Developments in noncommutative differential geometry

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A Thesis presented for the degree of

Doctor of Philosophy

Department of Mathematical Sciences
University of Durham

March 2002

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Abstract

One of the great outstanding problems of theoretical physics is the quantisation of gravity, and an associated description of quantum spacetime. It is often argued that, at short distances, the manifold structure of spacetime breaks down and is replaced by some sort of algebraic structure. Noncommutative geometry is a possible candidate for the mathematics of this structure. However, physical theories on noncommutative spaces are still essentially classical and need to be quantised.

We present a path integral formalism for quantising gravity in the form of the spectral action. Our basic principle is to sum over all Dirac operators. The approach is demonstrated on two simple finite noncommutative geometries (the two-point space and the matrix geometry $\mathrm{M}_2(\mathbb{C})$) and a circle. In each case, we start with the partition function and calculate the graviton propagator and Greens functions. The expectation values of distances are also evaluated. We find on the finite noncommutative geometries, distances shrink with increasing graviton excitations, while on a circle, they grow. A comparison is made with Rovelli's canonical quantisation approach, and with his idea of spectral path integrals. We also briefly discuss the quantisation of a general Riemannian manifold.

Included, is a comprehensive overview of the homological aspects of noncommutative geometry. In particular, we cover the index pairing between K-theory and K-homology, KK-theory, cyclic homology/cohomology, the Chern character and the index theorem. We also review the various field theories on noncommutative geometries.

Declaration

I declare that this thesis was composed by myself, and that no part of it has been submitted elsewhere for any other degree or qualification. The work in this thesis is based on the research I carried out at the Department of Mathematical Sciences, University of Durham.

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Chapter 1

Introduction

Noncommutative geometry significantly changes how we view spaces and what constitutes a space. It is very much the algebraic dual of the traditional geometry of points. The central theme is one of operator algebras and Hilbert spaces. In physical terms, noncommutative geometry looks at spaces through the eyes of fermions. The inverse of the fermion propagator, in the guise of the Dirac operator, plays a key role in the differential aspects of noncommutative geometry.

The use of operator algebras makes noncommutative geometry particularly attractive as the mathematics of quantum geometry. For example, it has been used in conformal field theory to describe stringy geometry [15, 31]. But, there has been little development of a corresponding theory of quantum gravity. The concept of a quantum field theory on a noncommutative space has yet to be worked out. Currently, field theories on noncommutative spaces are quantised on a case-by-case basis—there is no general formalism. We will take the first steps towards a general path integral formalism for quantising the spectral action.

The original reference for noncommutative geometry is Connes' book [10]. It is written at an advanced level and contains a wealth of information. More suitable for newcomers is the self-contained and up-to-date book [19] by Gracia-Bondia, Varilly and Figueroa. There is also Landi's book [27], which includes chapters on field theories and gravity. The standard model from a noncommutative geometric point of view is covered in [42, 43]. Connes' latest review and progress report on the subject is [12].

In chapter 2, we introduce the basics of noncommutative geometry. The axioms for a real spectral triple are stated and we describe the construction of noncommutative differential forms from the universal differential graded algebra. We then introduce the Dixmier trace as the noncommutative integral.

In chapter 3, we review the various field theories that can be formulated on noncommutative geometries. Using differential forms, we show how to construct the Yang-Mills and topological actions. Scalar field theories are also dealt with, including the Polyakov action. We then move onto spectral theories of gravity. In particular, we describe the spectral action, which we intend to quantise.

In chapter 4, we present our path integral formalism. We apply it to the two-point space, the matrix geometry $M_2(\mathbb{C})$ and a circle. In each case, the path integrals are standard finite dimensional integrals, so the technical difficulties associated with functional integration are avoided. We also make a comparison with the canonical quantisation approach taken by Rovelli, and with his idea of spectral path integrals. A brief discussion on the quantisation of a Riemannian manifold is included.

In chapter 5, we give an overview of the homological aspects of noncommutative geometry. The emphasis is on concepts rather than technical details. Amongst other things, we explain Poincaré duality in terms of K-theory/KK-theory and describe the index formula.

Chapter 2

Noncommutative Geometry

2.1 The Dictionary for Noncommutative Geometry

Noncommutative geometry, as developed by Connes [10], is founded on two theorems: the Gelfand-Naĭmark theorem and the Serre-Swan theorem. The Gelfand-Naĭmark theorem states that a locally compact Hausdorff space X is the same thing as the commutative C^*-algebra $C_0(X)$. All the topological information about a Hausdorff space is stored algebraically in the C^*-algebra of functions on it. A noncommutative C^*-algebra can therefore be regarded as an algebra of functions on a noncommutative space. This is the basis of noncommutative topology.

The Serre-Swan theorem states that a vector bundle over X is the same thing as a finitely generated projective module (or finite projective module for short) over $C^\infty(X)$. Specifically, any vector bundle is given by its space of smooth sections, which is a finite projective (right) module of the form $p(C^\infty(X))^n$, where $p \in \mathrm{M}_n(C^\infty(X))$ is a projection. This gives rise to the notion of a noncommutative vector bundle as a finite projective (right) module over a noncommutative pre-C^*-algebra. Noncommutative vector bundles capture the differential structure of a noncommutative space. They are necessary for the construction of physical theories on noncommutative spaces.

The noncommutative generalisations of these theorems lead to other dualities between algebra and geometry that also do not depend on commutativity in an essential way. All

Measure space	von Neumann algebra
Measure	Positive functional
Hausdorff space	C^*-algebra
Complex function	Operator
Compactification	Unitisation
Point	Pure state
Open subset	Ideal
Vector bundle	Finite projective module
Topological K-theory	Operator K-theory
Metric	Dirac operator
Differential form	Hochschild cycle
de Rham current	Hochschild cocycle
Integral	Dixmier trace
de Rham homology	Periodic cyclic cohomology
de Rham cohomology	Periodic cyclic homology

Table 2.1: The classical–quantum dictionary for geometry.

the key geometric notions have noncommutative counterparts, enabling the development of geometry for noncommutative spaces. The correspondence between the commutative (classical) and the noncommutative (quantum) can be summarised in a dictionary, table 2.1.

2.2 The Axioms for a Spectral Triple

A noncommutative geometry is fundamentally described by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Definition 2.2.1. A *spectral triple* (or *K-cycle*) $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive representation π of a pre-C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, together with an unbounded operator D on $\mathcal{H}$ such that $D = D^*$, $(D - \lambda \mathbb{1})^{-1} \in \mathbb{K}(\mathcal{H})$ for all $\lambda \notin \mathbb{R}$ (compact resolvent) and $[D, \pi(a)] \in \mathbb{B}(\mathcal{H})$ for all $a \in \mathcal{A}$.

This is the minimum amount of information required to define a differential structure on $\mathcal{A}$. For physical applications, it is also desirable to have some of the structure of a manifold. A noncommutative manifold is given by a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$. The axioms for a real spectral triple are given below [11].

2.2.1 Dirac operator

The Dirac operator D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent such that $[D, \pi(a)] \in \mathbb{B}(\mathcal{H})$ for all $a \in \mathcal{A}$. There exists an integer $m \geq 0$ such that $|D|^{-m}$ is an infinitesimal of order 1, i.e. $\mathrm{Tr}_\omega |D|^{-m} > 0$. The integer m is called the *dimension* of the spectral triple.

2.2.2 Real structure

The real structure J is an antiunitary operator ($J\lambda\psi = \overline{\lambda} J\psi$ and $\langle J\psi_1, J\psi_2\rangle = \langle \psi_2, \psi_1\rangle$) on $\mathcal{H}$ such that

$$J^2 = \varepsilon \mathbb{I}, \tag{2.1}$$

$$JD = \varepsilon' DJ, \tag{2.2}$$

$$J\Gamma = \varepsilon'' \Gamma J, \tag{2.3}$$

$$[\pi(a), \pi^{\mathrm{op}}(b)] = 0 \quad \text{(bimodule structure)}, \tag{2.4}$$

$$[[D, \pi(a)], \pi^{\mathrm{op}}(b)] = 0 \quad \text{(first order condition)}, \tag{2.5}$$

where $\pi^{\mathrm{op}}(b) = J\pi(b^*)J^{-1}$. The values of ε, ε' and ε'' are given in table 2.2. Condition (2.4) gives $\mathcal{H}$ the structure of an $\mathcal{A}$-$\mathcal{A}$-bimodule,

$$a\psi b := \pi(a)\pi^{\mathrm{op}}(b)\psi, \quad a, b \in \mathcal{A},\ \psi \in \mathcal{H}. \tag{2.6}$$

It substitutes the commutativity of a commutative algebra, $[a, b] = 0$, with the commutativity of the two representations π and π^{op}. In particular, the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is replaced by $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$. The first order condition (2.5) means D behaves like a first order differential operator. Note, it is symmetric in a and b due to (2.4).

m mod 8	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Table 2.2: The reduction of the real Clifford algebras.

2.2.3 $\mathbb{Z}_2$-grading and orientability

The $\mathbb{Z}_2$-grading Γ is a self-adjoint unitary operator ($\Gamma = \Gamma^*$ and $\Gamma^2 = \mathbb{1}$) on $\mathcal{H}$ such that

$$\begin{aligned} &\text{for } m \text{ even:} \\ \Gamma\pi(a) &= \pi(a)\Gamma \quad (\pi(a) \text{ is even}), \qquad (2.7) \\ \Gamma D &= -D\Gamma \quad (D \text{ is odd}), \qquad (2.8) \\ &\text{for } m \text{ odd:} \\ \Gamma &= \mathbb{1} \quad (\text{trivial grading}). \qquad (2.9) \end{aligned}$$

There exists a Hochschild m-cycle $c \in Z_m(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{\rm op})$ such that $\Gamma = \pi(c)$. Concretely, this means Γ is of the form

$$\Gamma = \sum \pi(a)\pi^{\rm op}(b)[D,\pi(a_1)][D,\pi(a_2)]\dots[D,\pi(a_m)]. \qquad (2.10)$$

Conceptually, Γ is the volume form.

2.2.4 Regularity and finiteness

Additionally, $a \in \mathcal{A}$ and $[D,a]$ are smooth vectors of the derivation $[|D|,-]$. The space of smooth vectors $\mathcal{H}^\infty$ is a finite projective left $\mathcal{A}$-module with a Hermitian structure $(-,-)$ given by

$$\mathrm{Tr}_\omega(\psi,\phi)|D|^{-m} := \langle\psi,\phi\rangle. \qquad (2.11)$$

2.2.5 Poincaré duality

There is an isomorphism between the K-theory and K-homology of $\mathcal{A}$, given by the K-homology fundamental class of the spectral triple.

2.3 Abstract Spectral Triples

Spectral triples can be formulated in more abstract terms. The basic structure is an associative algebra $(\mathcal{A}, \mathrm{d}s^{-1})$ generated by the elements of a pre-C^*-algebra $\mathcal{A}$ and a symbol $\mathrm{d}s^{-1}$ [11]. A homomorphism from $(\mathcal{A}, \mathrm{d}x^{-1})$ to $(\mathcal{B}, \mathrm{d}y^{-1})$ is just a *-homomorphism from $\mathcal{A}$ to $\mathcal{B}$, since there is a unique map $\mathrm{d}x^{-1} \to \mathrm{d}y^{-1}$. The involution on $\mathcal{A}$ can be extended to $(\mathcal{A}, \mathrm{d}s^{-1})$ by defining $(\mathrm{d}s^{-1})^* = \mathrm{d}s^{-1}$. There is also a natural $\mathbb{Z}_2$-grading Γ on $(\mathcal{A}, \mathrm{d}s^{-1})$ given by $\Gamma(a) = a$ for all $a \in \mathcal{A}$ and $\Gamma(\mathrm{d}s^{-1}) = -\mathrm{d}s^{-1}$.

A (odd) spectral triple for $\mathcal{A}$ is an involutive representation π of $(\mathcal{A}, \mathrm{d}s^{-1})$ on a Hilbert space $\mathcal{H}$ such that $D = \pi(\mathrm{d}s^{-1})$ satisfies the axioms for a Dirac operator. A spectral triple is even if π is a graded representation on a graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. A spectral triple is real if there is also an opposite representation π^{op} of $(\mathcal{A}, \mathrm{d}s^{-1})$ on $\mathcal{H}$ and the axioms given above are satisfied.

2.4 Points and Distances

Distances are the basic observable of any geometry. To define a notion of distance requires a notion of point and a metric.

2.4.1 Points

There are several possible notions of point for a noncommutative geometry, all coincide for commutative C^*-algebras. The natural dual to a point is a character.

Definition 2.4.1. A *character* on a C^*-algebra A is a *-homomorphism $\chi : A \to \mathbb{C}$. Equivalently, a multiplicative *-linear functional on A.

The characters of a C^*-algebra $C_0(X)$ are given by

$$\chi_x(f) := f(x), \quad f \in C_0(X). \tag{2.12}$$

Each one is labelled by a point $x \in X$, thus the space of characters is isomorphic to X. Any commutative C^*-algebra A can be realised as any algebra of functions by the

Gelfand transform. The Gelfand transform of an element $a \in A$ is the continuous function $f(\chi) := \chi(a)$ on the space of characters of A.

Another possible notion of point is a maximal ideal.

Definition 2.4.2. A *maximal ideal* of an algebra $\mathcal{A}$ is an ideal that is not contained in any other ideal of $\mathcal{A}$, apart from $\mathcal{A}$ itself.

A maximal ideal of $C_0(X)$ is a subalgebra of functions that vanish at a single point $x \in X$. Thus, the space of maximal ideals of $C_0(X)$ is isomorphic to X. Maximal ideals are particularly suitable as points from the standpoint of sheaf/topos theory, as the closed ideals of an algebra form a lattice. This is the approach of quantales [37]. For a commutative C^*-algebra, the space of maximal ideals is also isomorphic to the space of primitive ideals ($\mathcal{I} = \ker \pi$), which in turn is isomorphic to the space of irreducible representations.

The notion of point that extends most readily to noncommutative C^*-algebras is a pure state.

Definition 2.4.3. A *state* on a C^*-algebra A is a positive linear functional $\Psi : A \to \mathbb{C}$, $\Psi(a^*a) \geq 0$ for all $a \in A$ with unit norm. The norm of a positive linear functional is defined by

$$||\Psi|| := \sup_{a \in A}\{|\Psi(a)| : ||a|| \leq 1\}. \tag{2.13}$$

For a unital C^*-algebra, $||\Psi|| = \Psi(\mathbb{1})$.

Let Ψ_1 and Ψ_2 be states, then the convex combination

$$\lambda\Psi_1 + (1-\lambda)\Psi_2, \quad \lambda \in [0,1], \tag{2.14}$$

is also a state. Thus, the space of states is a convex set.

Definition 2.4.4. A state is *pure* if it is not a convex combination of two other states. Pure states are the extreme points of the convex set of states.

A pure state on a commutative C^*-algebra is the same thing as a character, hence it corresponds to a point. The advantage of pure states over characters is they are required only to be linear and not multiplicative functionals, so are not constrained by the commutativity of $\mathbb{C}$.

The pure states of a noncommutative C^*-algebra do not have an interpretation as points of an underlying space. Indeed, the underlying space is a noncommutative space. Instead, they can be thought of as the "delocalised positions of a delocalised point". The classical points of a commutative C^*-algebra are replaced by equivalence classes of the pure states of a noncommutative C^*-algebra. Noncommutative spaces are commonly referred to as fuzzy spaces because of their non-local nature.

When a C^*-algebra is represented on a Hilbert space $\mathcal{H}$, every unit vector $|\psi\rangle \in \mathcal{H}$ determines a (not necessarily pure) state in the form of an expectation value,

$$\Psi(a) = \langle\psi| \, a \, |\psi\rangle \, . \tag{2.15}$$

But the converse is not always true; not every state need be given by an expectation value. For example, delta functions (which are distributions not functions) give pure states on $C_0(X)$, but they do not correspond to any vector in a Hilbert space (such a vector would not be square-integrable). Although, it is common to formally introduce such a "vector" $|x\rangle$,

$$\Psi_x(f) = \langle x| \, f \, |x\rangle := \int f(y)\delta(x-y) \, \mathrm{d}y = f(x). \tag{2.16}$$

Further details about pure states can be found in [39].

Example 2.4.1 (States on $\mathbb{C} \oplus \mathbb{C}$)

The C^*-algebra $A = \mathbb{C} \oplus \mathbb{C}$ has two pure states,

$$\begin{aligned} \Psi_L(a) &= a_L, \\ \Psi_R(a) &= a_R, \end{aligned}$$

where $a = (a_L, a_R) \in A$ with $a_L, a_R \in \mathbb{C}$. It is isomorphic to the algebra of functions on two points. If A is represented on the Hilbert space $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}$, then the two pure states are given by the unit vectors

$$|L\rangle = (\mathrm{e}^{\mathrm{i}\alpha}, 0), \quad |R\rangle = (0, \mathrm{e}^{\mathrm{i}\beta}). \tag{2.17}$$

The other unit vectors give the mixed states $\Psi_\lambda = \lambda\Psi_L + (1-\lambda)\Psi_R$. So, all the states can be expressed as expectation values.

Example 2.4.2 (States on $\mathrm{M}_2(\mathbb{C})$)

Consider the C^*-algebra $A = \mathrm{M}_2(\mathbb{C})$ and its representation on the Hilbert space $\mathcal{H} = \mathbb{C}\oplus\mathbb{C}$. A general unit vector in $\mathcal{H}$ can be parameterised as

$$|\psi\rangle = \mathrm{e}^{\mathrm{i}\alpha}\left(\cos\left(\frac{\phi}{2}\right), \mathrm{e}^{\mathrm{i}\theta}\sin\left(\frac{\phi}{2}\right)\right). \tag{2.18}$$

Therefore, the state associated to it has the form

$$\Psi_{\phi,\theta}(a) = a_{11}\cos^2\left(\frac{\phi}{2}\right) + \left(a_{12}\,\mathrm{e}^{\mathrm{i}\theta} + a_{21}\,\mathrm{e}^{-\mathrm{i}\theta}\right)\frac{\sin\phi}{2} + a_{22}\sin^2\left(\frac{\phi}{2}\right). \tag{2.19}$$

It is easy to see that this is a pure state and that the space of pure states is isomorphic to $\mathbb{S}^2$. As A is noncommutative, the pure states cannot be interpreted as corresponding to points. Instead, they represent the delocalised positions of a delocalised point. If the Hilbert space is extended to $\mathcal{H} = \mathrm{M}_2(\mathbb{C})$, then the mixed states can also be obtained as expectation values.

2.4.2 Distances

The distance between any two states (pure or mixed) is defined as

$$d(\Psi,\Phi) := \sup_{a\in\mathcal{A}}\{|\Psi(a) - \Phi(a)| : ||[D,a]|| \leq 1\}. \tag{2.20}$$

Qualitatively, this selects a function a that varies in direct proportion to the coordinates—the coordinate function. The difference between the values of the coordinate function evaluated at two points is then the distance between the two points.

The operator norm $||T||$ of an operator T can be computed by taking the square root of the largest eigenvalue of T^*T,

$$||T|| = \sqrt{\max(\{\lambda_n \text{ of } T^*T\})}. \tag{2.21}$$

For commutative C^*-algebras, this is just the L^∞-norm,

$$||f||_\infty := \sup_{x\in M}|f(x)|. \tag{2.22}$$

Example 2.4.3 (Distances on $\mathbb{R}$)

The Dirac operator on $\mathbb{R}$ is just $D = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. Pure states are given by $\Psi_x(f) = f(x)$. So,

$$\begin{aligned} d(p,q) &= \sup_{f\in\mathcal{A}}\left\{|\Psi_p(f) - \Psi_q(f)| : ||[D,f]|| = \left|\left| -\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}\right|\right| = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| \leq 1\right\} \\ &= \sup_{f\in\mathcal{A}}\{|f(p) - f(q)| : f(x) = \lambda x + c,\ |\lambda| \leq 1\} \\ &= \sup_{|\lambda|\geq 0}\{|\lambda||p-q| : |\lambda| \leq 1\} = |p-q|. \end{aligned}$$

Example 2.4.4 (Distances on a Riemannian manifold)

For the Dirac operator on a Riemannian manifold, $||[D,f]|| = ||f||_{\mathrm{Lip}}$, where $||f||_{\mathrm{Lip}}$ is the Lipschitz norm. The Lipschitz norm is defined by

$$||f||_{\mathrm{Lip}} := \sup_{x\neq y}\frac{|f(x) - f(y)|}{d_{\mathrm{geo}}(x,y)}, \tag{2.23}$$

where $d_{\mathrm{geo}}(x,y)$ is the geodesic distance. So, if $||[D,f]|| \leq 1$, then

$$\frac{|f(x) - f(y)|}{d_{\mathrm{geo}}(x,y)} \leq ||f||_{\mathrm{Lip}} \leq 1,$$

for any x, y such that $x \neq y$. Thus,

$$\begin{aligned} d(p,q) &= \sup_{f\in\mathcal{A}}\{|f(p) - f(q)| : ||f||_{\mathrm{Lip}} \leq 1\} \\ &\leq d_{\mathrm{geo}}(p,q). \end{aligned}$$

Now let $f_q(x) = d_{\mathrm{geo}}(x,q)$, then $||[D,f_q]|| = ||f_q||_{\mathrm{Lip}} = 1$. Therefore,

$$\begin{aligned} d(p,q) &\geq |f_q(p) - f_q(q)| \\ &\geq d_{\mathrm{geo}}(p,q). \end{aligned}$$

Hence, $d(p,q) = d_{\mathrm{geo}}(p,q)$.

2.5 Differential Forms

For an introduction to differential forms, we recommend [42] or [18].

2.5.1 Universal differential forms

We start by introducing the universal differential graded algebra $\Omega\mathcal{A} := \bigoplus_{p\geq 0} \Omega^p\mathcal{A}$ for a unital associative algebra $\mathcal{A}$. The space of 0-forms is defined as $\Omega^0\mathcal{A} := \mathcal{A}$. Higher degree forms are generated by a differential δ which satisfies

$$\delta 1 = 0, \tag{2.24}$$

$$\delta(ab) = (\delta a)b + a\delta b, \tag{2.25}$$

$$\delta(a_0\delta a_1 \ldots \delta a_p) = \delta a_0\delta a_1 \ldots \delta a_p, \tag{2.26}$$

$$(\delta a)^* = -\delta a^*, \tag{2.27}$$

$$(a_0\delta a_1 \ldots \delta a_p)^* = (\delta a_p)^* \ldots (\delta a_1)^* a_0. \tag{2.28}$$

A p-form is given by a finite sum $\sum a_0\delta a_1 \ldots \delta a_p \in \Omega^p\mathcal{A}$.

Any differential graded algebra (Ω, d) with an algebra homomorphism $\rho : \mathcal{A} \to \Omega^0$ can be constructed as a unique homomorphism ρ_{d} from the universal differential graded algebra $(\Omega\mathcal{A}, \delta)$ by

$$\rho_{\mathrm{d}}(a_0\delta a_1 \ldots \delta a_n) := \rho(a_0)\mathrm{d}(\rho(a_1)) \ldots \mathrm{d}(\rho(a_n)). \tag{2.29}$$

2.5.2 Noncommutative differential forms

The differential forms for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ are given by a representation π of the universal forms in $\mathbb{B}(\mathcal{H})$,

$$\pi(a_0\delta a_1 \ldots \delta a_p) := a_0[D, a_1] \ldots [D, a_p]. \tag{2.30}$$

However, this is not a true homomorphism of differential graded algebras: $[D, -]$ is only a derivation, not a differential. This means there exists forms ω such that $\pi(\omega) = 0$ but $\pi(\delta\omega) \neq 0$. These are called *junk forms* and have to be quotiented out to get a differential graded algebra.

Example 2.5.1 (Differential graded algebra on $\mathbb{R}^m$)

We shall construct the differential graded algebra on $\mathbb{R}^m$ using the Dirac operator $D = -\mathrm{i}\partial\!\!\!/ = -\mathrm{i}\gamma^\mu\partial_\mu$:

0-forms: $a \in C^\infty(\mathbb{R}^m)$.

1-forms: $\sum a_0[D, a_1] = -\mathrm{i}\sum a_0 \partial\!\!\!/ a_1$.

2-forms: $\sum a_0[D, a_1][D, a_2] = -\sum a_0 \partial\!\!\!/ a_1 \partial\!\!\!/ a_2 = -\gamma^\mu\gamma^\nu \sum a_0 \partial_\mu a_1 \partial_\nu a_2$.

junk 2-forms: *Consider the universal 1-form $\omega = a\delta b - (\delta b)a$. We find*

$$\begin{aligned} \pi(\omega) &= \mathrm{i}(a\partial\!\!\!/ b - (\partial\!\!\!/ b)a) = \mathrm{i}(a\partial\!\!\!/ b - a\partial\!\!\!/ b) = 0 \\ \delta\omega &= \delta a\delta b - \delta((\delta b)a) = \delta a\delta b + \delta b\delta a \\ \pi(\delta\omega) &= -\gamma^\mu\gamma^\nu(\partial_\mu a\, \partial_\nu b + \partial_\mu b\, \partial_\nu a) \neq 0. \end{aligned}$$

The most general junk 2-form is constructed by taking linear combinations of this 2-form, i.e. $-\gamma^\mu\gamma^\nu \sum(\partial_\mu a\, \partial_\nu b + \partial_\mu b\, \partial_\nu a)$. It is symmetric in μ and ν, so using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_\gamma$ we can write it as $f\mathbb{1}_\gamma$. Quotienting out this junk leaves antisymmetric 2-forms, which are isomorphic to the usual de Rham differential 2-forms $f_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.

p-forms: $\sum a_0[D, a_1]\dots[D, a_p] = (-\mathrm{i})^p \sum a_0 \partial\!\!\!/ a_1 \dots \partial\!\!\!/ a_p$.

junk p-forms: *Similarly, these consist of symmetric combinations. Thus, all p-forms are antisymmetric, hence we get the de Rham differential graded algebra.*

Real spectral triples have a bimodule structure. So, a differential form is more generally given by

$$\pi^{\mathrm{op}}(b_0)\pi(a_0\delta a_1 \dots \delta a_p) := a_0 J b_0^* J^{-1}[D, a_1]\dots[D, a_p]. \tag{2.31}$$

2.6 Integration

The integral of an operator T is defined in terms of the Dixmier trace as

$$⨍ T \, \mathrm{d}s^m := \frac{m}{2}(4\pi)^{m/2}\,\Gamma(m/2)\,\mathrm{Tr}_\omega(T|D|^{-m}). \tag{2.32}$$

In the cases of interest (i.e. for measurable operators), the Dixmier trace is given by

$$\mathrm{Tr}_\omega T = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n, \tag{2.33}$$

where the eigenvalues λ_n of T are arranged in descending order. The Dixmier trace extracts the coefficient of the logarithmic divergence in the trace of an operator. For finite dimensional operators, it is proportional to the ordinary matrix trace.

