Optimization Models for Joint Airline Pricing and Seat Inventory Control: Multiple Products, Multiple Periods

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Abstract

Pricing and revenue management are two essential levers to optimize the sales of an airline's seat inventory and maximize revenues. Over the past few decades, they have generated a great deal of research but have typically been studied and optimized separately. On the one hand, the pricing process focused on demand segmentation and optimal fares, regardless of any capacity constraints. On the other hand, researchers in revenue management developed algorithms to set booking limits by fare product, given a set of fares and capacity constraints.

This thesis develops several approaches to solve for the optimal fares and booking limits jointly and simultaneously. The underlying demand volume in an airline market is modeled as a function of the fares. We propose an initial approach to the two-product, two-period revenue optimization problem by first assuming that the demand is deterministic. We show that the booking limit on sales of the lower-priced product is unnecessary in this case, allowing us to simplify the optimization problem.

We then develop a stochastic optimization model and analyze the combined impacts of fares and booking limits on the total number of accepted bookings when the underlying demand is uncertain. We demonstrate that this joint optimization approach can provide a 3-4% increase in revenues from a traditional pricing and revenue management practices.

The stochastic model is then extended to the joint pricing and seat inventory control optimization problem for booking horizons involving more than two booking periods, as is the case in reality. A generalized methodology for optimization is presented, and we show that the complexity of the joint optimization problem increases substantially with the number of booking periods. We thus develop three heuristics. Simulations for a three-period problem show that all heuristics outperform the deterministic optimization model. In addition, two of the heuristics can provide revenues close to those obtained with the stochastic model.

This thesis provides a basis for the integration of pricing and revenue management. The combined effects of fares and booking limits on the number of accepted bookings, and thus on the revenues, are explicitly taken into account in our joint optimization models. We showed that the proposed approaches can further enhance revenues.

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Chapter 1

Introduction

Pricing and revenue management are essential for an airline to maximize its revenues. The goal of pricing is to improve revenues by designing an appropriate set of fare products. A fare product is a combination of a price, also called a fare, and an assortment of travel constraints and service amenities. Each fare product is designed for and targeted toward a specific group of travellers. The probably most well-known two groups are the business travellers on one side and the leisure travellers on the other. By offering various fare products, the airline intends to force some passengers with travel constraints and a high willingness-to-pay to buy high revenue products, without deterring all other passengers from booking seats.

Revenue management, on the other hand, improves the airline's revenues by setting limits on the maximum number of seats to be sold for each fare product, especially the low revenue ones. The airline can only offer a fixed, predetermined, number of seats. Some booking requests will generally have to be rejected due to a lack of capacity, and it is therefore important for the airline to ensure that as few high-revenue requests are rejected as possible. To do so, limits on the number of low-revenue products that should be sold are set. This ensures that a minimum number of seats is saved for the high willingness-to-pay passengers arriving late in the booking process.

Over the past few decades, pricing and revenue management have generated a great deal of research but have typically been studied and optimized separately. Nevertheless, the two processes are interrelated and should ideally be considered and solved for as a single optimization problem. Both pricing and revenue management are based on an analysis of the booking patterns. Bookings are an indicator of the underlying demand characteristics, such as demand elasticity, and are used for market segmentation for example. On the other hand, the analysis of the demand volume per fare product or group of fare products is the basis for setting booking limits. In turn, both pricing and revenue management processes affect the choice set available to a potential passenger during the booking procedure. By setting fares and travel constraints, pricing defines the global set of options that could be available to a passenger. However, booking limits may render one or more of these options unavailable at the time of booking and therefore restrict the actual choice set of a passenger.

We could gain much insight on revenue maximization from a better understanding of the interactions between pricing and revenue management methods. The analysis of the combined impacts of fares and booking limits on a passenger's choice set, and thus on the final number of accepted bookings, would enable us to further improve the optimization process. This thesis investigates ways of analysing these two components' combined effects on the demand, with the objective of ultimately solving for the optimal fares and booking limits jointly and simultaneously. In our model, the underlying demand volume is modelled as a function of fares and we focus on determining how booking limits define the percentage of underlying demand that will finally be accepted. Based on this insight, we formulate a revenue maximization problem in which fares and booking limits are the decision variables.

This dissertation is divided into six chapters. We shall first detail the current pricing and seat allocation practices and describe our research objectives. In Chapter 2, we present the literature review and explain how this dissertation complements the existing body of work in joint pricing and seat allocation optimization. A two-product, two-period joint pricing and seat allocation problem is then formulated in

Chapter 3. We first propose a deterministic model and then introduce a stochastic approach. Finally, in Chapter 5, we suggest heuristics to tackle the multiple-time frame optimization problem. Chapter 6 summarizes the findings and presents possible directions for future work.

1.1 Background

Pricing and revenue management are part of airline planning, a lengthy and complex process. Nowadays, airline planning is usually broken down into five smaller and interdependent planning problems that are solved separately one after the other, as shown in Figure 1-1. Fleet planning, route evaluation and schedule development are the first steps. They focus on developing the airline's operations. These three steps determine the air transportation service to be offered by the airline and establish the capacity constraints. The inherent characteristics of the transportation service, e.g. the origin and destination airports, the departure time, the total travel time, the number of connections or the aircraft types, are all determined by the first three steps. The goal of the last two steps, pricing and revenue management, is to then generate and maximize revenues from the sale of the airline's seat inventory.

Pricing consists of the design of the different fare products that will be available to each origin-destination market served by the airline. Air travel demand is defined for an origin-destination (OD) market, consisting of a passenger's original departure city and his final trip destination. Several itineraries, with different connecting flights, may be possible for one OD market. During the pricing process, the total demand for each origin-destination market is analysed and segmented. A fare product is then designed for each identified segment. A fare product is associated with a combination of travel restrictions and service amenities, and a price. The product's fare is set according to the estimated willingness-to-pay of the segment's passengers. Different restrictions and services amenities are then used as fences to deter passengers from buying fare products with a lower fare than their estimated willingness-to-pay.

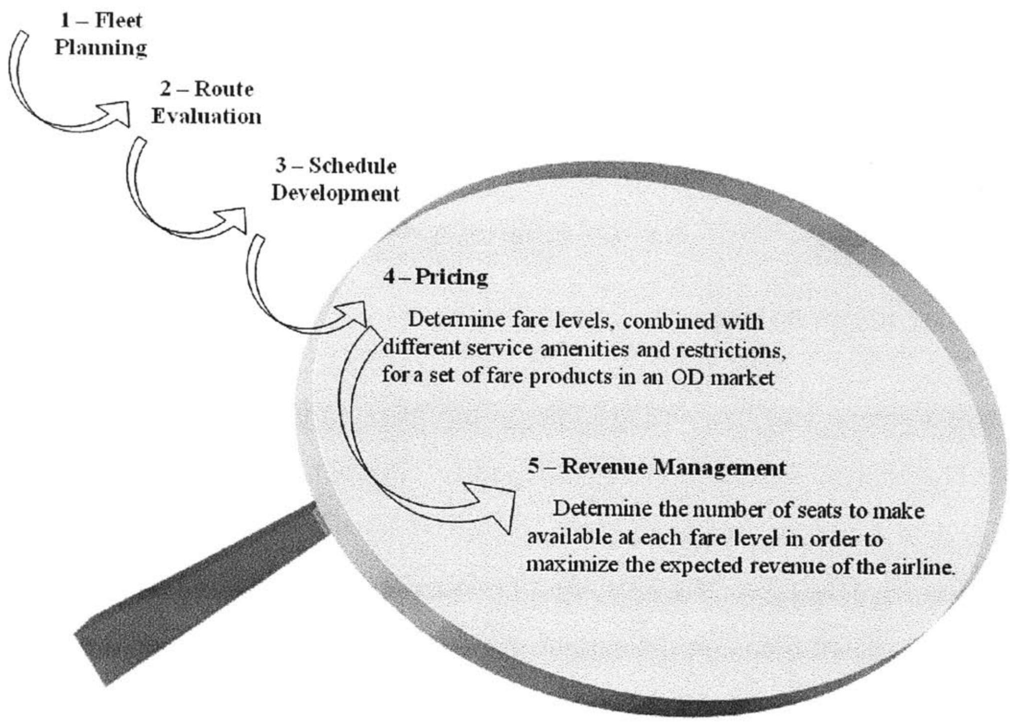


Figure 1-1: Airline planning, a sequential approach

Product differentiation or differential pricing enables the airline to extract additional consumer surplus from passengers. An example of a simple set of fare products, a fare structure, is shown in Table 1.1.

