}} \epsilon_{d\vec{\tau}\vec{\sigma}}] \Phi_{\vec{\mu}} \Phi_{\vec{\nu}} \Phi_{\vec{\rho}} \Phi_{\vec{\sigma}}.\end{aligned} \tag{A.18}$$

Using the epsilon identity (A.5) to move the index β around in the first two epsilons in the square bracket then

$$\begin{aligned}\epsilon_{d\vec{\kappa}\vec{\nu}} \epsilon_{\beta\vec{\tau}\vec{\sigma}} = {} & \epsilon_{\beta\vec{\kappa}\vec{\nu}} \epsilon_{d\vec{\tau}\vec{\sigma}} + \epsilon_{d\beta\kappa_2\ldots\kappa_r\vec{\nu}} \epsilon_{\kappa_1\vec{\tau}\vec{\sigma}} + \epsilon_{d\kappa_2\beta\ldots\kappa_r\vec{\nu}} \epsilon_{\kappa_2\vec{\tau}\vec{\sigma}} + \ldots + \epsilon_{d\kappa_1\ldots\kappa_{r-1}\beta\vec{\nu}} \epsilon_{\kappa_r\vec{\tau}\vec{\sigma}} \\ & + \epsilon_{d\vec{\kappa}\beta\nu_3\ldots\nu_n} \epsilon_{\nu_2\vec{\tau}\vec{\sigma}} + \epsilon_{d\vec{\kappa}\nu_2\beta\ldots\nu_n} \epsilon_{\nu_3\vec{\tau}\vec{\sigma}} + \ldots + \epsilon_{d\vec{\kappa}\nu_2\ldots\nu_{n-1}\beta} \epsilon_{\nu_n\vec{\tau}\vec{\sigma}}.\end{aligned} \tag{A.19}$$

The first term on the right hand side is just the other term in expression (A.18) so this will cancel. The last $n-1$ terms will all vanish when put into (A.18) due to symmetry conditions. This is because in the second epsilon, $\epsilon_{\nu_i\vec{\tau}\vec{\sigma}}$, ν_i and σ_i are antisymmetric, but in (A.18) ν_i and σ_i are symmetric due to the $\Phi_{\vec{\nu}}\Phi_{\vec{\sigma}}$ term, so these last $n-1$ terms vanish. This only leaves the middle terms. By swapping the labels κ_1 and κ_i and using antisymmetry of the epsilons to permute the κ_i with other indices we have

$$\begin{aligned}
\epsilon_{\nu\kappa_1\ldots\kappa_r\vec{\mu}}\epsilon_{d\kappa_1\kappa_2\ldots\beta\ldots\kappa_r\vec{\nu}}\epsilon_{\kappa_i\vec{\tau}\vec{\sigma}} &= \epsilon_{\nu\kappa_i\ldots\kappa_1\ldots\kappa_r\vec{\mu}}\epsilon_{d\kappa_i\kappa_2\ldots\beta\ldots\kappa_r\vec{\nu}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}} \\
&= \epsilon_{\nu\kappa_1\ldots\kappa_i\ldots\kappa_r\vec{\mu}}\epsilon_{d\beta\kappa_2\ldots\kappa_i\ldots\kappa_r\vec{\nu}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}} \\
&= \epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{d\beta\kappa_2\ldots\kappa_r\vec{\nu}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}. \qquad (A.20)
\end{aligned}$$

There are $r = d - n$ of these terms. Therefore,

$$B_{\nu d}B_{\alpha\beta} - B_{\nu\beta}B_{\alpha d} = r\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}. \qquad (A.21)$$

The epsilon identity (A.5) can be used again to swap subscript κ_1 about in the first two epsilons in the expression above

$$\begin{aligned}
\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}} = {} & \epsilon_{\nu\alpha\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\kappa_1\vec{\tau}\vec{\rho}} + \epsilon_{\nu\tau_1\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\kappa_1\tau_2\ldots\tau_r\vec{\rho}} + \ldots + \epsilon_{\nu\tau_r\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\tau_1\ldots\tau_{r-1}\kappa_1\vec{\rho}} \\
& + \epsilon_{\nu\rho_2\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\vec{\tau}\kappa_1\rho_3\ldots\rho_n} + \ldots + \epsilon_{\nu\rho_n\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\vec{\tau}\rho_2\ldots\rho_{n-1}\kappa_1}. \qquad (A.22)
\end{aligned}$$

The last $n-1$ terms will vanish when put into (A.21) due to symmetry considerations of the indices μ_i and ρ_i. By relabelling indices and using the antisymmetric property of the epsilons the middle terms become

$$\begin{aligned}
\epsilon_{\nu\tau_i\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\tau_1\ldots\kappa_1\ldots\tau_r\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}} &= \epsilon_{\nu\kappa_1\kappa_2\ldots\kappa_r\vec{\mu}}\epsilon_{\alpha\tau_1\ldots\tau_i\ldots\tau_r\vec{\rho}}\epsilon_{\tau_i\tau_1\ldots\kappa_1\ldots\tau_r\vec{\sigma}} \\
&= -\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}. \qquad (A.23)
\end{aligned}$$