Example 2.6.1 (Integrating over $\mathbb{S}^1$)
We shall demonstrate how to calculate the length of $\mathbb{S}^1$ by evaluating $⨍ \mathrm{d}s$. The Dirac operator on $\mathbb{S}^1$ is just $D = -\frac{\mathrm{i}}{R}\frac{\mathrm{d}}{\mathrm{d}\theta}$ and has eigenvalues n/R. Thus, $|D|^{-1}$ has eigenvalues $R/|n|$ with degeneracy 2, since $\pm n$ give the same eigenvalue. Arranging them in descending order, we have

$$R,\ R,\ R/2,\ R/2,\ R/3,\ R/3, \ldots \tag{2.34}$$

The degeneracy means that n does not uniquely label the eigenvalues. We can give a unique, but approximate labelling by averaging the eigenvalues over their degeneracy. The number of eigenvalues with a value less than or equal to Λ is

$$N = \sum_{n=1}^{R/\Lambda} 2 = \frac{2R}{\Lambda}, \tag{2.35}$$

so the averaged eigenvalues are

$$\overline{\lambda}_n = \frac{\Lambda}{n/N} = \frac{2R}{n}. \tag{2.36}$$

Therefore, the Dixmier trace gives

$$\begin{aligned}
\mathrm{Tr}_\omega |D|^{-1} &= \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \overline{\lambda}_n = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{2R}{n} \\
&= \lim_{N\to\infty} \frac{1}{\log N} \int_1^N \frac{2R}{n}\,\mathrm{d}n \\
&= 2R = \frac{1}{\pi}\int_M \mathrm{d}x.
\end{aligned}$$

Hence,

$$\fint \mathrm{d}s = \int_M \mathrm{d}x = 2\pi R. \tag{2.37}$$

Example 2.6.2 (Integrating over the flat torus $\mathbb{T}^m_{\text{flat}} \cong \mathbb{R}^m/(2\pi R\mathbb{Z})^m$)
We shall demonstrate how to calculate the volume of $\mathbb{T}^m_{\text{flat}}$ by evaluating $\fint \mathrm{d}s^m$. The Dirac operator is $D = -\mathrm{i}\partial\!\!\!/$ and has eigenvalues $\pm\sqrt{(n_1/R)^2 + \ldots + (n_m/R)^2}$, which for fixed $n_1, \ldots, n_m$ have a degeneracy of $g_\gamma = \frac{1}{2}\operatorname{tr}\mathbb{1}_\gamma$. Each eigenvalue lies on the surface of an $(m-1)$-sphere, so there is an additional degeneracy factor of $g_{\text{sphere}}(r) \approx \frac{2\pi^{m/2}}{\Gamma(m/2)} r^{m-1}$, where $r = \sqrt{n_1^2 + \ldots + n_m^2}$. Thus, the eigenvalues of $|D|^{-m}$ are $(R/r)^m$ with a total degeneracy of $2g_\gamma\, g_{\text{sphere}}(r)$. Counting the number of eigenvalues with a value less than or equal to Λ gives

$$\begin{aligned} N &\approx \int_1^{R/\Lambda^{1/m}} 2g_\gamma\, g_{\text{sphere}}(r)\,\mathrm{d}r = \frac{2\pi^{m/2}}{\Gamma(m/2)} \operatorname{tr}\mathbb{1}_\gamma \int_1^{R/\Lambda^{1/m}} r^{m-1}\,\mathrm{d}r \\ &\approx \frac{2\pi^{m/2}R^m}{m\Gamma(m/2)\Lambda} \operatorname{tr}\mathbb{1}_\gamma. \end{aligned}$$

Now, we can average the eigenvalues,

$$\overline{\lambda}_n = \frac{\Lambda}{n/N} = \frac{2\pi^{m/2}R^m}{m\Gamma(m/2)n} \operatorname{tr}\mathbb{1}_\gamma, \tag{2.38}$$

and calculate the Dixmier trace,

$$\begin{aligned} \operatorname{Tr}_\omega |D|^{-m} &= \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{2\pi^{m/2}R^m}{m\Gamma(m/2)n} \operatorname{tr}\mathbb{1}_\gamma \\ &= \lim_{N\to\infty} \frac{1}{\log N} \int_1^N \frac{2\pi^{m/2}R^m}{m\Gamma(m/2)n} \operatorname{tr}\mathbb{1}_\gamma\,\mathrm{d}n \\ &= \frac{2\pi^{m/2}R^m}{m\Gamma(m/2)} \operatorname{tr}\mathbb{1}_\gamma = \frac{2\operatorname{tr}\mathbb{1}_\gamma}{m(4\pi)^{m/2}\Gamma(m/2)} \int_M \mathrm{d}^m x. \end{aligned}$$

Hence,

$$\fint \mathrm{d}s^m = \operatorname{tr}\mathbb{1}_\gamma \int_M \mathrm{d}^m x = (2\pi R)^m \operatorname{tr}\mathbb{1}_\gamma. \tag{2.39}$$

Further examples can be found in [27, sec. 5].

Connes has suggested that one can think of the Dixmier trace as a way of extracting the classical part of an operator, i.e. the low momentum behaviour. He has further suggested that it could be used to obtain the classical world from the quantum one.

2.6.1 Scalar product of differential forms

The scalar product of forms is defined by

$$(\alpha, \beta) := \fint \alpha^* \beta \, \mathrm{d}s^m. \tag{2.40}$$

Example 2.6.3 (Scalar products on $\mathbb{R}^m$)

Consider the 1-forms $A = A_\mu \gamma^\mu$ and $B = B_\mu \gamma^\mu$. Then,

$$\begin{aligned}(A, B) &= \int_M \overline{A}_\mu B_\nu \, \mathrm{tr}(\gamma^\mu \gamma^\nu) \, \mathrm{d}^m x \\ &= \mathrm{tr}\, \mathbb{1}_\gamma \int_M \overline{A}_\mu B^\mu \, \mathrm{d}^m x \\ &= \mathrm{tr}\, \mathbb{1}_\gamma \int_M \overline{A} \wedge *B.\end{aligned}$$

Consider the 2-forms $F = F_{\mu\nu}\gamma^{\mu\nu}$ and $G = G_{\mu\nu}\gamma^{\mu\nu}$, where $\gamma^{\mu\nu} := \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Then,

$$\begin{aligned}(F, G) &= \int_M \overline{F}_{\mu\nu} G_{\alpha\beta} \, \mathrm{tr}(\gamma^{\nu\mu}\gamma^{\alpha\beta}) \, \mathrm{d}^m x \\ &= \frac{1}{2} \mathrm{tr}\, \mathbb{1}_\gamma \int_M \overline{F}_{\mu\nu} G^{\mu\nu} \, \mathrm{d}^m x \\ &= \mathrm{tr}\, \mathbb{1}_\gamma \int_M \overline{F} \wedge *G.\end{aligned}$$

(Trace formulas for γ matrices can be found in appendix B.2.)

There is also another possible definition for the scalar product [8],

$$(\alpha, \beta) := \frac{1}{2} \fint (\alpha + J\alpha J^{-1})^* (\beta + J\beta J^{-1}) \, \mathrm{d}s^m. \tag{2.41}$$

This symmetrised scalar product arises when considering gravity (it is induced by the spectral action). It gives identical results to (2.40) in the commutative case. We will keep to using the more traditional scalar product (2.40).

Chapter 3

Field Theories on Noncommutative Geometries

3.1 Yang-Mills Theory

Having defined differential forms and their scalar product for a noncommutative space, it is straightforward to write down a Yang-Mills action. The connection is a self-adjoint 1-form:

$$A = \pi(\textstyle\sum a\delta b) = \sum a[D, b], \tag{3.1}$$

with $\sum a[D, b] = -\sum [D, b^*]a^*$. Differentiating A gives the field strength,

$$\begin{aligned} F_{\mathcal{J}} &= \pi(\delta \textstyle\sum a\delta b) = \pi(\sum \delta a\delta b) \\ &= \textstyle\sum [D, a][D, b]. \end{aligned}$$

Since $F_{\mathcal{J}}$ is a 2-form, it is necessary to quotient out any contributions from junk forms. The true field strength, then, is $F = F_{\mathcal{J}}/\mathcal{J}$, where $\mathcal{J} := \pi(\delta \operatorname{Ker} \pi)$ is the space of junk forms. Taking the scalar product of F with itself gives the Yang-Mills action,

$$S_{\mathrm{YM}}[A] := \frac{1}{g^2}(F, F), \tag{3.2}$$

where g is the coupling constant.

It is invariant under the inner automorphisms of $\mathcal{A}$. The inner automorphism group, $\mathrm{Inn}(\mathcal{A})$, is the group of unitaries of $\mathcal{A}$. Under an inner automorphism, the Dirac operator transforms as

$$\begin{aligned} D \to UDU^* &= uJuJ^{-1}DJu^*J^{-1}u^* = uJuDu^*J^{-1}u^* \quad \text{using (2.2)} \\ &= uJ(D+u[D,u^*])J^{-1}u^* \\ &= uDu^* + J(J^{-1}uJ)u[D,u^*]J^{-1}u^* \quad \text{using (2.2)} \\ &= uDu^* + Ju[D,u^*](J^{-1}uJ)J^{-1}u^* \quad \text{using (2.4) \& (2.5)} \\ &= D + u[D,u^*] + Ju[D,u^*]J^{-1}, \end{aligned} \tag{3.3}$$

where $u \in \mathrm{Inn}(\mathcal{A})$. The connection transforms as

$$\begin{aligned} A \to UAU^* &= uJuJ^{-1}A(Ju^*J^{-1})u^* \\ &= uJuJ^{-1}(Ju^*J^{-1})Au^* \quad \text{as } [A, b^{\mathrm{op}}] = 0 \text{ by (2.4) \& (2.5)} \\ &= uAu^*, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} JAJ^{-1} \to UJAJ^{-1}U^* &= uJuAu^*(J^{-1}u^*J)J^{-1} \\ &= uJ(J^{-1}u^*J)uAu^*J^{-1} \quad \text{using } [A, b^{\mathrm{op}}] = 0 \text{ \& (2.4)} \\ &= JuAu^*J^{-1}. \end{aligned} \tag{3.5}$$

Thus, the transformation of the covariant Dirac operator is

$$D + A + JAJ^{-1} \to D + uAu^* + u[D,u^*] + JuAu^*J^{-1} + Ju[D,u^*]J^{-1}. \tag{3.6}$$

This corresponds to the gauge transformation

$$A \to uAu^* + u[D,u^*]. \tag{3.7}$$

Example 3.1.1 (Ordinary Yang-Mills)

Consider a four-dimensional matrix manifold $\mathcal{A} = C^\infty(M) \otimes \mathrm{M}_n(\mathbb{C})$ with Dirac operator $D = -\mathrm{i}\gamma^\mu(x)(\partial_\mu + \frac{1}{4}\omega_{bc\mu}(x)\gamma^b\gamma^c) \otimes \mathbb{1}_n$. The Yang-Mills field strength is $F = F_{\mu\nu}\gamma^\mu(x)\gamma^\nu(x)$. Thus, the Yang-Mills action functional is

$$\begin{aligned} S_{\mathrm{YM}}[A] &= -\frac{1}{g^2} ⨍ F^2 \,\mathrm{d}s^4 \\ &= -\frac{1}{g^2}\int_M \mathrm{tr}(F_{\alpha\beta}F_{\mu\nu})\,\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu)\,\sqrt{g}\,\mathrm{d}^4x \\ &= \frac{8}{g^2}\int_M \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,\sqrt{g}\,\mathrm{d}^4x. \end{aligned}$$

3.1.1 Scalar fields

Scalar fields naturally appear from noncommutative Yang-Mills theory as gauge bosons associated to discrete noncommutative spaces. The classic example is the Higgs field. Quite separate from this, there is also a scalar field action for noncommutative spaces,

$$S_\phi[\phi, \phi^*] := ([D, \phi], [D, \phi]). \tag{3.8}$$

Applying it to an $\mathrm{M}_n(\mathbb{C})$ matrix manifold yields n^2 complex scalar fields.

3.2 Topological Actions

In $2n$ dimensions, it is possible to define a noncommutative topological action,

$$S_\Gamma[A] := \fint \Gamma F^n \, \mathrm{d}s^{2n}. \tag{3.9}$$

This is an example of a Hochschild $2n$-cocycle. A Hochschild n-cocycle φ_n^D is an $(n+1)$-linear functional on $\mathcal{A}$ given by

$$\varphi_n^D(a_0, a_1, \ldots, a_n) := \frac{(-1)^n}{2} \operatorname{Tr}_\omega(\Gamma a_0 [D, a_1] \ldots [D, a_n] |D|^{-n}). \tag{3.10}$$

For a matrix manifold, the action reproduces the usual Chern numbers. In particular, the instanton number (the 2nd Chern number) in four dimensions,

$$S_\Gamma[A] = -4 \int_M \varepsilon_{\alpha\beta\mu\nu} \operatorname{tr}(F^{\alpha\beta} F^{\mu\nu}) \sqrt{g} \, \mathrm{d}^4 x, \tag{3.11}$$

and the magnetic monopole charge (the 1st Chern number) in two dimensions,

$$S_\Gamma[A] = -2\mathrm{i} \int_M \varepsilon_{\mu\nu} \operatorname{tr} F^{\mu\nu} \sqrt{g} \, \mathrm{d}^2 x. \tag{3.12}$$

For a finite geometry (zero dimensions), the action just gives

$$S_\Gamma = \operatorname{tr} \Gamma, \tag{3.13}$$

which is the fermion left-right asymmetry.

3.3 Fermions

The fermion action is constructed straightforwardly using the scalar product of the Hilbert space,

$$S_{\mathrm{F}}[\psi, \overline{\psi}, A] := \langle \psi, (D + A + \varepsilon' JAJ^{-1})\psi \rangle. \tag{3.14}$$

Under a gauge transformation, the fermions transform in the adjoint representation of the gauge group,

$$\psi \to U\psi = uJuJ^{-1}\psi = u\psi u^*, \quad u \in \mathrm{Inn}(\mathcal{A}). \tag{3.15}$$

However, the physical fermion fields actually transform in the fundamental representation, while the antifermion fields transform in the conjugate representation.

3.4 Polyakov Action

The Polyakov action has a natural formulation in terms of a scalar product of forms,

$$S_{\mathrm{P}}[X] = (\eta_{\mu\nu}\mathrm{d}X^\mu, \mathrm{d}X^\nu), \tag{3.16}$$

where $X : \Sigma \to \mathbb{R}^m$, with Σ a Riemann surface, and $\eta_{\mu\nu}$ is an Euclidean metric on $\mathbb{R}^m$. It can be generalised to noncommutative conformal manifolds by using the conformal equivalent of a spectral triple—a Fredholm module.

A Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ consists of a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a self-adjoint operator F. The operator F is required to satisfy $F^2 = \mathbb{1}$ and plays the role of a "conformal Dirac operator". A Fredholm module can be constructed from a spectral triple by defining $F = D|D|^{-1}$.

The noncommutative Polyakov action is thus defined to be

$$S_{\mathrm{P}}[X] := \int\!\!\!\!\!\!- \ \eta_{\mu\nu}[F, X^\mu][F, X^\nu], \tag{3.17}$$

where the $\eta_{\mu\nu}$ and the X^μ are self-adjoint elements of $\mathcal{A}$. It is conformally invariant by virtue of being a Hochschild 2-cocycle. Connes has used it [10, sec. IV.4.γ] to derive the following action for a four-dimensional conformal manifold Σ,

$$S_{\mathrm{P}}[X] = 2\int_\Sigma \eta_{\mu\nu}\bigg(\frac{R}{3}\,g(\mathrm{d}X^\mu,\mathrm{d}X^\nu) - \Delta\, g(\mathrm{d}X^\mu,\mathrm{d}X^\nu) + g(\nabla\mathrm{d}X^\mu,\nabla\mathrm{d}X^\nu) - \frac{1}{2}\,\Delta X^\mu\,\Delta X^\nu\bigg)\sqrt{g}\,\mathrm{d}^4x. \tag{3.18}$$

Note: for conformal manifolds with more than two dimensions, it is necessary to use the Wodzicki residue instead of the Dixmier trace to evaluate the integral (3.17).

Example 3.4.1 (The Polyakov action on the two-point space)
The Fredholm module of the two-point space $\mathcal{A} := \mathbb{C}\oplus\mathbb{C}$ is given by

$$\mathcal{H} := \mathbb{C}\oplus\mathbb{C} \quad \textit{with } \pi(a) := \begin{pmatrix} a_L & 0 \\ 0 & a_R \end{pmatrix},$$

$$F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define the metric $\eta_{\mu\nu}$ by

$$\pi(\eta_{\mu\nu}) := \begin{pmatrix} \eta_{L\mu\nu} & 0 \\ 0 & \eta_{R\mu\nu} \end{pmatrix}, \quad \eta_{L\mu\nu},\eta_{R\mu\nu}\in\mathbb{R}, \tag{3.19}$$

and the scalar fields X^μ by

$$\pi(X^\mu) := \begin{pmatrix} X^\mu_L & 0 \\ 0 & X^\mu_R \end{pmatrix}, \quad X^\mu_L, X^\mu_R \in\mathbb{R}. \tag{3.20}$$

So,

$$[F,X^\mu] = \begin{pmatrix} 0 & X^\mu_R - X^\mu_L \\ X^\mu_L - X^\mu_R & 0 \end{pmatrix}. \tag{3.21}$$

Thus,

$$S_{\mathrm{P}}[X] = \mathrm{tr}(\eta_{\mu\nu}[F,X^\mu][F,X^\nu]) \tag{3.22}$$

$$= -(\eta_{L\mu\nu}+\eta_{R\mu\nu})(X^\mu_L - X^\mu_R)(X^\nu_L - X^\nu_R) \tag{3.23}$$

$$= -(\eta_L+\eta_R)_{\mu\nu}(X_L-X_R)^\mu(X_L-X_R)^\nu. \tag{3.24}$$

3.5 Wodzicki Residue

The noncommutative Einstein-Hilbert action is given by

$$S_{\mathrm{EH}}[D] := \fint D^2 \, \mathrm{d}s^m. \tag{3.25}$$

But, this integral cannot be evaluated using the Dixmier trace, the operator $D^2|D|^{-m}$ lies outside its domain. Instead, one has to use the Wodzicki residue,

$$\operatorname{Wres} T := \int_M \int_{||k||=1} \operatorname{tr} p_{-m}(x,k) \, \mathrm{d}\Omega(k) \, \mathrm{d}^m x, \tag{3.26}$$

where $p_{-m}(x,k)$ is the symbol of order $-m$ of T. For pseudo-differential operators of order less than $-m$ or of non-integer order, the Wodzicki residue vanishes. (Pseudo-differential operators are described in appendix A.3.)

Example 3.5.1 (The Wodzicki residue of $|D|^{-m}$)
Consider the Dirac operator $D = -\mathrm{i}\partial\!\!\!/$ on the flat torus $\mathbb{T}^m_{\text{flat}}$. Its symbol of order 1 is $p_1(x,k) = \gamma^\mu k_\mu$. Thus, the symbol of order $-m$ of $|D|^{-m} = (D^2)^{-m/2}$ is $p_{-m}(x,k) = ||k||^{-m} \mathbb{1}_\gamma$. Hence,

$$\begin{aligned}
\operatorname{Wres} |D|^{-m} &= \int_M \int_{||k||=1} ||k||^{-m} \operatorname{tr} \mathbb{1}_\gamma \, \mathrm{d}\Omega(k) \, \mathrm{d}^m x \\
&= \operatorname{tr} \mathbb{1}_\gamma \int_M \int_{||k||=1} \mathrm{d}\Omega(k) \, \mathrm{d}^m x \\
&= \frac{2\pi^{m/2}}{\Gamma(m/2)} \operatorname{tr} \mathbb{1}_\gamma \int_M \mathrm{d}^m x.
\end{aligned} \tag{3.27}$$

The relation of the Wodzicki residue to the Dixmier trace is given by Connes' trace theorem.

Theorem 3.5.1 (Connes' trace theorem)
Let M be an m-dimensional compact Riemannian manifold. Let T be a pseudo-differential operator of order $-m$ (or lower) on M. Then,

$$\operatorname{Tr}_\omega T = \frac{1}{m(2\pi)^m} \operatorname{Wres} T. \tag{3.28}$$

The Wodzicki residue is the unique extension of the Dixmier trace to the algebra of classical pseudo-differential operators.

In terms of the Wodzicki residue then, the action functional (3.25) is given by

$$S_{\mathrm{EH}}[D] = \frac{\Gamma(m/2)}{2\pi^{m/2}} \operatorname{Wres} |D|^{2-m}. \tag{3.29}$$

It was shown in [24] that for a Dirac operator on an m-dimensional compact Riemannian manifold this yields the Einstein-Hilbert action

$$S_{\mathrm{EH}}[D] = \frac{m-2}{12} \operatorname{tr} \mathbb{1}_\gamma \int_M R \sqrt{g}\, \mathrm{d}^m x. \tag{3.30}$$

3.6 Spectral Action

The question was once asked whether there exists a space X such that $\mathrm{Diff}(X)$ is the (semi-direct) product of the diffeomorphism group of general relativity and the gauge group of the standard model, $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. If such a space exists, then it might be possible to obtain the standard model from a theory of gravity on it. The existence of such a manifold has been ruled out by a result in [36]. But, noncommutative geometry extends the concept of a space, so one can try asking whether there is a noncommutative space with this diffeomorphism group.

There is indeed such a noncommutative manifold, and Chamseddine and Connes have developed a theory of gravity for it [9]. Their theory of gravity is based on the spectral principle, which states physics depends only on the spectrum of the Dirac operator. This is a stronger requirement than diffeomorphism invariance: isometric manifolds are isospectral, but isospectral manifolds are not necessarily isometric (one cannot hear the shape of a drum [23]). The action functional for the theory is the spectral action

$$S[D] := \operatorname{Tr} \chi(D^2/\Lambda^2), \tag{3.31}$$

where χ is a cutoff function and Λ is a cutoff parameter. Here, the Dirac operator includes the internal fluctuations of the metric given by the gauge field A,

$$D := D_0 + A + \varepsilon' J A J^{-1}. \tag{3.32}$$

The spectral action is similar to the Wodzicki residue action (3.25). They are both some kind of regularised trace of D^2. The Wodzicki residue is regularised by the volume element

$|D|^{-m}$, while the spectral action uses the cutoff function χ to regularise the ordinary trace of operators. A further connection between the two actions is the spectral action implicitly contains the terms $\mathrm{Wres}\,|D|^{-m}$ and $\mathrm{Wres}\,|D|^{2-m}$.

The symmetrised gauge field $A+\varepsilon' JAJ^{-1}$ has the effect of removing a (commutative) U(1) factor from A. To show this, we rewrite A as $(G + A)$, where G is the noncommutative part and A is the commutative part. So,

$$\begin{aligned} D &= D_0 + (G + A) + \varepsilon' J(G + A)J^{-1} \\ &= D_0 + G + \varepsilon' JGJ^{-1} + A + \varepsilon' JAJ^{-1}. \end{aligned}$$

The commutative part A is given by a finite sum of 1-forms,

$$A = \sum_i a_i[D_0, b_i], \tag{3.33}$$

where the a_i and b_i are elements in the centre of $\mathcal{A}$. For such elements, $a^{\mathrm{op}} = Ja^*J^{-1} = a$. So,

$$\begin{aligned} Ja_i[D_0, b_i]J^{-1} &= Ja_iJ^{-1}[JD_0J^{-1}, Jb_iJ^{-1}] = \varepsilon' Ja_iJ^{-1}[D_0, Jb_iJ^{-1}] \quad \text{using (2.2)} \\ &= \varepsilon' Ja_iJ^{-1}[D_0, b_i^*] = \varepsilon'[D_0, b_i^*]Ja_iJ^{-1} \quad \text{using (2.5)} \\ &= \varepsilon'[D_0, b_i^*]a_i^* = -\varepsilon'(a_i[D_0, b_i])^*. \end{aligned}$$

Thus,

$$\begin{aligned} A + \varepsilon' JAJ^{-1} &= \sum_i a_i[D_0, b_i] + \varepsilon' \sum_i Ja_i[D_0, b_i]J^{-1} \\ &= \sum_i a_i[D_0, b_i] - \sum_i (a_i[D_0, b_i])^* \\ &= A - A^* = 0. \end{aligned}$$

Hence, the commutative part of the gauge field is removed.

The consequence of this is on a Riemannian manifold there are no gauge fields, only gravity. Similarly, on a matrix manifold, $\mathcal{A} = C^\infty(M) \otimes \mathrm{M}_n(\mathbb{C})$, there is an $\mathrm{SU}(n)$, not $\mathrm{U}(n)$, gauge field. To obtain a U(1) gauge field, one needs to consider a tensor product like $\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathrm{M}_n(\mathbb{C}))$.

Note: for a truly noncommutative manifold (e.g. the noncommutative torus), the tensor product with a matrix algebra does yield a $\mathrm{U}(n)$ gauge field as the algebra has a trivial centre.

The spectral action can be evaluated using the heat kernel expansion. We begin by writing the cutoff function as a Laplace transform,

$$\chi(u) = \int_0^\infty X(t)\, \mathrm{e}^{-tu}\, \mathrm{d}t. \tag{3.34}$$

The action then becomes

$$S[D] := \operatorname{Tr} \chi(D^2/\Lambda^2) = \int_0^\infty X(t)\, \operatorname{Tr} \mathrm{e}^{-tD^2/\Lambda^2}\, \mathrm{d}t.$$

We can now apply the heat kernel expansion (C.9) to obtain

$$S[D] = \int_0^\infty X(t) \sum_{n=0}^\infty (t/\Lambda^2)^{\frac{n-m}{2}}\, a_n(D^2)\, \mathrm{d}t = \sum_{n=0}^\infty \Lambda^{m-n} \int_0^\infty X(t)\, t^{\frac{n-m}{2}}\, \mathrm{d}t\, a_n(D^2).$$

Using the Mellin transform (C.16), we have

$$\begin{aligned} \int_0^\infty u^k\, \chi(u)\, \mathrm{d}u &= \int_0^\infty u^k \int_0^\infty X(t)\, \mathrm{e}^{-tu}\, \mathrm{d}t\, \mathrm{d}u = \int_0^\infty X(t) \int_0^\infty u^k\, \mathrm{e}^{-tu}\, \mathrm{d}u\, \mathrm{d}t \\ &= \Gamma(k+1) \int_0^\infty X(t)\, t^{-(k+1)}\, \mathrm{d}t. \end{aligned} \tag{3.35}$$

Thus,

$$\begin{aligned} S[D] &= \sum_{n=0}^\infty \Lambda^{m-n} \frac{1}{\Gamma\left(\frac{m-n}{2}\right)} \int_0^\infty u^{\frac{m-n-2}{2}}\, \chi(u)\, \mathrm{d}u\, a_n(D^2) \\ &= \sum_{n=0}^\infty \Lambda^{m-n}\, f_n\, a_n(D^2), \end{aligned} \tag{3.36}$$

where the coefficients f_n are given by

$$f_n = \frac{1}{\Gamma\left(\frac{m-n}{2}\right)} \int_0^\infty u^{\frac{m-n-2}{2}}\, \chi(u)\, \mathrm{d}u. \tag{3.37}$$

The action depends only weakly on χ; it merely determines the coefficients f_n. In this sense, the spectral action is universal.