	Restrictions		
Fare Product	Saturday-night stay required	Purchase 21-day before departure	Fare ($)
1	No	No	700
2	No	Yes	450
3	Yes	Yes	300

Table 1.1: Example of a fare structure

Once the complete set of fare products is defined, the revenue management process determines the maximum number of bookings to accept for each fare product in

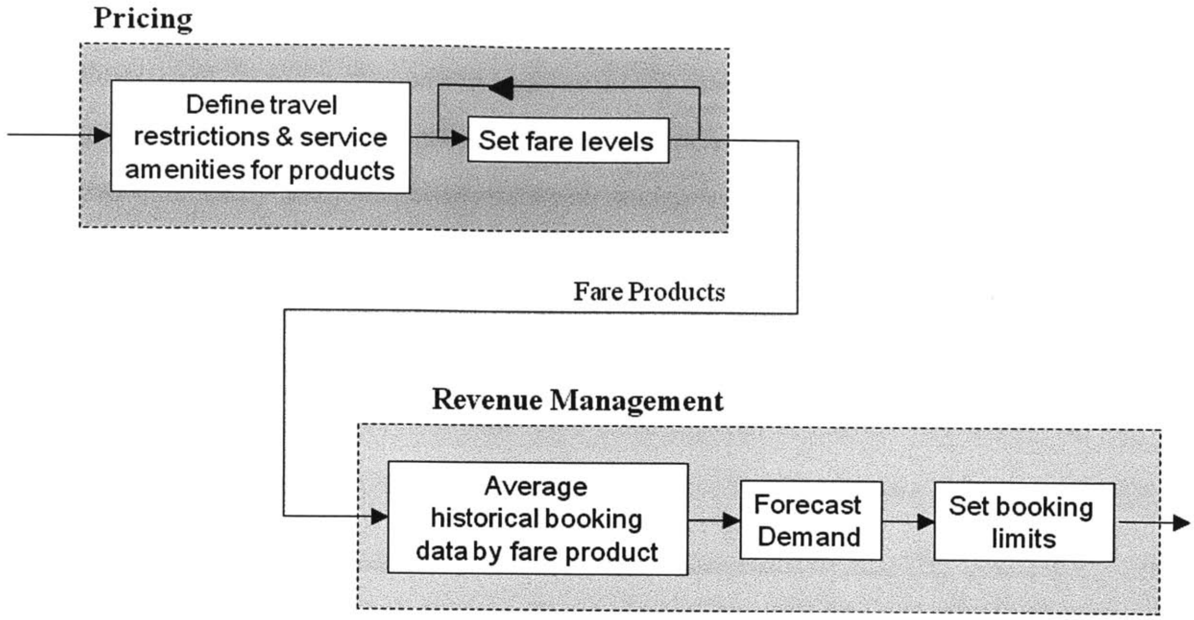


Figure 1-2: Overview of the current approach to pricing and revenue management

order to maximize the airline's expected revenues, as shown in Figure 1-2. Revenue management systems can forecast the demand for each fare product and compute the different booking limits. Several optimization techniques have been developed to calculate the appropriate levels of protection.

One aspect of revenue management is overbooking. It was first developed in the 1960's and enabled airlines to account for the no-show behaviour of a proportion of passengers. With overbooking, more bookings than can physically be accommodated are accepted, in an attempt the reduce the impacts of the very few passengers that may decide not to travel at the very last minute. Another aspect of revenue management is seat allocation. If too many seats are sold at a discounted fare and the plane fills up, passengers with the willingness to pay the traditional full fare can be denied their booking request, which represents a revenue loss for the airline. The need to protect seats for later booking but higher revenue passengers triggered research on seat allocation. The latest research developments in the field concern demand forecasting and seat allocation, the main focus of this study. An example of a set of booking limits is shown in Table 1.2. In this example, as soon as the 30^{th} Fare

Product 3 booking is accepted, the product becomes unavailable. Similarly, only 40 bookings can be accepted for Fare Product 2. On the other hand, Fare Product 1 remains available throughout the entire selling period, unless the plane fills up.

	Restrictions			Nested
	Saturday-night	Purchase 21-day		Booking
Fare Product	stay required	before departure	Fare ($)	Limits
1	No	No	700	100
2	No	Yes	450	60
3	Yes	Yes	300	20

Table 1.2: Example of booking limits for a fare structure (capacity = 100 seats)

Pricing and revenue management have traditionally been considered separately, both from a theoretical and practical point of view. These two fields have indeed generated much interest from the research community over the years but few researchers have considered the two fields jointly. At the same time, the pricing department and the revenue management department of an airline are usually two distinct departments. Although many airlines have tried to integrate these two departments, most still have two distinct systems in place.

1.2 The Evolution of Pricing and Seat Allocation Practices

Before the deregulation of the industry in the United States in 1978, airlines were not able to freely set their own prices. The US Civil Aeronautics Board (CAB) was the entity setting the fares of the interstate airlines. To do so, the CAB used a simple cost-based approach: prices were computed in order to cover the operating cost of the airlines and ensure a profit margin. Fares for the different origin-destination markets were based on distance. Only one fare product was offered for a given flight in the economy class.

In 1977, shortly before deregulation, American Airlines was allowed by the CAB to introduce an additional fare product, the Super Saver Fare. The new fare product was first tested on flights between New York and both San Francisco and Los Angeles, but it soon became available on most domestic routes. This product was substantially cheaper than the normal economy fare but came with a 30-day advance purchase requirement and a minimum stay of at least seven days (Bailey et al., 1985). The average load factor of the industry in the 1970's was low, around 50% (Ben-Yosef, 2005), and legacy carriers were facing a competitive threat from charters. The expectation was that cheaper fares would increase the load factor and enable the airline to compete with lower-priced alternatives.

However, cheaper fares could also affect the revenues adversely. Indeed, discounting has two effects: stimulation and diversion (Krajewski and Ritzman, 1990). Lower prices will stimulate a previously unmet demand from price-sensitive customers. However, lower prices are also attractive to the non-price-sensitive customers who would have flown with the airline at the non-discounted fare anyway. In order words, with the introduction of cheaper fares, passengers can book seats at a fare level lower than the one they would have been prepared to pay. They are thus diverted from the original product, creating a loss in revenue (McGill and Ryzin, 1999).

The advance purchase restriction and minimum stay requirement were a first attempt at lessening the impact of diversion. The restrictions created two market segments. People who can book their flight tickets more than a month in advance and stay at their destination for more than a week are likely to be leisure travellers with a lower willingness-to-pay than last-minute passengers. With the restrictions enforced by American Airlines, late booking passengers had no choice but to buy the full fare. The new fare product enabled the airline to sell seats that would have otherwise remained empty while still retaining the high-revenue passengers who could not fulfil all the restrictions.

Nevertheless, this pricing strategy of implementing restrictions cannot, alone, ensure an increase in revenue. The demand for the discounted fare could potentially fill up a flight long before departure, leaving no seats to late booking higher-fare passengers. Some seats should therefore be protected for full fare passengers who book later and generate more revenues. There were no simple rules to determine the right number of seats to save for full-fare passengers. The introduction of this second fare product marked the beginning of the development of seat allocation models.

Yet, despite the fact that the effectiveness of a fare structure depends partly on the ability to set good booking limits, the progress made in the revenue management field was never the driver for change and innovation in the airlines' pricing strategies.

In 1985, American Airlines announced the introduction of a new fare product, the "Ultimate Supersaver Fare". The ultimate supersaver fare offered up to a 74% discount from the regular coach fare, an unprecedented discount. The new fares were subject to a new restriction, on top of the then usual 30-day advance purchase requirement and a Saturday-night minimum stay. 50% of the ultimate supersaver fare was non-refundable. This latter restriction enabled the legacy carrier to further segment the domestic market. American Airlines was facing the rapid and successful growth of the low cost carrier People Express. The introduction of the ultimate supersaver fare was the traditional airline's response to the threat of the low cost competitor (Ben-Yosef, 2005).

In the following years, the complexity of the fare structures put in place by the airlines kept increasing. This trend was hastened by the development of computer reservation systems, enhancing airlines' ability to record and analyse booking data. Different types of restrictions were implemented as fences to prevent high-revenue passengers from buying lower fares. Advance purchase requirements were combined with minimum stay restrictions, cancellation policies and non-refundable conditions. The pricing strategy of the legacy carriers grew more and more complex in an attempt to refine the demand segmentation.

This trend was interrupted by the rapid growth of a new generation of low-cost carriers. These new entrants challenged the pricing practices imposed by the legacy carriers over the years. While the low-cost carriers still offered a large number of fares, they significantly reduced their price range. In addition, they entered the market with simplified fare structures. While most tickets on low-cost carriers were non-refundable and had rescheduling fees, they did not require a Saturday-night stay. In fact, most of the low-cost airlines removed the minimum stay requirement altogether, effectively offering one-way tickets only. The new entrants chose to base their customer segmentation strategy mainly on the advance purchase requirement, an easy to understand rule for the travellers.

Faced with the rapid and threatening expansion of the low-cost carriers, some legacy airlines reviewed their strategy. In 2005, Delta Air Lines announced the removal of the minimum stay requirement on all its US domestic markets altogether. The legacy carrier was struggling and the removal of the restriction was an attempt at recapturing passengers (Shaw, 2007).

A new trend was initiated in 2007 with the emergence of the fare structure now called "Fare Families". A few airlines chose to reduce the number of fare products they offered altogether. A handful of products with clearly differentiated restrictions and service amenities were identified and then branded. The branded products are the same across the markets. However, their prices can vary over time and by markets.

Over the years, the airlines have applied marketing principles and segmented the demand as much as possible by combining various types of restrictions and amenities. The major transitions were always the responses to a competitive threat.

Revenue management was most likely another factor of success, even if progress in the seat allocation field was never the driver for pricing changes. Pricing evolved irrespective of, and usually more quickly than, revenue management. Similarly, revenue management was unaffected by the changes in the fare structures offered by the airlines. As airlines implemented fare families, for example, very few new revenue management models were developed.

The objective of revenue management is to maximize the revenues of an airline by making some seats available at a lower fare to passengers who would not have travelled otherwise, while protecting seats for passengers with a higher willingness-to-pay but a later time arrival in the booking system. Revenue management is required only if there are at least two fare products competing for the same resource, a finite number of seats.

From the outset, technology did not permit to assess each request separately, one at a time. Durham (1995) reported that a computer reservations system may have to handle more than five thousand booking requests per second at certain peak times. The individual real time assessment of each request was infeasible and revenue management focused on recommending booking limits by fare product for periods of a few hours to a few days. The status of each fare product was updated on a periodic basis.