Since there are r of these terms we can write

$$\begin{aligned}
\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}\Phi_{\vec{\mu}}\Phi_{\vec{\rho}} &= (\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{\kappa_1\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}} - r\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}})\Phi_{\vec{\mu}}\Phi_{\vec{\rho}} \\
\Rightarrow \quad (1+r)\epsilon_{\nu\vec{\kappa}\vec{\mu}}\epsilon_{\alpha\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}\Phi_{\vec{\mu}}\Phi_{\vec{\rho}} &= \epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{\kappa_1\vec{\tau}\vec{\rho}}\epsilon_{\kappa_1\vec{\tau}\vec{\sigma}}\Phi_{\vec{\mu}}\Phi_{\vec{\rho}}. \qquad (A.24)
\end{aligned}$$

which gives

$$B_{\nu d}B_{\alpha\beta} - B_{\nu\beta}B_{\alpha d} = \frac{r}{r+1}B_{\kappa\kappa}[\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}]. \qquad (A.25)$$

Substituting this into the expression (A.17) we find

$$\begin{aligned}
& B_{\tau\tau}[B_{\mu d}\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}} + B_{\mu\alpha}\epsilon_{\nu d\vec{\kappa}'\vec{\mu}}\epsilon_{\beta d\vec{\kappa}'\vec{\nu}}]\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\phi_\mu\phi_\nu \\
&= B_{\tau\tau}[\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{d\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}} - \epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{\alpha\vec{\tau}\vec{\sigma}}\epsilon_{\nu d\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}]\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\mu\phi_\nu \\
&= B_{\tau\tau}[\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}(\epsilon_{d\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}} - \epsilon_{\alpha\vec{\tau}\vec{\sigma}}\epsilon_{\nu d\vec{\kappa}'\vec{\mu}})]\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\mu\phi_\nu. \qquad (A.26)
\end{aligned}$$

Now, using (A.5) to move subscript d,

$$\epsilon_{d\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\vec{\kappa}'\vec{\mu}} = \epsilon_{\nu\vec{\tau}\vec{\sigma}}\epsilon_{d\alpha\vec{\kappa}'\vec{\mu}} + \epsilon_{\alpha\vec{\tau}\vec{\sigma}}\epsilon_{\nu d\vec{\kappa}'\vec{\mu}} + \epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha d\kappa_3\ldots\kappa_r\vec{\mu}} + \ldots + \epsilon_{\kappa_r\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\kappa_2\ldots\kappa_{r-1}d\vec{\mu}}$$
$$+ \epsilon_{\mu_2\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\vec{\kappa}'d\mu_3\ldots\mu_n} + \ldots + \epsilon_{\mu_n\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\vec{\kappa}'\mu_2\ldots\mu_{n-1}d}. \tag{A.27}$$

The second term will cancel with the term in (A.26). The last $n-1$ terms will vanish due to symmetry considerations for the indices μ_i and σ_i. For the middle terms, by relabelling and using antisymmetry,

$$\epsilon_{\kappa_i\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha\kappa_2\ldots d\ldots\kappa_r\vec{\mu}}\epsilon_{d\beta\kappa_2\ldots\kappa_i\ldots\kappa_r\vec{\nu}} = \epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha d\vec{\kappa}''\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}. \tag{A.28}$$

There are $(r-1) = (d-n-1)$ of these terms. We now have

$$\frac{r}{r+1}B_{\tau\tau}[\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{\nu\vec{\tau}\vec{\sigma}}\epsilon_{d\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}} + (r-1)\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha d\vec{\kappa}''\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}]\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\mu\phi_\nu. \tag{A.29}$$

Again, rewriting the epsilons from the second part of (A.29), this time moving subscript κ_2,

$$\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}} = \epsilon_{\kappa_2\vec{\tau}\vec{\rho}}\epsilon_{d\beta\mu\vec{\kappa}''\vec{\nu}} + \epsilon_{\mu\kappa_2\tau_2\ldots\tau_r\vec{\rho}}\epsilon_{d\beta\tau_1\vec{\kappa}''\vec{\nu}} + \ldots + \epsilon_{\mu\tau_1\ldots\tau_{r-1}\kappa_2\vec{\rho}}\epsilon_{d\beta\tau_r\vec{\kappa}''\vec{\nu}}$$
$$+ \epsilon_{\mu\vec{\tau}\kappa_2\rho_3\ldots\rho_n}\epsilon_{d\beta\rho_2\vec{\kappa}''\vec{\nu}} + \ldots + \epsilon_{\mu\vec{\tau}\rho_2\ldots\rho_{n-1}\kappa_2}\epsilon_{d\beta\rho_n\vec{\kappa}''\vec{\nu}}. \tag{A.30}$$