In four dimensions ($m = 4$), the coefficients reduce to

$$\begin{aligned} f_0 &= \int_0^\infty u\, \chi(u)\, \mathrm{d}u \\ f_2 &= \int_0^\infty \chi(u)\, \mathrm{d}u \\ f_4 &= \chi(0) \\ f_{2(n+2)} &= (-1)^n\, \chi^{(n)}(0). \end{aligned}$$

The simplest choice for the cutoff function is the characteristic function of the unit interval,

$$\chi(u) := \begin{cases} 1 & |u| \leq 1, \\ 0 & |u| \geq 1. \end{cases} \tag{3.38}$$

With this, the spectral action just counts the number of Dirac eigenvalues with an absolute value less than Λ. The values of the coefficients are:

$$f_0 = \frac{1}{2}, \quad f_2 = 1, \quad f_4 = 1, \quad f_{2(n+3)} = 0. \tag{3.39}$$

For a four-dimensional Riemannian manifold, the action becomes

$$\begin{aligned} S[D] &= \frac{1}{48\pi^2}\int_M (6\Lambda^4 + \Lambda^2 R)\sqrt{g}\,\mathrm{d}^4x \\ &+ \frac{1}{120}\int_M (5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 12R_{;\mu}{}^{\mu})\sqrt{g}\,\mathrm{d}^4x. \end{aligned} \tag{3.40}$$

The $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term can be expressed in terms of the Weyl tensor $C_{\mu\nu\rho\sigma}$,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2. \tag{3.41}$$

This gives us

$$\begin{aligned} S[D] &= \frac{1}{48\pi^2}\int_M (6\Lambda^4 + \Lambda^2 R)\sqrt{g}\,\mathrm{d}^4x \\ &+ \frac{1}{120}\int_M \left(-7C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{22}{3}R^2 - 22R_{\mu\nu}R^{\mu\nu} + 12R_{;\mu}{}^{\mu}\right)\sqrt{g}\,\mathrm{d}^4x. \end{aligned}$$

We can also use the Euler characteristic, which in four dimensions is given by

$$\chi_{\text{Euler}}(M) = \frac{1}{32\pi^2}\int_M (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\sqrt{g}\,\mathrm{d}^4x. \tag{3.42}$$

Replacing the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ with the Weyl tensor and rearranging yields

$$\begin{aligned} \int_M \left(\frac{22}{3}R^2 - 22R_{\mu\nu}R^{\mu\nu}\right)\sqrt{g}\,\mathrm{d}^4x &= 352\pi^2\chi_{\text{Euler}}(M) \\ &- \int_M 11C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{g}\,\mathrm{d}^4x. \end{aligned}$$

This can be directly substituted into the action to give,

$$\begin{aligned} S[D] &= \frac{1}{48\pi^2}\int_M \left(6\Lambda^4 + \Lambda^2 R - \frac{3}{20}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{10}R_{;\mu}{}^{\mu}\right)\sqrt{g}\,\mathrm{d}^4x \\ &+ \frac{11}{180}\chi_{\text{Euler}}(M). \end{aligned}$$

Finally, we can neglect the surface terms,

$$S[D] = \frac{1}{48\pi^2} \int_M \left(6\Lambda^4 + \Lambda^2 R - \frac{3}{20} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \sqrt{g}\, \mathrm{d}^4 x. \tag{3.43}$$

It is possible to remove the huge cosmological constant term by slightly modifying the cutoff function [28]:

$$\widetilde{\chi}(u) := \chi(u) - \epsilon^2 \chi(\epsilon u), \quad \epsilon \ll 1. \tag{3.44}$$

The values of the coefficients then change to

$$f_0 = 0, \quad f_2 = 1 - \epsilon, \quad f_4 = 1 - \epsilon^2, \quad f_{2(n+3)} = 0. \tag{3.45}$$

For the noncommutative manifold of the standard model (see appendix E.3), the spectral action gives the standard model action in addition to the gravitational action above.

It is remarkable that the action of the fundamental theories of physics can be obtained from simply counting eigenvalues/states. In [5], it is pointed out that this count of states can be directly related to the Bekenstein-Hawking entropy. Indeed, $\mathrm{d}s^2 = D^{-2}$, so one is counting area eigenstates with eigenvalues larger than the cutoff area $1/\Lambda^2$. The spectral action may not be a theory of *everything*, but it does fit on a T-shirt[1]!

[1] "We hope to explain the entire universe in a single, simple formula that you could wear on your T-shirt.", Leon Lederman, director of Fermilab.

Chapter 4

Quantisation and Noncommutative Geometry

4.1 Quantum Theory

From a physical perspective, a spectral triple describes a geometry in terms of its fermion geodesics. The Dirac operator is the inverse of the fermion propagator,

$$D = \left(\times\!\!\longrightarrow\!\!\times \right)^{-1}, \tag{4.1}$$

and the Hilbert space is the space of spinors (the one-particle subspace of the fermion Fock space). Moreover, the fermion geodesic equation is

$$D\psi = 0, \quad \psi \in \mathcal{H}. \tag{4.2}$$

There is, thus, a close relationship between noncommutative geometry and quantum field theory.

For some quantum field theories, it is possible to construct a spectral triple from their field content. The boson (gauge) fields determine the algebra, the kinematics of the fermion (matter) fields determine the Hilbert space, and the fermion dynamics determine the Dirac operator. Though, in general, it is not always possible to reconstruct the quantum field theory from the spectral triple. For instance, it may not be possible to recover the precise form of the action functional. One important theory that can be recovered from its spectral triple is the standard model. Its field content can be arranged to give a real spectral triple (see appendix E.3), which can then be used as a starting point to derive the standard model action. There have been many papers [5, 35, 7] investigating the phenomenological consequences of constructing the standard model from a spectral triple.

4.2 Quantum Mechanics

Operator algebras play an important role in both noncommutative geometry and quantum mechanics. It is, therefore, interesting to see to what extent they are related. The obvious connection between the two is the phase space of quantum mechanics is a noncommutative geometry. But, we are more interested in how the real spectral triple of a manifold M is related to the quantum phase space of a particle moving in M.

The quantum phase space is described by the algebra $\mathcal{F}_\hbar$ of functions of the operators x_μ and p_μ, which satisfy $[x_\mu, p_\nu] = \mathrm{i}\hbar\eta_{\mu\nu}$. It contains the algebra of functions on M, $C^\infty(M)$, as a subalgebra. The usual method of obtaining the quantum phase space is to quantise the classical phase space T^*M. However, real spectral triples can give us a more direct route if we can find a way to extend $C^\infty(M)$ to $\mathcal{F}_\hbar$.

A straightforward way to do this, is to find the operators p_μ. The components D_μ of the Dirac operator $D = \gamma^\mu D_\mu$ are obvious candidates, since $[D_\mu, x_\nu] = -\mathrm{i}\eta_{\mu\nu}$. Therefore, we define

$$p_\mu := \hbar D_\mu, \tag{4.3}$$

and generate $\mathcal{F}_\hbar$ from $C^\infty(M)$ and $\{f(D_\mu) : f \in C^\infty(\mathbb{R}^m)\}$. A similar construction is discussed in [14]. Further, we can define creation and annihilation operators by

$$a_\mu := \frac{1}{\sqrt{2}}(x_\mu + \mathrm{i}D_\mu), \tag{4.4}$$

$$a_\mu^\dagger := \frac{1}{\sqrt{2}}(x_\mu - \mathrm{i}D_\mu). \tag{4.5}$$

Spectral triples shift the emphasis away from the canonical commutation relations as the basis of quantum mechanics to the momentum-Dirac relation $\gamma^\mu p_\mu = \hbar D$. The canonical commutation relations naturally appear as a consequence of the geometry of M. This makes a lot of sense, since momentum should be the generator of translations, and the generator of translations should be determined by the geometry. To summarise,

$$\begin{array}{ccccc} \text{Geometry} & + & \text{Physics} & = & \text{Quantum Mechanics} \\ [D_\mu, x_\nu] = -\mathrm{i}\eta_{\mu\nu} & & p_\mu := \hbar D_\mu & & [p_\mu, x_\nu] = -\mathrm{i}\hbar\eta_{\mu\nu} \end{array}$$

We would like to generalise the above construction to arbitrary real spectral triples. It is important that the spectral triples be real as the Dirac operator, and hence momentum, should be first order differential operators. The quantum phase space algebra $\mathcal{F}_\hbar$ can be generated as before, with $C^\infty(M)$ replaced by the noncommutative algebra $\mathcal{A}$ of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$. But, there is a problem with defining creation/annihilation operators as there is no obvious choice for the operators $x_\mu \in \mathcal{A}$. One can either try to impose $[a_\mu, a_\nu^\dagger] = \eta_{\mu\nu}$ and derive x_μ, or try appealing to the distance formula (2.20) for a definition of x_μ.

Even spectral triples $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ are particularly interesting from a quantum mechanical point of view, as they provide examples of $N = 1$ supersymmetric quantum mechanics (see [14]). The supercharge is given by

$$Q = D, \tag{4.6}$$

and is odd with respect to the $\mathbb{Z}_2$-grading Γ. It generates the supersymmetry transformation

$$u(\theta) = \mathrm{e}^{-\mathrm{i}\theta Q}, \tag{4.7}$$

where $\theta = \sum a[Q, b]$ is a 1-form. The Hamiltonian is fixed by the supersymmetry algebra,

$$H = Q^2. \tag{4.8}$$

The case of infinite dimensional even spectral triples and supersymmetric quantum field theory is dealt with in [10, sec. IV.9.β].

4.3 Quantisation of Noncommutative Geometries

The question of how to quantise a field theory on a general noncommutative geometry remains largely unresolved. Conventional techniques work on Riemannian-like manifolds and have been used on noncommutative extensions, such as almost commutative geometries (the tensor product of a Riemannian manifold with a finite noncommutative geometry) and the noncommutative torus [25]. Beyond this, most efforts have focused on quantising a particular noncommutative geometry [34, 21, 22].

We shall develop a path integral approach, based on our work in [20], that is applicable to any noncommutative geometry. The focus will be on quantising the spectral action, which is the natural geometric action for a noncommutative geometry. The Dirac operator is the dynamical variable of the spectral action, and plays the role of the metric. A path integral should therefore be some sort of "sum over Dirac operators". We will try to define what this might mean by appealing to the conventional path integral formalism. Our approach will build on, and complement, the work done by Rovelli in [41].

4.3.1 Path integral quantisation

We have chosen to develop a path integral approach, rather than a canonical approach, because it requires knowledge of only the fields, and not their dynamics. To be able to canonical quantise a noncommutative geometry, we would need a general procedure for finding the phase space, and constructing a symplectic structure on it. Conventionally, this amounts to finding the canonical momenta and using the Poisson bracket. In contrast, path integrals need a (gauge invariant) measure on the space of histories. Deciding how to parameterise this space is thus an important consideration. The advantage lies in that this does not depend on the details of the action, unlike finding the phase space. The only things that really matter are the fields, because they determine the measure. One of the other benefits of using path integrals is they are explicitly covariant.

A good starting point for developing a path integral formalism for noncommutative geometry is the conventional formalism. It has lead to standard model predictions that agree spectacularly with experiment, so it should be incorporated as a special case. Since the standard model action can be expressed in the form of a spectral action, a dictionary can be set up between noncommutative geometry and quantum field theory. This makes it apparent that the (gauge) fields parameterise the Dirac operator. So, the space of histories of the fields is equivalent to the space of histories of the Dirac operator. From the noncommutative geometry point of view then, the degrees of freedom of the Dirac operator correspond to the fields in the spectral action, and hence give the path integration measure. Thus, in principle, we can path integral quantise a general spectral action. Schematically, the general partition function can be written as

$$Z := \int \mathcal{D}D \; \mathrm{e}^{-\operatorname{Tr}\chi(D^2/\Lambda^2)}, \tag{4.9}$$

where D is the Dirac operator. The function χ and parameter Λ are the cutoffs for the spectral action.

4.3.2 The two-point space and Higgs gravity

The two-point space is the simplest example of a noncommutative space. It consists of just two points which we label L and R. The spectral triple is given by

$$\begin{aligned} \mathcal{A} &:= \mathbb{C} \oplus \mathbb{C} = \left\{ f := \begin{pmatrix} f_L & 0 \\ 0 & f_R \end{pmatrix} \right\}, \\ \mathcal{H} &:= \mathbb{C} \oplus \mathbb{C}, \\ D &:= \frac{1}{\hbar} \begin{pmatrix} 0 & m \\ \overline{m} & 0 \end{pmatrix}, \end{aligned} \tag{4.10}$$

where m is a complex constant which fixes the distance between the two points. It can almost be made into a real spectral triple; there is an obvious grading $\Gamma := \mathrm{diag}(1, -1)$ and a real structure J given by complex conjugation. However, they do not satisfy all of Connes' axioms. The two-point space can best be described as a "scaled" even Fredholm module.

Some may be unsettled by the appearance of $\hbar$ in the Dirac operator before quantisation. It is used only to follow the convention that m has units of mass, rather than inverse length, and so can be omitted. Alternatively, one could view $\hbar$ as the noncommutative geometriy version of c. In the same way that c relates space and time on a Lorentzian manifold, $\hbar$

relates space and (inverse) mass on a noncommutative geometry ("*spacemass*"). No $\hbar$ is required for quantisation as the spectral action is naturally dimensionless. We, however, will take our actions to have the usual dimensions of $\hbar$.

To move from a static (flat) space to a dynamic (curved) space, we promote the constant m to a variable ϕ, which will play the role of the gravitational field. This is the analogue of moving from $\eta_{\mu\nu}$ to $g_{\mu\nu}(x)$ on a Lorentzian manifold. In fact, ϕ is really a connection, so it plays the role of a vierbein/spin connection rather than a metric. In the context of the standard model, ϕ is interpreted as the Higgs field, hence we refer to this as Higgs gravity.

The spectral action is taken to be

$$S := \frac{1}{G} \operatorname{tr} D^2 = \frac{2l_\mathrm{P}^2}{\hbar}|\phi|^2, \tag{4.11}$$

where G is the gravitational coupling constant, and $l_\mathrm{p} := 1/\sqrt{\hbar G}$ is the Planck length. It has a U(1) symmetry which comes from $\mathrm{Inn}(\mathcal{A})$, the inner automorphism group of $\mathcal{A}$. For the two-point space, $\mathrm{Inn}(\mathcal{A}) \cong \mathrm{U}(1) \times \mathrm{U}(1)$, which acts on ϕ via the U(1) transformations given by the homomorphism $\mathrm{U}(1) \times \mathrm{U}(1) \to \mathrm{U}(1) : (g, h) \to gh^{-1}$. The inner automorphisms are analogous to the diffeomorphisms of general relativity. They are often referred to as internal diffeomorphisms.

Varying the action, the equations of motion are simply

$$\phi = 0, \quad \overline{\phi} = 0. \tag{4.12}$$

Using Connes' distance formula (2.20), the distance between the two points is

$$d(L, R) = \sup_{f \in \mathcal{A}} \{|f_L - f_R| : \frac{|\phi|^2}{\hbar^2}|f_L - f_R|^2 \leq 1\} = \frac{\hbar}{|\phi|} = \frac{m_\mathrm{P}}{|\phi|}l_\mathrm{P}, \tag{4.13}$$

where m_p is the Planck mass. So, classically, the metric structure D vanishes and the distance is infinite.

Now, we quantise by doing path integrals over ϕ and $\overline{\phi}$, the degrees of freedom of D. The partition function is thus

$$Z := \int \mathrm{d}D \; \mathrm{e}^{-S/\hbar} = \int \mathrm{d}\overline{\phi} \, \mathrm{d}\phi \; \exp\left(-\frac{2|\phi|^2}{m_\mathrm{p}^2}\right). \tag{4.14}$$

Since the action has a U(1) symmetry, we shall employ some gauge-fixing. This involves nothing more than switching to polar coordinates (r, θ), and dropping the irrelevant θ integration. Note: as the number of gauge degrees of freedom is finite, gauge-fixing is not strictly necessary (the θ integration does not give an infinite contribution). After integrating out gauge equivalent Dirac operators, the partition function reduces to

$$Z = \int_0^\infty \mathrm{d}\phi\, \phi \exp\left(-\frac{2\phi^2}{m_\mathrm{p}^2}\right) = \frac{m_\mathrm{p}^2}{4}, \tag{4.15}$$

where ϕ is now used to denote the positive real field $|\phi|$.

Expectation values are calculated in the usual fashion. For example,

$$\langle \phi \rangle = \frac{1}{Z}\int_0^\infty \mathrm{d}\phi\, \phi^2 \exp\left(-\frac{2\phi^2}{m_\mathrm{p}^2}\right) = \frac{\sqrt{2\pi}}{4} m_\mathrm{p}, \tag{4.16}$$

$$\langle d(L,R) \rangle = \frac{1}{Z}\int_0^\infty \mathrm{d}\phi\, m_\mathrm{p} l_\mathrm{p} \exp\left(-\frac{2\phi^2}{m_\mathrm{p}^2}\right) = \sqrt{2\pi}\, l_\mathrm{p}. \tag{4.17}$$

Here, we see that in the vacuum state, ϕ has acquired a v.e.v., and the distance has become finite. Though, the classical distance relation (4.13) no longer holds.

In general,

$$\int_0^\infty \mathrm{d}\phi\, \phi^n \exp\left(-\frac{2\phi^2}{m_\mathrm{p}^2}\right) = \frac{1}{2}\Gamma\left(\tfrac{n+1}{2}\right)\left(\frac{m_\mathrm{p}}{\sqrt{2}}\right)^{n+1}. \tag{4.18}$$

Thus, the Greens functions are

$$\langle \phi^n \rangle = \Gamma\left(\tfrac{n+2}{2}\right)\left(\frac{m_\mathrm{p}}{\sqrt{2}}\right)^{n}. \tag{4.19}$$

In particular, the propagator functions can be expressed as

$$\langle (\phi\phi)^n \rangle = n!\left(\frac{m_\mathrm{p}^2}{2}\right)^{n} \tag{4.20}$$

for $n \in \mathbb{Z}$. These reproduce the usual propagator combinatorics (i.e. Wick contractions) for a complex scalar field.

In an excited state, the distance $d(L,R)$ is given by its expectation value in a background of propagators. So, for the Nth particle state,

$$\langle d(L,R) \rangle_N = \frac{1}{Z_N}\langle \phi^N d(L,R) \phi^N \rangle, \tag{4.21}$$

where $Z_N := \langle(\phi\bar{\phi})^N\rangle$. This evaluates to

$$\langle d(L,R)\rangle_N = \frac{\Gamma(N+\frac{1}{2})}{\Gamma(N+1)}\sqrt{2}\,l_{\mathrm{p}}. \tag{4.22}$$

The distance thus gets successively smaller as the number of gravitons (Higgs particles) is increased. Using Stirling's formula, we find that the distance shrinks to zero in the $N \to \infty$ limit, and so the two points merge into one. The metric D correspondingly becomes infinite, since the description of the geometry as two points is no longer valid. This resembles the behaviour of a high curvature limit, i.e. gravitational collapse to a black hole.

The spectral action can be supplemented with the fermionic term

$$S_F := \langle\overline{\psi}, D\psi\rangle = \overline{\psi}_L\phi\psi_R + \overline{\psi}_R\overline{\phi}\psi_L, \tag{4.23}$$

which is invariant under the full $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry. Note that this is purely an interaction term—the fermions are fixed at the points and do not propagate. Quantising as before, we write down the partition function,

$$Z = \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\mathrm{d}\overline{\psi}\,\mathrm{d}\psi\,\exp\left(-\frac{2|\phi|^2}{m_{\mathrm{p}}^2} - \langle\overline{\psi}, D\psi\rangle\right). \tag{4.24}$$

Remember that the Hilbert space is complex, and not Grassmann, so

$$Z = \int \mathrm{d}\overline{\phi}\,\mathrm{d}\phi\,\frac{1}{\det D}\exp\left(-\frac{2|\phi|^2}{m_{\mathrm{p}}^2}\right) = -\int_0^\infty \mathrm{d}\phi\,\frac{1}{\phi}\exp\left(-\frac{2\phi^2}{m_{\mathrm{p}}^2}\right) = \infty. \tag{4.25}$$

This makes the v.e.v. $\langle d(L,R)\rangle$ ill-defined, while both $\langle\phi\rangle$ and the propagator $\langle\phi\bar{\phi}\rangle$ will be zero. For the excited states $(N \geq 1)$, the expectation values continue to be well-behaved. The effect of the fermions is to shield out the gravitational field, by lowering the states by one. If we were to take the tensor product of the Hilbert space with a spinor Hilbert space $L^2(\mathrm{spin}(M))$, then the fermions would enhance the gravitational field, by raising the states.

Note: for a generic finite noncommutative geometry, the fermion contribution will be $(\det D)^{-k}$ where k is the number of fermion generations fixed by the Hilbert space.

4.3.3 Matrix geometries and gauge gravity

Next, we look at the quantisation of the simplest matrix geometry, $\mathrm{M}_2(\mathbb{C})$. Its spectral triple is

$$\begin{aligned}
\mathcal{A} &:= \mathrm{M}_2(\mathbb{C}) = \left\{ f := \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \right\}, \\
\mathcal{H} &:= \mathrm{M}_2(\mathbb{C}), \\
D &:= \frac{1}{\hbar} \begin{pmatrix} A_1 & A_2 \\ \overline{A}_2 & -A_1 \end{pmatrix},
\end{aligned} \tag{4.26}$$

where D is an SU(2) gauge field, with A_1 real and A_2 complex. This is a reduction of the even spectral triple obtained by tensoring the representation with the Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^2)$,

$$\begin{aligned}
\mathcal{A}' &:= \mathcal{A}, \\
\mathcal{H}' &:= \mathcal{H} \oplus \mathcal{H} \quad \text{with } f' := f \otimes \mathbb{I}_2, \\
D' &:= D \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Gamma' &:= \mathbb{I}_2 \otimes \mathrm{diag}(1, -1).
\end{aligned} \tag{4.27}$$

Moreover, this itself is the point-reduction of the real spectral triple with $\mathcal{A}'' := C^\infty(\mathbb{R}^2) \otimes \mathcal{A}$, $\mathcal{H}'' := L^2(\mathrm{spin}(\mathbb{R}^2)) \otimes \mathcal{H}$ and $D'' := -\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu)$. The C^*-algebra $\mathrm{M}_2(\mathbb{C})$ can be thought of as being that of the fuzzy sphere $\mathbb{S}^2_{(n=1)}$ [33], which only has the north and south poles as distinguishable points.

The spectral action evaluates to

$$S := \frac{1}{G} \operatorname{tr} D^2 = \frac{2l_\mathrm{P}^2}{\hbar} \left({A_1}^2 + |A_2|^2 \right), \tag{4.28}$$

which is invariant under SU(2) gauge transformations. Like the two-point space, the inner automorphisms $\mathrm{Inn}(\mathcal{A}) \cong \mathrm{U}(2)$ act on D via a homomorphism, $\mathrm{U}(2) \to \mathrm{SU}(2)$. The homomorphism removes the trivial U(1) factor that commutes with D.

As before, we shall quantise by first gauge-fixing the action. This is most easily accomplished by changing to spherical polar coordinates. So, after dropping irrevelant factors, the partition function reads

$$Z = \int_0^\infty \mathrm{d}\phi\, \phi^2 \exp\left(-\frac{2\phi^2}{m_\mathrm{p}^2}\right) = \frac{\sqrt{2\pi}}{16} m_\mathrm{p}^3, \tag{4.29}$$

where $\phi := \sqrt{A_1{}^2 + |A_2|^2}$. Effectively, we have chosen a gauge-fixing condition such that

$$D = \frac{1}{\hbar} \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix}. \tag{4.30}$$

This gauge can be obtained from any other by performing an SU(2) gauge transformation

$$D \to uDu^\dagger = D + u[D, u^\dagger] \tag{4.31}$$

with

$$u = \frac{1}{2\sqrt{\phi(\phi - A_1)}} \begin{pmatrix} \phi - A_1 + \overline{A}_2 & \phi - A_1 - A_2 \\ -(\phi - A_1 - \overline{A}_2) & \phi - A_1 + A_2 \end{pmatrix}. \tag{4.32}$$

The Greens functions for ϕ are

$$\langle \phi^n \rangle = \frac{2}{\sqrt{\pi}} \Gamma\left(\tfrac{n+3}{2}\right) \left(\frac{m_\mathrm{p}}{\sqrt{2}}\right)^n. \tag{4.33}$$

As one would expect, they reflect the combinatorics of a field that can propagate through either a real mode ($A_1 \to A_1$) or a complex one ($A_2 \to \overline{A}_2$).

The distance between the poles of the fuzzy sphere,

$$d(1,4) = \sup_{f \in \mathcal{A}} \{|f_1 - f_4| : ||[D, f]|| \leq 1\}, \tag{4.34}$$

is not as straightforward to calculate as the distance between the points of the two-point space. Evaluating the condition $||[D, f]|| \leq 1$ gives

$$\frac{\hbar}{\phi} \geq \begin{cases} |(f_1 - f_4) + (f_2 - f_3)| \\ |(f_1 - f_4) - (f_2 - f_3)| \end{cases} \text{depending on which is larger.} \tag{4.35}$$

This can be simplified by expressing it in terms of "distances" and phases,

$$\frac{\hbar}{\phi} \geq |d_{14}\, \mathrm{e}^{\mathrm{i}\alpha} \pm d_{23}\, \mathrm{e}^{\mathrm{i}\beta}\, |, \tag{4.36}$$

where $d_{14}\, \mathrm{e}^{\mathrm{i}\alpha} := (f_1 - f_4)$ and $d_{23}\, \mathrm{e}^{\mathrm{i}\beta} := (f_2 - f_3)$. Squaring up both sides, it is then easy to determine the larger lower bound,

$$\begin{aligned} \frac{\hbar^2}{\phi^2} &\geq d_{14}{}^2 \pm 2 d_{14} d_{23} \cos(\alpha - \beta) + d_{23}{}^2 \\ &\geq d_{14}{}^2 + 2 d_{14} d_{23} |\cos(\alpha - \beta)| + d_{23}{}^2. \end{aligned} \tag{4.37}$$

Hence, the upper bound on d_{14} is

$$d_{14} \leq -d_{23}|\cos(\alpha-\beta)| + \sqrt{\frac{\hbar^2}{\phi^2} - d_{23}{}^2\sin^2\theta}. \tag{4.38}$$

Taking the supremum, the distance is therefore

$$d(1,4) = \frac{\hbar}{\phi} = \frac{m_{\mathrm{P}}}{\phi} l_{\mathrm{P}}. \tag{4.39}$$

Similarly, we also find

$$d(2,3) = \frac{\hbar}{\phi} = \frac{m_{\mathrm{P}}}{\phi} l_{\mathrm{P}}. \tag{4.40}$$

(We should clarify that there are no states $\langle\psi_2|\, f\,|\psi_2\rangle = f_2$ and $\langle\psi_3|\, f\,|\psi_3\rangle = f_3$, but there are two (pure) states $|\psi_2\rangle$ and $|\psi_3\rangle$ such that $|\,\langle\psi_2|\, f\,|\psi_2\rangle - \langle\psi_3|\, f\,|\psi_3\rangle\,| = |f_2 - f_3|$.)

The expectation value of the distances, in the Nth particle state, is

$$\langle d\rangle_N = \frac{\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}\sqrt{2}\, l_{\mathrm{P}}. \tag{4.41}$$

Just like the two-point space, the distances shrink to zero in the $N \to \infty$ limit. However, the nature of this collapse is rather different. The K-groups of the fuzzy sphere do not change as it collapses to a point, indeed $K_*(\mathrm{M}_2(\mathbb{C})) \cong K_*(\mathbb{C})$. Whereas this is not the case for the two-point space, for which $K_*(\mathbb{C}\oplus\mathbb{C}) \cong K_*(\mathbb{C}) \oplus K_*(\mathbb{C}) \ncong K_*(\mathbb{C})$. So, the collapse of the fuzzy sphere involves a change in commutativity, rather than topology.

From a K-theory perspective, the fuzzy sphere is more like a (noncommutative) point than a sphere. It is referred to as a sphere because of its SU(2) symmetry. In fact, the space of pure states of $\mathrm{M}_2(\mathbb{C})$ is a 2-sphere. Incidentally, the K-groups of a 2-sphere are actually isomorphic to those of the two-point space.

The fermion action for the fuzzy sphere is

$$\begin{aligned} S_F &:= \operatorname{tr}\Psi^\dagger D\Psi \\ &= \overline{\psi}_1 A_1\psi_1 + \overline{\psi}_2 A_1\psi_2 - \overline{\psi}_3 A_1\psi_3 - \overline{\psi}_4 A_1\psi_4 \\ &\quad + \overline{\psi}_1 A_2\psi_3 + \overline{\psi}_3\overline{A}_2\psi_1 + \overline{\psi}_2 A_2\psi_4 + \overline{\psi}_4\overline{A}_2\psi_2. \end{aligned} \tag{4.42}$$

It contains twice as many fermions as (4.23) due to the larger Hilbert space. The contribution to the partition function will thus be $(\det D)^{-2} = \phi^{-4}$. This will have the effect of lowering the states by two.

