Mean Curvature Flow Self-Shrinkers with Genus and Asymptotically Conical Ends

by

Niels Martin Møller

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Author ....
Department of Mathematics
April 19th, 2012

Certified by ....
Professor Tobias H. Colding
Levinson Professor of Mathematics
Thesis Supervisor

Accepted by ....
Professor Bjorn Poonen
Chairman, Department Committee on Graduate Theses

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Abstract

This doctoral dissertation is on the theory of Minimal Surfaces and of singularities in Mean Curvature Flow, for smooth submanifolds Σ^n in an ambient Riemannian $(n+1)$-manifold N^{n+1}, including:

(1) New asymptotically conical self-shrinkers with a symmetry, in $\mathbb{R}^{n+1}$.

(1′) Classification of complete embedded self-shrinkers with a symmetry, in $\mathbb{R}^{n+1}$, and of asymptotically conical ends with a symmetry.

(2) Construction of complete, embedded self-shrinkers $\Sigma_g^2 \subseteq \mathbb{R}^3$ of genus g, with asymptotically conical infinite ends, via minimal surface gluing.

(3) Construction of closed embedded self-shrinkers $\Sigma_g^2 \subseteq \mathbb{R}^3$ with genus g, via minimal surface gluing.

In the work there are two central geometric and analytic themes that cut across (1)-(3): The notion of asymptotically conical infinite ends in (1)–(1′) and (2), and in (2) and (3) the gluing methods for minimal surfaces which were developed by Nikolaos Kapouleas.

For the completion of (2) it was necessary to initiate the development of a stability theory in a setting with unbounded geometry, the manifolds in question having essentially singular (worse than cusp-like) infinities. This was via a Schauder theory in weighted Hölder spaces for the stability operator, which is a Schrödinger operator of Ornstein-Uhlenbeck type, on the self-shrinkers viewed as minimal surfaces. This material is, for the special case of graphs over the plane, included as part of the thesis.

The results in (1)-(1′) are published as the joint work [KMø11] with Stephen Kleene, and the result in (2) was proven in collaboration with Kleene-Kapouleas, and appeared in [KKMø10]. The results in (3) are contained in the preprint [Mø11].

Thesis Supervisor: Professor Tobias H. Colding
Title: Levinson Professor of Mathematics

To my mother and father, Bente & Mogens,
and to Ejvind, Lisbeth, Martin, Sonja, Onkel Peter & Eli.

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Finally, I would be looking here at no very mathematical thesis without some persistent nudges from science TAs such as Peter Thostrup, and from friends such as Jimi Lee Truelsen, who suggested in my 2nd-3rd years in Århus that I ought to try out Mathematics, and not follow early desires of becoming a chemist or physicist. I end with the best teacher I've had, in high school chemistry, Stig Carlsson.

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Chapter 1

Introduction

1.1 Singularities in Mean Curvature Flow

1.1.1 Introduction to Solitons in Geometric Flows

In Riemannian geometry a geometric flow is a well-defined process that modifies a geometric object via a nonlinear partial differential equation of heat equation type. The possibility of *solitons*, that is fixed profile evolutions, arises when this diffusive nature can be counterbalanced by the focusing nature of the nonlinearity.

We consider a smooth family of immersions $X(\cdot,t)$ of the manifold M^n into $\mathbb{R}^{n+1}$ (or more generally N^{n+1}),

$$X : M^n \times [0,T) \to \mathbb{R}^{n+1}, \tag{1.1}$$

and let $\Sigma_t = X(M,t)$. Then the family of surfaces $\{\Sigma_t\}$ is said to move by Mean Curvature Flow (in co-dimension one), if

$$\frac{\partial X(p,t)}{\partial t} = -H_{X(p,t)}\nu_{X(p,t)}, \tag{1.2}$$

where ν_X is a normal unit length vector field to Σ_t and $H_X = \sum_{i=1}^n \kappa_i$ denotes the mean curvature (= sum of principal curvatures κ_i, by convention positive when the surface curves away from ν_X).

The smooth flow generally meets a singularity in finite time, as the result of the surface disappearing in a point, by thin neck formations (as for the dumbbell examples) and many other more complicated possible behaviors. See Figure 1-1. As a general principle, these singularities correspond to the self-similar surfaces, i.e. to soliton solutions to the flow.

1.1.2 Self-shrinkers in Mean Curvature Flow

Singularities will, under appropriate geometric circumstances (the so-called Type I condition) that allow for parabolic rescalings, lead to a hypersurface $\Sigma^n \subseteq \mathbb{R}^{n+1}$ satisfying the self-shrinking soliton equation, which is the following (where ν_Σ is

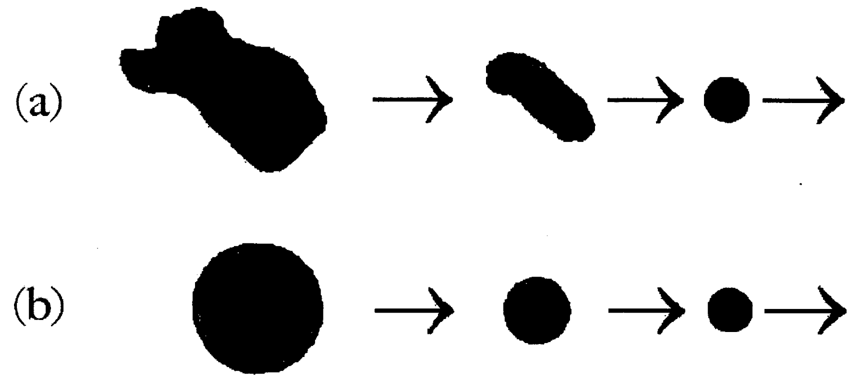

Figure 1-1: (a) Under the curve shortening flow, a curve embedded in the Euclidean plane $\mathbb{R}^2$ shrinks to a point as time passes - the curve becomes a round circle in the limit. (b) Correspondingly: The round circle is a self-shrinking curve.

a unit field normal to Σ^n):

$$H_X(\Sigma) = \frac{\langle X, \nu_\Sigma \rangle}{2}, \quad X \in \Sigma^n \subseteq \mathbb{R}^{n+1}. \tag{1.3}$$

See for example Klaus Ecker's book [Ec04] for the proof of this, from Huisken monotonicity formula, and for further background of mean curvature flow.

The equation in 1.3 is a nonlinear (quasi-linear) elliptic partial differential equation involving the mean curvature. Conversely, every solution Σ to (1.3) moves under the mean curvature flow in a solitary manner, shrinking by homotheties via $\Sigma_t = \sqrt{-t}\Sigma_{-1}$ towards the point $0 \in \mathbb{R}^{n+1}$ after time duration $T = 1$ (but the non-compact examples of course not becoming just a point in the limit).

Despite this role as "atoms" for the singularity theory played by the self-shrinkers, the exhaustive list of the rigorously known examples of complete, embedded self-shrinkers in $\mathbb{R}^{n+1}$ reads:

(i) $\mathbb{R}^{n-k} \times \mathbb{S}^k$, where $0 \leq k \leq n$ (flat planes times round spheres).

(ii) $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, embedded via Angenent's rotational maps.

(iii) Lagrangian examples by Anciaux in certain even dimensions (e.g. [Anc06]).

Although this list contains all known examples in codimension one, a classical expectation in the field is the following:

Conjecture 1.1.1 (Conjecture 1.2 in [Il95]). *Any smooth complete embedded self-shrinker $\Sigma^2 \subseteq \mathbb{R}^3$ with at most quadratic area growth has a finite number of infinite ends, each of which are either asymptotic to a smooth cone or a cylinder. Furthermore, there are no cylindrical ends, unless Σ is the cylinder.*

There exists an extensive and expanding literature on the topic of mean curvature self-shrinkers. Some of the most important considerations and significant developments can be found in, to name a few, the papers by Gerhard Huisken [Hu84]-[Hu93] (self-shrinkers as blow-up models, classification results), Angenent [An89] (the shrinking torus), Colding-Minicozzi [CM7]-[CM9] (compactness, entropy, generic flows), Le-Sesum [LS] (gap theorems), Lu Wang [Wa09]-[Wa11] (uniqueness theorems) and Ding-Xin [DX11] (volume growth).

1.1.3 New Examples and Classification of Self-shrinkers in $\mathbb{R}^{n+1}$ With a Symmetry

In Chapter 2 (which appeared as the paper [KMø11]), a new 1-parameter family of non-compact smooth rotationally symmetric, embedded, positive mean curvature, asymptotically conical self-shrinking ends $\Sigma^n \subseteq \mathbb{R}^{n+1}$ with boundary was found. Namely, for each symmetric cone $\mathscr{C} \subseteq \{x_1 \geq 0\} \subseteq \mathbb{R}^{n+1}$ with vertex at 0, of slope $\sigma > 0$, there is a unique rotationally symmetric self-shrinker Σ_σ (lying outside of $\mathscr{C}$) which is asymptotic to $\mathscr{C}$ as $x_1 \to \infty$.

The discovery of the rotationally symmetric ends in Theorem 2.3.1 caused us surprise, especially in light of Conjecture 1.1.1, since the self-shrinker equation (1.3) has been considered by many authors since the 1980's without mention of such examples, while analogous examples in [AIC] in the case of self-expanders were known. The proofs of both existence and uniqueness follow from a simple new integral identity for the solutions, which is derived by freezing the nonlinearities and solving the linear equation skeleton. Here the central theme of cones is immediately apparent.

In Chapter 2 (also contained in [KMø11]), the Theorem 2.3.1 is used to further provide a classification of the complete, embedded self-shrinkers with a symmetry. The earlier papers by Huisken [Hu90], Soner-Souganidis [SS93], and the newer Giga-Giga-Saal [GGS10], were concerned with (different proofs of) weaker versions of the uniqueness of self-shrinkers given by entire cylindrical graphs. In Chapter 2 (and in [KMø11]) is contained in particular a slightly more geometric proof of Huisken's uniqueness, by using the new asymptotically conical ends, and Lemma 2.3.4 in Chapter 2 (Lemma 1 in [KMø11]) removes the assumption of $H \geq 0$ from all the previous results.

The methods used for the uniqueness proof were interesting in their own right: Since Equation (1.3) is only $O(n+1)$-symmetric with respect to $0 \in \mathbb{R}^{n+1}$, Alexandroff's techniques or the method of moving planes fail. However, scrutinizing the equation reveals how some relative positions of parts of a self-shrinker Σ and some corresponding directions of translation of the parts do in fact lead to the correct sign of the 2nd order derivatives being compared, allowing for application of a maximum principle. This, together with a 3rd order derivative argument (note that this is natural: Since one is fixing the shrinker center and time, Equation (1.3) can alternatively be thought of as a 3rd order elliptic PDE with more symmetries), to rule out S-shaped geodesics, the classification of all infinite ends with symmetry and finally a combinatorial exhaustion lead to a proof of Theorem 2.2.2.

Note that Theorem 2.3.1, and a result also proven in Chapter 2 (and [KMø11]) stating that the asymptotically conical ends cannot be extended smoothly to complete, embedded surfaces was afterwards used by Lu Wang (via non-trivial generalizations of the uniqueness statement for such infinite ends with fixed asymptotic cones), for proving the following:

Corollary 1.1.2 ([KMø11] + [Wa11]). *Let $\mathscr{C} \subseteq \mathbb{R}^{n+1}$ be a symmetric regular cone with vertex at 0. Then there does not exist any smooth complete embedded self-shrinker in $\mathbb{R}^{n+1}$ with an end asymptotic to $\mathscr{C}$, unless $\mathscr{C}$ is a hyperplane (in which case the unique such shrinker is the hyperplane itself).*

1.1.4 Minimal Surfaces & Self-Shrinkers in $\mathbb{R}^{n+1}$

Self-shrinkers are minimal surfaces in Euclidean space with respect to a conformally changed Gaussian metric g:

$$\begin{aligned} \Sigma^n \subseteq \mathbb{R}^{n+1} \quad \text{is a self-shrinker} \quad &\Leftrightarrow \quad H_{g_{ij}}(\Sigma) = 0, \\ g_{ij} = \frac{\delta_{ij}}{\exp\left(\frac{1}{2n}|X|^2\right)}, &\quad X \in \mathbb{R}^{n+1}. \end{aligned} \tag{1.4}$$

The classical minimal surfaces $H_{\mathbb{R}^{n+1}}(\Sigma) = 0$ in Euclidean space have a century-long and rich history starting with such prominent characters as Euler, Scherk, Riemann and many more, with a wealth of examples with intricate topologies. By analogy via (1.4) it was long expected that a similar picture could be drawn for the self-shrinking solitons in mean curvature flow, and indeed many such examples were conjectured based on computer simulations (see [Il95]).

The ambient space in (1.4), that is Euclidean space with the Gaussian metric in (1.4) is of course not complete, and does not have bounded curvatures. See the picture in the 2-dimensional example, in Figure 1-2.

1.1.5 Gluing Constructions for Self-Shrinkers with Asymptotically Conical Infinite Ends

In Chapter 3 (contained in [KKMø10]) we extend the list of known self-shrinkers (i)-(iii) above in Section 1.1.2 to an infinite family of higher genus surfaces, giving examples of complete embedded self-shrinkers with non-trivial asymptotically conical infinite ends as in Conjecture 1.1.1 (see the illustration in Figure 3-1).

Namely, for every large enough number $g \in \mathbb{N}$ there exists a complete, properly embedded, orientable, smooth surface $\Sigma_g \subseteq \mathbb{R}^3$, which is a mean curvature self-shrinker of genus g. Each Σ_g has the dihedral symmetry group with $4g+4$ elements, has one non-compact end, and separates $\mathbb{R}^3$ into two connected components. The ends are outside a Euclidean ball a graph over a plane $\mathcal{P} \subseteq \mathbb{R}^3$ (through the origin), and asymptotic to the regular cone on a non-zero smooth (4g+4)-symmetric (vertical) graph over a great circle in $\mathbb{S}^2$ (hence the visual appearance of a "wobbling sheet"). The family of surfaces constructed is such that

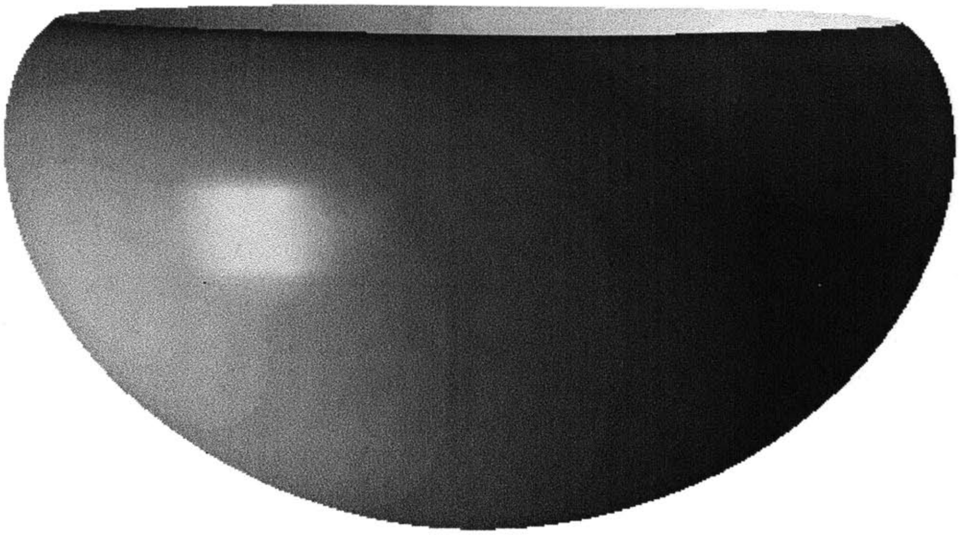

Figure 1-2: Ambient space with a very curly infinity: Showing part of Euclidean 2-space with the shrinker metric $\left(\mathbb{R}^2, \exp\left(-|X|^2/2\right)\delta_{ij}\right)$ immersed isometrically into $\mathbb{R}^3$ as a surface. The point $0 \in \mathbb{R}^2$ is at the very bottom, and the portion drawn is out to $|X| = 2$, with $\mathrm{dist}(0, X) \simeq 1.5$, while $\mathrm{dist}(0, \infty) = \sqrt{\pi} \simeq 1.7$.

inside any fixed ambient ball $B_R(0) \subseteq \mathbb{R}^3$, the sequence $\{\Sigma_g\}$ converges in Hausdorff sense to the union $\mathbb{S}^2 \cup \mathcal{P}$ (and in C^∞-sense away from $\mathbb{S}^2 \cap \mathcal{P}$), where $\mathcal{P}$ is a fixed plane through the origin in $\mathbb{R}^3$.

The proof uses the constructive methods developed by Nikolaos Kapouleas ([Ka90]-[Ka12], and earlier work by Schoen [Sch88], see also Traizet [Tr96]), to desingularize the union of the round sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ and a flat plane $\mathcal{P} \subseteq \mathbb{R}^3$. One constructs an initial surface by excision of the singular curve, obtaining an embedded but disconnected surface with boundary, and replacement of it by a surface with genus modeled on Scherk's singly periodic surface (see Figure 1-4), a classical minimal surface family. Historically, the motivating examples for the desingularizations were the Costa-Hoffman-Meeks minimal surfaces (see Figure 1-3), where in the two different limits as $g \to \infty$, one obtains either (1) The singular union of a catenoid and a plane (when fixing the size of one catenoidal end), and (2) Scherk's singly-periodic surface (when fixing the size of one of the handles). One then seeks to reverse the combined limiting behaviors in (1)+(2), by suturing a surface from these constituents.

Next, one introduces deformations in order to avoid small eigenvalues of the stability operator $\mathcal{L}$, which the Scherk surfaces are responsible for and which a priori would ruin the estimates on the inverse operator $\mathcal{L}^{-1}$. The main analysis idea is thereafter to consider small normal graphs over the initial surface, and solve the corresponding non-linear PDE problem via Newton's method, or more accurately using Schauder's fixed point theorem. For this, the central object of

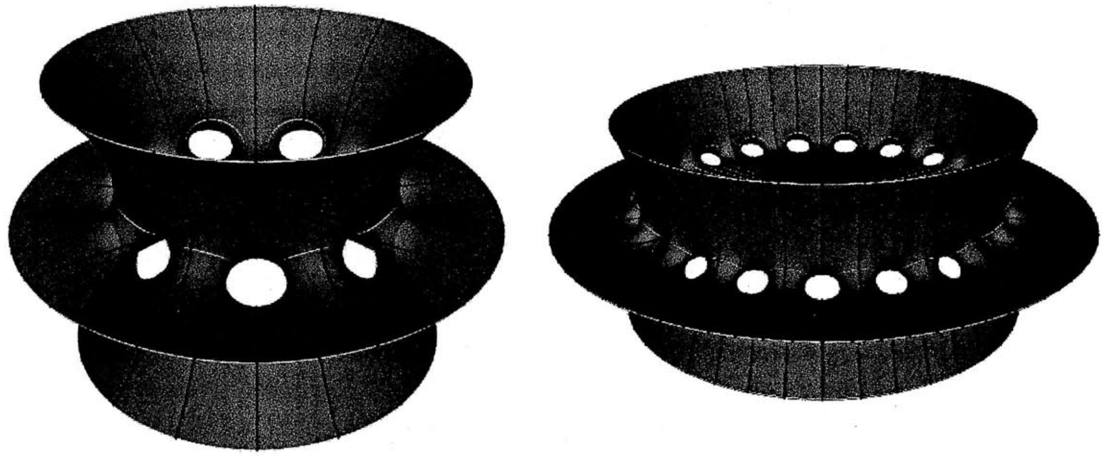

Figure 1-3: The Costa-Hoffman-Meeks minimal surface family Σ_g that in the limit as $g \to \infty$ motivates the gluing constructions (Mathematica. Courtesy of Matthias Weber)

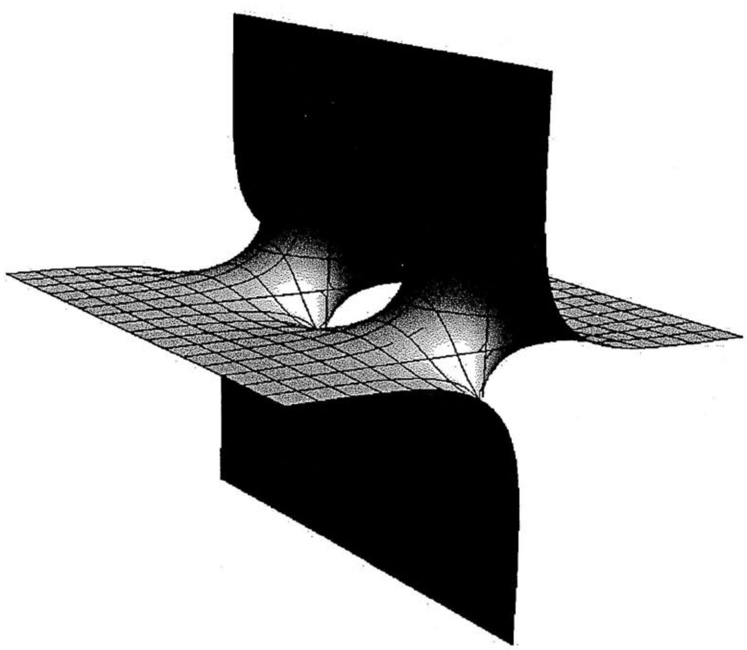

Figure 1-4: $\Sigma[\theta]$: Scherk's singly-periodic minimal surfaces ($H = 0$). Showing angle parameter $\theta = \frac{\pi}{4}$ (Mathematica).

study is the stability operator, or linearized operator of the equation (1.3), which in general for $\Sigma^n \subseteq \mathbb{R}^{n+1}$ reads:

$$\mathcal{L}_\Sigma = \Delta_\Sigma + |A_\Sigma|^2 - \tfrac{1}{2}X \cdot \nabla_\Sigma + \tfrac{1}{2}, \tag{1.5}$$

where X is the distance vector from the origin to the point $X \in \Sigma \subseteq \mathbb{R}^{n+1}$.

In the specific case for Theorem 3.1.1, one introduces a smooth family of self-shrinking cap-shaped surfaces Cap$[\theta]$ with boundary, depending on a small parameter θ. When $\theta = 0$ the surface Cap$[\theta]$ is a round spherical cap of radius 2 (cutting the sphere at the plane $\mathcal{P}$), while for $\theta > 0$ (resp. $\theta < 0$) intersects the plane $\mathcal{P}$ at the angle $\frac{\pi}{2} - \theta$, and in a circle of radius $r[\theta] \neq 2$.

The next step, which presents the major conceptual and technical issue in the understanding of the stability operator $\mathcal{L}$, is to construct an inverse and provide resolvent estimates.

The operator $\mathcal{L}$ in (1.5) is an elliptic second-order Ornstein-Uhlenbeck operator, which hence on any infinite end has unbounded coefficients. The complication comes directly from the fact that the metric g_{ij} in (1.4) gives unbounded geometry, for example on a plane through 0 the Ricci curvature is:

$$\mathrm{Ric}^g(\nu,\nu) = e^{\frac{|X|^2}{4}}\left(1 - \frac{|X|^2}{16}\right) \to -\infty \quad \text{(w.r.t. Gaussian unit normal } \nu) \tag{1.6}$$

See the illustration in Figure 1-2 for an impression of what this ambient space looks like.

Construction of $\mathcal{L}^{-1}$ on a non-compact set $\Omega \subseteq M$ therefore firstly calls for the identification of appropriate Banach spaces $\mathscr{B}^i = \mathscr{B}^i(\Omega)$ such that $\mathcal{L} : \mathscr{B}^2(\Omega) \to \mathscr{B}^0(\Omega)$. Most crucially, to apply Schauder's method one desires a compact embedding hierarchy $\mathscr{B}^{2,\alpha'}(\Omega) \hookrightarrow \mathscr{B}^{2,\alpha}(\Omega)$, for $1 > \alpha' > \alpha > 0$, similar to for the standard Hölder spaces $C^{k,\alpha}$ which were used in the less involved asymptotically flat $H = 0$ cases (see [Ka97]), as in many other well-known applications in non-compact but bounded geometry. Namely, let for $X_u = X_0 + u\nu_0$ the map on $\mathscr{B}^{2,\alpha}(\Omega)$ be given by

$$\Phi(u) = (H_u - \langle X_u, \nu_u\rangle) - (H_0 - \langle X_0, \nu_0\rangle) - \mathcal{L}u. \tag{1.7}$$

One looks at a chain of maps

$$\mathscr{B}^{2,\alpha'}(\Omega) \xrightarrow{\Phi} \mathscr{B}^{0,\alpha'}(\Omega) \xrightarrow{\mathcal{L}^{-1}} \mathscr{B}^{2,\alpha'}(\Omega) \hookrightarrow \mathscr{B}^{2,\alpha}(\Omega), \quad 0 < \alpha < \alpha' < 1. \tag{1.8}$$

with the last embedding of Banach spaces w.r.t. the α-grading needing to be compact.

Indeed in the $H = 0$ cases, the ends compactify fully. E.g. for asymptotically planar ends of the Costa-Hoffman-Meeks surfaces, via a circle inversion map $x \mapsto R_0\frac{x}{|x|^2}$, and analysis on the non-compact ends amounts in such cases to properties of the usual Hölder spaces $C^{k,\alpha}(B_1(0))$ and the Taylor series at 0 of a harmonic function w/ dihedral symmetry on the unit disk $B_1(0)$.

From counterexamples to a Schauder theory with polynomial weights by Priola [Pr], for the Dirichlet problem of a closely related Ornstein-Uhlenbeck operator with unbounded coefficients, on a half-plane and additionally without the extra symmetry that the self-shrinker equation has, the availability of such results fails in general.

However, the stability operator $\mathcal{L}$ in Equation (1.5) has two key special properties:

(A) *Mixed homogeneity* of $\mathcal{L}$. Namely, the usual minimal surface operator $\Delta_\Sigma + |A|^2$ is of homogeneity -2, while the extra terms $-X\cdot\nabla_\Sigma + 1$ are of homogeneity 0.

(B) *Conical functions are annihilated* by the homogeneity 0 term $-X\cdot\nabla_\Sigma + 1$.

The properties (A) and (B) allow for appropriate Hölder spaces $C^{k,\alpha}_{\mathrm{hom}}(\Omega,|x|^{-1})$ with homogenous weights, then anisotropic versions $C^{k,\alpha}_{\mathrm{an}}(\Omega,|x|^{-1})$ and finally cone space $\mathcal{CS}^{k,\alpha}(\Omega,|x|^{-1})$ adapted to the problem, to be defined.

By compactness of the curve $\partial\Omega = \partial B_R(0)$ on which we model the conical infinities, and by decay of all weights on v in the pair $(\sigma, v) \in \mathcal{CS}^{2,\alpha'}(\Omega,|x|^{-1})$, we get the straightforward but important consequence, that the following yields a compact inclusion hierarchy:

$$\mathcal{CS}^{2,\alpha'}(\Omega,|x|^{-1}) = C^{2,\alpha'}(\partial\Omega)\times C^{2,\alpha'}_{\mathrm{an}}(\Omega,|x|^{-1}) \hookrightarrow C^{2,\alpha}(\partial\Omega)\times C^{2,\alpha}_{\mathrm{an}}(\Omega,|x|^{-1}),$$

whenever $0<\alpha<\alpha'<1$.

The reason the Hölder cone spaces and the above property become useful is our corresponding main result in the Liouville-type theorem giving decomposition into "regular cone plus lower order", which shows that there is a natural inverse operator $\mathcal{L}^{-1}$ between the (smaller) product spaces we have defined.

Remark 1.1.3. *The spaces $C^{2,\alpha}_{\mathrm{hom}}(\Omega,|x|)$, with linear growth, do not have the compact hierarchy property, and hence have no use in a Schauder argument.*

This is evidenced by the lack of compactness in the abundant counterexamples of scaling sequences:

$$u_j(x) := j u^0(x/j), \quad j\in\mathbb{N}, \quad u^0\in C^\infty_c(\Omega). \tag{1.9}$$

These all have $\|u_j\|_{C^{2,\alpha'}_{\mathrm{hom}}(\Omega,|x|)} = \|u^0\|_{C^{2,\alpha'}_{\mathrm{hom}}(\Omega,|x|)}$, but no limit in $C^{2,\alpha}_{\mathrm{hom}}(\Omega,|x|)$.

Hence one has to appeal to the stability equation $\mathcal{L}u = E$ and establish extra structure of the solution space, via the Liouville-type decomposition.

After our work in Chapter 3 was completed, X.H. Nguyen [Ng11] announced a result very similar to Theorem 3.1.1. The spaces that were tried for a linear theory in [Ng11] are equivalent to the spaces mentioned in Remark 1.1.3. See also the same author previously [Ng06]-[Ng07] with a different approach to a proof of Theorem 3.1.1.

1.1.6 Gluing of Closed Self-Shrinkers: Sphere and Torus

In the final Chapter 4 (which was contained in [Mø11]), we further extend the list of complete, embedded self-shrinkers in Euclidean space, by appending an infinite family of higher genus closed surfaces.

Namely, for every large enough even integer $g = 2k$, $k \in \mathbb{N}$, there exists a compact, embedded, orientable, smooth surface without boundary $\Sigma_g^2 \subseteq \mathbb{R}^3$, which is a mean curvature self-shrinker of genus g. The proof of this, in Theorem 4.1.1, also uses the techniques from [Ka90]-[Ka12], similarly to what was described in the previous section, by desingularizing the immersed union of a sphere $\mathbb{S}^2$ and one of Angenent's self-shrinking tori $\mathbb{T}^2$ by insertion of two circular necks modelled again on Scherk's singly-periodic surface in Figure 1-4.

Since there are in this case no infinite ends to deal with, the central difficulty in the proof is the following: A torus $\mathbb{T}^2$ is not known to exist by any other means than abstractly (and uniqueness is not known), as some closed geodesic in the upper half-plane endowed with a certain singular Gaussian metric. Yet precise estimates on the eigenvalues of $\mathcal{L}$ with Dirichlet and Neumann (as appropriate in each separate case) on the natural subdomains of $\mathbb{T}^2 \cup \mathbb{S}^2$ are needed, in order to perform the construction. This on the other hand corresponds to analysis of the Jacobi fields in Angenent's upper half-plane metric.

Chapter 2

Self-Shrinkers with a Rotational Symmetry: New Examples and Classification

2.1 Summary

In this paper we present a new family of non-compact properly embedded, self-shrinking, asymptotically conical, positive mean curvature ends $\Sigma^n \subseteq \mathbb{R}^{n+1}$ that are hypersurfaces of revolution with S^{n-1} boundaries. These hypersurface families interpolate between the plane and half-cylinder in $\mathbb{R}^{n+1}$, and any rotationally symmetric self-shrinking non-compact end belongs to our family. The proofs involve the global analysis of a cubic-derivative quasi-linear ODE.

We also prove the following classification result: a given complete, embedded, self-shrinking hypersurface of revolution Σ^n is either a hyperplane $\mathbb{R}^n$, the round cylinder $\mathbb{R} \times S^{n-1}$ of radius $\sqrt{2(n-1)}$, the round sphere S^n of radius $\sqrt{2n}$, or is diffeomorphic to an $S^1 \times S^{n-1}$ (i.e. a "doughnut" as in [An89], which when $n = 2$ is a torus). In particular for self-shrinkers there is no direct analogue of the Delaunay unduloid family. The proof of the classification uses translation and rotation of pieces, replacing the method of moving planes in the absence of isometries.

2.2 Introduction and Statement of Results

We consider smooth n-dimensional hypersurfaces $\Sigma^n \subseteq \mathbb{R}^{n+1}$, $n \geq 2$, possibly with boundary $\partial\Sigma \neq \emptyset$, satisfying the (extinction time $T = 1$) self-shrinker equation for mean curvature flow, away from $\partial\Sigma$,

$$H = \frac{\langle \vec{X}, \vec{\nu} \rangle}{2}, \tag{2.1}$$

where $\vec{\nu}$ is the unit normal such that $\vec{H} = -H\vec{\nu}$.

Theorem 2.2.1. *In $\mathbb{R}^{n+1}$ there exists a 1-parameter family of non-compact smooth rotationally symmetric, embedded, positive mean curvature, asymptotically conical self-shrinking ends Σ^n with boundary.*

In fact for each rotationally symmetric cone $\mathscr{C}$ in $\{x_1 \geq 0\} \subseteq \mathbb{R}^{n+1}$ with tip at the origin, of slope $\sigma > 0$, there is a unique such self-shrinker Σ_σ, lying outside of $\mathscr{C}$, which is asymptotic to $\mathscr{C}$ as $x_1 \to \infty$.

Theorem 2.2.2. *Let $\Sigma^n \subseteq \mathbb{R}^{n+1}$ be a complete, embedded, self-shrinking hypersurface of revolution. Then Σ^n is one of the following:*

(1) n-dimensional hyperplane $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$,

(2) round cylinder $\mathbb{R} \times S^{n-1}$ of radius $\sqrt{2(n-1)}$,

(3) round sphere S^n of radius $\sqrt{2n}$,

(4) a smooth embedded $S^1 \times S^{n-1}$.

Remark 2.2.3. *Note that the list (1)–(3) together with Angenent's torus (in $\mathbb{R}^3$, and more general his specific $S^1 \times S^{n-1}$-solutions) gives all the presently known examples of complete, embedded self-shrinkers.*

In case (4), our assertion is only that Σ is generated by a closed, smooth, embedded curve. We conjecture however that geometrically such Σ must be symmetric with respect to $x_1 \mapsto -x_1$ and in fact coincide with Angenent's torus in [An89].

By combining Theorem 2.2.2 with a result by Anciaux [Anc09] we have the following corollary in 3-space.

Corollary 2.2.4. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a self-shrinker of genus zero which is foliated by circles. Then Σ is either: a plane, a round cylinder of radius $\sqrt{2}$, or a round sphere of radius 2.*

The study of the self-shrinker equation $H = \frac{1}{2}\langle \vec{X}, \vec{\nu}\rangle$ is motivated by the regularity theory for mean curvature flow. In particular, type I singularities are governed by (2.1), as Huisken showed in [Hu90]. Huisken in [Hu93] classified the possible singularities for the flow of a positive mean curvature initial surface, under the assumption of a bound on $|A|^2$. Currently, very few complete solutions of Equation (2.1) are known, embedded or otherwise, with the sphere, plane, cylinder, and Angenent's Torus (constructed in [An89]) being the only known examples. There is however numerical evidence for many more. David Chopp in [Ch94] (and see [AIC]) numerically computed a large number of interesting (apparently) self-similar solutions, and Angenent in [An89] gave numerical evidence for immersed topological spheres, although none of them have actually been rigorously demonstrated. The methods in [Ka97] of Kapouleas for producing examples of complete embedded minimal surfaces in Euclidean space by desingularization promise to be successful in the context of Equation (2.1); in particular, X. H. Nguyen in [Ng09] has had success in providing examples of self-translating (not self-shrinking) surfaces under mean curvature flow.

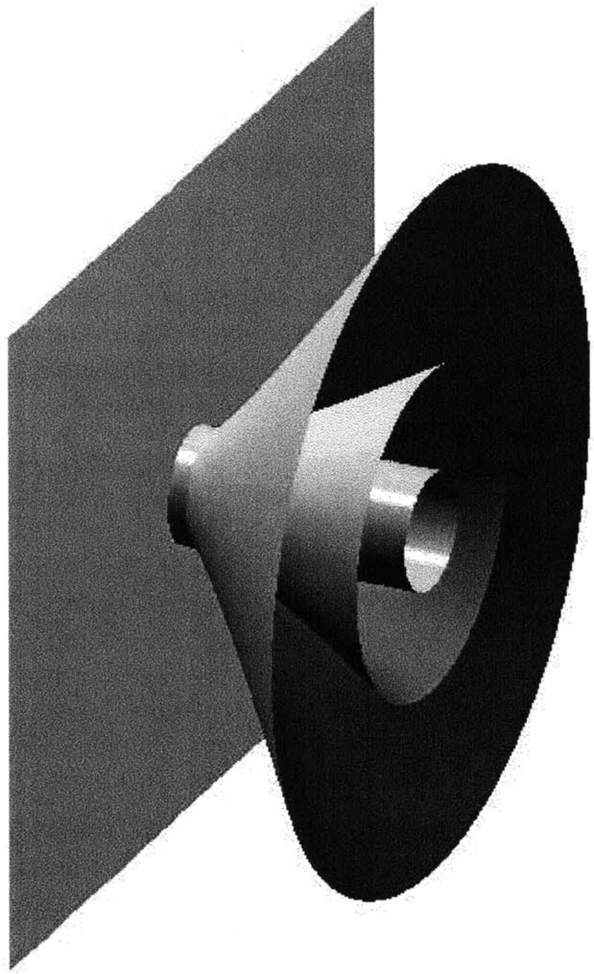

Figure 2-1: Examples of the asymptotically conical self-shrinking "trumpet" ends in Theorem 2.2.1, interpolating between the flat plane and round cylinder (MATLAB).

The numerical evidence cited above suggests that, in general, the singularity profile for mean curvature flow can be quite exotic and wildly behaved, and the classification of solutions to (2.1) seems impossible in general, however in dimension 2 the methods of Colding-Minicozzi in [CM1]–[CM8] offer a possibility. For a generic initial surface, one expects to find a rather tame singularity singularity profile, due to the highly unstable nature of most solutions of (2.1). In fact, this is a long-standing conjecture of Huisken, which was recently answered positively by Colding and Minicozzi in [CM8].

The study of Equation (2.1) turns out to be a variational problem. Namely, the solutions are actually minimal hypersurfaces in the conformal metric (see [Hu90])

$$g = e^{-\frac{|\vec{X}|^2}{2n}} \sum_{i=1}^{n+1} dx_i^2 \tag{2.2}$$

on $\mathbb{R}^{n+1}$. If Σ_γ is a hypersurface of revolution determined by a planar curve γ, then Σ_γ is minimal in the metric g if and only if the curve γ is a geodesic in the upper half-plane with non-complete conformal metric (c.f. [An89])

$$g_{\mathrm{Ang}} = r^{2(n-1)} e^{-\frac{(x^2+r^2)}{2}} \left\{dx^2 + dr^2\right\}, \tag{2.3}$$

where (x,r), $r > 0$ are Euclidean coordinates on the upper half-plane. The idea of reducing the problem of finding closed minimal sub-manifolds to the search for closed geodesics on a related manifold with a singular metric goes back at least as far as the paper [HL71], where Hsiang and Lawson produced closed minimal submanifolds of S^n invariant under a subgroup of the full isometry group. Mean curvature flow restricted to the rotational class is not a new venture either. For example, in addition to [An89] the paper [AAG95] considered regularity of viscosity solutions for mean curvature flow within the class of rotational surfaces.

The geodesic equation for curves parametrized by arc length in the upper half plane with metric g_{Ang} as given above in (2.3) is given by the following system of equations (see [An89]):

$$\begin{cases} \dot{x} = \cos\theta \\ \dot{r} = \sin\theta \\ \dot{\theta} = \frac{x}{2}\sin\theta + \left(\frac{n-1}{r} - \frac{r}{2}\right)\cos\theta, \end{cases} \tag{2.4}$$

where θ is the angle that $\dot{\gamma}$ makes with the x-axis, and where "$\cdot$" denotes derivation in the arc length parameter. We will use this notation throughout the article.

Thus, for a hypersurface of revolution generated by a graph $u \in C^2(I)$ over an interval on the x_1-axis, $u : I \to \mathbb{R}^+$, the self-shrinker equation is

$$H(u(x)) = \frac{\langle \vec{X}, \vec{\nu} \rangle}{2} = \frac{1}{2}\frac{u(x) - xu'(x)}{(1+(u')^2)^{\frac{1}{2}}}, \tag{2.5}$$

which is equivalent to the following ODE

$$u''(x) = \left[\frac{xu'(x) - u(x)}{2} + \frac{n-1}{u(x)}\right]\left(1 + (u'(x))^2\right), \tag{2.6}$$

which is a cubic-derivative quasi-linear elliptic second-order equation of the following form

$$u'' - xp(x, u'(x))u' + p(x, u'(x))u = g(u(x), u'(x)),$$

for appropriately defined functions p and g. For the graph of a function f over the r-axis, the equation becomes

$$f''(r) = \left\{\left(\frac{r}{2} - \frac{n-1}{r}\right)f'(r) - \frac{f(r)}{2}\right\}\left(1 + (f'(r))^2\right). \tag{2.7}$$

For such equations, containing a non-linearity of the form $(u')^3$, the general existence results by Nagumo and others do not apply (see for instance the survey [CH]) and we will be developing an approach from scratch. Furthermore, note that the sign of the terms in (2.6) are such that one does not have the best possible maximum and convexity principles, but instead an oscillating behavior (e.g. as is the case for the linear equations $u''+bu'+cu = 0$ when $c > 0$ is positive), contrasting for example the situation one would have had for self-expanders. Much of the intricacy concerning Equation (2.6) stems from this fact, and also from the lack of known exact symmetries.

The reader will notice that, in the proof of Theorem 2.2.2 (e.g. in Proposition 2.4.6), solutions to Equation (2.4) are translated to get contradictions via a maximum principle, as in the method of moving planes. However, here translation is not an isometry for the geometric problem in (2.1), and correspondingly is not an invariance for (2.4). In certain situations, depending on the relative position of solutions and signed direction of translation, it is even "better" than an exact symmetry, a key special feature which we exploit repeatedly in our maximum principle arguments.

Few known examples of complete embedded hypersurfaces in $\mathbb{R}^n$ satisfying the self-shrinker equation, and indeed several non-existence results are known. In the paper [Hu90], Huisken showed that the only positive mean curvature $H \geq 0$ rotationally symmetric hypersurface Σ^n, defined by revolution of an entire graph over the x_1-axis is the cylinder.

However, without the curvature assumption $H \geq 0$ it is still an open question as to whether there could exist for example non-standard embedded self-shrinking spheres, planes or cylinders. Note in this connection that Angenent in [An89] gave numerical evidence for many non-round immersed spheres with a rotational symmetry axis. Our Theorem 2.2.2 answers the question negatively under the assumption of embeddedness as well as a rotational symmetry axis of the hypersurface. Thus there are no analogues of the members of the rotationally symmetric Delaunay unduloid family of embedded, complete, singly periodic constant mean curvature surfaces that in the $H \equiv 1$ case interpolates between the round cylinder and a string of round spheres touching at antipodal points (see [De] or [Ke]). However, as Theorem 2.2.1 demonstrates there does exist a family of self-shrinkers (with boundary) interpolating between the flat plane and round cylinder orthogonal to the plane.