The initial revenue management systems focused on optimizing the revenues for each flight leg independently. The systems computed the booking limits for each fare product on a leg. They were developed for the first type of fare structures of the airline industry: a limited number of fare products with clearly differentiated restrictions. In these revenue management systems, the demand for each fare product was modelled by a Gaussian distributed random variable and the demands were assumed to be independent. The possible diversion of a passenger from the full fare product to a lower fare product was not considered. The assumptions were intended for mathematical and programming convenience, but nonetheless led to impressive revenue gains for the airlines.

The success of airline revenue management was widely reported. American Airlines estimated the benefit of such techniques at $1.4 billion over a period of three years (Smith et al., 1992), further stimulating the development and implementation of revenue management methods. More advanced systems were designed to address the broader issue of network revenue optimization.

As airlines developed hubs and expanded their network, the number of connecting passengers increased tremendously. The origin-destination fare product sold to a connecting passenger depends on the availability of seats on each of the connecting flights. This creates an interdependence among flights. Considering the capacity of different flight legs independently to optimize the airline's total revenue becomes suboptimal. Flight leg heuristics were extended to origin-destination control mechanisms to account for some network effects.

Later on, researchers suggested modifications to take into account the possible diversion of passengers to a lower fare product when determining the booking limits. However, despite the tremendous evolution of the pricing strategies over the years, the research on seat allocation optimization models remained heavily based on the initial fully restricted fare structures.

1.3 Research Motivation and Objectives

The need for seat allocation controls materialized as soon as the airlines started offering more than one fare product to their passengers and the possibility that the demand may exceed capacity arose. As the airlines kept widening their array of fare products, the problem of setting booking limits on each fare product became a real challenge and drew the attention of the research community.

The first revenue management approaches suggested by researchers were designed for the contemporary restricted fare structure. This fare structure relied on many different types of travel restrictions, such as the advance purchase requirement, the minimum length of stay restriction or the Saturday night stay requirement, combined with different fare levels. The demands for each fare product was assumed to be independent.

When the minimum stay requirement was temporarily removed in 2005 by Delta Air Lines, the inherent characteristics of the fare structure changed. The choice of fare products for the potential passengers was not restricted by the length of stay anymore and this change affected the consumers' booking patterns. Most passengers who used not to be able to meet the length of stay requirement and thus used to have to buy less restricted more expensive fare products were suddenly able to book much lower fare products. A large part of the demand for the higher fare products was diverted to the lower ones, dramatically changing the booking patterns which are the basis for the revenue management analysis. The revenue management process incorporated the updated average fares and some adjustments were implemented to take into account the possible passenger diversion between fare products. Nevertheless, the underlying models remained mainly unchanged. Similarly, as airlines implemented fare families, very few new revenue management models were developed.

The following question is then raised: can the pricing and revenue management processes be addressed jointly to further maximize revenues? While fares do constitute an important input for seat allocation optimizers, they are determined during the pricing process. Fare levels are frequently updated, but are changed irrespective of the work done downstream to set the booking limits. On the other hand, the fares are used as fixed inputs by the seat allocation systems. The booking limit for each fare product is determined and implemented assuming fixed, averaged, fares. Introducing fares as an additional decision variable in revenue management and solving jointly for fares and booking limits could further increase revenues. The goal of this research is to develop a method to jointly optimize pricing and seat allocation.

1.4 Thesis Outline

The remainder of this thesis is organised as follows: Chapter 2 presents a literature review of pricing and revenue management methods. We shall first start with the description of the evolution of pricing techniques. We will then turn to the developments in the revenue management field, by focusing first on forecasting methodologies and then seat allocation mechanisms. The third part of the chapter is dedicated to the recent work on joint pricing and seat allocation optimization.

In Chapter 3, we formulate the two-product, two-period joint pricing and seat allocation problem. We propose a first approach to solving this problem by first assuming that the demand is deterministic. The deterministic model is discussed and its performance analysed through a numerical example.

Chapter 4 presents a stochastic approach to the joint optimization problem. The demand is assumed to be a uniformly distributed random variable and a geometrical analogy is used to express the objective revenue function. The stochastic approach simultaneously computes the fares and the booking limit that maximize the total revenues. Simulations show that the proposed approach performs well when compared to a traditional revenue management approach.

We then extend the approach to the multiple-time frame joint pricing and seat allocation optimization problem. A generalized methodology is presented in Chapter 5. In light of the complexity of the objective function arising from the consideration of multiple time frames, we derive and test three heuristics to the joint optimization problem.

Chapter 6 summarizes the findings and draws conclusions from them. We will also discuss practical applications and possible future research directions.

Chapter 2

Literature Review

A considerable body of work exists on airline pricing and seat allocation optimization. Whitin published a single period pricing model in 1955 and Littlewood first considered the two-class, single leg, seat allocation problem in 1972. Nevertheless, it is only in the mid nineties that the problem of joint pricing and seat allocation for perishable goods raised the attention of researchers.

In this chapter, we first review the pricing literature. An overview of the revenue management research on forecasting and seat allocation is then provided. Finally, we discuss the recent developments in joint pricing and seat allocation.

2.1 Pricing

Research on airline pricing has been undertaken from many different perspectives. Economists, marketing scientists and operations researchers have all studied the subject.

The existing body of work from an economic point of view is very large. Schmalensee (1981) studied the impact of an airline's pricing strategy on social welfare. Borenstein (1985) used spatial models to analyse price discrimination in oligopolistic markets. Borenstein and Rose (1991) suggested price dispersion as an indicator of price discrimination in the US airline industry.

Studies of pricing strategies in the context of revenue optimization stemmed from research on production-pricing problems. The airline pricing problem was first considered as a special case of the classical newsvendor problem where the production cost was fixed and the product was perishable with no penalty cost or salvage value. Whitin (1955),Mills (1959, 1962), Karlin and Carr (1962), Zabel (1970) and Hempenius (1970) worked on single-period newsvendor models that included both the price and the inventory level as a decision variable. Their studies only concerned the single-product case. Petruzzi and Dada (1999) presented an overview of the research on this field.

Karlin and Carr (1962), Nevins (1966), Zabel (1972), Thomas (1974), Thowsen (1975), Petruzzi and Dada (1999), Federgruen and Heching (1999) extended the scope of the research to a multi-period stocking and pricing problem. The planning period was segmented and the unsold products at the end of a time period were available to meet the demand of the subsequent time periods. Each time period was associated with a pricing and stocking decision. The studies remained focus on the single-product case.

Gallego and van Ryzin (1994) and Zhao and Zheng (2000) addressed the problem of dynamically pricing a given inventory of a single product. The demand was price sensitive and stochastic and the objective was to maximize the expected revenues. Optimal prices were functions of the inventory level and the length of the planning period. Chatwin (2000) and Feng and Gallego (2000) restricted the number of allowable prices to a finite set. Gallego and van Ryzin (1997), Paschalidis and Tsitsiklis (2000) suggested to extend the dynamic pricing models to the multi-product case. The various products shared the same supply of resources. The two groups of researchers proposed heuristics based on deterministic models.

2.2 Revenue Management

Revenue management commonly encompasses overbooking, forecasting and seat allocation. The focus of this dissertation is forecasting and seat allocation.

2.2.1 Forecasting

Work on airline forecasting for revenue management dates back to the 1960's. The first efforts focused on modelling the demand distribution and the passenger arrival processes. Beckmann and Bobkoski (1958), Beckmann (1958), Lyle, Belobaba (1987a) discussed the reasonable fit of the Poisson, negative binomial, gamma or normal probability distributions.

Later on, researchers proposed remedial measures to the truncation of the observed historical demand due to booking limits and capacity constraints in the airline industry. The distinction between aggregate and disaggregate forecasting emerged. Taneja (1978) and Sa (1987) provided an overview of aggregate forecasting techniques. Lee (1990) discussed issues in disaggregate forecasting. Later work by Weatherford et al. (1993) took into account the possible diversion of a passenger in the two-product case. Some passengers might indeed be willing to pay the higher fare product but will always buy the lower available fare product. There was diversion when those passengers ended up buying the higher fare product because the other one was not available.

2.2.2 Seat Allocation

Littlewood (1972) first studied the seat allocation problem for the two-class, single flight leg case and proposed a simple rule to accept or reject bookings. A discount fare booking should only be accepted if the revenue it generates exceeds the expected revenue of a future request for the full fare. Belobaba (1987b, 1989, 2002) extended the principle to the multiple-fare products problem with the expected marginal seat revenue heuristic (ESMRa and EMSRb). This method did not yield the optimal booking limits, except in the two-class problem, but provided reasonable approximations in typical situations. However, the heuristic could realistically be implemented and was therefore widely adopted by airlines. Other approaches to derive the optimal booking limits for single-leg flights were developed by Curry (1990), Wollmer (1992), Brumelle and McGill (1993) and Robinson (1995). These methods were referred to as leg-based approaches in the literature. They aimed at optimizing the expected revenue from seat allocation for each flight leg independently.

In the 1980's, airlines rapidly expanded their use of the hub-and-spoke network model and it soon supplanted the point-to-point service model. The number of connecting passengers increased significantly, creating network effects: each flight leg carried a mix of local and connecting passengers. The yield of a connecting passenger might on a particular flight leg be lower than that of the passenger sitting next to him and yet the total itinerary contribution to the airline might be greater. On the other hand, a connecting passenger may displace two local passengers who could have generated more total revenue for the airline. The conventional leg-based approach to revenue management failed to address these two fundamental network issues.