4.3.4 Comparison with Rovelli's canonical quantisation

We can try to compare our path integral approach with Rovelli's canonical approach (see [41] for details). In his example, the spectral action is modified to obtain non-trivial equations of motion. The action he uses is

$$\begin{aligned} S &:= \frac{1}{2}\operatorname{tr} D\widetilde{M}D \\ &= \frac{1}{2G}\left(\overline{m}_1 m_1 + \mathrm{e}^{-\mathrm{i}\theta}\,\overline{m}_1 m_2 + \mathrm{e}^{\mathrm{i}\theta}\,\overline{m}_2 m_1 + \overline{m}_2 m_2\right). \end{aligned} \tag{4.43}$$

But, this can be factorised as

$$\begin{aligned} &= \frac{1}{2G}\left(\overline{m}_1 + \mathrm{e}^{\mathrm{i}\theta}\,\overline{m}_2\right)\left(m_1 + \mathrm{e}^{-\mathrm{i}\theta}\,m_2\right) \\ &= \frac{|m|^2}{2G}, \end{aligned} \tag{4.44}$$

where $m := m_1 + \mathrm{e}^{-\mathrm{i}\theta}\,m_2$. Thus, we actually end up with a much simpler action and set of equations of motion. Canonical quantisation in this variable is a very different problem from the one considered by Rovelli.

Physically, the interaction terms in (4.43) allow the particles m_1 and m_2 to spontaneously change into one another. This is like a mixing term, so m_1 and m_2 will not make good eigenstates. As we have seen in (4.44), the linear combination given by m will make a good eigenstate.

Although the action (4.43) is not spectral *per se*, we can in fact still quantise it with our path integral approach. We begin by rewriting the action in terms of an effective Dirac operator, D', so it is spectral:

$$S = \frac{1}{2}\operatorname{tr} D\widetilde{M}D = \frac{1}{2}\operatorname{tr} D^\dagger P^\dagger P D = \frac{1}{2G}\operatorname{tr} D'^\dagger D' \tag{4.45}$$

Solving $P^\dagger P = \widetilde{M}$ gives

$$P = \frac{1}{\sqrt{2G}}\begin{pmatrix} 1 & \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 1 & \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.46}$$

thus

$$D' = \sqrt{G}PD = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & m \\ 0 & 0 & m \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.47}$$

Further, a self-adjoint operator D'' can be constructed by

$$D'^{\dagger}D' = \left(\frac{D'^{\dagger}+D'}{\sqrt{2}}\right)^2 = D''^2 = \frac{1}{4}\begin{pmatrix} 0 & 0 & m \\ 0 & 0 & m \\ \overline{m} & \overline{m} & 0 \end{pmatrix}^2 , \tag{4.48}$$

since D' is nilpotent. The degrees of freedom of D'' are m and $\overline{m}$, just as we have proposed. Quantising this, we end up with path integrals equivalent to those for the two-point space.

The problem with trying to canonically quantise spectral actions for finite noncommutative geometries is they have no phase space as such. This could be taken to mean that they simply cannot be quantised, but we have shown otherwise using path integrals. Perhaps some generalisation of phase space is needed (like tangent groupoids, see [44, sec. 6]), or maybe the path integral approach is just more fundamental.

4.3.5 Path integral quantisation of Rovelli's geometry

Having quantised Rovelli's modified spectral action (4.43) using path integrals, we shall now do the same for the un-modified spectral action

$$S := \frac{1}{G}\operatorname{tr}(D + JDJ^{-1})^2, \tag{4.49}$$

where

$$D := \frac{1}{\hbar}\begin{pmatrix} 0 & 0 & \phi_1 \\ 0 & 0 & \phi_2 \\ \overline{\phi}_1 & \overline{\phi}_2 & 0 \end{pmatrix}. \tag{4.50}$$

Unlike the geometries we have used in our examples, the geometry used by Rovelli does satisfy all the axioms for a real spectral triple. The eigenvalues and eigenvectors of $D + JDJ^{-1}$ are

$$\lambda = 0 \quad : \quad \begin{pmatrix} |\phi_2|^2 & -\phi_1\overline{\phi}_2 & 0 \\ -\overline{\phi}_1\phi_2 & |\phi_1|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.51}$$

$$\lambda = \pm\frac{2}{\hbar}\sqrt{|\phi_1|^2+|\phi_2|^2} \quad :$$
$$\begin{pmatrix} |\phi_1|^2 & \phi_1\overline{\phi}_2 & \pm\sqrt{|\phi_1|^2+|\phi_2|^2}\,\phi_1 \\ \overline{\phi}_1\phi_2 & |\phi_2|^2 & \pm\sqrt{|\phi_1|^2+|\phi_2|^2}\,\phi_2 \\ \pm\sqrt{|\phi_1|^2+|\phi_2|^2}\,\overline{\phi}_1 & \pm\sqrt{|\phi_1|^2+|\phi_2|^2}\,\overline{\phi}_2 & |\phi_1|^2+|\phi_2|^2 \end{pmatrix}, \tag{4.52}$$

so

$$S = \frac{8l_{\rm p}^2}{\hbar}\left(|\phi_1|^2 + |\phi_2|^2\right). \tag{4.53}$$

This has a U(2) symmetry under the $\mathrm{Inn}(\mathcal{A}) \cong \mathrm{U}(2) \times \mathrm{U}(1)$ gauge transformations

$$D \to (uJuJ^{-1})D(uJuJ^{-1})^\dagger = D + u[D, u^\dagger] + Ju[D, u^\dagger]J^{-1}. \tag{4.54}$$

An overall factor of U(1) acts trivially because it commutes with D.

Quantising the action, we get the gauge-fixed partition function

$$Z = \int_0^\infty \mathrm{d}\phi\, \phi^3 \exp\left(-\frac{8\phi^2}{m_{\rm p}^2}\right) = \frac{m_{\rm p}^4}{128}, \tag{4.55}$$

where $\phi := \sqrt{|\phi_1|^2 + |\phi_2|^2}$. From this, we find the Greens functions to be

$$\langle \phi^n \rangle = \Gamma\left(\tfrac{n+4}{2}\right)\left(\frac{m_{\rm p}}{\sqrt{8}}\right)^n. \tag{4.56}$$

For the distance used in [41, eqn. 22],

$$d = \frac{\hbar}{\sqrt{|\phi_1|^2 + |\phi_2|^2}} = \frac{m_{\rm p}}{\phi} l_{\rm p}, \tag{4.57}$$

the expectation values are

$$\langle d \rangle_N = \frac{\Gamma(N + \frac{3}{2})}{\Gamma(N + 2)}\sqrt{8}\, l_{\rm p}. \tag{4.58}$$

4.3.6 Spectral integrals

A proposal for a path integral approach is also put forward in [41]. It suggests that the integration measure should be given by the eigenvalues of the Dirac operator. This complements the spectral invariance of the spectral action. We shall refer to such path integrals as *spectral integrals.*

Spectral integrals differ from our path integrals in the way they integrate over the space of Dirac operators. The starting point for both is the space of self-adjoint operators, which can be partitioned into unitary equivalence classes. In our approach, we quotient out all those operators that have a non-zero trace, to leave only traceless self-adjoint operators. We then remove any degrees of freedom belonging to the center of the C^*-algebra $\mathcal{A}$. This has the effect of reducing the unitary equivalence classes down to $\mathrm{Inn}(\mathcal{A})$ equivalence classes. The space we are left with is the space of Dirac operators that we integrate over. We use gauge-fixing to perform the integration, so path integrals separate into a contribution from the gauge orbits and an integral along a section.

In contrast, spectral integrals just integrate over the orbit space of the unitary group action on the space of self-adjoint operators. The orbit space has the operator eigenvalues as cartesian coordinates, so there is no dependence on $\mathrm{Inn}(\mathcal{A})$. (To be precise, one should order the eigenvalues, $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, or include a symmetry factor in the integrals.) This means that different finite geometries with representations of the same dimension will have the same spectral integrals.

As a case in point, take the two-point space and the matrix geometry $\mathrm{M}_2(\mathbb{C})$. Both have two-dimensional representations and so two Dirac operator eigenvalues. Their spectral integrals will therefore be identical, making it impossible to distinguish between the two geometries using expectation values alone. For example, both geometries have the distance v.e.v.

$$\langle d \rangle = \frac{1}{Z} \int \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \, \frac{\sqrt{2}\, l_{\mathrm{p}}^2}{\sqrt{\lambda_1{}^2 + \lambda_2{}^2}} \exp\left(-\frac{\lambda_1{}^2 + \lambda_2{}^2}{l_{\mathrm{p}}^2} \right) = \sqrt{2\pi}\, l_{\mathrm{p}}. \tag{4.59}$$

It should be remembered that spectral integrals are, so far, just an idea, and we have interpreted it literally. An obvious refinement that could be made is to impose a traceless condition on the eigenvalues.

4.3.7 Spectral gravity on a circle

So far, we have only quantised the spectral action on simple finite noncommutative geometries. These provide nice toy models, but are far from being physically realistic. To obtain more interesting models, we shall try quantising some Riemannian manifolds. We will begin with a circle.

The real spectral triple for a circle is given by

$$\begin{aligned} \mathcal{A} &:= C^\infty(\mathbb{S}^1), \\ \mathcal{H} &:= L^2(\mathbb{S}^1), \\ D &:= -\frac{\mathrm{i}}{e(\theta)} \frac{\mathrm{d}}{\mathrm{d}\theta}, \\ J &:= 1 \circ {}^-, \end{aligned}$$

where $\theta \in [0, 2\pi]$ and $e(\theta)$ is a function with period 2π. Usually, one fixes $e(\theta)$ to be 1, but we are interested in having a dynamical metric. Note, varying the metric alters the Hochschild 1-cycle represented by Γ. Under the action of an inner automorphism, the Dirac operator transforms trivially, $D \to uJuJ^{-1}DJ\overline{u}J^{-1}\overline{u} = D$.

To evaluate the spectral action, we need to calculate the eigenvalues of the Dirac operator.

$$\begin{aligned}
-\frac{\mathrm{i}}{e(\theta)}\frac{\mathrm{d}\psi}{\mathrm{d}\theta} &= \lambda\psi \\
\frac{\mathrm{d}\psi}{\psi} &= \mathrm{i}\lambda e(\theta)\,\mathrm{d}\theta \\
\psi &= A\exp\left(\mathrm{i}\lambda\int_0^\theta e(\theta')\,\mathrm{d}\theta'\right).
\end{aligned}$$

The periodicity of $e(\theta)$ implies

$$\lambda\int_\theta^{\theta+2\pi} e(\theta')\,\mathrm{d}\theta' = \lambda\int_0^{2\pi} e(\theta')\,\mathrm{d}\theta' = 2n\pi. \tag{4.60}$$

The quantity $\int_0^{2\pi} e(\theta)\,\mathrm{d}\theta$ is just the circumference L of the circle ($e = \sqrt{g}$). So, the eigenvalues are

$$\lambda_n[e] = \frac{2n\pi}{L[e]}. \tag{4.61}$$

As there are an infinite number of eigenvalues, we need to insert a cutoff function χ into the spectral action:

$$S[e] := \frac{1}{G}\operatorname{Tr}\chi(D^2), \tag{4.62}$$

where

$$\chi(u) := \begin{cases} l_{\mathrm{p}}^{-2} & |u| \leq l_{\mathrm{p}}^{-2}, \\ 0 & |u| \geq l_{\mathrm{p}}^{-2}. \end{cases} \tag{4.63}$$

Evaluating it gives

$$\begin{aligned}
S[e] &= \frac{2}{G}\sum_{n=1}^{\infty}\chi(\lambda_n^2) = \frac{2}{G}\sum_{n=1}^{\frac{L[e]}{2\pi l_{\mathrm{p}}}} l_{\mathrm{p}}^{-2} && (4.64) \\
&= \frac{\hbar}{\pi l_{\mathrm{p}}}L[e] = \frac{\hbar}{\pi l_{\mathrm{p}}}\int_0^{2\pi} e(\theta)\,\mathrm{d}\theta. && (4.65)
\end{aligned}$$

We have ignored the zero mode, which just give a constant contribution (of $\hbar$).

Alternatively, we can evaluate the spectral action using the heat kernel expansion,

$$S[e] = \hbar\sum_{n=0}^{\infty} l_{\mathrm{p}}^{n-1} f_n\, a_n(D^2), \tag{4.66}$$

where

$$f_n = \frac{1}{\Gamma\left(\frac{1-n}{2}\right)} \int_0^\infty u^{-\frac{n+1}{2}} \chi(u)\,\mathrm{d}u. \tag{4.67}$$

The Seeley-DeWitt coefficients for a circle are

$$a_0(\theta, D^2) = \frac{1}{\sqrt{4\pi}}, \quad a_{n>0}(\theta, D^2) = 0. \tag{4.68}$$

Thus, we need only evaluate f_0,

$$\begin{aligned} f_0 &= \frac{1}{\sqrt{\pi}} \int_0^\infty u^{-\frac{1}{2}} \chi(u)\,\mathrm{d}u & (4.69)\\ &= \frac{1}{\sqrt{\pi}} \int_0^1 u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{2}{\sqrt{\pi}}. & (4.70) \end{aligned}$$

Hence,

$$S[e] = \frac{2\hbar}{\sqrt{\pi}\, l_\mathrm{p}} \int_0^{2\pi} \frac{e(\theta)}{\sqrt{4\pi}}\,\mathrm{d}\theta = \frac{\hbar}{\pi l_\mathrm{p}} \int_0^{2\pi} e(\theta)\,\mathrm{d}\theta. \tag{4.71}$$

The action consists solely of a cosmological constant term. It can be removed by replacing χ with

$$\widetilde{\chi}(u) := \chi(u) - \epsilon\chi(\epsilon^2 u), \quad \epsilon \ll 1. \tag{4.72}$$

But, this is undesirable as we want a non-zero action. We can expand $e(\theta)$ as a Fourier series,

$$e(\theta) = \sum_p e_p\, \mathrm{e}^{\mathrm{i}p\theta}. \tag{4.73}$$

The action then becomes

$$\begin{aligned} S[e] &= \frac{\hbar}{\pi l_\mathrm{p}} \sum_p e_p \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta}\,\mathrm{d}\theta \\ &= \frac{\hbar}{\pi l_\mathrm{p}} \left(2\pi e_0 + \sum_{p\neq 0} e_p \int_0^{2\pi} \mathrm{e}^{\mathrm{i}p\theta}\,\mathrm{d}\theta \right) \\ &= \frac{2\hbar}{l_\mathrm{p}} e_0. & (4.74) \end{aligned}$$

Therefore, we need only consider functions of the form $e(\theta) = R$. This is not too surprising as the circumference L of a circle is completely determined by its radius R ($L = 2\pi R$). So, much like finite noncommutative geometries, the circle only has a finite number of degrees of freedom. The equations of motion are simply

$$R = 0. \tag{4.75}$$

This differs from finite noncommutative geometries, which have dynamics corresponding to infinite distances.

Applying our quantisation procedure, the partition function is

$$Z = \int_0^\infty \mathrm{d}R\, \mathrm{e}^{-2R/l_\mathrm{P}} = \frac{l_\mathrm{P}}{2}. \tag{4.76}$$

So, the Greens functions are given by

$$\langle R^N \rangle = \frac{1}{Z} \int_0^\infty \mathrm{d}R\, R^N\, \mathrm{e}^{-2R/l_\mathrm{P}} = N! \left(\frac{l_\mathrm{P}}{2}\right)^N. \tag{4.77}$$

From these, we get the propagators,

$$\langle (RR)^N \rangle = \frac{1}{Z} \int_0^\infty \mathrm{d}R\, R^{2N}\, \mathrm{e}^{-2R/l_\mathrm{P}} = (2N)! \left(\frac{l_\mathrm{P}}{2}\right)^{2N}. \tag{4.78}$$

They have unusual combinatorics due to the action being linear in R. They are not so much propagators as they are sets of an even number of vertices.

The distance between two points θ_1 and θ_2 is

$$d(\theta_1, \theta_2) = \sup_{f \in \mathcal{A}} \{ |f(\theta_1) - f(\theta_2)| : \left| \frac{\mathrm{d}f}{\mathrm{d}\theta} \right| \leq R \} = |\theta_1 - \theta_2| R. \tag{4.79}$$

Classically, $d(\theta_1, \theta_2) = 0$. Their distance v.e.v. evaluates to

$$\langle d(\theta_1, \theta_2) \rangle = \frac{1}{Z} \int_0^\infty \mathrm{d}R\, |\theta_1 - \theta_2| R\, \mathrm{e}^{-2R/l_\mathrm{P}} = |\theta_1 - \theta_2| \frac{l_\mathrm{P}}{2}. \tag{4.80}$$

As distance depends linearly on R, the classical distance relation (4.79) still holds for v.e.v.s,

$$\langle d(\theta_1, \theta_2) \rangle = |\theta_1 - \theta_2| \langle R \rangle. \tag{4.81}$$

In the Nth particle state,

$$\langle d(\theta_1, \theta_2) \rangle_N = \frac{1}{Z_N} \int_0^\infty \mathrm{d}R\, R^N |\theta_1 - \theta_2| R R^N\, \mathrm{e}^{-2R/l_\mathrm{P}} = (2N+1) |\theta_1 - \theta_2| \frac{l_\mathrm{P}}{2}, \tag{4.82}$$

where

$$Z_N := \int_0^\infty \mathrm{d}R\, R^{2N}\, \mathrm{e}^{-2R/l_\mathrm{P}} = (2N)! \left(\frac{l_\mathrm{P}}{2}\right)^{2N+1}. \tag{4.83}$$

Thus, the distance between any two points grows with the number of gravitons—gravity acts repulsively. This is the type of behaviour one would expect from a cosmological constant. To obtain actions with local degrees of freedom, it is necessary to consider manifolds with four or more dimensions.

4.3.8 Riemannian manifolds

We now outline how our approach might work for less trivial Riemannian manifolds. The Dirac operator for a Riemannian manifold is

$$D := -\mathrm{i}\gamma^a e_a^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \frac{1}{4}\omega_{bc\mu}(x)\gamma^b\gamma^c\right), \tag{4.84}$$

where e_μ^a is the vierbein and ω_μ^{ab} is the spin connection. (Note: $\mathrm{Tr}\, D = 0$, as each term contains an odd number of gamma matrices.) As shown in section 3.6, the spectral action yields the Einstein-Hilbert action (ignoring higher order terms).

Usually, the metric, $g_{\mu\nu}$, is considered as the dynamical field and hence gives the measure for path integrals. In our approach, the vierbein and spin connection would be used instead, these being the degrees of freedom of the Dirac operator. This resembles the conventional connection-based way of quantising Yang-Mills theories. So, one might hope that this will make things more tractable.

We can go further. Let us now use a Dirac operator with a self-dual spin connection A_μ^{ab}. Since we work in an Euclidean signature, A_μ^{ab} is real as $A_\mu^{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}A_\mu^{cd}$ (it is complex in a Lorentzian signature). Applying this constraint to the spectral action will give the Einstein-Hilbert action with a self-dual curvature. This is essentially the Ashtekar formulation of general relativity.

The canonical quantisation, with respect to the spin connection, proceeds by performing a $3+1$ ADM decomposition. From this, the conjugate momentum, π_{ab}^μ, can be determined. It is self-dual and related to the vierbein. Making use of the self-duality, one can define the variables

$$A_\mu^i := A_\mu^{0i}, \quad \pi_i^\mu := \pi_{0i}^\mu, \tag{4.85}$$

where $i = 1, 2, 3$ is a space index. Their Poisson bracket is

$$\{A_\mu^i(x), \pi_j^\nu(y)\} = \delta_\mu^\nu \delta_j^i \delta^3(x-y). \tag{4.86}$$

This is very much like the Yang-Mills situation, with i and j as the SO(3) group indices. There are also constraint equations, the most notorious of which, is the Hamiltonian constraint. The quantisation of the constraints is dealt with by using loop representations [16]. This is the origin of loop quantum gravity.

The path integral quantisation is related to spin foams. It is possible to write the Einstein-Hilbert action in the form of a BF theory,

$$S := \int_M e_a^\mu e_b^\nu F_{\mu\nu}^{ab} e \, \mathrm{d}^4 x = \int_M B_{ab}^{\mu\nu} F_{\mu\nu}^{ab} \sqrt{g} \, \mathrm{d}^4 x, \tag{4.87}$$

where $F_{\mu\nu}^{ab}$ is the self-dual curvature, and $B_{ab}^{\mu\nu} = e_a^\mu e_b^\nu$ is a constraint. Path integrals over the spin connection and vierbein then resemble the quantisation of BF theory. To make the path integrals well-defined, they can be discretised by triangulating the manifold. In BF theory, this gives rise to the concept of spin foams [2], the spin network equivalent of Feynman diagrams.

Chapter 5

Homological Aspects of Noncommutative Geometry

5.1 Noncommutative Topology

The foundation of noncommutative topology is the Gelfand-Naĭmark duality between the category **LocCmpctHaus** of locally compact Hausdorff spaces and the category **CommC*-Alg** of commutative C^*-algebras. Every morphism in **CommC*-Alg** is dual to a morphism in **LocCmpctHaus**. For instance, a character $\chi_x : C_0(X) \to \mathbb{C}$ is dual to a point $x : \mathrm{pt} \to X$, where pt is the one-point space. Note, $\mathbb{C}$ is an initial object for the unital *-homomorphisms in **CommC*-Alg**, and pt is a terminal object for the continuous maps in **LocCmpctHaus**.

The functor from **CommC*-Alg** to **LocCmpctHaus** is given by

$$\hom(-, C(\mathrm{pt})) = \hom(-, \mathbb{C}). \tag{5.1}$$

It takes a C^*-algebra $C_0(X)$ to its space of characters $\hom(C_0(X), \mathbb{C})$, which is isomorphic to X. Conversely, the inverse functor from **LocCmpctHaus** to **CommC*-Alg** is given by

$$\hom(-, U(\mathbb{C})) = \hom(-, \mathbb{R}^2), \tag{5.2}$$

where $\hom(X, \mathbb{R}^2)$ is the (underlying set of the) C^*-algebra of functions from $X \to \mathbb{R}^2 \cong \mathbb{C}$.

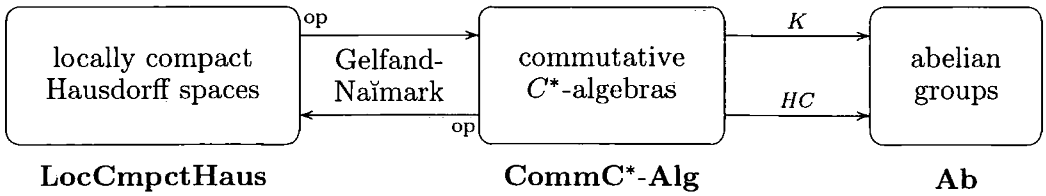

Figure 5.1: Noncommutative topology.

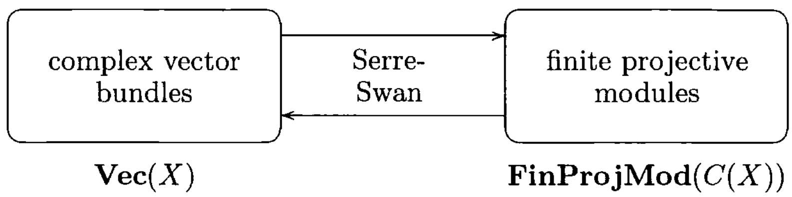

Figure 5.2: The Serre-Swan theorem.

There is also an equivalence between **CmpctHaus** and the category **CmpctRegLoc** of compact regular locales. This can be interpreted as a Gelfand-Naĭmark duality between **CmpctRegLoc** and the category of unital commutative C^*-algebras. A locale is the lattice of open sets of a space (meets distribute over joins). The noncommutative generalisation of a locale is a quantale [37]. However, the category **Qu** of quantales is not equivalent to the category of unital C^*-algebras [26]. But, there is a faithful functor Max from the category of unital C^*-algebras to a subcategory of **Qu**. So at least in some sense, a noncommutative C^*-algebra is the algebra of functions on the noncommutative space made up of the open sets of a quantale.

The Serre-Swan theorem is another equivalence of categories; namely between the category of complex vector bundles over a compact Hausdorff space X and the category of finite projective modules over $C(X)$. It is the justification for treating a finite projective module over a noncommutative C^*-algebra as a noncommutative vector bundle. The concept of a noncommutative vector bundle is less ambiguous than that of a noncommutative space. A finite projective module *is* a noncommutative vector bundle, whereas a C^*-algebra is only dual to a noncommutative space.

Finite projective modules are an important source of topological invariants in noncommutative geometry. They are the subject of K-theory—a cohomology theory for C^*-algebras.

Equally important is cyclic homology/cohomology. Before embarking on a tour of the various homology and cohomology theories for C^*-algebras, or their pre-C^*-algebras, we shall first outline homology theory in general.

5.2 Homology and Cohomology

Homology and cohomology theories are essentially functors from some category of interest to **Ab**. Homology functors are covariant, while cohomology functors are contravariant. They are generally constructed in two stages. The first stage is to find a functor from the category of interest to an abelian category of simplicial objects. A common way of creating a simplicial object is to build a comonad from a pair of adjoint functors. The adjoint functors are usually a free construction and its forgetful functor.

The second stage is to use the standard techniques of homological algebra [46] to obtain a series of functors from the abelian category of simplicial objects to **Ab**. Each of these functors gives rise to a homology or cohomology group. The homology functors map an abelian simplicial object X to the abelian simplicial object $X \otimes A$, where A is a constant abelian simplicial object (the coefficient object), and then take the homology of the associated chain complex. Similarly, the cohomology functors map an abelian simplicial object X to the abelian cosimplicial object $\mathrm{Hom}(X, A)$, and then take the cohomology of the associated cochain complex.

This is all formalised by the concept of derived functors. The homology functors are the derived functors $\mathrm{Tor}_n(X, A)$ (torsion products) of $X \otimes A$, and the cohomology functors are the derived functors $\mathrm{Ext}^n(X, A)$ (group extensions) of $\mathrm{Hom}(X, A)$. In particular, $\mathrm{Tor}_0(X, A) = X \otimes A$ and $\mathrm{Ext}^0(X, A) = \mathrm{Hom}(X, A)$. Note, $\otimes$ and Hom are adjoint functors, $\hom(A \otimes B, C) \cong \hom(A, \mathrm{Hom}(B, C))$.

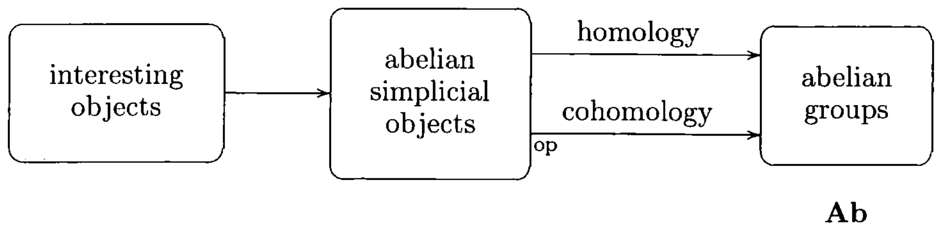

Figure 5.3: Homological algebra.