The last $n-1$ terms will vanish due to symmetry considerations of ν_i and ρ_i. For the middle terms, again by relabelling,

$$\epsilon_{\mu\tau_1\ldots\kappa_2\ldots\tau_r\vec{\rho}}\epsilon_{d\beta\tau_i\vec{\kappa}''\vec{\nu}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}} = -\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}. \tag{A.31}$$

There are $r = d-n$ of these terms so,

$$(r+1)\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}} = \epsilon_{\kappa_2\vec{\tau}\vec{\rho}}\epsilon_{d\beta\mu\vec{\kappa}''\vec{\nu}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}. \tag{A.32}$$

Therefore, we now have

$$\frac{r}{r+1}B_{\tau\tau}[\epsilon_{\mu\vec{\tau}\vec{\rho}}\epsilon_{\nu\vec{\tau}\vec{\sigma}}\epsilon_{d\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}} + \frac{r-1}{r+1}\epsilon_{\kappa_2\vec{\tau}\vec{\rho}}\epsilon_{\kappa_2\vec{\tau}\vec{\sigma}}\epsilon_{\nu\alpha d\vec{\kappa}''\vec{\mu}}\epsilon_{d\beta\vec{\kappa}''\vec{\nu}}]\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\mu\phi_\nu\phi_{\alpha\beta}. \tag{A.33}$$

Rewriting this in terms of the Lagrangians and Jacobians then this becomes

$$\begin{aligned}
&\frac{r}{r+1}B_{\tau\tau}[\epsilon_{d\alpha\vec{\kappa}'\vec{\mu}}\epsilon_{d\beta\vec{\kappa}'\vec{\nu}}\Phi_{\vec{\mu}}\Phi_{\vec{\nu}}r!\mathcal{L} - \frac{r-1}{r+1}B_{\kappa\kappa}J_{d\alpha\vec{\kappa}''}J_{d\beta\vec{\kappa}''}]\phi_{\alpha\beta} \\
= \;&\frac{rr!}{r+1}\left(\frac{\partial J_{\vec{\mu}}}{\partial\phi_\tau}\frac{\partial J_{\vec{\mu}}}{\partial\phi_\tau}\right)\left[\left(\frac{\partial J_{d\vec{\kappa}'}}{\partial\phi_\alpha}\frac{\partial J_{d\vec{\kappa}'}}{\partial\phi_\beta}\right)\mathcal{L} - \frac{r-1}{(r+1)!}\left(\frac{\partial J_{\vec{\nu}}}{\partial\phi_\kappa}\frac{\partial J_{\vec{\nu}}}{\partial\phi_\kappa}\right)J_{d\alpha\vec{\kappa}''}J_{d\beta\vec{\kappa}''}\right]\phi_{\alpha\beta}.
\end{aligned} \tag{A.34}$$

This completes the calculation for the terms involving $\phi_{\alpha\beta}$. A very similar calculation can be carried out to rewrite the coefficients of $\phi^j_{\alpha\beta}$ ($j \neq 1$) from (A.14).

Set $\phi^j = \psi$ and this time define $B_{\nu\rho}$ to be

$$B_{\nu\rho} = \epsilon_{\nu\nu_2\mu_1\ldots\mu_{d-n}\nu_3\ldots\nu_n}\epsilon_{\rho_1\rho\mu_1\ldots\mu_{d-n}\rho_3\ldots\rho_n}\phi_{\rho_1}\psi_{\rho_2}\phi^3_{\nu_3}\ldots\phi^n_{\nu_n}\phi^3_{\rho_3}\ldots\phi^n_{\rho_n}. \tag{A.35}$$

We choose $j = 2$ to make notation easier but j could be chosen to be any value $j = 2, 3, \ldots, n$. The calculation is then almost identical to the one given above, and the term involving $\psi_{\alpha\beta}$ is found to be

$$\frac{rr!}{r+1}\left(\frac{\partial J_{\bar{\mu}}}{\partial \phi_\tau}\frac{\partial J_{\bar{\mu}}}{\partial \psi_\tau}\right)\left[\left(\frac{\partial J_{d\bar{K}'}}{\partial \psi_\alpha}\frac{\partial J_{d\bar{K}'}}{\partial \psi_\beta}\right)\mathcal{L} - \frac{r-1}{(r+1)!}\left(\frac{\partial J_{\bar{\nu}}}{\partial \psi_\kappa}\frac{\partial J_{\bar{\nu}}}{\partial \psi_\kappa}\right)J_{d\alpha\bar{K}''}J_{d\beta\bar{K}''}\right]\psi_{\alpha\beta}. \tag{A.36}$$

When the condition that the Lagrangian vanishes is put into the equations of motion, they can be rearranged to give

$$J_{d\alpha\bar{K}''}J_{d\beta\bar{K}''}\phi^i_{\alpha\beta} = 0, \tag{A.37}$$

as required. Comparing (A.37) with (A.11), these are the equations of motion for the Lagrangian involving a square root (A.9) in $(d-1)$ dimensions.