Researchers worked on developing origin-destination control mechanisms for networks. The first approaches were based on conventional leg-based methods. Instead of setting booking limits for fare products, controls were set for "value classes". All the itineraries that comprised the same particular flight leg were ranked, and the fare products of these itineraries were then mapped to buckets according to their value to the network. Each bucket was a value class. The leg-based heuristics were then used to compute the booking limits of each value class. American Airlines called the clustering process "virtual nesting". Belobaba (1989), Smith and Penn (1988) and Williamson (1992) presented different mapping techniques. Virtual nesting addressed the first network issue by taking into account the possibility that a low yield connecting passenger on a flight leg may in fact be generating more revenue overall than a local passenger.

More advanced network optimization techniques were then developed to tackle the second network issue mentioned previously, which is the displacement of two local passengers by a connecting passenger. Glover et al. (1982) and Curry (1990) provided mathematical programming formulations for network flow models. These results were used to estimate the displacement cost of an itinerary. Smith and Penn (1988), Simpson (1989) and Williamson (1992) developed bid-price methods in which a booking is accepted only if its total net contribution is greater than the estimated displacement costs, also called the bid-price.

2.3 Joint Pricing and Seat Allocation

To our knowledge, the literature on joint pricing and seat allocation for perishable assets is very sparse.

Weatherford (1997) emphasized the importance of considering prices as part of the overall optimization problem and suggests including them as decision variables in the seat allocation problem. The study focused on the case of a single flight leg with at least two fare products. The demand for each product was assumed to be normally distributed, with a mean a linear function of the product's own fare as well as the next higher and lower fares. The selling period was not divided into subintervals which greatly reduced the complexity of the problem. For example, there were only three decision variables in the case of the two fare product problem: two prices and one booking limit.

The analysis was divided into three parts. First, Weatherford assumed that the inventory was partitioned and that there were no diversion: passengers with the willingness-to-pay the higher fare do not consider the lower fare product. These two assumptions are released, one after the other, in the two subsequent parts of the analysis. This was the only case in which the closed form expression of the expected revenue could be derived. In the last part, the more realistic behavior of a high willingness-to-pay passenger buying a lower fare product if it is available was taken into account. However, it was assumed that the demand for the different products follows a strict arrival order, with the lowest fare product demand arriving first. Numerical methods were used to find the optimal solutions and the analysis was restricted to two and three fare products, due to the increased complexity of the problem.

Kuyumcu and Garcia-Diaz (2000) tackled the joint pricing and seat allocation problem with the objective of broadening the scope of the problem by taking into account the entire airline network in the decision making process. The analysis was not focused on the revenues generated by one single flight leg, but by all the airline's origin-destination (OD) itineraries that constituted the network.

The demands for each OD and fare class combinations were assumed to be mutually independent and normally distributed. There were no explicit hypothesis regarding the relationship between the demands and fares, or any other fare product characteristics. The demands were known, through the analysis of historical data, and the objective function was the system-wide expected marginal seat revenue. A graph theoretical approach was proposed to determine the subgroup of fare products that would maximize the revenues, given an initial set of fare products. The seat allocation technique used was the standard leg-based revenue management method EMSR, even if the focus was the network.

The fares remained exogenous variables to the decision process in this study. The approach simplified the optimization problem by reducing the number of fare products available, and then relied on standard EMSR to set booking limits for each OD itinerary.

Bertsimas and de Boer (2002) also analyzed the joint pricing and seat allocation optimization problem in a network. In this study too, it was assumed that the demand for each fare product was uncertain and that its expectation only depended on its own price. In the first part of the paper, the selling time period was not divided into subintervals. The analysis was first reduced to the case of a single fare product. It was shown that the objective function could be concave with respect to the production level for a certain set of probability distributions, including the uniform and normal distributions. The authors proposed an iterative non-linear optimization algorithm to determine the optimal fares and allocation policies for all fare products.

The second part of the analysis focused on the multi-period optimization problem. Three heuristics were presented. In the first one, the demand for a product was first aggregated over all the periods considered, which reduced the problem to the single time period problem. In the second heuristic, the inventory was partitioned, which also reduced the problem to a larger single time period problem. In the third heuristic, the allocation policy was set first and the optimization then only concerned pricing.

Cote et al. (2003) proposed a model with the capability of jointly solving the pricing and seat allocation problem in a network with a competitor. The approach was based on bilevel programming: the airline was assumed to know how its competitor would react and integrate this behavior in its decision process. The main objective of the study was to propose an approach to determine the optimal fares across the network, knowing how the competitor would react. The main variables of interest were the fares, not the booking limits. The demand for each fare product and origin-destination itinerary combination was assumed to be fully known: there was no randomness or hypothesis regarding the relationship between the fare products' characteristics. The inventory was partitioned and the model does not allow for diversion. The booking limits are obtained *a posteriori*. Indeed, once the two airlines pricing policy was known, the booking limits were simply set equal to difference between the flight's capacity and the higher fare products' demands.

Chew et al. (2009) developed a joint optimization approach for a single product with a two period lifetime. The demand for the product was assumed to be uncertain, and its expectation was a linear function of its fare. Based on the concave properties of the objective revenue function, the authors suggested an iterative procedure to compute the optimal fare and booking limit. The extension to the multiple time period problem was then considered. As the concavity of the expected revenue function did not always hold, three heuristics were suggested. In the first heuristic, the inventory was no longer nested but partitioned. In the other two heuristics, the time intervals were grouped into two time periods. The two-period procedure was then applied.

Figure 2-1 summarizes the main aspects of joint pricing and seat allocation optimization privileged by each one of the mentioned studies and thus provides a visual help to identify the gaps in the literature. Weatherford (1997) was the only one to consider the demands for different fare products to be mutually dependent and to solve for the optimal fares and booking limits simultaneously. Kuyumcu and Garcia-Diaz (2000), Whitin (1955), and Cote et al. (2003) chose to approach the optimization at the network level instead and the focus of Cote et al. (2003) was mostly the competitive effect. None of these three studies extended the work of Weatherford (1997) by introducing dependent demands. For Chew et al. (2009), the emphasized aspect is the ability to update the fare of a product several times over the booking process.

Figure 2-1 also shows how this dissertation complements the existing body of work on joint pricing and seat allocation optimization. Our intent is not to directly analyze the impacts of fares and booking limits on the demand at a network level or in a competitive environment. Instead, we revert to the base case, the single-flight leg case, but assume that the demands for the fare products considered are dependent. Furthermore, we divide the selling period into sub-intervals to model the fact that airlines can increase their revenues by changing their fares several times over the booking period. Our study combines the joint pricing and seat allocation aspects that were the focus of Weatherford (1997) and Chew et al. (2009). This dissertation could then be the starting point for more comprehensive research encompassing network or competitive effects.

	Numb. of products $m \geq 2$	Demand: Mutually Dependent	Demand: Type	Numb. of time periods $t \geq 2$	Multiple Flight Legs	Competitor	Simultaneous optimization
(Weatherford, 1997)	✓	✓	Linear function of price with cross-elasticities	✗	✗	✗	✓
(Kuyumcu and Garcia-Diaz, 2000)	✓	✗	Normally distributed	✗	✓	✗	✗
(Bertsimas and de Boer, 2002)	✓	✗	Non-negative function of price, non-increasing, twice continuously differentiable	✓	✓	✗	✗
(Cote, Marcotte & Savard, 2003)	✓	✗	Constant	✗	✓	✓	✗
(Chew, Lee and Liu, 2008)	✗	✗	Linear function of price	✓	✗	✗	✗
This research	✓	✓	Function of price & other product's price	✓	✗	✗	✓

Figure 2-1: Gap in the joint pricing and seat allocation literature

Chapter 3

Deterministic Approach to Joint Pricing and Seat Allocation Optimization

In this chapter, we develop a deterministic model of joint pricing and seat allocation optimization. In this approach, the prices of different fare products offered by the airline are no longer considered as exogenous variables to the seat inventory problem. Instead, the demand for fare products is modelled as a function of the fare products' prices, which thus become decision variables, as booking limits are.

This newly formulated optimization problem involves a large number of variables. We initially facilitate an analysis by assuming that the demand is deterministic. This highly simplifying assumption, which is relaxed later, allows us to gain insight into the optimization problem. We will show that without demand uncertainty, the fares can be used as the sole lever to match inventory and demand. The deterministic assumption will therefore allow us to set aside the booking limit in a first time and find a simple solution to the optimization problem. This shall constitute a starting point to observe, thanks to a numerical example, the impact of the flight capacity on the observed demand and pave the way for the following chapter.

We open up the chapter with a description of the problem's scope. The fare products and their characteristics are defined. The underlying selling mechanism used by the airline is explained. The notations used are then decrypted in Section 3.2, which allows us to then introduce the model chosen for the demand functions. In Section 3.4, we give a detailed analysis of the resulting objective function. Finally, we illustrate the approach with three sets of numerical simulations.

3.1 Scope of the Problem

Traditional revenue management techniques focus exclusively on booking limits for each fare class. The objective of these techniques is to improve the total expected revenues by determining the best booking limit for each product. In the formulation of the traditional revenue management optimization problem, the prices of different fare products are known and fixed and are considered as input to the problem. The booking limits are determined given the demand estimated at those price points.

In reality, however, prices do affect the demand and thus the revenues. We believe that the airline's revenues could be further improved by modifying the formulation of the seat inventory optimization problem and including prices as decision variables.

To do so, we place ourselves in a single carrier, single flight, single origin-destination (OD) market environment. The flight has a fixed capacity of C seats. This simple context enables us to reduce the complexity of the analysis by removing competitive or network effects.