5.2.1 The simplicial category

The *simplicial category* Δ is defined as the small category whose objects are the totally ordered finite sets

$$[n] = \{0 < 1 < 2 < \ldots < n\}, \quad n \geq 0, \tag{5.3}$$

and whose morphisms are monotonic non-decreasing (order-preserving) maps. It is generated by two families of morphisms:

$$\begin{aligned} \delta_i^n &: [n-1] \to [n] \quad \text{is the injection missing } i \in [n], \\ \sigma_i^n &: [n+1] \to [n] \quad \text{is the surjection such that } \sigma_i^n(i) = \sigma_i^n(i+1) = i \in [n]. \end{aligned}$$

The δ_i^n morphisms are called *face maps*, and the σ_i^n morphisms are called *degeneracy maps.* They satisfy the following relations,

$$\delta_j^{n+1}\,\delta_i^n = \delta_i^{n+1}\,\delta_{j-1}^n \quad \text{for } i < j, \tag{5.4}$$

$$\sigma_j^{n-1}\,\sigma_i^n = \sigma_i^{n-1}\,\sigma_{j+1}^n \quad \text{for } i \leq j, \tag{5.5}$$

$$\sigma_j^n\,\delta_i^{n+1} = \begin{cases} \delta_i^n\,\sigma_{j-1}^{n-1} & \text{if } i < j, \\ \mathrm{id}_n & \text{if } i = j \text{ or } i = j+1, \\ \delta_{i-1}^n\,\sigma_j^{n-1} & \text{if } i > j+1. \end{cases} \tag{5.6}$$

All morphisms $[n] \to [0]$ factor through σ_0^0, so $[0]$ is terminal.

There is a bifunctor $+ : \Delta \times \Delta \to \Delta$ defined by

$$[m] + [n] = [m+n+1], \tag{5.7}$$

$$(f+g)(i) = \begin{cases} f(i) & \text{if } 0 \leq i \leq m, \\ g(i-m-1) + m' + 1 & \text{if } m < i \leq (m+n+1), \end{cases} \tag{5.8}$$

where $f : [m] \to [m']$ and $g : [n] \to [n']$. Sometimes, the simplicial category is defined to include the empty set $[-1] = \emptyset$, which provides an initial object for the category. We will denote this category by $\Delta^\emptyset$. This makes $\Delta^\emptyset$ a strict monoidal category as $\emptyset$ is a unit for the bifunctor: $\emptyset + [n] = [n] = [n] + \emptyset$ and $\mathrm{id}_\emptyset + f = f = f + \mathrm{id}_\emptyset$. Further, $\Delta^\emptyset$ is actually the free monoidal category on a monoid object (the monoid object being $[0]$, with product $\sigma_0^0 : [0] + [0] \to [0]$).

Example 5.2.1 (Morphisms in Δ)

Here are some examples of morphisms in the simplicial category:

$$\begin{aligned}
\delta_1^4 &: (0,1,2,3) \to (0,2,3,4), \\
\sigma_2^4 &: (0,1,2,3,4,5) \to (0,1,2,2,3,4), \\
\delta_1^4\,\delta_0^3 = \delta_0^4\,\delta_0^3 &: (0,1,2) \to (2,3,4), \\
\sigma_1^1\,\sigma_1^2 = \sigma_1^1\,\sigma_2^2 &: (0,1,2,3) \to (0,1,1,1), \\
\sigma_4^4\,\delta_2^5 = \delta_2^4\,\sigma_3^3 &: (0,1,2,3,4) \to (0,1,3,4,4).
\end{aligned}$$

(Composition is performed from right-to-left.)

Definition 5.2.1. A *simplicial object* in a category C is a contravariant functor from Δ to C. Such a functor X is uniquely specified by the morphisms $X(\delta_i^n) : [n] \to [n-1]$ and $X(\sigma_i^n) : [n] \to [n+1]$, which satisfy

$$X(\delta_i^{n-1})\,X(\delta_j^n) = X(\delta_{j-1}^{n-1})\,X(\delta_i^n) \quad \text{for } i < j, \tag{5.9}$$

$$X(\sigma_i^{n+1})\,X(\sigma_j^n) = X(\sigma_{j+1}^{n+1})\,X(\sigma_i^n) \quad \text{for } i \le j, \tag{5.10}$$

$$X(\delta_i^{n+1})\,X(\sigma_j^n) = \begin{cases} X(\sigma_{j-1}^{n-1})\,X(\delta_i^n) & \text{if } i < j, \\ \mathrm{id}_n & \text{if } i = j \text{ or } i = j+1, \\ X(\sigma_j^{n-1})\,X(\delta_{i-1}^n) & \text{if } i > j+1. \end{cases} \tag{5.11}$$

In particular, a *simplicial set* is a simplicial object in **Set**. Equivalently, one could say that a simplicial set is a presheaf on Δ. The object $X([n])$ of a simplicial set is a set of n-simplices, and is called the n-skeleton.

Definition 5.2.2. An *augmented simplicial object* in a category C is a contravariant functor from $\Delta^\emptyset$ to C.

Any augmented simplicial object is of course also a simplicial object by composition with the inclusion functor $\Delta \hookrightarrow \Delta^{\emptyset}$. A monoid object in a strict monoidal category B determines a functor $\Delta^{\emptyset} \to B$. This in turn determines a functor $(\Delta^{\emptyset})^{\mathrm{op}} \to B^{\mathrm{op}}$ and hence an augmented simplicial object in B^{op}. In other words, a comonoid object in B^{op} determines a simplicial object in B^{op}.

The *nerve* of a category C is the simplicial set $\hom(i(-), C)$, where $i : \Delta \to \mathbf{Cat}$ is the inclusion functor that takes each ordered set $[n]$ to the pre-order $\mathbf{n+1}$. The pre-order $\mathbf{n}$ is the category consisting of n partially-ordered objects, with one morphism $a \to b$ iff $a \leq b$. A functor between two categories induces a natural transformation between their nerves. So, the nerve defines a functor $\mathbf{Cat} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$.

Geometric realisation is a functor $| - | : \mathbf{Set}^{\Delta^{\mathrm{op}}} \to \mathbf{Top}$. Composed together with the nerve, it gives a functor $B : \mathbf{Cat} \to \mathbf{Top}$, which associates to each category C its *classifying space* BC.

Example 5.2.2 (Classifying space of a discrete group)
The nerve of a discrete group G is the simplicial set S with objects

$$\begin{aligned} S([0]) &= \{1\}, \\ S([1]) &= \{g_1\} = G, \\ S([2]) &= \{(g_1, g_2)\} = G \times G, \\ &\vdots \\ S([n]) &= \{(g_1, \ldots, g_n)\} = G^n, \end{aligned}$$

and with morphims

$$\begin{aligned} S(\delta_i^n)(g_1, \ldots, g_n) &= \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & \text{if } 0 < i < n, \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n, \end{cases} \\ S(\sigma_i^n)(g_1, \ldots, g_n) &= (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n). \end{aligned}$$

The classifying space BG is the Eilenberg-Mac Lane space $K(G, 1)$, that is, a connected space such that $\pi_1(BG) = G$ and $\pi_n(BG) = 0$ for $n \neq 1$. In particular, $B\mathbb{Z} = K(\mathbb{Z}, 1) = \mathbb{S}^1$ and $B\mathrm{U}(1) = K(\mathbb{Z}, 2) = \mathbb{C}P^{\infty}$.

5.2.2 Chain complexes and homology groups

A simplicial object $X : \Delta^{\mathrm{op}} \to A$ in an abelian category A determines a chain complex $([n], \partial_n)$ where

$$\partial_n := \sum_{i=0}^{n} (-1)^i X(\delta_i^n) : [n] \to [n-1]. \tag{5.12}$$

Definition 5.2.3. A *chain complex* (C_n, ∂_n) is a sequence of abelian groups or R-modules C_n and boundary morphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_{n-1}\partial_n = 0$. The elements of C_n are called *n-chains.*

The homology groups of a chain complex (C_n, ∂_n) are defined by

$$H_n(C_n, \partial_n) := \frac{Z_n(C_n, \partial_n)}{B_n(C_n, \partial_n)}, \tag{5.13}$$

where $Z_n(C_n, \partial_n) := \operatorname{Ker} \partial_n$ is the group of *n-cycles* and $B_n(C_n, \partial_n) := \operatorname{Im} \partial_{n+1}$ is the group of *n-boundaries.* Put simply,

$$H_n = \frac{n\text{-cycles}}{n\text{-boundaries}}, \tag{5.14}$$

with boundaries $\subset$ cycles $\subset$ chains.

Dually, a cosimplicial object $X : \Delta \to A$ in an abelian category A determines a cochain complex $([n], \delta^n)$ where

$$\delta^n := \sum_{i=0}^{n} (-1)^i X(\delta_i^n) : [n] \to [n+1]. \tag{5.15}$$

Definition 5.2.4. A *cochain complex* (C^n, δ^n) is a sequence of abelian groups or R-modules C^n and coboundary morphisms $\delta^n : C^n \to C^{n+1}$ such that $\delta^{n+1}\delta^n = 0$. The elements of C^n are called *n-cochains.*

The cohomology groups of a cochain complex (C^n, δ^n) are defined by

$$H^n(C^n, \delta^n) := \frac{Z^n(C^n, \delta^n)}{B^n(C^n, \delta^n)}, \tag{5.16}$$

where $Z^n(C^n, \delta^n) := \operatorname{Ker} \delta^n$ is the group of *n-cocycles* and $B^n(C^n, \delta^n) := \operatorname{Im} \delta^{n-1}$ is the group of *n-coboundaries.*

The hom-bifunctor can be used to turn a simplicial object into a cosimplicial object, or vice versa, by $\hom(X(-), a)$ for a fixed object a. Two particular examples of homology are singular homology and group homology. We outline their constructions below.

5.2.3 Singular homology

Each ordered set $[n]$ in Δ can be considered as a standard n-simplex. This defines an inclusion functor $i : \Delta \to \mathbf{Top}$. The hom-bifunctor on $\mathbf{Top}$ then gives a simplicial set $\hom(i(-), X)$ for a topological space X. The simplicial set functor $X \to \hom(i(-), X) : \mathbf{Top} \to \mathbf{Set}^{\Delta^{\mathrm{op}}}$ is the right adjoint of geometric realisation. Composition with the free construction functor $F_{\mathbb{Z}} : \mathbf{Set} \to \mathbf{Ab}$ creates a simplicial abelian group. This gives a functor $\mathbf{Top} \to \mathbf{Ab}^{\Delta^{\mathrm{op}}}$. The homology and cohomology of X, with coefficients in an abelian group A, is then given by

$$H_n(X, A) := \mathrm{Tor}_n(S(X), A), \tag{5.17}$$

$$H^n(X, A) := \mathrm{Ext}^n(S(X), A), \tag{5.18}$$

where $S(X)$ is the simplicial abelian group $F_{\mathbb{Z}}(\hom(i(-), X))$. The coefficient group A is to be understood as the constant simplicial abelian group $A([n]) = A$. Singular cohomology can also be expressed in terms of homotopy classes,

$$H^n(X, A) := [X, K(A, n)], \tag{5.19}$$

where $K(A, n)$ is an Eilenberg-Mac Lane space.

Theorem 5.2.1 (de Rham's theorem)
The singular cohomology groups $H^n(X, \mathbb{R})$ are isomorphic to the de Rham cohomology groups $H^n_{\mathrm{dR}}(X)$.

5.2.4 Group homology

Every group G gives a comonad $L_G := \mathbb{Z}G \otimes U(-)$ in $\mathbb{Z}G$-**Mod** via the adjoint functors $U : \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ and $\mathbb{Z}G \otimes - : \mathbf{Ab} \to \mathbb{Z}G\text{-}\mathbf{Mod}$. So, every $\mathbb{Z}G$-module M determines a simplicial module $S_{L_G}(M) : \Delta^{\mathrm{op}} \to \mathbb{Z}G\text{-}\mathbf{Mod}$. Consider the simplicial module given by $M = \mathbb{Z}$, the trivial $\mathbb{Z}G$-module, for a group G. Then, the homology and cohomology of G, with coefficients in a $\mathbb{Z}G$-module A, is given by

$$H_n(G, A) := \mathrm{Tor}_n(S_{L_G}(\mathbb{Z}), A), \tag{5.20}$$

$$H^n(G, A) := \mathrm{Ext}^n(S_{L_G}(\mathbb{Z}), A), \tag{5.21}$$

where $S_{L_G}(\mathbb{Z})$ is the simplicial $\mathbb{Z}G$-module $S_{L_G}(\mathbb{Z})([n]) = L_G^{n+1}(\mathbb{Z})$, and A is the constant simplicial $\mathbb{Z}G$-module $A([n]) = A$. There is an isomorphism between group homology/cohomology and singular homology/cohomology,

$$H_n(G, A) \cong H_n(BG, A), \tag{5.22}$$

$$H^n(G, A) \cong H^n(BG, A). \tag{5.23}$$

5.3 Homotopy Theory

Homotopy theory is concerned with the deformation of maps between topological spaces.

Definition 5.3.1. A *homotopy* between two maps $f, g : X \to Y$ is a continuous map $F : X \times [0,1] \to Y$ such that $F_0 = f$ and $F_1 = g$, where $F_t(x) := F(x,t) \in Y$.

Homotopies are a very general type of map. The homotopy groups of a (pointed) topological space X are defined by

$$\pi_n(X) := [\mathbb{S}^n, X], \tag{5.24}$$

where $[X, Y]$ denotes the set of homotopy classes of maps from X to Y. (The cohomotopy groups are defined by $\pi^n(X) := [X, \mathbb{S}^n]$.) From the perspective of category theory, there is a fundamental n-groupoid functor π_n from the category of pointed topological spaces to the category of n-groupoids. Its right adjoint is the classifying space functor (the category of groupoids is a full subcategory of **Cat**).

Using the Gelfand-Naĭmark duality, homotopy theory can be translated to the C^*-algebraic setting.

Definition 5.3.2. A *homotopy* between two *-homomorphisms $\eta, \phi : A \to B$ is a *-homomorphism $\psi : A \to C([0,1], B) = C([0,1]) \otimes B$ such that $\psi_0 = \eta$ and $\psi_1 = \phi$, where $\psi_t(a) := (\psi(a))(t) \in B$.

Connes has suggested [10, sec. II.A] the following generalisation of the homotopy groups for a unital C^*-algebra A,

$$\pi_{n,k}(A) := [A, \mathrm{M}_k(C(\mathbb{S}^n))]_{\mathbb{1}}, \tag{5.25}$$

where $[A, B]_{\mathbb{1}}$ is the set of homotopy classes of unital *-homomorphisms from A to B.

$$\begin{tikzcd}[column sep=huge] \boxed{R\text{-modules}} \arrow[r, shift left, "S\text{-}R\text{-bimodule}\otimes_R -"] & \boxed{S\text{-modules}} \arrow[l, shift left, "R\text{-}S\text{-bimodule}\otimes_S -"] \\ R\textbf{-Mod} & S\textbf{-Mod} \end{tikzcd}$$

Figure 5.4: Morita equivalence of rings.

5.4 Morita Equivalence

When working with any type of noncommutative ring, it is hard to avoid the notion of Morita equivalence. Essentially, two rings are Morita equivalent if their categories of representations are equivalent. In the context of noncommutative topology, it provides the means of comparing noncommutative spaces "up to noncommutativity". The topology of a noncommutative space should clearly not depend on the commutativity of its coordinates. So, topological invariants should be Morita invariant.

Definition 5.4.1. Two rings R and S are said to be *Morita equivalent* if there is an R-S-bimodule P and an S-R-bimodule Q such that $P \otimes_S Q \cong R$ and $Q \otimes_R P \cong S$ as bimodules. A representation of R is an R-module (or equivalently an R-$\mathbb{Z}$-bimodule). The bimodule P defines a functor $P \otimes_S - : S\textbf{-Mod} \to R\textbf{-Mod}$ from the representation category of S to the representation category of R. Similarly, $Q \otimes_R - : R\textbf{-Mod} \to S\textbf{-Mod}$ is a functor from the representation category of R to the representation category of S. This gives an equivalence of categories since $P \otimes_S Q \otimes_R - \cong R \otimes_R -$ and $Q \otimes_R P \otimes_S - \cong S \otimes_S -$ are naturally isomorphic to identity functors.

Morita equivalence has a natural interpretation as the notion of equivalence in the weak 2-category **Bimod** with rings as objects, bimodules as morphisms and bimodule homomorphisms as 2-morphisms [29]. An R-S-bimodule is a morphism from S to R and composition is given by the bimodule tensor product. A morphism $P : S \to R$ which is invertible up to 2-isomorphism is exactly a Morita equivalence.

Note, an R-S-bimodule can also be thought of as a generalised homomorphism from R to S. Since, a homomorphism $\rho : R \to S$ determines an R-S-bimodule given by S as a right S-module with left R-action $rs := \rho(r)s$, $r \in R$, $s \in S$. The composition of (generalised)

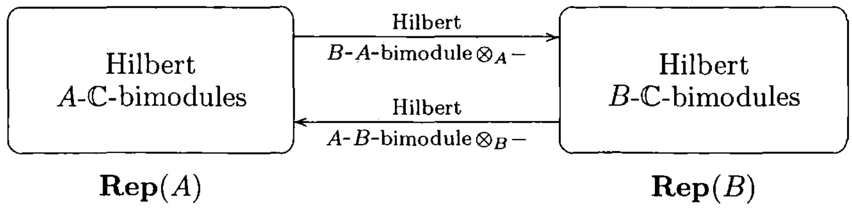

Figure 5.5: Strong Morita equivalence of C^*-algebras.

homomorphisms is the opposite tensor product of bimodules. In other words, there is a contravariant functor from **Rng** to **Bimod**.

For C^*-algebras, there is the more refined notion of strong Morita equivalence. It is Morita equivalence using Hilbert bimodules.

Definition 5.4.2. A *Hilbert module* (or *C^*-module*) over A is a right A-module $\mathcal{E}$ equipped with an A-valued inner product $\langle -,- \rangle : \mathcal{E} \times \mathcal{E} \to A$. The norm of an element $v \in \mathcal{E}$ is defined by $||v|| := \sqrt{||\langle v, v \rangle||}$.

Example 5.4.1 (Common Hilbert modules)

A Hilbert $\mathbb{C}$-module is just a Hilbert space. Any C^*-algebra A is a Hilbert A-module with inner product $\langle a, b\rangle := a^*b$ for all $a, b \in A$. The direct sum $A^n = A \oplus \ldots \oplus A$ of n copies of A is a Hilbert A-module with module action $(a_1, \ldots, a_n)b = (a_1 b, \ldots, a_n b)$ and inner product $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n)\rangle = \sum_{i=1}^n a_i^* b_i$, for all $a_i, b, b_i \in A$.

Definition 5.4.3. A *Hilbert A-B-bimodule* is a Hilbert module $\mathcal{E}$ over B together with a *-homomorphism π from A to $\mathrm{End}(\mathcal{E})$.

Definition 5.4.4. Two C^*-algebras A and B are said to be *strongly Morita equivalent* if there is a Hilbert A-B-bimodule $\mathcal{E}$ and a Hilbert B-A-bimodule $\mathcal{F}$ such that $\mathcal{E} \otimes_B \mathcal{F} \cong A$ and $\mathcal{F} \otimes_A \mathcal{E} \cong B$ as Hilbert bimodules. A representation of A is a *-homomorphism $\pi : A \to \mathbb{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. This is exactly the same thing as a Hilbert A-$\mathbb{C}$-bimodule. The Hilbert bimodules $\mathcal{E}$ and $\mathcal{F}$ define an equivalence of categories between the representation categories $\mathbf{Rep}(A)$ and $\mathbf{Rep}(B)$. If X is a Hilbert A-$\mathbb{C}$-bimodule, then the inner product on the Hilbert B-$\mathbb{C}$-bimodule $Y := \mathcal{F} \otimes_A X$ is given by $\langle f' \otimes x', f \otimes x\rangle_Y := \langle \langle f, f'\rangle_{\mathcal{F}} x', x\rangle_X$.

Completely analogous to **Bimod**, there is a weak 2-category **HilbBimod** with C^*-algebras as objects, Hilbert bimodules as morphisms and adjointable linear maps as 2-morphisms [29]. An equivalence of objects in **HilbBimod** is a strong Morita equivalence.

Theorem 5.4.1 (Brown-Green-Rieffel)
Two separable C^*-algebras A and B are strongly Morita equivalent iff they are stably equivalent ($A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$).

5.5 K-theory

There are two distinct flavours of K-theory: topological (or operator) K-theory and algebraic K-theory. The most relevant for noncommutative geometry is topological K-theory. Topological K-theory [45, 4] is a generalised cohomology theory on the category **C*-Alg** of separable C^*-algebras. Its functors are covariant; that is, contravariant with respect to topological spaces (by composition with the Gelfand-Naĭmark functor). It classifies noncommutative spaces (C^*-algebras) by their vector bundles (finite projective modules).

The functor underlying K-theory is the Grothendieck K-functor from the category of abelian semigroups to the category of abelian groups. It assigns to each abelian semigroup G, the smallest abelian group containing G. For example, $K(\mathbb{N}) = \mathbb{Z}$. The K-theory groups are just the Grothendieck groups of vector bundles over spaces.

The K_0 functor is defined by

$$K_0(A) := K([\mathbf{FinProjMod}(A)]), \tag{5.26}$$

where A is a C^*-algebra and $[C]$ denotes the set of isomorphism classes of objects in a category C (the decategorification of C). A finite projective (right) module pA^n is completely determined by the projection $p \in \mathrm{M}_n(A)$, for a fixed C^*-algebra A. The direct sum $pA^n \oplus qA^m$ of two finite projective modules pA^n and qA^m induces an addition

$$[p] + [q] := [p \oplus q] = [\mathrm{diag}(p, q)] \tag{5.27}$$

of the homotopy/equivalence classes of the projections p and q. So, the elements of the group $K_0(A)$ are the formal differences of the homotopy classes of the projections in $\mathrm{M}_\infty(A)$.

The K_1 functor is defined in terms of K_0 by

$$K_1(A) := K_0(SA), \tag{5.28}$$

where $SA := A \otimes C_0(\mathbb{R})$ is the suspension of A. Suspension is, obviously, a functor $S : \mathbf{C^*\text{-}Alg} \to \mathbf{C^*\text{-}Alg}$. Since $A \otimes C_0(\mathbb{R}) \cong \{f : \mathbb{S}^1 \to A \mid f(1) = 0\}$, it is often helpful to think of SA as an algebra of certain loops in A. The elements of the group $K_1(A)$ are the formal differences of the homotopy classes of the unitaries/invertibles in $\mathrm{M}_\infty(A)$. Note, unitaries and invertibles are homotopically equivalent as every $z \in \mathrm{GL}(A)$ is connected to $u = z|z|^{-1} \in \mathrm{U}(A)$ by the homotopy $t \to z|z|^{-t}$.

Similarly, higher K-groups can be defined by repeated suspensions. It turns out, however, that

$$K_2(A) := K_1(SA) \cong K_0(A), \tag{5.29}$$

so there are effectively only two K-groups. This is the Bott periodicity theorem. (Real K-theory has a period of 8 instead of 2.)

Example 5.5.1 ($K_i(\mathbb{C})$, $K_i(\mathrm{M}_n(\mathbb{C}))$)
A projection in $\mathrm{M}_k(\mathbb{C})$ *is given by* $p_j = \mathrm{diag}(\mathbb{1}_j, 0, \ldots, 0)$ *up to unitary equivalence. So, the abelian semigroup of projections in* $\mathrm{M}_\infty(\mathbb{C})$ *is isomorphic to* $\mathbb{N} \cup \{0\}$*. Hence,* $K_0(\mathbb{C}) = \mathbb{Z}$*. The unitary group* $\mathrm{U}_k(\mathbb{C})$ *is connected for all* $k > 0$*, hence* $K_1(\mathbb{C}) = 0$*.*

A projection in $\mathrm{M}_k(\mathrm{M}_n(\mathbb{C}))$ *is just a projection in* $\mathrm{M}_{kn}(\mathbb{C})$*, hence* $K_0(\mathrm{M}_n(\mathbb{C})) = \mathbb{Z}$*. The unitary group* $\mathrm{U}_k(\mathrm{M}_n(\mathbb{C}))$ *is isomorphic to* $\mathrm{U}_{kn}(\mathbb{C})$*, hence* $K_1(\mathrm{M}_n(\mathbb{C})) = 0$*.*

The main properties of K_i are:

$$K_i(A \oplus B) = K_i(A) \oplus K_i(B), \tag{5.30}$$

$$K_i(\mathrm{M}_n(A)) = K_i(A) \quad \text{(Morita invariance)}, \tag{5.31}$$

$$K_i(A \otimes \mathbb{K}) = K_i(A) \quad \text{(stability)}, \tag{5.32}$$

$$K_{i+2}(A) = K_i(A) \quad \text{(Bott periodicity)}. \tag{5.33}$$

There is also a cup product

$$\cup : K_i(A) \times K_j(B) \to K_{i+j}(A \otimes B). \tag{5.34}$$

A	$K_0(A)$	$K_1(A)$	Notes
$\mathbb{C}$	$\mathbb{Z}$	0	point
$\mathrm{M}_n(\mathbb{C})$	$\mathbb{Z}$	0	noncommutative point
$\mathbb{K}$	$\mathbb{Z}$	0	
$C_0(\mathbb{R}^{2n})$	$\mathbb{Z}$	0	$C_0(\mathbb{R}^k) \cong S^k\mathbb{C}$
$C_0(\mathbb{R}^{2n+1})$	0	$\mathbb{Z}$	
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$	
$C(\mathbb{S}^{2n})$	$\mathbb{Z}^2$	0	$C(\mathbb{S}^k) \cong C_0(\mathbb{R}^k) + \mathbb{C}\mathbb{I}$
$C(\mathbb{S}^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$	
A_θ	$\mathbb{Z}^2$	$\mathbb{Z}^2$	noncommutative torus

Table 5.1: K-groups of some common C^*-algebras [45].

Theorem 5.5.1 (Axiomatic K-theory)

Let K be a continuous, stable, homotopy invariant, half-exact functor from **C*-Alg** *to* **Ab**. *If $K(\mathbb{C}) = \mathbb{Z}$ and $K(S\mathbb{C}) = 0$, then $K(A) \cong K_0(A)$ for A in a large subcategory of* **C*-Alg**. *If $K(\mathbb{C}) = 0$ and $K(S\mathbb{C}) = \mathbb{Z}$, then $K(A) \cong K_1(A)$.*

5.5.1 Algebraic K-theory

Algebraic K-theory gives invariants for rings, so is also applicable to C^*-algebras [40]. The algebraic K_0^{alg} functor is identical to the topological K_0^{top} functor, but the same is not true for the higher algebraic K-functors. Indeed, there is no Bott periodicity in algebraic K-theory. However, there is a natural transformation (the *comparison map*) $K_n^{\mathrm{alg}} \to K_n^{\mathrm{top}}$, and both K-theories are isomorphic for stable C^*-algebras. (Stability is essential for Bott periodicity in algebraic K-theory.)

The algebraic K-groups are defined by

$$K_n^{\mathrm{alg}}(R) := \pi_n(B\mathrm{GL}(R)^+), \quad n \geq 1, \tag{5.35}$$

where $\mathrm{GL} : \mathbf{Rng} \to \mathbf{Grp}$ is the functor that takes a ring R to its general linear group $\mathrm{GL}(R)$ (with the discrete topology). This compares with a similar definition for the topological K-groups,

$$K_n^{\mathrm{top}}(A) := \pi_n(B\mathrm{GL}^{\mathrm{top}}(A)), \quad n \geq 1, \tag{5.36}$$

where $\mathrm{GL}^{\mathrm{top}} : \mathbf{C^*\text{-}Alg} \to \mathbf{Grp}$ is the functor that takes a C^*-algebra A to its topological general linear group $\mathrm{GL}^{\mathrm{top}}(A)$. Equivalently, $K_n^{\mathrm{top}}(A) := \pi_{n-1}(\mathrm{GL}^{\mathrm{top}}(A))$, since $\pi_n(BG) = \pi_{n-1}(G)$ for any topological group G.