Therefore, it has been proved that the equations of motion for the companion Lagrangian without a square root when subject to some constraints are equivalent to the equations of motion for the companion Lagrangian with a square root in one less dimension.

Appendix B

Equations of Motion for the Companion Lagrangian

In this appendix we show that the equations of motion for the companion Lagrangian with a square root ,

$$\mathcal{L} = \sqrt{\det\left|\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x_\mu}\right|}, \tag{B.1}$$

can be written as

$$J_{\mu\vec{K}'}J_{\nu\vec{K}'}\phi^i_{\mu\nu} = 0. \tag{B.2}$$

Remember that these equations of motion are sums of the Universal Field Equations which take the form given in (1.40).

Again, we make extensive use of the epsilon identity

$$\begin{aligned}\epsilon_{\mu\nu_2\nu_3\dots\nu_d}\epsilon_{\rho_1\rho_2\dots\rho_d} = \epsilon_{\rho_1\nu_2\nu_3\dots\nu_d}\epsilon_{\mu\rho_2\dots\rho_d} + \epsilon_{\rho_2\nu_2\nu_3\dots\nu_d}\epsilon_{\rho_1\mu\rho_3\dots\rho_d} + \dots \\ \dots + \epsilon_{\rho_d\nu_2\nu_3\dots\nu_d}\epsilon_{\rho_1\rho_2\dots\rho_{d-1}\mu}.\end{aligned} \tag{B.3}$$

A fuller explanation of this identity can be found in Appendix C. The equation of motion with respect to field ϕ^i is

$$\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\phi^j_{\mu\nu} = 0. \tag{B.4}$$

Notation

Indices with arrows above them represent several indices which can be thought of as a vector. We use the following notation:
$\vec{\alpha}$, $\vec{\gamma}$, $\vec{\rho}$, $\vec{\sigma}$ denote $n-1$ component vectors. *E.g.* $\vec{\alpha}$ denotes $\{\alpha_2, \alpha_3, \ldots, \alpha_n\}$.
$\vec{\kappa}$, $\vec{\tau}$ denote $d-n$ component vectors. *E.g.* $\vec{\kappa}$ denotes $\{\kappa_1, \kappa_2, \ldots, \kappa_{d-n}\}$.
$\vec{\kappa}'$ denotes the $d-n-1$ component vector $\{\kappa_2, \kappa_3, \ldots, \kappa_{d-n}\}$.
Also $\Phi_{\vec{\alpha}}$ denotes the product of $n-1$ fields $\phi^2_{\alpha_2}\phi^3_{\alpha_3}\cdots\phi^n_{\alpha_n}$.
We set $r = d-n$.

First part of calculation

Start by setting field $\phi^1 = \phi$. Consider the equation of motion (B.4) with respect to ϕ. The Jacobians and their derivatives can be written as

$$J_{\vec{\kappa}} = \epsilon_{\vec{\kappa}\nu_1\vec{\nu}}\phi_{\nu_1}\Phi_{\vec{\nu}}, \qquad \frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu} = \epsilon_{\vec{\kappa}\mu\vec{\nu}}\Phi_{\vec{\nu}}, \qquad \frac{\partial^2 J_{\vec{\kappa}}}{\partial\phi_\mu\partial\phi_\nu} = 0. \tag{B.5}$$

Up to a numerical factor the Lagrangian is

$$\mathcal{L} = \sqrt{J_{\vec{\kappa}}J_{\vec{\kappa}}}. \tag{B.6}$$

Therefore,

$$\frac{\partial\mathcal{L}}{\partial\phi_\mu} = \frac{1}{\mathcal{L}}\frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu}J_{\vec{\kappa}}, \tag{B.7}$$

so,

$$\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu}\phi_{\mu\nu} = \frac{1}{\mathcal{L}^3}\left(\frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu}\frac{\partial J_{\vec{\kappa}}}{\partial\phi_\nu}J_{\vec{\tau}}J_{\vec{\tau}} - \frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu}J_{\vec{\kappa}}\frac{\partial J_{\vec{\tau}}}{\partial\phi_\nu}J_{\vec{\tau}}\right)\phi_{\mu\nu}. \tag{B.8}$$

This is the term in the equation of motion with respect to ϕ which contains $\phi_{\mu\nu}$. Using the definition of the Jacobian (B.5), the numerator of this can be written as

$$\begin{aligned}
&\left(\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\kappa}\nu\vec{\gamma}}\epsilon_{\vec{\tau}\gamma\vec{\rho}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}} - \epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\kappa}\gamma\vec{\gamma}}\epsilon_{\vec{\tau}\nu\vec{\rho}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}}\right)\Phi_{\vec{\alpha}}\Phi_{\vec{\gamma}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\alpha\phi_\gamma\phi_{\mu\nu} \\
= \; & \epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}}\left(\epsilon_{\vec{\kappa}\nu\vec{\gamma}}\epsilon_{\vec{\tau}\gamma\vec{\rho}} - \epsilon_{\vec{\kappa}\gamma\vec{\gamma}}\epsilon_{\vec{\tau}\nu\vec{\rho}}\right)\Phi_{\vec{\alpha}}\Phi_{\vec{\gamma}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\alpha\phi_\gamma\phi_{\mu\nu}.
\end{aligned} \tag{B.9}$$