In the OD market considered, two fare products, Fare Product *1* and Fare Product *2*, are offered by the airline. The two products provide exactly the same in-flight service; nevertheless, these products are associated with two distinct sets of purchase restrictions and rules, and are therefore priced differently. In our notations, Fare Product *1* represents the more expensive, less restricted product. The other product, Fare Product *2*, is priced lower, but in return, has additional restrictions and rules.

Since the very beginning of revenue management, both researchers and airline managers have acknowledged that passengers booking first exhibit different behaviors from those booking last. They have different trip purposes, different willingness-to-pay and different tolerance regarding fare products' rules. The implementation of restrictions, such as advance purchase requirements, was an attempt at segmenting the demand based on those changing characteristics. A few days or weeks before the flight departure date, the lower discounted fare would no longer be available as most of the passengers arriving later have a willingness-to-pay closer to the higher price.

Our fare structure does not include advance purchase requirements. The two fare products remain available throughout the entire selling period unless the flight sells out. Instead, to take into account the changing characteristics of the passengers, we divide the selling horizon into subintervals which allows the prices to change from one period to the other, while keeping the restrictions unchanged. We start by dividing the booking period into just two time frames. Increasing the number of subintervals would result in additional decision variables and would greatly increase the complexity of the problem. We will explore the multiple-time period optimization problem in Chapter 5.

Bookings start to be accepted at the beginning of the first time frame, *TF1*. The flight departs at the end of the second and last time frame, *TF2*. The prices of the two products can be modified at the start of each new time period. In the notation, x_t and y_t represent, for each TFt, the prices of Fare Product *1* and Fare Product *2*, respectively. These price points are decision variables. The set of restrictions and rules associated with each fare product are set by the airline. These restrictions remain the same throughout the entire booking period: they are constant characteristics of the two fare products.

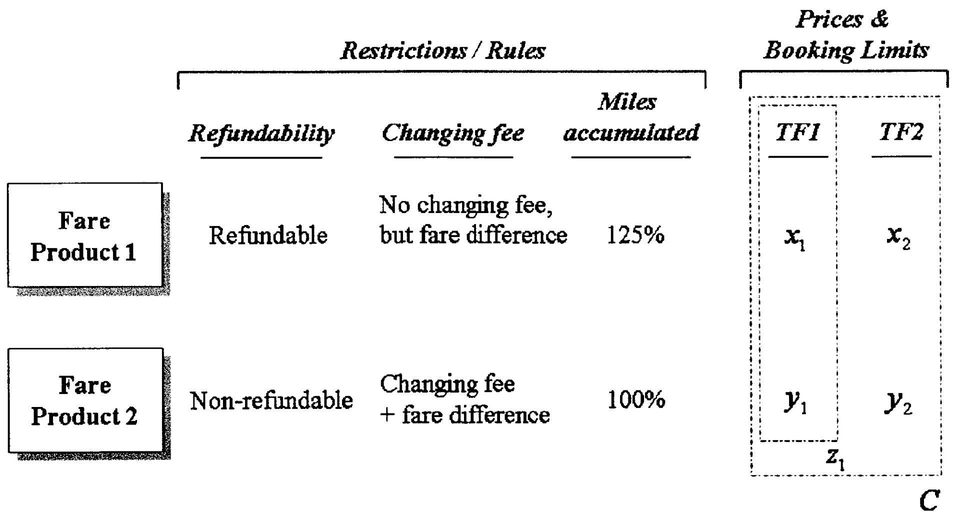


Figure 3-1: Example of the two fare products, their restrictions, fares and booking limit

In addition, the airline can limit the total number of seats to be sold in the first time frame. This should enable the airline to protect a minimum number of seats for $TF2$ passengers, which are usually less price sensitive and therefore more likely to accept to pay a higher fare. We assume that the total capacity is nested between the two time frames: unsold seats from the first time frame are available for booking in the second time frame. The flight capacity caps the total number of bookings that can be accepted over the course of the two time frames. The booking limit imposed by the airline on the two products in the first time frame is noted z_1 and is the fifth decision variable of our optimization problem. The flight capacity C is, on the other hand, fixed.

An example of the two fare products and their possible restrictions is provided in Figure 3-1.

The joint pricing and seat allocation problem modeled here consists of maximizing the total revenues generated by the sale of the two fare products during the two time periods by optimizing the four price points and the first time frame's booking limit.

3.2 Notations

The following notations are used throughout this and the next chapters:

- C is the capacity of the flight.
- x_t is the price of Fare Product *1* in TFt
- y_t is the price of Fare Product *2* in TFt. We impose that for all t, $y_t \leq x_t$.
- z_1 is the booking limit corresponding to $TF1$, i.e. the total number of seats that the airline is willing to sell in the first time frame. The number of seats available for booking limit in the second time frame $TF2$ is at least equal to $C - z_1$.
- $n_{x,t}$ is the demand for Fare Product *1* in TFt.
- $n_{y,t}$ is the demand for Fare Product *2* in TFt.
- $n_{total,t}$ is the combined demand for fare products *1* and *2* in TFt.
- p_t is the probability that a random passenger chooses Fare Product *1* in TFt.
- R_t is the total revenues generated by the combined sale of the two fare products in TFt.
- R_{total} is the total revenues generated by the sale of the two fare products over the entire booking period.

3.3 Modeling the Demand for Two Fare Products

The total demand for the two fare products is approximated by a linear function of the lower available price:

$$n_{total,i}(x_i, y_i) = \alpha_i - \beta_i y_i$$
$$\text{with } \alpha_i, \beta_i \geq 0$$
$$\text{and } y_i \in \left[0, \frac{\alpha_i}{\beta_i}\right]$$

The assumption that the demand is a linear function of the price is very common in the literature (Weatherford, 1997; Chew et al., 2009). The second part of the assumption on the lower available price is a good approximation for large Fare Product 2 demand relative to Fare Product 1 demand, as we will show.

We allow for diversion: if passengers can meet all the restrictions attached to the lower fare product, they will buy it, even if they have the willingness-to-pay for the higher-end fare product, regardless of the product preference. Hence, passengers book the fare product with the lower available price and the most restrictions they can accept. The passengers buying Fare Product *1* are therefore mostly "business" travellers and are relatively price inelastic. The passengers buying Fare Product *2* are, in comparison, more price sensitive.

We further postulate that the demand of both fare products is a linear function of the fare product's price. Examples of demand curves for the two fare products are shown in Figure 3-2.

Let the demand function for Fare Product *2* be:

$$n_{y,i}(y_i) = \alpha_{y,i} - \beta_{y,i} y_i$$
$$\text{with } \alpha_{y,i}, \beta_{y,i} \geq 0$$
$$\text{and } y_i \in \left[0, \frac{\alpha_{y,i}}{\beta_{y,i}}\right]$$

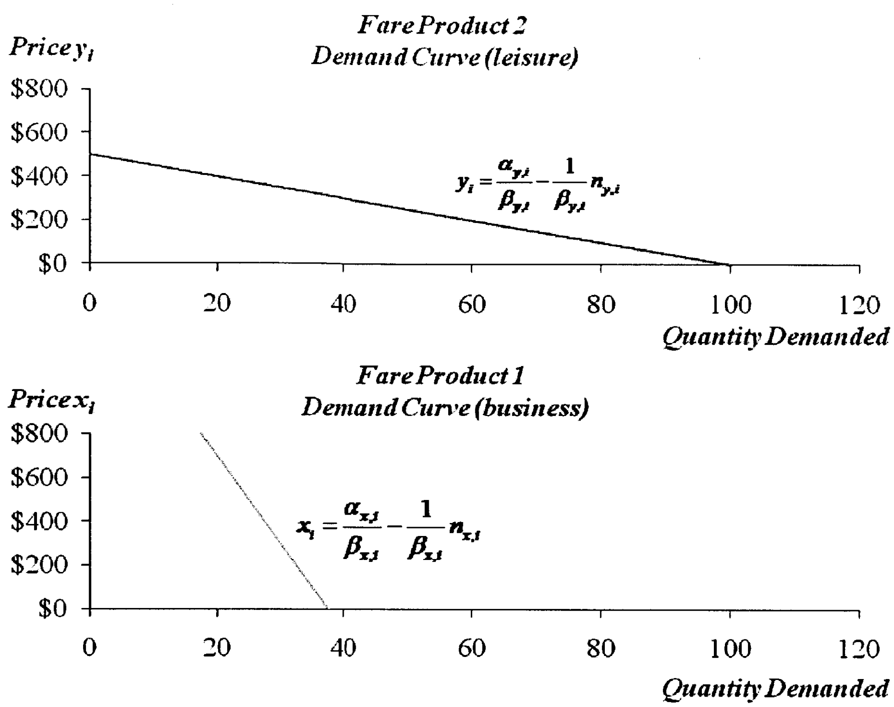


Figure 3-2: Demand curves for the two fare products

The demand function for the higher Fare Product *1* is given by:

$$n_{x,i}(x_i) = \alpha_{x,i} - \beta_{x,i} x_i$$
$$\text{with } \alpha_{x,i}, \beta_{x,i} \geq 0$$
$$\text{and } x_i \in \left[0, \frac{\alpha_{x,i}}{\beta_{x,i}}\right]$$

In addition, $\beta_{x,i} x_i \sim \varepsilon(o)$.