Notice that the existence of the "trumpet" family of self-shrinkers as in Theorem 2.2.1 (and its precise version in the below Theorem 2.3.1) along with the maximum principle for Equation (2.1) places certain crude a priori restrictions on what the non-compact ends of a general self-shrinker can be. Without investigating such issues further at present, let us remind the reader that this is related to announced work by Tom Ilmanen [Il] stating that self-shrinkers have ends that are (in Hausdorff sense) asymptotically conical.

As mentioned in the introduction, there has been recent interest in applying the desingularizing methods of Kapouleas to the construction of complete embedded self-shrinkers with genus. Apart from its use in the proof of Theorem 2.2.2, the perspective of such constructions is one of the main interests of Theorem 2.2.1.

We note that simultaneously with our work for this paper, the recent monograph Giga-Giga-Saal [GGS10] was also concerned with different proofs of well-known weaker versions of the uniqueness of self-shrinkers given by entire cylindrical graphs, which dates back to Huisken [Hu90] (see also Soner-Souganidis [SS93] and Altschuler-Angenent-Giga [AAG95]), a special case of the present paper. Note also that the below Lemma 2.3.4 alone removes the assumption of $H \geq 0$ from all such results, see Corollary 2.3.2.

2.3 Proof of Theorem 2.2.1: An Integral Identity for Graphs

The version of Theorem 2.2.1 we will prove is more precisely stated as follows:

Theorem 2.3.1 (:= Theorem 1′). *Let $n \geq 2$. For each fixed ray from the origin,*

$$r_\sigma(x) = \sigma x, \quad r_\sigma : (0,\infty) \to \mathbb{R}^+, \quad \sigma > 0,$$

there exists a unique smooth graphical solution $u_\sigma : [0,\infty) \to \mathbb{R}^+$, of (2.6), asymptotic to r_σ.

Also, for $d > 0$, any solution $u : (d,\infty) \to \mathbb{R}^+$ to (2.6) is either the cylinder $u \equiv \sqrt{2(n-1)}$, or is one of the u_σ for some $\sigma = \sigma(u) > 0$.

Furthermore, the following properties hold for u_σ when $\sigma > 0$:

(i) $u_\sigma > r_\sigma$, *and* $u_\sigma(0) < \sqrt{2(n-1)}$,

(ii) $|u_\sigma(x) - \sigma x| = O(\frac{1}{x})$, *and* $|u'_\sigma(x) - \sigma| = O(\frac{1}{x^2})$ *as* $x \to +\infty$,

(iii) Σ_σ *generated by* u_σ *has mean curvature* $H(\Sigma_\sigma) > 0$,

(iv) u_σ *is strictly convex, and* $0 < u'_\sigma < \sigma$ *holds on* $[0,\infty)$,

(v) γ_σ, the maximal geodesic containing the graph of u_σ, is not embedded.

This immediately gives the following corollary, where as an improvement over [Hu90] (where $H \geq 0$ was required) we do not need any curvature assumption.

Corollary 2.3.2. *Let Σ^n be a smooth self-shrinking hypersurface of revolution, which is generated by rotating an entire graph around the x_1-axis. Then Σ^n is the round cylinder $\mathbb{R} \times S^{n-1}$ of radius $\sqrt{2(n-1)}$ in $\mathbb{R}^{n+1}$.*

Proof of Corollary. Any entire graph is a graph over the right half axis. Theorem 2.3.1 characterizes all such graphs, and in particular says that none are embedded excepting the cylinder solution. □

Note that our Theorem 2.3.1 amounts to the following interesting geometric fact, which we get since instead of $(n-1)$ we may take an arbitrary number $\alpha > 0$ everywhere in our proofs.

Corollary 2.3.3. *In the (non-complete) generalized Gaussian upper half-plane*

$$\mathbb{G}_\alpha = (\mathbb{R}_x \times \mathbb{R}_r^+, g_{ij} = r^{2\alpha} e^{-\frac{x^2+r^2}{4}} \delta_{ij}),$$

for any $\alpha \geq 0$, there exists for each $\sigma > 0$ a unique geodesic ray u_σ, with the properties in Theorem 2.3.1. Note that for the usual Gaussian metric (where $\alpha = 0$), we have $u_\sigma \equiv r_\sigma$, i.e. straight lines through the origin.

We will need the following lemma, which observes sufficient conditions for solutions to become non-graphical.

Lemma 2.3.4. *If $x_0 \in (0, \infty)$, and (x_0, x_∞) is a maximally extended solution to the initial value problem*

$$\begin{cases} u'' = \left[\frac{x}{2}u' + \frac{n-1}{u} - \frac{u}{2}\right]\left(1 + (u')^2\right), \\ u(x_0) = \sigma x_0, \\ u'(x_0) \geq \sigma, \end{cases} \tag{2.8}$$

where $\sigma > 0$. Then $x_\infty < \infty$, and if $u(x_0) \geq \sqrt{2(n-1)}$, then $x_\infty \leq (1 + \frac{1}{n-1})x_0$. Geometrically these initial conditions mean that $H(u(x_0)) \leq 0$ at the point on the hypersurface Σ.

Proof of Lemma 2.3.4. Defining $\Psi(x) := xu'(x) - u(x)$, we note that the initial conditions are equivalent to $\Psi(x_0) \geq 0$. Since

$$\Psi' = \left(\frac{\Psi}{2} + \frac{n-1}{u}\right)(1 + (u')^2) > \frac{\Psi}{2} \geq 0, \tag{2.9}$$

we see that $\Psi' > 0$ and $\Psi \geq 0$, so $u'(x) \geq u(x)/x > 0$, for $x \geq x_0$. Thus in particular there always exists $x_0' \geq x_0$ such that $u(x_0') \geq \sqrt{2(n-1)}$ and $\Psi(x_0') \geq 0$, and we can without loss of generality assume $u(x_0) \geq \sqrt{2(n-1)}$.

If we define for u the quantity

$$\Phi(x) := \frac{x}{2}u' + \frac{n-1}{u} - \frac{u}{2},$$

then by (2.8) we have $\Phi(x_0) \geq \frac{n-1}{\sigma x_0}$. We claim that in fact $\Phi(x) \geq \frac{n-1}{\sigma x_0}$ for all $x \geq x_0$. Namely assuming this holds up to x we have for $x \geq x_0$,

$$\begin{aligned}\frac{d}{dx}\left(\frac{x}{2}u' + \frac{n-1}{u} - \frac{u}{2}\right) &= \frac{x}{2}u'' - (n-1)\frac{u'}{u^2} = \frac{x}{2}\left(1 + (u')^2\right)\Phi - (n-1)\frac{u'}{u^2} \\ &\geq \frac{x}{2}\frac{n-1}{\sigma x_0}\left(1 + (u')^2\right) - (n-1)\frac{u'}{u^2} \geq \frac{n-1}{2\sigma} > 0,\end{aligned}$$

assuming that both $u(x) \geq \sqrt{2(n-1)}$ and $u'(x) \geq \sigma$. In particular $u'' \geq 0$, and hence the set of conditions

$$\begin{cases} \Phi(x) \geq \frac{n-1}{\sigma x_0}, \\ u'(x) \geq \sigma, \\ u(x) \geq \sqrt{2(n-1)}, \end{cases} \tag{2.10}$$

are simultaneously preserved by the self-shrinker ODE in (2.6) as $x \geq x_0$ increases.

Using $\Phi(x) \geq \frac{n-1}{\sigma x_0}$, we get that $u'' \geq \frac{n-1}{\sigma x_0}(1 + (u')^2)$, for $x \geq x_0$, and integrating this inequality gives

$$u'(x) \geq \tan\left[(n-1)\frac{x - x_0}{\sigma x_0} + \arctan\sigma\right],$$

which finally leads to $x_\infty < \frac{\sigma x_0}{n-1}\left(\frac{\pi}{2} - \arctan\sigma\right) \leq \frac{n}{n-1}x_0 + x_0 < \infty$. □

Remark 2.3.5. *Incidentally Lemma 2.3.4 also removes the $H \geq 0$ assumption, yielding a different proof of Corollary 2.3.2.*

Lemma 2.3.6 (Integral identity). *Any solution $u : (d, \infty) \to \mathbb{R}^+$ to (2.6), where $d \geq 0$, satisfies for some $\sigma = \sigma(u) \geq 0$ the identity*

$$u(x) = 2(n-1)x\int_x^\infty \frac{1}{t^2}\left\{\int_t^\infty \frac{s}{2}\frac{1 + (u'(s))^2}{u(s)}e^{-\frac{1}{2}\int_t^s z(1+(u'(z))^2)dz}ds\right\}dt + \sigma x, \tag{2.11}$$

when $x \in (d, \infty)$.

Proof. Suppose first that we are given a solution $u : (d, a) \to \infty$ over an interval (d, a). We can regard the solution u as solving an inhomogeneous linear equation determined by freezing the coefficients at u,

$$u'' - \left(1 + (\varphi')^2\right)\frac{x}{2}u' + \left(1 + (\varphi')^2\right)\frac{u}{2} = (n-1)\frac{\left(1 + (\varphi')^2\right)}{\varphi}, \tag{2.12}$$

where we have set $u = \varphi$. We can solve the resulting linear equation with variable coefficients, for $x \in (d, a)$, by making the observation that a pair of spanning solutions of the homogeneous linear equation are

$$u_1(x) = x, \quad \text{and} \quad u_2(x) = x\int_x^a \frac{e^{-\frac{1}{2}\int_s^a z(1+(\varphi')^2)dz}}{s^2}ds, \tag{2.13}$$

Then computing the Wronskian $W(s) = e^{-\frac{1}{2}\int_s^a z(1+(\varphi'(z))^2)dz}$, and matching the initial conditions gives

$$\begin{aligned} u(x) &= \frac{u(a)}{a}x + (u(a)-u'(a)a)x\int_x^a \frac{e^{-\frac{1}{2}\int_s^a z(1+(u')^2)dz}}{s^2}ds \\ &+ (n-1)x\int_x^a \frac{1}{t^2}\left\{\int_t^a \frac{s\left(1+(u'(s))^2\right)}{u(s)}e^{-\frac{1}{2}\int_t^s z(1+(u'(z))^2)dz}ds\right\}dt. \end{aligned} \tag{2.14}$$

To complete the proof of (2.11), we will show that for some limit $\sigma \geq 0$,

$$\frac{u(a)}{a} \to \sigma, \quad \text{for} \quad a \to \infty \tag{2.15}$$

$$(u(a)-u'(a)a)x\int_x^a \frac{e^{-\frac{1}{2}\int_s^a z(1+u'^2)dz}}{s^2}ds \to 0, \quad \text{for} \quad a \to \infty. \tag{2.16}$$

Recall that by Lemma 2.3.4, for any solution $u : (d,\infty) \to \mathbb{R}^+$ the quantity $\Psi(x) = xu'(x) - u(x)$ is pointwise negative. Thus the ratio $\frac{u(a)}{a}$ is monotonically decreasing in a, and hence converges to some limit $\sigma \geq 0$. The negativity of Ψ also implies that

$$u(x) \geq (n-1)x\int_x^a \frac{1}{t^2}\left\{\int_t^a \frac{s\left(1+(u'(s))^2\right)}{u(s)}e^{-\frac{1}{2}\int_t^s z(1+(u'(z))^2)dz}ds\right\}dt + \sigma x. \tag{2.17}$$

By this it follows that there exists a sequence $\{a_k\}$ increasing to infinity such that $u(a_k) \geq \sqrt{2(n-1)}$. Namely, otherwise one would have that $u(x) < \sqrt{2(n-1)}$ for large enough x. With (2.17) we get for such large x that

$$u(x) \geq \frac{2(n-1)}{\sqrt{2(n-1)}} - R(a) \to \sqrt{2(n-1)}, \quad \text{for} \quad a \to \infty,$$

where $R(a)$ is an explicit error term, yielding the contradiction $u(x) \geq \sqrt{2(n-1)}$.

Moreover, we can modify the sequence $\{a_k\}$ to satisfy in addition $u'(a_k) \geq 0$. This is easily seen to follow from Equation (2.6) and the mean value theorem, using that $u(a_k) \geq \sqrt{2(n-1)}$ on the original sequence. Thus we have

$$0 < u(a_k) - u'(a_k)a_k < \sqrt{2(n-1)},$$

so that this term is bounded, and since

$$\int_x^{a_k} \frac{e^{-\frac{1}{2}\int_s^{a_k} z(1+(u'(z))^2)dz}}{s^2} ds \le \frac{e^{-\frac{a_k^2}{4}}}{x^2} \int_x^{a_k} e^{\frac{s^2}{4}} ds \to 0, \quad \text{for} \quad a_k \to \infty.$$

we see that inserting the sequence $a_k \to \infty$ in (2.14) leads to (2.11). □

As an immediate consequence of Lemma 2.3.6 we see that $u_\sigma(x) > \sigma x$, i.e. $u_\sigma > r_\sigma$, which leads to the following L^∞-estimates.

Lemma 2.3.7. *Let $u : (d, \infty) \to \mathbb{R}^+$ be as in Lemma 2.3.6, with $\sigma > 0$. Then*

$$\sup_{s \in (x,\infty)} |u(s) - \sigma s| \le \frac{2(n-1)}{\sigma x}, \tag{2.18}$$

$$\sup_{s \in (x,\infty)} |u'(s) - \sigma| \le \frac{2(n-1)}{\sigma x^2}, \tag{2.19}$$

for $x \in (d, \infty)$. In particular u extends to $u_\sigma : (0, \infty) \to \mathbb{R}^+$.

Proof. We can estimate using $u > r_\sigma$,

$$\begin{aligned} |u(x) - \sigma x| \le & \frac{2(n-1)}{\sigma} \int_x^\infty \frac{1}{t^2} \left\{ \int_t^\infty \frac{s}{2}(1 + (u'(s))^2) e^{-\int_t^s \frac{z}{2}(1+(u'(z))^2)dz} ds \right\} dt \\ \le & \frac{2(n-1)}{\sigma x} \end{aligned}$$

and by similar reasoning obtain the estimate for u'. □

We can thus assume without loss of generality that $d = 0$.

To prove existence of a solution u_σ for any σ, we find it illustrative to construct each solution u_σ as a limit of approximating solutions. More specifically, fixing a $\sigma > 0$, we solve the initial value problem

$$\begin{cases} u'' = \left[\frac{x}{2}u' + \frac{n-1}{u} - \frac{u}{2}\right]\left(1 + (u')^2\right), \\ u(a) = a\sigma, \\ u'(a) = \sigma. \end{cases} \tag{2.20}$$

for a positive. Denoting the solution $u_{\sigma,a}$, one derives the analogous identity

$$u_{\sigma,a}(x) = (n-1)x \int_x^a \frac{1}{t^2} \left\{ \int_t^a \frac{\left(1 + u'_{\sigma,a}(s)^2\right)}{u_{\sigma,a}(s)} e^{-\frac{1}{2}\int_t^s z(1+u'_{\sigma,a}(z)^2)dz} ds \right\} dt + \sigma x, \tag{2.21}$$

for $x < a$. The lack of terms in this expression corresponding to the homogeneous equation is a special property of the initial conditions. One derives uniform estimates analogous to (2.18)–(2.19) for the solutions,

$$\sup_{s\in(x,a)} |u_{\sigma,a}(s)-\sigma s| \le \frac{2(n-1)}{\sigma x}, \tag{2.22}$$

$$\sup_{s\in(x,a)} |u'_{\sigma,a}(s)-\sigma| \le \frac{2(n-1)}{\sigma x^2}, \tag{2.23}$$

for any $x < a$. This gives that each solution $u_{\sigma,a}$ extends to $(0,a)$, and by compactness that the family $\{u_a\}_{a>0}$ converges to a limiting solution u_σ on $(0,\infty)$, uniformly in the C^2-norm on compact sub-intervals.

Note however, that each approximate solution is really only approximate: Lemma 2.3.4 implies that they do not remain graphical for values of x much larger than a, but bend upwards with $u'_{\sigma,a}(x) \to \infty$ as $x \to x_\infty < \infty$.

We next prove that any solution $u_\sigma : (0,\infty) \to \mathbb{R}^+$ asymptotic to the ray r_σ is unique. Recall that we have shown that, given a σ, any solution u_σ must satisfy

$$u_\sigma(x) > \sigma x \tag{2.24}$$

as well as the L^∞-estimates in (2.18)–(2.19). Consider for $b > 0$ the Banach space

$$C_0^1([b,\infty)) = \{v : [b,\infty) \to \mathbb{R} \mid v, v' \in C_0([b,\infty))\}$$

of continuously differentiable functions v such that $|v(x)| \to 0$ and $|v'(x)| \to 0$ as $x \to +\infty$, endowed with the uniform C^1-norm $\|v\|_{C^1} = \|v\|_\infty + \|Dv\|_\infty$, where the supremum is taken over $[b,\infty)$.

Also, for $b, \sigma > 0$ we can for example consider the open subsets

$$Y_{\sigma,b} := \left\{ v \in C_0^1([b,\infty)) \,\middle|\, v(x) > 0, \quad |v'(x)| < \frac{4(n-1)}{\sigma x^2} \right\},$$

so that by the estimates (2.18)–(2.19) the solutions to (2.6) - σx are in $Y_{\sigma,b}$.

Then we will show that the non-linear mapping T_σ on $Y_{\sigma,b}$ given by

$$[T_\sigma v](x) = 2(n-1)x \int_x^\infty \frac{1}{t^2} \left\{ \int_t^\infty \frac{1+(v'(s)+\sigma)^2}{v(s)+\sigma s} \frac{s}{2} e^{-\int_t^s \frac{z}{2}(1+(v'(z)+\sigma)^2)dz} ds \right\} dt$$

is a contraction, if $b = b(n,\sigma)$ is chosen large enough. Note that if u is a solution to the equation (2.6), then by the integral identity in Lemma 2.3.6

$$[\tilde{T}_\sigma u](x) := [T_\sigma(s \mapsto u(s) - \sigma s)](x) + \sigma x = u(x),$$

and conversely, so that $v(x) + \sigma x$ solves equation (2.6) if and only if $T_\sigma v = v$.

In fact $T_{\sigma,b}$ is well-defined, and we get the mapping property

$$T_\sigma : Y_{\sigma,b} \to Y_{\sigma,b},$$

as follows similarly to the proofs of the estimates in Lemma 2.3.7 and of the properties (2.24), using the integral identity in Lemma 2.3.6.

Proposition 2.3.8. *There exists $b_0 = b_0(n,\sigma)$ such that T_σ is a contraction for the norm $\|\cdot\|_{C^1}$ on the set of functions $Y_{\sigma,b}$ for $b \geq b_0$.*

Proof. For two functions $v_1, v_2 \in Y_{\sigma,b}$ we may write $u_i(x) = v_i(x) + \sigma x$ and get for $\tilde{T}_\sigma$ the expression:

$$\begin{aligned}
\tilde{T}_\sigma u_2 - \tilde{T}_\sigma u_1 &= 2(n-1)x\int_x^\infty \frac{1}{t^2}\int_t^\infty \left(\frac{1}{u_2}-\frac{1}{u_1}\right)\frac{s}{2}(1+(u_2')^2)e^{-\int_t^s \frac{1}{2}z(1+(u_2'(z))^2)} \\
&+ 2(n-1)x\int_x^\infty \frac{1}{t^2}\int_t^\infty \frac{\frac{s}{2}(1+(u_2')^2)}{u_1}\left(e^{-\int_t^s \frac{z}{2}(1+(u_2')^2)} - e^{-\int_t^s \frac{z}{2}(1+(u_1')^2)}\right) \\
&+ 2(n-1)x\int_x^\infty \frac{1}{t^2}\int_t^\infty \frac{1}{u_1}\frac{s}{2}\left((u_2')^2-(u_1')^2\right)e^{-\int_t^s \frac{z}{2}(1+(u_1')^2)} \\
&=: A+B+C.
\end{aligned}$$

We estimate the term A by

$$A \leq 2(n-1)\frac{\|u_2-u_1\|_\infty}{\sigma^2 x^2}.$$

The term C may be estimated by

$$\begin{aligned}
C &\leq \|u_2'-u_1'\|_\infty \|u_2'+u_1'\|_\infty \frac{n-1}{\sigma}\int_x^\infty \frac{1}{t^2}\left\{\int_t^\infty s e^{\frac{1}{4}(t^2-s^2)}ds\right\}dt \\
&= \|u_2'-u_1'\|_\infty \|u_2'+u_1'\|_\infty \frac{2(n-1)}{\sigma x}.
\end{aligned}$$

To estimate the term B, we note that, for real numbers $x, y \leq c$, one has $|e^y - e^x| \leq e^c|y-x|$ so that we may estimate term B as follows:

$$\begin{aligned}
B &\leq \|u_2'+u_1'\|_\infty \|u_2'-u_1'\|_\infty \|1+(u_2')^2\|_\infty \frac{(n-1)x}{\sigma}\int_x^\infty \frac{1}{t^2}\left\{\int_t^\infty \frac{1}{4}(s^2-t^2)e^{\frac{1}{4}(t^2-s^2)}ds\right\}dt \\
&\leq \|u_2'+u_1'\|_\infty \|u_2'-u_1'\|_\infty \|1+(u_2')^2\|_\infty \frac{(n-1)x}{2\sigma}\int_x^\infty \frac{1}{t^3}\left\{\int_t^\infty \frac{1}{2}(s^2-t^2)e^{\frac{1}{4}(t^2-s^2)}s\,ds\right\}dt \\
&= \|u_2'+u_1'\|_\infty \|u_2'-u_1'\|_\infty \|1+(u_2')^2\|_\infty \frac{(n-1)x}{2\sigma}\int_x^\infty \frac{1}{t^3}\left\{\int_0^\infty \tau e^{-\tau}d\tau\right\}dt \\
&\leq \|u_2'+u_1'\|_\infty \|u_2'-u_1'\|_\infty \|1+(u_2')^2\|_\infty \frac{(n-1)}{2\sigma x}.
\end{aligned}$$

Also, we may write

$$(\tilde{T}_\sigma u)' = \frac{\tilde{T}_\sigma(u)}{x} - \frac{2(n-1)}{x}\int_x^\infty \frac{\frac{s}{2}\left(1+(u'(s))^2\right)}{u(s)}e^{-\int_x^s \frac{z}{2}(1+(u'(z))^2)dz}ds + \sigma,$$

and from this representation formula similarly get, for $p_i(s) = \frac{s}{2}(1 + (u_i'(s))^2)$:

$$\begin{aligned}
(\tilde{T}_\sigma u_2)' - (\tilde{T}_\sigma u_1)' &= \frac{1}{x}(T_\sigma u_2 - T_\sigma u_1) \\
&- \frac{2(n-1)}{x}\int_x^\infty \left(\frac{1}{u_2(s)} - \frac{1}{u_1(s)}\right) p_2(s) e^{-\int_x^s p_2(z)dz} \\
&- \frac{2(n-1)}{x}\int_x^\infty \frac{p_2(s)}{u_1(s)}\left(e^{-\int_x^s p_2(z)dz} - e^{-\int_x^s p_1(z)dz}\right) \\
&- \frac{2(n-1)}{x}\int_x^\infty \frac{1}{u_1(s)}(p_2(s) - p_1(s))\, e^{-\int_x^s p_1(z)dz} \\
&= \frac{1}{x}(T_\sigma u_2 - T_\sigma u_1) - A' - B' - C'.
\end{aligned}$$

Then the terms A', B', and C' may be treated similarly to the terms A, B, and C before.

Thus we see, going back to v_i and to T_σ, that:

$$\|T_\sigma v_2 - T_\sigma v_1\|_{C^1} < \tau \|v_2 - v_1\|_{C^1}, \tag{2.25}$$

for some $0 < \tau < 1$, if we choose b_0 large enough, and with the C^1-norm taken over (b_0, ∞). Thus $T_\sigma : Y_{\sigma,b_0} \to Y_{\sigma,b_0}$ is a contraction.

Note also that a family version of the Proposition follows, that is if $0 < \sigma_i < \infty$ are given, then b_0 and τ can be chosen so that (2.25) holds uniformly for $\sigma \in [\sigma_1, \sigma_2]$ □

Applying the Proposition shows the claimed uniqueness for graphs over half-lines satisfying Equation (2.6). Namely, let two solutions u_1 and u_2 to the equation for the same σ-value be given. Then for b_0 chosen large enough we have $u_1, u_2 \in Y_{\sigma,b_0}$ and the result follows.

Remark 2.3.9. *Since the map T_σ is a contraction for large enough x-values, one can also prove the existence part (for large x) of Theorem 2.2.1 using a fixed point principle.*

The graphs of the functions u_σ constructed above are eventually graphical over the r-axis as well (since they are eventually increasing), and are given by functions $f_{1/\sigma} : [r_{1/\sigma}, \infty] \to \mathbb{R}$ on some maximal domain $(r_{1/\sigma}, \infty)$. The functions $f_{1/\sigma}$ then satisfy equation (2.7), and an analysis similar to that in the proof of Lemma 2.3.6 gives that the $f_{1/\sigma}$ satisfy the identity

$$S_\sigma f_{1/\sigma} = f_{1/\sigma}, \tag{2.26}$$

where the map S_σ given by

$$f \mapsto \frac{r}{\sigma} - (n-1)r\int_r^\infty \frac{1}{t^2}\int_t^\infty f'(s)\left(1 + (f'(s))^2\right) e^{-\int_t^s \frac{z}{2}(1+(f'(z))^2)dz} ds dt, \tag{2.27}$$

which is then similarly shown to be a contraction mapping. The fixed points of the maps $S_{1/\sigma}$ and T_σ then determine a complete geodesic γ_σ in the upper half plane. We now show that the γ_σ depend smoothly on the parameter σ in the C^k topology. For this, we will need the following general fact, proved in the Appendix.

Lemma 2.3.10. *Let $\Phi_\sigma : Y \to Y$ be a smooth one parameter family of smooth contraction mappings on a fixed open subset Y of a Banach space X. Then the fixed points x_σ (assumed to exist) are smooth functions of σ.*

Thus, the solutions γ_σ depend smoothly in C^1 on the parameter σ. However, the geodesic equation gives that the dependence is smooth in C^k for any k.

Lemma 2.3.11. *The map $\sigma \mapsto f_\sigma : \mathbb{R}^+ \to C^k$ is smooth for any k.*

Proof. The equation

$$f_\sigma''(r) = \left\{\left(\frac{r}{2} - \frac{n-1}{r}\right) f_\sigma'(r) - \frac{f_\sigma(r)}{2}\right\}(1 + (f_\sigma')^2), \tag{2.28}$$

immediately gives that the second derivative f_σ'' is differentiable in σ. Differentiating (2.7) in r then gives that all higher derivatives $f_\sigma^{(k)}$ are differentiable in σ as well. □

By Lemma 2.3.11, the function $F(\sigma, r)$ given by

$$F(\sigma, r) = f_\sigma(r) \tag{2.29}$$

is smooth on its domain of definition. Note that, as $\sigma \to \infty$ the functions $f_{1/\sigma}$ converge to the function $f_0(r) \equiv 0$ uniformly in C^k on compact subsets of $(0, \infty)$ for any k. Thus, defining $\hat{\sigma} = 1/\sigma$, it follows that the function $g(r) = \frac{df}{d\hat{\sigma}}|_{\hat{\sigma}=0}(r)$ is defined on $(0, \infty)$ and satisfies the linearized equation

$$g''(r) = \left(\frac{r}{2} - \frac{n-1}{r}\right) g'(r) - \frac{g(r)}{2}. \tag{2.30}$$

To analyze solutions of the linearized equation, we again prove an integral identity.

Lemma 2.3.12. *The solution to the linearized equation g above satisfies the identity*

$$g(r) = r - (n-1)r \int_r^\infty \frac{1}{t^2} \int_t^\infty g'(s) e^{(t^2 - s^2)/4} ds dt. \tag{2.31}$$

Proof. Differentiating identity (2.26) (with $\hat{\sigma} = 1/\sigma$), we obtain

$$f_{\hat{\sigma}}'(r) = f_{\hat{\sigma}}/r + \frac{n-1}{r} \int_r^\infty f_{\hat{\sigma}}'(s)(1 + (f_{\hat{\sigma}}'(s))^2) e^{-\int_r^s \frac{z}{2}(1 + (f_{\hat{\sigma}}'(z))^2) dz} ds. \tag{2.32}$$

Thus, for $r > 2(n-1)$, we get

$$f_{\hat{\sigma}}'(r)/\hat{\sigma} < (1 - 2(n-1)/r^2)^{-1}. \tag{2.33}$$

Now, since

$$f_{\hat{\sigma}}'(s)/\hat{\sigma}\left(1+(f_{\hat{\sigma}}'(s))^2\right)e^{-\int_t^s \frac{z}{2}(1+(f_{\hat{\sigma}}'(z))^2)dz} \to g'(s)e^{(t^2-s^2)/4}, \quad \text{as} \quad \hat{\sigma}\to 0. \tag{2.34}$$

the above estimate (2.33) gives convergence of the equation (2.26) divided by $\hat{\sigma} = 1/\sigma$, as $\hat{\sigma} \to 0$ to (2.31) by, for example, the dominated convergence theorem. □

Corollary 2.3.13. *The solution g to the linearized equation assumes both positive and negative values on $(0,\infty)$.*

Proof. Assume first that $g > 0$ everywhere on $(0,\infty)$. Note that we must then also have $g' > 0$ everywhere. For suppose $g'(r_0) \leq 0$ at some r_0. Then appealing to equation (2.30), we see that $g'(r) < 0$ for all $r > r_0$. In particular, for $r > \sqrt{2(n-1)}$, we get $g''(r) < 0$, which implies that the graph of g will eventually intersect the r-axis, a contradiction.

Thus we have $g' > 0$ on $(0,\infty)$. However, the identity (2.31) then gives the contradiction

$$\lim_{r\to 0} g(r) = -(n-1)\int_0^\infty g'(s)e^{-s^2/4}ds < 0. \tag{2.35}$$

Since the equation (2.31) is linear homogeneous, it follows that $g < 0$ cannot hold either. □

Lemma 2.3.14. *The functions u_σ have positive slope on $[0,\infty)$ for $\sigma > 0$ sufficiently large.*

Proof. As before, take $\hat{\sigma} = 1/\sigma$. Then the graphs $f_{\hat{\sigma}}$ are defined on the maximal interval $(r_{\hat{\sigma}},\infty)$ (that is, $\lim_{r\to r_{\hat{\sigma}}^+} f_{\hat{\sigma}}'(r) \to \infty$). Note that $r_{\hat{\sigma}} \to 0$ as $\hat{\sigma} \to 0$, since the graphs $f_{\hat{\sigma}}$ converge uniformly to 0 in any C^k on compact subsets of $(0,\infty)$.

Now, let r_0 be a point such that $\frac{\partial f}{\partial \sigma}(r_0) = g(r_0) < 0$. Then, choosing $\hat{\sigma}$ sufficiently small so that $r_{\hat{\sigma}} < r_0$, we get that

$$f_{\hat{\sigma}}(r_0) = g(r)\hat{\sigma} + O(\hat{\sigma}^2) < 0, \tag{2.36}$$

for $\hat{\sigma}$ small enough. Since each $f_{\hat{\sigma}}$ is eventually positive, we see that there is a largest point $m_{\hat{\sigma}}$ such that $f_{\hat{\sigma}}(m_{\hat{\sigma}}) = 0$. Thus $f_{\hat{\sigma}}'(m_{\hat{\sigma}}) > 0$. Then the graph of $f_{\hat{\sigma}|[m_{\hat{\sigma}},\infty]}$ is also graphical over the x-axis, and defines the solutions u_σ for $\hat{\sigma} = 1/\sigma$. Thus, for σ sufficiently large, the function u_σ is increasing. □

As corollaries to the above, we now obtain the properties (i) and (iv) listed in Theorem 2.3.1.

Proof of Theorem 2.3.1(iv). Firstly we prove the second part of Theorem 2.3.1(iv), namely that the functions $u_\sigma : [0,\infty) \to \mathbb{R}^+$ are strictly increasing for any $\sigma > 0$.

By Lemma 2.3.14 this is true for large enough $\sigma > 0$. Assume there exists a $\sigma > 0$, and hence a largest $\sigma_0 > 0$, such that this is not true. Then there is a point $x_0 > 0$ such that $u_{\sigma_0}'(x_0) = 0$ and since σ_0 is the largest such, then by continuity of the solution in σ, we must have $u_{\sigma_0}(x_0) = \sqrt{2(n-1)}$ unless $x_0 = 0$ (since else by (2.6) there would be a point $x_0' \neq x_0$ such that $u_\sigma'(x_0') < 0$ violating the maximality). Thus u_{σ_0} is in that case is the cylinder, a contradiction. Since in the other case $u_{\sigma_0}'(0) = 0$, we get by reflection a smooth, entire graphical surface of revolution with $H \geq 0$ and thus by [Hu90] we get that u_{σ_0} is the cylinder, again a contradiction. Thus Lemma 2.3.14 extends to all $\sigma > 0$.

As a corollary, we get the convexity in Theorem 2.3.1(iv), i.e. that u_σ is strictly convex on $[0,\infty)$ for $\sigma > 0$. Namely, differentiating (2.11) twice, we obtain

$$\frac{u_\sigma''}{1+(u_\sigma')^2} = (n-1)\left[\frac{1}{u_\sigma(x)} - \int_x^\infty \frac{\frac{s}{2}(1+(u_\sigma'(s))^2)}{u(s)} e^{-\int_x^s \frac{z}{2}(1+(u_\sigma'(z))^2)dz} ds\right],$$

and hence $u_\sigma'' > 0$ on $[0,\infty)$, since $u_\sigma(s) > u_\sigma(x)$ for $s > x$. □

We also get the second property in Theorem 2.3.1(i).

Proof of Theorem 2.3.1(i). Using the integral identity in Lemma 2.3.6 for u_σ gives the following sharp bound on the value of $u(0)$, using l'Hôpital's rule:

$$u(0) \leq \frac{2(n-1)}{u(0)} \int_0^\infty \frac{s}{2}(1+(u'(s))^2) e^{-\int_0^s \frac{z}{2}(1+(u'(z))^2)dz} ds \leq \frac{2(n-1)}{u(0)},$$

with sharp inequality unless $u \equiv u(0)$, so that

$$u(0) \leq \sqrt{2(n-1)},$$

with equality if and only if u is the cylinder solution. □

2.4 Classification of Self-Shrinkers with Rotational Symmetry

In this section we prove Theorem 2.2.2, which we restate here for the convenience of the reader in the context of geodesics in the upper half plane (H^+, g_{Ang}).

Theorem 2.4.1. *Let γ be a complete embedded geodesic for the metric g_{Ang} in the upper half plane H^+. Then the following statement hold.*

(1) If γ is closed, it is a curve that intersects the r-axis exactly twice.

(2) If γ is not closed, it is either the r-axis, the line $r = \sqrt{2(n-1)}$, or the sphere $x^2 + r^2 = 2n$.

Corollary 2.4.2 (= Theorem 2.3.1(v)). *In particular this implies the remaining part (v) in Theorem 2.3.1, that the asymptotically conical ends are not parts of complete, embedded self-shrinkers.*

To facilitate the discussion, we say that a point in a smooth curve is "vertical" if the tangent line at that point is parallel to e_r, and "horizontal" if parallel to e_x, where $\{e_x, e_r\}$ is the unit basis corresponding to the Euclidean coordinates (x, r) on H^+. By the first and second quadrants, we as usual mean the sets $\{(x,r) \mid x, r > 0\}$ and $\{(x,r) \mid x < 0, r > 0\}$ contained in H^+, respectively. For a smooth curve $\gamma(t) = (x(t), r(t))$ parametrized by Euclidean arc length, we denote

$$\theta(t) = \arccos \dot{x}(t) = \arctan(\dot{r}(t)/\dot{x}(t)),$$

and we say that $\gamma(t)$ is a solution to (2.4), if the triple $(x(t), r(t), \theta(t))$ solves (2.4). We occasionally refer to such curves γ as "geodesics" for the metric g_{Ang} in H^+, although this is a slight abuse of terminology as solutions to (2.4) are parametrized by Euclidean arc length, not arc length with respect to g_{Ang}. We will make frequent use of the following elementary observation.

Lemma 2.4.3. *Let $\gamma(t) = (x(t), r(t))$ be a solution to (2.4). Then the functions $x(t)$ and $r(t) - \sqrt{2(n-1)}$ have neither positive minima nor negative maxima, and in particular these functions have different signs at successive critical points.*

Remark 2.4.4. *We remind the reader that the reflection $(x,r) \mapsto (-x,r)$ is a symmetry of the equation, a fact that will be used often in the below.*

The following lemma is of fundamental importance for our proof. Included in the statement of (2) is the (geometrically unsurprising) fact that geodesics for the metric g_{Ang} that leave the upper half plane through the x-axis do so orthogonally.

Lemma 2.4.5. *Let $\gamma : (a,b) \to H^+$ be a solution to (2.4), maximally extended as a graph over the x-axis. Then*

(1) There is $t \in (a,b)$ such that $x(t) = 0$.

(2) Assuming the existence and finiteness of the limit

$$x_b := \lim_{t \to b^-} x(t) < \infty,$$

the curve γ extends smoothly to $(a,b]$, with $\gamma(b)$ a vertical point. If $r(b) = 0$, the curvature of γ at $\gamma(b)$ (signed w.r.t. the orientation out of the half-plane) is $-\frac{x_b}{2n}$.

(3) There is at least one horizontal point in γ.

Proof. Assume the orientation of the curve is such that $\cos\theta = \dot{x}(t) > 0$ for $t \in (a,b)$. Set

$$\Lambda(t) := x(t)\sin\theta(t) - r(t)\cos\theta(t) = -\langle \gamma(t), \nu(t) \rangle, \tag{2.37}$$

where $\nu(t) = (-\sin\theta(t), \cos\theta(t))$ is the (leftward pointing w.r.t $\dot{\gamma}$) unit normal to γ. Then (2.4) becomes

$$\dot{\theta} = \frac{1}{2}\Lambda + \frac{n-1}{r}\cos\theta, \tag{2.38}$$

and Λ satisfies the equation

$$\dot{\Lambda} = \frac{1}{2}\Lambda\langle\gamma,\dot{\gamma}\rangle + \frac{n-1}{r}\cos\theta\langle\gamma,\dot{\gamma}\rangle. \tag{2.39}$$

We now investigate the oscillation behavior. Picking some $(x_0, u(x_0))$ on γ and integrating by parts in (2.6) gives

$$\arctan u' \,|_{x_0}^{x} = xu(x) - x_0 u(x_0) + \int_{x_0}^{x}\left[\frac{n-1}{u(s)} - u(s)\right]ds, \tag{2.40}$$

so that if we assume a lower (resp. upper) bound on $r(t) = u(x)$, as $x \to x_b$, it leads to a uniform upper (resp. lower) bound on $u'(x)$. Therefore by the mean value theorem (recall that by Lemma 2.4.3 successive points where $u'(x) = 0$ must occur on either side of the line $r = \sqrt{2(n-1)}$), such points must either eventually stop occurring as $t \to b$, or the limit $r(t) \to \sqrt{2(n-1)}$ must hold. But if $u'(x)$ eventually has a fixed sign, then the limit $\lim_{t\to b} r(t)$ also exists.

Thus if we denote by r_b^+ (resp. r_b^-) the limsup (resp. liminf) of $r(t)$ as $t \to b$, then we have shown that either:

(i) There is a limit: $\lim r_b = r_b^+ = r_b^-$, or

(ii) Both $r_b^+ = \infty$ and $r_b^- = 0$.

But the second situation does not happen: Case (ii) implies that the straight line segment $\{(x_b, t) : t > 0\}$ satisfies (2.38), and thus we conclude $x_b = 0$. But from (ii) we thus also obtained a positive solution $g(r)$ to the linearized equation at the r-axis (2.30), which gives a contradiction similarly to in Corollary 2.3.13.

Now, it is easy to see that the limit r_b is finite: If $x_b \le 0$, then assuming both $r(t) > \sqrt{2(n-1)}$ and $\dot{r}(t) > 0$ then (2.38) gives that $\dot{\theta}(t) < 0$, which immediately bounds r_b away from ∞.

On the other hand, assuming still $r_b = +\infty$ but $x_b > 0$, then (again by Lemma 2.4.3) eventually $\dot{r}(t) > 0$ as $t \to b$, and hence eventually $\langle\gamma,\dot{\gamma}\rangle > 0$. There are also choices of $t_0 \in (a,b)$ arbitrarily close to b with $\Lambda(t_0) > 0$, since else for some fixed $x^0 < x_b$ we would have had $xu'(x) - u(x) < 0$ for x in an interval (x^0, x_b), leading to the contradictory bound:

$$r_b = \lim_{x\to x_b} u(x) \le x_b \frac{u(x^0)}{x^0} < \infty. \tag{2.41}$$

Now, since $\langle\gamma(t),\dot{\gamma}(t)\rangle > 0$, the property $\Lambda(t) > 0$ is propagated on $t \in (t_0, b)$, by (2.39). Dividing (2.39) by Λ and integrating over (t_0, t) gives that

$$\Lambda(t) > e^{|\gamma(t)|^2/4 - |\gamma(t_0))|^2/4}\Lambda(t_0). \tag{2.42}$$

However, combining (2.42) with (2.38) and $|\gamma(t)| \to +\infty$ gives that

$$\theta(t) - \theta(t_0) \to +\infty \tag{2.43}$$

as $t \to b$, contradicting that γ is graphical over the x-axis.

Thus the limit r_b exists and is a non-negative real number. If it is positive, then $\gamma(t)$ remains in a relatively compact subset of the upper half plane H^+ as $t \to b$. Equation (2.38) then gives uniform C^k-bounds on $\gamma(t)$ and the desired smooth extension to a vertical endpoint, giving the conclusion (2) in that case. Lemma 2.4.3 then implies that $x_b > 0$.