Consider the first two epsilons inside the bracket. Using the epsilon identity to swap index γ,

$$\epsilon_{\vec{\kappa}\nu\vec{\gamma}}\epsilon_{\vec{\tau}\gamma\vec{\rho}} = \sum_{i=1}^{r}\epsilon_{\kappa_1\kappa_2\ldots\gamma\ldots\kappa_r\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_i\vec{\rho}} + \epsilon_{\vec{\kappa}\gamma\vec{\gamma}}\epsilon_{\vec{\tau}\nu\vec{\rho}} + \sum_{i=2}^{n}\epsilon_{\vec{\kappa}\nu\gamma_2\gamma_3\ldots\gamma\ldots\gamma_n}\epsilon_{\vec{\tau}\gamma_i\vec{\rho}}. \tag{B.10}$$

The last $n-1$ terms will all vanish due to the antisymmetry of the epsilons and the symmetry of $\Phi_{\vec{\gamma}}\Phi_{\vec{\rho}}$. The middle term is just the second term in (B.9). The first $r=d-n$ terms, when multiplied by $\epsilon_{\vec{\kappa}\mu\vec{\alpha}}$, are all the same. This is seen by swapping the labels of κ_1 and κ_i with each other and then rearranging the indices.

$$\begin{aligned}\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\kappa_1\kappa_2\ldots\gamma\ldots\kappa_r\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_i\vec{\rho}} &= \epsilon_{\kappa_i\kappa_2\ldots\kappa_1\ldots\kappa_r\mu\vec{\alpha}}\epsilon_{\kappa_i\kappa_2\ldots\gamma\ldots\kappa_r\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}} \\ &= \epsilon_{\kappa_1\kappa_2\ldots\kappa_i\ldots\kappa_r\mu\vec{\alpha}}\epsilon_{\gamma\kappa_2\ldots\kappa_i\ldots\kappa_r\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}} \\ &= \epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\gamma\vec{\kappa}'\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}. \end{aligned} \tag{B.11}$$

Therefore, the expression in (B.9) can now be written as

$$r\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}}\epsilon_{\gamma\vec{\kappa}'\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}\Phi_{\vec{\alpha}}\Phi_{\vec{\gamma}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\alpha\phi_\gamma\phi_{\mu\nu}. \tag{B.12}$$

Looking at the first two epsilons, we can use the epsilon identity again to move the index κ_1 around,

$$\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}} = \sum_{i=1}^{r}\epsilon_{\tau_i\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\tau_1\tau_2\ldots\kappa_1\ldots\tau_r\alpha\vec{\sigma}} + \epsilon_{\alpha\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\vec{\sigma}} + \sum_{i=2}^{n}\epsilon_{\sigma_i\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\sigma_2\sigma_3\ldots\kappa_1\ldots\sigma_n}. \tag{B.13}$$

The last $n-1$ terms will vanish due to the symmetry of $\Phi_{\vec{\alpha}}\Phi_{\vec{\sigma}}$. The first r terms, when multiplied by $\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}$, can be rewritten by swapping the labels κ_1 and τ_i and then rearranging the indices, as follows,

$$\begin{aligned}\epsilon_{\tau_i\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\tau_1\tau_2\ldots\kappa_1\ldots\tau_r\alpha\vec{\sigma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}} &= \epsilon_{\kappa_1\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\tau_1\tau_2\ldots\tau_i\ldots\tau_r\alpha\vec{\sigma}}\epsilon_{\tau_1\tau_2\ldots\kappa_1\ldots\tau_r\tau_i\vec{\rho}} \\ &= -\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}. \end{aligned} \tag{B.14}$$

This has the same form as the left hand side of (B.13) (when it is also multiplied by $\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}$) so we have

$$(r+1)\epsilon_{\vec{\kappa}\mu\vec{\alpha}}\epsilon_{\vec{\tau}\alpha\vec{\sigma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}\Phi_{\vec{\alpha}}\Phi_{\vec{\sigma}} = \epsilon_{\alpha\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\vec{\sigma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}\Phi_{\vec{\alpha}}\Phi_{\vec{\sigma}}. \tag{B.15}$$