5.6 K-homology

K-homology is the dual homology theory of K-theory. Its functors are contravariant on the category of separable C^*-algebras. Whereas K-theory classifies vector bundles, K-homology classifies the elliptic pseudo-differential operators acting on the vector bundles.

Abstract elliptic pseudo-differential operators are represented by Fredholm modules.

Definition 5.6.1. An *odd Fredholm module* $(\mathcal{H}, F)$ over a C^*-algebra A is given by an involutive representation π of A on a Hilbert space $\mathcal{H}$, together with an operator F on $\mathcal{H}$ such that $F = F^*$, $F^2 = \mathbb{1}$ and $[F, \pi(a)] \in \mathbb{K}(\mathcal{H})$ for all $a \in A$.

Definition 5.6.2. An *even Fredholm module* $(\mathcal{H}, F, \Gamma)$ is given by an odd Fredholm module $(\mathcal{H}, F)$ together with a $\mathbb{Z}_2$-grading Γ on $\mathcal{H}$, $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{1}$, such that $\Gamma\pi(a) = \pi(a)\Gamma$ and $\Gamma F = -F\Gamma$.

Definition 5.6.3. A Fredholm module is called *degenerate* if $[F, \pi(a)] = 0$ for all $a \in A$. Degenerate Fredholm modules are homotopic to the 0-module.

The K-homology group $K^0(A)$ is defined as the abelian group of homotopy classes of even Fredholm modules over A,

$$K^0(A) := [(\mathcal{H}, F, \Gamma)]. \tag{5.37}$$

Addition is given by the direct summation of Fredholm modules. The inverse of an even Fredholm module $(\mathcal{H}, F, \Gamma)$ is the even Fredholm module $(\mathcal{H}, -F, -\Gamma)$.

Just as with K-theory, higher K-homology groups can be defined by suspension, $K^n(A) := K^0(S^n A)$. As one would expect, Bott periodicity also holds for K-homology. The elements of $K^1(A)$ are the homotopy classes of odd Fredholm modules over A. Briefly, the relation to Brown-Douglas-Fillmore extension theory [6] is $K^1(A) = Ext(A) := Ext(A, \mathbb{K})$, for A nuclear.

Example 5.6.1 ($K^i(\mathbb{C})$, $K^i(\mathrm{M}_n(\mathbb{C}))$)
A non-degenerate even Fredholm module $(\mathcal{H}_k, F_k, \Gamma_k)$ *over* $\mathbb{C}$ *is given by*

$$\begin{aligned}
\mathcal{H}_k &:= \mathbb{C}^k \oplus \mathbb{C}^k \quad \text{with } \pi_k(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_k, \\
F_k &:= \begin{pmatrix} 0 & \mathbb{1}_k \\ \mathbb{1}_k & 0 \end{pmatrix}, \\
\Gamma_k &:= \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & -\mathbb{1}_k \end{pmatrix}.
\end{aligned}$$

The set of (homotopy classes of) all such Fredholm modules, their inverses and the 0-module is isomorphic to $\mathbb{Z}$. *Hence,* $K^0(\mathbb{C}) = \mathbb{Z}$. *All odd Fredholm modules over* $\mathbb{C}$ *are (up to homotopy) of the form* $(\mathbb{C}^k, \mathbb{1}_k)$ *with* $\pi(a) := a \otimes \mathbb{1}_k$. *They are clearly degenerate, hence* $K^1(\mathbb{C}) = 0$.

A non-degenerate even Fredholm module $(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k})$ *over* $\mathrm{M}_n(\mathbb{C})$ *is given by*

$$\begin{aligned}
\mathcal{H}_{n,k} &:= \mathcal{H}_{nk} \quad \text{with } \pi_{n,k}(a) := \pi_k(a), \\
F_{n,k} &:= F_{nk}, \\
\Gamma_{n,k} &:= \Gamma_{nk}.
\end{aligned}$$

Thus, as with even Fredholm modules over $\mathbb{C}$, $K^0(\mathrm{M}_n(\mathbb{C})) = \mathbb{Z}$. *All odd Fredholm modules over* $\mathrm{M}_n(\mathbb{C})$ *are (up to homotopy) of the form* $(\mathbb{C}^{nk}, \mathbb{1}_{nk})$ *with* $\pi(a) := a \otimes \mathbb{1}_k$. *Hence,* $K^1(\mathrm{M}_n(\mathbb{C})) = 0$.

The main properties of K^i are:

$$K^i(A \oplus B) = K^i(A) \oplus K^i(B), \tag{5.38}$$

$$K^i(\mathrm{M}_n(A)) = K^i(A) \quad \text{(Morita invariance)}, \tag{5.39}$$

$$K^i(A \otimes \mathbb{K}) = K^i(A) \quad \text{(stability)}, \tag{5.40}$$

$$K^{i+2}(A) = K^i(A) \quad \text{(Bott periodicity)}. \tag{5.41}$$

A	$K^0(A)$	$K^1(A)$
$\mathbb{C}$	$\mathbb{Z}$	0
$\mathrm{M}_n(\mathbb{C})$	$\mathbb{Z}$	0
$\mathbb{K}$	$\mathbb{Z}$	0
$C_0(\mathbb{R}^{2n})$	$\mathbb{Z}$	0
$C_0(\mathbb{R}^{2n+1})$	0	$\mathbb{Z}$
$C(\mathbb{T}^n)$	$\mathbb{Z}^{2^{n-1}}$	$\mathbb{Z}^{2^{n-1}}$
$C(\mathbb{S}^{2n})$	$\mathbb{Z}^2$	0
$C(\mathbb{S}^{2n+1})$	$\mathbb{Z}$	$\mathbb{Z}$
A_θ	$\mathbb{Z}^2$	$\mathbb{Z}^2$

Table 5.2: K-homology groups of some common C^*-algebras.

There is also a product map

$$K^i(A) \times K^j(B) \to K^{i+j}(A \otimes B). \tag{5.42}$$

5.7 Pairing between K-theory and K-homology

The pairing between K-theory and K-homology (also known as the cap product) is given by the indices of certain Fredholm operators constructed from the elements of the K-theory and K-homology groups.

Definition 5.7.1. The *index* of a Fredholm operator F is defined by

$$\operatorname{Ind} F := \dim \operatorname{Ker} F - \dim \operatorname{Ker} F^*. \tag{5.43}$$

5.7.1 Pairing between K^0 and K_0

The pairing between an even Fredholm module $(\mathcal{H}, F, \Gamma)$ over $\mathrm{M}_n(A)$ and a projection $p \in \mathrm{M}_n(A)$ is given by

$$\langle [(\mathcal{H}, F, \Gamma)], [p] \rangle := \operatorname{Ind}(\pi(p) F^+ \pi(p)), \tag{5.44}$$

where $\pi(p)F^+\pi(p) : \pi(p)\frac{\mathbb{1}+\Gamma}{2}\mathcal{H} \to \pi(p)\frac{\mathbb{1}-\Gamma}{2}\mathcal{H}$ is a Fredholm operator and $F^+ := \frac{\mathbb{1}-\Gamma}{2} F \frac{\mathbb{1}+\Gamma}{2}$.

5.7.2 Pairing between K^1 and K_1

The pairing between an odd Fredholm module $(\mathcal{H}, F)$ over $\mathrm{M}_n(A)$ and a unitary $u \in \mathrm{U}_n(A)$ is given by

$$\langle [(\mathcal{H}, F)], [u] \rangle := \mathrm{Ind}(P\pi(u)P), \tag{5.45}$$

where $P\pi(u)P : P\mathcal{H} \to P\mathcal{H}$ is a Fredholm operator and $P := \frac{\mathbb{1}+F}{2}$ is a projection.

Example 5.7.1 (The pairing between $K^i(\mathbb{C})$ and $K_i(\mathbb{C})$)
Consider an even Fredholm module $(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k})$ over $\mathrm{M}_n(\mathbb{C})$ and a projection $p_j = \mathrm{diag}(\mathbb{1}_j, 0, \ldots, 0) \in \mathrm{M}_n(\mathbb{C})$. Then,

$$\begin{aligned} \pi(p_j) &= \begin{pmatrix} p_j & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_k, \\ F^+ &= \begin{pmatrix} 0 & 0 \\ \mathbb{1}_{nk} & 0 \end{pmatrix}. \end{aligned}$$

So, $\pi(p)F^+\pi(p) : \mathbb{C}^{jk} \to 0$ and $(\pi(p)F^+\pi(p))^* : 0 \to \mathbb{C}^{jk}$. Hence,

$$\langle [(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k})], [p_j] \rangle = jk - 0 = jk. \tag{5.46}$$

Consider an odd Fredholm module $(\mathbb{C}^{nk}, \mathbb{1}_{nk})$ over $\mathrm{M}_n(\mathbb{C})$ and a unitary $u \in \mathrm{U}_n(\mathbb{C})$. Then, $P = \mathbb{1}_{nk}$ and $\pi(u) = u \otimes \mathbb{1}_k$. Hence,

$$\langle [(\mathbb{C}^{nk}, \mathbb{1}_{nk})], [u] \rangle = \mathrm{Ind}(u \otimes \mathbb{1}_k) = 0, \tag{5.47}$$

as u is invertible.

5.7.3 Pairing with K-cycles

The pairing can be extended from Fredholm modules to K-cycles (spectral triples) by taking $F = D|D|^{-1}$. So, for an even real K-cycle $(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J, \Gamma)$ and projections $p, q \in \mathrm{M}_n(\mathcal{A})$,

$$\langle [(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J, \Gamma)], [p \otimes q^{\mathrm{op}}] \rangle = \mathrm{Ind}(pJq^*J^{-1}D^+pJq^*J^{-1}), \tag{5.48}$$

using the homotopy $t \to D|D|^{-t}$ from D to $D|D|^{-1}$. Similarly, for an odd real K-cycle $(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J)$ and unitaries $u, v \in \mathrm{U}_n(\mathcal{A})$,

$$\langle [(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J)], [u \otimes v^{\mathrm{op}}] \rangle = \mathrm{Ind}(PuJv^* J^{-1} P), \tag{5.49}$$

where $P = \frac{\mathbb{1} + D|D|^{-1}}{2}$ is a projection.

5.7.4 The intersection form and Poincaré duality

Given a real K-cycle, the *intersection form* defines a pairing on K-theory. The K_0 pairing is given by

$$\langle [p], [q] \rangle_D := \langle [(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J, \Gamma)], [p \otimes q^{\mathrm{op}}] \rangle, \tag{5.50}$$

and the K_1 pairing is given by

$$\langle [u], [v] \rangle_D := \langle [(\mathrm{M}_n(\mathcal{A}), \mathcal{H}, D, J)], [u \otimes v^{\mathrm{op}}] \rangle. \tag{5.51}$$

An important property of a differential manifold M is Poincaré duality, $H^k(M) \cong H_{m-k}(M)$. Poincaré duality holds if there is a non-degenerate bilinear pairing $\langle -, - \rangle : H^k(M) \times H^{m-k}(M) \to \mathbb{R}$ on the de Rham cohomology of M. Using the classical Chern character $\mathrm{ch}^i : K^i(M) \otimes \mathbb{Q} \to H^i(M, \mathbb{Q})$, this translates to a non-degenerate intersection form. Therefore, a noncommutative space is considered to be a noncommutative manifold if it satisfies Poincaré duality in the sense of having a non-degenerate intersection form, i.e. its K-theory is isomorphic to its K-homology.

5.8 KK-theory

K-theory and K-homology are both special cases of a more general bivariant theory known as KK-theory. KK-theory is a bifunctor from the category of separable C^*-algebras to the category of abelian groups,

$$KK : \mathbf{C^*\text{-}Alg}^{\mathrm{op}} \times \mathbf{C^*\text{-}Alg} \to \mathbf{Ab}. \tag{5.52}$$

The contravariant argument represents K-homology and the covariant argument represents K-theory:

$$KK(A, \mathbb{C}) = K^0(A), \tag{5.53}$$

$$KK(\mathbb{C}, A) = K_0(A). \tag{5.54}$$

In essence, KK-theory is K-homology with Fredholm bimodules. What we suggestively refer to as a Fredholm bimodule is commonly know as a Kasparov bimodule.

Definition 5.8.1. An *odd Kasparov A-B-bimodule* $(\mathcal{E}, F)$ is given by a Hilbert A-B-bimodule $\mathcal{E}$, and an operator F on $\mathcal{E}$ such that $(F - F^*)\pi(a) \in \mathbb{K}(\mathcal{E})$, $(F^2 - \mathbb{1})\pi(a) \in \mathbb{K}(\mathcal{E})$ and $[F, \pi(a)] \in \mathbb{K}(\mathcal{E})$ for all $a \in A$.

Definition 5.8.2. An *even Kasparov A-B-bimodule* $(\mathcal{E}, F, \Gamma)$ is given by an odd Kasparov A-B-bimodule $(\mathcal{E}, F)$ together with a $\mathbb{Z}_2$-grading Γ on $\mathcal{E}$, $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{1}$, such that $\Gamma\pi(a) = \pi(a)\Gamma$ and $\Gamma F = -F\Gamma$.

Definition 5.8.3. A Kasparov bimodule is called *degenerate* if $(F - F^*)\pi(a) = 0$, $(F^2 - \mathbb{1})\pi(a) = 0$ and $[F, \pi(a)] = 0$ for all $a \in A$.

Definition 5.8.4. A Kasparov bimodule is called *normalised* if $F = F^*$ and $F^2 = \mathbb{1}$. It is possible to normalise any Kasparov bimodule.

Kasparov A-B-bimodules can be thought of as generalised *-homomorphisms from A to B.

Example 5.8.1 (The Kasparov bimodule for a *-homomorphism)
*A *-homomorphism $\phi : A \to B$ defines an even Kasparov A-B-bimodule $(\mathcal{E}, F, \Gamma)$, where*

$$\begin{aligned}
\mathcal{E} &:= B \oplus B \quad \textit{with } \pi(a) := \begin{pmatrix} \phi(a) & 0 \\ 0 & 0 \end{pmatrix}, \\
F &:= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \\
\Gamma &:= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.
\end{aligned}$$

The elements of $KK(A,B)$ are the homotopy classes of even Kasparov A-B-bimodules,

$$KK(A,B) := [(\mathcal{E},F,\Gamma)]. \tag{5.55}$$

There are several strengths of homotopy that can be used. They all coincide for separable C^*-algebras. We state the weakest (most general).

Definition 5.8.5. A *homotopy* between two even Kasparov A-B-bimodules $(\mathcal{E}_0, F_0, \Gamma_0)$ and $(\mathcal{E}_1, F_1, \Gamma_1)$ is an even Kasparov A-$C([0,1],B)$-bimodule $(\mathcal{E},F,\Gamma)$ such that $(\mathcal{E}(0), F(0), \Gamma(0)) = (\mathcal{E}_0, F_0, \Gamma_0)$ and $(\mathcal{E}(1), F(1), \Gamma(1)) = (\mathcal{E}_1, F_1, \Gamma_1)$, where the even Kasparov A-B-bimodule $(\mathcal{E}(t), F(t), \Gamma(t))$ is given by the evaluation homomorphism from $C([0,1],B)$ to B.

It is possible to work just with homotopies where only the operator F varies (*operator homotopies*).

The homotopy classes form an abelian group with addition given by the direct summation of Kasparov bimodules. Degenerate Kasparov bimodules are homotopic to the 0-bimodule, and the inverse of an even Kasparov bimodule $(\mathcal{E},F,\Gamma)$ is the even Kasparov bimodule $(\mathcal{E},-F,-\Gamma)$. A normalised even Kasparov A-$\mathbb{C}$-bimodule is just an even Fredholm module over A, hence $KK(A,\mathbb{C}) = K^0(A)$.

Higher KK-groups can be defined by suspending one of the arguments,

$$KK_n(A,B) := KK(S^nA,B) = KK(A,S^nB). \tag{5.56}$$

The group $KK_1(A,B)$ is the abelian group of homotopy classes of odd Kasparov A-B-bimodules. Bott periodicity in K-theory and K-homology means that $KK_{i+2}(A,B) = KK_i(A,B)$.

The properties of KK-theory are generalisations of those of K-theory and K-homology:

$$KK(A_1 \oplus A_2, B) = KK(A_1,B) \oplus KK(A_2,B), \tag{5.57}$$

$$KK(A, B_1 \oplus B_2) = KK(A,B_1) \oplus KK(A,B_2), \tag{5.58}$$

$$KK(\mathrm{M}_m(A), \mathrm{M}_n(B)) = KK(A,B) \quad \text{(Morita invariance)}, \tag{5.59}$$

$$KK(A \otimes \mathbb{K}, B) = KK(A,B) \quad \text{(stability)}, \tag{5.60}$$

$$KK(A, B \otimes \mathbb{K}) = KK(A,B) \quad \text{(stability)}, \tag{5.61}$$

$$KK(SA,B) = KK(A,SB) \quad \text{(suspension)}, \tag{5.62}$$

$$KK(SA,SB) = KK(A,B) \quad \text{(Bott periodicity)}. \tag{5.63}$$

The most important is the Kasparov intersection product

$$\otimes_B : KK(A,B) \times KK(B,C) \to KK(A,C). \tag{5.64}$$

This incorporates the index pairing between K-theory and K-homology,

$$\begin{aligned} \cap : KK(\mathbb{C},A) \times KK(A,\mathbb{C}) &\to KK(\mathbb{C},\mathbb{C}) \\ K_0(A) \times K^0(A) &\to \mathbb{Z} \\ K_1(A) \times K^1(A) &\to \mathbb{Z} \quad \text{via suspension.} \end{aligned}$$

The most general form of the intersection product is

$$\otimes_D : KK(A_1, B_1 \otimes D) \times KK(D \otimes A_2, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2). \tag{5.65}$$

In the words of [3, sec. 19.8], "This product is associative and functorial in all possible senses. The product generalizes composition and tensor product of *-homomorphisms, cup and cap products, tensor product of elliptic pseudodifferential operators, and the pairing between K-theory and K-homology." Poincaré duality is just the pairing

$$\cap : KK(\mathbb{C},A) \times KK(A \otimes A^{\mathrm{op}},\mathbb{C}) \to KK(A^{\mathrm{op}},\mathbb{C}) \cong KK(A,\mathbb{C}) \tag{5.66}$$

$$K_i(A) \times KR^i(A \otimes A^{\mathrm{op}}) \to K^i(A) \tag{5.67}$$

given by the KR-homology class $\mu \in KR^i(A \otimes A^{\mathrm{op}})$ of a real K-cycle $(\mathcal{A}, \mathcal{H}, D, J, \Gamma)$.

In many respects, the KK bifunctor is like a hom-bifunctor: it is bivariant and the intersection product defines an "associative composition of morphisms". In fact, it is possible to construct an additive category **KK** whose objects are separable C^*-algebras and whose hom-sets are the KK-groups. **KK** is the universal enveloping category of **C*-Algh**, where **C*-Algh** is the category whose objects are separable C^*-algebras and whose morphisms are homotopy classes of stable *-homomorphisms. A *stable *-homomorphism* between two C^*-algebras A and B is a *-homomorphism between A and $B \otimes \mathbb{K}$. An isomorphism in **KK** is known as a KK-equivalence.

5.9 E-theory

KK-theory has a reputation for being technically difficult. A simpler, related theory is E-theory [3, sec. 25].

Definition 5.9.1. An *asymptotic morphism* between two C^*-algebras A and B is a family of maps $T = \{T_{\hbar\in(0,\hbar_0]} : A \to B\}$, for some $\hbar_0 > 0$, such that $\hbar \to T_\hbar(a)$ is norm-continuous for every $a \in A$, and for any $a, b \in A$ and $\lambda \in \mathbb{C}$:

$$\begin{aligned}
\lim_{\hbar\to 0} ||T_\hbar(a) + \lambda T_\hbar(b) - T_\hbar(a+\lambda b)|| &= 0, \\
\lim_{\hbar\to 0} ||T_\hbar(a)^* - T_\hbar(a^*)|| &= 0, \\
\lim_{\hbar\to 0} ||T_\hbar(ab) - T_\hbar(a)T_\hbar(b)|| &= 0.
\end{aligned}$$

Example 5.9.1 (The asymptotic morphism for a *-homomorphism)
*A *-homomorphism* $\phi : A \to B$ *defines an asymptotic morphism* $T = \{T_\hbar := \phi\} : A \to B$.

An important example of an asymptotic morphism is given by the Moyal quantisation map $Q_\hbar : C_0(T^*\mathbb{R}^n) \to \mathbb{K}(L^2(\mathbb{R}^n))$. This defines an asymptotic morphism from the C^*-algebra of classical observables to the C^*-algebra of quantum observables.

The E-theory groups are defined by

$$E(A,B) := [\![SA, SB\otimes\mathbb{K}]\!] \cong [\![SA\otimes\mathbb{K}, SB\otimes\mathbb{K}]\!], \tag{5.68}$$

where $[\![A,B]\!]$ denotes the set of homotopy classes of asymptotic morphisms from A to B. For separable nuclear C^*-algebras, E-theory is isomorphic to KK-theory. But unlike KK-theory, it is half-exact in both arguments. The main properties of E-theory are:

$$E(\mathbb{C}, A) = K_0(A), \tag{5.69}$$

$$E(\mathrm{M}_m(A), \mathrm{M}_n(B)) = E(A,B) \quad \text{(Morita invariance)}, \tag{5.70}$$

$$E(A\otimes\mathbb{K}, B) = E(A,B) \quad \text{(stability)}, \tag{5.71}$$

$$E(A, B\otimes\mathbb{K}) = E(A,B) \quad \text{(stability)}, \tag{5.72}$$

$$E(SA, B) = E(A, SB) \quad \text{(suspension)}, \tag{5.73}$$

$$E(SA, SB) = E(A,B) \quad \text{(Bott periodicity)}. \tag{5.74}$$

$$\begin{tikzcd}
\mathbf{C^*\text{-}Alg} \arrow[r, "\text{universal}", "\text{envelope}"'] \arrow[d] & \mathbf{E} \\
\mathbf{C^*\text{-}Algh} \arrow[r, "\text{universal}", "\text{envelope}"'] & \mathbf{KK} \arrow[u]
\end{tikzcd}$$

Figure 5.6: Universal enveloping categories of C^*-algebras.

An additive category **E** can be constructed whose objects are separable C^*-algebras and whose hom-sets are the E-groups. The composition of asymptotic morphism is well-defined up to homotopy using $[\![A, B\otimes\mathbb{K}]\!] \cong [\![A\otimes\mathbb{K}, B\otimes\mathbb{K}]\!]$. **E** is the universal enveloping category of **C*-Alg**.

5.10 Cyclic Homology and Cohomology

Cyclic homology [10, 32, 4] acts as the noncommutative generalisation of de Rham cohomology. Unlike other homology theories, it is constructed from cyclic objects rather than simplicial objects. There is a simplicial version, called Hochschild homology.

5.10.1 The cyclic category

The *cyclic category* Λ is the small category with objects Λ_n, $n \geq 0$, and morphisms generated by face maps $\delta_i^n : \Lambda_{n-1} \to \Lambda_n$, degeneracy maps $\sigma_i^n : \Lambda_{n+1} \to \Lambda_n$ and (anti)cyclic maps $\tau_n : \Lambda_n \to \Lambda_n$, satisfying (5.4), (5.5), (5.6) and

$$\tau_n\,\delta_0^n = \delta_n^n, \tag{5.75}$$

$$\tau_n\,\delta_i^n = \delta_{i-1}^n\,\tau_{n-1} \quad \text{for } 1 \leq i \leq n, \tag{5.76}$$

$$\tau_n\,\sigma_0^n = \sigma_n^n\,(\tau_{n+1})^2, \tag{5.77}$$

$$\tau_n\,\sigma_i^n = \sigma_{i-1}^n\,\tau_{n+1} \quad \text{for } 1 \leq i \leq n, \tag{5.78}$$

$$(\tau_n)^{n+1} = \mathrm{id}_n. \tag{5.79}$$

It has the simplicial category as a subcategory and any morphism from Λ_n to Λ_m can be uniquely written as the product ϕg, where $\phi \in \hom_\Delta([n],[m])$ and $g \in \mathbb{Z}_{n+1}$. For this reason, Λ is sometimes denoted by ΔC, where C is the disconnected groupoid whose automorphism sets are the cyclic groups, $\hom_C(n,n) = \mathbb{Z}_n$. If Λ_n is identified with $[n]$, then τ_n is the map defined by $\tau_n(0) = n$ and $\tau_n(i) = i-1$ for $i \neq 0$, $i \in [n]$.

The cyclic category is also self-dual: there is a contravariant functor $^* : \Lambda \to \Lambda^{\mathrm{op}}$ which gives an isomorphism $\Lambda^{\mathrm{op}} \cong \Lambda$. This can be seen by constructing an extra degeneracy map

$$\sigma^n_{n+1} := \sigma^n_0 \, (\tau_{n+1})^{n+1}. \tag{5.80}$$

Then, $(\delta^n_i)^* = \sigma^{n-1}_i$ (use the extra degeneracy map for $i = n$), $(\sigma^n_i)^* = \delta^{n+1}_{i+1}$ and $(\tau_n)^* = (\tau_n)^n$.

Definition 5.10.1. A *cyclic object* in a category C is a contravariant functor from Λ to C. Such a functor X is uniquely specified by the morphisms $X(\delta^n_i) : [n] \to [n-1]$, $X(\sigma^n_i) : [n] \to [n+1]$ and $X(\tau_n) : [n] \to [n]$, which satisfy (5.9), (5.10), (5.11) and

$$\begin{aligned}
X(\delta^n_0)\, X(\tau_n) &= X(\delta^n_n), && (5.81)\\
X(\delta^n_i)\, X(\tau_n) &= X(\tau_{n-1})\, X(\delta^n_{i-1}) \quad \text{for } 1 \le i \le n, && (5.82)\\
X(\sigma^n_0)\, X(\tau_n) &= X(\tau_{n+1})^2 X(\sigma^n_n), && (5.83)\\
X(\sigma^n_i)\, X(\tau_n) &= X(\tau_{n+1})\, X(\sigma^n_{i-1}) \quad \text{for } 1 \le i \le n, && (5.84)\\
X(\tau_n)^{n+1} &= \mathrm{id}_n. && (5.85)
\end{aligned}$$

Any cyclic object is also a simplicial object by composition with the inclusion functor $\Delta \hookrightarrow \Lambda$.

5.10.2 Cyclic modules

For any unital algebra $\mathcal{A}$ over a field k, there is a functor $\mathcal{A}^\natural : \Lambda \to k$-**Mod** defined by

$$\begin{aligned}
\mathcal{A}^\natural(\Lambda_n) &:= \mathcal{A}^{\otimes(n+1)}\\
&= \mathcal{A} \otimes \mathcal{A} \otimes \ldots \otimes \mathcal{A} \quad (n+1) \text{ terms},\\
\mathcal{A}^\natural(f)\,(a_0 \otimes \ldots \otimes a_n) &:= b_0 \otimes \ldots \otimes b_m \quad \text{for } f \in \hom(\Lambda_n, \Lambda_m),
\end{aligned}$$

where $b_j = \prod_{l \in f^{-1}(j)} a_l$ with $f^{-1}(j) = \{i \in \Lambda_n : f(i) = j\}$ and $b_j = 1$ when $f^{-1}(j) = \emptyset$. A cyclic k-module $C(\mathcal{A})$ is then obtained by composing $\mathcal{A}^\natural$ with $^* : \Lambda^{\mathrm{op}} \cong \Lambda$. But since * is an isomorphism, it is possible to work directly with either $C(\mathcal{A})$ or $\mathcal{A}^\natural$.