If on the other hand $r_b = 0$, then we claim that $\theta(t)$ decreases to $-\pi/2$ monotonically as t increases to b. To see this, note first that $\theta(t)$ cannot remain bounded away from $-\pi/2$ as $t \to b$, since otherwise (2.38) and $r(t) \to 0$ give that

$$\dot\theta(t) \geq \frac{\delta}{r(t)} - 2x_b, \quad \text{where} \quad \delta := \inf_{t \nearrow b}(\cos\theta(t)), \tag{2.44}$$

for t close enough to b. This, after using that $\dot r(t) \geq -1$ and integrating, gives

$$\theta(t_2) - \theta(t_1) \geq \log\left(\delta\frac{r(t_1)}{r(t_2)}\right) - 2x_b(t_2 - t_1),$$

for any $t_2 > t_1$, and implicitly bounds $r(t)$ away from zero as $t \to b$, a contradiction.

In particular there must be points arbitrarily close to b s.t. $\dot\theta > 0$. Now, $r(t) \to 0$ and Lemma 2.4.3 imply that $\dot r(t) < 0$ for $b - t$ sufficiently small, and differentiating (2.38) gives that

$$\ddot\theta(t) = -\frac{n-1}{r^2}\dot r(t)\cos\theta(t), \tag{2.45}$$

at times t for which $\dot\theta(t) = 0$, if there were any. Thus it follows that in fact $\dot\theta(t) < 0$ for all $b - t$ sufficiently small, and we have proved that $\theta(t) \searrow -\pi/2$ as $t \nearrow b$.

Finally, applying l'Hôpital's rule to (2.38) gives that

$$\lim_{t \to b^-} \dot\theta(t) = -\frac{x_b}{2n}.$$

so that $\gamma(t)$ extends with two derivatives to $(a, b]$ with $x_b > 0$. The higher regularity then follows immediately, giving (2) also in this case.

In all cases, we see that $x_b > 0$. By symmetry, we get that also $x_a < 0$, which gives claim (1).

To see (3), suppose first that (a, b) is a bounded interval. Note that (3) is clear if $\lim\theta(t) \in \{\pm\pi/2\}$ is different at the two endpoints. We may thus assume, with our chosen orientation, that $\lim_{t\to a+}\theta(t) = \lim_{t \to b-}\theta(t) = \pi/2$, and we argue by contradiction.

By (1), there is a $t_0 \in (a, b)$ so that $x(t_0) = 0$. If $r(t_0) > \sqrt{2(n-1)}$, then (2.38) gives that $\dot\theta(t_0) < 0$. Differentiating (2.38) and evaluating at a point t for which $\dot\theta(t) = 0$ gives that

$$\ddot\theta(t) = -\frac{n-1}{r^2(t)}\dot r(t)\cos\theta(t) < 0, \tag{2.46}$$

so that $\theta(t)$ is bounded away from $\pi/2$ as $t \to b$, a contradiction. If $r(t_0) < \sqrt{2(n-1)}$, then (2.38) gives $\theta(t_0) > 0$ and we apply a similar argument as before to contradict the assumption $\lim_{t\to a+}\theta(t) = \pi/2$. In the case of equality $r(t_0) = \sqrt{2(n-1)}$, we have $\dot{\theta}(t_0) = 0$, and we refer to (2.46) to obtain that $\dot{\theta}(t) < 0$ for $t > t_0$, from which as before we obtain a contradiction.

If $b = \infty$ and a is finite, then Theorem 2.3.1 gives that γ contains the graph of a function u_σ for some $\sigma > 0$. In particular, we have that, with t_0 as before, $r(t_0) < \sqrt{2(n-1)}$, and we argue as before that $\dot{r}(t) = 0$ for some $t \in (a, t_0)$.

Finally, if $(a,b) = \mathbb{R}$, then Theorem 2.3.1 gives that γ coincides with the line $r = \sqrt{2(n-1)}$ for which (3) clearly holds. □

Proposition 2.4.6. *Let γ be a complete solution to (2.4), such that one of the following statements hold:*

(1) γ contains 7 vertical points.

(2) γ is closed and contains two vertical points in the first quadrant.

(3) γ is not closed and contains one interior vertical point.

Then γ is not embedded.

Proof of Proposition 2.4.6(1). Consider a segment of γ containing seven consecutive vertical points, which we identify with the interval $[1,7] \subset \mathbb{R}$ such that the vertical points correspond to integer values of the parameter. The vertical points will thus be denoted by $(x(k), r(k))$ for $k = 1,\dots,7$. Then by Lemma 2.4.3, after possibly reflecting γ through the r-axis, we can assume that $x(k)$ is positive for k odd and negative otherwise. Lemma 2.4.5(3) then gives the existence of a horizontal point in each segment $[k, k+1]$, $k = 1,\dots,6$, which we identify with the points $k + \frac{1}{2}$, $k = 1,\dots,6$. Lemma 2.4.3 implies that both the segments $[2 + \frac{1}{2}, 3 + \frac{1}{2}]$ and $[4 + \frac{1}{2}, 5 + \frac{1}{2}]$ intersect the line $r = \sqrt{2(n-1)}$, so assume, after possibly reversing orientation that $[2 + \frac{1}{2}, 3 + \frac{1}{2}]$ intersects to the left of $[4 + \frac{1}{2}, 5 + \frac{1}{2}]$. Take γ_1 to be the segment $[2+\frac{1}{2}, 3+\frac{1}{2}]$ and take γ_2 to be the segment $[3+\frac{1}{2}, 6+\frac{1}{2}]$. Note that on γ_1 the outward pointing unit tangent is $-e_x$ at each endpoint, while on γ_2 the outward pointing unit tangent is e_x.

We now translate the curve γ_1 in the positive e_x direction until a point of first contact with γ_2. Note that such a point occurs, since both segments intersect the line $r = \sqrt{2(n-1)}$, with γ_1 intersecting to the left of γ_2, and that this point of first contact occurs away from the endpoints of both segments, and more generally does not occur at any horizontal point (since in particular the convexity near such a point is preserved under translation, it could not be a first intersection). Let $\hat{\gamma}_1 = \gamma_1 + ce_x$ denote the segment for which first point of contact occurs. Appealing to system (2.4) we get that

$$\dot{\theta}_{\gamma_2}(\hat{p}) - \dot{\theta}_{\hat{\gamma}_1}(\hat{p}) = \frac{c}{2}\sin\theta \tag{2.47}$$

holds at the point of first contact $\hat{p}$, and where $\theta = \theta_{\gamma'}(\hat{p}) = \theta_{\hat{\gamma}_1}(\hat{p})$, which is a contradiction. □

Proof of Proposition 2.4.6(2). For simplicity of description, we identify γ with the unit circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Suppose now that there are two vertical points in γ in the first quadrant, which after possibly reparametrizing we identify with the points $[0]$ and $[\pi]$ in $\mathbb{S}^1$. By assumption, we have that $x([0])$ and $x([\pi])$ are positive. Lemma 2.4.3 then gives an additional vertical point on each arc $([0],[\pi])$ and $([\pi],[0])$, which we identify with the points $[\frac{\pi}{2}]$ and $[\frac{3\pi}{2}]$ respectively. Now Lemma 2.4.5(3) gives that there are horizontal points along the arcs $([0],[\frac{\pi}{2}])$, $([\frac{\pi}{2}],[\pi])$, $([\pi],[\frac{3\pi}{2}])$ and $([\frac{3\pi}{2}],[0])$, which we identify with the four points $[\frac{(2k-1)\pi}{4}]$, $k = 1,\dots 4$. Lemma 2.4.3 gives that both arcs $[[\frac{7\pi}{4}],[\frac{\pi}{4}]]$ and $[[\frac{3\pi}{4}],[\frac{5\pi}{4}]]$ intersect the line $r = \sqrt{2(n-1)}$, and after possibly relabeling, we can assume that $[[\frac{7\pi}{4}],[\frac{\pi}{4}]]$ intersects to the left of $[[\frac{3\pi}{4}],[\frac{5\pi}{4}]]$. Assume now that the arc $[[\frac{7\pi}{4}],[\frac{\pi}{4}]]$ contains no vertical points other than $[0]$, and take $\gamma_1 = [[\frac{7\pi}{4}],[\frac{\pi}{4}]]$ and $\gamma_2 = [[\frac{\pi}{4}],[\frac{7\pi}{4}]]$. We then translate γ_1 until a point of first contact with γ_2 and, arguing as in the proof of Proposition 2.4.6(1) obtain a contradiction. □

Proof of Proposition 2.4.6(3). Identify γ with an interval (a,b) under a Euclidean arc length parametrization, and assume first that a and b are finite. Proposition 2.4.6(2) gives that γ contains a finite number of vertical points. Lemma 2.4.5 then gives that γ extends to the closed interval $[a,b]$ with vertical endpoints, and the assumption that γ is complete in H^+ gives that these endpoints are contained in the x-axis.

Now, suppose γ contains an interior vertical point $c \in (a,b)$, and assume that it is in the second quadrant. By Lemma 2.4.5(3) the arcs $[a,c]$ and $[c,b]$ each contain horizontal points p_1 and p_2, respectively, and consequently both intersect the line $r = \sqrt{2(n-1)}$. Assume $[a,c]$ intersects to the left of $[c,b]$, and assume $[p_1,p_2]$ contains no vertical points other than c. Then set $\gamma_1 = [a,p_1]$ and set $\gamma_2 = [p_1,b]$.

Note that γ_1 and γ_2 both intersect the line $r = \sqrt{2(n-1)}$, and are compact. Moreover, the outward pointing tangent to γ_1 at p_1 is $-e_x$, and the outward pointing tangent to γ_2 at p_1 is e_x. As before, we translate γ_1 in the positive e_x direction until a point of first contact with γ_2. By construction, this point of first contact cannot occur at p_1 (or its translated version). Moreover, by Lemma 2.4.5(2) it cannot occur at the endpoints of γ_1 and γ_2 contained in the x-axis. Hence, it is interior and non-transversal, and we obtain a contradiction as in the proof of Proposition 2.4.6(1) and (2).

Assume now that both a and b are infinite and identify γ with the real line under a Euclidean arc length parametrization. Assume that 0 is a vertical point in the second quadrant. Then by the completeness of γ, the arcs $(-\infty,0]$, $[0,\infty)$ contain geodesic segments, maximally extended as graphs over the x-axis, and by Lemma 2.4.5 both contain horizontal points p_1 and p_2, respectively. Assume as before that $[p_1,p_2]$ contains no vertical points other than 0. By Proposition 2.4.6(2), γ has a finite number of vertical points, and thus decomposes into a finite number of geodesic segments, maximally extended as graphs over the x-axis. Then since $(-\infty,0]$ and $[0,\infty)$ have infinite Euclidean length, Lemma 2.4.5 and Theorem 2.3.1

imply that they contain the segments

$$\{(x, u_{\sigma_i}(x)) | x \geq 0\}, \quad i = 1, 2,$$

for distinct positive σ_1 and σ_2, respectively, after possibly reflecting through the r-axis. Thus, both $(-\infty, 0]$ and $[0, \infty)$ intersect the line $r = \sqrt{2(n-1)}$ by Theorem 2.3.1, so assume that $(-\infty, 0]$ does so to the left of $[0, \infty)$. We then set $\gamma_1 := (-\infty, p_1]$ and $\gamma_2 = [p_1, \infty)$. As before γ_1 and γ_2 intersect the line $r = \sqrt{2(n-1)}$, the outward pointing tangent to γ_1 at p_1 is $-e_x$, the outward pointing tangent to γ_2 at p_1 is e_x, and both curves γ_1 and γ_2 are properly embedded and separated by a positive distance (since σ_1 and σ_2 are distinct). We then translate γ_1 until a point of first contact with γ_2 and obtain a contradiction as in the previous case.

Finally, the case where the $(a, b) = [0, \infty)$, is handled exactly as in the previous cases, and consequently we omit the details. □

We can now prove Theorem 2.4.1.

Proof of Theorem 2.4.1. Note that by Proposition 2.4.6 any non-closed embedded geodesic γ different from the r-axis cannot contain any interior vertical points and thus is globally given by the graph of a function $u(x)$ satisfying (2.6) on an open interval I away from its endpoints. Let Σ denote the surface of revolution determined by γ. Then Σ is smooth and embedded and satisfies the self-shrinker equation (2.1). Lemma 2.3.4 gives that $u'(x)x - u(x) < 0$, for all $x \in I$. This is in turn equivalent to the positivity of the mean curvature of Σ with respect to the downward pointing unit normal (with respect to the axis of rotation). Huisken's classification of mean convex self-shrinkers [Hu90] then implies that Σ is either the round sphere of radius $\sqrt{2n}$, or the cylinder of radius $\sqrt{2(n-1)}$.

If γ is closed, then Proposition 2.4.6 gives that it has at most two vertical points, and Lemma 2.4.3 says that each is in a different quadrant of H^+. This concludes the proof. □

2.5 Appendix

We include for completeness a proof of the smoothness of fixed points that we used in Lemma 2.3.10.

Proof of Lemma 2.3.10. Let σ be fixed, and let $x_{\sigma+h}, x_\sigma$ be fixed points for Φ_σ and $\Phi_{\sigma+h}$. Then

$$\begin{aligned} |x_{\sigma+h} - x_\sigma| &= |\Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_\sigma)| \\ &\leq |\Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_{\sigma+h})| + |\Phi_\sigma(x_{\sigma+h}) - \Phi_\sigma(x_\sigma)| \\ &\leq \left|\frac{\partial \Phi_\sigma}{\partial \sigma}(\sigma, x_{\sigma+h})\right| h + \tau |x_{\sigma+h} - x_\sigma| + o(h). \end{aligned}$$

This gives that the x_σ are at least Lipshitz continuous functions of σ. To show differentiability, we again write

$$
\begin{aligned}
x_{\sigma+h} - x_\sigma &= \Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_\sigma) \\
&= \Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_{\sigma+h}) + \Phi_\sigma(x_{\sigma+h}) - \Phi_\sigma(x_\sigma) \\
&= D_{x_\sigma}\Phi_\sigma(x_{\sigma+h} - x_\sigma) + O(|x_{\sigma+h} - x_\sigma|^2) + \Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_{\sigma+h}).
\end{aligned}
$$

Rearranging terms, we see that

$$
\left(I - D_{x_\sigma}\Phi_\sigma - O(|x_{\sigma+h} - x_\sigma|)\right)(x_{\sigma+h} - x_\sigma) = \Phi_{\sigma+h}(x_{\sigma+h}) - \Phi_\sigma(x_{\sigma+h}).
$$

Dividing by h above and sending $h \to 0$, we get

$$
\frac{dx_\sigma}{d\sigma} = \left(I - D_{x_\sigma}\Phi_\sigma\right)^{-1}\frac{\partial\Phi}{\partial\sigma}(\sigma, x_\sigma). \tag{2.48}
$$

Note that the operator $A = I - D_{x_\sigma}\Phi_\sigma$ is invertible, since the fact that Φ_σ is a contraction gives $\|D_x\Phi\| < 1$.

Note that the formula for the derivative (2.48) gives that the fixed points x_σ depend smoothly on the parameter σ, since the right hand side of σ may be differentiated in σ if the mappings Φ_σ are smooth. □

Chapter 3

Non-Compact Complete Embedded Mean Curvature Flow Self-Shrinkers With Asymptotically Conical Ends

We give the first rigorous construction of complete, embedded self-shrinking hypersurfaces under mean curvature flow, since Angenent's torus in 1989. The surfaces exist for any sufficiently large prescribed genus g, and are non-compact with one end. Each has $4g+4$ symmetries and comes from desingularizing the intersection of the plane and sphere through a great circle, a configuration with very high symmetry.

Each is at infinity asymptotic to the cone in $\mathbb{R}^3$ over a $2\pi/(g+1)$-periodic graph on an equator of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$, with the shape of a periodically "wobbling sheet". This is a dramatic instability phenomenon, with changes of asymptotics that break much more symmetry than seen in minimal surface constructions.

The core of the proof is a detailed understanding of the linearized problem in a setting with unbounded geometry, leading to special PDEs of Ornstein-Uhlenbeck type with fast growth on coefficients of the gradient terms. This involves identifying new, adequate weighted Hölder spaces of asymptotically conical functions in which the operators invert, via a Liouville-type result with precise asymptotics.

3.1 Introduction

In studying the flow of a hypersurface by mean curvature in Euclidean n-space as well as in general ambient Riemannian n-manifolds (M^n, g), $n \geq 3$, the basic "atoms" of singularity theory are the self-similar surfaces in $\mathbb{R}^n$, viz. solitons moving by an ambient conformal Killing field, and of these the self-shrinkers are the most important. Taking center stage when identified by Huisken in 1988 (and the compact $H \geq 0$ case classified: Round spheres; see [Hu90]) as the surfaces for which equality holds in his celebrated monotonicity formula, the self-shrinkers arise as blow-up limits when assuming natural curvature bounds.

It is notable that even when $n = 3$ only a few complete, embedded self-shrinking surfaces in $\mathbb{R}^3$ are to this date rigorously known: Flat planes, round cylinders, round spheres and a (not round-profile) torus of revolution discovered by Angenent in [An89] (this list exhausts the rotationally symmetric examples, although the uniqueness of the torus is still open; see [KMø11]). Note also that several results involving self-shrinkers in some generality have appeared, most prominently a smooth compactness theorem (for closed, fixed genus surfaces [CM7]) and a theory of generic singularities of Colding-Minicozzi, including classification of all $H \geq 0$ complete hypersurfaces (see [CM8] and [DX11]). See also [LS] and [Wa11] for other uniqueness results.

The self-shrinker equation is a nonlinear partial differential equation of mean curvature type, indeed the self-shrinkers are minimal with respect to a certain Gaussian metric on Euclidean space, and as such the current status of known examples can be likened to the situation before Scherk's, Riemann's and Enneper's minimal surface examples, when only rotationally symmetric surfaces were known. In recent years, several authors ([Tr96], [Ka97], [Ka05], [Ka11]) have, via singular perturbation techniques, greatly expanded upon the list of rigorously known minimal surfaces in $\mathbb{R}^3$. Since the local considerations involved in the constructions would work in some generality (see [Ka05] and [Ka11]), it has long been expected that such constructions could work for self-similar surfaces under mean curvature flow, and indeed there are constructions for the self-translating case in the interesting work by X.H. Nguyen (see [Ng09]-[Ng10]).

The existence of self-shrinkers with the topology we consider in this paper was conjectured by Tom Ilmanen in 1995 (from numerics, using Brakke's surface evolver; see [Il95]), while their asymptotic geometry was not clear at that point.

Our main theorem is the following:

Theorem 3.1.1. *For every large enough integer g there exists a complete, embedded, orientable, smooth surface $\Sigma_g \subseteq \mathbb{R}^3$, with the properties:*

- *(i) Σ_g is a mean curvature self-shrinker of genus g.*
- *(ii) Σ_g is invariant under the dihedral symmetry group with $4g+4$ elements.*
- *(iii) Σ_g has one non-compact end, and separates $\mathbb{R}^3$ into two connected components.*
- *(iv) The end is outside some Euclidean ball a graph over a plane, asymptotic to the cone on a non-zero vertical smooth (4g+4)-symmetric graph over a great circle in $\mathbb{S}^2$ (hence the visual appearance of a "wobbling sheet").*
- *(v) Inside any fixed ambient ball $B_R(0) \subseteq \mathbb{R}^3$, the sequence $\{\Sigma_g\}$ converges in Hausdorff sense to the union $\mathbb{S}^2 \cup \mathcal{P}$, where $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^3$. In fact, the bounds*

$$d_H\Big[\Sigma_g \cap B_R(0), (\mathbb{S}^2 \cup \mathcal{P}) \cap B_R(0)\Big] \leq C\frac{R}{g}, \tag{3.1}$$

on the Hausdorff distance d_H hold for some constant $C > 0$. The convergence is furthermore locally smooth away from the intersection circle.

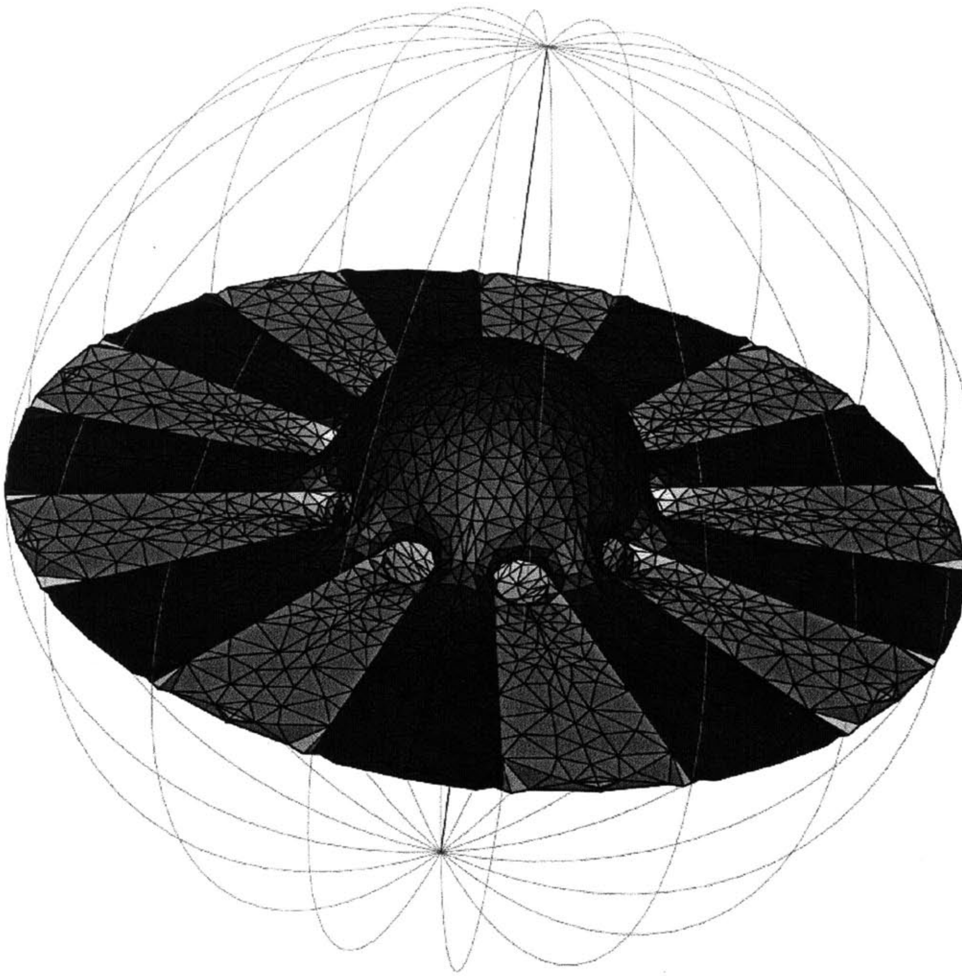

Figure 3-1: Tom Ilmanen's conjectural shrinker of genus 8 with 18 Scherk handles.

Corollary 3.1.2. *Euclidean flat cylinders over Σ_g are shrinkers. So, in any fixed dimension $n \geq 2$ we obtain self-shrinking hypersurfaces $\Sigma_g^n = \Sigma_g \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^{n+1}$, with arbitrary large first Betti number $b_1(\Sigma_g \times \mathbb{R}^{n-2}) = b_1(\Sigma_g) = g$.*

The general approach of this article is the same as that of [Ka97], which follows the general methodology developed in [Ka95]. Our construction is analogous to a specific instance of the main theorem in [Ka97], the case of a catenoid intersecting a plane through its waist, which is simpler than the general case because of the extra symmetry. On the other hand, we must contend with major analytic difficulties arising from the unbounded nature of the self-shrinker equation, which do not arise in minimal and constant mean curvature constructions.

To look further into the analytical difficulties faced here, it is instructive to use the mentioned characterization of self-shrinkers (which shrink towards the origin, with scaling factor $\sqrt{2(1-t)}$): Minimal surfaces $S \subseteq \mathbb{R}^3$ w.r.t. the conformal metric

$g_{ij} = e^{-|x|^2/4}\delta_{ij}$, where $|x|$ is the distance to the origin and δ_{ij} is the Euclidean standard metric. All previous desingularization constructions – and indeed much of geometric analysis – rely on some kind of reasonably bounded geometry such as for example geodesic completeness, curvature bounds, or even stronger assumptions such as asymptotic flatness. We must however here face that the metric is geodesically incomplete (non-extendible: the distance to infinity is finite) and the Ricci curvature of a plane through the origin in the unit normal direction, respectively the Gauss curvature of the induced metric on such a plane, are (see Appendix C):

$$\mathrm{Ric}(\vec{\nu},\vec{\nu}) = e^{|x|^2/4}(1-|x|^2/16) \to -\infty, \quad \text{for} \quad |x| \to \infty,$$
$$K_{(\mathbb{R}^2,g)} = \frac{1}{2}e^{|x|^2/4} \to +\infty, \quad \text{for} \quad |x| \to \infty.$$

It should hence come as no surprise that the analysis we need to perform could not follow from any very general principle, and in fact this paper also gives the first successful example of a construction for such an unbounded geometry. Our new (anisotropically) weighted Hölder spaces and accompanying Liouville-type result and global Schauder-type estimates for the exterior linear problem of Ornstein-Uhlenbeck type, that are pivotal to the completion of the construction, arise from homogeneity properties of the linearized operator, which in turn lend their origin to the parabolic self-similar nature: It is the sum of homogeneous operators, with a homogeneity zero term which annihilates cones. We consider the problem of solving the equation for homogeneous functions and find good (sharp) choices for weighted Hölder norms, and then proceed for general functions with those very same spaces.

Note that the global Schauder estimates have no obvious extensions to general Laplace-type operators under the same growth rates on the coefficients, and there are counterexamples by Priola for a very similar equation (see [Pr]).

It is fruitful to compare our construction with that of desingularizing, in the $H \equiv 0$ case, the intersection of a catenoid with a plane through the waist, leading to the Costa-Hoffmann-Meeks surfaces (of high genus). In that construction the plane remains flat, and one automatically gets improved power of decay of the constructed minimal surfaces back to the original plane, namely the decay rate is $1/|x|^{g+1}$ as $|x| \to \infty$. In our construction no such improvement shows up, the self-shrinkers constructed have regardless of g the (likely sharp) asymptotics:

$$\sigma(\theta)|x| + O(|x|^{-1}), \quad |x| \to \infty. \tag{3.2}$$

Another difference from the previously known constructions for minimal surfaces is that the surfaces we construct must be entropy unstable (since by [CM8] the only stables ones are of the form $S^{n-k} \times \mathbb{R}^k, k = 0,\ldots,n$), and this is another way of viewing some of the complications that arise here. However, it is from the desingularization viewpoint not presence of the instability per se that is the problem, it is the severe way in which it happens, witnessed by Equation (3.2): Imposing ever so much dihedral symmetry never renders it negligible.

Finally, we will mention that X.H. Nguyen via nonlinear parabolic methods has studied a related, truncated nonlinear exterior problem for the self-shrinker equation and obtained existence results (see [Ng06]-[Ng07]). Also, L. Wang has announced interesting existence and uniqueness result for exterior graphs with prescribed cones at infinity (see [Wa11]), which provides separate evidence of the dramatic change of asymptotics of the non-compact ends, i.e. that our examples are not asymptotic to planes.

After this work was completed, we learned of a preprint by X.H. Nguyen [Ng11] which announces results very similar to ours.

3.2 Overview of the paper

The basic philosophy of the desingularization procedure is as follows: Consider the initial configuration of a plane intersecting a sphere through a great circle. For each τ with $\tau^{-1} = k \in \mathbb{N}$ a positive integer, define a one parameter family of surfaces $\mathcal{M}[\tau, \theta]$ that serve as approximate solutions to the self-shrinker equation. The surfaces $\mathcal{M}[\tau, \theta]$ are invariant under the action of the dihedral group with $4k$ elements, and under various normalizations converge either to the initial configuration or to Scherk's singly-periodic surface Σ_0 as the parameters τ and θ tend to zero.

On each of these surfaces, we consider graphs of small functions u, and produce via an incarnation of Newton's method, here Schauder's fixed point theorem, a pair (θ^*, u^*) such that the graph over $\mathcal{M}[\tau, \theta^*]$ by u^* solves the self-shrinker equation exactly. Naturally, to apply the Schauder fixed point theorem, one needs to first understand the linearized equation on these surfaces, and to do this one needs to understand the linearized equation on the limits under both normalizations; that is to say on the initial configuration and on Scherk's surface. That is, we need to solve the equation $\mathcal{L}u = E$ on the initial surface $\mathcal{M}[\tau, \theta]$ with reasonable estimates, where $\mathcal{L}$ is the linearized operator for the self-shrinker equation (note that the study of this operator played an important role in [CM7]-[CM9]) and the function E is the initial error in the self-shrinker equation on $\mathcal{M}[\tau, \theta]$.

On the pieces of the initial configuration (that is, the surfaces with boundary determined by the intersection circle), we prove that the linearized equation is always solvable with Dirichlet boundary conditions (and here we are, on the outer plane, forced to allow a dramatic change of asymptotics to include conical functions that are oscillatory in the angular variable). Near the intersection circle, the linearized equation turns out to be a perturbation of the stability operator on Scherk's surface.

The linearized equation on Scherk's surface is not solvable, with appropriate bounds on the norm of the inverse, in any bounded function space, in general, due to the persistence of a one-dimensional kernel spanned by a translational Killing field. But as long as the inhomogeneous term E is "orthogonal" to this kernel, we can solve the equation in a weighted Hölder space with exponential decay. The decay then allows a solution to be patched up globally to a solution on the entire initial surface. The role of the parameter θ in the surfaces $\mathcal{M}[\tau,\theta]$ is then to arrange for the initial error term E to be orthogonal to the kernel. As θ changes, two of the pieces of $\mathcal{M}[\tau,\theta]$ move within a family of perturbed cap-shaped self-shrinkers near the round spherical caps. Note therefore that the role of the chosen θ^* in this problem is of a more technical nature (unlike for example the case of catenoidal ends for the $H \equiv 0$ constructions in [Ka97], where it entails an important global change of asymptotics in itself).

The paper is structured as follows:

Section 3 sets notation and conventions for frequently used basic objects, while Section 4 discusses basic properties of the self-shrinker equation and its linearization.

In Section 5, the initial surfaces $\mathcal{M}[\tau,\theta]$ are introduced, and their basic properties – smoothness in parameters, symmetries – are established.

Section 6 gives necessary estimates for the mean curvature of the desingularizing surfaces $\Sigma[\tau,\theta]$ and its variation under the θ parameter.

In Section 7, the linearized operator $\mathcal{L}$ on the curled up Scherk belt $\Sigma[\tau,\theta]$ is studied. We prove that the operator is invertible as a map between Hölder spaces with decay, modulo a one-dimensional cokernel, and we show that this cokernel can indeed be geometrically generated by varying the θ parameter.

In Section 8, we study the exterior Ornstein-Uhlenbeck problem and identify the correct weighted Hölder cone spaces which have all desired properties (such as a compact inclusion hierarchy), and in which we invert the linearized operator.

In Section 9, the patching up of solutions of the linear problem on the various pieces of the initial surfaces $\mathcal{M}[\tau,\theta]$ to a global solution is undertaken.

In Section 10, we verify the important fact that the nonlinear part of the problem closes up in the norms from Section 8, that is we prove the quadratic improvement required for Newton's method to be applicable.

Finally, in Section 11 we then complete the argument by setting up and carrying out the Schauder fixed point procedure. The Appendix at the end records various computations which were needed throughout.

3.3 Notation and Conventions

Throughout $\mathbb{R}^3$ will denote Euclidean 3-space, $\vec{X}$ will denote a point in $\mathbb{R}^3$, (x,y,z) the Cartesian coordinates of the point, and $\{\vec{e}_x,\vec{e}_y,\vec{e}_z\}$ the associated standard basis, so that $\vec{X}=(x,y,z)=x\vec{e}_x+y\vec{e}_y+z\vec{e}_z$. We denote by $\mathcal{P}_{xy}$, $\mathcal{P}_{yz}$, and $\mathcal{P}_{xz}$ the xy-, yz-, and xz-coordinate planes respectively.

We adopt the convention in this article that for a surface S, all associated geometric objects and quantities will bear "S" as a subscript, with the exception of Scherk's singly-periodic surface Σ_0 and the surfaces $\Sigma[\tau,\theta]$ defined in Section 3.5 . Objects associated with Σ_0 will at times simply bear the subscript "0". In most cases, the surfaces $\Sigma[\tau,\theta]$ will appear with the τ and θ arguments suppressed - so, for example, as simply Σ - and their associated quantities will be identified without subscript. The reader should take care to distinguish subscripts from superscripts, as "0" will appear throughout the article as superscript as well.

We denote by $\vec{\nu}_S$ the Gauss map of an oriented surface S. Given a function $f : S \to \mathbb{R}$ on a surface S, we use the shorthand $\{S : f \leq 0\}$ to denote the set $\{p \in S : f(p) \leq 0\} \subset S$, and likewise for "$\geq$". Note that under appropriate assumptions on f, $\{S : f \leq 0\}$ is a smooth surface with smooth (possibly empty) boundary, and we view $\{S : f \leq 0\}$ as inheriting all geometric quantities from S – i.e. first and second fundamental forms – via the inclusion mapping. Also, for a function f, we denote by S_f the normal graph of f over S. Note that when f and S are class $C^{k,\alpha}$ and f is sufficiently small, then S_f is a $C^{k-1,\alpha}$ surface naturally parametrized by S.

Geometric objects defined on any of the surfaces Σ given in Section 3.5 may be viewed as objects on Σ_0 via the map $\mathcal{Z} : \Sigma_0 \to \Sigma$.

We denote by H^+ the upper half plane $\{(s,z) : s > 0\}$ and by $\mathcal{C}$ its quotient (a cylinder) under the action $z \mapsto z + 2\pi$. Throughout this article, we fix a smooth, non-decreasing function $\psi_0 : \mathbb{R} \to \mathbb{R}$ which vanishes on $(-\infty, 1/3)$ and has $\psi_0 \equiv 1$ on $(2/3, \infty)$. Also, we let $\psi[a,b] : \mathbb{R} \to [0,1]$ be

$$\psi[a,b](s) := \psi_0\left(\tfrac{s-a}{b-a}\right),$$

so that $\psi[a,b]$ transitions from 0 at a to 1 at b.

We will for the compact pieces in our construction work in the usual weighted Hölder spaces $C^{k,\alpha}(S, g_S, f)$ on Riemannian surfaces (S, g_S), defined by finiteness of the corresponding norms

$$\left\| u : C^{k,\alpha}(S, g_S, f) \right\| := \sup_{x \in S} \frac{1}{f(x)} \left\| u : C^{k,\alpha}(S \cap B(x), g_S) \right\|, \tag{3.3}$$

with weight function $f : S \to \mathbb{R}$, where g_S is the metric for which the usual $C^{k,\alpha}$-norm is taken and $B(x)$ the geodesic ball of radius 1 centered at x. When the metric is understood, we sometimes drop it from the notation writing $C^{k,\alpha}(S, f) = C^{k,\alpha}(S, g_S, f)$.

3.4 The Self-Shrinker Equation

Recall that the PDE to be satisfied for a smooth oriented surface $S \subseteq \mathbb{R}^3$ to be a self-shrinker (shrinking towards the origin with singular time $T = 1$) is

$$H_S(\vec{X}) - \tfrac{1}{2}\vec{X} \cdot \vec{\nu}_S(\vec{X}) = 0, \tag{3.4}$$

for each $\vec{X} \in S$, where by convention $H_S = \sum_1^n \kappa_i$ is the sum of the signed principal curvatures w.r.t. the chosen normal $\vec{\nu}_S$ (i.e. $H = 2$ for the sphere with outward pointing $\vec{\nu}$). Such surfaces shrink by homothety towards the origin under flow by the (orientation-independent) mean curvature vector $\vec{H} = -H\vec{\nu}$, by the factor $\sqrt{2(1-t)}$. In particular, we have normalized Equation (3.4) so that $T = 1$ is the singular time.

The surface $\tilde{S}$ obtained by dilating a self-shrinker S about the origin by a factor of τ^{-1} satisfies the corresponding rescaled equation

$$H_{\tilde{S}}(\vec{X}) - \tfrac{1}{2}\tau^2 \vec{X} \cdot \vec{\nu}_{\tilde{S}}(\vec{X}) = 0. \tag{3.5}$$

For a smooth normal variation $\vec{X}_t$ determined by a function u via $X_t = X_0 + tu\vec{\nu}_{\tilde{S}}$, where $\vec{X}_0$ parametrizes $\tilde{S}$, the pointwise linear change in (minus) the quantity on the left hands side in (3.5) is given by the stability operator (see the Appendix, and also [CM7]-[CM8] for more properties of this operator)

$$\mathcal{L}_{\tilde{S}} u = \Delta_{\tilde{S}} u + |A_{\tilde{S}}|^2 u - \tfrac{1}{2}\tau^2 \left(\vec{X} \cdot \nabla_{\tilde{S}} u - u\right). \tag{3.6}$$

Because at times we want to treat Equation (3.5) as a perturbation of the mean curvature equation, we isolate the part of the linear change due to varying the mean curvature of S and set

$$\mathcal{L}_S^0 = \Delta_S + |A_S|^2. \tag{3.7}$$

Note that Equation (3.4) and its dilated version (3.5) are invariant under the orthogonal group $O(3)$.

3.5 The Initial Surfaces

In this section we describe in detail the construction of the initial surfaces $\mathcal{M}[\tau,\theta]$, depending on parameters τ and θ which we assume satisfy

$$0 < \tau \le \delta_\tau, \quad |\theta| \le \delta_\theta,$$

throughout for appropriate constants that will later be chosen. The surfaces are approximate solutions to Equation (3.4), and by means of a fixed point argument we will for each small enough τ produce a function on them (for appropriately chosen θ) whose graph satisfies Equation (3.4) exactly. The basic ingredients are the singly periodic Scherk's singly-periodic surface Σ_0 and a family of half surfaces $\mathcal{K}[\theta]$ that are rotationally symmetric (about the y-axis) perturbations of the round hemisphere of radius 2. The crucial properties of the half-surfaces $\mathcal{K}[\theta]$ are that they satisfy Equation (3.4) exactly, intersect the plane $\mathcal{P} = \mathcal{P}_{xz}$ at the angle $\pi/2 - \theta$ and when θ vanishes agree with the hemisphere $\mathbb{S}^2(2) \cap \{y \ge 0\}$.

Let $\mathcal{C}[\theta]$ denote the configuration consisting of the plane $\mathcal{P}$ together with $\mathcal{K}[\theta]$ and a copy of $\mathcal{K}[\theta]$ reflected through $\mathcal{P}$ and let $c[\theta]$ denote their circle of intersection. For each τ with τ^{-1} an integer, the surfaces $\mathcal{M}[\tau,\theta]$ outside of a neighborhood of $c[\theta]$ of uniformly fixed radius will agree with $\mathcal{C}[\theta]$. Inside this neighborhood they will consist, loosely speaking, of τ^{-1} fundamental domains of Σ_0, rescaled by a factor of τ that have been "curled" and appropriately smoothed out to replace the singular intersection circle in the configuration. The analysis is simplified by identifying the symmetries preserved by this procedure and then imposing these from the beginning.

Definition 3.5.1. *Let G_τ be the subgroup generated by $\omega_\tau, \xi_\tau \in O(3)$, where:*

(1) ω_τ is the rotation about the y-axis by a positive angle $\pi\tau$ followed by the reflection $y \mapsto -y$.

(2) ξ_τ, is the reflection through a plane $\mathcal{P}_\tau$, which is $\{z = 0\}$ rotated an angle of $(\pi/2)\tau$ around the y-axis.

Denote also by $\sigma_\tau = \omega_\tau^2$ the rotation about the y-axis by a positive angle $2\pi\tau$.

We will construct the surfaces $\mathcal{M}[\tau,\theta]$ so that they are invariant under G_τ, with σ_τ orientation preserving and ω_τ orientation reversing. We assume implicitly that τ^{-1} is a positive integer. These symmetries will be reflected in the analysis by working with functions on $\mathcal{M}[\tau,\theta]$ that are invariant under σ_τ and ξ and anti-invariant under ω_τ. As the parameter $\tau \to 0$, the surfaces $\mathcal{M}[\tau,\theta]$ converge, under an appropriate renormalization, to a surface $\Sigma[\theta]$, singly periodic in the direction of the z-axis and invariant under the action of a group G_0, as follows:

Definition 3.5.2. *Let G_0 be the group generated by the Euclidean isometries ω_0 and ξ_0, where:*

(1) ω_0 is the translation $z \mapsto z + \pi$ followed by the reflection $y \mapsto -y$.

(2) ξ_0 is the reflection through the plane $\{z = \pi/2\}$.

Denote also by $\sigma_0 = \omega_0^2$ the translation $z \mapsto z + 2\pi$.

The geometrically correct notion of symmetric functions is as in the next definition, the point being to ensure that normal graphs (using the fixed unit normal giving the orientation) over the symmetric surface inherit the symmetries.

Definition 3.5.3. *Let S be an oriented surface invariant under G_τ (resp. G_0). By the G_τ-equivariant (resp. G_0-invariant) functions we will mean all $f : S \to \mathbb{R}$ such that*

$$\beta^* f = \langle \vec{\nu}_S, \beta\vec{\nu}_S \rangle f, \quad \forall \beta \in G_\tau \quad \text{(resp. } G_0\text{)}.$$

Now, recall Scherk's minimal surface Σ_0 (cf. [Ka97] p. 101–106) with angle $\frac{\pi}{2}$ between the asymptotic planes:

$$\Sigma_0 = \{(x,y,z) \subseteq \mathbb{R}^3 | \sinh x \sinh y - \sin z = 0\}. \tag{3.8}$$

In addition to G_0, the isometries of Σ_0 include reflection in the planes $\{x = y\}$ and $\{x = -y\}$. The regions $\Sigma_0 \cap \{\pm x > 0\}$ and $\Sigma_0 \cap \{\pm y > 0\}$ are graphs over $\mathcal{P}_{xz}$ and $\mathcal{P}_{yz}$ respectively, and the symmetries of Σ_0 give that it is globally determined by the graph of a single function

$$f : H^+ \to \mathbb{R}. \tag{3.9}$$

where $H^+ = \{(s,z)|s>0\}$. That is, in the half space $I = \{(x,y,z)|x>0\})$ we have

$$\Sigma_0 \cap I = \{(x, f(x,z), z)\}$$

with function $f(s,z)$ satisfying the estimate

$$\|f : C^5(\{H^+ : s \geq 1\}, e^{-s})\| \leq C. \tag{3.10}$$

A simple rephrasing of this estimate is as follows: Let $\mathrm{Proj}_{\{x\cdot y=0\}} : \mathbb{R}^3 \to \{x \cdot y = 0\} = \mathcal{P}_{xz} \cup \mathcal{P}_{yz}$ denote the nearest point projection to this closed set. Then $\mathrm{Proj}_{\{x\cdot y=0\}}$ is well defined away from the planes $\{x = \pm y\}$ and its restriction to Σ_0 satisfies the estimate $\|\mathrm{Proj}_{\{x\cdot y=0\}}{}^{-1} - \mathrm{Id}\| : C^5(\{H^+ : s \geq 1\}, e^{-s})\| \leq C$. On Σ_0 we define the function s by

$$s((x,y,z)) = \max\{|x|, |y|\}. \tag{3.11}$$

Note that since Σ_0 is minimal, $\Sigma_0/\langle\sigma_0\rangle$ is conformal under the Gauss map $\vec{\nu}_{\Sigma_0}$ with conformal factor $\frac{1}{2}|A_{\Sigma_0}|^2$ to the punctured sphere $\{S^2 : x \geq 0\}\backslash\{(\pm 1,0,0),(0,\pm 1,0)\}$.