The total, combined, demand for the two fare products thus is

$$n_{total,i}(x_i, y_i) = \alpha_{y,i} + \alpha_{x,i} - \beta_{x,i} x_i - \beta_{y,i} y_i$$

where $\beta_{x,i} x_i \sim \varepsilon(o)$. The total demand function can therefore approximated as:

$$n_{total,i}(x_i, y_i) = \alpha_i - \beta_y y_i$$

The total combined demand is represented by the full blue line in Figure 3-3. The

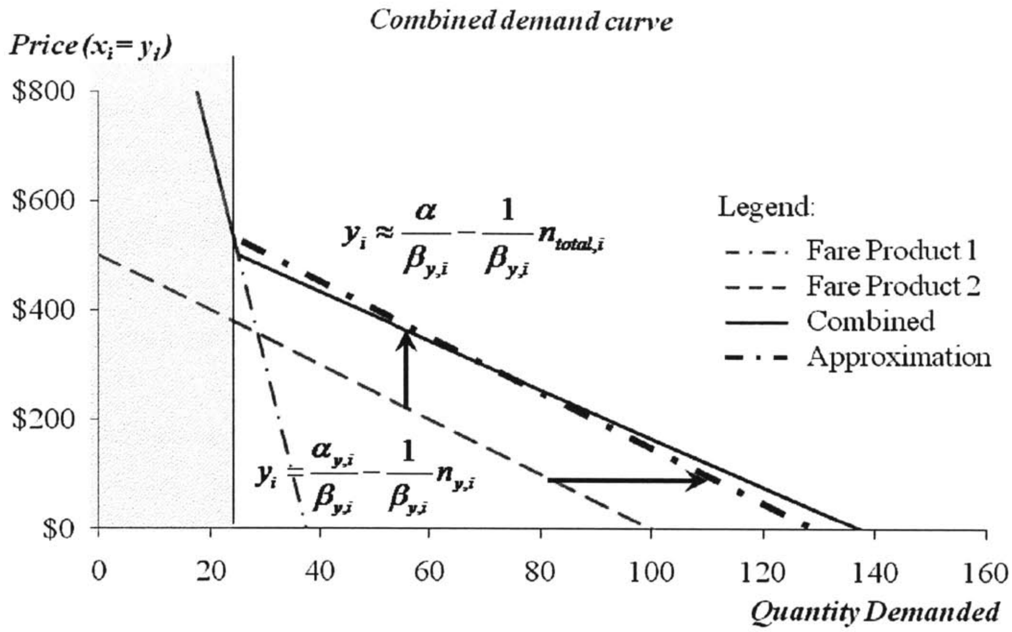


Figure 3-3: Combined demand curve and its approximation

approximation that we use is the blue dotted line.

The higher-end fare product passengers are taken into account through the constant α_i.

For simplifying purposes, most of the previous studies addressing the multiple-product problem assume that the demands for the different products are independent of each other (Kuyumcu and Garcia-Diaz, 2000; Bertsimas and de Boer, 2002; Cote et al., 2003; Chew et al., 2009). In reality, the demand for a product depends not only on the product's own price, but also on the other product's price: there is diversion between the products. A change in the price of one of the fare products affects not only its own demand but also the demands for the other products. It is therefore important in the joint pricing and seat allocation approach to take this dependence into account.

In our model, we allow for sell-up: passengers able to meet all the restrictions of Fare Product *2* may choose to buy the higher fare product with less restrictions. Let p_i be the probability that a passenger chooses the less restricted product: $p_i = \frac{n_{x,i}}{n_{total,i}}$.

We choose to express this choice probability by the binary logit model:

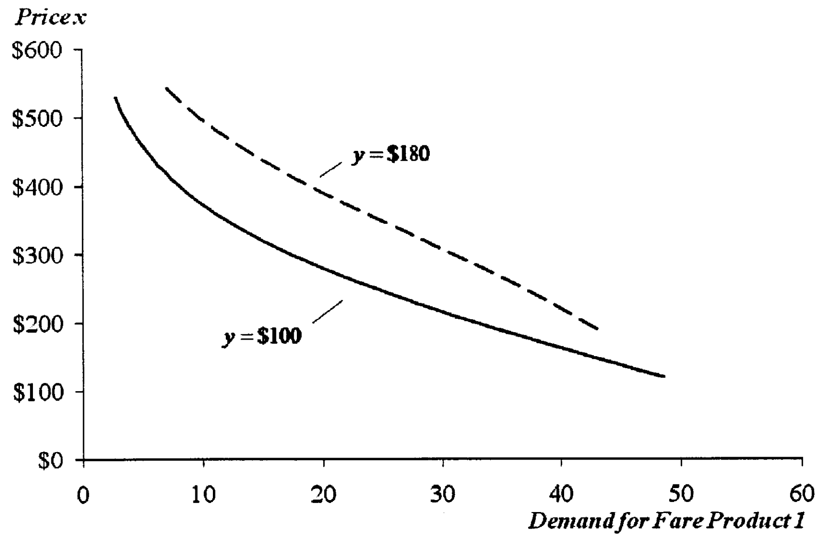
$$p_i\left(x_i, y_i\right) = \frac{1}{1 + e^{a_i - b_i y_i + c_i x_i}}$$
$$\text{with } b_i, c_i \geq 0$$

The parameters a_i, b_i and c_i should be such that for equal and reasonable values of x_i and y_i, the probability is 100%.

The resulting demand for each fare product is a non-linear function of the two prices. Both the lower and higher fares y_i and x_i have impacts on individual demands. Previous works on the joint pricing and seat allocation optimization problem have either assumed that the demand for a product only depended on its own price, or that the demand for a product was a linear function of the other products' prices. Figures 3-4 and 3-5 are examples of the two fare products' demand curves, given $n_{total,i}$ and p_i. There is substitution between the two fare products.

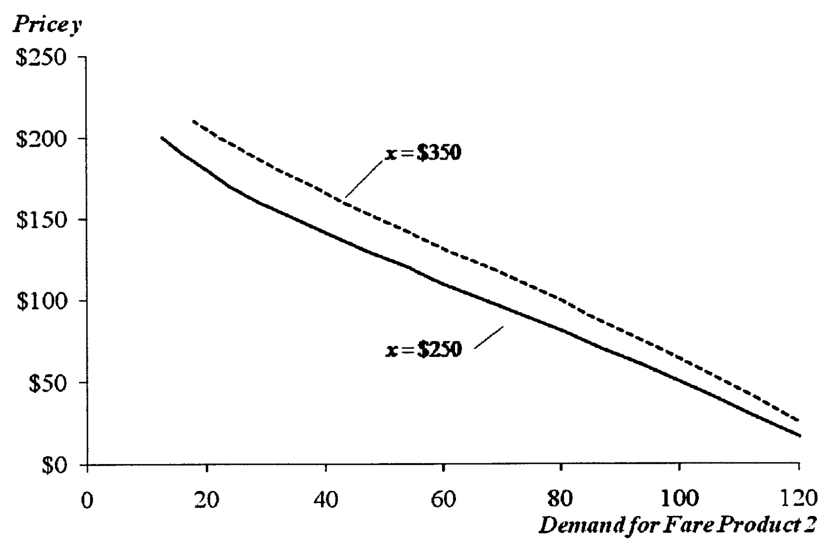
The demands in the different time frames are assumed to be independent of each other. This is a very common assumption in the literature.

In this chapter, we also assume that both the total demand and the probability of a passenger choosing the less restricted product are deterministic. Here, neither equation includes a random variable that would take into account the stochasticity of the demand. This assumption greatly simplifies the joint optimization problem, with the demand for each product entirely defined by the fares. Once the two fares are



For $n_{total} = 135 - 0.44y$ and $p = \frac{1}{1+e^{0.86-0.02y+0.01x}}$

Figure 3-4: Example of demand curve for Fare Product 1



For $n_{total} = 135 - 0.44y$ and $p = \frac{1}{1+e^{0.86-0.02y+0.01x}}$

Figure 3-5: Example of demand curve for Fare Product 2 set, the demands are known, without any uncertainty or variability. This assumption will be relaxed in the next chapter.

The booking limit z_1 and flight capacity C affect the total number of bookings that can be accepted in the two time frames. The bookings accepted in $TF1$ and $TF2$ are $min\left(n_{total,1}, z_1\right)$ and $min\left[n_{total,2}, C - min\left(n_{total,1}, z_1\right)\right]$ respectively.

3.4 Objective Function

The objective function is the total revenue generated by the sale of the two products over the course of the two time frames that constitute the entire selling period:

$$R_{total} = R_1 + R_2$$

$$\text{with } R_1 = \begin{cases} n_{total,1}p_1x_1 + n_{total,1}\left(1-p_1\right)y_1, & \text{if } n_{total,1} < z_1; \\ z_1p_1x_1 + z_1\left(1-p_1\right)y_1, & \text{otherwise.} \end{cases}$$

$$R_2 = \begin{cases} n_{total,2}p_2x_2 + n_{total,2}\left(1-p_2\right)y_2, & \text{if } n_{total,2} < C - min\left(n_{total,1}; z_1\right); \\ \left(C - min\left(n_{total,1}; z_1\right)\right)\left[p_2x_2 + \left(1-p_2\right)y_2\right], & \text{otherwise.} \end{cases}$$

The objective in this section is to demonstrate that the booking limit is unnecessary in the deterministic case.

Time Frame 1

Consider the first time frame, $TF1$. We can show that the third variable z_1 is a redundant variable in the deterministic case. We will also analyse the revenue function.

In this section, we will drop the index 1 to simplify the notation.