5.10.3 Derived functors

The cyclic homology and cohomology groups are defined by the derived functors

$$HC_n(\mathcal{A}) := \mathrm{Tor}_n(\mathcal{A}^\natural, k^\natural), \tag{5.86}$$

$$HC^n(\mathcal{A}) := \mathrm{Ext}^n(\mathcal{A}^\natural, k^\natural). \tag{5.87}$$

Equivalently, in terms of cyclic modules,

$$HC_n(\mathcal{A}) := \mathrm{Tor}_n(C(\mathcal{A}), C(k)), \tag{5.88}$$

$$HC^n(\mathcal{A}) := \mathrm{Ext}^n(C(\mathcal{A}), C(k)). \tag{5.89}$$

The Hochschild homology and cohomology groups are completely analogous,

$$HH_n(\mathcal{A}) := \mathrm{Tor}_n(S(\mathcal{A}), S(k)), \tag{5.90}$$

$$HH^n(\mathcal{A}) := \mathrm{Ext}^n(S(\mathcal{A}), S(k)), \tag{5.91}$$

with simplicial modules instead of cyclic modules.

Both cyclic homology/cohomology and Hochschild homology/cohomology are Morita invariant,

$$HC_n(\mathrm{M}_k(\mathcal{A})) = HC_n(\mathcal{A}), \quad HC^n(\mathrm{M}_k(\mathcal{A})) = HC^n(\mathcal{A}), \tag{5.92}$$

$$HH_n(\mathrm{M}_k(\mathcal{A})) = HH_n(\mathcal{A}), \quad HH^n(\mathrm{M}_k(\mathcal{A})) = HH^n(\mathcal{A}). \tag{5.93}$$

Cyclic homology has a product,

$$HC_n(\mathcal{A}) \otimes HC_m(\mathcal{B}) \to HC_{n+m+1}(\mathcal{A} \otimes \mathcal{B}), \tag{5.94}$$

and a coproduct. The coproduct corresponds to the cup product of cyclic cohomology,

$$\cup : HC^n(\mathcal{A}) \otimes HC^m(\mathcal{B}) \to HC^{n+m}(\mathcal{A} \otimes \mathcal{B}). \tag{5.95}$$

5.10.4 Cycles and cyclic cocycles

Cyclic cohomology is easier to work with than cyclic homology. The elements of the cyclic cohomology groups are called cyclic cocycles. Cyclic cocycles are the characters of cycles over an algebra $\mathcal{A}$.

Definition 5.10.2. An *n-dimensional cycle* $(\Omega, \mathrm{d}, \int)$ over $\mathcal{A}$ is given by a differential graded algebra (Ω, d), where $\Omega = \overset{n}{\underset{p=0}{\oplus}} \Omega^p$ and $\mathrm{d}^2 = 0$, and a closed graded trace $\int : \Omega^n \to \mathbb{C}$ ($\int \mathrm{d}\omega_{n-1} = 0$ and $\int \omega_p \, \omega_q = (-1)^{pq} \int \omega_q \, \omega_p$), together with a homomorphism $\rho : \mathcal{A} \to \Omega^0$.

There are also related notions of chain and boundary.

Definition 5.10.3. An *$(n+1)$-dimensional chain* $(\Omega, \partial\Omega, \mathrm{d}, \int)$ is given by an $(n+1)$-dimensional differential graded algebra (Ω, d) and an n-dimensional differential graded algebra $(\partial\Omega, \mathrm{d}')$, with a surjective homomorphism $r : \Omega \to \partial\Omega$, and a graded trace $\int :$ $\Omega^{n+1} \to \mathbb{C}$ such that $\int \mathrm{d}\omega = 0$ for all $\omega \in \Omega^n$ with $r(\omega) = 0$.

Definition 5.10.4. The *boundary* of a chain $(\Omega, \partial\Omega, \mathrm{d}, \int)$ is the cycle $(\partial\Omega, \mathrm{d}, \int')$ where $\int' \omega' := \int \mathrm{d}\omega$, $\omega' \in (\partial\Omega)^n$, for any $\omega \in \Omega^n$ with $r(\omega) = \omega'$.

The character of an n-dimensional cycle over $\mathcal{A}$ is the $(n+1)$-linear functional on $\mathcal{A}$ given by

$$\tau_n(a_0, \ldots, a_n) := \int \rho(a_0)\mathrm{d}(\rho(a_1)) \ldots \mathrm{d}(\rho(a_n)). \tag{5.96}$$

This is a cyclic n-cocycle and any cyclic cocycle is the character of some cycle. In particular, a cyclic 0-cocycle is a trace on $\mathcal{A}$, thus $HC^0(\mathcal{A}) := \mathrm{Hom}(\mathcal{A}^\natural, k^\natural) = \{\text{traces on } \mathcal{A}\}$.

Example 5.10.1 ($HC^0(\mathrm{M}_k(\mathbb{C}))$)
All traces on $\mathrm{M}_k(\mathbb{C})$ are of the form $\mathrm{tr}_z(A) = z \, \mathrm{tr}\, A$, where $z \in \mathbb{C}$. Hence, $HC^0(\mathrm{M}_k(\mathbb{C})) =$ $\mathbb{C}$.

Example 5.10.2 ($HC^0(C^\infty(M))$)
All traces on $C^\infty(M)$ are of the form $\mathrm{Tr}_g(f) = \int_M g(x) f(x) \sqrt{g}\, \mathrm{d}^m x$, where $g \in C^\infty(M)$. Hence, $HC^0(C^\infty(M)) = C^\infty(M)$. More generally, any closed de Rham current is a cyclic cocycle.

The pairing $HC^n \times HC_n \to k$ between a cyclic n-cocycle τ_n and a cyclic n-cycle c_n is given by

$$\langle \tau_n, c_n \rangle := \int \rho(c_n), \tag{5.97}$$

where $c_n = \sum a_0 \otimes a_1 \otimes \ldots \otimes a_n$ and $\rho(c_n) = \sum \rho(a_0)\mathrm{d}(\rho(a_1)) \ldots \mathrm{d}(\rho(a_n))$. This generalises the pairing between de Rham currents and differential forms.

Cyclic homology is closely related to $\mathbb{S}^1$-equivariant homology as $B\Lambda = BU(1)$. Specifically,

$$HC_n(k[X]) \cong H_n^{\mathbb{S}^1}(|X|, k), \tag{5.98}$$

where $k[X]$ is the free cyclic k-module on a cyclic set X (c.f. the isomorphism between simplicial and singular homology, $H_n(k[X]) \cong H_n(|X|, k)$, where X is a simplicial set).

5.10.5 Periodic cyclic homology and cohomology

The cyclic homology groups are connected by the *periodicity map* $S : HC_n(\mathcal{A}) \to HC_{n-2}(\mathcal{A})$ ($S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ in the case of the cyclic cohomology groups). Each group is thus the start of a sequence

$$HC_n(\mathcal{A}) \xleftarrow{S} HC_{n+2}(\mathcal{A}) \xleftarrow{S} HC_{n+4}(\mathcal{A}) \xleftarrow{S} \ldots$$

($HC^n(\mathcal{A}) \xrightarrow{S} HC^{n+2}(\mathcal{A}) \xrightarrow{S} HC^{n+4}(\mathcal{A}) \xrightarrow{S} \ldots$). These sequences are used to define periodic cyclic homology HP_i (periodic cyclic cohomology HP^i). For smooth algebras, the periodic cyclic homology/cohomology groups are the inductive limits

$$\begin{aligned} HP_i(\mathcal{A}) &= \lim_{\xleftarrow{S}} HC_{2n+i}(\mathcal{A}), && (5.99) \\ HP^i(\mathcal{A}) &= \lim_{\xrightarrow{S}} HC^{2n+i}(\mathcal{A}). && (5.100) \end{aligned}$$

(In general, there is an extra term.) The periodic cyclic homology/cohomology groups are periodic with period 2,

$$\begin{aligned} HP_{i-2}(\mathcal{A}) &= HP_i(\mathcal{A}), && (5.101) \\ HP^{i+2}(\mathcal{A}) &= HP^i(\mathcal{A}), && (5.102) \end{aligned}$$

hence their name.

5.11 The Chern Character

The Chern character is a natural transformation from K-theory to cyclic homology, or by duality, a natural transformation from K-homology to cyclic cohomology:

$$\mathrm{ch}_{i,n} : K_i(A) \to HC_{2n+i}(A), \tag{5.103}$$

$$\mathrm{ch}^{i,n} : K^i(A) \to HC^{2n+i}(A). \tag{5.104}$$

A	$HC_0(A)$	$HC_1(A)$	$HC_{n\geq 2}(A)$	$HP_0(A)$	$HP_1(A)$
$\mathbb{C}$	$\mathbb{C}$	0	$\mathbb{C}$ for n even, 0 for n odd	$\mathbb{C}$	0
$\mathrm{M}_k(\mathbb{C})$	$\mathbb{C}$	0	$\mathbb{C}$ for n even, 0 for n odd	$\mathbb{C}$	0
$C^\infty(M)$	$\Omega^0 M = C^\infty(M)$	$\frac{\Omega^1 M}{\mathrm{d}\Omega^0 M}$	$\frac{\Omega^n M}{\mathrm{d}\Omega^{n-1} M} \oplus H_{\mathrm{dR}}^{n-2}(M) \oplus H_{\mathrm{dR}}^{n-4}(M) \oplus \ldots \oplus H_{\mathrm{dR}}^{0}(M)$ for n even $\frac{\Omega^n M}{\mathrm{d}\Omega^{n-1} M} \oplus H_{\mathrm{dR}}^{n-2}(M) \oplus H_{\mathrm{dR}}^{n-4}(M) \oplus \ldots \oplus H_{\mathrm{dR}}^{1}(M)$ for n odd (Note: $\frac{\Omega^n M}{\mathrm{d}\Omega^{n-1} M} = \operatorname{Coker} \mathrm{d}_{n-1}$.)	$H_{\mathrm{dR}}^{\mathrm{even}}(M)$	$H_{\mathrm{dR}}^{\mathrm{odd}}(M)$
$\mathcal{A}_\theta$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$

Table 5.3: Cyclic homology groups of some common pre-C^*-algebras.

A	$HC^0(A)$	$HC^1(A)$	$HC^{n\geq 2}(A)$	$HP^0(A)$	$HP^1(A)$
$\mathbb{C}$	$\mathbb{C}$	0	$\mathbb{C}$ for n even, 0 for n odd	$\mathbb{C}$	0
$\mathrm{M}_k(\mathbb{C})$	$\mathbb{C}$	0	$\mathbb{C}$ for n even, 0 for n odd	$\mathbb{C}$	0
$C^\infty(M)$	$Z_0^{\mathrm{dR}}(M) = C^\infty(M)$	$Z_1^{\mathrm{dR}}(M)$	$Z_n^{\mathrm{dR}}(M) \oplus H^{\mathrm{dR}}_{n-2}(M) \oplus H^{\mathrm{dR}}_{n-4}(M) \oplus \ldots \oplus H_0^{\mathrm{dR}}(M)$ for n even $Z_n^{\mathrm{dR}}(M) \oplus H^{\mathrm{dR}}_{n-2}(M) \oplus H^{\mathrm{dR}}_{n-4}(M) \oplus \ldots \oplus H_1^{\mathrm{dR}}(M)$ for n odd (Note: $Z_n^{\mathrm{dR}}(M) = \operatorname{Ker} \partial_n$.)	$H^{\mathrm{dR}}_{\mathrm{even}}(M)$	$H^{\mathrm{dR}}_{\mathrm{odd}}(M)$
$\mathcal{A}_\theta$	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$

Table 5.4: Cyclic cohomology groups of some common pre-C^*-algebras.

5.11.1 Homological Chern character

The Chern map $\mathrm{ch}_{0,n} : K_0(A) \to HC_{2n}(A)$ takes a projection p to the cyclic $2n$-cycle

$$\mathrm{ch}_{0,n}(p) := \frac{(2n)!}{n!}\,\mathrm{tr}\left(\left(p-\frac{1}{2}\right)(\mathrm{d}p)^{2n}\right), \tag{5.105}$$

where $\mathrm{tr}\,p$ is the trace over the matrix indices of $p \in \mathrm{M}_k(A)$. (The $-\frac{1}{2}$ is a matter of convention.) Similarly, the Chern map $\mathrm{ch}_{1,n} : K_1(A) \to HC_{2n+1}(A)$ takes a unitary $u \in \mathrm{M}_k(A)$ to the cyclic $(2n+1)$-cycle

$$\mathrm{ch}_{1,n}(u) := n!\,\mathrm{tr}\left(u^{-1}\mathrm{d}u\left(\mathrm{d}u^{-1}\mathrm{d}u\right)^n\right). \tag{5.106}$$

5.11.2 Cohomological Chern character

The n-dimensional cycle associated to a Fredholm module is defined by

$$\begin{aligned}
\Omega^p &:= \left\{\omega_p = \sum a_0[F,a_1]\ldots[F,a_p]\right\},\\
\mathrm{d}\omega_p &:= [F,\omega_p]_s := F\omega_p - (-1)^p\omega_p F,\\
\int \omega_n &:= \mathrm{Tr}(\Gamma\omega_n),
\end{aligned}$$

where $\Gamma = \mathbb{I}$ for odd Fredholm modules. (The domain of $\int$ can be extended with the definition $\int \omega_n := \frac{1}{2}\mathrm{Tr}(\Gamma F[F,\omega_n]_s)$.) Note, $F[F,\omega_p]_s = -(-1)^p[F,\omega_p]_s F$. Unlike the derivation $[D,-]$, the supercommutator $[F,-]_s$ is a differential as $F^2 = \mathbb{I}$.

The character of a Fredholm module is the cyclic n-cocycle

$$\begin{aligned}
\tau_n^F(a_0,\ldots,a_n) &:= (-1)^n \int a_0\mathrm{d}a_1\ldots\mathrm{d}a_n\\
&= (-1)^n\,\mathrm{Tr}(\Gamma a_0[F,a_1]\ldots[F,a_n]),
\end{aligned} \tag{5.107}$$

where $\Gamma = \mathbb{I}$ for odd Fredholm modules. This defines the Chern maps

$$\mathrm{ch}^{0,n}(\mathcal{H},F,\Gamma) := \tau_{2n}^F, \tag{5.108}$$

$$\mathrm{ch}^{1,n}(\mathcal{H},F) := \tau_{2n+1}^F. \tag{5.109}$$

5.11.3 Chern-Connes pairing

The homological Chern character can be used to define a pairing $HC^{2n+i}(A) \times K_i(A) \to k$ between cyclic cohomology and K-theory,

$$\langle \tau_{2n}, p \rangle := \langle \tau_{2n}, \mathrm{ch}_{0,n}(p) \rangle, \tag{5.110}$$

$$\langle \tau_{2n+1}, u \rangle := \langle \tau_{2n+1}, \mathrm{ch}_{1,n}(u) \rangle, \tag{5.111}$$

where τ_n is a cyclic cocycle. Likewise, the cohomological Chern character defines a pairing $K^i(A) \times HC_{2n+i}(A) \to k$ between K-homology and cyclic homology given by

$$\langle (\mathcal{H}, F, \Gamma), c_{2n} \rangle := \langle \mathrm{ch}^{0,n}(\mathcal{H}, F, \Gamma), c_{2n} \rangle, \tag{5.112}$$

$$\langle (\mathcal{H}, F), c_{2n+1} \rangle := \langle \mathrm{ch}^{1,n}(\mathcal{H}, F), c_{2n+1} \rangle, \tag{5.113}$$

where c_n is a cyclic cycle.

5.11.4 The index formula

The index formula gives a way of calculating the pairing between K-theory and K-homology using the Chern character. Specifically,

$$\langle \mathrm{ch}^{0,n}(\mathcal{H}, F, \Gamma), \mathrm{ch}_{0,n}(p) \rangle = \langle [(\mathcal{H}, F, \Gamma)], [p] \rangle, \tag{5.114}$$

$$\langle \mathrm{ch}^{1,n}(\mathcal{H}, F), \mathrm{ch}_{1,n}(u) \rangle = \langle [(\mathcal{H}, F)], [u] \rangle. \tag{5.115}$$

This generalises the Atiyah-Singer index theorem. The pairing between Chern characters is given by

$$\langle \mathrm{ch}^{0,n}(\mathcal{H}, F, \Gamma), \mathrm{ch}_{0,n}(p) \rangle = \frac{(2n)!}{n!} \mathrm{Tr} \left(\Gamma \left(p - \frac{1}{2} \right) [F, p]^{2n} \right), \tag{5.116}$$

$$\langle \mathrm{ch}^{1,n}(\mathcal{H}, F), \mathrm{ch}_{1,n}(u) \rangle = -n! \, \mathrm{Tr} \left(u^{-1}[F, u] \left([F, u^{-1}][F, u] \right)^n \right). \tag{5.117}$$

For finite dimensional algebras, the Chern character pairing is just

$$\langle \mathrm{ch}^{0,0}(\mathcal{H}, F, \Gamma), \mathrm{ch}_{0,0}(p) \rangle = \mathrm{Tr}(\Gamma p). \tag{5.118}$$

$$\begin{tikzcd}
K^i(A) \times K_i(A) \arrow[rr, "\langle -,- \rangle_K"] \arrow[d, "\mathrm{ch}^{i,n} \times \mathrm{ch}_{i,n}"'] & & \mathbb{Z} \arrow[d, hook] \\
HC^{2n+i}(A) \times HC_{2n+i}(A) \arrow[rr, "\langle -,- \rangle_{HC}"] & & k
\end{tikzcd}$$

Figure 5.7: The index formula [32].

Example 5.11.1 (The index formula for $\mathbb{C}$)

Consider an even Fredholm module $(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k})$ over $\mathrm{M}_n(\mathbb{C})$ and a projection $p_j = \mathrm{diag}(\mathbb{I}_j, 0, \ldots, 0) \in \mathrm{M}_n(\mathbb{C})$. Then, $\pi(p_j) = \mathrm{diag}(\mathbb{I}_{jk}, 0, \ldots, 0) \in \mathrm{M}_{nk}(\mathbb{C})$, so

$$\begin{aligned}
\langle \mathrm{ch}^{0,0}(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k}), \mathrm{ch}_{0,0}(p_j) \rangle &= \mathrm{Tr}(\Gamma\pi(p_j)) \\
&= jk. \qquad (5.119)
\end{aligned}$$

This agrees with the pairing $\langle [(\mathcal{H}_{n,k}, F_{n,k}, \Gamma_{n,k})], [p_j] \rangle$ calculated earlier.

Theorem 5.11.1 (Connes' character formula)

For every Hochschild n-cycle $c_n \in Z_n(\mathcal{A}, \mathcal{A})$,

$$\langle \varphi_n^D, c_n \rangle = \langle \tau_n^F, c_n \rangle. \qquad (5.120)$$

Example 5.11.2 (The character formula for $\mathbb{S}^1$)

The K-cycle for $\mathbb{S}^1$ is $\left(C^\infty(\mathbb{S}^1), L^2(\mathbb{S}^1), -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\right)$. Its pairing with the unitary $u = \mathrm{e}^{\mathrm{i}k\theta}$ is given by

$$\mathrm{Ind}(PuP) = -\,\mathrm{Tr}\left(u^{-1}[F,u]\right) = -\frac{1}{2}\,\mathrm{Tr}_\omega\left(u^{-1}[D,u]|D|^{-1}\right). \qquad (5.121)$$

The projection $P = \frac{\mathbb{I}+D|D|^{-1}}{2}$ maps a function $\psi(\theta) = \sum c_n \mathrm{e}^{\mathrm{i}n\theta} \in L^2(\mathbb{S}^1)$ to

$$(P\psi)(\theta) = \sum_{n\geq 0}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta}. \qquad (5.122)$$

So,

$$\begin{aligned}
PuP &: \sum_{n\geq 0}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta} \to \sum_{n\geq 0}^{\infty} c_n \mathrm{e}^{\mathrm{i}(n+k)\theta}, \qquad (5.123)\\
Pu^*P &: \sum_{n\geq 0}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\theta} \to \sum_{n\geq k}^{\infty} c_n \mathrm{e}^{\mathrm{i}(n-k)\theta}. \qquad (5.124)
\end{aligned}$$

Thus, $\mathrm{Ind}(PuP) = 0 - k = -k$. *Alternatively,* $u^{-1}[D,u] = k$, *so*

$$\begin{aligned}\langle \varphi_1^D, u^{-1} \otimes u \rangle &= -\frac{k}{2}\,\mathrm{Tr}_\omega\, |D|^{-1} \\ &= -k. \end{aligned} \tag{5.125}$$

5.11.5 The bivariant Chern character

Just as KK-theory generalises K-theory and K-homology, there is a bivariant cyclic theory which generalises cyclic homology and cohomology,

$$HC_n(A,\mathbb{C}) = HC^n(A), \tag{5.126}$$

$$HC_n(\mathbb{C},A) = HC_n(A). \tag{5.127}$$

It also has a product like KK-theory,

$$HC_n(A,B) \times HC_m(B,C) \to HC_{n+m}(A,C). \tag{5.128}$$

The Chern character $\mathrm{ch}_{i,n} : K_i(A) \to HC_{2n+i}(A)$ can be extended to a bivariant Chern character $\mathrm{ch}_{i,n} : KK_i(A,B) \to HC_{2n+i}(A,B)$, which is compatible with the Kasparov intersection product. Not only is the bivariant Chern character a (bi)natural transformation between bifunctors, it is also a functor from **KK** to **HC**. Actually, KK-theory and bivariant cyclic theory are not defined on compatible categories of algebras. So, it is necessary to use either a variant of KK-theory, such as topological KK-theory for locally convex algebras (kk), or a variant of bivariant cyclic theory, such as bivariant local cyclic homology for complete bornological algebras (HE^{loc}). More details can be found in [13].

Chapter 6

Conclusion

We have developed a path integral approach to quantise the spectral action. In principle, it can be applied to any noncommutative geometry. We have successfully used it on the two-point space, the matrix geometry $M_2(\mathbb{C})$ and a circle.

In the case of the two finite noncommutative geometries, we found graviton excitations have the effect of shrinking distances. Intuitively, this is what one would expect, given gravity is attractive. The two geometries behave in quite different ways as they collapse to a point. The two-point space undergoes a topological change, which is suggestive of the formation of something like a black hole (an apt term would be "*black point*"). Whereas, the matrix geometry maintains its topology, but loses its noncommutativity instead. We expect the shrinking of distances by gravitons to be a general feature of quantised finite noncommutative geometries. The introduction of fermions onto the geometries had the effect of shielding out the gravitational field. All the graviton states are lowered by an amount equal to the number of fermion generations.

Comparing our approach with Rovelli's, led us to question the validity of his results. We found his equations of motion could be expressed in much simpler terms, which result in a smaller phase space. This will alter his canonical quantisation. Despite this, both approaches seem to support the qualitative result that distances shrink with increasing graviton excitations.

In the case of a circle, we found graviton excitations have the effect of increasing distances. Again, this is what one would expect, given the spectral action is a cosmological constant. A circle is too trivial a geometry for there to be any interesting effects. Effectively, it consists of an infinite number of two-point spaces. To obtain new phenomena, it is probably necessary to quantise a torus or sphere. This would not be an easy task. Two-dimensional quantum gravity has been researched, so there would also be the opportunity to compare results.

The idea of spectral integrals is very appealing as it is consistent with the philosophy of spectral invariance. But, we have concerns over the possible lack of any topological dependence. The K-groups should somehow restrict the space of eigenvalues to integrate over. We want to integrate over all Dirac operators, not all self-adjoint operators. Of course, the definition of a Dirac operator is given by the axioms for a spectral triple. So, to develop spectral integrals further, it is necessary to formulate the axioms in terms of the Dirac operator eigenvalues. This problem also arises when the eigenvalues are considered as the variables of the classical spectral action [28]. On Riemannian manifolds, our path integral approach coincides with the conventional one, by construction. It would be interesting to see how spectral integrals differ from this.

Appendix A

C^*-algebras and Operators

A.1 C^*-algebras and Hilbert Spaces

We recall some basic definitions regarding C^*-algebras and Hilbert spaces.

A.1.1 Vector spaces

Definition A.1.1. A *normed vector space* is a vector space V with a map $||-|| : V \to \mathbb{R}$ satisfying the following properties:

$$
\begin{aligned}
||v|| &\geq 0 \quad \text{with } ||v|| = 0 \text{ iff } v = 0 \text{ (positive definite)}, && \text{(A.1)}\\
||\lambda v|| &= |\lambda|\,||v|| \quad \forall \lambda \in \mathbb{C}, && \text{(A.2)}\\
||u+v|| &\leq ||u|| + ||v|| \quad \text{(triangle inequality)}. && \text{(A.3)}
\end{aligned}
$$

Definition A.1.2. A *Banach space* is a complete normed vector space.

Definition A.1.3. A *Hilbert space* $\mathcal{H}$ is a Banach space with a scalar product $\langle -, - \rangle$ such that $||v|| = \sqrt{\langle v, v \rangle}$ for all $v \in \mathcal{H}$.

Definition A.1.4. A *linear map* between two vector spaces V and W is a map $L : V \to W$ that satisfies $L(\lambda u + \mu v) = \lambda L(u) + \mu L(v)$ for all $\lambda, \mu \in \mathbb{C}$ and for all $u, v \in V$.

A.1.2 Algebras

Definition A.1.5. A *Banach algebra* is a Banach space with a multiplication law compatible with the norm, i.e. $||ab|| \leq ||a||\,||b||$ (product inequality).

Definition A.1.6. A *Banach *-algebra* is a Banach algebra with an involution * satisfying the following properties:

$$\begin{aligned} a^{**} &= a, && \text{(A.4)} \\ (ab)^* &= b^*a^*, && \text{(A.5)} \\ (\lambda a + \mu b)^* &= \overline{\lambda} a^* + \overline{\mu} b^* \quad \forall \lambda, \mu \in \mathbb{C}, && \text{(A.6)} \\ ||a^*|| &= ||a||. && \text{(A.7)} \end{aligned}$$

Definition A.1.7. A *C^*-algebra* A is a Banach *-algebra such that $||a^*a|| = ||a||^2$ for all $a \in A$.

Definition A.1.8. A *pre-C^*-algebra* $\mathcal{A}$ is a dense *-subalgebra of a C^*-algebra A that is stable under the holomorphic functional calculus.

Definition A.1.9. A **-homomorphism* between two C^*-algebras A and B is a linear map $\phi : A \to B$ that satisfies

$$\begin{aligned} \phi(ab) &= \phi(a)\phi(b), && \text{(A.8)} \\ \phi(a^*) &= \phi(a)^*. && \text{(A.9)} \end{aligned}$$

Definition A.1.10. A *unital *-homomorphism* is a *-homomorphism ϕ between two unital C^*-algebras A and B that satisfies $\phi(\mathbb{I}_A) = \mathbb{I}_B$.

A.2 Operators on Hilbert Spaces

We now focus our attention on C^*-algebras of operators acting on Hilbert spaces.

Definition A.2.1. An operator T acting on a Hilbert space $\mathcal{H}$ is said to be *bounded* if there exists a $c \in \mathbb{R}$ such that

$$||Tv|| \leq c||v|| \quad \forall v \in \mathcal{H}. \tag{A.10}$$

If T is bounded, then the smallest such c is called the *operator norm* of T and is denoted $||T||$. The operator norm can equivalently be defined as

$$||T|| := \sup_{v \neq 0} \frac{||Tv||}{||v||} = \sup_{||v|| \leq 1} ||Tv||. \tag{A.11}$$

The set of bounded operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra and is denoted $\mathbb{B}(\mathcal{H})$.