Let ω_0^* and ξ_0^* denote the Euclidean isometries given by $(x,y,z) \mapsto (-x,y,-z)$ and $(x,y,z) \mapsto (x,y,-z)$, respectively. By computing the gradient of the function defining Σ_0 we obtain the intertwining relations

$$\begin{aligned} \vec{\nu}_{\Sigma_0} \circ \omega_0(\vec{X}) &= \omega_0^* \circ \vec{\nu}_{\Sigma_0}(\vec{X}), \\ \vec{\nu}_{\Sigma_0} \circ \xi_0(\vec{X}) &= \xi_0^* \circ \vec{\nu}_{\Sigma_0}(\vec{X}). \end{aligned} \tag{3.12}$$

Thus, functions on Σ_0 that are invariant under ξ_0 and anti-invariant under ω_0 (i.e. G_0-equivariant) push forward under the Gauss map to functions that are invariant under ξ_0^* and anti-invariant with respect to the inversion ω_0^*. Since the Gauss map will be the fundamental tool in understanding the linear operator $\mathcal{L}_{\Sigma_0}$ on Σ_0 we record the following lemma.

Lemma 3.5.4. *The kernel of the operator $\Delta_{S^2} + 2$ on the unit sphere in the space of L^2-functions that are invariant under ξ_0^* and anti-invariant under ω_0^* is one-dimensional, spanned by the ambient coordinate function x.*

Proposition 3.5.5. *For $|\theta| \leq \delta_\theta$ with δ_θ sufficiently small, there is a smooth one parameter family of surfaces $\mathcal{K}[\theta]$, with the following properties:*

- *(0) Each $\mathcal{K}[\theta]$ satisfies Equation (3.4).*
- *(1) $\mathcal{K}[0]$ is the upper hemisphere of radius 2 and the surfaces $\mathcal{K}[\theta]$ are given as normal graphs over $\mathcal{K}[0]$.*
- *(2) The surfaces $\mathcal{K}[\theta]$ are invariant with respect to rotations about the y-axis.*
- *(3) The boundary $\partial\mathcal{K}[\theta]$ is a circle in the plane $\mathcal{P}_{xz}$ of radius $r[\theta]$, and the inward pointing co-normal η_θ to $\partial\mathcal{K}[\theta]$ at the x-axis satisfies*

$$\eta_\theta \cdot \vec{e}_x = \sin(\theta).$$

(4) *There are conformal parametrizations*

$$\kappa[\theta] : \mathcal{C} \mapsto \mathcal{K}[\theta] \setminus \{y\text{-}axis\}$$

of the surfaces $\mathcal{K}[\theta]$, *where* $\mathcal{C} = H^+/\{z \mapsto z + 2\pi\}$ *is the flat cylinder of radius* 1 *such that:*

(i) $\kappa[\theta](\{(s,z) : s = const.\})$ *is a circle with center on the y-axis parallel to the xz plane.*

(ii) $\kappa[\theta](\{s = 0\}) = \partial\mathcal{K}[\theta]$.

(iii) *The conformal factor is* $\rho^2_{\kappa[\theta]}(s,z) = x^2(s,z) + z^2(s,z)$.

(iv) *There are bounds*

$$|\nabla^k \kappa[\theta]|, |\nabla^k \dot{\kappa}[\theta]| \leq C(k) \tag{3.13}$$

where "·" denotes derivation in the θ *parameter.*

Proof. See Appendix. □

Definition 3.5.6. *We denote by* $\mathcal{K}[\tau,\theta]$ *the surface* $\mathcal{K}[\theta]$ *dilated by the factor* τ^{-1} *and* $\kappa[\tau,\theta] : H^+ \to \mathcal{K}[\tau,\theta]$ *the map given by*

$$\kappa[\tau,\theta](s,z) = \tau^{-1}\kappa[\theta](\tau s, \tau z).$$

Definition 3.5.7. *Let* $\psi = \psi[1/2,1]$. *Then define the maps* $\mathcal{B}[\tau,\theta] : \mathbb{R}^3 \to \mathbb{R}^3$ *and* $\mathcal{Z}[\tau,\theta] : \mathbb{R}^3 \to \mathbb{R}^3$ *by*

$$\mathcal{B}[\tau,\theta](x,y,z) = r[\theta]\tau^{-1}e^{\tau x}(\cos\tau z, 0, \sin\tau z) + r[\theta]y\vec{e}_y,$$

and

$$\mathcal{Z}[\tau,\theta](x,y,z) = \psi(y)(\kappa[\tau,\theta](y,z) + r[\theta]x\vec{\nu}_{\kappa[\tau,\theta]}(y,z)) + (1-\psi(y))\mathcal{B}[\tau,\theta](x,y,z).$$

Proposition 3.5.8. *The maps* $\mathcal{Z}[\tau,\theta]$ *have the following properties:*

(1) *They depend smoothly on the parameters* τ *and* θ *with bounds*

$$\left|\nabla^k \mathcal{Z}[\tau,\theta]\right|, \left|\nabla^k \dot{\mathcal{Z}}[\tau,\theta]\right| \leq C\tau^{k-1}, \quad k > 1.$$

(2) *We have that*

$$\mathcal{Z}[\theta] := \lim_{\tau\to 0} \mathcal{Z}[\tau,\theta] - \tau^{-1}r[\theta]\vec{e}_x = r[\theta](\psi R_\theta + (1-\psi)\mathrm{Id})$$

where $R_\theta \in SO(3)$ *is the rotation determined by*

$$\begin{aligned} \vec{e}_x &\mapsto \cos\theta\vec{e}_x - \sin\theta\vec{e}_y \\ \vec{e}_z &\mapsto \vec{e}_z \\ \vec{e}_y &\mapsto \cos\theta\vec{e}_y + \sin\theta\vec{e}_x, \end{aligned}$$

In particular $\mathcal{Z}[0]$ is globally the identity transformation.

Proof. Claim (1) follows directly from the estimates 3.13 recorded in Proposition 3.5.5. Part (2) can be seen by applying l'Hôpital's rule. □

We now are ready to define the "desingularizing" and "initial" surfaces, and to set notation for various distinguished subsurfaces. For technical reasons, we work with a family of cut-off Scherk surfaces that agree with the asymptotic planes $\mathcal{P}_{xz}$ and $\mathcal{P}_{yz}$ outside of a cylinder around the line $\{x = y = 0\}$ and of a fixed radius proportional to τ^{-1}. The reason for this is that the image of these cut-off surfaces under the maps $\tau\mathcal{Z}[\tau,\theta]$, outside of a tubular neighborhood (of fixed radius independent of τ and θ) of the circle $c[\theta]$, is thus contained in the initial configuration $\mathcal{C}[\theta]$.

Proposition 3.5.9. *We obtain "desingularizing" surfaces $\Sigma[\tau,\theta]$ as follows:*

1. *For a constant $\delta_s > 0$ to be determined later, assume $\tau \le \delta_s$ and define the immersion $\varphi_\tau : \Sigma_0 \to \mathbb{R}^3$ by*

$$\varphi_\tau(\vec{X}) = \psi[3\delta_s\tau^{-1}, 4\delta_s\tau^{-1}]\vec{X} + (1 - \psi[3\delta_s\tau^{-1}, 4\delta_s\tau^{-1}])\,\mathrm{Proj}_{\{x\cdot y=0\}}(\vec{X}),$$

   where the cut-off function is evaluated at $s = s(\vec{X})$.

2. *The surface $\Sigma[\tau,\theta]$ is*

$$\Sigma[\tau,\theta] := \mathcal{Z}[\tau,\theta] \circ \varphi_\tau(\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}),$$

   which with sufficiently small $\delta_\theta, \delta_\tau > 0$ is well-defined, smooth and embedded for $\tau < \delta_\tau$ and $|\theta| \le \delta_\theta$.

The set $\mathcal{T} := \{\tau\Sigma[\tau,\theta] : \frac{4\delta_s}{\tau} \le s \le \frac{5\delta_s}{\tau}\}$, where the transition happens, consists of four connected components each of which is by construction a subregion of either a top/bottom spherical cap $\mathcal{K}[\theta]$ or of the plane $\mathcal{P}$.

Considering the singular initial configuration $\mathcal{C}[\theta]$, the set $\mathcal{C}[\theta] \setminus \mathcal{T}$ therefore has 5 connected components. One is the central piece containing the curve $c[\theta]$, but this singular component is now discarded and replaced by the smooth desingularizing surface $\Sigma[\tau,\theta]$ to obtain the initial surface:

Definition 3.5.10. *The initial surface $\mathcal{M}[\tau,\theta]$ is the union of $\tau\Sigma[\tau,\theta]$ with the four components of $\mathcal{C}[\theta] \setminus \mathcal{T}$ that do not contain the singular curve $c[\theta]$.*

Since $\tau\Sigma[\tau,\theta]$ overlaps with $\mathcal{C}[\theta]$ in the set $\mathcal{T}$, and we have excised the set containing the singular curve $c[\theta]$, the surfaces $\mathcal{M}[\tau,\theta]$ are smooth. The constructed surfaces are orientable, but notice the topology is such that if we orient, say, the top sphere with outward pointing normal then the bottom sphere has inwards pointing normal.

Proposition 3.5.11. *For $\delta_\theta, \delta_\tau > 0$ chosen sufficiently small, the surfaces $\mathcal{M}[\tau,\theta]$ are smooth, embedded, oriented and invariant under the action of G_τ.*

Remark 3.5.12. *Note that when $\tau^{-1} = k \in \mathbb{N}$, we have replaced a great circle by $2k$ Scherk handles. Hence, as computing the Euler characteristic reveals, the initial surface $\mathcal{M}[\tau,\theta]$ has topological genus $g = k-1$ and $4k$ symmetries. Thus we have:*

$$\tau = \frac{1}{g+1} \quad \text{and} \quad |G_\tau| = 4g+4. \tag{3.12}$$

Definition 3.5.13. *We define the function s on the surfaces $\Sigma[\tau,\theta]$ and $\mathcal{M}[\tau,\theta]$ as follows.*

(1) On $\Sigma[\tau,\theta]$, we take s to be the push forward by $\mathcal{Z}[\tau,\theta]\cdot\phi_\tau$ of the function s defined on Σ_0.

(2) s is then extended continuously to a constant on the remainder of $\mathcal{M}[\tau,\theta] \supset \Sigma[\tau,\theta]$.

Remark 3.5.14. *The reader will note that the surfaces $\Sigma[\tau,\theta]$ are by construction diffeomorphic to $\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}$ under the map $\mathcal{Z}[\tau,\theta]\circ\varphi_\tau$. We will, throughout this article, identify functions, tensors, and operators on $\Sigma[\tau,\theta]$ with their pull-backs by $\mathcal{Z}[\tau,\theta]\circ\varphi_\tau$, and vice versa.*

3.6 Geometric Quantities on the Initial Surfaces

Proposition 3.6.1. *Let $0 < \gamma < 1$. Then on $\{\Sigma[\tau,\theta] : s \ge 1\}$ we have:*

$$\left\| H_\Sigma - \tfrac{1}{2}\tau^2 \vec{X}\cdot\vec{\nu}_\Sigma : C^2(\{\Sigma[\tau,\theta] : s\ge 1\}, e^{-\gamma s}) \right\| \le C\tau,$$

and

$$\left\| \frac{\partial}{\partial\theta}\left\{ H_\Sigma - \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{\nu}_\Sigma \right\} : C^1(\{\Sigma[\tau,\theta] : s\ge 1\}, e^{-\gamma s}) \right\| \le C\tau.$$

for $\tau > 0$ sufficiently small.

Proof. In the following we let $\delta = \delta_{ij}$ denote the flat standard metric on the upper half plane H^+. Note that the surface $\Sigma[\tau,\theta]\cap\{y \ge 1\}$ has the parametrization $\varphi : \{H^+ : y \ge 1\} \to \mathbb{R}^3$ given by

$$\varphi(s,z) = \kappa[\tau,\theta](s,z) + \psi[4\delta_s\tau^{-1}, 3\delta_s\tau^{-1}](s) f(s,z) r[\theta]\vec{\nu}_\kappa(s,z),$$

We will in the rest of this proof denote $\kappa = \kappa[\tau,\theta]$. When $s \ge 4\delta_s/\tau$ the estimates are trivially satisfied since $\varphi \equiv \kappa[\tau,\theta]$ in this region. For s belonging to the interval $[3\delta_s/\tau, 4\delta_s/\tau]$, we have that

$$H_\Sigma - \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{\nu}_\Sigma = -\Delta_\mathcal{K}\hat{f} - |A_\mathcal{K}|^2\hat{f} + \tfrac{1}{2}\tau^2\left(\nabla_\mathcal{K}\hat{f}\cdot\vec{X} - \hat{f}\right) + Q_{\hat{f}} + \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{Q}_{\hat{f}},$$

where $\hat{f} = \psi[4\delta_s\tau^{-1}, 3\delta_s\tau^{-1}](s)f(s,z)$ and $Q_{\hat{f}}$ and $\vec{Q}_u$ denote terms that are at least quadratic in $\hat{f}$ and its derivatives. In this region, we may (since $\gamma < 1$) arrange that

$e^{-s} < \tau e^{-\gamma s}$ by taking τ sufficiently small in terms of γ. The estimate then follows by observing that $|\nabla_\delta^k \hat{f}| \le Ce^{-s}$, $k = 0,1,2$, that $\rho_\kappa^{-2}\Delta_\delta = \Delta_{\mathcal{K}}$, and that both ρ_κ and $|A_{\mathcal{K}}|^2$ are uniformly bounded in this region.

We now treat the case $s \le 3\delta_s/\tau$ as follows. Since $\{\Sigma_0 : y \ge 1\}$ is a graph over H^+ which is itself minimal, and since dilations preserve minimality, we have from the variation formula (3.71) in the Appendix the relation

$$\Delta_\delta f = r_\theta Q_f. \tag{3.13}$$

We then estimate the error term on $\Sigma = \{\Sigma[\tau,\theta] : s \le 3\delta_s/\tau\}$, using that it is a graph over $\mathcal{K} = \mathcal{K}[\tau,\theta]$, as follows:

$$\begin{aligned} H_\Sigma - \tfrac{1}{2}\tau^2 \vec{X}\cdot\vec{\nu}_\Sigma &= -r_\theta \mathcal{L}_{\mathcal{K}} f + Q_{r_\theta f} + \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{Q}_{r_\theta f} \\ &= -r_\theta\Delta_{\mathcal{K}} f - |A_{\mathcal{K}}|^2 r_\theta f + \tfrac{1}{2}\tau^2 r_\theta\left(\vec{X}\cdot\nabla_{\mathcal{K}} f - f\right) + Q_{r_\theta f} + \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{Q}_{r_\theta f} \\ &= -|A_{\mathcal{K}}|^2 r_\theta f + \tfrac{1}{2}\tau^2 r_\theta\left(\vec{X}\cdot\nabla_{\mathcal{K}} f - f\right) + r_\theta^2(Q_f - \rho_\kappa^{-2}Q_f) + \tfrac{1}{2}\tau^2\vec{X}\cdot\vec{Q}_{r_\theta f}, \end{aligned}$$

where in the last equality we have used (3.13).

Note that as a consequence of the estimates for $\mathcal{Z}[\tau,\theta]$ recorded in (3.5.8) the terms $|A_{\mathcal{K}}|^2 r_\theta f$ and $\frac{1}{2}\tau^2 r_\theta(\vec{X}\cdot\nabla_{\mathcal{K}} f - f)$ appearing above and their variations by θ satisfy the desired estimates, so it remains to estimate the terms $\mathcal{R} := r_\theta^2(Q_f - \rho_\kappa^{-2}Q_f)$. At $\tau = 0$ one has that $\mathcal{R} \equiv 0$, and since one may verify that the map $(\tau,\theta) \mapsto \mathcal{R}(\cdot)$ is C^1 in the parameters $\tau \ge 0$ and θ as a map into $C^2(H^+,\delta,e^{-\gamma s})$, we get the claimed estimates by one-sided Taylor expansion. □

3.7 The Linearized Equation Away From the End

Definition 3.7.1. *Set $\Sigma[\theta] = \mathcal{Z}[\theta](\Sigma_0)$ and let the function $\dot{H}_{\Sigma[\theta]} : \Sigma[\theta] \to \mathbb{R}$ denote the variation under θ of the mean curvature $H_{\Sigma[\theta]}$ of $\Sigma[\theta]$, that is for all $x_0 \in \Sigma_0$*

$$\dot{H}_{\Sigma[\theta]} \circ \mathcal{Z}[\theta](x_0) := \frac{\partial}{\partial\theta}\left[H_{\Sigma[\theta]} \circ \mathcal{Z}[\theta](x_0)\right]$$

Then the function $w : \Sigma[\tau,\theta] \to \mathbb{R}$ is given by

$$w[\tau,\theta] = \dot{H}_{\Sigma[\theta]} \circ \mathcal{Z}[\theta] \circ (\mathcal{Z}\circ\varphi_\tau)^{-1}[\tau,\theta].$$

where we are viewing $\mathcal{Z}[\tau,\theta]\circ\varphi_\tau$ as a diffeomorphism of $\{\Sigma_0 : s \le 5\delta_s/\tau\}$ onto $\Sigma[\tau,\theta]$.

Lemma 3.7.2. *The function w has the following properties:*

(1) w is supported on $\{\Sigma[\tau,\theta] : s \le 1\}$.

(2) The estimate

$$\left\| \frac{\partial}{\partial\theta}\left\{H_\Sigma - \tfrac{1}{2}\tau^2(\vec{X}\cdot\vec{\nu}_\Sigma)\right\} - w : C^1(\Sigma, g, e^{-\gamma s}) \right\| \le C\tau,$$

holds for all sufficiently small θ and τ.

(3) When $\tau = 0$ and $\theta = 0$ it holds

$$\int_{\Sigma_0/\langle\sigma_0\rangle} w_0(\vec{e}_x \cdot \vec{\nu}) d\mu_{\Sigma_0} = 8\pi.$$

where $w_0 \equiv w[0,0]$.

Proof. (1) and (2) follow directly from Definition 3.7.1 and Proposition 3.6.1. To see (3), set $S_c = \{\Sigma_0 : s \le c\}/\langle\sigma_0\rangle$. We then we have

$$\int_{S_c} w(\vec{e}_x \cdot \vec{\nu}_{\Sigma_0}) = \int_{S_c} u\mathcal{L}_{\Sigma_0}(\vec{e}_x \cdot \vec{\nu}_{\Sigma_0}) + \int_{\partial S_c} \left[(\vec{e}_x \cdot \vec{\nu})(\nabla u \cdot \vec{\eta}) - u\vec{\eta} \cdot \nabla(\vec{e}_x \cdot \vec{\nu}) \right]$$

where $\nabla = \nabla_{\Sigma_0}$, $\vec{\eta}$ is the co-normal at the boundary of S_c, and $u = \frac{\partial}{\partial\theta}\mathcal{Z}[\theta]\big|_{\theta=0} \cdot \vec{\nu}_{\Sigma_0}$, so that $\mathcal{L}_{\Sigma_0} u = w_0$. The claim then follows by taking c to ∞ and noting that $|\nabla(\vec{e}_x \cdot \vec{\nu}_{\Sigma_0})(s,z)| \le Ce^{-s}$, $|\vec{\eta} - \vec{e}_y| \le Ce^{-c}$, and $|(\nabla u(s,z) - 2\vec{e}_y)| \le Ce^{-s}$. □

By Proposition 3.6.1, the quantity $E = H_\Sigma - \frac{1}{2}\tau^2(\vec{X} \cdot \vec{\nu}_\Sigma)$ and its variations under θ lie in the weighted Hölder spaces $C^{0,\alpha}(\Sigma, g, e^{-\gamma s})$. The symmetries of Σ give that E is G_τ-equivariant, and that its pull-back to Σ_0 by $\mathcal{Z}$ is G_0-equivariant. For the remainder of this article, all functions defined on $\Sigma[\tau,\theta]$ are assumed to be invariant under the symmetry group G_τ.

Proposition 3.7.3. *Given any $E \in C^{0,\alpha}(\Sigma, g, e^{-\gamma s})$, there is a constant $b = b_E$ and a function $v = v_E$ such that*

$$\begin{aligned} \mathcal{L}_\Sigma v &= E - bw, \\ v &= 0, \quad \text{on} \quad \partial\Sigma, \end{aligned}$$

$$|b|,\ \|v : C^{2,\alpha}(\Sigma, g, e^{-\gamma s})\| \le C\|E : C^{0,\alpha}(\Sigma, g, e^{-\gamma s})\|.$$

Moreover, the pair (v_E, b_E) depends continuously on the parameters τ and θ (see Remark (3.5.14))

We prove first Proposition 3.7.3 in the limiting case $\tau = 0$, $\theta = 0$, and handle the general case as a perturbation.

Proposition 3.7.4. *Given any $E \in C^{0,\alpha}(\Sigma_0, g_0, e^{-\gamma s})$, there is a constant $b = b_E$ and a function $v = v_E$ such that*

$$\mathcal{L}_{\Sigma_0} v = E - bw_0, \tag{3.14}$$

and such that

$$|b|,\ \|v : C^{2,\alpha}(\Sigma_0, g_0, e^{-\gamma s})\| \le C\|E : C^{0,\alpha}(\Sigma_0, g_0, e^{-\gamma s})\|$$

Proof. Let $E \in C^{0,\alpha}(\Sigma_0, e^{-\gamma s})$ be a given G_0-equivariant function, and assume for the moment that E is supported on $\{\Sigma_0 : s \leq a\}$ where $a > 1$ is a large constant. Recall that the Gauss map

$$\nu_{\Sigma_0} : \Sigma_0 \to \mathbb{S}^2$$

is a conformal covering which descends to a diffeomorphism from Σ_0/σ_0 onto the punctured sphere $\mathbb{S}^2 \setminus \{(\pm 1, 0, 0), (0, \pm 1, 0)\}$, with the four punctures corresponding to the four asymptotic ends of Σ_0. The function $\bar{E} = \vec{\nu}_{\Sigma_0 *}(E/|A_{\Sigma_0}|^2) \in L^2(\mathbb{S}^2, g_{\mathbb{S}^2})$ is then well-defined and satisfies

$$\|\bar{E}\|_{L^2(\mathbb{S}^2)} \leq C\|E\|_{C^{0,\alpha}(\Sigma_0, e^{-\gamma s})}$$

where the constant C depends on a. It is easily verified that since E is G_0-equivariant, the function $\bar{E}$ satisfies the identities (3.12), which then give that $\bar{E}$ is L^2 orthogonal to the functions y and z on $\mathbb{S}^2$ from Lemma 3.5.4. Now, (3) in Lemma 3.7.2 gives that

$$\int_{\mathbb{S}^2} \bar{w}x = 8\pi,$$

where $\bar{w} = \vec{\nu}_{\Sigma_0 *}(w_0/|A_\Sigma|^2)$. Thus, we may find a constant b such that $\bar{E} - b\bar{w}$ is L^2-orthogonal to x. We then get a function $v : \mathbb{S}^2 \to \mathbb{R}$ satisfying

$$(\Delta_{\mathbb{S}^2} + 2)v = \bar{E} - \bar{w}$$

and the identities (3.12), from which we conclude that $v(1,0,0) = -v(-1,0,0)$, while $v(0, \pm 1, 0) = 0$. Define then the G_0-equivariant function $u : \Sigma_0 \to \mathbb{R}$ by

$$u = \vec{\nu}^*_{\Sigma_0}(v - v(1,0,0)x).$$

We then get immediately that u satisfies

$$\mathcal{L}_{\Sigma_0} u = E - b w_0.$$

That u has the appropriate decay, i.e. lies in the space $C^{2,\alpha}(\Sigma_0, g_0, e^{-\gamma s})$, follows by observing that the operator $\mathcal{L}_{\Sigma_0}$ is asymptotically a perturbation of the Laplace operator on the flat cylinder C, for which the decay estimates hold. To conclude the proof, note that we may reduce to the case that E is supported in $\{\Sigma_0 : s \leq a\}$ as follows: Recall that each component of $\{\Sigma_0 : s \geq a\}$ is given by the graph of a small function $f : H^+ \to \mathbb{R}$ with f satisfying (3.10). For a sufficiently large, the operator $\mathcal{L}_{\Sigma_0}$ on $\{\Sigma_0 : s \geq a - 1\}$ is then a perturbation of the Laplace operator Δ_{H^+} on the flat half cylinder H^+. Proposition 3.15.1 then gives a function u' on $\{\Sigma_0 : s \geq a\}$ satisfying

$$\begin{aligned} \mathcal{L}_{\Sigma_0} u' &= E, \\ u' &= c, \quad \text{on} \quad \partial\{\Sigma_0 : s \geq a - 1\}. \end{aligned}$$

for a constant c with $|c| \leq C\|E\|$. Define the smooth cutoff function $\psi = \psi[a-1, a]$.

We then get that $\psi u'$ is defined on all of Σ_0 and satisfies

$$\mathcal{L}_{\Sigma_0}(\psi u') = \psi E + \mathcal{E}$$

where $\mathcal{E}$ is an error term introduced by smoothing out u' on the boundary of $\{\Sigma_0 : s \geq a-1\}$. The function $F = (1-\psi)E - \mathcal{E}$ is then supported on $\{\Sigma_0 : s \leq a\}$. This concludes the proof. □

Remark 3.7.5. *The reader will note that the highly symmetric nature of our construction, in contrast with the general situation and in particular the construction in [Ka97], allow us to obtain a solution with the appropriate decay with a single parameter.*

In particular, the function v satisfying the equivalent problem on the sphere has opposite values at $(\pm 1, 0, 0)$, which allows simultaneous cancellation of both values by a single multiple of the kernel element x.

Corollary 3.7.6. *Given*

$$E \in C^{0,\alpha}(\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\}, g_0, e^{-\gamma s}),$$

there is a constant $b \in \mathbb{R}$ and a function

$$v \in C^{0,\alpha}(\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\}, g_0, e^{-\gamma s})$$

such that

$$\begin{aligned} \mathcal{L}_{\Sigma_0} v &= E - b w_0 \text{ on } \{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\} \\ v &= 0 \text{ on } \partial\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\} \end{aligned}$$

with the bounds

$$\|v : C^{2,\alpha}(\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\}, g_0, e^{-\gamma s})\| \leq C\|E : C^{0,\alpha}(\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\}, g_0, e^{-\gamma s})\|$$

and

$$|b| \leq C\|E : C^{0,\alpha}(\{\Sigma_0 : s \leq 5\delta_s \tau^{-1}\}, g_0, e^{-\gamma s})\|.$$

Proof. First, we apply Proposition 3.7.4 to obtain a function v_1 satisfying (3.14). Now, note that for a large constant $a > 0$, the operator $\mathcal{L}_{\Sigma_0} = \Delta_{\Sigma_0} + |A_{\Sigma_0}|^2$ on $\{\Sigma_0 : a \leq s \leq 5\delta_s \tau^{-1}\}$ is a perturbation of the Laplacian on a long cylinder. This allows us (see Proposition 3.15.1) to solve the following Dirichlet problem, with ∂_a and ∂_τ denoting the boundary components of $\{\Sigma_0 : a \leq s \leq 5\delta_s \tau^{-1}\}$ in the obvious way:

$$\begin{aligned} \mathcal{L}_{\Sigma_0} v_2 &= 0 \\ v_2 &= v_1 + c_1 \text{ on } \partial_\tau \\ v_2 &= 0 \text{ on } \partial_a \end{aligned} \tag{3.15}$$

with the bounds

$$\begin{aligned}|c_1|, \|v_2 : C^{2,\alpha}(\{\Sigma_0 : a \le s \le 5\delta_s\tau^{-1}\}, g_0)\| &\le C\|v_1 : C^{2,\alpha}(\partial_\tau, g_0)\| \\ &\le Ce^{-5\gamma\delta_s\tau^{-1}}\|E : C^{0,\alpha}(\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}, g_0, e^{-\gamma s})\|.\end{aligned}$$

The function $v = v_1 - \psi[a, a+1](v_2 - c_1)$ then solves

$$\begin{aligned}\mathcal{L}_{\Sigma_0} v &= E - bw_0 + \mathcal{E} \text{ on } \{\Sigma_0 : s \le 5\delta_s\tau^{-1}\} \\ v &= 0 \text{ on } \partial\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}.\end{aligned}$$

for an error term $\mathcal{E}$ and has the required bounds on the norm, and by taking τ sufficiently small and using that $|A_{\Sigma_0}|^2 < Ce^{-s}$ (a consequence of (3.10)) we get that

$$\|\mathcal{E} : C^{0,\alpha}(\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}, g_0, e^{-\gamma s})\| \le 1/2\|E : C^{0,\alpha}(\{\Sigma_0 : s \le 5\delta_s\tau^{-1}\}, g_0, e^{-\gamma s})\|.$$

We then iterate this process to obtain an exact solution. □

We now prove Proposition 3.7.3 in full generality.

Proof. Recall that $\mathcal{Z} \circ \varphi_\tau : \{\Sigma_0 : s \le 5\delta_s/\tau\} \to \Sigma$ is a diffeomorphism. By referring to the derivative bounds on the maps $\mathcal{Z}[\tau, \theta]$ recorded in Proposition 3.5.8 it is clear that we can arrange so that

$$\|g_{\Sigma_0} - (\mathcal{Z} \circ \varphi_\tau)^* g_\Sigma : C^{2,\alpha}(\Sigma_0, g_0)\| < \epsilon$$

by choosing the constant δ_s sufficiently small for arbitrary positive ϵ. Now, by choosing a sufficiently large and τ sufficiently small, we can arrange that

$$(\mathcal{Z} \circ \varphi_\tau)^* |A_\Sigma|^2, |A_{\Sigma_0}|^2 < \epsilon \tag{3.16}$$

on $\{\Sigma_0 : a \le s \le 5\delta_s/\tau\}$. It follows that the operator norm of $(\mathcal{Z} \circ \varphi_\tau)^* \mathcal{L}_\Sigma - \mathcal{L}_{\Sigma_0} : C^{2,\alpha}(\Sigma_0) \to C^{0,\alpha}(\Sigma_0)$ can be made arbitrarily small. The proposition then follows by formally treating $(\mathcal{Z} \circ \varphi_\tau)^* \mathcal{L}_\Sigma$ as a perturbation of $\mathcal{L}_{\Sigma_0}$.

□

Lemma 3.7.7. *For any $\gamma \in (0,1)$ there exists $C = C(\gamma)$ such that*

$$\begin{aligned}&\left\| H_\Sigma - \tfrac{1}{2}\tau^2 \vec{X} \cdot \vec{\nu}_\Sigma - \theta w : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s}) \right\| \\ &\le C(\tau + |\theta|^2),\end{aligned}$$

where H_Σ is the mean curvature of Σ.

Proof. This is a consequence of the smooth dependence of the surfaces Σ on the parameters θ, τ and the definition of w. □

3.8 An Exterior Linear Problem of Ornstein-Uhlenbeck Type

On a flat plane $\mathcal{P}$ through the origin, with the induced standard Euclidean metric, the Dirichlet problem for the linearized operator $\mathcal{L}_{\mathcal{P}}$ in (3.6) at unit scale becomes:

$$\begin{cases} \mathcal{L}_{\mathcal{P}} u = \Delta u - \frac{1}{2}\left(\vec{X} \cdot \nabla u - u\right) = E, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.17}$$

for $u : \Omega \longrightarrow \mathbb{R}$, where the domain $\Omega = \mathbb{R}^2 \setminus B_R(0)$ is the exterior of a disk with radius $R \simeq 2$. The Laplacian and gradient are taken with respect to the standard Euclidean metric on the plane. The function E is implicitly assumed to be G_τ-equivariant.

The operator $\mathcal{L}_{\mathcal{P}}$ is of Ornstein-Uhlenbeck type (such operators are related to Brownian motion and number operators in quantum mechanics). It is of course clear that the local theory for this equation is classic, using for example standard Schauder estimates. On the non-compact exterior domain however, with such fast growth on the gradient term, there is generally no reasonable global elliptic theory available (see for example the counterexamples [Pr]) and it is not a priori clear even what spaces to study the problem in. There exists in fact a vast literature on Ornstein-Uhlenbeck operators for various restrictive assumptions on the coefficients and corresponding choices of function spaces (see for example [CV87] and [DL95]), but since remarkably there is nothing in the literature that is adequate for our construction, we must develop our theory from scratch.

Firstly, note that the connection with the stability operator as a minimal surface in the Gaussian metric (see (3.74) in the Appendix), is via the following conjugation identity,

$$\mathcal{L}_{\mathcal{P}} u = \Delta u - \tfrac{1}{2}(\vec{X} \cdot \nabla - 1)u = e^{|x|^2/8}\left(\Delta - \frac{|x|^2}{16} + 1\right)e^{-|x|^2/8}u, \tag{3.18}$$

where the exponential functions act by multiplication.

The operator in the parentheses in (3.18) is of course nothing but the Hamilton operator for the two-dimensional quantum harmonic oscillator, plus a constant. Rescaling coordinates, it has the expression

$$\hat{H} = \tfrac{1}{2}\Delta - \tfrac{1}{2}|x|^2 + 2. \tag{3.19}$$

This connection to the harmonic oscillator turns out to be about as misleading as it is helpful, for as we will see below, it is certainly not a natural point of departure for our applications, because of the involved conjugation with the Gaussian densities.

We get however from (3.18) the following elementary lemma. The notation $H^s(\mathbb{R}^2)$ refers to the Sobolev space of functions with s derivatives in $L^2(\mathbb{R}^2)$.

Lemma 3.8.1. *Given G_τ-equivariant $E \in e^{|x|^2/8}L^2(\mathbb{R}^2)$ and assuming $\tau \leq \frac{1}{3}$, there is a* *unique G_τ-equivariant $u \in e^{|x|^2/8}H^2(\mathbb{R}^2)$ such that $\mathcal{L}_{\mathcal{P}}u = E$.*

Furthermore, there is a uniform constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}^2} |u(x)| \le C\Big(\sup_{x\in\mathbb{R}^2}(1+|x|)|E(x)|\Big)(1+|x|), \tag{3.20}$$

for all $E \in C^0(\mathbb{R}^2)$ s.t. $\sup_{x\in\mathbb{R}^2}(1+|x|)|E(x)| < \infty$ (hence $E \in e^{|x|^2/8}L^2(\mathbb{R}^2)$).

The same statements hold if we replace $\mathbb{R}^2$ by $\Omega = \mathbb{R}^2 \setminus B_R(0)$ and add the condition $u_{|\partial\Omega} = 0$.

Proof. Since the L^2-eigenvalues of $\hat{H}$ are $\lambda_{(n_1,n_2)}(\hat{H}) = n_1 + n_2 + 1$, for $n_i \ge 0$, and the well-known L^2-basis for $\hat{H}$ consists of Hermite functions, the $e^{|x|^2/8}L^2(\mathbb{R}^2)$-kernel of $\mathcal{L}_{\mathcal{P}}$ thus corresponds to the first excited eigenmodes,

$$\ker \mathcal{L}_{\mathcal{P}} = \operatorname{span}\{x_1, x_2\},$$

which thus disappears under the assumption of G_τ-equivariance (given we insert at least $2\tau^{-1} = 2k \ge 2$ handles). Hence there is a well-defined inverse map $\mathcal{L}_{\mathcal{P}}^{-1} : L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$, which by isometry invariance of $\mathcal{L}_{\mathcal{P}}$ preserves the imposed symmetries.

If we consider the disk $B_{\sqrt{17}}(0) = \{|x|^2 \le 17\}$, then if $v \in H^2(\mathbb{R}^2)$ satisfies $\hat{H}v \ge 0$ and $v \le 0$ on $\partial B_{\sqrt{17}}(0)$, we conclude the simple maximum principle result that $v \le 0$ on $\Omega = \mathbb{R}^2 \setminus B_{\sqrt{17}}(0)$. This is standard, but we briefly sketch the proof. Namely, let $w := \max(0, v)$ so that

$$w\Delta v \ge (\tfrac{1}{2}|x|^2 - 2)w^2 \ge 0 \tag{3.21}$$

Then by Green's first identity, which is justified since $v \in H^2(\mathbb{R}^2)$ and $w \in H^1(\mathbb{R}^2)$,

$$-\int_{\mathbb{R}^2\setminus B_{\sqrt{17}}(0)} |\nabla w|^2 \ge 0, \tag{3.22}$$

where we used $w_{|\partial B_{\sqrt{17}}(0)} = 0$. Thus $w = 0$ which proves the claim.

We now take, for numbers $A, B > 0$ to be determined below, the test functions

$$v(x_1, x_2) := e^{-|x|^2/8}\Big(u - Ax_1 + \frac{B}{2}\frac{x_1}{|x|^2}\Big).$$

We consider a fundamental domain $\theta \in [-\pi/k, \pi/k]$ positioned inside the support of the test functions and compute:

$$\begin{aligned}\hat{H}v &= e^{-|x|^2/8}\Big(E + B\frac{x_1}{|x|^2}\Big) \ge e^{-|x|^2/8}\left(B - \left|\frac{|x|E}{\cos(\frac{\pi}{k})}\right|\right)\frac{x_1}{|x|^2}\\ &\ge e^{-|x|^2/8}\Big(B - 2\sup_{x\in\mathbb{R}^2}(1+|x|)|E(x)|\Big)\frac{x_1}{|x|^2},\end{aligned}$$

where we have used that $k \ge 3$, so that $\cos(\pi/k) \ge \frac{1}{2}$. Thus we get that picking

$B := 2\sup_{x\in\mathbb{R}^2}(1+|x|)|E(x)|$ ensures $\hat{H}v \geq 0$ (on the fundamental domain). By picking A large depending linearly on B and on $\|u_{|\partial B_{\sqrt{17}}}\|_\infty$, we arrange $v \leq 0$ on (a fundamental domain of) $\partial B_{\sqrt{17}}$, and hence the result follows by the above maximum principle combined with the estimate

$$\begin{aligned}\|u_{|\partial B_{\sqrt{17}}}\|_\infty &\leq C\|e^{-|x|^2/8}u\|_{H^2(B_5(0))} \leq C\|e^{-|x|^2/8}E(x)\|_{L^2(\mathbb{R}^2)} \\ &\leq C\left(\int_{\mathbb{R}^2}\frac{e^{-|x|^2/4}}{(1+|x|)^2}dx\right)^{1/2}\sup_{x\in\mathbb{R}^2}(1+|x|)|E(x)|,\end{aligned}$$

using the Sobolev inequality on the larger disk $B_5(0)$. Hence the estimate (3.20) follows. The argument in the case with Ω instead of $\mathbb{R}^2$ is similar. □

However nice such simple lemmas may appear, the truth is that the spaces $e^{|x|^2/8}L^2(\Omega)$ are not well-suited for our geometric analysis purposes, in particular they do not have any good compact embedding properties, because of the Gaussian (and linear) growth involved. What we would like is to separate out the conical asymptotics and obtain sharp, uniform control in adequate weighted spaces, with second order derivative bounds, in such a way that we can proceed with our geometric construction. To accomplish this, we first introduce in the next section the appropriate new cone spaces.

3.8.1 Hölder Cone Spaces for the Exterior Problem

In this section we define the weighted Hölder spaces suitable for working with homogeneous functions. Note that these are different from the standard spaces considered in Equation (3.3), although they could be naturally rephrased as such with a different metric (in fact the pull-back metric under the projection from any fixed symmetric cone) on the plane.

Definition 3.8.2 (Homogeneously weighted Hölder spaces). *We define the appropriate weighted spaces of Hölder functions for decay rate $k \in \mathbb{N}$,*

$$C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-k}) = \{f \in C^{0,\alpha}_{\text{loc}}(\Omega) : \|f : C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-k})\| < \infty\},$$

with norms

$$\|f : C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-k})\| := [f]_{\Omega,\alpha,-k-\alpha} + \sup_{x\in\Omega}|x|^k|f(x)|,$$

where the weighted Hölder coefficients of decay rate $-k-\alpha$ are defined as:

$$[f]_{\Omega,\alpha,-k-\alpha} := \sup_{x,y\in\Omega}\frac{1}{|x|^{-k-\alpha}+|y|^{-k-\alpha}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We then let:

$$C^{2,\alpha}_{\text{hom}}(\Omega,|x|^{-1}) := \{f \in C^{2,\alpha}_{\text{loc}}(\Omega) : D_\beta f \in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1}),\ |\beta| \leq 2\},$$

where β ranges over all multiindices, with norm given by

$$\|f : C^{2,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\|^2 := \sum_{|\beta|\le 2} \|D_\beta f : C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\|^2. \tag{3.23}$$

Definition 3.8.3. *The anisotropically homogeneously weighted Hölder spaces are the following:*

$$C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1}) := \{f \in C^{2,\alpha}_{\text{hom}}(\Omega,|x|^{-1}) : \vec{X}\cdot\nabla f \in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\},$$

with norms

$$\|f : C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1})\|^2 := \|f : C^{2,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\|^2 + \|\vec{X}\cdot\nabla f : C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\|^2.$$

The definition of the homogeneously weighted spaces are motivated partly by the following lemma. Note also that $C^{0,\alpha}_{\text{hom}}(\Omega,|x|^k) \subseteq e^{|x|^2/8}L^2(\Omega)$.