The revenue function is given by:

$$R(x,y,z) = \begin{cases} n_{total}px + n_{total}(1-p)\,y, & \text{if } n_{total} < z, \\ zpx + z(1-p)\,y, & \text{otherwise.} \end{cases} \tag{3.1}$$

$$\text{with } \begin{cases} n_{total}(x,y) = \alpha - \beta y \\ p(x,y) = \dfrac{1}{1+e^{a-by+cx}} \end{cases}$$

where $\alpha, \beta, b, c \geq 0$,

and $y \in \left[0, \dfrac{\alpha}{\beta}\right]$

The second part of Equation 3.1 represents the case in which the total demand is greater than the booking limit:

$$R(x,y,z) = zpx + z(1-p)\,y$$

By isolating the variable z in the revenue function, we get:

$$\begin{aligned} R(x,y,z) &= z\,[px + (1-p)\,y] \\ &= z \cdot g(x,y) \end{aligned}$$

where the function g represents the average fare in the time frame considered:

$$g(x,y) = px + (1-p)\,y$$
$$\text{with } p(x,y) = \frac{1}{1+e^{a-by+cx}}$$

The revenue function's maximum is not reached.

$$\forall k > 0,\ p\left(x + \frac{k}{c}, y + \frac{k}{b}\right) = p(x,y)$$

The function g increases in x and y. Thus:

$$R\left(x+\frac{k}{c}, y+\frac{k}{b}, z\right) > R(x, y, z)$$

Thus, for all booking limits z, the maximum of the revenue function is reached on the boundary of the domain of definition, for $n_{total} = z$. In other words, the maximum of the revenue function is reached for $y = \frac{\alpha - z}{\beta}$.

We shall now turn to the other part of Equation 3.1, representing the case in which the total demand is below the booking limit:

$$R(x, y, z) = n_{total} p x + n_{total} (1-p) y$$

This function does not depend on z, but solely on x and y. Figure 3-6 is an illustration of revenues as a function of those two price points. The function is also twice differentiable. It is neither concave nor convex.

Let (x^*, y^*) be the optimal solution to Equation (3.1). Then we can set z to be equal to $\alpha - \beta y^*$.

In both considered cases, the booking limit resulting in the optimal revenues is given by $z = \alpha - \beta y^*$. The booking limit is a redundant variable. Since the demand is deterministic, it can simply be adjusted by changing the fares. We can ensure that the demand does not reach the upper bound that the booking limit represents by increasing the lower fare y_1.

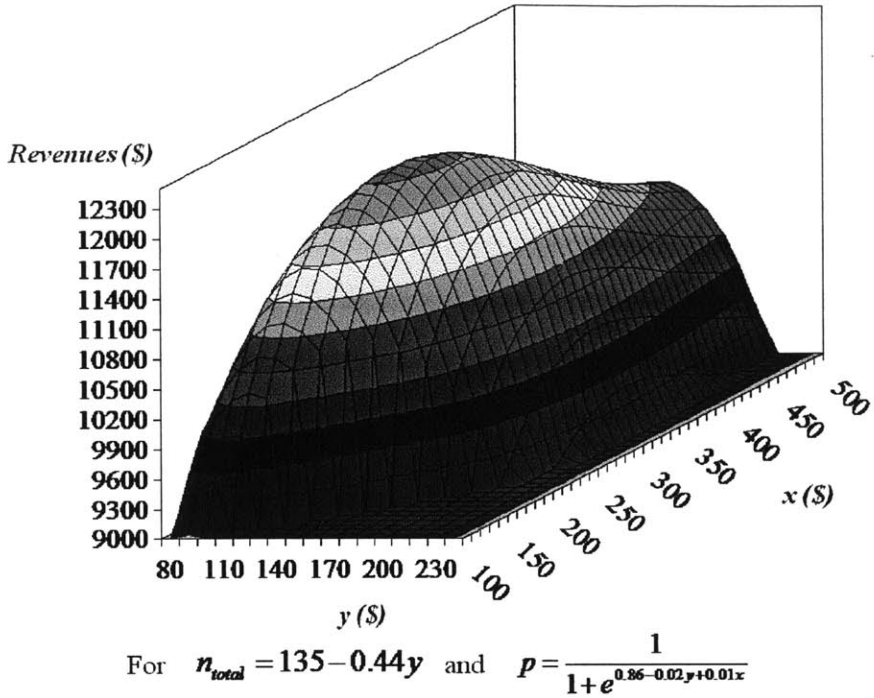


Figure 3-6: Example of the revenue as a function of the two fares, without a booking limit

The equivalent revenue function for $TF1$ is:

$$R(x,y) = n_{total}px + n_{total}(1-p)y$$

$$\text{with } \begin{cases} n_{total}(x,y) = \alpha - \beta y \\ p(x,y) = \dfrac{1}{1+e^{a-by+cx}} \end{cases}$$

where $\alpha, \beta, b, c \geq 0$,

and $y \in \left[0, \dfrac{\alpha}{\beta}\right]$

Total revenues over the two time frames

These results can be generalized and applied to the second time frame. The objective function becomes:

$$R_{total} = R_1 + R_2$$
$$\text{with} \begin{cases} R_1 = n_{total,1} p_1 x_1 + n_{total,1} (1 - p_1) y_1 \\ R_2 = n_{total,2} p_2 x_2 + n_{total,2} (1 - p_2) y_2 \end{cases}$$

The total revenue function is also concave and we therefore have a concave optimization problem:

$$\text{Maximize } R = R_1 (x_1, y_1, z_1) + R_2 (x_2, y_2)$$
$$\text{Subject to } \alpha_1 - \beta_1 y_1 + \alpha_2 - \beta_2 y_2 - C \leq 0$$
$$\text{with } z_1 = \alpha_1 - \beta_1 y_1$$

This is a non-linear constrained maximization problem. It is possible to use the interior point method to find a solution.

3.5 Numerical Results

A numerical example is used to illustrate the proposed approach and gain more insights into the benefits of joint pricing and seat allocation optimization.

The assumed parameters for the demands in the two time frames considered are given in Table 3.1. The flight capacity is 100 seats. The demand level in the second time frame is lower and less price elastic than in the first time frame, as it is usually observed in the airline industry. The probability that a passenger will buy the higher fare product, for a given pair of fares, increases with time, all else being equal.

The optimal deterministic solution to this joint pricing and seat allocation problem

	TF1	*TF2*
Total Demand $n_{total,t}$	$\alpha_1 = 135$ $\beta_1 = 0.44$	$\alpha_2 = 85$ $\beta_2 = 0.20$
Probability p_t	$a_1 = 0.864$ $b_1 = -0.020$ $c_1 = 0.009$	$a_2 = -0.038$ $b_2 = -0.016$ $c_2 = 0.008$

Table 3.1: Parameters for the demand functions

is obtained by running an interior point algorithm in Matlab. The maximum total revenue is reached with the following fares:

	TF1	*TF2*
Fare Product 1	$x_1^* = \$349.1$	$x_2^* = \$462.4$
Fare Product 2	$y_1^* = \$173.3$	$y_2^* = \$223.2$
Implied booking limit	$z_1^* = 60$	

Table 3.2: Optimal solution to the deterministic joint pricing and seat allocation problem

Based on the assumed parameters, we can deduce the deterministic demands and revenues expected at those fare levels, as shown in Table 3.3.

	TF1	*TF2*	*Total*
Average Fare ($)	244	336	
Probability p_t	p_1^* =40.3%	p_2^* =47.0%	
Total Demand	$n_{total,1}^*$ =59.6	$n_{total,2}^*$ =40.4	n_{total}^* =100.0
Revenues	R_1^* = **14,559**	R_2^* = **13,544**	R_{total}^* = **28,103**

Table 3.3: Deterministic demand and revenues

The deterministic model results in a total number of booking requests matching the flight capacity exactly. The optimal lower fares are such that 60 passengers shall requests bookings in the first time frame, and 40 in the second time frame. The total revenue generated by the two fare products is $28.1k.

3.6 Simulations

In the previous sections, the optimization problem was formulated with a demand assumed to be deterministic. We showed that this hypothesis implies that the booking limit is z_1 is a redundant variable. The demand for air travel in reality is known to be stochastic by nature. We therefore test our solution to the optimization problem in a stochastic environment. In this last part of the chapter, we run simulations in which the demand generated is a random variable. The set of simulations shall enable us to assess the importance of a booking limit in a stochastic setting. We shall run a first set of simulations without enforcing any booking limit and then a second set of simulations with a booking limit set to $z_1^* = \alpha_1 - \beta_1 y_1^* = 60$. Lastly, we will compare the proposed joint pricing and seat allocation approach with a more traditional and well established leg-based seat allocation method.

3.6.1 Simulations with no booking limit

In the first set of simulations, we use the fares corresponding to the deterministic optimal solution (see Table 3.2) without enforcing the implied booking limit of $z_1 = 60$ seats. The resulting maximum number of bookings accepted in $TF1$ and $TF2$ are $min\left(n_{total,1}, C\right)$ and $min\left[n_{total,2}, C - min\left(n_{total,1}, C\right)\right]$ respectively.