Theorem A.2.1 (Gelfand-Naĭmark representation theorem)
*Every C^*-algebra is isomorphic to a C^*-subalgebra (closed *-subalgebra) of some $\mathbb{B}(\mathcal{H})$. In particular, every finite dimensional C^*-algebra is isomorphic to a direct sum of matrix algebras.*

Definition A.2.2. A bounded operator is *compact* if it is the norm limit of finite rank operators.

The set of compact operators on a Hilbert space $\mathcal{H}$ is a C^*-algebra and is denoted $\mathbb{K}(\mathcal{H})$. All operators on a finite dimensional Hilbert space are compact. In fact, compact operators behave similarly to finite dimensional operators.

Example A.2.1 (Integral operators)
Let T be an integral operator on $C([0,1])$ defined by

$$(Tf)(x) := \int_0^1 K(x,y) f(y)\,\mathrm{d}y, \quad f \in C([0,1]), \tag{A.12}$$

where $K(x,y)$ is the kernel. Then, T is a compact operator.

Closely related to C^*-algebras are von Neumann algebras.

Definition A.2.3. A *von Neumann algebra* (or *W^*-algebra*) is a weakly closed C^*-subalgebra of $\mathbb{B}(\mathcal{H})$.

Theorem A.2.2 (Double commutant theorem)
Let A be a C^*-subalgebra of $\mathbb{B}(\mathcal{H})$ containing the identity operator $\mathbb{1}_{\mathcal{H}}$. Then A is a von Neumann algebra iff $A = A''$, where $A' = \{T \in \mathbb{B}(\mathcal{H}) : Ta = aT\ \forall a \in A\}$ is the commutant of A.

A.3 Pseudo-differential Operators

Let E be a vector bundle over an m-dimensional manifold M, with space of smooth sections $\Gamma(M,E)$.

Definition A.3.1. A *pseudo-differential operator of order* d is an operator $P:\Gamma(M,E)\to\Gamma(M,E)$ of the form

$$\begin{aligned}(Pf)(x) &= \frac{1}{(2\pi)^m}\int p(x,k)\,\mathrm{e}^{\mathrm{i}k\cdot x}\,F(k)\,\mathrm{d}^m k\\ &= \frac{1}{(2\pi)^m}\iint p(x,k)\,\mathrm{e}^{\mathrm{i}k\cdot(x-y)}\,f(y)\,\mathrm{d}^m y\,\mathrm{d}^m k,\end{aligned}\tag{A.13}$$

where $f(x)=\frac{1}{(2\pi)^m}\int \mathrm{e}^{\mathrm{i}k\cdot x}\,F(k)\,\mathrm{d}^m k$ is expressed as a Fourier transform, and the *total symbol* $p(x,k)$ is a matrix of smooth functions.

We are mainly interested in classical pseudo-differential operators.

Definition A.3.2. A pseudo-differential operator P is said to be *classical* if its total symbol has an asymptotic expansion of the form

$$p(x,k)\sim\sum_{n=0}^{\infty}p_{d-n}(x,k),\tag{A.14}$$

where $p_n(x,k)$ is a symbol of order n. The *principal symbol* is defined as $\sigma_d(P)=p_d(x,k)$.

Definition A.3.3. A pseudo-differential operator P of order $-\infty$ is called a *smoothing operator*, and has the integral representation

$$(Pf)(x)=\int K(x,y)f(y)\,\mathrm{d}^m y,\tag{A.15}$$

where the kernel $K(x,y)$ is a smooth function.

Of particular importance are elliptic pseudo-differential operators. These include operators such as Dirac operators and Fredholm operators.

Definition A.3.4. A pseudo-differential operator is said to be *elliptic* if its principal symbol is invertible (modulo smoothing operators).

Example A.3.1 (The symbol of a Dirac operator)

The Dirac operator

$$D=-\mathrm{i}\gamma^a\,e_a^\mu(x)\left(\frac{\partial}{\partial x^\mu}+\frac{1}{4}\omega_{bc\mu}(x)\,\gamma^b\gamma^c\right)\tag{A.16}$$

is a pseudo-differential operator of order 1. Its total symbol is given by

$$p(x,k) = p_1(x,k) + p_0(x,k), \tag{A.17}$$

where

$$\begin{aligned} p_1(x,k) &= \gamma^a\, e_a^\mu(x) k_\mu, & \text{(A.18)} \\ p_0(x,k) &= -\frac{\mathrm{i}}{4} e_a^\mu(x)\, \omega_{bc\mu}(x)\, \gamma^a \gamma^b \gamma^c. & \text{(A.19)} \end{aligned}$$

The principal symbol of D is thus $\sigma_1(D) = p_1(x,k) = \gamma^a\, e_a^\mu(x) k_\mu$. The inverse of this matrix is $\gamma^a\, e_a^\mu(x) k_\mu / (g^{\alpha\beta} k_\alpha k_\beta)$, hence D is an elliptic pseudo-differential operator.

Appendix B

Clifford Algebras

B.1 Definitions

Clifford algebras are heavily used in the spin geometry of Riemannian manifolds. In this appendix, we have gathered together some useful definitions and results.

Definition B.1.1. A (complex) *Clifford algebra* is the associative algebra generated by the elements of a (complex) vector space with the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_\gamma, \tag{B.1}$$

where $g_{\mu\nu}$ is the metric of the vector space.

Clifford algebras with a Euclidean metric are C^*-algebras.

Definition B.1.2. The *chirality element* of a Clifford algebra with an m-dimensional Euclidean metric is defined by

$$\gamma^{m+1} := \mathrm{i}^{[m/2]}\gamma^0 \dots \gamma^{m-1}. \tag{B.2}$$

B.2 Trace Formulas

Here are some useful trace formulas for γ matrices:

$$\begin{aligned}
\operatorname{tr}(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\operatorname{tr}\mathbb{1}_\gamma && \text{(B.3)}\\
\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) &= (g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha})\operatorname{tr}\mathbb{1}_\gamma && \text{(B.4)}\\
\operatorname{tr}(\text{odd no. of } \gamma \text{ matrices}) &= 0 && \text{(B.5)}\\
\operatorname{tr}(\gamma^{m+1}\gamma^{\mu_1}\dots\gamma^{\mu_m}) &= (-\mathrm{i})^{[m/2]}\varepsilon^{\mu_1\dots\mu_m}\operatorname{tr}\mathbb{1}_\gamma && \text{(B.6)}
\end{aligned}$$

B.3 The Exterior Algebra Representation

A Clifford algebra has a natural representation in terms of differential forms. Define the Clifford product of 1-forms by

$$\alpha \vee \beta := \alpha\wedge\beta + g(\alpha,\beta). \tag{B.7}$$

(Often, $\alpha\vee\beta$ is written simply as $\alpha\beta$.) Then, the elements of a Clifford algebra can be represented by forms using the *symbol map* $\sigma : \mathbb{C}\mathrm{l}(V) \to \Lambda V$,

$$\sigma(\omega_{\mu_1\dots\mu_n}\gamma^{\mu_1}\dots\gamma^{\mu_n}) := \omega_{\mu_1\dots\mu_n}\mathrm{d}x^{\mu_1}\vee\dots\vee\mathrm{d}x^{\mu_n}. \tag{B.8}$$

The symbol map is an isomorphism of vector spaces. Its inverse is the *quantisation map* $Q : \Lambda V \to \mathbb{C}\mathrm{l}(V)$,

$$Q(\alpha_\mu\mathrm{d}x^\mu\wedge\beta_\nu\mathrm{d}x^\nu) := \alpha_\mu\beta_\nu\gamma^\mu\gamma^\nu - g^{\mu\nu}\alpha_\mu\beta_\nu. \tag{B.9}$$

B.4 Two-Dimensional Euclidean Space

The complex Clifford algebra for $\mathbb{R}^2$ is $\mathbb{C}\mathrm{l}(\mathbb{R}^2)\cong \mathrm{M}_2(\mathbb{C})$. Its irreducible representation (which is faithful) is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma^1 = \begin{pmatrix} 0 & \mathrm{i}\\ -\mathrm{i} & 0\end{pmatrix}.$$

The chirality element is

$$\gamma^3 = \mathrm{i}\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},$$

which gives a $\mathbb{Z}_2$-grading.

B.5 Three-Dimensional Euclidean Space

The complex Clifford algebra for $\mathbb{R}^3$ is $\mathbb{C}\mathrm{l}(\mathbb{R}^3) \cong \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$. Its irreducible representation (which is not faithful) is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The chirality element is

$$\gamma^4 = \mathrm{i}\gamma^0\gamma^1\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which gives a trivial grading.

B.6 Four-Dimensional Euclidean Space

The complex Clifford algebra for $\mathbb{R}^4$ is $\mathbb{C}\mathrm{l}(\mathbb{R}^4) \cong \mathrm{M}_4(\mathbb{C})$. Its irreducible representation (which is faithful) is given by

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}.$$

The chirality element is

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which gives a $\mathbb{Z}_2$-grading.

Appendix C

The Heat Equation and the Zeta Function

C.1 The Heat Kernel

Let E be a vector bundle over an m-dimensional manifold M, with smooth sections $\Gamma(M, E)$. The heat equation is

$$\left(\frac{\partial}{\partial t} + P\right) f(x,t) = 0, \quad \text{for } t \geq 0, \tag{C.1}$$

where $P : \Gamma(M, E) \to \Gamma(M, E)$ is an elliptic self-adjoint pseudo-differential operator, with eigenfunctions given by $P\phi_n(x) = \lambda_n \phi_n(x)$. It has the formal solution

$$f(x,t) = \mathrm{e}^{-tP} f(x), \tag{C.2}$$

where $f(x) = f(x, 0)$ is the initial condition. We proceed by expanding $f(x)$ in terms of the orthonormal basis of eigenfunctions,

$$f(x) = \sum_n c_n \phi_n(x), \tag{C.3}$$

with the coefficients given by

$$c_n = (\phi_n, f) := \int_M \overline{\phi}_n(x) f(x) \sqrt{g}\, \mathrm{d}^m x. \tag{C.4}$$

The solution can then be written as

$$f(x,t) = \sum_n \mathrm{e}^{-t\lambda_n}\, c_n \phi_n(x). \tag{C.5}$$

Next, we define the heat kernel,

$$K(t,x,y) := \sum_n \mathrm{e}^{-t\lambda_n}\, \phi_n(x) \otimes \overline{\phi}_n(y), \tag{C.6}$$

so that

$$\begin{aligned} f(x,t) &= \int_M K(t,x,y) f(y)\, \sqrt{g}\, \mathrm{d}^m y \\ &= \sum_n \mathrm{e}^{-t\lambda_n}\, \phi_n(x) \int_M \overline{\phi}_n(y) f(y)\, \sqrt{g}\, \mathrm{d}^m y \\ &= \sum_n \mathrm{e}^{-t\lambda_n}\, c_n \phi_n(x). \end{aligned} \tag{C.7}$$

Thus,

$$\mathrm{Tr}\, \mathrm{e}^{-tP} = \sum_n \mathrm{e}^{-t\lambda_n} = \int_M K(t,x,x)\, \sqrt{g}\, \mathrm{d}^m x. \tag{C.8}$$

It can be shown [17] that the heat kernel has the asymptotic ($t \to 0^+$) expansion

$$\mathrm{Tr}\, \mathrm{e}^{-tP} \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} a_n(P), \tag{C.9}$$

where d is the order of P. The $a_n(P) = \int_M a_n(x,P)\sqrt{g}\, \mathrm{d}^m x$ are the Seeley-DeWitt coefficients, which are zero for n odd. For $P = D^2$, where D is the Dirac operator on a m-dimensional Riemannian manifold, the first three non-zero coefficients are [9]:

$$a_0(x, D^2) = \frac{\mathrm{tr}\, \mathbb{1}_\gamma}{(4\pi)^{m/2}}, \tag{C.10}$$

$$a_2(x, D^2) = \frac{\mathrm{tr}\, \mathbb{1}_\gamma}{(4\pi)^{m/2}} \frac{R}{12}, \tag{C.11}$$

$$a_4(x, D^2) = \frac{\mathrm{tr}\, \mathbb{1}_\gamma}{(4\pi)^{m/2}} \frac{1}{1440} (5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 12R_{;\mu}{}^{\mu}), \tag{C.12}$$

where $\mathbb{1}_\gamma$ is the identity matrix for the Clifford algebra. The Ricci tensor and scalar curvature are defined by

$$R_{\mu\nu} = R_{\mu\rho}{}^{ab}\, e^\rho_b\, e_{a\nu}, \tag{C.13}$$

$$R = R_{\mu\nu}{}^{ab}\, e^\mu_a\, e^\nu_b. \tag{C.14}$$

C.2 The Zeta Function

The heat kernel can be related to the zeta function,

$$\zeta(s,P) := \operatorname{Tr} P^{-s} = \sum_n \lambda_n^{-s}, \tag{C.15}$$

using the Mellin transform,

$$\int_0^\infty t^{s-1}\,\mathrm{e}^{-t\lambda_n}\,\mathrm{d}t = \int_0^\infty \lambda^{-(s-1)}(t\lambda)^{s-1}\,\mathrm{e}^{-t\lambda_n}\,\frac{\mathrm{d}(t\lambda_n)}{\lambda_n} = \lambda_n^{-s}\,\Gamma(s), \tag{C.16}$$

where

$$\Gamma(s) := \int_0^\infty t^{s-1}\,\mathrm{e}^{-t}\,\mathrm{d}t. \tag{C.17}$$

So,

$$\zeta(s,P) = \frac{1}{\Gamma(s)}\sum_n \int_0^\infty t^{s-1}\,\mathrm{e}^{-t\lambda_n}\,\mathrm{d}t = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\operatorname{Tr}\mathrm{e}^{-tP}\,\mathrm{d}t. \tag{C.18}$$

At $s = 0, -1, -2, \ldots$, a non-positive integer, $\Gamma(s)$ has isolated simple poles. The zeta function is then regular at these values,

$$\zeta(s,P) = a_n(P)\,\operatorname*{Res}_{s=\frac{m-n}{d}}\Gamma(s). \tag{C.19}$$

It is also worth mentioning that

$$\left.\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\sum_n \mathrm{e}^{-s\ln\lambda_n}\Big|_{s=0} = -\sum_n \ln\lambda_n. \tag{C.20}$$

Thus, the determinant of P can be written as

$$\det P = \prod_n \lambda_n = \prod_n \mathrm{e}^{\ln\lambda_n} = \mathrm{e}^{\sum_n \ln\lambda_n} = \mathrm{e}^{-\zeta'(0,P)} \tag{C.21}$$

(zeta function regularisation).

Appendix D

Category Theory

D.1 Categories

We introduce the relevant category theory language. For a definitive account of category theory, see [30]. A good set of expository writings (on this and other subjects) can be found in [1].

Definition D.1.1. A *category* C consists of a class of objects, a set $\hom(a,b)$ of morphisms for every ordered pair (a,b) of objects, an identity morphism $\mathrm{id}_a \in \hom(a,a)$ for each object a, and an associative composition map $\hom(b,c) \times \hom(a,b) \to \hom(a,c)$ for every ordered triple (a,b,c) of objects. Since the identity morphisms are uniquely determined by the objects, a category is completely specified by its morphisms.

Definition D.1.2. A morphism $m : a \to b$ is *monic* (or *left cancellable*) in a category C when for any two parallel morphisms $f_1, f_2 : d \to a$, the equality $mf_1 = mf_2$ implies $f_1 = f_2$. Injections are monic.

Definition D.1.3. A morphism $e : a \to b$ is *epi* (or *right cancellable*) in a category C when for any two parallel morphisms $f_1, f_2 : b \to c$, the equality $f_1\, e = f_2\, e$ implies $f_1 = f_2$. Surjections are epi.

Universal properties are a central theme in category theory. Most arise in the form of limits or colimits. The fundamental example of a limit is a terminal object, and the fundamental example of a colimit is an initial object.

Definition D.1.4. An object s in a category C is *initial* if for every object a in C there is exactly one morphism $s \to a$ (there can be any number of morphisms $a \to s$).

Definition D.1.5. An object t in a category C is *terminal* if for every object a in C there is exactly one morphism $a \to t$ (there can be any number of morphisms $t \to a$).

Definition D.1.6. An object z that is both initial and terminal is called a *zero object.* The composite morphism $a \to z \to b$ is called the *zero morphism* from a to b.

D.2 Functors

A morphism of categories is a functor.

Definition D.2.1. A *functor* $T : C \to D$ from a category C to a category D is a map which assigns an object $T(a)$ of D to each object a of C, and a morphism $T(f)$ of D to each morphism f of C, preserving the identity morphisms ($T(\mathrm{id}_a) = \mathrm{id}_{T(a)}$) and composition ($T(gf) = T(g)\,T(f)$).

An important bifunctor (a functor on the product of two categories) from any category C to **Set** is the hom-bifunctor

$$\hom : C^{\mathrm{op}} \times C \to \mathbf{Set}. \tag{D.1}$$

It has the composition map $\hom(b, c) \times \hom(a, b) \to \hom(a, c)$.

D.3 Natural Transformations

A morphism of functors is a natural transformation.

Definition D.3.1. A *natural transformation* $\tau : S \Rightarrow T$ from a functor $S : C \to D$ to a functor $T : C \to D$ is a map which assigns a morphism $\tau_a : S(a) \to T(a)$ of D to each object a of C in such a way that $T(f)\,\tau_a = \tau_b\,S(f)$ for every morphism $f : a \to b$ in C.

Definition D.3.2. Let C and D be categories with functors $L : C \to D$ and $R : D \to C$. Then, L is *left adjoint* to R and R is *right adjoint* to L if there is a natural isomorphism

$$\hom_D(L(c), d) \cong \hom_C(c, R(d)) \tag{D.2}$$

for every object c of C and d of D.

Definition D.3.3. Two categories C and D are *equivalent* if there are functors $F : C \to D$ and $G : D \to C$ with natural isomorphisms $FG \cong \mathrm{id}_D$ and $GF \cong \mathrm{id}_C$. Note, F is left adjoint to G and G is right adjoint to F as

$$\hom(F(c), d) \xrightarrow{G} \hom(GF(c), G(d)) \longleftrightarrow \hom(c, G(d)), \tag{D.3}$$

and G is left adjoint to F and F is right adjoint to G as

$$\hom(G(d), c) \xrightarrow{F} \hom(FG(d), F(c)) \longleftrightarrow \hom(d, F(c)). \tag{D.4}$$

Definition D.3.4. Let A and B be categories. The *functor category* B^A is the category whose objects are the functors from A to B, and whose morphisms are the natural transformations between them (the composition of morphisms is the vertical composition of natural transformations).

The hom-sets of **Cat** are functor categories: $\hom_{\mathbf{Cat}}(A, B) = B^A$. A specific example of a functor category is the category of representations of a group G, k-$\mathbf{Mod}^G$, where G is considered as a category with one object and isomorphisms.

D.4 Duality

Any categorical construction can be dualised by reversing the direction of the morphisms.

Definition D.4.1. The *opposite category* C^{op} of a category C is the category obtained by reversing the direction of the morphisms of C, $\hom_{C^{\mathrm{op}}}(a, b) = \hom_C(b, a)$.

Definition D.4.2. A *contravariant functor* (or *cofunctor* for short) $T : C^{\mathrm{op}} \to D$ from a category C to a category D is a (covariant) functor from C^{op} to D (or equivalently from C to D^{op}).

D.5 Monoidal Categories

We need only consider strict monoidal categories since every monoidal category is equivalent to a strict one.

Definition D.5.1. A *strict monoidal category* $(M, \Box, e)$ is a category M together with a bifunctor $\Box : M \times M \to M$ which is associative, and an object e which is a unit for $\Box$.

The functor category C^C is a strict monoidal category: $\Box$ is given by the composition of functors and the horizontal composition of natural transformations, and e is the identity functor.

Definition D.5.2. A *monoid object* in a monoidal category $(M, \Box, e)$ is an object m of M together with an associative product $m \,\Box\, m \to m$ and a unit $e \to m$.

Definition D.5.3. A *monad* (or *triple*) on a category C is a monoid object in C^C.

D.6 Abelian Categories

Abelian categories feature heavily in homological algebra.

Definition D.6.1. An *Ab-category* is a category in which every hom-set is an additive abelian group and for which composition is bilinear.

Definition D.6.2. An *additive category* is an Ab-category which has a zero object and a biproduct (the product is isomorphic to the coproduct) for each pair of its objects.

Any additive category is a symmetric monoidal category.

Definition D.6.3. A *kernel* of a morphism $f : a \to b$ is a morphism $k : d \to a$ such that $fk = 0$, and every morphism h such that $fh = 0$ factors uniquely through k. Every kernel is monic. In terms of sets, $\operatorname{Im} k = \operatorname{Ker} f$ ($\operatorname{Im} h \subset \operatorname{Ker} f$).

Definition D.6.4. A *cokernel* of a morphism $f : a \to b$ is a morphism $u : b \to c$ such that $uf = 0$, and every morphism h such that $hf = 0$ factors uniquely through u. Every cokernel is epi. In terms of sets, $\operatorname{Im} u = \operatorname{Coker} f$ ($\operatorname{Im} h \subset \operatorname{Coker} f$).

Definition D.6.5. An *abelian category* is an additive category in which every morphism has a kernel and cokernel, and every monic is a kernel and every epi is a cokernel.

Clearly, if A is an abelian category, then A^{op} is an abelian category. If A is an abelian category and C is any category, then A^C is an abelian category. The most common abelian categories are: the category of abelian groups (**Ab**), and the category of R-modules (R-**Mod**).

D.7 Presheaves and Topoi

Sheaves and topos theory play a key role in algebraic geometry.

Definition D.7.1. A *presheaf* on a category C is a contravariant functor $C^{\mathrm{op}} \to \mathbf{Set}$.

Loosely speaking, a *topos* is a category which has similar properties to those of **Set** (cartesian closed with a subobject classifier). Any **Set**-valued functor category is a topos. An important example of a topos is the category of sheaves (or presheaves) on a category C.

To define the notion of a sheaf for a noncommutative space, one must turn to quantales [38]. (A quantale is the noncommutative generalisation of a locale.)

Appendix E

Spectral Triple Reference

E.1 Riemannian Manifold

The canonical real spectral triple for a Riemannian spin manifold:

$$\begin{aligned}
\mathcal{A} &:= C^\infty(M), && \text{(E.1)}\\
\mathcal{H} &:= L^2(\mathrm{spin}(M)), && \text{(E.2)}\\
D &:= -\mathrm{i}\gamma^a\, e_a^\mu(x)\left(\frac{\partial}{\partial x^\mu} + \frac{1}{4}\,\omega_{bc\mu}(x)\,\gamma^b\gamma^c\right), && \text{(E.3)}\\
J &:= \gamma^0\gamma^2 \circ {}^-, && \text{(E.4)}\\
\Gamma &:= \gamma_5. && \text{(E.5)}
\end{aligned}$$

Dimension: 4 (straightforward generalisation to arbitrary dimensions).

E.2 Matrix Manifold

The real spectral triple for a matrix manifold (the manifold underlying the Yang-Mills action):

$$\begin{aligned}
\mathcal{A} &:= C^\infty(M) \otimes \mathrm{M}_n(\mathbb{C}), && \text{(E.6)}\\
\mathcal{H} &:= L^2(\mathrm{spin}(M)) \otimes \mathrm{M}_n(\mathbb{C}), && \text{(E.7)}\\
D &:= -\mathrm{i}\gamma^a\, e_a^\mu(x)\left(\left(\frac{\partial}{\partial x^\mu} + \frac{1}{4}\,\omega_{bc\mu}(x)\,\gamma^b\gamma^c\right) \otimes \mathbb{1}_n + \mathrm{i}g A_\mu^a(x)\, T_a\right), && \text{(E.8)}
\end{aligned}$$

$$J := \gamma^0\gamma^2 \otimes \mathbb{1}_n \circ {}^\dagger, \tag{E.9}$$

$$\Gamma := \gamma_5 \otimes \mathbb{1}_n. \tag{E.10}$$

Dimension: 4 (straightforward generalisation to arbitrary dimensions).

E.3 Standard Model Manifold

The real spectral triple for the noncommutative geometry of the standard model:

$$\mathcal{A} := C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus \mathrm{M}_3(\mathbb{C})), \tag{E.11}$$

$$\mathcal{H} := L^2(\mathrm{spin}(M)) \otimes (\mathbb{C}^{24} \oplus \mathbb{C}^{21} \oplus \mathbb{C}^{24} \oplus \mathbb{C}^{21}), \tag{E.12}$$

$$D := -\mathrm{i}\gamma^a\, e_a^\mu(x) \left(\frac{\partial}{\partial x^\mu} + \frac{1}{4}\,\omega_{bc\mu}(x)\,\gamma^b\gamma^c\right) \otimes \mathbb{1}_{90} + \gamma_5 \otimes D_m, \tag{E.13}$$

$$D_m := \begin{pmatrix} 0 & M & 0 & 0 \\ M^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{M} \\ 0 & 0 & \overline{M}^\dagger & 0 \end{pmatrix}, \tag{E.14}$$

$$M := \begin{pmatrix} \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} \otimes \mathbb{1}_3 & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix}, \tag{E.15}$$

$$M_u := \mathrm{diag}(m_u, m_c, m_t), \tag{E.16}$$

$$M_d := V_{\mathrm{CKM}}\,\mathrm{diag}(m_d, m_s, m_b), \tag{E.17}$$

$$M_e := \mathrm{diag}(m_e, m_\mu, m_\tau), \tag{E.18}$$

$$J := \gamma^0\gamma^2 \otimes \begin{pmatrix} 0 & 0 & \mathbb{1}_{24} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{21} \\ \mathbb{1}_{24} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{21} & 0 & 0 \end{pmatrix} \circ {}^-, \tag{E.19}$$

$$\Gamma := \gamma_5 \otimes \begin{pmatrix} -\mathbb{1}_{24} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{21} & 0 & 0 \\ 0 & 0 & -\mathbb{1}_{24} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{21} \end{pmatrix}. \tag{E.20}$$

Dimension: 4 (straightforward generalisation to arbitrary dimensions).

E.4 Noncommutative Torus

The real spectral triple for the noncommutative torus $\mathbb{T}^2_\theta$:

$$\begin{aligned}
\mathcal{A} &:= \mathcal{A}_\theta, && \text{(E.21)}\\
\mathcal{H} &:= L^2(\mathcal{A}_\theta) \oplus L^2(\mathcal{A}_\theta), && \text{(E.22)}\\
D &:= -\mathrm{i}\begin{pmatrix} 0 & \delta_1 + \mathrm{i}\delta_2 \\ \delta_1 - \mathrm{i}\delta_2 & 0 \end{pmatrix}, && \text{(E.23)}\\
J &:= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \circ {}^*, && \text{(E.24)}\\
\Gamma &:= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. && \text{(E.25)}
\end{aligned}$$

Dimension: 2.

E.5 Simple Finite Noncommutative Geometry

The real spectral triple of a simple finite noncommutative geometry (used by Rovelli in [41]):

$$\begin{aligned}
\mathcal{A} &:= \mathrm{M}_2(\mathbb{C}) \oplus \mathbb{C}, && \text{(E.26)}\\
\mathcal{H} &:= \mathrm{M}_3(\mathbb{C}), && \text{(E.27)}\\
D &:= D_0 + JD_0J^{-1}, && \text{(E.28)}\\
& \quad D_0 := \begin{pmatrix} 0 & 0 & m_1 \\ 0 & 0 & m_2 \\ \overline{m}_1 & \overline{m}_2 & 0 \end{pmatrix}, && \text{(E.29)}\\
J &:= \mathbb{1}_3 \circ {}^\dagger, && \text{(E.30)}\\
\Gamma &:= \gamma J \gamma J^{-1}, && \text{(E.31)}\\
& \quad \gamma := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. && \text{(E.32)}
\end{aligned}$$

Dimension: 0.

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