Lemma 3.8.4. *Let $h(x) = c(\frac{x}{|x|})|x|^k$ be homogeneous of degree $k \in \mathbb{Z}$, where $c \in C^{2,\alpha}(S^1)$, then*

$$\begin{aligned} &(\nabla)^l h \in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{k-l}), \quad l = 0,1,2, \quad \textit{with} \\ &\|(\nabla h)^l\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{k-l})} \le \|c\|_{C^{l,\alpha}(S^1)}. \end{aligned} \tag{3.24}$$

When $k = 1$, then we have the property

$$\mathcal{L}_{\mathcal{P}} h \in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1}).$$

Furthermore $\mathcal{L}_{\mathcal{P}} : C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1}) \to C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})$ is a bounded operator.

Proof. The first claim for homogeneous functions h is elementary from the definitions, using scaling.

When $k = 1$, $\mathcal{L}_{\mathcal{P}} h = \Delta h - \frac{1}{2}(\vec{X}\cdot\nabla - 1)h = \Delta h$ is a sum of homogeneous functions, namely one of degree -1 and one of degree -2, and the second and third result also follow. □

Definition 3.8.5. *The (anisotropic homogeneous) Hölder cone space of functions asymptotic to graphical cones over the plane, are:*

$$\mathcal{CS}^{0,\alpha}(\Omega,|x|^{-1}) := C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1}), \tag{3.25}$$

$$\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1}) := C^{2,\alpha}(\partial\Omega) \times C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1}), \tag{3.26}$$

the latter equipped with the product norm

$$\|(c,f) : \mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})\|^2 := \|c\|^2_{C^{2,\alpha}(S^1)} + \|f : C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1})\|^2.$$

Remark 3.8.6.

(i) *The pairs (c,f) injectively model graphs $u:\Omega\to\mathbb{R}$ as follows,*

$$u = u_{(c,f)}(r,\theta) := c(\theta)r + f(r,\theta), \tag{3.27}$$

in polar coordinates, and by abuse of notation we write $u=(c,f)$.

(ii) *An important consequence in this context, is that our linearized operator in (3.17) induces a well-defined bounded map $(c,f)\mapsto\mathcal{L}_{\mathcal{P}}(u_{(c,f)})$,*

$$\mathcal{L}_{\mathcal{P}}:\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})\to\mathcal{CS}^{0,\alpha}(\Omega,|x|^{-1})=C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1}), \tag{3.28}$$

as opposed to second order operators generally (e.g. $\Delta+1$).

Proposition 3.8.7. *The spaces $C^{k,\alpha}_{\text{hom}}(\Omega,|x|^{-1})$, $C^{k,\alpha'}_{\text{an}}(\Omega,|x|^{-1})$ and $\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})$ are Banach, and the natural inclusions for $0<\alpha<\alpha'<1$,*

$$C^{k,\alpha'}_{\text{hom}}(\Omega,|x|^{-1})\hookrightarrow C^{k,\alpha}_{\text{hom}}(\Omega,|x|^{-1}), \tag{3.29}$$
$$C^{k,\alpha'}_{\text{an}}(\Omega,|x|^{-1})\hookrightarrow C^{k,\alpha}_{\text{an}}(\Omega,|x|^{-1}), \tag{3.30}$$
$$\mathcal{CS}^{2,\alpha'}(\Omega,|x|^{-1})\hookrightarrow\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1}), \tag{3.31}$$

are compact.

Proof. It is a standard exercise to verify that these spaces are complete with the norms we have defined.

Since Ω is non-compact, it is for the compactness of the embeddings (3.29)-(3.31) to be true crucial that: (A) We have arranged that the weight functions on all derivatives are decaying, and (B) Cones are modeled by functions on a compact curve in $\mathbb{S}^2$, here on $\partial\Omega=\mathbb{S}^1$. Note that it is an important special feature of the operator $\mathcal{L}_{\mathcal{P}}$ that the property (B) can be brought into play (see the Liouville result in Proposition 3.8.9).

Namely, for any bounded domain $D\subset\subset\mathbb{R}^n$ the embeddings $C^{k,\alpha'}(D)\hookrightarrow C^{k,\alpha}(D)$, of the usual Hölder spaces, are compact if $0<\alpha<\alpha'<1$, as follows from the Arzelà-Ascoli theorem. This fact along with a standard cut-off argument and the property (A) shows that the embeddings in (3.29) and (3.30) are compact.

From the compactness of (3.30) and the property (B), i.e. compactness of

$$C^{k,\alpha'}(\partial\Omega)\hookrightarrow C^{k,\alpha}(\partial\Omega),$$

it now finally follows that also the product

$$C^{2,\alpha'}(\partial\Omega)\times C^{2,\alpha'}_{\text{an}}(\Omega,|x|^{-1})\hookrightarrow C^{2,\alpha}(\partial\Omega)\times C^{2,\alpha}_{\text{an}}(\Omega,|x|^{-1}) \tag{3.32}$$

is indeed a compact embedding hierarchy, completing the proof of (3.31). □

3.8.2 Homogeneously Weighted Hölder Estimates

In this section we prove the second derivative Schauder estimates in the weighted Hölder spaces. Recall that we take $\Omega = \mathbb{R}^2 \setminus B_R(0)$ to be a domain exterior to a disk.

Proposition 3.8.8. *If $E \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$ and $v \in e^{|x|^2/8}H^2(\Omega) \cap C^{2,\alpha}_{\text{loc}}(\Omega)$ is a solution to $\mathcal{L}_{\mathcal{P}}v = \Delta v - \frac{1}{2}(\vec{X}\cdot\nabla - 1)v = E$, then*

$$D_{x_ix_j}v \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1}),$$

and if $v_{|\partial\Omega} = 0$ there is a constant $C = C(\alpha) > 0$ such that

$$\|D_{x_ix_j}v\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})} \leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})}. \tag{3.33}$$

Proof. There are several routes one may take to prove such a result, for example the resolvents can be found in the form of contour integrals by summing up the eigenfunctions via Mehler's formula.

However, using the well-known connection to parabolic equations (and whence this problem came, of course) is less involved. Namely, the equation

$$\mathcal{L}_{\mathcal{P}}u = \Delta u - \tfrac{1}{2}(\vec{X}\cdot\nabla - 1)u = E,$$

is the elliptic equation describing a backwards self-similar solution to the flat space heat equation, but with a modified source term.

It is convenient to consider a fixed extension map $v \mapsto \tilde{v} \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ with the property

$$\|\tilde{v}\|_{C^{2,\alpha}(B_R(0))} \leq C\|v\|_{C^{2,\alpha}(B_{R+1}(0)\setminus B_R(0))}, \tag{3.34}$$

where the constant is independent of v. Then letting $\tilde{E} = \mathcal{L}_{\mathcal{P}}\tilde{v}$ we see that $\tilde{E} \in C^{0,\alpha}_{\text{hom}}(\mathbb{R}^2, |x|^{-1})$ and

$$\begin{aligned}
\|\tilde{E}\|_{C^{0,\alpha}_{\text{hom}}(\mathbb{R}^2,|x|^{-1})} &\leq \|E\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})} + C\|\tilde{E}\|_{C^{0,\alpha}(B_R(0))} \\
&\leq \|E\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})} + C\|v\|_{C^{2,\alpha}(B_{R+1}(0)\setminus B_R(0))} \\
&\leq \|E\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})} + C\Big[\|E\|_{C^{0,\alpha}(B_{R+1}(0)\setminus B_R(0))} + \sup_{x\in\Omega}(1+|x|)|E(x)|\Big] \\
&\leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})},
\end{aligned}$$

where in the second to last estimate we used Schauder estimates (such as Theorem 10.2.1-10.2.2 in [Jo02]), using the fact that $v_{|\partial B_R} = 0$ and the bounds on v from the second part of Lemma 3.20. Now, since also automatically

$$\|D_{x_ix_j}v\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})} \leq \|D_{x_ix_j}\tilde{v}\|_{C^{0,\alpha}_{\text{hom}}(\mathbb{R}^2,|x|^{-1})}, \tag{3.35}$$

we see that it is enough to prove the estimate (3.33) for $\tilde{v}$ and $\tilde{E}$, so we assume without loss of generality that v and E are defined on $\mathbb{R}^2$.

The elliptic equation is now, as mentioned above, easily rewritten to the condition

$$v(x,t) := \sqrt{1-t}\,u\Big(\tfrac{x}{\sqrt{1-t}}\Big) \tag{3.36}$$

solves the following heat equation

$$\begin{cases} \partial_t v - \Delta v = F(x,t), & (t,x) \in (0,1)\times \mathbb{R}^2, \\ v(x,0) = u(x), & x \in \mathbb{R}^2, \end{cases} \tag{3.37}$$

where the correspondingly transformed source term now reads:

$$F(x,t) := -\frac{E\Big(\frac{x}{\sqrt{1-t}}\Big)}{\sqrt{1-t}}. \tag{3.38}$$

Now, recall the heat kernel in Euclidean space,

$$\Phi(x-y,t-s) := \frac{1}{4\pi(t-s)} e^{-\frac{|x-y|^2}{4(t-s)}}.$$

Note that we have the following representation formula which allows us to use standard methods of proof (e.g. the standard, non-weighted Schauder theory for the heat equation. See for example [La])

$$\begin{aligned} D_{x_i x_j} v(x,t) &= \int_{\mathbb{R}^2} D_{x_i x_j}\Phi(x-y,t)u(y)dy \\ &\quad - \int_0^t \int_{\mathbb{R}^2} D_{x_i x_j}\Phi(x-y,t-s)\Big[F(x,s)-F(y,s)\Big]dyds, \end{aligned} \tag{3.39}$$

where

$$D_{x_i x_j}\Phi(x-y,t-s) = \Big[\frac{(x_i-y_i)(x_j-y_j)}{4(t-s)^2} - \frac{\delta_{ij}}{2(t-s)}\Big]\Phi(x-y,t-s), \tag{3.40}$$

and we have subtracted a term which is zero. The expression is well-defined when F is Hölder in the x-variable, and justified by inserting a cut-off $\chi_h(t)$, supported away from $t=1$, then differentiating under the integral and finally letting $h \to 0$.

Note from (3.40) the useful inequality (for constants $A, B > 0$):

$$|D_{x_i x_j}\Phi(x-y,t-s)| \le A(t-s)^{-2}e^{-B\frac{|x-y|^2}{t-s}} \tag{3.41}$$

and note also that since $E \in C^{0,\alpha}_{\mathrm{hom}}(|x|^{-1})$

$$\Big|F(x,s)-F(y,s)\Big| \le \frac{\|E\|_{C^{0,\alpha}_{\mathrm{hom}}(|x|^{-1})}}{2}|x-y|^\alpha\Big(|x|^{-1-\alpha}+|y|^{-1-\alpha}\Big),$$

from the way in which we have defined $C^{0,\alpha}_{\text{hom}}(|x|^{-1})$.

Let us first prove that with $E \in C^{0,\alpha}_{\text{hom}}(|x|^{-1})$, we have

$$\sup_{x \in \mathbb{R}^2 \setminus B_R(0)} (1+|x|)|D_{x_i x_j} u(x)| \leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})} \tag{3.42}$$

Note that by virtue of the scaling in the definition of v, it suffices for (3.42) to establish that

$$\sup_{t \in (t_R,1)} \sup_{|x|=1} |D_{x_i x_j} v(x,t)| \leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})},$$

where $t_R := 1 - \frac{1}{2R^2}$. Let us fix $R = 2$, such that $t_R = \frac{7}{8}$. We see that for $|x| = 1$ we have from Equation (3.39)

$$\begin{aligned} |D_{x_i x_j} v(x,t)| &\leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})} \int_{\mathbb{R}^2} e^{-B'|y|^2}(1+|y|)dy \\ &+ C'\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})} \int_0^1 \int_{\mathbb{R}^2} |x-y|^\alpha \Big(|x|^{-1-\alpha} + |y|^{-1-\alpha}\Big)(1-s)^{-2} e^{-B\frac{|x-y|^2}{1-s}} dyds \\ &= C''\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}, \end{aligned}$$

where we used $|u(x-y)| \leq C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}(1+|y|)$ as well as (3.41) to estimate the first term, and where of course the integral

$$\int_0^1 \int_{\mathbb{R}^2} |y|^\alpha (1-s)^{-2} e^{-B\frac{|y|^2}{1-s}} dyds < \infty, \quad \text{for any } \alpha > 0.$$

Again, by the scaling in (3.36) and our definition of the weighted spaces, the desired estimate for the Hölder coefficients will follow if we can show that

$$\sup_{t\in(t_R,0)}\sup_{|x_0|\le|x|=1}|D_{x_ix_j}v(x,t)-D_{x_ix_j}v(x_0,t)|\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x_0|^{\alpha}.$$

Hence we compute for $|x_0|\le|x|\le 1$:

$$\begin{aligned}
&D_{x_ix_j}v(x,t)-D_{x_ix_j}v(x^0,t)=\\
&\int_{\mathbb{R}^2}\Big(D_{x_ix_j}\Phi(x-y,t)-D_{x_i^0x_j^0}\Phi(x^0-y,t)\Big)u(\tfrac{y}{\sqrt{2}})dy\\
&+\int_0^t\int_{|x-y|\le 2|x-x^0|}D_{x_ix_j}\Phi(x-y,t-s)\Big[F(y,s)-F(x,s)\Big]dyds\\
&-\int_0^t\int_{|x-y|\le 2|x-x^0|}D_{x_i^0x_j^0}\Phi(x^0-y,t-s)\Big[F(y,s)-F(x^0,s)\Big]dyds\\
&+\int_0^t\int_{|x-y|\ge 2|x-x^0|}\Big(D_{x_ix_j}\Phi(x-y,t-s)-D_{x_i^0x_j^0}\Phi(x^0-y,t-s)\Big)\Big[F(y,s)-F(x,s)\Big]dyds\\
&+\int_0^t\int_{|x-y|\ge 2|x-x^0|}D_{x_i^0x_j^0}\Phi(x^0-y,t-s)\Big[F(x,s)-F(x^0,s)\Big]dyds\\
&=:I_1+\ldots+I_5.
\end{aligned}\tag{3.43}$$

In this expression, the first term is estimated using the mean value principle, such that for

$$|D_{x_ix_j}\Phi(x-y,t)-D_{x_i^0x_j^0}\Phi(x^0-y,t)|\le|y+\xi|\frac{|x-x^0|}{t}\Phi(\xi+y,t)\le C'e^{-B'\frac{|y+\xi|^2}{t}}|x-x^0|^{\alpha},$$

for some point ξ on the line between the points x^0 and x, so $|\xi|\le 2$, and some constants $B',C'=C(t_R)$ independent of $|x|,|x^0|\le 1$. Hence one gets the estimate

$$|I_1|\le C|x-x^0|^{\alpha}\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}\int_{\mathbb{R}^2}e^{-B'\frac{|y|^2}{t}}(|y|+2)dy.\tag{3.44}$$

The terms I_2 and I_3 are of course symmetric in $x\leftrightarrow x^0$ and have similar estimates. For I_2 we get:

$$\begin{aligned}
|I_2|&\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}\int_0^t\int_{|x-y|\le 2|x-x^0|}(t-s)^{-2}e^{-B\frac{|x-y|^2}{t-s}}|x-y|^{\alpha}dyds\\
&\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}\int_{|x-y|\le 2|x-x'|}|x-y|^{-2+\alpha}\\
&\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x'|^{\alpha}.
\end{aligned}\tag{3.45}$$

For the term I_4 we use the estimate

$$|D_{x_ix_j}\Phi(x-y,t)-D_{x_i^0x_j^0}\Phi(x^0-y,t)|\le c|x-x^0|(t-s)^{-5/2}e^{-B\frac{|x-y|^2}{t-s}},$$

which holds whenever $|x-y|\ge 2|x-x^0|$. Hence we see

$$\begin{aligned}|I_4|&\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x^0|\int_0^t\int_{|x-y|\ge 2|x-x^0|}(t-s)^{-5/2}e^{-B\frac{|x-y|^2}{t-s}}|x-y|^\alpha dyds\\&\le C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x^0|^\alpha.\end{aligned}\tag{3.46}$$

For the last term, we rewrite it as

$$I_5=-\int_0^t\int_{|x-y|=2|x-x^0|}\frac{\partial\Phi(x^0-y,t-s)}{\partial x_j^0}\Big[F(x,s)-F(x^0,s)\Big](\vec{e}_i\cdot\vec{\nu})dM(y)ds,$$

where $\vec{e}_i$ the ith unit vector in $\mathbb{R}^2$, $dM(y)$ is the line element and $\vec{\nu}$ the outward pointing unit normal to the disk of radius $2|x-x^0|$. Since

$$\frac{\partial\Phi(x^0-y,t-s)}{\partial x_j^0}=-\frac{x_j^0-y_j}{8\pi(t-s)^2}e^{-\frac{|x^0-y|^2}{4(t-s)}},\tag{3.47}$$

we finally get

$$\begin{aligned}|I_5|&\le\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x^0|^{1+\alpha}\int_0^t\int_{|x-y|=2|x-x^0|}\frac{e^{\frac{-|x-x^0|^2}{4(t-s)}}}{4\pi(t-s)^2}dM(y)ds\\&=C\|E\|_{C^{0,\alpha}_{\text{hom}}(|x|^{-1})}|x-x^0|^\alpha.\end{aligned}$$

□

Using the second derivative bound we can now proceed to our final proposition of this section, which is a Liouville-type structure theorem in that we prove solutions are homogeneous degree one polynomials in x plus a remainder belonging to the space $C^{2,\alpha}_{\text{an}}(|x|^{-1})$. This detailed analysis of the solutions – completing our separation of the conical part – is exactly what will make our construction work.

Theorem 3.8.9 (Liouville-type result). *There is a constant $C>0$ s.t. for any G_τ-equivariant $E\in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})$ there exists a unique G_τ-equivariant $u=(c,f)\in\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})$ such that*

$$\mathcal{L}_{\mathcal{P}}\Big[c(\theta)\cdot|x|+f(x)\Big]=\mathcal{L}_{\mathcal{P}}u=E,$$

where $u=u_{(c,f)}$ and $u=0$ on $\partial\Omega$. Furthermore we have the estimate

$$\|(c,f):\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})\|\le C\|E:C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})\|.\tag{3.48}$$

Proof. Let $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ be a solution to $\mathcal{L}u = E$. It follows from the weighted Hölder estimates in Proposition 3.8.8 that

$$D_{x_i x_j} u \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1}), \tag{3.49}$$

and hence we have $\Delta u \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$ and hence $w := -\vec{X} \cdot \nabla u + u = E - \Delta u \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$. Solving for u in polar coordinates ($|x| = r$), we get after imposing initial conditions $u_{|\partial\Omega} = 0$ that (normalize here for simplicity the radius R of $\partial\Omega$ to 1):

$$u(r,\theta) = c(\theta)r + v_0, \tag{3.50}$$

$$c(\theta) := -\int_1^\infty \frac{w(s,\theta)}{s^2} ds = \lim_{r\to\infty} \frac{u(r,\theta)}{r}, \tag{3.51}$$

$$v_0(x) := r \int_r^\infty \frac{w(s,\theta)}{s^2} ds. \tag{3.52}$$

By (3.49) and (3.50), and the lemma for homogeneous functions (3.24), we see that also $D_{x_i x_j} v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$.

Now, $-\vec{X}\cdot\nabla v_0 + v_0 = -\vec{X}\cdot\nabla u + u = w \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$ from above. It follows easily from the formula (3.52) for v_0 that $v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$, and hence we see that also $\vec{X} \cdot \nabla v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$.

It remains to show that the full gradient satisfies

$$\nabla v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1}), \tag{3.53}$$

Note for this, that

$$\Delta v_0 = E - \Delta(c(\theta)|x|) + \vec{X} \cdot \nabla v_0 - v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1}). \tag{3.54}$$

Equation (3.53) follows now easily from this with $v_0 \in C^{0,\alpha}_{\text{hom}}(\Omega, |x|^{-1})$, by standard use of the Green's function for the ordinary flat Laplacian on the plane (see for example the estimate (10.1.30) in [Jo02]).

Hence we have shown that there is the desired Liouville decomposition, and the corresponding estimates follow. □

3.9 Linearized Equation on the Initial Surface $\mathcal{M}[\tau, \theta]$

We let $\underline{a} := 8|\log\tau|$ and then $\mathcal{N}^{\pm}_y, \mathcal{N}^{\pm}_x$ are used to denote the connected components of $\{\mathcal{M}[\tau,\theta] : s \geq \underline{a}\}$. Let also $\mathcal{S} := \mathcal{H}(\Sigma)$, where we denote by $\mathcal{H}$ the homothety by a factor of τ.

Definition 3.9.1. *Let $v \in C^{k,\alpha}_{\text{loc}}(\mathcal{M})$. We identify v with its restrictions to Σ, $\mathcal{N}^{\pm}_y$ and $\mathcal{N}^{\pm}_x$. Then for $k = 0, 2$ we define the norm $\|v\|_{\mathcal{X}\mathcal{S}^{k,\alpha}}$ to be the maximum of the following quantities, where $b_0 = e^{-5\delta_s/\tau}$ and $b_2 = e^{-5\delta_s/\tau}/\tau^{10}$:*

(1) $\tau^{1-k}\|v\circ\mathcal{H}\|_{C^{k,\alpha}(\Sigma,e^{-\gamma s},g_\Sigma)}$, *and*

(2) $b_k^{-1}\|v\|_{\mathcal{C}\mathcal{S}^{k,\alpha}(\mathcal{N}_x^{+}\backslash\mathcal{S},|x|^{-1})}$, *as given in Definition 3.8.5.*

(3) $b_k^{-1}\|v\|_{C^{k,\alpha}(\mathcal{N}_y^{\pm}\backslash\mathcal{S},g_{\mathcal{N}_y^{\pm}})}$, *and* $b_k^{-1}\|v\|_{C^{k,\alpha}(\mathcal{N}_x^{-}\backslash\mathcal{S},g_{\mathcal{N}_y^{\pm}})}$.

We let be $\mathcal{X}\mathcal{S}^{k,\alpha}(\mathcal{M})$ be the space of functions v for which $\|v\|_{\mathcal{X}\mathcal{S}^{k,\alpha}}<\infty$.

Lemma 3.9.2. *Let $\mathcal{N}_i$ stand for any of the ends $\mathcal{N}_y^{\pm}$, $\mathcal{N}_x^{\pm}$. Then for $\tau>0$ sufficiently small the Dirichlet operator, for zero initial value on $\partial\mathcal{N}_i$,*

$$\mathcal{L}_{\mathcal{N}_i}:\mathcal{X}\mathcal{S}_0^{2,\alpha}(\mathcal{N}_i)\to\mathcal{X}\mathcal{S}^{0,\alpha}(\mathcal{N}_i)$$

is invertible, with operator norm of the inverse bounded uniformly in $\tau>0$.

Proof. For the exterior flat domain, this is what was proved in Section 8. For the flat disk and round spherical cap, we check the invertibility by computing the Dirichlet spectrum of the stability operator $\mathcal{L}$ on these surfaces, using a perturbation argument to extend the property to the θ-family of spherical caps (by possibly taking δ_θ smaller). These spectrum computations can be found in the Appendix.

Note however that we are considering the region of $\tau\Sigma[\tau,\theta]$, very near the removed circle, and here the initial surface $\mathcal{M}[\tau,\theta]$ and hence the ends $\mathcal{N}$, do not exactly coincide with the subsets of the configuration $\mathcal{C}[\theta]$. The difference is on each piece a small normal graph with compact support, coming from the function $f(s,z)$ describing Scherk's surface as a graph over its four asymptotic planes. But by construction and the estimates (3.10) we verify that the cut-off $\underline{a}=8\log\tau$ is appropriately large, since for the two induced metrics in question,

$$\|g_{\mathcal{N}_i}-g_{\mathcal{C}[\theta]}:C^3(\{\tau\Sigma[\tau,\theta]:s\geq\underline{a}\},g_{\mathcal{C}[\theta]})\|\leq C\tau^{-2}e^{-\underline{a}}=C\tau^6,$$

and similarly for the induced second fundamental forms $|A|^2$, and hence the lemma follows for small enough $\tau>0$ by a perturbation within the compact domain $\{\tau\Sigma[\tau,\theta]:s\geq\underline{a}\}$, for the quantities used in the definition of $\mathcal{L}$. □

Note that the property (3.28) extends so that also $\mathcal{L}_{\mathcal{M}}$, the linearized operator of $H-\langle\vec{X},\vec{\nu}\rangle$ over $\mathcal{M}$, is a bounded map from the Hölder cone space.

Definition 3.9.3. *Let $\Theta:[-\delta_\theta,\delta_\theta]\to C^\infty(\mathcal{M})$ be given by*

$$\Theta(\theta)=\frac{1}{\tau}\mathcal{H}^*(\theta w),$$

where $\theta\in[-\delta_\theta,\delta_\theta]$.

Theorem 3.9.4. *Given $E\in\mathcal{X}\mathcal{S}^{0,\alpha}(\mathcal{M})$, there exist $v_E\in\mathcal{X}\mathcal{S}^{2,\alpha}(\mathcal{M})$ and $\theta_E\in\mathbb{R}$, such that*

$$\mathcal{L}_{\mathcal{M}}v_E=E+\Theta(\theta_E),$$

and

$$\|v_E\|_{\mathcal{XS}^{2,\alpha}} \leq C\|E\|_{\mathcal{XS}^{0,\alpha}}, |\theta_E| \leq C\|E\|_{\mathcal{XS}^{0,\alpha}},$$

Proof. Let the cut-off functions $\psi := \psi[5\delta_s/\tau, 5\delta_s/\tau - 1]\circ s$ as well as $\psi' := \psi[\underline{a}, \underline{a}+1]\circ s$ be given on $\mathcal{M}$, and let $\underline{a} = 8|\log\tau|$.

The starting point of our iteration is $E_0 := E$. Applying Proposition 3.7.3 to $\Sigma = \Sigma[\tau,\theta] = \mathcal{H}^{-1}(\mathcal{S})$ with the cut-off source term $E' := \tau(\psi E_{n-1})\circ\mathcal{H}$. From the corresponding v_E we get $v := \tau\mathcal{H}_*(v_E)$ and we let the $\theta_n := \theta_{E'}$. By construction we have thus on $\mathcal{S}$ that

$$\mathcal{L}_{\mathcal{M}} v = \psi E_{n-1} + \Theta(\theta_n).$$

We now feed the new source term $E'' = (1-\psi^2)E_{n-1} - [\mathcal{L}_{\mathcal{M}}, \psi]v$ into the equation on the union of the ends $\mathcal{N}_{\cup} := \mathcal{N}_y \cup \mathcal{N}_x^- \cup \mathcal{N}_x^+$ (here the commutator is by definition $[\mathcal{L}_{\mathcal{M}}, \psi]f := \mathcal{L}_{\mathcal{M}}(\psi)f - \psi(\mathcal{L}_{\mathcal{M}} f)$), and obtain a solution $v_{E''}$ which we call v',

$$\mathcal{L}_{\mathcal{M}} v' = (1-\psi^2)E_{n-1} - [\mathcal{L}_{\mathcal{M}}, \psi]v.$$

We then finally define

$$v_n := \psi v + \psi' v'.$$

By considering the supports of ψ, ψ' and $[\mathcal{L}_{\mathcal{M}}, \psi]$, we see that

$$\mathcal{L}_{\mathcal{M}} v_n = E_{n-1} + [\mathcal{L}_{\mathcal{M}}, \psi']v' + \Theta(\theta_n). \tag{3.55}$$

We then also define the new source term $E_n = -[\mathcal{L}_{\mathcal{M}}, \psi']v'$. Again, we use the fact that $[\mathcal{L}_{\mathcal{M}}, \psi']$ is supported on $[\underline{a}, \underline{a}+1]$, use Lemma 3.9.2, and estimate (for τ sufficiently small),

$$\begin{aligned}
\|E_n\|_{\mathcal{XS}^{0,\alpha}} &= \tau\|E_n \circ \mathcal{H} : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s})\| \\
&\leq \tau e^{\gamma(\underline{a}+1)}\|[\mathcal{L}_{\mathcal{M}}, \psi']v' \circ \mathcal{H} : C^{0,\alpha}(\Sigma_{[\underline{a},\underline{a}+1]}, g_\Sigma)\| \\
&\leq C\tau^{-p_0} e^{\gamma(\underline{a}+1)}\|v' \circ \mathcal{H} : C^{2,\alpha}(\Sigma_{[\underline{a},\underline{a}+1]}, g_\Sigma)\| \\
&\leq C\tau^{-p_0'} e^{\gamma(\underline{a}+1)} e^{-\left(\frac{5\delta_s}{\tau}-1\right)}\|(1-\psi^2)E_{n-1} - [\mathcal{L}_{\mathcal{M}}, \psi]v\|_{\mathcal{XS}^{0,\alpha}},
\end{aligned}$$

where we used in the third line the uniform control of the geometry of Σ in the strips $s \in [\underline{a}, \underline{a}+1]$, and in the third line the Definition 3.9.1, and the fact that the term considered in the last line has support in $s \in [\frac{5\delta_s}{\tau} - 1, \frac{5\delta_s}{\tau}]$, we thus get

$$\begin{aligned}
\|E_n\|_{\mathcal{XS}^{0,\alpha}} &\leq C\tau^{-p_0'} e^{\gamma(\underline{a}+1)} e^{-\left(\frac{5\delta_s}{\tau}-1\right)}\|E_{n-1}\|_{\mathcal{XS}^{0,\alpha}} \\
&\leq Ce^{-\frac{\delta_s}{\tau}}\|E_{n-1}\|_{\mathcal{XS}^{0,\alpha}}.
\end{aligned}$$

We define $v_E := \sum_{n=1}^\infty v_n$ and $\theta_E := \sum_{n=1}^\infty \theta_n$. The first sum converges in the Banach space $\mathcal{XS}^{2,\alpha}(\mathcal{M})$, the second converges to some real number which is the θ_E, with the desired estimates. The function v_E then satisfies $\mathcal{L}_{\mathcal{M}} v_E = E + \Theta(\theta_E)$. □

Definition 3.9.5. *Let $\mathcal{S}$ be a smooth surface (possibly with boundary). For a function $v \in C^{2,\alpha}(\mathcal{S})$ for which $\mathcal{S}_v$ is a $C^{2,\alpha}$-surface, we define on $\mathcal{S}$:*

$$\mathcal{F}_{\mathcal{S}}(v) := H_{\mathcal{S}_v} - \tfrac{1}{2}\langle \vec{X}, \vec{\nu}_{\mathcal{S}_v}\rangle,$$

and denote $\mathcal{F}_{\mathcal{S}} := \mathcal{F}_{\mathcal{S}}(0)$.

Corollary 3.9.6. *There are $v_{\mathcal{F}} \in \mathcal{XS}^{2,\alpha}(\mathcal{M})$ and $\theta_{\mathcal{F}}$ such that*

$$\begin{gathered}\mathcal{L}_{\mathcal{M}} v_{\mathcal{F}} = \mathcal{F}_{\mathcal{M}} + \Theta(\theta_{\mathcal{F}}) w,\\ |\theta_{\mathcal{F}} - \theta| \le C\tau, \quad \|v_{\mathcal{F}}\|_{\mathcal{XS}^{2,\alpha}} \le C\tau,\end{gathered}$$

where $\mathcal{M} = \mathcal{M}[\tau, \theta]$.

3.10 The Nonlinear Terms in Hölder Cone Spaces

Proposition 3.10.1. *Given $v \in \mathcal{XS}^{2,\alpha}(\mathcal{M})$ with $\|v\|_{\mathcal{XS}^{2,\alpha}}$ smaller than a suitable constant, we have that the graph $\mathcal{M}_v$ over $\mathcal{M}$, is a smooth immersion and moreover*

$$\mathcal{F}_{\mathcal{M}}(v) - \mathcal{F}_{\mathcal{M}} - \mathcal{L}_{\mathcal{M}} v \in \mathcal{XS}^{0,\alpha}(\mathcal{M}),$$

with the quadratic improvement bounds:

$$\|\mathcal{F}_{\mathcal{M}}(v) - \mathcal{F}_{\mathcal{M}} - \mathcal{L}_{\mathcal{M}} v\|_{\mathcal{XS}^{0,\alpha}} \le C\|v\|^2_{\mathcal{XS}^{2,\alpha}}. \tag{3.56}$$

Proof. We first deal with the argument needed on the exterior plane $\Omega = \mathbb{R}^2 \setminus B_R(0)$. Note that $\vec{\nu} \equiv \vec{e_3}$ and the terms for the equation (3.4) read

$$\mathcal{F}_{\Omega}(v) = -\frac{\operatorname{Hess} v(\nabla v, \nabla v)}{(1+|\nabla v|^2)^{3/2}} + \frac{\mathcal{L}_{\mathcal{P}} v}{\sqrt{1+|\nabla v|^2}}, \tag{3.57}$$

where $\mathcal{L}_{\mathcal{P}}$ is again the linearized operator from (3.17).

Thus we see that for the exterior plane Ω we have

$$\begin{aligned}\mathcal{F}_{\Omega}(v) - \mathcal{F}_{\Omega} - \mathcal{L}_{\Omega} v &= -\frac{\operatorname{Hess} v(\nabla v, \nabla v)}{(1+|\nabla v|^2)^{3/2}} + \left[(1+|\nabla v|^2)^{-1/2} - 1\right]\mathcal{L}_{\mathcal{P}} v\\ &=: T_1 + T_2.\end{aligned}$$

Let us first estimate the weighted sup-norm. By the Bernoulli inequalities we have the quadratic bounds:

$$\begin{aligned}\left|(1+|\nabla v|^2)^{-1/2} - 1\right| &\le \tfrac{1}{2}|\nabla v|^2,\\ \left|(1+|\nabla v|^2)^{-3/2} - 1\right| &\le \tfrac{3}{2}|\nabla v|^2.\end{aligned} \tag{3.58}$$

We can now estimate the supremum part of the weighted norms:

$$\begin{aligned}|x|\,|(\mathcal{F}_\Omega(v)-\mathcal{F}_\Omega(0)-\mathcal{L}_\Omega v)(x)| &\le |x|\,|\operatorname{Hess} v|\,|\nabla v|^2+\tfrac{1}{2}|x|\,|\nabla v|^2|\mathcal{L}_{\mathcal{P}}v|\\ &\le \|v\|^3_{\mathcal{CS}^{2,\alpha}}+\tfrac{1}{2}\|v\|^3_{\mathcal{CS}^{2,\alpha}}=\tfrac{3}{2}\|v\|^3_{\mathcal{CS}^{2,\alpha}},\end{aligned}$$

on the exterior of the disk, where we used again the crucial mapping property (3.28) on the Hölder cone spaces.

Similar but slightly more involved computations now show that the Hölder coefficients in the norm are also estimated as claimed. For example it follows by (3.58) that

$$\begin{aligned}\frac{\left|(1+|\nabla v(x)|^2)^{-1/2}-(1+|\nabla v(y)|^2)^{-1/2}\right|}{|x-y|^\alpha} &\le \frac{|\nabla v(x)|+|\nabla v(y)|}{2(1+|\nabla v(y)|^2)^{3/2}}\frac{\left|\nabla v(x)-\nabla v(y)\right|}{|x-y|^\alpha}\\ &\le [|x|^{-\alpha}+|y|^{-\alpha}]\,\|v\|^2_{\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})}.\end{aligned}$$

It follows easily that

$$T_2:=\left[\frac{1}{\sqrt{1+|\nabla v|^2}}-1\right]\mathcal{L}_{\mathcal{P}}v\in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1}), \tag{3.59}$$

since by assumption $\mathcal{L}_{\mathcal{P}}v\in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})$ and also $\nabla v\in C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{0})$. We get the corresponding higher order bounds as follows. Assume without loss of generality that $|x|\ge|y|$, which is reflected in how we distribute terms, and recall the estimates (3.24) in Lemma 3.8.4:

$$\begin{aligned}\frac{|T_2(x)-T_2(y)|}{|x-y|^\alpha} &\le \frac{\left|(1+|\nabla v(x)|^2)^{-1/2}-(1+|\nabla v(y)|^2)^{-1/2}\right|}{|x-y|^\alpha}|(\mathcal{L}_{\mathcal{P}}v)(x)|\\ &\quad+\tfrac{1}{2}|\nabla v|^2\frac{|(\mathcal{L}_{\mathcal{P}}v)(x)-(\mathcal{L}_{\mathcal{P}}v)(y)|}{|x-y|^\alpha}\\ &\le [|x|^{-\alpha}+|y|^{-\alpha}]\,\|v\|^2_{\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})}\frac{\|\mathcal{L}_{\mathcal{P}}v\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})}}{|x|}\\ &\quad+\tfrac{1}{2}\left[|x|^{-1-\alpha}+|y|^{-1-\alpha}\right]\|v\|^2_{\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})}\|\mathcal{L}_{\mathcal{P}}v\|_{C^{0,\alpha}_{\text{hom}}(\Omega,|x|^{-1})}\\ &\le \frac{3}{2}\left[|x|^{-1-\alpha}+|y|^{-1-\alpha}\right]\|v\|^3_{\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})}.\end{aligned}$$

As for the term T_1, we write $\operatorname{Hess} v(\nabla v,\nabla v)=\sum_{i,j}(D_{x_ix_j}v)(D_{x_i}v)(D_{x_j}v)$. We again have an estimate

$$\begin{aligned}\frac{\left|(1+|\nabla v(x)|^2)^{-3/2}-(1+|\nabla v(y)|^2)^{-3/2}\right|}{|x-y|^\alpha} &\le \frac{3}{2}\frac{|\nabla v(x)|+|\nabla v(y)|}{(1+|\nabla v(y)|^2)^{5/2}}\frac{\left|\nabla v(x)-\nabla v(y)\right|}{|x-y|^\alpha}\\ &\le 3\,[|x|^{-\alpha}+|y|^{-\alpha}]\,\|v\|^2_{\mathcal{CS}^{2,\alpha}(\Omega,|x|^{-1})}.\end{aligned}$$

We find by the above, since we may again assume $|x| \geq |y|$,

$$\begin{aligned}
\frac{|T_1(x)-T_1(y)|}{|x-y|^\alpha} &\leq \frac{|(D_{x_ix_j}v)(x)-(D_{x_ix_j}v)(y)|}{|x-y|^\alpha}\frac{|\nabla v(y)|^2}{(1+|\nabla v(y)|^2)^{3/2}} \\
&+|(D_{x_ix_j}v)(x)|\frac{|\nabla v(x)-\nabla v(y)|}{|x-y|^\alpha}\frac{|\nabla v(y)|}{(1+|\nabla v(y)|^2)^{3/2}} \\
&+|(D_{x_ix_j}v)(x)|\frac{|\nabla v(x)-\nabla v(y)|}{|x-y|^\alpha}\frac{|\nabla v(x)|}{(1+|\nabla v(y)|^2)^{3/2}} \\
&+|(D_{x_ix_j}v)(x)||\nabla v(x)|^2\frac{\left|(1+|\nabla v(x)|^2)^{-3/2}-(1+|\nabla v(y)|^2)^{-3/2}\right|}{|x-y|^\alpha} \\
&\leq \left[|x|^{-1-\alpha}+|y|^{-1-\alpha}\right]\|v\|_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\|v\|^2_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})} \\
&+2|x|^{-1}\|v\|_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\left[|x|^{-\alpha}+|y|^{-\alpha}\right]\|v\|_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\|v\|_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})} \\
&+3|x|^{-1}\|v\|_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\|v\|^2_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\left[|x|^{-\alpha}+|y|^{-\alpha}\right]\|v\|^2_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})} \\
&\leq 3\left[|x|^{-1-\alpha}+|y|^{-1-\alpha}\right]\left(\|v\|^3_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}+\|v\|^5_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\right).
\end{aligned}$$

Collecting these estimates, we have shown:

$$\|\mathcal{F}_\Omega(v)-\mathcal{F}_\Omega(0)-\mathcal{L}_\Omega v : C^{0,\alpha}_{\mathrm{hom}}(\Omega,|x|^{-1})\| \leq C\left(\|v\|^3_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}+\|v\|^5_{\mathcal{C}\mathcal{S}^{2,\alpha}(\Omega,|x|^{-1})}\right).$$

Picking now $\tau > 0$ small enough in terms of δ_s to ensure $b_0 > 1$ (and hence also $b_2 > 1$) in the Definition 3.9.1 of $\mathcal{X}\mathcal{S}^{2,\alpha}(\mathcal{M})$, we see that taking $\|v\|_{\mathcal{X}\mathcal{S}^{2,\alpha}(\mathcal{M})} \leq 1$, we finally obtain:

$$\|\mathcal{F}_\Omega(v)-\mathcal{F}_\Omega(0)-\mathcal{L}_\Omega v : \mathcal{X}\mathcal{S}^{0,\alpha}(\mathcal{M})\| \leq C\|v : \mathcal{X}\mathcal{S}^{2,\alpha}(\mathcal{M})\|^2. \tag{3.60}$$

For the core piece $\Sigma[\theta,\tau]$, the argument follows closely the one in [Ka97]. Namely, using the uniform control on the geometry $\|A : C^3(\Sigma, g_\Sigma)\| \leq C$ and $\|\tau^2\vec{X}\cdot\vec{\nu} : C^0(\Sigma)\| \leq \tau$ with the expression for the quadratic term in Equation (3.72), one obtains again that when $\|f : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s}+b_2)\|$ is small enough,

$$\begin{aligned}
&\|\mathcal{F}_\Sigma(f)-\mathcal{F}_\Sigma-\tau^2\mathcal{L}_\Sigma f : C^{0,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s})\| \\
&\qquad \leq C\|f : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s}+b_2)\|^2.
\end{aligned}$$

For the central disk and the top and bottom spherical caps the proofs are again the same, by uniform control of the geometry and (3.72). □

3.11 Fixed Point Argument: Existence of the Self-Shrinkers

We consider for any fixed $0 < \alpha' < \alpha < 1$ the corresponding Banach space

$$\mathcal{X}\mathcal{S}^{2,\alpha'} := \mathcal{X}\mathcal{S}^{2,\alpha'}(\mathcal{M}[\tau,0]),$$

from the family that we previously studied, and take the subsets

$$\Xi = \{(\theta, u) \in [-\delta_\theta, \delta_\theta] \times \mathcal{XS}^{2,\alpha'} : |\theta| \leq \zeta\tau, \|u\|_{\mathcal{XS}^{2,\alpha}} \leq \zeta\tau\}.$$

We state the following lemma (as in [Ka97]), whose easy proof we omit.