Our expression (B.12) is now

$$\begin{aligned}&\frac{r}{r+1}\epsilon_{\alpha\vec{\kappa}'\mu\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\vec{\sigma}}\epsilon_{\gamma\vec{\kappa}'\nu\vec{\gamma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}\Phi_{\vec{\alpha}}\Phi_{\vec{\gamma}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}\phi_\alpha\phi_\gamma\phi_{\mu\nu} \\ &= \frac{r}{r+1}[\epsilon_{\vec{\tau}\kappa_1\vec{\sigma}}\epsilon_{\vec{\tau}\kappa_1\vec{\rho}}\Phi_{\vec{\rho}}\Phi_{\vec{\sigma}}][\epsilon_{\mu\vec{\kappa}'\alpha\vec{\alpha}}\epsilon_{\nu\vec{\kappa}'\gamma\vec{\gamma}}\Phi_{\vec{\alpha}}\Phi_{\vec{\gamma}}\phi_\alpha\phi_\gamma]\phi_{\mu\nu} \\ &= \frac{r}{r+1}\frac{\partial J_{\vec{\tau}}}{\partial\phi_{\kappa_1}}\frac{\partial J_{\vec{\tau}}}{\partial\phi_{\kappa_1}}J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\phi_{\mu\nu}. \end{aligned} \tag{B.16}$$

Now we need to consider the coefficients of $\phi^i_{\mu\nu}$ in the equation of motion, where $i\neq 1$.

Slight Change in Notation

From now on,

$\vec{\alpha}$, $\vec{\gamma}$, $\vec{\rho}$, $\vec{\sigma}$ denote $n-2$ component vectors. *E.g.* $\vec{\alpha}$ denotes $\{\alpha_3, \alpha_4, \ldots, \alpha_n\}$.

$\Psi_{\vec{\alpha}}$ denotes the product of $n-2$ fields $\phi^3_{\alpha_3}\phi^4_{\alpha_4}\cdots\phi^n_{\alpha_n}$.

All other notation is the same.

Second part of calculation

Without any loss of generality, we consider the term containing $\phi^2_{\mu\nu}$. For convenience, set $\phi^2 = \psi$. The calculation follows similar lines to the one above. We now write the Jacobian and its derivatives as

$$J_{\vec{\kappa}'} = \epsilon_{\vec{\kappa}'\rho_1\rho_2\vec{\rho}}\phi_{\rho_1}\psi_{\rho_2}\Psi_{\vec{\rho}}, \tag{B.17}$$

$$\frac{\partial J_{\vec{\kappa}'}}{\partial \phi_\mu} = \epsilon_{\vec{\kappa}'\mu\rho_2\vec{\rho}}\psi_{\rho_2}\Psi_{\vec{\rho}}, \qquad \frac{\partial J_{\vec{\kappa}'}}{\partial \psi_\nu} = \epsilon_{\vec{\kappa}'\rho_1\nu\vec{\rho}}\phi_{\rho_1}\Psi_{\vec{\rho}}. \tag{B.18}$$

For the term in the equation of motion containing $\psi_{\mu\nu}$ we find

$$\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\psi_\nu}\psi_{\mu\nu} = \frac{1}{\mathcal{L}^3}\left(\frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu}\frac{\partial J_{\vec{\kappa}}}{\partial\psi_\nu}J_{\vec{\tau}}J_{\vec{\tau}} - \frac{\partial J_{\vec{\kappa}}}{\partial\phi_\mu}J_{\vec{\kappa}}\frac{\partial J_{\vec{\tau}}}{\partial\psi_\nu}J_{\vec{\tau}}\right)\psi_{\mu\nu}. \tag{B.19}$$

There are no second derivatives of the Jacobians because these terms vanish due to the symmetry of $\psi_{\mu\nu}$. Putting these new expressions for the Jacobians into (B.19), we find

$$\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\vec{\sigma}}\left(\epsilon_{\vec{\kappa}\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\rho_2\vec{\rho}} - \epsilon_{\vec{\kappa}\gamma_1\rho_2\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\nu\vec{\rho}}\right)\phi_{\gamma_1}\phi_{\rho_1}\phi_{\sigma_1}\psi_{\alpha_2}\psi_{\rho_2}\psi_{\sigma_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\gamma}}\Psi_{\vec{\rho}}\Psi_{\vec{\sigma}}\psi_{\mu\nu}. \tag{B.20}$$

Using the epsilon identity to move the ρ_2 index about,

$$\begin{aligned}\epsilon_{\vec{\kappa}\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\rho_2\vec{\rho}} = &\sum_{i=1}^{r}\epsilon_{\kappa_1\kappa_2\ldots\rho_2\ldots\kappa_r\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\kappa_i\vec{\rho}} + \epsilon_{\vec{\kappa}\rho_2\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\gamma_1\vec{\rho}} \\ &+ \epsilon_{\vec{\kappa}\gamma_1\rho_2\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\nu\vec{\rho}} + \sum_{i=3}^{n}\epsilon_{\vec{\kappa}\gamma_1\nu\gamma_3\gamma_4\ldots\rho_2\ldots\gamma_n}\epsilon_{\vec{\tau}\rho_1\gamma_i\vec{\rho}}.\end{aligned} \tag{B.21}$$

The first r terms will give $r\epsilon_{\rho_2\vec{\kappa}'\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}$ in the expression from a similar argument to that used in (B.11). The next term will vanish due to the symmetry of $\phi_{\rho_1}\phi_{\gamma_1}$. The next term appears in (B.20). The last $n-2$ terms will vanish due to the symmetry of $\Psi_{\vec{\rho}}\Psi_{\vec{\gamma}}$. Our expression (B.20) is now

$$r\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\vec{\sigma}}\epsilon_{\rho_2\vec{\kappa}'\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\phi_{\gamma_1}\phi_{\rho_1}\phi_{\sigma_1}\psi_{\alpha_2}\psi_{\rho_2}\psi_{\sigma_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\gamma}}\Psi_{\vec{\rho}}\Psi_{\vec{\sigma}}\psi_{\mu\nu}. \tag{B.22}$$

Using the epsilon identity to move the κ_1 index around in the first two epsilons, then

$$\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\vec{\sigma}} = \sum_{i=1}^{r}\epsilon_{\tau_i\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\tau_1\tau_2\ldots\kappa_1\ldots\tau_r\sigma_1\sigma_2\vec{\sigma}} + \epsilon_{\sigma_1\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\sigma_2\vec{\sigma}}$$
$$+ \epsilon_{\sigma_2\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\kappa_1\vec{\sigma}} + \sum_{i=3}^{n}\epsilon_{\sigma_i\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\sigma_3\ldots\kappa_1\ldots\sigma_n}. \tag{B.23}$$

The last $n-2$ terms will vanish due to the symmetry of $\Psi_{\vec{\alpha}}\Psi_{\vec{\sigma}}$. The term before this will vanish due to the symmetry of $\psi_{\alpha_2}\psi_{\sigma_2}$. The first r terms can each be rewritten as $-\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\vec{\sigma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}$ in the same manner as in (B.14), so that

$$(r+1)\epsilon_{\vec{\kappa}\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\sigma_1\sigma_2\vec{\sigma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\psi_{\alpha_2}\psi_{\sigma_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\sigma}} = \epsilon_{\sigma_1\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\sigma_2\vec{\sigma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\psi_{\alpha_2}\psi_{\sigma_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\sigma}}. \tag{B.24}$$

The expression is now

$$\begin{aligned}&\frac{r}{r+1}\epsilon_{\sigma_1\vec{\kappa}'\mu\alpha_2\vec{\alpha}}\epsilon_{\vec{\tau}\kappa_1\sigma_2\vec{\sigma}}\epsilon_{\rho_2\vec{\kappa}'\gamma_1\nu\vec{\gamma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\phi_{\gamma_1}\phi_{\rho_1}\phi_{\sigma_1}\psi_{\alpha_2}\psi_{\rho_2}\psi_{\sigma_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\gamma}}\Psi_{\vec{\rho}}\Psi_{\vec{\sigma}}\psi_{\mu\nu}\\ =\ &\frac{r}{r+1}[\epsilon_{\vec{\tau}\kappa_1\sigma_2\vec{\sigma}}\epsilon_{\vec{\tau}\rho_1\kappa_1\vec{\rho}}\phi_{\rho_1}\Psi_{\vec{\rho}}\psi_{\sigma_2}\Psi_{\vec{\sigma}}][\epsilon_{\mu\vec{\kappa}'\sigma_1\alpha_2\vec{\alpha}}\epsilon_{\nu\vec{\kappa}'\gamma_1\rho_2\vec{\gamma}}\phi_{\sigma_1}\psi_{\alpha_2}\phi_{\gamma_1}\psi_{\rho_2}\Psi_{\vec{\alpha}}\Psi_{\vec{\gamma}}]\psi_{\mu\nu}.\end{aligned} \tag{B.25}$$

In terms of Jacobians, this is equal to

$$\frac{r}{r+1}\frac{\partial J_{\vec{\tau}}}{\partial\phi_{\kappa_1}}\frac{\partial J_{\vec{\tau}}}{\partial\psi_{\kappa_1}}J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\psi_{\mu\nu}. \tag{B.26}$$

Therefore, in general, the equations of motion can be written as

$$\frac{\partial^2\mathcal{L}}{\partial\phi^i_\mu\partial\phi^j_\nu}\phi^j_{\mu\nu} = \frac{r}{r+1}\frac{1}{\mathcal{L}^3}\frac{\partial J_{\vec{\tau}}}{\partial\phi^i_{\kappa_1}}\frac{\partial J_{\vec{\tau}}}{\partial\phi^j_{\kappa_1}}J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\phi^j_{\mu\nu} = 0. \tag{B.27}$$

By simple rearranging, it can be seen that this is equivalent to

$$J_{\mu\vec{\kappa}'}J_{\nu\vec{\kappa}'}\phi^j_{\mu\nu} = 0. \tag{B.28}$$

Appendix C

Useful Epsilon Identity

The proofs in appendices A and B make use of an epsilon identity. This appendix explains how the identity works. It is easiest to consider a totally antisymmetric epsilon tensor with only two indices, ϵ_{ij}, where $i, j = 1, 2$ and $\epsilon_{12} = +1$.