Lemma 3.11.1. *There is for $\theta \in [-\delta_\theta, \delta_\theta]$ a smooth family of diffeomorphisms*

$$D_\theta : \mathcal{M}[\tau, 0] \to \mathcal{M}[\tau, \theta], \quad \textit{with}$$

$$\|f_1 \circ D_\theta^{-1}\|_{\mathcal{XS}^{2,\alpha}} \leq C\|f_1\|_{\mathcal{XS}^{2,\alpha}}, \quad \|f_2 \circ D_\theta\|_{\mathcal{XS}^{2,\alpha}} \leq C\|f_2\|_{\mathcal{XS}^{2,\alpha}}, \tag{3.61}$$

for all $f_1 \in C^{2,\alpha}(\mathcal{M}[\tau, 0])$ and $f_2 \in C^{2,\alpha}(\mathcal{M}[\tau, \theta])$.

The problem stated in Theorem 3.9.4 is then continuous in τ and θ in the sense that, for fixed $E \in \mathcal{XS}^{0,\alpha}(\mathcal{M}[\tau, 0])$, the pair

$$(v_{E \circ D_\theta^{-1}} \circ D_\theta, \theta_{E \circ D_\theta^{-1}}) \in \mathcal{XS}^{2,\alpha}(\mathcal{M}[\tau, 0]) \times \mathbb{R}$$

depends continuously on τ and θ.

We define the map $\mathcal{J} : \Xi \to [-\delta_\theta, \delta_\theta] \times \mathcal{XS}^{2,\alpha'}$ as follows: Let $(\theta, u) \in \Xi$ and let $v := u \circ D_\theta^{-1} - v_{\mathcal{F}}$, where $v_{\mathcal{F}}$ comes from an application of Corollary 3.9.6, and the function $\mathcal{F} = \mathcal{F}_{\mathcal{M}[\tau,\theta]}(0)$ as before is defined on $\mathcal{M}[\tau, \theta]$. We thus have

$$\|v\|_{\mathcal{XS}^{2,\alpha}} \leq C(\zeta + 1)\tau.$$

Now, we use Proposition 3.10.1 to get that M_v is well-defined, and

$$\|\mathcal{F}_{\mathcal{M}}(v) - \mathcal{F}_{\mathcal{M}} - \mathcal{L}_{\mathcal{M}} v\| \leq C(\zeta + 1)^2\tau^2.$$

Inserting therefore $E = \mathcal{F}_{\mathcal{M}}(v) - \mathcal{F}_{\mathcal{M}} - \mathcal{L}_{\mathcal{M}} v$ into Theorem 3.9.4 gives a v_E and θ_E. We obtain, for some appropriate constant C_0 that:

$$\mathcal{F}_{\mathcal{M}}(v) = \mathcal{L}_{\mathcal{M}}\left(u \circ D_\theta^{-1} + v_E\right) - \Theta(\theta_{\mathcal{F}} + \theta_E), \tag{3.62}$$

$$|\theta - \theta_{\mathcal{F}} - \theta_E| \leq C_0\left(\tau + (\zeta + 1)^2\tau^2\right), \tag{3.63}$$

$$\|v_E\|_{\mathcal{XS}^{2,\alpha}} \leq 2C_0(\zeta + 1)^2\tau^2. \tag{3.64}$$

Then the definition of $\mathcal{J}$ is taken to be

$$\mathcal{J}(\theta, u) = \left(\theta - \theta_{\mathcal{F}} - \theta_E, -v_E \circ D_\theta\right). \tag{3.65}$$

Thus, by assuming ζ large enough and $\tau > 0$ small enough, we arrange that $\mathcal{J}(\Xi) \subseteq \Xi$. By the properties of our weighted spaces in Proposition 3.8.7 and $\alpha' < \alpha$, Ξ is a compact subset of $[-\delta_\theta, \delta_\theta] \times \mathcal{XS}^{2,\alpha'}$ and is also convex. The map $\mathcal{J}$ is continuous by Definition 3.11.1 and Proposition 3.9.4. Finally, by Schauder's fixed point theorem, we get existence of the desired fixed point $(\theta^*, u^*) \in \Xi$, and we see that the corresponding $\mathcal{M}[\tau,\theta^*]_{u^*}$ is an immersed self-shrinker. The rest of the proof of the main theorem now follows easily.

For example, embeddedness is assured by our setup: By construction, there is some fixed ball B_{R_0} such that the end of every Σ_g is graphical outside that ball, and hence embedded. Now, above one could pick $\zeta = 2C_0$ independent of τ, one concludes for all $\tau > 0$ small enough in terms of this that $\|u^*\|_{\mathcal{X}\mathcal{S}^{0,\alpha}} \leq C\tau^2$, and since also by construction the normal injectivity radius of a compact piece the initial surface, say $B_{R_0+1}(0)$, can be assumed bounded below as $\mathrm{inj}_\perp(\mathcal{M}[\tau,\theta] \cap B_{R_0+1}(0)) \geq c\tau$, for some $c > 0$, it follows that (for possibly even smaller $\tau > 0$) the constructed surfaces Σ_g are embedded. The Hausdorff convergence statement (v) in Theorem 3.1.1 also follows immediately from the definitions of the norms.

It also follows easily that each surface Σ_g is geodesically complete. Namely, a curve that leaves every compact set must have infinite length, as follows by projecting it onto the plane and estimating the arc length from below, again since the ends are graphical outside some ball.

3.12 Appendix 3.A: The Building Blocks of the Initial Surfaces

The proof of Proposition 3.5.5 follows from the following lemma.

Lemma 3.12.1. *There exist $\delta, \varepsilon_0, \varepsilon_1 > 0$ such that there is a smooth map $h \mapsto \rho_h$ for $h \in (2-\delta, 2+\delta)$ such that*

- *For each h, the function ρ_h is a generates a curve contained in the set $\{(x,0,z) : x, z \geq 0\}$.*
- *$\rho_0(\varphi) \equiv 2$, with $\varphi = \arctan(x/z)$, and $\rho_h(0) = 2+h, \rho_h'(0) = 0$.*
- *The following are orientation-reversing diffeomorphisms:*

  (1) $h \mapsto \rho_h(0) : (2-\delta, 2+\delta) \longrightarrow (2-\varepsilon_0, 2+\varepsilon_0)$,

  (2) $h \mapsto \rho_h'(\pi/2) : (2-\delta, 2+\delta) \longrightarrow (-\varepsilon_1, \varepsilon_1)$.
- *The graph $(\rho_h(\varphi), \varphi)$, $\varphi \in (0, \pi/2)$, in the xz-plane gives by revolution a self-shrinker.*

Proof of Lemma 3.12.1. A curve $(\rho, \varphi), \varphi = \arctan(x/z)$ in the xz-plane generated by a function $\rho(\varphi)$ that generates a smooth solution to the self-shrinker equation (3.4) satisfies:

$$\rho''(\varphi) = \frac{1}{\rho}\left\{\rho^2 + 2(\rho')^2 + \left[1 - \frac{\rho^2}{2} - \frac{\rho'}{\rho\tan\varphi}\right]\left(\rho^2 + (\rho')^2\right)\right\}. \tag{3.66}$$

The Taylor-expansion in the Banach space of C^2 functions of the solution in the h-parameter is

$$\rho_h(\varphi) = 2 + (h-2)w_1(\varphi) + \frac{(h-2)^2}{2}w_2(\varphi) + O\left((h-2)^3\right),$$

where w_i are smooth functions. The w_i satisfy the conditions $w_1(0) = 1$ and $w_1'(0) = 0$ (and $w_2(0) = w_2'(0) = 0$ and similarly for higher corrections), and as is easily computed w_1 satisfies the linear equation

$$w_1'' + \frac{1}{\tan\varphi}w_1' + 4w_1 = 0, \tag{3.67}$$

while w_2 satisfies a linear equation where the w_1 enters into the coefficients.

The claims (1) and (2) follow from the following two properties

$$w_1(\tfrac{\pi}{2}) < 0, \tag{3.68}$$
$$w_1'(\tfrac{\pi}{2}) < 0, \tag{3.69}$$

for the solution to (3.67) having $w_1(0) = 1$ and $w_1'(0) = 0$.

In fact since if we subsitute $x = \cos(\varphi)$ in the equation (3.67) to obtain Legendre's differential equation, the explicit general solution to this initial value problem is of course well-understood, namely

$$w_1(\varphi) = C_1 P_l(\cos\varphi) + C_2 Q_l(\cos\varphi),$$

where P_l and Q_l are respectively the Legendre functions of the first and second kind, and $l = (\sqrt{17}-1)/2$ is the positive solution to $l(l+1) = 4$. Here we see $C_2 = 0$, since $Q_l(\cos\varphi)$ has a pole at $\varphi = 0$, and $C_1 = 1$ since $P_l(1) = 1$. Thus the properties are easily verified and the lemma follows. □

Proof of Proposition 3.5.5. Given ρ_h constructed above, set $\theta = \tan\{\rho_h'(\pi/2)\}$. We then take $\mathcal{K}_\theta$ to be the surface immersed by the map κ_θ given by

$$\kappa_\theta(s,z) = r(\varphi(s))(\cos z, \sin z, 0) + (0, 0, z(\varphi(s)))$$

where $r(\varphi) = \rho_h(\varphi)\sin(\varphi)$, $z(\varphi) = \rho_h\cos(\varphi)$, and the map $s \mapsto \varphi(s)$ satisfying

$$s_\varphi = \frac{\sqrt{r_\varphi^2 + x_{3,\varphi}^2}}{r(\varphi)}, \quad s(\pi/2) = 0.$$

That (0)-(3) are satisfied by the family $\mathcal{K}_\theta$ are clear by construction. Likewise, once it is checked that $s(\varphi)$ is a conformal parameter, (4)i - iv are easy to verify. □

3.13 Appendix 3.B: Variation Formulae

Let $\vec{X}: M \to \mathbb{R}^3$ be a C^2-immersion of a surface. We denote by $\vec{X}_u : M \to \mathbb{R}^3$ the surface $\vec{X}_u = \vec{X} + u\vec{\nu}$. Then denoting by H_u and $\vec{\nu}_u$ etc. the quantities for $\vec{X}_u$, we get (see [Ka97] and [Ng09])

$$\vec{\nu}_u = \vec{\nu} - \nabla u + \vec{Q}_u^\nu, \tag{3.70}$$

$$H_u = H - (\Delta u + |A|^2 u) + Q_u, \tag{3.71}$$

$$\begin{aligned} H_u - \tfrac{1}{2}\tau^2 \vec{X}_u \cdot \vec{\nu}_u = H &- \tfrac{1}{2}\tau^2 \vec{X} \cdot \vec{\nu} \\ &- \left[\Delta u + |A|^2 u - \tfrac{1}{2}\tau^2(\vec{X} \cdot \nabla u - u)\right] \\ &+ Q_u + \tfrac{1}{2}\tau^2 \vec{X} \cdot \vec{Q}_u, \end{aligned} \tag{3.72}$$

where the quantities Q_u and $\vec{Q}_u$ are quadratic.

3.14 Appendix 3.C: Stability Operators

Let $M^2 \hookrightarrow N := (\mathbb{R}^3, h = e^{2\omega} h_0)$ be an immersion into a conformally changed Euclidean space, where $\omega : N \to \mathbb{R}$. Here h_0 will denote the standard metric $h_0 = \delta_{ij}$.

Denote by g_0 and g the metrics induced on M^2 from respectively h_0 and g by the immersion.

Here we have the conventions:

$$\Delta f = \operatorname{div}(\nabla f) = \operatorname{tr}(\nabla_i \partial_j f),$$
$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$
$$\operatorname{Ric}(X,Y) = \operatorname{tr}(Z \mapsto R(X,Z)Y),$$

so that the Ricci curvature of the standard round sphere is positive.

Then we have the following Lemma.

Lemma 3.14.1. *Assume M^2 is an oriented minimal surface in N. Then the stability operator of M^2 is the operator on functions on M^2 given by:*

$$\begin{aligned} L_g &= \Delta_g + |A_h|_g^2 + \operatorname{Ric}^h(\vec{\nu},\vec{\nu}) \\ &= e^{-2\omega}\Big[\Delta_{g_0} + e^{-2\omega}|A_h|_{g_0}^2 - \operatorname{Hess}_{h_0}(\vec{\nu}_0,\vec{\nu}_0)\omega + (\vec{\nu}_0.\omega)^2 - \Delta_{h_0}\omega - \|\nabla_{h_0}\omega\|_{h_0}^2\Big], \end{aligned}$$

where $\vec{\nu}_0$ is the unit normal vector w.r.t the metric h_0.

From this formula, we get the stability operators:

Proposition 3.14.2.

(i) The stability operator of the sphere $\mathbb{S}^2$ of radius 2 in $\mathbb{R}^3$ as a minimal surface in the metric $g = e^{\frac{-|x|^2}{4}}\delta$ is:

$$L = e\Big(\Delta_{\mathbb{S}_2^2} + 1\Big). \tag{3.73}$$

In particular $\ker(L) = \{0\}$ *on $\mathbb{S}_2^2$, as well as on the hemispheres of radius 2 with Dirichlet boundary conditions.*

(ii) The stability operator on a flat plane through the origin is

$$L = e^{\frac{|x|^2}{4}}\Big(\Delta_{\mathbb{R}^2} - \frac{|x|^2}{16} + 1\Big), \tag{3.74}$$

where $\Delta_{\mathbb{R}^2}$ is the usual flat Laplacian in $(\mathbb{R}^2, \delta_{ij})$. In particular, on both the disk of radius $\sqrt{2}$, and of radius 2, $\ker(L) = \{0\}$ *when we impose Dirichlet boundary conditions.*

Proof. Recall that by definition $A(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y$, where $\bar{\cdot}$ means a smooth extension to a neighborhood in N, and

$$|A|^2 = g^{ij}g^{kl}a_{ik}a_{jl} = e^{-4\omega}g_0^{ij}g_0^{kl}a_{ik}a_{jl}.$$

We also recall the conformal changes of the Levi-Civitas:

$$\nabla^h_{\bar X}\bar Y=\nabla^{h_0}_{\bar X}\bar Y+(\bar X.\omega)Y+(\bar Y.\omega)\bar X-h_0(\bar X,\bar Y)\nabla^{h_0}\omega \tag{3.75}$$

$$\nabla^g_X Y=\nabla^{g_0}_X Y+(X.\omega)Y+(Y.\omega)X-g_0(X,Y)\nabla^{g_0}\omega. \tag{3.76}$$

This gives that

$$A_h(X,Y)=A_{h_0}(X,Y)-g_0(X,Y)\left\{\nabla^{h_0}\omega-\nabla^{g_0}\omega\right\} \tag{3.77}$$

The Ricci curvature changes in dimension $n=3$ when $h_0=\delta$ according to:

$$\begin{aligned}\mathrm{Ric}^h&=\mathrm{Ric}^{h_0}-(n-2)\left[\nabla^{h_0}\,\mathrm{d}\,\omega-\mathrm{d}\,\omega\otimes\mathrm{d}\,\omega\right]+\left[-\Delta^{h_0}\omega-(n-2)\|\nabla^{h_0}\omega\|^2_{h_0}\right]h_0,\\&=-\mathrm{Hess}^{h_0}\,\omega+\mathrm{d}\,\omega\otimes\mathrm{d}\,\omega-\left[\Delta^{h_0}\omega+\|\nabla^{h_0}\omega\|^2_{h_0}\right]h_0.\end{aligned}$$

Here we have used that for $h_0=\delta$ we have

$$\nabla^{h_0}\,\mathrm{d}\,\omega=\mathrm{Hess}\,\omega,$$

and that $\vec\nu=e^{-\omega}\nu_0$ is the new unit normal.

Recall also that in 2 dimensions the Laplacian is conformally covariant:

$$\Delta_g=e^{-2\omega}\Delta_{g_0}. \tag{3.78}$$

Using these formulae, the Lemma follows.

To prove the proposition, we need $\omega=-\frac{|x|^2}{8}$. Thus we have

$$\begin{aligned}&\nabla^{h_0}\omega=-\frac{1}{4}x,\quad \mathrm{Hess}\,\omega(\partial_i,\partial_j)=-\frac{1}{4}\delta_{ij},\\&\Delta\omega=-\frac{3}{4},\quad \|\nabla^{\mathbb{R}^3}\omega\|^2_{\mathbb{R}^3}=\frac{1}{16}|x|^2.\end{aligned}$$

And we get on $\mathbb{R}^3$ that

$$\begin{aligned}(\mathrm{Ric}_h)_{ij}&=-\mathrm{Hess}_{h_0}\,\omega(\partial_i,\partial_j)+(\partial_i\omega)(\partial_j\omega)-\left[\Delta^{\mathbb{R}^3}\omega+\|\nabla^{\mathbb{R}^3}\omega\|^2\right]\delta_{ij}\\&=\frac{1}{4}\delta_{ij}+\frac{1}{16}x_ix_j+\frac{3}{4}\delta_{ij}-\frac{|x|^2}{16}\delta_{ij}=\delta_{ij}+\frac{1}{16}x_ix_j-\frac{|x|^2}{16}\delta_{ij}.\end{aligned}$$

Thus we get

$$\mathrm{Ric}^h(\vec\nu,\vec\nu)=e^{-2\omega}\left[1+\frac{|x\cdot\nu|^2}{16}-\frac{|x|^2}{16}\right], \tag{3.79}$$

so that on the round sphere of radius 2,

$$\mathrm{Ric}^h(\vec\nu,\vec\nu)=e^{-2\omega}. \tag{3.80}$$

Now, we pull back the induced metric g_2 on $\mathbb{S}^2_2$ of radius 2 by the map $\Phi(x) = 2x$ taking $\mathbb{S}^2_1 \to \mathbb{S}^2_2$ to get the isometry $(\mathbb{S}^2_1, \Phi^* g_2) \simeq (\mathbb{S}^2_2, g_2)$. Then note that for $X, Y \in T\mathbb{S}^2_1$ we have $\Phi^* g_2(X,Y) = g_{\mathbb{R}^3}(d\Phi(X), d\Phi(Y)) = 4g_{\mathbb{R}^3}(X,Y) = 4g_1$. Thus by the covariance in Equation (3.78), the spectrum of the operator L is the same as that of $\Delta_{\mathbb{S}^2_1} + 4$ on the sphere of radius 1.

Now, since the eigenvalues of Δ on the unit sphere $\mathbb{S}^2 = \mathbb{S}^2_1$ are

$$\lambda_k = -k(k+1),$$

we see that $\Delta_{\mathbb{S}^2_1} + 4$ is invertible on the sphere. The eigenvalues for the Dirichlet problem for Δ on the hemispheres are the same, but with smaller multiplicity (and in particular 0 is not an eigenvalue). Thus $\Delta_{\mathbb{S}^2_1} + 4$ is also invertible there.

Considering the plane $\{z = 0\}$, one gets similarly $A = 0$, and

$$\mathrm{Ric}^h(\vec{\nu}, \vec{\nu}) = e^{-2\omega}\left(1 - \frac{|x|^2}{16}\right). \tag{3.81}$$

Recall that for the Dirichlet problem for the harmonic oscillator on the unit disk, we have that $\lambda_k = -k^2$, where $k = 1, 2, 3, \ldots$ are the integers. Thus $\lambda_k = -\frac{k^2}{2}$ on the disk of radius $\sqrt{2}$, while $\lambda_k = -\frac{k^2}{4}$ on the disk of radius 2. Thus in either case the corresponding stability operator L is invertible. $\square$

3.15 Appendix 3.D: Laplacians on Flat Cylinders

We here recall a simple analytical result on (Ω, g_0) the flat cylinder $\Omega = H^+_{\leq l}/G$ equipped with the standard metric $g_0 = ds^2 + dz^2$, where G is the group generated by $(s,z) \to (s, z + 2\pi)$, and $l \in (10, \infty)$ is called the length of the cylinder. We have $\partial\Omega = \partial_0 \cup \partial_l$ where ∂_0 and ∂_l are the boundary circles $\{s = 0\}$ and $\{s = l\}$ respectively.

Let $\mathcal{L}$ on the flat cylinder (Ω, g_0) be given by

$$\mathcal{L}v = \Delta_\chi v + \mathbf{A} \cdot \nabla v + B \cdot v, \tag{3.82}$$

where χ is a C^2 Riemannian metric, $\mathbf{A} \in C^1(\Omega, \mathbb{R}^2)$ is a vector field, and $B \in C^1(\Omega)$. We define

$$N(\mathcal{L}) := \left\{ \|\chi - g_0 : C^2(\Omega, g_0)\| + \|\mathbf{A} : C^1(\Omega, g_0)\| + \|B : C^1(\Omega, g_0)\| \right\}$$

Proposition 3.15.1. *Given $\gamma \in (0,1)$ and $\varepsilon > 0$, if $N(\mathcal{L})$ is small enough in terms of α, γ and ε, then there is a bounded linear map*

$$\underline{\mathcal{R}} : C^{2,\alpha}(\partial_0, g_0) \times C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \to C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$$

such that for (f, E) in the domain of $\underline{\mathcal{R}}$ and $v = \underline{\mathcal{R}}(f, E)$, the following properties are true, where the constants C depend only on α and γ:

1. *$\mathcal{L}v = E$ on Ω.*

2. *$v = f - \text{avg}_{\partial_0} f + B(f,E)$ on ∂_0, where $B(f,E)$ is a constant on ∂_0 and $\text{avg}_{\partial_0} f$ denotes the average of f over ∂_0.*

3. *$v \equiv 0$ on ∂_l.*

4. $\|v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\|$
   $\leq C\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|.$

5. $|B(f,E)| \leq \varepsilon\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0, g_0)\| + C\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|.$

6. *If E vanishes, then*
$$\|v : C^0(\Omega)\| \leq 2\|v : C^0(\partial_0)\|.$$

Moreover, the function v depends continuously on $\mathcal{L}$.

Proof. For a metric $\chi = (\chi_{ij})$ in local coordinates, we can write the Laplace operator Δ_χ as
$$\Delta_\chi = \chi^{ij}\partial_{ij} + \chi^{ij}_{,i}\partial_j + \chi^{kj}\Gamma^i_{ik}\partial_j \tag{3.83}$$
where $\chi^{-1} = (\chi^{ij})$ is the inverse matrix for (χ_{ij}), and where
$$\Gamma^k_{ij} = \frac{1}{2}\chi^{lk}\left(\chi_{li,j} + \chi_{jl,i} - \chi_{ij,l}\right) \tag{3.84}$$
are the Christoffel symbols for the Riemannian connection for χ. By (3.83) we have that
$$\|\mathcal{L} - \Delta_{g_0}\| \leq C\mathcal{N}(\mathcal{L})$$
where $\|\mathcal{L}-\Delta_{g_0}\|$ denotes the operator norm of $\mathcal{L}-\Delta_{g_0}$ as a map from $C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})$ to $C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})$. Thus by taking $\mathcal{N}(\mathcal{L})$ sufficiently small we can arrange so that
$$\|\mathcal{L} - \Delta_{g_0}\| < \delta \tag{3.85}$$
for any $\delta > 0$. Despite the presence of small L^2 eigenvalues for the flat laplacian Δ_{g_0} on a long cylinder we can still define a uniformly bounded inverse as follows: Given a function $E \in C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})$, write $E(s,z) = E_0(s,z) + e_0(z)$, with $e_0(z) = \frac{1}{2\pi}\int_{\sigma=z} E(s,\sigma)ds$ the radial average of E. Then for any function $f \in C^{2,\alpha}(\partial_0)$ we can solve
$$\begin{aligned} \Delta_{g_0} U_0 &= E_0 \\ U_0 &= f - \text{avg}_{\partial_0} f \text{ on } \partial_0 \\ U_0 &= 0 \text{ on } \partial_l \end{aligned}$$
with
$$\|U_0 : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| \leq C\|E_0 : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\| + C\|f - \text{avg}_{\partial_0} f : C^{2,\alpha}(\partial_0)\|.$$

The radial part which projects onto the small eigenvalues is then directly integrated by setting $u_0(z) = \int_z^l \int_s^l e_0(t)dtds$. We then have

$$\begin{aligned}\mathcal{L}(U_0 + u_0) &= E + (\mathcal{L} - \Delta_{g_0})(U_0 + u_0) := E + E_1\\ U_0 + u_0 &= c_0 \text{ on } \partial_0\\ U_0 + u_0 &= 0 \text{ on } \partial_l\end{aligned}$$

where E_1 is defined by the equality above and satisfies

$$\begin{aligned}\|E_1 : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\| &\leq \delta\|U_0 + u_0 : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\|\\ &\leq \delta(C_0 + 1)\|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\|\end{aligned}$$

where C_0 denotes the operator norm of $\Delta_{g_0}^{-1}$ in the space of L^2 functions with zero radial average. The process is then iterated to obtain a sequence $\{(U_k, u_k)\}_{k=1}^{\infty}$ satisfying

$$\begin{aligned}\Delta_{g_0} U_k &= (\mathcal{L} - \Delta_{g_0})(U_{k-1} + u_{k-1}) - e_k,\\ U_k &= 0 \text{ on } \partial\Omega,\end{aligned}$$

with

$$e_k(z) = \int_{\sigma=z} (\mathcal{L} - \Delta_{g_0})(U_{k-1} + u_{k+1})(s, \sigma)ds$$

and

$$u_k = \int_z^l \int_s^l e_k(t)dtds \tag{3.86}$$

Choosing δ so that $\delta C_0 = \epsilon' < 1$, we then have that

$$\|U_k : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\|, \|u_k : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s})\| < \epsilon'^k \|E : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s})\| \tag{3.87}$$

The alternating partial sums $v_k = \Sigma_{i=0}^{k}(-1)^i(U_i + u_i)$ then converge to a function v satisfying $(1) - (6)$ above. The continuous dependence on $\mathcal{L}$ follows directly by construction.

□

Remark 3.15.2. *The reader will note that a similar proposition was first recorded in [Ka97] and [Ka95], and is a fundamental part of the linear theory in both these articles. The proposition recorded here differs from the previous versions in that we allow a much broader class of perturbations at the expense of a uniqueness claim.*

Chapter 4

Closed Embedded Self-Shrinkers via the Torus

We construct many closed, embedded mean curvature self-shrinking surfaces $\Sigma_g^2 \subseteq \mathbb{R}^3$ of high genus $g = 2k$, $k \in \mathbb{N}$.

Each of these shrinking solitons has isometry group equal to the dihedral group on $2g$ elements, and comes from the "gluing", i.e. desingularizing of the singular union, of the two known closed embedded self-shrinkers in $\mathbb{R}^3$: The round 2-sphere $\mathbb{S}^2$, and Angenent's self-shrinking 2-torus $\mathbb{T}^2$ of revolution. This uses the results and methods N. Kapouleas developed for minimal surfaces in [Ka97]–[Ka12].

4.1 Introduction

Recall that a smooth surface $\Sigma^2 \subseteq \mathbb{R}^3$ is a mean curvature self-shrinker if it satisfies the corresponding nonlinear elliptic self-shrinking soliton PDE:

$$H(\Sigma) = \frac{\langle X, \nu_\Sigma \rangle}{2}, \quad X \in \Sigma^2 \subseteq \mathbb{R}^3, \tag{4.1}$$

where H denotes the mean curvature, X the position vector and ν_Σ is a unit length vector field normal to $\Sigma^2 \subseteq \mathbb{R}^3$.

While important as singularity models in mean curvature flow (see e.g. [Hu90]-[Hu93] and [CM6]-[CM8]), the list of known closed, embedded surfaces satisfying Equation (4.1) is short:

- The round 2-sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$,
- Angenent's (non-circular) 2-torus, in [An89].

Apart from these, there is numerical evidence for the existence of a self-shrinking "fattened wire cube" in [Ch94].

There is a rigorous construction of closed, embedded, smooth mean curvature self-shrinkers with high genus g, embedded in Euclidean space $\mathbb{R}^3$. The theorem is the following:

Main Theorem 4.1.1. *For every large enough even integer $g = 2k$, $k \in \mathbb{N}$, there exists a compact, embedded, orientable, smooth surface without boundary $\Sigma_g^2 \subseteq \mathbb{R}^3$, with the properties:*

(i) Σ_g is a mean curvature self-shrinker of genus g.

(ii) Σ_g is invariant under the dihedral symmetry group with $2g = 4k$ elements.

(iii) The sequence $\{\Sigma_g\}$ converges in Hausdorff sense to the union $\mathbb{S}^2 \cup \mathbb{T}^2$, where $\mathbb{T}^2$ is a rotationally symmetric self-shrinking torus in $\mathbb{R}^3$. The convergence is locally smooth away from the two intersection circles constituting $\mathbb{S}^2 \cap \mathbb{T}^2$.

This paper consists of: (1) Brief account of the construction and its important components, and (2) Proofs of the central explicit estimates of functions that for some small $\delta, \varepsilon > 0$ in an appropriate sense are δ-close to being eigenfunctions of the stability operator (corresp. δ-Jacobi fields), on surfaces that are ε-close to being self-shrinkers (corresp. ε-geodesics) near a candidate for the self-shrinking torus.

The implications of such estimates are: The existence of a self-shrinking torus $\mathbb{T}^2$ (via existence of a smooth closed geodesic loop) with useful quantitative estimates of its geometry. Hence it gives conclusions about, the Dirichlet and Neumann problems of the stability operator $\mathcal{L}$ on this "quantitative torus", leading to the main technical result below in Theorem 4.2.7 which is sufficient to prove Theorem 4.1.1.

A thorough treatment of the background and details relating to this problem, and the construction as developed for general compact minimal hypersurfaces in 3-manifolds by Nikolaos Kapouleas in [Ka97]-[Ka05] can be found in the recent [Ka11], the contents of which will not be described here (it should be noted that the highly symmetric special case enjoys significant simplifications compared to the full theory). Note that also the references [Ng06]-[Ng07], and more recently [KKMø10] and [Ng11], were concerned with gluing problems for (non-compact) self-shrinkers.

Self-shrinkers are minimal surfaces in Euclidean space with respect to a conformally changed Gaussian metric g:

$$\begin{aligned} \Sigma^n \subseteq \mathbb{R}^{n+1} \quad &\text{is a self-shrinker} \quad \Leftrightarrow \quad H_{g_{ij}}(\Sigma) = 0, \\ g_{ij} &= \frac{\delta_{ij}}{\exp(|X|^2/2n)}, \quad X \in \mathbb{R}^{n+1}. \end{aligned} \tag{4.2}$$

Recall the constructions by Nicos Kapouleas (in [Ka97]-[Ka11]), concerning desingularization of a finite collection of compact minimal surfaces in a general ambient Riemannian 3-manifold (M^3, g). The conditions for the construction to work, in our situation, are the following where the collection is identified with one immersed surface $\mathcal{W}$ with intersections along the (smooth) curve $\underline{\mathcal{C}}$, which can have several connected components.

Conditions 4.1.1 ([Ka05]-[Ka11]).

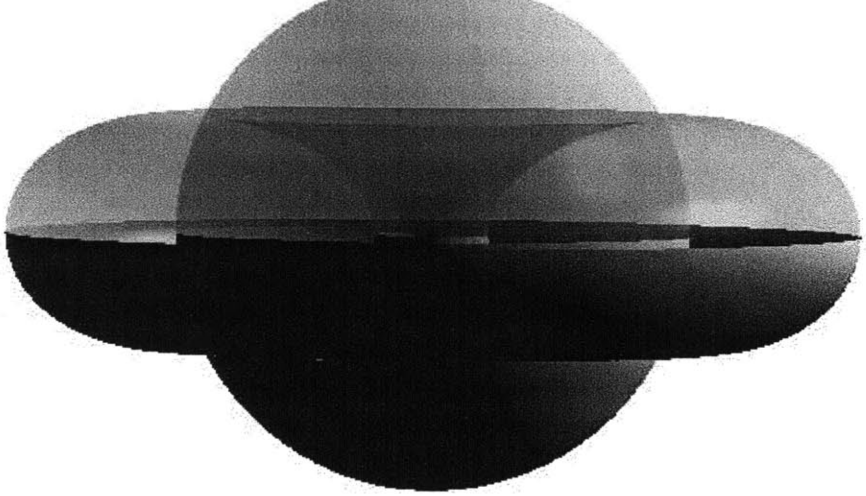

Figure 4-1: The closed, embedded self-similar "torusspheres" $\Sigma_g^2 \subseteq \mathbb{R}^3$ of genus g in Theorem 1. *Showing*: Immersed singular configuration, before handle insertion along two intersection curves (MATLAB).

(I) *There are no points of triple intersection, all intersections are transverse and* $\underline{\mathcal{C}} \cap \partial\mathcal{W} = \emptyset$ *holds.*

(N1) *The kernel for the linearized operator*

$$\mathcal{L} = \Delta + |A|^2 + \mathrm{Ric}(\nu, \nu) \quad on \quad \mathcal{W}, \tag{4.3}$$

with Dirichlet conditions on $\partial\mathcal{W}$*, is trivial (unbalancing condition).*

(N2) *The kernel for the linearized operator* $\mathcal{L}$ *on* $\hat{\mathcal{W}}$*, with Dirichlet conditions on* $\partial\hat{\mathcal{W}}$*, is trivial (flexibility condition).*

Instead of (N1) one may substitute:

(N1') *The kernel for the linearized operator* $\mathcal{L}$ *on* $\mathcal{W}$*, with Neumann conditions on* $\partial\mathcal{W}$*, is trivial (unbalancing condition).*

The Neumann version (N1') of the non-degeneracy conditions will be used for the construction of the closed, embedded self-shrinkers where one solves the self-shrinker equations for graphs with the Neumann conditions over the circle of intersection of the torus and symmetry plane $\mathbb{T}^2 \cap \mathcal{P}$, where $\mathbb{T}^2$ denotes a self-shrinking torus. Then the closed surfaces are obtained via doubling them by reflection through this plane.

An important version of the above conditions, is the one obtained by imposing symmetries, say under a (discrete) subgroup $G \subseteq O(3)$, throughout the construction. Indeed, one may then restrict to verifying the non-degeneracy conditions (N1)–(N2) under the additional assumption of the symmetries in G.

While some properties of the Jacobi fields can be deduced from the known eigenvalues and -functions for the stability operator $\mathcal{L}$ (see e.g. [CM2]) one does not obtain enough accurate information for our purposes. We show how to estimate the quantities using the (generalized) Bellman-Grönwall's inequalities for second order Sturm-Liouville problems, and explicit test functions.

The care one needs to exercise when estimating the explicit constants, as well as the number and complexity of the barriers and test functions one needs to choose, becomes non-trivial owing to mainly two factors: 1) The large Lipshitz constants of the PDE system (relative to the scale of the self-shrinking torus). The geometric reason this happens is that the Gauß curvature in the metric on Angenent's upper half-plane has a maximal value of around 30 along the candidate torus (with the maximum occurring at the point nearest to the origin in $\mathbb{R}^3$), giving naive characteristic conjugate distances of down to $\frac{\pi}{\sqrt{K}} \sim 0.5$, while the circumference of the torus is around ~ 7, in the metric. Hence the solutions to Jacobi's equation can be expected to, and indeed does, oscillate several times around the circumference of $\mathbb{T}^2$, in a non-uniform way and yet fail to "match up". Furthermore: 2) The location of the conjugate points on $\mathbb{S}^2$ for the appropriate Jacobi equation is furthermore very near to the singular curve $\underline{\mathcal{C}}$ (i.e. the boundaries of the connected components of $\mathcal{M} \setminus \underline{\mathcal{C}}$), which requires also requires tighter estimates.

Hence it takes work to strengten the estimates to a useful form. One device to do this is what we call the "sesqui"-shooting problem in Definition 4.2.4, for identifying the position of the torus, where "sesqui" refers to the fact that it is a double shooting problem but with a compatibility condition linking the two: That each pair of curves always meet at a simple, explicitly known solution. Here we take the round cylinder of radius $\sqrt{2}$ as reference. Since the errors are exponential in the integral of the Lipshitz constants, one may by virtue of the sesqui-shooting roughly take the square-root of the errors, which allows us to obtain bounds with explicit constants of the order of 10^1–10^2 instead of 10^4.

Listing our explicit choices of test functions (f.ex. adequate piecewise polynomial choices can easily be found using Taylor expansions at a few selected points) to use with the key estimates in Section 4.2 is likely not in itself very enlightening for the reader at this point, and with further simplification of such yet to be worked out, we postpone describing them in detail, being confident to later be able to supply a collection for which the properties can be checked manually with minimal time consumption.

4.2 ε-Geodesics, δ-Jacobi Fields: A Quantitative Self-Shrinking Torus $\mathbb{T}^2$

4.2.1 Existence of $\mathbb{T}^2$, With Geometric Estimates

In order to later understand precisely the kernel of the stability operators, we need to establish a detailed quantitative version of the existence of Angenent's torus.

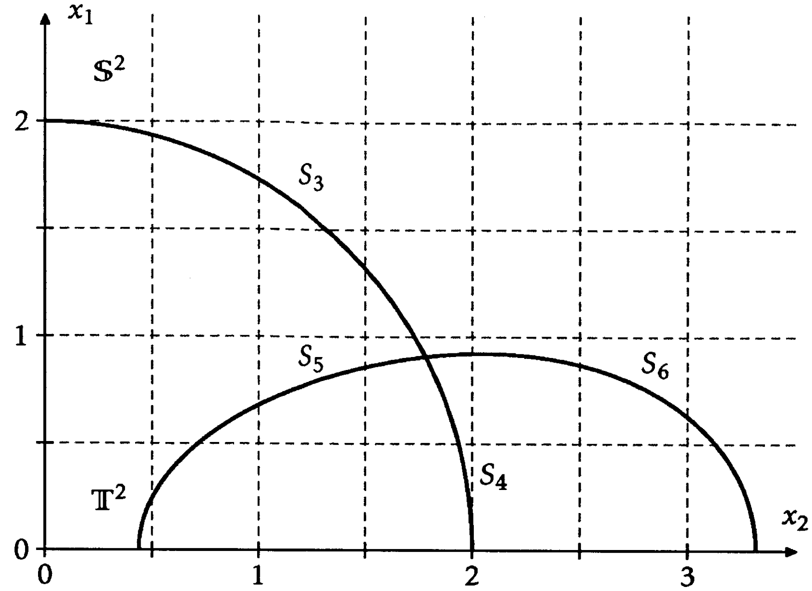

Figure 4-2: The connected components of $\hat{\mathcal{W}} = (\mathbb{S}^2 \cup \mathbb{T}^2) \setminus (\mathbb{S}^2 \cap \mathbb{T}^2)$. The self-intersecting initial surface is generated by rotation around the x_1-axis and reflection in the x_2-axis.

Towards the end of this section we will arrive at the basic, explicit estimates on the location and geometric quantities of such a torus, but we will first work out the explicit estimates for also the Jacobi equation before finalizing the choices in the estimate, in order to require as little as possible of the test functions.

We note that much less complicated estimates would ensure the mere existence, leading to a different proof of the self-shrinking "doughnut" existence result from [An89], in the 3-dimensional case. But as explained, here we will need very precise estimates of several aspects of the geometry of $\mathbb{T}^2$.

Proposition 4.2.1. *Let $\varepsilon_{gap} = 10^{-3}$. There exists a closed, embedded, self-shrinking torus of revolution $\mathbb{T}^2$ with the following properties:*

(1) The torus is $(x_1 \mapsto -x_1)$-symmetric.

(2) $\mathbb{T}^2$ intersects the x_2-axis orthogonally at two heights $a^+ > a^- > 0$,

$$\tfrac{4034}{1217} - \tfrac{5}{2}\varepsilon_{gap} < a^+ < \tfrac{4034}{1217} + \tfrac{5}{2}\varepsilon_{gap}, \tag{4.4}$$

$$\tfrac{7}{16} - \tfrac{3}{98} < a^- < \tfrac{7}{16} + \tfrac{3}{98}. \tag{4.5}$$

(3) $\mathbb{T}^2$ intersects the sphere $\mathbb{S}^2$ of radius 2 at two points $p^\pm = (\pm x_{\mathbb{S}^2}, y_{\mathbb{S}^2})$, where:

$$\left\| p^\pm - \left(\pm\tfrac{29}{32}, \tfrac{41}{23}\right)\right\|_{\mathbb{R}^2} \le 5\varepsilon_{gap}. \tag{4.6}$$

Correspondingly, the angles $\angle(\pm\vec{e}_1, p^\pm)$ from the x_1-axis satisfy:

$$\left|\angle(\pm\vec{e}_1, p^\pm) - \tfrac{11}{10}\right| \leq \tfrac{1}{20}. \tag{4.7}$$

Remark 4.2.2. *Note that we do not assert or prove uniqueness of $\mathbb{T}^2$ with these properties, even though this is expected to be true.*

Proof. We first need to describe the double shooting problem, or rather "sesqui"-shooting problem since we remove one parameter. For this we need an elementary lemma, which can either be proved (in a weaker version) using the approximation methods described later in this section, or by analysis directly of the ODE in (4.36).

Lemma 4.2.3. *For each $d \in [0, \sqrt{2}]$ let $\gamma_d : [0,\infty) \to \mathbb{R}^+ \times \mathbb{R}$ be the geodesic starting at $(0,d)$ with initial derivative $\gamma_d'(0) = (1,0)$ and consider the first time $t^0(d)$ such that γ_d intersects the cylinder geodesic $\{x_2 = \sqrt{2}\}$. Then the function $I : [0,\sqrt{2}] \to [0,\sqrt{2}]$ given by*

$$I(d) := x_1(\gamma_s(t^0(d))),$$

is continuous and strictly increasing. It satisfies the bounds:

$$1 \leq \sup_{d\in[\frac{13}{32},\frac{15}{32}]} I'(d) \leq \frac{5}{4}.$$

Definition 4.2.4 ("Sesqui"-Shooting Problem).

(i) Fix $a > \sqrt{2}$.

(ii) Let $\gamma_a : [0,t_a] \to \mathbb{R}^+ \times \mathbb{R}$ be the fully extended geodesic contained in $\{x_1 \geq 0\}$, with initial conditions $\gamma_a(0) = (0,a)$ and $\gamma_a'(0) = (1,0)$.