The epsilon identity in this case is

$$\epsilon_{ij}\epsilon_{ab} = \epsilon_{aj}\epsilon_{ib} + \epsilon_{bj}\epsilon_{ai}. \tag{C.1}$$

This amount to swapping the index i with each of the indices a and b in the second epsilon on the left hand side to give a sum of two terms on the right hand side. To see that this is true we need to consider several cases.

Case 1

$\epsilon_{ij}\epsilon_{ab} = +1$ so $i = a$, $j = b$ but $i \neq j, b$.
Therefore,
$\epsilon_{aj}\epsilon_{ib} = +1$ since $a = i$, $j = b$ but $a \neq j, b$, as above, and
$\epsilon_{bj}\epsilon_{ai} = 0$ since $i = a$.
The right hand side of (C.1) equals the left hand side so for this case the identity is true.

Case 2

$\epsilon_{ij}\epsilon_{ab} = -1$ so $i = b$, $j = a$ but $i \neq j, a$.
Therefore,
$\epsilon_{aj}\epsilon_{ib} = 0$ since $a = j$, and
$\epsilon_{bj}\epsilon_{ai} = -1$ since $b = j$, $a = i$ but $b \neq i, a$.
The right hand side of (C.1) equals the left side so for this case the identity is true.

Case 3

$\epsilon_{ij}\epsilon_{ab} = 0.$

In this case then there are three possibilities.

(i) $i = j$, $a \neq b$ or $i \neq j$, $a = b$. Three of the four indices must be the same so $\epsilon_{aj}\epsilon_{ib} = 0$, $\epsilon_{bj}\epsilon_{ai} = 0$, since in each case, for one of the epsilons both indices are the same. Therefore, the right hand side of the identity is zero as required.

(ii) $i = j = a = b$. All four indices are the same so $\epsilon_{aj}\epsilon_{ib} = 0$, $\epsilon_{bj}\epsilon_{ai} = 0$. Therefore the right hand side of the identity is zero as required.

(iii) $i = j$, $a = b$ but $i \neq a, b$. Therefore, $\epsilon_{aj}\epsilon_{ib} = -\epsilon_{bj}\epsilon_{ai}$, so the right hand side of the identity vanishes. Therefore the identity is true in this case.

So, it has been shown that in all cases the epsilon identity is true when there are two indices. Similar arguments can be used to show that the identity works for higher numbers of indices.

The identity for three indices, using the convention $\epsilon_{123} = +1$ is

$$\epsilon_{ijk}\epsilon_{abc} = \epsilon_{ajk}\epsilon_{ibc} + \epsilon_{bjk}\epsilon_{aic} + \epsilon_{cjk}\epsilon_{abi}. \tag{C.2}$$

The identity for four indices, using the convention $\epsilon_{1234} = +1$ is

$$\epsilon_{ijkl}\epsilon_{abcd} = \epsilon_{ajkl}\epsilon_{ibcd} + \epsilon_{bjkl}\epsilon_{aicd} + \epsilon_{cjkl}\epsilon_{abid} + \epsilon_{djkl}\epsilon_{abci}. \tag{C.3}$$

In general

So, in general, the useful epsilon identity for a totally antisymmetric epsilon tensor with n indices, using the convention $\epsilon_{12\ldots n} = +1$ is

$$\begin{aligned}\epsilon_{ij_2j_3\ldots j_n}\epsilon_{a_1a_2\ldots a_n} &= \epsilon_{a_1j_2j_3\ldots j_n}\epsilon_{ia_2\ldots a_n} + \epsilon_{a_2j_2j_3\ldots j_n}\epsilon_{a_1ia_3\ldots a_n} + \cdots + \epsilon_{a_nj_2j_3\ldots j_n}\epsilon_{a_1a_2\ldots a_{n-1}i} \\ &= \sum_{r=1}^{n} \epsilon_{a_rj_2j_3\ldots j_n}\epsilon_{a_1a_2\ldots i\ldots a_n}.\end{aligned} \tag{C.4}$$

The index i from the first epsilon on the left hand side is swapped with each index from the second epsilon on the left hand side to give a sum of n terms on the right hand side.

Appendix D

Octonion Matrices

The seven 7×7 matrices on the next page are solutions to the algebra created from the Nahm equations. These matrices were constructed using the octonionic structure constants, c_{ijk}, which are taken to be

$$c_{127} = c_{631} = c_{541} = c_{532} = c_{246} = c_{347} = c_{567} = 1. \tag{D.1}$$

These are totally antisymmetric. All other c_{ijk} are zero. The jk^{th} entry of the matrix B_i is given by $[B_i]_{jk} = c_{ijk}$. The matrices are used in the solutions which have been found for the Moyal-Nahm equations in eight dimensions.

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \qquad B_6 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_7 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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