(iii) By the maximum principle for (4.36), we have $\gamma_a \cap \{x_2 = \sqrt{2}\} \neq \emptyset$. Assume that the first point of γ_a crossing $\sqrt{2}$ belongs to $\{0 < x_1 \leq \sqrt{2}\}$, and denote the time it happens by $t^0(a)$.

(iv) By the preceding Lemma, there exists a unique $b(a) \in [0,\sqrt{2}]$ and $s^0(a)$ such that for γ_b on $[0,s^0]$ with $\gamma_b(0) = (0,b)$ and $\gamma_b'(0) = (1,0)$, we have

$$\gamma_a(t^0(a)) = \gamma_b(s^0(a)).$$

(iv) Define $\Phi(a)$ by

$$\Phi(a) := \frac{\vec{e}_3 \cdot (\gamma_a'(t^0(a)) \times \gamma_b'(s^0(a))}{|\gamma_a'(t^0(a))||\gamma_b'(s^0(a))|},$$

or equivalently the oriented angle between the tangents of γ_a and γ_b at the intersection point.

The following lemma is clear from the definition:

Lemma 4.2.5. *The function Φ in Definition 4.2.4 is well-defined and continuous, on the open connected set of values $a > \sqrt{2}$ with the property that the first point of intersection of γ_a with $\{x_2 = \sqrt{2}\}$ belongs to $\{x_1 \in (0, \sqrt{2})\}$.*

The strategy for the proof of the proposition will now be to prove for two different nearby pairs (a^+, b^+) and (a^-, b^-), that will be chosen such that they are geodesics broken at the cylinder $\{x_2 = \sqrt{2}\}$, that:

$$\Phi(a^+) > 0, \tag{4.8}$$

$$\Phi(a^-) < 0. \tag{4.9}$$

Existence then follows, from the intermediate value theorem, of a pair (a^0, b^0) such that:

$$\Phi(a^0) = 0, \tag{4.10}$$

$$a^- < a^0 < a^+, \tag{4.11}$$

$$b^- < b^0 < b^+. \tag{4.12}$$

The estimates leading to the proof of (4.8)–(4.9) will then lead to the estimates in the proposition.

Consider now a curve (parametrized by any parameter t),

$$\gamma(t) = (x(t), y(t)).$$

Then the equation for γ to generate a self-shrinker by rotation reads:

$$x''y' - y''x' = \left[\frac{yx' - xy'}{2} - \frac{x'}{y}\right]\left((x')^2 + (y')^2\right).$$

We need the following simple lemma:

Lemma 4.2.6. *If we define the operator $\mathcal{M}_1$ acting on C^2-functions $u : I \to \mathbb{R}$, for some interval $I \subseteq [0, \infty]$, by*

$$\mathcal{M}_1(u, p) := \left[\frac{x_1 p - u}{2} + \frac{1}{u}\right]\left(1 + p^2\right), \tag{4.13}$$

then the function $u(x_1)$, a graph over the x_1-axis, generates a self-shrinker by rotation (around x_1) if and only if

$$u'' = \mathcal{M}_1(u, u'). \tag{4.14}$$

Likewise for graphs over the x_2-axis, defining

$$\mathcal{M}_2(f, q) := \left[\left(\frac{x_2}{2} - \frac{1}{x_2}\right)q - \frac{f}{2}\right]\left(1 + q^2\right), \tag{4.15}$$

again characterizes such solutions via

$$f'' = \mathcal{M}_2(f, f'). \tag{4.16}$$

We compute that:

$$\frac{\partial}{\partial u}\mathcal{M}_1(u,p) = \left[-\frac{1}{2} - \frac{1}{u^2}\right](1+p^2), \tag{4.17}$$

$$\frac{\partial}{\partial p}\mathcal{M}_1(u,p) = \frac{x_1}{2}(1+p^2) + 2p\left[\frac{x_1 p - u}{2} + \frac{1}{u}\right], \tag{4.18}$$

$$\frac{\partial}{\partial f}\mathcal{M}_2(f,q) = -\frac{1}{2}\left(1+q^2\right), \tag{4.19}$$

$$\frac{\partial}{\partial q}\mathcal{M}_2(f,q) = \left(\frac{x_2}{2} - \frac{1}{x_2}\right)(1+3q^2) - fq. \tag{4.20}$$

We will first consider graphs $u : [0, \frac{3}{5}] \to \mathbb{R}$. Now, consider an approximate solution U, and we fix the quantity ε_1^T, later to be chosen, which reflects the order of magnitude of the precision we wish to determine the position of $\mathbb{T}^2$ with on this interval.

$$\mathcal{M}_1(U,U') - \mathcal{M}_1(u,u') = (U-u)\frac{\partial}{\partial u}\mathcal{M}_1(\xi,u') + (U'-u')\frac{\partial}{\partial p}\mathcal{M}_1(U,\xi'), \tag{4.21}$$

for some $\xi \in [U(x_1), u(x_1)]$ and $\xi' \in [U'(x_1), u'(x_1)]$.

Top of the torus: x_1-graph

Assume we have the uniform estimates:

$$|U'' - \mathcal{M}_1(U,U')| \leq \varepsilon_1^T,$$

for some ε_1^T. Assume also, for the argument (and later show this to be self-consistent, i.e. propagated by the equation), the following bounds:

$$\int_0^{x_1} \left|\frac{1}{2} + \frac{1}{\xi^2}\right| (1 + (u'(x_1))^2) dx_1 \leq \frac{4}{5}x_1, \quad x_1 \in [0, \tfrac{3}{5}], \quad |\xi - U(x_1)| \leq \varepsilon_0,$$

$$\int_0^{x_1} \left|\frac{x_1}{2}(1+(\xi')^2) + 2\xi'\left[\frac{x_1\xi' - U(x_1)}{2} + \frac{1}{U(x_1)}\right]\right| dx_1 \leq \frac{8}{3}x_1^2, \quad x_1 \in [0, \tfrac{3}{5}], \quad |\xi' - U'(x_1)| \leq \varepsilon_0.$$

If we let $\varphi(x_1) := |U'(x_1) - u'(x_1)|$, we can now estimate (recall that $\|f'|'| = |f''|$

almost everywhere, as an easy consequence of Sard's Theorem):

$$\begin{aligned}\varphi'(x_1) &\le \big||U'-u'|'\big| \overset{\text{a.e}}{=} |U''-u''| \\ &\le |\mathcal{M}_1(U,U')-\mathcal{M}_1(u,u')|+\varepsilon_1^T \\ &\le |\partial_u\mathcal{M}_1(\xi,u')||U-u|+|\partial_p\mathcal{M}_1(U,\xi')||U'-u'|+\varepsilon_1^T \\ &\le |\partial_p\mathcal{M}_1(U,\xi')|\varphi(x_1)+|\partial_u\mathcal{M}_1(\xi,u')|\int_0^{x_1}\varphi(s)ds+|\partial_u\mathcal{M}_1(\xi,u')||U(0)-u(0)|+\varepsilon_1^T.\end{aligned} \tag{4.22}$$

Thus we integrate this inequality, which holds almost everywhere with respect to the Lebesgue measure, and get:

$$\begin{aligned}\varphi(x_1) &\le \varphi(0)+\int_0^{x_1}|\partial_p\mathcal{M}_1(U,\xi')(s)|\varphi(s)ds+\int_0^{x_1}|\partial_u\mathcal{M}_1(\xi,u')(t)|\int_0^t\varphi(s)dsdt \\ &\quad +\tfrac{4}{5}x_1|U(0)-u(0)|+\varepsilon_1^Tx_1 \\ &\le \varphi(0)+\tfrac{4}{5}x_1|U(0)-u(0)|+\varepsilon_1^Tx_1+\int_0^{x_1}\Big[|\partial_p\mathcal{M}_1(U,\xi')(s)|+\tfrac{4}{5}x_1\Big]\varphi(s)ds.\end{aligned}$$

We are now ready to use the integral form of Grönwall-Bellman's inequality, namely for $\alpha(t)$ a non-decreasing function:

$$\forall t\in I:\varphi(t)\le\alpha(t)+\int_a^t\beta(s)\varphi(s)ds \quad\Longrightarrow\quad \forall t\in I:\varphi(t)\le\alpha(t)\exp\left\{\int_a^t\beta(s)ds\right\}.$$

Here we thus conclude from the above, that

$$\varphi(x_1)\le\Big[\varphi(0)+\tfrac{4}{5}x_1|U(0)-u(0)|+\varepsilon_1^Tx_1\Big]\exp\Big\{\tfrac{8}{3}x_1^2+\tfrac{4}{5}x_1^2\Big\}, \tag{4.23}$$

so that here we obtain the estimates (for $\varphi(0)=0$):

$$\begin{aligned}|U'(x_1)-u'(x_1)| &\le \tfrac{3}{5}\exp(\tfrac{156}{125})(\varepsilon_1^T+\tfrac{4}{5}|U(0)-u(0)|), \\ |U(x_1)-u(x_1)| &\le |U(0)-u(0)|+(\varepsilon_1^T+\tfrac{4}{5}|U(0)-u(0)|)\int_0^{\frac{3}{5}}s\exp\Big(\tfrac{52}{15}s^2\Big)ds \\ &= |U(0)-u(0)|+\tfrac{15}{104}\Big(\exp\Big(\tfrac{52}{15}(\tfrac{3}{5})^2\Big)-1\Big)(\varepsilon_1^T+\tfrac{4}{5}|U(0)-u(0)|) \\ &\le \tfrac{23}{80}|U(0)-u(0)|+\tfrac{9}{25}\varepsilon_1^T.\end{aligned}$$

Top of the torus: x_2-graph to the sphere

We continue with the next part, which is graphical over the x_2-axis. In this region we again let $\psi(x_2):=|F'-f'|$. Here we will assume the estimate

$$|F''-\mathcal{M}_2(F,F')|\le\varepsilon_2^T, \tag{4.24}$$

and furthermore the (the second one to later be proven consistent) estimates on

$x_2 \in [y_{S^2}, u_T(3/5)]$

$$I_u^{(2)} = \int_{x_2}^{u_T(3/5)} \left|\frac{1}{2}(1+(F')^2)\right| dx_2 \le \tfrac{13}{8} - \tfrac{x_2}{2},$$
$$I_p^{(2)} = \int_{x_2}^{u_T(3/5)} \left|\left(\frac{x_2}{2} - \frac{1}{x_2}\right)(1+3(\xi')^2) - F(x_2)\xi'\right| dx_2 \le \tfrac{7}{4} - \tfrac{9}{10}(x_2 - y_{S^2})^2.$$

We can then estimate as follows:

$$\begin{aligned}\psi'(x_2) &\le |\partial_u \mathcal{M}_2||F-f| + \left|\partial_p \mathcal{M}_2\right||F'-f'| + \varepsilon_2^T \\ &\le |\partial_u \mathcal{M}_2| \int_{x_2}^{a} \psi(s)ds + \left|\partial_p \mathcal{M}_2\right| \psi(x_2) + |\partial_u \mathcal{M}_2||F(a)-f(a)| + \varepsilon_2^T,\end{aligned}$$

which integrates to

$$\psi(x_2) \le \int_{x_2}^{u_T(3/5)} \left[|\partial_p \mathcal{M}_2| + \tfrac{13}{8} - \tfrac{x_2}{2}\right]\psi(s)ds + (\tfrac{13}{8} - \tfrac{x_2}{2})|F(u_T(3/5)) - f(u_T(3/5))| \tag{4.25}$$
$$+ \varepsilon_2^T(u_T(3/5) - x_2) + \psi(a), \tag{4.26}$$

so that, again by Grönwall-Bellman,

$$\begin{aligned}\psi(x_2) \le &\left[(\tfrac{13}{8} - \tfrac{x_2}{2})|F(a) - f(a)| + |F'(a) - f'(a)| + \varepsilon_2^T(u_T(3/5) - x_2)\right]\times \\ &\exp\left\{\tfrac{7}{4} - \tfrac{9}{10}(x_2 - y_{S^2})^2 + \left(\tfrac{13}{8} - \tfrac{x_2}{2}\right)(u_T(3/5) - x_2)\right\}.\end{aligned}$$

The endpoint estimates are:

$$\begin{aligned}\psi(y_{S^2}) &\le \tfrac{59}{4}|F(u_T(3/5)) - f(u_T(3/5))| + \tfrac{54}{5}|F'(u_T(3/5)) - f'(u_T(3/5))| + \tfrac{377}{20}\varepsilon_2^T \\ &\le \tfrac{59}{4}\frac{|U(F(u_T(3/5))) - u(f(u_T(3/5)))|}{|U'(F(\xi))|} + \tfrac{54}{5}\frac{|U'(F(u_T(3/5))) - u'(f(u_T(3/5)))|}{|U'(F(u_T(3/5)))||u'(f(u_T(3/5)))|} + \tfrac{377}{20}\varepsilon_2^T \\ &\le \tfrac{59}{4}\tfrac{10}{11}\left(\tfrac{9}{25}\varepsilon_1^T + \tfrac{23}{80}|U(0) - u(0)|\right) + \tfrac{54}{5}\left(\tfrac{10}{11}\right)^2 \tfrac{3}{5}\exp(\tfrac{156}{125})(\varepsilon_1^T + \tfrac{4}{5}|U(0) - u(0)|) + \tfrac{377}{20}\varepsilon_2^T \\ &\le 27|U(0) - u(0)| + 34\varepsilon_1^T + 19\varepsilon_2^T.\end{aligned}$$

Integrating the above estimate for $\psi(x_2)$, we also get:

$$\begin{aligned}|F(y_{S^2}) - f(y_{S^2})| &\le (1 + \tfrac{63}{8})|F(u_T(3/5)) - f(u_T(3/5))| + \tfrac{83}{20}|F'(u_T(3/5)) - f'(u_T(3/5))| + \tfrac{137}{20}\varepsilon_2^T \\ &\le 12|U(0) - u(0)| + 15\varepsilon_1^T + 7\varepsilon_2^T.\end{aligned}$$

Top of the torus, x_2-graph from sphere to cylinder

We consider the region between the sphere and cylinder, and let again $\psi(x_2) :=$

$|F' - f'|$. Here we will assume the estimate

$$|F'' - \mathcal{M}_2(F, F')| \le \varepsilon_3^T, \tag{4.27}$$

and furthermore the estimates on $x_2 \in [y_{S^2}, \sqrt{2}]$

$$I_u^{(2)} = \int_{x_2}^{y_{S^2}} \left|\frac{1}{2}(1 + (F')^2)\right| dx_2 \le \tfrac{1}{4}(y_{S^2} - x_2),$$
$$I_p^{(2)} = \int_{x_2}^{y_{S^2}} \left|\left(\frac{x_2}{2} - \frac{1}{x_2}\right)(1 + 3(\xi')^2) - F(x_2)\xi'\right| dx_2 \le \tfrac{27}{50}(y_{S^2} - x_2),$$

which again should later be checked hold in the regime we consider.

We estimate as follows:

$$\begin{aligned}\psi'(x_2) &\le |\partial_u \mathcal{M}_2||F - f| + \left|\partial_p \mathcal{M}_2\right||F' - f'| + \varepsilon_2^T \\ &\le |\partial_u \mathcal{M}_2| \int_{x_2}^{y_{S^2}} \psi(s) ds + \left|\partial_p \mathcal{M}_2\right| \psi(x_2) + |\partial_u \mathcal{M}_2||F(y_{S^2}) - f(y_{S^2})| + \varepsilon_3^T,\end{aligned}$$

which integrates to:

$$\psi(x_2) \le \psi(y_{S^2}) + \int_{x_2}^{y_{S^2}} \left[|\partial_p \mathcal{M}_2| + \tfrac{1}{4}(y_{S^2} - x_2)\right] \psi(s) ds + \tfrac{1}{4}(y_{S^2} - x_2)|F(y_{S^2}) - f(y_{S^2})| + \varepsilon_3^T(y_{S^2} - x_2), \tag{4.28}$$

so that, again by Grönwall-Bellman,

$$\begin{aligned}\psi(x_2) \le &\left[\tfrac{1}{4}(y_{S^2} - x_2)|F(y_{S^2}) - f(y_{S^2})| + |F'(y_{S^2}) - f'(y_{S^2})| + \varepsilon_3^T(y_{S^2} - x_2)\right] \times \\ &\exp\left\{\tfrac{27}{50}(y_{S^2} - x_2) + \tfrac{1}{4}(y_{S^2} - x_2)^2\right\}.\end{aligned}$$

Inserting the previous estimate gives:

$$\begin{aligned}\psi(\sqrt{2}) &\le \tfrac{1}{8}|F(y_{S^2}) - f(y_{S^2})| + \tfrac{4}{3}|F'(y_{S^2}) - f'(y_{S^2})| + \tfrac{1}{2}\varepsilon_3^T \\ &\le \tfrac{1}{8}\left(12|U(0) - u(0)| + 15\varepsilon_1^T + 7\varepsilon_2^T\right) + \tfrac{4}{3}\left(27|U(0) - u(0)| + 34\varepsilon_1^T + 19\varepsilon_2^T\right) + \tfrac{1}{2}\varepsilon_3^T.\end{aligned}$$

Hence, we finally obtain the estimate:

$$|F'(\sqrt{2}) - f'(\sqrt{2})| \le 36|U(0) - u(0)| + 45\varepsilon_1^T + 25\varepsilon_2^T + \tfrac{1}{2}\varepsilon_3^T.$$

Integrating the estimates, we also get:

$$\begin{aligned}|F(y_{S^2}) - f(y_{S^2})| &\le (1 + \tfrac{1}{50})|F(y_{S^2}) - f(y_{S^2})| + \tfrac{33}{80}|F'(y_{S^2}) - f'(y_{S^2})| + \tfrac{2}{25}\varepsilon_3^T \\ &\le 24|U(0) - u(0)| + 30\varepsilon_1^T + 15\varepsilon_2^T + \tfrac{2}{25}\varepsilon_3^T.\end{aligned}$$

Bottom of the torus: x_1-graph from the plane

Assume uniform estimates:

$$|U'' - \mathcal{M}_1(U, U')| \le \varepsilon_1^B,$$

for some ε_1^B. Assume also, for the argument (and later show this to be self-consistent, i.e. propagated by the equation), the following bounds (for $x_1 \in [0, \frac{1}{2}]$):

$$\int_0^{x_1} \left|\frac{1}{2} + \frac{1}{\xi^2}\right| (1 + (u'(x_1))^2) dx_1 \le \frac{29}{5} x_1 + \frac{1}{20}, \quad |\xi - U(x_1)| \le \varepsilon_0,$$
$$\int_0^{x_1} \left|\frac{x_1}{2}(1 + (\xi')^2) + 2\xi' \left[\frac{x_1 \xi' - U(x_1)}{2} + \frac{1}{U(x_1)}\right]\right| dx_1 \le 4x_1^2 + \frac{1}{5} x_1, \quad |\xi' - U'(x_1)| \le \varepsilon_0.$$

As always, we get with $\varphi(x) = |U'(x) - u'(x)|$:

$$\begin{aligned}
\varphi(x_1) \le \varphi(0) &+ \int_0^{x_1} |\partial_p \mathcal{M}_1(U, \xi')(s)| \varphi(s) ds + \int_0^{x_1} |\partial_u \mathcal{M}_1(\xi, u')(t)| \int_0^t \varphi(s) ds dt \\
&+ (\tfrac{29}{5} x_1 + \tfrac{1}{20}) |U(0) - u(0)| + \varepsilon_1^B x_1 \\
&\le \varphi(0) + (\tfrac{29}{5} x_1 + \tfrac{1}{20}) |U(0) - u(0)| + \varepsilon_1^B x_1 + \int_0^{x_1} \left[|\partial_p \mathcal{M}_1(U, \xi')(s)| + \tfrac{29}{5} x_1 + \tfrac{1}{20} \right] \varphi(s) ds.
\end{aligned}$$

Using once again Grönwall-Bellman's inequality,

$$\varphi(x_1) \le \left[\varphi(0) + (\tfrac{29}{5} x_1 + \tfrac{1}{20}) |U(0) - u(0)| + \varepsilon_1^B x_1\right] \exp\left\{\tfrac{29}{5} x_1^2 + \tfrac{x_1}{20} + 4x_1^2 + \tfrac{1}{5} x_1\right\}, \quad (4.29)$$

and we obtain the estimates (for $\varphi(0) = 0$):

$$\begin{aligned}
|U'(x_1) - u'(x_1)| &\le \left[(\tfrac{29}{5} x_1 + \tfrac{1}{20}) |U(0) - u(0)| + \varepsilon_1^B x_1\right] \exp\left\{\tfrac{49}{5} x_1^2 + \tfrac{x_1}{4}\right\} \\
|U'(\tfrac{1}{2}) - u'(\tfrac{1}{2})| &\le 39 |U(0) - u(0)| + \tfrac{28}{5} \varepsilon_1^B, \\
|U(x_1) - u(x_1)| &\le |U(0) - u(0)| \left(1 + \int_0^{x_1} (\tfrac{29}{5} s + \tfrac{1}{20}) \exp\left\{\tfrac{49}{5} s^2 + \tfrac{s}{4}\right\} ds\right) \\
&\quad + \varepsilon_1^B \int_0^{x_1} s \exp\left\{\tfrac{49}{5} s^2 + \tfrac{s}{4}\right\} ds \\
|U(\tfrac{1}{2}) - u(\tfrac{1}{2})| &\le \tfrac{23}{5} |U(0) - u(0)| + \tfrac{3}{5} \varepsilon_1^B.
\end{aligned}$$

Bottom: x_2-graph to cylinder

We continue with the next part, which is graphical over the x_2-axis. In this region we again let $\psi(x_2) = \psi_B(x_2) := |F' - f'|$. Here we will assume the estimate

$$|F'' - \mathcal{M}_2(F, F')| \le \varepsilon_2^B, \quad (4.30)$$

and furthermore the (to later be proven consistent) estimates on $x_2 \in [a_0, \sqrt{2}]$ (where

$|a_0 - \frac{28}{39}| \le \varepsilon_0)$.

$$I_u^{(2)} = \int_{a_0}^{x_2} \left|\frac{1}{2}(1+(F')^2)\right| dx_2 \le \frac{16}{25}x_2 - \frac{3}{7},$$
$$I_p^{(2)} \doteq \int_{a_0}^{x_2} \left|\left(\frac{x_2}{2} - \frac{1}{x_2}\right)(1+3(\xi')^2) - F(x_2)\xi'\right| dx_2 \le \frac{9}{10} - \frac{11}{10}\left(x_2 - \frac{3}{2}\right)^2.$$

We can then estimate as follows:

$$\begin{aligned}\psi'(x_2) &\le |\partial_u \mathcal{M}_2||F-f| + \left|\partial_p \mathcal{M}_2\right||F'-f'| + \varepsilon_2^B \\ &\le |\partial_u \mathcal{M}_2| \int_a^{x_2} \psi(s)ds + \left|\partial_p \mathcal{M}_2\right|\psi(x_2) + |\partial_u \mathcal{M}_2||F(a) - f(a)| + \varepsilon_2^B,\end{aligned}$$

which integrates to

$$\psi(x_2) \le \int_{a_0}^{x_2} \left[|\partial_p \mathcal{M}_2| + \tfrac{16}{25}x_2 - \tfrac{3}{7}\right]\psi(s)ds + (\tfrac{16}{25}x_2 - \tfrac{3}{7})|F(a_0) - f(a_0)| + \varepsilon_2^B(x_2 - a_0) + \psi(a). \tag{4.31}$$

By Grönwall-Bellman,

$$\begin{aligned}\psi(x_2) \le& \left[(\tfrac{16}{25}x_2 - \tfrac{3}{7})|F(a) - f(a)| + |F'(a) - f'(a)| + \varepsilon_2^B(x_2 - a_0)\right]\times \\ & \exp\left\{\tfrac{9}{10} - \tfrac{11}{10}\left(x_2 - \tfrac{3}{2}\right)^2 + (x_2 - a_0)\left(\tfrac{16}{25}x_2 - \tfrac{3}{7}\right)\right\} \\ &\le \tfrac{13}{8}|F(a_0) - f(a_0)| + \tfrac{17}{5}|F'(a_0) - f'(a_0)| + \tfrac{69}{29}\varepsilon_2^B \\ &\le \tfrac{13}{8}\frac{|U(F(a_0)) - u(f(a_0))|}{|U'(F(\xi))|} + \tfrac{17}{5}\frac{|U'(F(a_0)) - u'(f(a_0))|}{|U'(F(a_0))||u'(f(a_0))|} + \tfrac{69}{29}\varepsilon_2^B \\ &\le \tfrac{13}{8}\tfrac{10}{12}\left(\tfrac{3}{5}\varepsilon_1^B + \tfrac{23}{5}|U(0) - u(0)|\right) + \tfrac{17}{5}\left(\tfrac{10}{12}\right)^2(\tfrac{28}{5}\varepsilon_1^B + 39|U(0) - u(0)|) + \tfrac{69}{29}\varepsilon_2^B \\ &\le 100|U(0) - u(0)| + 15\varepsilon_1^B + \tfrac{5}{2}\varepsilon_2^B.\end{aligned}$$

Integrating the first line of this estimate, we also get:

$$\begin{aligned}|F(\sqrt{2}) - f(\sqrt{2})| \le& (1 + \tfrac{23}{50})|F(a_0) - f(a_0)| + \tfrac{14}{9}|F'(a_0) - f'(a_0)| + \tfrac{2}{3}\varepsilon_2 \\ \le& 48|U(0) - u(0)| + 7\varepsilon_1^B + \tfrac{2}{3}\varepsilon_2^B.\end{aligned}$$

Hence we will arrange that for test functions (curves) $\gamma_{\text{up}}, \gamma_{\text{low}} : [0,1] \to \mathbb{R}\times\mathbb{R}_+$:

$$\gamma_{\text{down}}(0) = (0, \tfrac{4034}{1217} - \tfrac{5}{2}\varepsilon_{\text{ud}}) \tag{4.32}$$
$$\gamma_{\text{up}}(0) = (0, \tfrac{4034}{1217} + \tfrac{5}{2}\varepsilon_{\text{ud}}). \tag{4.33}$$
$$\tag{4.34}$$

Then, as explained earlier, the intermediate value theorem applied to the sesqui-shooting problem implies the existence of the torus with the estimates (4.4)–(4.5).

To see Property (3), we recall that

$$|F(y_{\mathbb{S}^2}) - f(y_{\mathbb{S}^2})| \leq \tfrac{317}{25}|U(0) - u(0)| + 11\varepsilon_1 + \tfrac{27}{5}\varepsilon_2, \tag{4.35}$$

from which the estimate (4.6) follows.

□

4.2.2 The Stability Operator $\mathcal{L}$ on Subsets of $\mathbb{S}^2$ and $\mathbb{T}^2$

Recall the self-shrinker equation for a smooth oriented surface $S \subseteq \mathbb{R}^3$ to be a self-shrinker (shrinking towards the origin with singular time $T = 1$) is

$$H_S(\vec{X}) - \tfrac{1}{2}\vec{X} \cdot \vec{\nu}_S(\vec{X}) = 0, \tag{4.36}$$

for each $\vec{X} \in S$, where by convention $H_S = \sum_1^n \kappa_i$ is the sum of the signed principal curvatures w.r.t. the chosen normal $\vec{\nu}_S$ (i.e. $H = 2$ for the sphere with outward pointing $\vec{\nu}$). We have normalized Equation (4.36) so that $T = 1$ is the singular time.

For a smooth normal variation $\vec{X}_t$ determined by a function u via $X_t = X_0 + tu\vec{\nu}_S$, where $\vec{X}_0$ parametrizes S, the pointwise linear change in (minus) the quantity on the left hand side in (4.36) is given by the stability operator (see the Appendix, and also [CM6]-[CM7] for more properties of this operator)

$$\mathcal{L}_S u = \Delta_S u + |A_S|^2 u - \tfrac{1}{2}\left(\vec{X} \cdot \nabla_S u - u\right). \tag{4.37}$$

We are now ready to prove the main technical theorem in this paper, concerning the kernel on the connected components (with boundaries) of

$$(\mathbb{S}^2 \cup \mathbb{T}^2) \setminus (\mathbb{S}^2 \cap \mathbb{T}^2),$$

where $\mathbb{T}^2$ is the accurately estimated "quantitative" torus.

Theorem 4.2.7. *There exists $N > 0$ large enough that for the below six surfaces with boundary $S_1, \ldots S_6$,*

$$\ker \mathcal{L}_{S_i} = \{0\} \quad \textit{for} \quad i = 1, \ldots, 6 \quad \textit{[w/ indicated boundary conditions]},$$

when imposing at least N-fold rotational symmetry:

(1) The surfaces $S_1 = \mathbb{S}^2 \cap \{x_1 \geq 0\}$ and $S_2 = \mathbb{T}^2 \cap \{x_1 \geq 0\}$ [Neumann conditions].

(2) The components S_3 and S_4 of $\{x_1 \geq 0\} \cap \mathbb{S}^2 \setminus (\mathbb{S}^2 \cap \mathbb{T}^2)$ [Neumann conditions on $\{x_1 = 0\}$, Dirichlet conditions elsewhere].

(3) The components S_5 and S_6 of $\{x_1 \geq 0\} \cap \mathbb{T}^2 \setminus (\mathbb{S}^2 \cap \mathbb{T}^2)$ [Neumann conditions on $\{x_1 = 0\}$, Dirichlet conditions elsewhere].

Recall first the variational characterization of shrinkers, as critical points of the functional

$$A(\Sigma) = \int_\Sigma e^{-|X(p)|^2/4} dA(p). \tag{4.38}$$

As exploited in [KKMø10] in the planar case, the conjugation identity connecting the stability operator in the Gaussian density to the linearized operator in (4.37) can be useful for the analysis. The general identity on any surface in $\mathbb{R}^3$ is stated in the following lemma.

Lemma 4.2.8. *Let $\Sigma^2 \subseteq \mathbb{R}^3$ be a smooth self-shrinker. Then the following conjugation identity holds for the stability operator on Σ:*

$$\mathcal{L} = e^{\frac{|X|^2}{8}}\left(\Delta + |A|^2 - \frac{|X|^2 + |X^\perp|^2}{16} + 1\right)e^{\frac{-|X|^2}{8}}. \tag{4.39}$$

Proof of Lemma 4.2.8. By an elementary computation using Appendix C in [KKMø10] and noting that $|X^\perp| = 2|H|$. □

We let $\omega = (x')^2 + (y')^2$ and express functions on the surface of rotation in coordinates as $v = v(t,\theta)$ and get for the intrinsic Laplacian that

$$\Delta = \frac{1}{y\sqrt{\omega}}\frac{\partial}{\partial t}\left(\frac{y}{\sqrt{\omega}}\frac{\partial}{\partial t}\right) + \frac{1}{y^2}\frac{\partial^2}{\partial \theta^2}.$$

The square of the second fundamental form is

$$|A|^2 = \frac{1}{\omega^3}\left[(x''y' - y''x')^2 + \frac{(x')^2\omega^2}{y^2}\right] \tag{4.40}$$

$$= \frac{1}{\omega}\left[\left(\frac{yx' - xy'}{2} - \frac{x'}{y}\right)^2 + \frac{(x')^2}{y^2}\right], \tag{4.41}$$

Recall that

$$K_{e^{2w}g_0} = e^{-2\omega}\left(-\Delta_{g_0}\omega + K_{g_0}\right),$$

so that the Gauß curvature of Angenent's metric is

$$K_{\mathrm{Ang}} = \frac{e^{\frac{x^2+y^2}{2}}}{y^2}\left(1 + \frac{1}{y^2}\right)$$

The remaining terms in the expression for the stability operator give:

$$-\tfrac{1}{2}X \cdot \nabla v + \tfrac{1}{2}v = -\tfrac{1}{2}\frac{xx' + yy'}{\omega}\frac{\partial v}{\partial t} + \tfrac{1}{2}v.$$

By virtue of rotational symmetry, the equation $\mathcal{L}v = 0$ separates, and we expand

v by its Fourier series on each radial circle:

$$v(t,\theta) = \sum_{m\in\mathbb{Z}} v_m(t)e^{im\theta}, \tag{4.42}$$

and thus the equations we study are

$$\mathcal{L}_m v_m = 0, \tag{4.43}$$

for the appropriate boundary conditions (e.g. Dirichlet or Neumann), where

$$\mathcal{L}_m v_m = \frac{1}{y\sqrt{\omega}}\frac{\partial}{\partial t}\left(\frac{y}{\sqrt{\omega}}\frac{\partial}{\partial t}\right)v_m - \frac{1}{2}\frac{xx'+yy'}{\omega}v_m' + \left(|A|^2 + \frac{1}{2} - \frac{m^2}{y^2}\right)v_m. \tag{4.44}$$

Observing how the size of the 0th order coefficient improves with increasing symmetry, we thus immediately conclude the following proposition:

Proposition 4.2.9. *On the compact surface of revolution S_γ with boundary, generated by the curve γ, we let:*

$$M_0(\gamma) := \sup_{p\in\gamma}\left[y(p)\sqrt{\tfrac{1}{2}+|A(p)|^2}\right]. \tag{4.45}$$

Then the unique solutions to each of the above Sturm-Liouville problems (with the above-mentioned appropriate boundary condition for each case) are:

$$v_m \equiv 0, \quad \textit{for all} \quad m \geq M_0(\gamma). \tag{4.46}$$

Proof. This follows immediately from an application of the usual maximum principle in one variable. □

The usage of the preceding proposition is now via the assumption of k-dihedral symmetry, which is built into the entire construction. It then follows, that only such Fourier modes m in the separation Equation (4.42) satisfying $k \mid m$ may appear in the decomposition. Hence by restricting to large enough values of k,

$$k \geq \max_i M_0(\gamma_i), \tag{4.47}$$

and correspondingly large genus g, then by virtue of Proposition 4.2.9, there can be assumed to be no $m > 0$ modes in the decomposition. Note of course, that since $m = 0$ modes are k-symmetric for any k (which amounts to $k \mid 0$), the argument does not apply to $m = 0$.

We now therefore focus solely on the $m = 0$ mode. Recalling the second variation formula for the energy of a curve, we get:

$$\frac{d^2}{ds^2}A(\Sigma) = \frac{d^2}{ds^2}\int_\Sigma e^{-\frac{|x|^2}{4}}dA_{\mathbb{R}^3} \tag{4.48}$$

$$= -\int (f\mathcal{L}f)e^{\frac{|x|^2}{4}}dA_{\mathbb{R}^3} \tag{4.49}$$

$$= \frac{d^2}{dv^2}\left(\mathrm{vol}(\mathbb{S}^{n-1})\mathrm{Length}_{\mathrm{Ang}}\right) \tag{4.50}$$

$$= \mathrm{vol}(\mathbb{S}^{n-1})\int\left\langle -\frac{DV}{dt} - R(\gamma', V)\gamma', V\right\rangle dt \tag{4.51}$$

$$= -\mathrm{vol}(\mathbb{S}^{n-1})\int_a^b g(g'' + K_{\mathrm{Ang}}g)dt. \tag{4.52}$$

Note that we can therefore rewrite the $m = 0$ equation in terms of $\tilde{v}_0 = \sqrt{\omega}v_0$, if we simultaneousely parametrize by unit length w.r.t. to the Angenent half-plane metric, to obtain:

$$\tilde{v}_0'' + K_{\mathrm{Ang}}\tilde{v}_0 = \frac{1}{\sqrt{\omega}}\mathcal{L}_0 v_0 = 0. \tag{4.53}$$

That is, of course, the well-known Jacobi equation for the geodesic γ. Since $\sqrt{\omega} > 0$, Dirichlet conditions for v_0 correspond exactly to Dirichlet conditions for $\tilde{v}_0$. Note that since, by symmetry,

$$\frac{\partial\sqrt{\omega}}{\partial t}(t_0) = 0, \quad \text{when} \quad \gamma(t_0) \in \{x_1 = 0\}, \tag{4.54}$$

imposing Neumann conditions on $\{x_1 = 0\}$ is also equivalent for v_0 and $\tilde{v}_0$.

Lemma 4.2.10. *For graphs of the form $(x_1, u(x_1))$ the operator $\mathcal{L}_0$ specializes to $\omega = 1 + (u')^2$, and*

$$\begin{aligned}\omega\mathcal{L}_0 v &= v'' - \frac{x_1}{2}\left(1 + (u')^2\right)v' + \left[\left(\frac{u - x_1u'}{2} - \frac{1}{u}\right)^2 + \frac{1}{u^2} + \frac{1 + (u')^2}{2}\right]v,\\ &=: v'' + P(x_1, u, u')v' + Q(x_1, u, u')v,\end{aligned}$$

when u is a solution to the shrinker equation.

For solution graphs of the form $(f(x_2), x_2)$ we have $\omega = 1 + (f')^2$ and the formula is:

$$\begin{aligned}\omega\mathcal{L}_0 g &= g'' + \left[\frac{1}{x_2} - \frac{x_2}{2}\right]\left(1 + (f')^2\right)g' + \left[\left(\frac{x_2f' - f}{2} - \frac{f'}{x_2}\right)^2 + \frac{(f')^2}{x_2^2} + \frac{1 + (f')^2}{2}\right]g\\ &=: g'' + R(x_2, f, f')g' + S(x_2, f, f')g.\end{aligned}$$

Let us as a preliminary consideration note that the mean curvature H has rota-

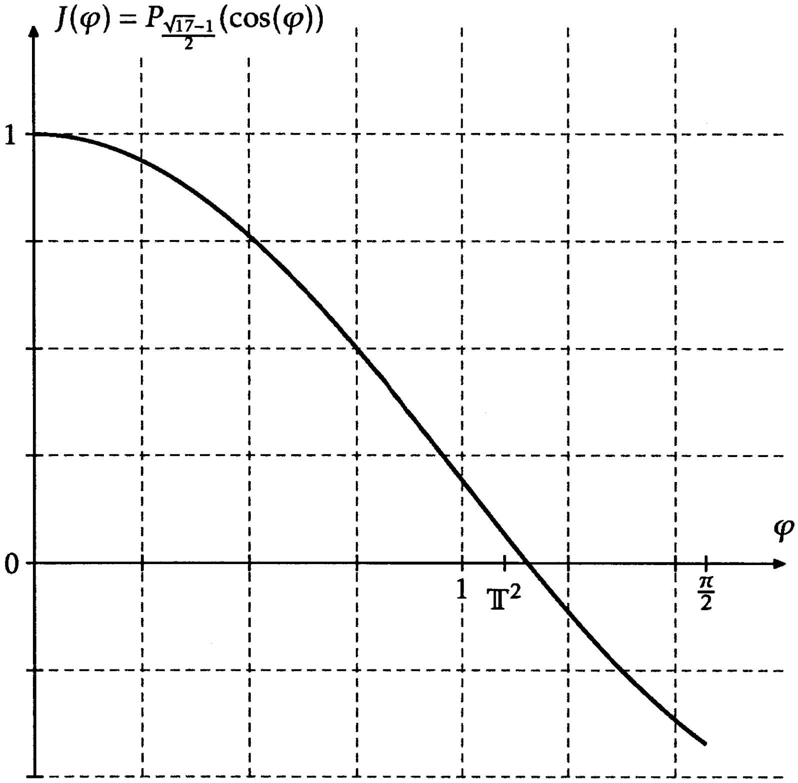

Figure 4-3: Jacobi field J on $S_3 \subseteq \mathbb{S}^2$, with Neumann condition on $\{x = 0\}$. Drawn w.r.t. polar coordinates around $(0, 1)$, where φ is the angle out from the x_2-axis in Figure 4.2.1. Point of intersection with $\mathbb{T}^2$ indicated.

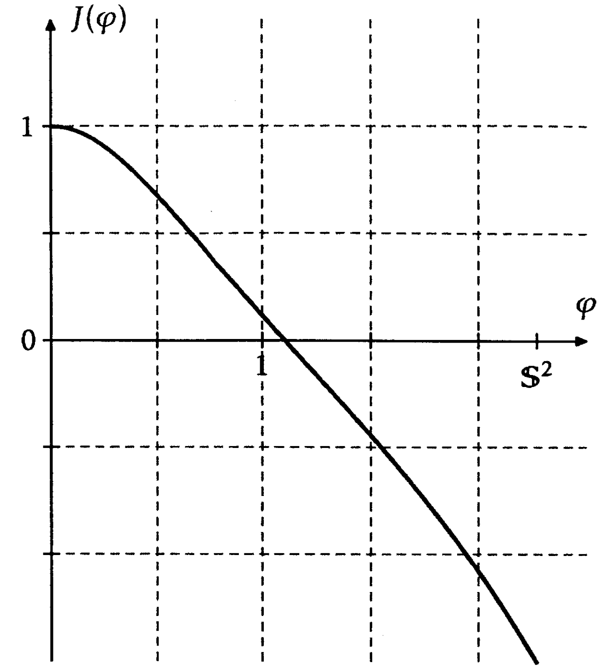

Figure 4-4: Jacobi field J on $S_5 \subseteq \mathbb{T}^2$, with Neumann condition on $\{x = 0\}$. Drawn w.r.t. polar coordinates around $(0,1)$, where φ is the angle out from the x_2-axis in Figure 4.2.1. Point of intersection with $\mathbb{S}^2$ indicated.

tional symmetry, and is an eigenfunction with eigenvalue 1 (see [CM2]):

$$\mathcal{L}H = H, \tag{4.55}$$

and Neumann conditions on $\{x_1 = 0\}$. The profile of $\mathbb{T}^2$ is convex as shown in [KMø11], and thus since the sign of the mean curvature changes only at points of tangential contact with a straight line from the origin, we see that on $\mathbb{T}^2$ the function $H \circ \gamma$ has exactly one zero.

By Sturm-Liouville theory, we now conclude from (4.55) that a solution to the Neumann problem for $\mathcal{L}_0$, if it exists, needs to have at least two zeros in the interval. However, it of course turns out there are no such non-trivial fields with Neumann conditions, which is what we now will apply more detailed analysis to show.

Surfaces contained in $\mathbb{S}^2$

For surfaces contained in $\mathbb{S}^2$ it is of natural convenience to use polar coordinates. Recall that a curve $(\rho, \varphi), \varphi = \arctan(x/z)$ in the xz-plane generated by a function $\rho(\varphi)$ that generates a smooth solution to the self-shrinker equation (4.36) satisfies:

$$\rho''(\varphi) = \frac{1}{\rho}\left\{\rho^2 + 2(\rho')^2 + \left[1 - \frac{\rho^2}{2} - \frac{\rho'}{\rho \tan\varphi}\right]\left(\rho^2 + (\rho')^2\right)\right\}. \tag{4.56}$$

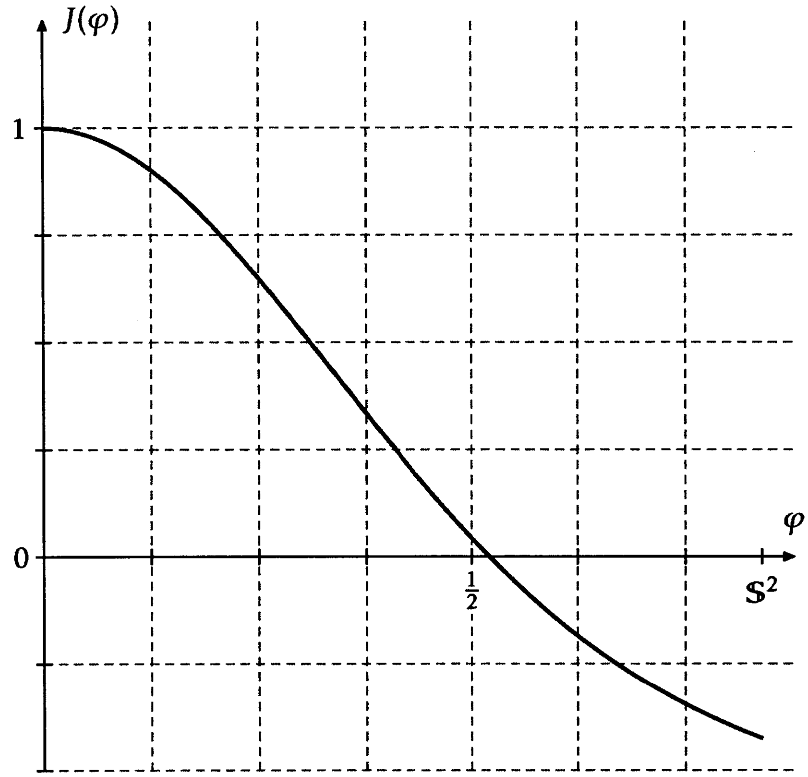

Figure 4-5: Jacobi field J on $S_6 \subseteq \mathbb{T}^2$, with Neumann condition on $\{x = 0\}$. Drawn w.r.t. polar coordinates around $(0, 1)$, where φ is the angle out from the x_2-axis in Figure 4.2.1. Point of intersection with $\mathbb{S}^2$ indicated.

A function w giving a (via a unit normal w.r.t. Euclidean length) variation field, must thus on $\mathbb{S}^2$ satisfy the equation (see Appendix A in [KKMø10])

$$w'' + \frac{1}{\tan\varphi} w' + 4w = 0, \tag{4.57}$$

with appropriate boundary conditions. The substitution $x = \cos(\varphi)$ in (4.57) gives Legendre's differential equation, and the solution is:

$$w(\varphi) = C_1 P_{l_0}(\cos\varphi) + C_2 Q_{l_0}(\cos\varphi),$$

where P_l and Q_l are respectively the Legendre functions of the first and second kind, and $l_0 = (\sqrt{17}-1)/2$ is the positive root of $l(l+1) = 4$.

For the surface $S_1 \subseteq \mathbb{S}^2$, which is generated by rotation of the radius 2 quarter-circle, the boundary conditions in the theorem are $w'(0) = 0$ and $w'(\frac{\pi}{2}) = 0$. But since $Q_{l_0}(\cos\varphi)$ has a pole at $\varphi = 0$, we see that $C_2 = 0$. Thus, if we normalize w so that $C_1 = 1$, we have by the first condition that $w = P_{l_0}(\cos\varphi)$. However,

$$\frac{dP_{l_0}(\cos\varphi)}{d\varphi}_{|\varphi=\frac{\pi}{2}} = \frac{\sqrt{\pi}}{2} \frac{\sqrt{17}+1}{\Gamma\left(\frac{1-\sqrt{17}}{4}\right)\Gamma\left(\frac{5+\sqrt{17}}{4}\right)} \neq 0, \tag{4.58}$$

and hence there is no such Neumann mode. By the preceding, we therefore conclude that under N-fold symmetry, for a large enough $N > 0$,

$$\ker \mathcal{L}_{S_1} = \{0\}, \quad \text{on} \quad S_1 = \mathbb{S}^2 \cap \{x_1 \geq 0\} \quad \text{[Neumann conditions]}. \tag{4.59}$$

As for the surface $S_3 \subseteq \mathbb{S}^2$, which we let for definiteness be the component such that $(2,0) \in S_3$), again $C_1 = 1$ and $C_2 = 0$. Now, since by (4.7),

$$P_{l_0}(\cos(\angle(\vec{e_1}, p^+))) \geq P_{l_0}(\cos(\tfrac{11}{10} + \tfrac{1}{20})) \geq \tfrac{1}{200} > 0, \tag{4.60}$$

we conclude again

$$\ker \mathcal{L}_{S_3} = \{0\} \quad \text{[Dirichlet conditions]}. \tag{4.61}$$

For $S_4 \subseteq \mathbb{S}^2$, the component with $(0,2) \in S_4$, we see that with $w'(\frac{\pi}{2}) = 0$ and fixing $w(\frac{\pi}{2}) = 1$,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} P_{l_0}(\cos(\frac{\pi}{2})) & Q_{l_0}(\cos(\frac{\pi}{2})) \\ \frac{d}{d\varphi} P_{l_0}(\cos(\frac{\pi}{2})) & \frac{d}{d\varphi} P_{l_0}(\cos(\frac{\pi}{2})) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

such that with these constants

$$w(\angle(\vec{e_1}, p^+)) \geq w(\tfrac{11}{10}) - 100\varepsilon_{\text{gap}} > \tfrac{1}{2} > 0.$$

Thus on the surface S_4 with Neumann and Dirichlet conditions as in the statement of the theorem, we also conclude

$$\ker \mathcal{L}_{S_4} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{Dirichlet on } S_4 \cap \mathbb{T}^2]. \tag{4.62}$$

Surfaces Contained in $\mathbb{T}^2$

In this section we will show that the Jacobi fields with Neumann conditions $u_0'(0) = 0$ (and normalized to $u_0(0) = 1$), from respectively the top at bottom of the torus $\mathbb{T}^2$ from Section 4.2 have the following end point values at point where $\mathbb{T}^2$ intersects the round 2-sphere of radius 2:

$$\begin{pmatrix} u_0(t_{\text{top}}) \\ u_0'(t_{\text{top}}) \end{pmatrix} = \frac{1}{50}\begin{pmatrix} -22 \pm \varepsilon_{\text{gap}} \\ -37 \pm \varepsilon_{\text{gap}} \end{pmatrix} \tag{4.63}$$

and

$$\begin{pmatrix} u_0(t_{\text{bot}}) \\ u_0'(t_{\text{bot}}) \end{pmatrix} = \frac{1}{50}\begin{pmatrix} -77 \pm \varepsilon_{\text{gap}} \\ -84 \pm \varepsilon_{\text{gap}} \end{pmatrix} \tag{4.64}$$

This means firstly that the Dirichlet problems on each part (w/ Neumann conditions on $\mathcal{P}$) have trivial kernel, that is:

$$\ker \mathcal{L}_{S_5} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{Dirichlet on } S_5 \cap \mathbb{S}^2], \tag{4.65}$$

$$\ker \mathcal{L}_{S_6} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}, \text{Dirichlet on } S_6 \cap \mathbb{S}^2]. \tag{4.66}$$

Secondly, note that non-triviality of the kernel of $\mathcal{L}$ on $\mathbb{T}^2 \cap \{x_1 \geq 0\}$ with Neumann conditions has now been reduced to the conditions

$$\alpha u_0(t_{\text{top}}) = \beta u_0(t_{\text{top}}),$$
$$\alpha u_0'(t_{\text{top}}) = -\beta u_0(t_{\text{bot}}),$$

for a non-zero pair (α, β) or in other words, singularity of the matrix $\mathcal{N}$:

$$\mathcal{N} := \begin{pmatrix} u_0(t_{\text{top}}) & -u_0(t_{\text{bot}}) \\ u_0'(t_{\text{top}}) & u_0'(t_{\text{bot}}) \end{pmatrix}. \tag{4.67}$$

But in fact from (4.63)–(4.64) we see

$$\det \mathcal{N} \geq \tfrac{9}{5} > 0, \tag{4.68}$$

so we finally conclude that also

$$\ker \mathcal{L}_{S_2} = \{0\} \quad [\text{Neumann on } \{x_1 = 0\}.$$

To show the required estimates of the Jacobi fields, we consider first an approximate solution V to the linearized equation on the approximate curve Γ from the previous section.

Top: x_1-graph (Jacobi Equation)

Let us consider the first part of Γ, graphical over the x_1-axis, on $x_1 \in [0, \frac{3}{5}]$. Here, we have the expressions:

$$\begin{aligned}
&\left|\partial_2 P(x_1, \xi, U')\right| = 0, \\
&\left|\partial_3 P(x_1, u, \xi')\right| = |x_1 \xi'|, \\
&\left|\partial_2 Q(x_1, \xi, U')\right| = 2\left|\left(\frac{\xi - x_1 U'}{2} - \frac{1}{\xi}\right)\left(\frac{1}{2} + \frac{1}{\xi^2}\right) - \frac{1}{\xi^3}\right|, \\
&\left|\partial_3 Q(x_1, u, \xi')\right| = \left|x_1\left(\frac{u - x_1 \xi'}{2} - \frac{1}{u}\right) - \xi'\right|.
\end{aligned}$$

Assume for a small $\delta_1^T > 0$ the uniform bounds:

$$\left|V'' + P(x_1, U, U')V' + Q(x_1, U, U')V\right| \le \delta_1^T, \tag{4.69}$$

$$\left|\partial_3 P(x_1, u, \xi')\right||v'| + \left|\partial_3 Q(x_1, u, \xi')\right||v| \le x_1(2x_1^2 + 3), \tag{4.70}$$

$$\left|\partial_2 Q(x_1, \xi, U')\right||v| \le \tfrac{8}{5} - \tfrac{3}{2}x_1^2. \tag{4.71}$$

We let $\Phi(x_1) := |V'(x_1) - v'(x_1)|$ and estimate:

$$\begin{aligned}
\Phi'(x_1) &\le \left||V' - v'|'\right| \overset{\text{a.e}}{=} |V'' - v''| \\
&\le \left|P(x_1, U, U')V' - P(x_1, u, u')v'\right| + \left|Q(x_1, U, U')V - Q(x_1, u, u')v\right| + \delta_1^T \\
&\le \left|P(x_1, U, U')\right||V' - v'| + \left|P(x_1, U, U') - P(x_1, u, u')\right||v'| \\
&\quad + \left|Q(x_1, U, U')\right||V - v| + \left|Q(x_1, U, U') - Q(x_1, u, u')\right||v| + \delta_1^T \\
&\le \left|P(x_1, U, U')\right||V' - v'| + \left|Q(x_1, U, U')\right||V - v| \\
&\quad + \left(\left|\partial_3 P(x_1, u, \xi')\right||v'| + \left|\partial_3 Q(x_1, u, \xi')\right||v|\right)|U' - u'| \\
&\quad + \left(\cancel{\left|\partial_2 P(x_1, \xi, U')\right|}|v'| + \left|\partial_2 Q(x_1, \xi, U')\right||v|\right)|U - u| + \delta_1^T \\
&\le \left|P(x_1, U, U')\right|\Phi(x_1) + \left|Q(x_1, U, U')\right|\int_0^{x_1} \Phi(s)ds + \left|Q(x_1, U, U')\right||V(0) - v(0)| \\
&\quad + x_1(2x_1^2 + 3)|U'(x_1) - u'(x_1)| + (\tfrac{8}{5} - \tfrac{3}{2}x_1^2)|U(x_1) - u(x_1)| + \delta_1^T.
\end{aligned}$$

We integrate on $[0, x_1]$ to get:

$$\begin{aligned}
\Phi(x_1) \le &\int_0^{x_1}\left[\left|P(s, U, U')\right| + \int_0^{x_1}\left|Q(t, U, U')\right|dt\right]\Phi(s)ds + \left(\int_0^{x_1}\left|Q(s, U, U')\right|ds\right)|V(0) - v(0)| \\
&+ \int_0^{x_1} s(2s^2 + 3)|U'(s) - u'(s)|ds + \int_0^{x_1}(\tfrac{8}{5} - \tfrac{3}{2}s^2)|U(s) - u(s)|ds + \delta_1^T x_1.
\end{aligned}$$

Recall the estimates for $|U'(s)-u'(s)|$, which lead to:

$$\int_0^{x_1} s(2s^2+3)|U'(s)-u'(s)|ds \le \left[\tfrac{4}{5}|U(0)-u(0)|+\varepsilon_1^T\right]\int_0^{x_1} s^2(2s^2+3)\exp\left\{\tfrac{52}{15}s^2\right\}ds$$
$$\le \tfrac{9}{20}|U(0)-u(0)|+\tfrac{14}{25}\varepsilon_1^T.$$

Note also that from the estimates for $|U-u|$ from $|U'-u'|$, we have already once estimated the integral of the latter. We now need the sizes of these elementary Gaussian double integrals:

$$\int_0^{\frac{3}{5}}(\tfrac{8}{5}-\tfrac{3}{2}x_1^2)\left(1+\int_0^{x_1}\tfrac{4s}{5}\exp\left\{\tfrac{52}{15}s^2\right\}ds\right)dx_1 \le \frac{23}{25},$$
$$\int_0^{\frac{3}{5}}(\tfrac{8}{5}-\tfrac{3}{2}x_1^2)\int_0^{x_1} s\exp\left\{\tfrac{52}{15}s^2\right\}dsdx_1 \le \frac{7}{100}.$$

Thus

$$\int_0^{\frac{3}{5}}(\tfrac{8}{5}-\tfrac{3}{2}s^2)|U(s)-u(s)|ds \le \tfrac{23}{25}|U(0)-u(0)|+\tfrac{7}{100}\varepsilon_1^T.$$

Now, we will furthermore assume the following bounds on the test functions, pertaining to the approximation by ε-geodesics:

$$\int_0^{x_1}|P(s,U,U')|ds \le \int_0^{x_1} s(\tfrac{7}{5}s^2+\tfrac{7}{10})ds \le \tfrac{7}{20}x_1^2(x_1^2+1), \tag{4.72}$$
$$\int_0^{x_1}\left|Q(s,U,U')\right|ds \le \int_0^{x_1}(\tfrac{16}{5}s^2+\tfrac{5}{2})ds \le \tfrac{16}{15}x_1^3+\tfrac{5}{2}x_1. \tag{4.73}$$

Applying Grönwall-Bellman again, to these new estimates, we see, using also that $V(0)=v(0)=1$ by assumption:

$$\Phi(x_1) \le \left[\int_0^{x_1} s(2s^2+3)|U'(s)-u'(s)|ds+\int_0^{x_1}(\tfrac{8}{5}-\tfrac{3}{2}s^2)|U(s)-u(s)|ds+\delta_1^T x_1\right]\times$$
$$\exp\left\{\int_0^{x_1}\left|P(s,U,U')\right|ds+x_1\int_0^{x_1}\left|Q(s,U,U')\right|ds\right\}$$
$$\le\left[\tfrac{137}{100}|U(0)-u(0)|+\tfrac{63}{100}\varepsilon_1^T+\tfrac{3}{5}\delta_1^T\right]\times$$
$$\exp\left\{\tfrac{7}{20}x_1^2(x_1^2+1)+\tfrac{16}{15}x_1^4+\tfrac{5}{2}x_1^2\right\}.$$

Thus

$$|V'(\tfrac{3}{5})-v'(\tfrac{3}{5})| \le \tfrac{23}{5}|U(0)-u(0)|+\tfrac{11}{5}\varepsilon_1^T+\tfrac{51}{25}\delta_1^T,$$
$$|V(\tfrac{3}{5})-v(\tfrac{3}{5})| \le \tfrac{13}{10}|U(0)-u(0)|+\tfrac{3}{5}\varepsilon_1^T+\tfrac{11}{50}\delta_1^T.$$

Here, the last estimate followed by integrating the estimates above, and using again that $V(0) = v(0)$.

Top: x_2-graph to the sphere (Jacobi Equation)

We consider the next part of Γ, graphical over the x_2-axis over approximately the region $[y_{S^2}, u_T(3/5)]$ (in the backwards direction). Here, we have the expressions:

$$\begin{aligned}
\left|\partial_2 R(x_2,\xi,F')\right| &= 0,\\
\left|\partial_3 R(x_2,f,\xi')\right| &= 2\left|\frac{1}{x_2}-\frac{x_2}{2}\right||\xi'|,\\
\left|\partial_2 S(x_2,\xi,F')\right| &= \left|\left(\frac{1}{x_2}-\frac{x_2}{2}\right)F'+\frac{\xi}{2}\right|,\\
\left|\partial_3 S(x_2,f,\xi')\right| &= \left|\left(\frac{1}{x_2}-\frac{x_2}{2}\right)(x_2\xi'-f)+\frac{4\xi'}{x_2^2}\right|.
\end{aligned}$$

Assume for a small $\delta_2^T > 0$ the uniform bounds:

$$\left|G''+R(x_2,F,F')G'+S(x_2,F,F')G\right| \le \delta_2^T, \tag{4.74}$$

$$\left|\partial_3 R(x_2,u,\xi')\right||g'|+\left|\partial_3 S(x_2,f,\xi')\right||g| \le \tfrac{3}{10}+\tfrac{33}{20}(x_2-y_{S^2})^5 \tag{4.75}$$

$$\left|\partial_2 S(x_2,\xi,U')\right||g| \le \eta(x_2), \tag{4.76}$$

where

$$\eta(x_2) = \begin{cases} \frac{41}{200}-\frac{3}{10}(x_2-y_{S^2})^2, & x_2\in[y_{S^2},\frac{5}{2}],\\ \frac{3}{4}-\frac{5}{2}(x_2-y_{S^2})+\frac{21}{10}(x_2-y_{S^2})^2, & x_2\in[\frac{5}{2},u_T(3/5)]. \end{cases} \tag{4.77}$$

We let $\Psi(x_2) := |G'(x_2)-g'(x_2)|$ and estimate:

$$\begin{aligned}
\Psi'(x_2) &\le \left||G'-g'|'\right| \overset{\text{a.e}}{=} |G''-g''|\\
&\le \left|R(x_2,F,F')G'-R(x_2,f,f')g'\right|+\left|S(x_2,F,F')G-S(x_2,f,f')g\right|+\delta_2^T\\
&\le \left|R(x_2,F,F')\right||G'-g'|+\left|R(x_2,F,F')-R(x_2,f,f')\right||g'|\\
&\quad+\left|S(x_2,F,F')\right||G-g|+\left|S(x_2,F,F')-S(x_2,f,f')\right||g|+\delta_2^T\\
&\le \left|R(x_2,F,F')\right||G'-g'|+\left|S(x_2,F,F')\right||G-g|\\
&\quad+\left(\left|\partial_3 R(x_2,f,\xi')\right||g'|+\left|\partial_3 S(x_2,f,\xi')\right||g|\right)|F'-f'|\\
&\quad+\left|\partial_2 S(x_2,\xi,F')\right||g||F-f|+\delta_2^T\\
&\le \left|R(x_2,F,F')\right|\Psi(x_2)+\left|S(x_2,F,F')\right|\left|\int_0^{x_2}\Psi(s)ds\right|+\left|S(x_2,F,F')\right||G(u_T(3/5))-g(u_T(3/5))|\\
&\quad+\left[\tfrac{3}{10}+\tfrac{33}{20}(x_2-y_{S^2})^5\right]|F'(x_2)-f'(x_2)|+\eta(x_2)|F(x_2)-f(x_2)|+\delta_2^T.
\end{aligned}$$

We integrate on $[x_2, u_T(3/5)]$ to get:

$$\Psi(x_2) \le \int_{x_2}^{u_T(3/5)} \left[\left| R(s,F,F') \right| + \int_{x_2}^{u_T(3/5)} \left| S(t,F,F') \right| dt \right] \Psi(s) ds$$
$$+ \left(\int_{x_2}^{u_T(3/5)} \left| S(s,F,F') \right| ds \right) |G(u_T(3/5)) - g(u_T(3/5))|$$
$$+ \int_{x_2}^{u_T(3/5)} \left[\tfrac{3}{10} + \tfrac{33}{20}(s - y_{S^2})^5 \right] |F'(s) - f'(s)| ds + \int_{x_2}^{u_T(3/5)} \eta(s)|F(s) - f(s)| ds$$
$$+ |G'(u_T(3/5)) - g'(u_T(3/5))| + \delta_2^T (u_T(3/5) - x_2).$$

Recall, we have above shown the estimates:

$$|F'(x_2) - f'(x_2)| \le \Big[(\tfrac{13}{8} - \tfrac{x_2}{2}) |F(u_T(3/5)) - f(u_T(3/5))| + |F'(u_T(3/5)) - f'(u_T(3/5))|$$
$$+ \varepsilon_2^T (u_T(3/5) - x_2) \Big] \exp\left\{ \tfrac{7}{4} - \tfrac{9}{10}(x_2 - y_{S^2})^2 + \left(\tfrac{13}{8} - \tfrac{x_2}{2}\right)(u_T(3/5) - x_2) \right\},$$
$$|F(u_T(3/5)) - f(u_T(3/5))| \le \tfrac{10}{12} \left(\tfrac{9}{25} \varepsilon_1^T + \tfrac{23}{80} |U(0) - u(0)| \right),$$
$$|F'(u_T(3/5)) - f'(u_T(3/5))| \le \left(\tfrac{10}{12} \right)^2 (\varepsilon_1^T + \tfrac{4}{5} |U(0) - u(0)|).$$

We therefore get the bound:

$$\int_{y_{S^2}}^{u_T(3/5)} \left[\tfrac{3}{10} + \tfrac{33}{20}(s - y_{S^2})^5 \right] |F'(s) - f'(s)| ds \le \tfrac{80}{25} |U(0) - u(0)| + 6\varepsilon_1^T + \tfrac{27}{10} \varepsilon_2^T.$$

Again, we will need the sizes of some elementary Gaussian double integrals:

$$\int_{y_{S^2}}^{u_T(3/5)} \eta(x_2) \left(1 + \int_{x_2}^{u_T(3/5)} (\tfrac{13}{8} - \tfrac{s}{2}) \exp\left\{ \tfrac{7}{4} - \tfrac{9}{10}(s - y_{S^2})^2 + \left(\tfrac{13}{8} - \tfrac{s}{2}\right)(u_T(3/5) - s) \right\} ds \right) dx_2 \le \tfrac{29}{50}$$
$$\int_{y_{S^2}}^{u_T(3/5)} \int_{x_2}^{u_T(3/5)} \eta(x_2) \exp\left\{ \tfrac{7}{4} - \tfrac{9}{10}(s - y_{S^2})^2 + \left(\tfrac{13}{8} - \tfrac{s}{2}\right)(u_T(3/5) - s) \right\} ds dx_2 \le \tfrac{29}{50},$$
$$\int_{y_{S^2}}^{u_T(3/5)} \int_{x_2}^{u_T(3/5)} \eta(x_2)(u_T(3/5) - s) \exp\left\{ \tfrac{7}{4} - \tfrac{9}{10}(s - y_{S^2})^2 + \left(\tfrac{13}{8} - \tfrac{s}{2}\right)(u_T(3/5) - s) \right\} ds dx_2 \le \tfrac{19}{50}.$$

Thus

$$\int_{y_{S^2}}^{u_T(3/5)} \eta(x_2) |F(x_2) - f(x_2)| dx_2 \le \tfrac{1}{2} |U(0) - u(0)| + \tfrac{29}{50} \varepsilon_1^T + \tfrac{19}{50} \varepsilon_2^T.$$

Assume once again bounds for the ε-geodesics:

$$\int_{x_2}^{u_T(3/5)} |R(s,F,F')|ds \le -\tfrac{16}{25}x_2^2 + \tfrac{22}{10}x_2 - \tfrac{18}{25}, \tag{4.78}$$

$$\int_{x_2}^{u_T(3/5)} \left|S(s,F,F')\right|ds \le -\tfrac{27}{100}x_2^2 + \tfrac{7}{25}x_2 - \tfrac{19}{10}. \tag{4.79}$$

By Grönwall-Bellman with the new estimates, we see:

$$\begin{aligned}\Psi(x_2) \le \Bigg[&\left(\left(\int_{x_2}^{u_T(3/5)} \left|S(s,F,F')\right|ds\right)|G(u_T(3/5)) - g(u_T(3/5))|\right. \\ &+ \int_{x_2}^{u_T(3/5)} \left[\tfrac{3}{10} + \tfrac{33}{20}(x_2 - y_{S^2})^5\right]|F'(s) - f'(s)|ds \\ &+ \int_{x_2}^{u_T(3/5)} \eta(x_2)|F(s) - f(s)|ds + \delta_2^T(u_T(3/5) - x_2) + |G'(u_T(3/5)) - g'(u_T(3/5))|\Bigg] \times \\ &\exp\left\{\int_{x_2}^{u_T(3/5)} \left|R(s,F,F')\right|ds + (u_T(3/5) - x_2)\int_{x_2}^{u_T(3/5)} \left|S(s,F,F')\right|ds\right\}.\end{aligned}$$

Thus we have as before estimates:

$$|G'(y_{S^2}) - g'(y_{S^2})| \le$$

and by integration

$$|G(y_{S^2}) - g(y_{S^2})|$$

Bottom: x_1-graph (Jacobi Equation)

Let us consider the first part of Γ, graphical over the x_1-axis. Here, we have the expressions:

$$\begin{aligned}\left|\partial_2 P(x_1,\xi,U')\right| &= 0, \\ \left|\partial_3 P(x_1,u,\xi')\right| &= |x_1\xi'|, \\ \left|\partial_2 Q(x_1,\xi,U')\right| &= 2\left|\left(\frac{\xi - x_1U'}{2} - \frac{1}{\xi}\right)\left(\frac{1}{2} + \frac{1}{\xi^2}\right) - \frac{1}{\xi^3}\right|, \\ \left|\partial_3 Q(x_1,u,\xi')\right| &= \left|x_1\left(\frac{u - x_1\xi'}{2} - \frac{1}{u}\right) - \xi'\right|.\end{aligned}$$

Assume for a small $\delta_1^B > 0$ the uniform bounds:

$$\left|V'' + P(x_1,U,U')V' + Q(x_1,U,U')V\right| \le \delta_1^B, \tag{4.80}$$

$$\left|\partial_3 P(x_1,u,\xi')\right||v'| + \left|\partial_3 Q(x_1,u,\xi')\right||v| \le 2, \tag{4.81}$$

$$\left|\partial_2 Q(x_1,\xi,U')\right||v| \le 41 - 80x_1. \tag{4.82}$$

We let $\Phi(x_1) := |V'(x_1) - v'(x_1)|$ and estimate:

$$\begin{aligned}
\Phi'(x_1) &\le \left||V'-v'|'\right| \overset{\text{a.e}}{=} |V''-v''| \\
&\le \left|P(x_1,U,U')V' - P(x_1,u,u')v'\right| + \left|Q(x_1,U,U')V - Q(x_1,u,u')v\right| + \delta_1^B \\
&\le \left|P(x_1,U,U')\right||V'-v'| + \left|P(x_1,U,U') - P(x_1,u,u')\right||v'| \\
&\quad + \left|Q(x_1,U,U')\right||V-v| + \left|Q(x_1,U,U') - Q(x_1,u,u')\right||v| + \delta_1^B \\
&\le \left|P(x_1,U,U')\right||V'-v'| + \left|Q(x_1,U,U')\right||V-v| \\
&\quad + \left(\left|\partial_3 P(x_1,u,\xi')\right||v'| + \left|\partial_3 Q(x_1,u,\xi')\right||v|\right)|U'-u'| \\
&\quad + \left|\partial_2 Q(x_1,\xi,U')\right||v||U-u| + \delta_1^B
\end{aligned}$$

Hence:

$$\begin{aligned}
&\le \left|P(x_1,U,U')\right|\Phi(x_1) + \left|Q(x_1,U,U')\right|\int_0^{x_1}\Phi(s)ds + \left|Q(x_1,U,U')\right||V(0)-v(0)| \\
&\quad + 2|U'(x_1)-u'(x_1)| + (41-80x_1)|U(x_1)-u(x_1)| + \delta_1^B.
\end{aligned}$$

We integrate on $[0,x_1]$ to get:

$$\begin{aligned}
\Phi(x_1) &\le \int_0^{x_1}\left[\left|P(s,U,U')\right| + \int_0^{x_1}\left|Q(t,U,U')\right|dt\right]\Phi(s)ds + \left(\int_0^{x_1}\left|Q(s,U,U')\right|ds\right)|V(0)-v(0)| \\
&\quad + 2\int_0^{x_1}|U'(s)-u'(s)|ds + \int_0^{x_1}(41-80s)|U(s)-u(s)|ds + \delta_1^B x_1.
\end{aligned}$$

Recall the estimates:

$$\begin{aligned}
2\int_0^{x_1}|U'(s)-u'(s)|ds &\le 2|U(0)-u(0)|\int_0^{x_1}\left(\tfrac{29}{5}s+\tfrac{1}{20}\right)\exp\left\{\tfrac{49}{5}s^2+\tfrac{s}{4}\right\}ds \\
&\quad + 2\varepsilon_1^B\int_0^{x_1}s\exp\left\{\tfrac{49}{5}s^2+\tfrac{s}{4}\right\}ds \\
&\le \tfrac{36}{5}|U(0)-u(0)| + \tfrac{6}{5}\varepsilon_1^B.
\end{aligned}$$

Note also that from the estimates for $|U-u|$ from $|U'-u'|$, we have already once estimated the integral of the latter. We now need the sizes of these elementary Gaussian double integrals:

$$\begin{aligned}
&\int_0^{\frac{1}{2}}(41-80x_1)\left(1+\int_0^{x_1}\left(\tfrac{29}{5}s+\tfrac{1}{20}\right)\exp\left\{\tfrac{49}{5}s^2+\tfrac{s}{4}\right\}ds\right)dx_1 \le \frac{67}{5}, \\
&\int_0^{\frac{1}{2}}\int_0^{x_1}(41-80x_1)s\exp\left\{\tfrac{49}{5}s^2+\tfrac{s}{4}\right\}dsdx_1 \le \frac{12}{25}.
\end{aligned}$$

Thus

$$\int_0^{\frac{1}{2}} (41-80x_1)|U(x_1)-u(x_1)|dx_1 \le \tfrac{67}{5}|U(0)-u(0)| + \frac{12}{25}\varepsilon_1^B.$$

Now, we will furthermore assume the bounds pertaining to the ε-geodesics:

$$\int_0^{x_1} |P(s,U,U')|ds \le \frac{4}{9}x_1^2, \tag{4.83}$$

$$\int_0^{x_1} \left|Q(s,U,U')\right| ds \le 4-16(x_1-\tfrac{1}{2})^2. \tag{4.84}$$

Again, by Grönwall-Bellman we see (with $V(0)=v(0)$):

$$\begin{aligned}\Phi(x_1) \le & \left[2\int_0^{x_1} |U'(s)-u'(s)|ds + \int_0^{x_1}(41-80s)|U(s)-u(s)|ds + \delta_1^B x_1\right]\times \\ & \exp\left\{\int_0^{x_1} \left|P(s,U,U')\right| ds + x_1\int_0^{x_1} \left|Q(s,U,U')\right| ds\right\} \\ \le & \left[\tfrac{103}{5}|U(0)-u(0)| + \tfrac{42}{25}\varepsilon_1^B + \delta_1^B x_1\right]\times \\ & \exp\left\{\tfrac{4}{9}x_1^2 + 4x_1 - 16x_1(x_1-\tfrac{1}{2})^2\right\}.\end{aligned}$$

Thus

$$|V'(\tfrac{1}{2})-v'(\tfrac{1}{2})| \le 171|U(0)-u(0)| + 14\varepsilon_1^B + \tfrac{38}{9}\delta_1^B,$$
$$|V(\tfrac{1}{2})-v(\tfrac{1}{2})| \le 31|U(0)-u(0)| + \tfrac{51}{20}\varepsilon_1^B + \tfrac{11}{5}\delta_1^B.$$

Here, the last estimate followed by integration and using again $V(0)=v(0)$.

Bottom: x_2-graph to cylinder (Jacobi Equation)

We consider the next part of Γ, graphical over the x_2-axis over $[a_0,\sqrt{2}]$. Here, we have the expressions:

$$\left|\partial_2 R(x_2,\xi,F')\right| = 0,$$
$$\left|\partial_3 R(x_2,f,\xi')\right| = 2\left|\frac{1}{x_2}-\frac{x_2}{2}\right||\xi'|,$$
$$\left|\partial_2 S(x_2,\xi,F')\right| = \left|\left(\frac{1}{x_2}-\frac{x_2}{2}\right)F' + \frac{\xi}{2}\right|,$$
$$\left|\partial_3 S(x_2,f,\xi')\right| = \left|\left(\frac{1}{x_2}-\frac{x_2}{2}\right)(x_2\xi'-f) + \frac{4\xi'}{x_2^2}\right|.$$

Assume for a small $\delta_2^B > 0$ the uniform bounds:

$$\left|G'' + R(x_2,F,F')G' + S(x_2,F,F')G\right| \le \delta_2^B, \tag{4.85}$$

$$\left|\partial_3 R(x_2,u,\xi')\right||g'| + \left|\partial_3 S(x_2,f,\xi')\right||g| \le \frac{4}{5} + 8\left(x_2 - \sqrt{2}\right)^2, \tag{4.86}$$

$$\left|\partial_2 S(x_2,\xi,U')\right||g| \le \frac{24}{50} - \frac{3}{4}\left(x_2 - \sqrt{2}\right)^2. \tag{4.87}$$

We let $\Psi(x_2) := |G'(x_2) - g'(x_2)|$ and estimate:

$$\begin{aligned}
\Psi'(x_2) &\le \left||G'-g'|'\right| \overset{\text{a.e}}{=} |G''-g''| \\
&\le \left|R(x_2,F,F')G' - R(x_2,f,f')g'\right| + \left|S(x_2,F,F')G - S(x_2,f,f')g\right| + \delta_2^B \\
&\le \left|R(x_2,F,F')\right||G'-g'| + \left|R(x_2,F,F') - R(x_2,f,f')\right||g'| \\
&\quad + \left|S(x_2,F,F')\right||G-g| + \left|S(x_2,F,F') - S(x_2,f,f')\right||g| + \delta_2^B \\
&\le \left|R(x_2,F,F')\right||G'-g'| + \left|S(x_2,F,F')\right||G-g| \\
&\quad + \left(\left|\partial_3 R(x_2,f,\xi')\right||g'| + \left|\partial_3 S(x_2,f,\xi')\right||g|\right)|F'-f'| \\
&\quad + \left|\partial_2 S(x_2,\xi,F')\right||g||F-f| + \delta_2^B \\
&\le \left|R(x_2,F,F')\right|\Psi(x_2) + \left|S(x_2,F,F')\right|\int_0^{x_2}\Psi(s)ds + \left|S(x_2,F,F')\right||G(a_0)-g(a_0)| \\
&\quad + \left[\frac{4}{5} + 8\left(x_2-\sqrt{2}\right)^2\right]|F'(x_2)-f'(x_2)| + \left[\frac{24}{50} - \frac{3}{4}\left(x_2-\sqrt{2}\right)^2\right]|F(x_2)-f(x_2)| + \delta_2^B.
\end{aligned}$$

We integrate on $[a_0,x_2]$ to get:

$$\begin{aligned}
\Psi(x_2) &\le \int_{a_0}^{x_2}\left[\left|R(s,F,F')\right| + \int_{a_0}^{x_2}\left|S(t,F,F')\right|dt\right]\Psi(s)ds + \left(\int_{a_0}^{x_2}\left|S(s,F,F')\right|ds\right)|G(a_0)-g(a_0)| \\
&\quad + \int_{a_0}^{x_2}\left[\tfrac{4}{5} + 8(s-\sqrt{2})^2\right]|F'(s)-f'(s)|ds + \int_{a_0}^{x_2}\left[\tfrac{24}{50} - \tfrac{3}{4}(s-\sqrt{2})^2\right]|F(s)-f(s)|ds \\
&\quad + |G'(a_0)-g'(a_0)| + \delta_2^B(x_2-a_0).
\end{aligned}$$

Recall, we have above shown the estimates:

$$\begin{aligned}
|F'(x_2)-f'(x_2)| &\le \left[\left(\tfrac{16}{25}x_2 - \tfrac{3}{7}\right)|F(a_0)-f(a_0)| + |F'(a_0)-f'(a_0)| + \varepsilon_2^B(x_2-a_0)\right] \\
&\qquad \times \exp\left\{\tfrac{9}{10} - \tfrac{11}{10}\left(x_2-\tfrac{3}{2}\right)^2 + (x_2-a_0)\left(\tfrac{16}{25}x_2 - \tfrac{3}{7}\right)\right\}, \\
|F(a_0)-f(a_0)| &\le \tfrac{10}{12}\left(\tfrac{3}{5}\varepsilon_1^B + \tfrac{23}{5}|U(0)-u(0)|\right) \\
|F'(a_0)-f'(a_0)| &\le \left(\tfrac{10}{12}\right)^2\left(\tfrac{28}{5}\varepsilon_1^B + 39|U(0)-u(0)|\right).
\end{aligned}$$

We therefore get the bound:

$$\int_{a_0}^{\sqrt{2}} \left[\tfrac{4}{5} + 8(s-\sqrt{2})^2\right] |F'(s) - f'(s)| ds \leq 78|U(0) - u(0)| + 12\varepsilon_1^B + \tfrac{9}{10}\varepsilon_2^B.$$

Again, we will need the sizes of some elementary Gaussian double integrals:

$$\int_{a_0}^{\sqrt{2}} \left[\tfrac{24}{50} - \tfrac{3}{4}(x_2-\sqrt{2})^2\right]\left(1 + \int_{a_0}^{x_2} (\tfrac{16}{25}s - \tfrac{3}{7})\exp\left\{\tfrac{9}{10} - \tfrac{11}{10}\left(s-\tfrac{3}{2}\right)^2 + (s-a_0)\left(\tfrac{16}{25}s - \tfrac{3}{7}\right)\right\} ds\right) dx_2$$
$$\leq \frac{3}{10},$$
$$\int_{a_0}^{\sqrt{2}} \int_{a_0}^{x_2} \left[\tfrac{24}{50} - \tfrac{3}{4}(x_2-\sqrt{2})^2\right] \exp\left\{\tfrac{9}{10} - \tfrac{11}{10}\left(x_2-\tfrac{3}{2}\right)^2 + (x_2-a_0)\left(\tfrac{16}{25}x_2 - \tfrac{3}{7}\right)\right\} ds dx_2 \leq \tfrac{1}{5},$$
$$\int_{a_0}^{\sqrt{2}} \int_{a_0}^{x_2} \left[\tfrac{24}{50} - \tfrac{3}{4}(x_2-\sqrt{2})^2\right](s-a_0) \exp\left\{\tfrac{9}{10} - \tfrac{11}{10}\left(x_2-\tfrac{3}{2}\right)^2 + (x_2-a_0)\left(\tfrac{16}{25}x_2 - \tfrac{3}{7}\right)\right\} ds dx_2 \leq \frac{7}{125}.$$

Thus

$$\int_{a_0}^{\sqrt{2}} \left[\tfrac{24}{50} - \tfrac{3}{4}(x_2-\sqrt{2})^2\right] |F(x_2) - f(x_2)| dx_2 \leq \tfrac{197}{30}|U(0) - u(0)| + \tfrac{47}{50}\varepsilon_1^B + \tfrac{7}{125}\varepsilon_2^B.$$

Assume again bounds for the ε-geodesics:

$$\int_{a_0}^{x_2} |R(s,F,F')| ds \leq \tfrac{9}{20} - \tfrac{3}{4}(x_2-\sqrt{2})^2, \tag{4.88}$$
$$\int_{a_0}^{x_2} \left|S(s,F,F')\right| ds \leq x_2 - \tfrac{2}{5}. \tag{4.89}$$

By Grönwall-Bellman with the new estimates, we see:

$$\Psi(x_2) \leq \Bigg[\left(\int_{a_0}^{x_2} \left|S(s,F,F')\right| ds\right)|G(a_0) - g(a_0)| + \int_{a_0}^{x_2} \left[\tfrac{4}{5} + 8(s-\sqrt{2})^2\right]|F'(s) - f'(s)| ds$$
$$+ \int_{a_0}^{x_2} \left[\tfrac{24}{50} - \tfrac{3}{4}(x_2-\sqrt{2})^2\right]|F(s) - f(s)| ds + \delta_2^B(x_2 - a_0) + |G'(a_0) - g'(a_0)|\Bigg] \times$$
$$\exp\left\{\int_{a_0}^{x_2} \left|R(s,F,F')\right| ds + (x_2 - a_0)\int_{a_0}^{x_2} \left|S(s,F,F')\right| ds\right\}$$
$$\leq \tfrac{63}{20}\Bigg[31|U(0) - U(0)| + \tfrac{51}{20}\varepsilon_1^B + \tfrac{11}{5}\delta_1^B + 78|U(0) - u(0)| + 12\varepsilon_1^B + \tfrac{9}{10}\varepsilon_2^B + \tfrac{197}{30}|U(0) - u(0)|$$
$$+ \tfrac{47}{50}\varepsilon_1^B + \tfrac{7}{125}\varepsilon_2^B + (\sqrt{2} - a_0)\delta_2^B + \frac{171|U(0) - u(0)| + 14\varepsilon_1^B + \tfrac{38}{9}\delta_1^B}{|u'(\tfrac{1}{2})|}\Bigg]$$

Thus

$$|G'(\tfrac{1}{2}) - v'(\tfrac{1}{2})| \le 813|U(0) - u(0)| + 86\varepsilon_1^B + 3\varepsilon_2^B + \tfrac{95}{27}\delta_1^B + (\sqrt{2} - a_0)\delta_2^B,$$
$$|G(\tfrac{1}{2}) - v(\tfrac{1}{2})| \le 522|U(0) - u(0)| + 56\varepsilon_1^B + 2\varepsilon_2^B + 11\delta_1^B + \tfrac{9}{10}\delta_2^B.$$

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