The total demand for the two fare products is a random variable generated at the beginning of each time period. We will test two possible probability density functions: a uniform and a Gaussian probability density function. Thus, for each considered scenario, two distinct sets of simulations will be run. In the first set of simulations, the demand is uniformly distributed in both time frames. In a second set of simulations, the demand is normally distributed in both time frames. The demands for the two time frames are independent. For both type of demands, we generate 1,000 samples. The mean total demand is a linear function of the lower fare, y_i, and we use the same parameters, displayed in Table 3.1, to model these expected values. The stan-

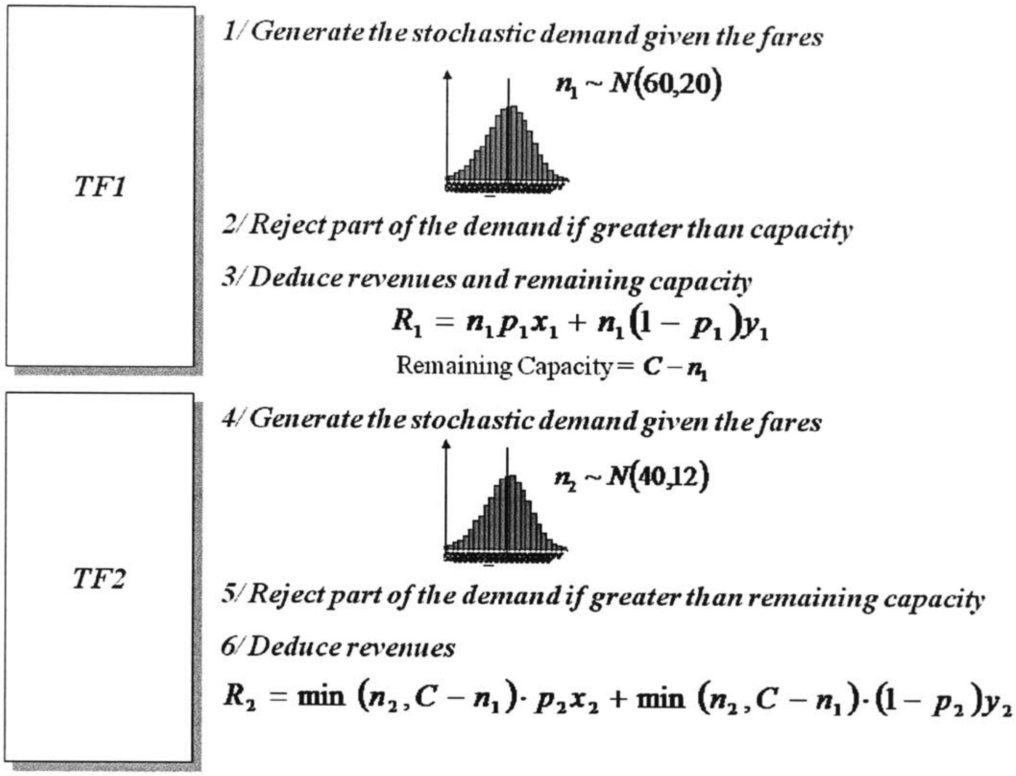


Figure 3-7: Overview of the simulation process

dard deviations of the demands in $TF1$ and $TF2$ are 20 and 12 respectively, which corresponds to about a third of the deterministic demand for each time frame. For the normal distribution, the demand is truncated in order to prevent any instance of negative demand. Finally, the flight capacity does not affect the probability that a passenger chooses the higher fare product.

The simulation process is summarized in Figure 3-7. We first generate demand in $TF1$ and decide how many booking requests can be accepted given the flight capacity. Then we generate the $TF2$ demand, and given the remaining capacity, we determine the number of bookings that can be accepted.

The average demands and revenues resulting from the two different sets of simulations are shown in the two tables, Table 3.4 and Table 3.5. The cumulative distributions of the revenues for the two types of demand, are shown in Figure 3-8. They

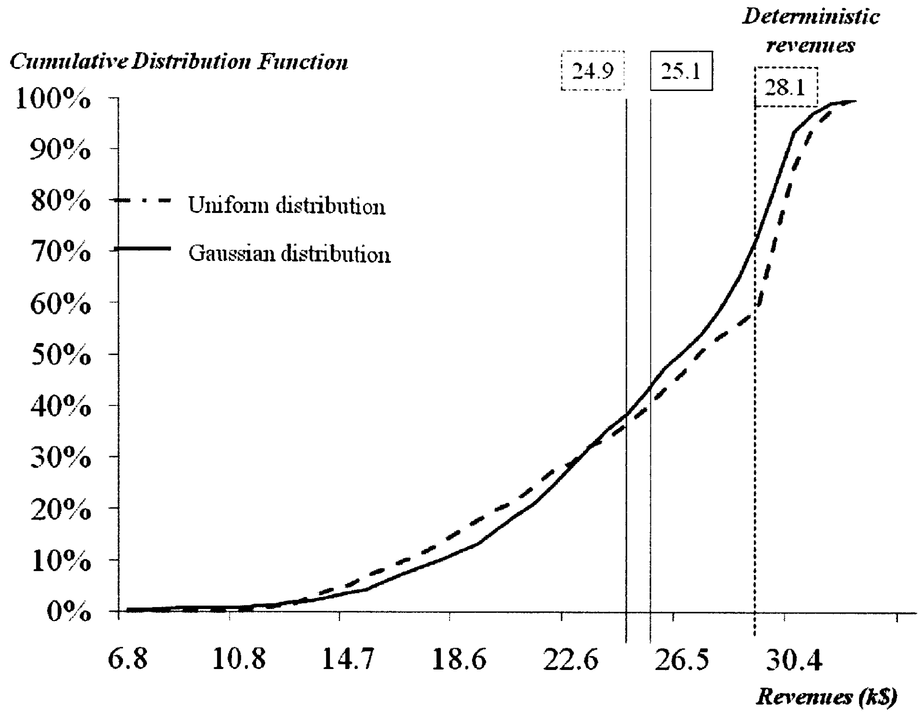


Figure 3-8: Cumulative distribution functions of the revenues when no booking limit is enforced

are fairly similar.

	TF1	*TF2*	*Total*
Accepted Bookings	59.1	31.1	90.2
Difference with theory	*-1%*	*-23%*	*-10%*
Average Fare ($)	244	336	276
Revenues ($)	**14,430**	**10,426**	**24,856**

Table 3.4: Simulation results when no booking limit is enforced - uniform distribution

For both types of distributions, the accepted demand in the first time frame matches the predicted deterministic demand. There is only a -1 to 1% relative difference with the theoretical demand of 59.6. However, in the second time frame, we observe a large difference between the accepted demand and the predicted, underlying, demand. The average accepted demand in $TF2$ is about 31.1, which is 23% lower than the mean underlying demand. The resulting load factor averages 90-91%.

	TF1	*TF2*	*Total*
Accepted Bookings	60.1	31.1	91.2
Difference with theory	*1%*	*-23%*	*-9%*
Average Fare ($)	244	336	276
Revenues ($)	**14,684**	**10,423**	**25,107**

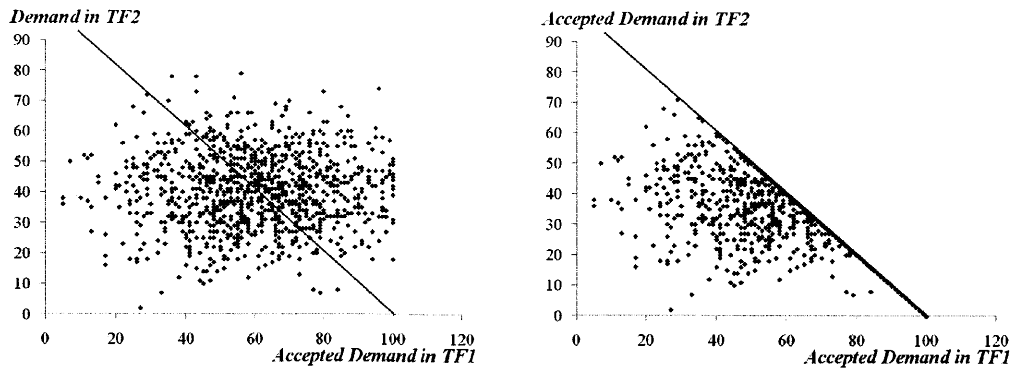
Table 3.5: Simulation results when no booking limit is enforced - Gaussian distribution

This gap in the accepted demand, not observed in the first time frame, is due to the flight capacity and the combined demand variability. Figure 3-9(a) shows the distribution of the demand in $TF2$ given the accepted demand in $TF1$. In about 50% of the samples generated, the combined demand of $TF1$ and $TF2$ exceeds the flight capacity, represented on the figure by a red line. Thus, in about 50% of the samples generated, part of the $TF2$ underlying demand was not accommodated and only the demand matching the remaining capacity was accepted, as shown in Figure 3-9(b). The average accepted demand is displaced and lowered by the constraint represented by the flight capacity. This translates into a 23% loss in the second time frame's revenues, which represents a -9% decrease in revenue overall.

The average fare in the second time frame is nevertheless higher than that of the first time frame. We could mitigate the loss in revenue by displacing part of the rejected demand to the first time frame. To do so, we would need to enforce a booking limit in the first time frame, thus protecting a minimum number of seats for the second time frame. This case will be illustrated with the next set of simulations, where we set of limit to the number of bookings accepted in $TF1$.

3.6.2 Simulations with a booking limit

In the second test, we enforce the booking limit: the number of bookings accepted in the first time frame is at most $z_1^* = 60$. For a more accurate comparison with the previous scenario, we do not regenerate the underlying demand. Only the number of accepted bookings changes. The fares still correspond to the deterministic optimal



(a) Underlying Demand in TF2 (b) Accepted Demand in TF2

Figure 3-9: Impact of the flight capacity on the accepted demands

solution.

The results are summarized in Table 3.6. Enforcing the booking limit in the first time frame results in a 0.5-0.7% increase in the estimated average total revenues. The results are further detailed in the Table 3.7 and Table 3.8. The revenue improvement is driven by the increase in the accepted demand in $TF2$. Enforcing the booking limit z_1 in the first time frame lowered the estimated expected $TF1$ constrained demand from 59 to 51, a 14% decrease, which resulted in a \$1950 loss in expected revenues for the uniform distribution. However, the few seats saved in the first time frame became available for the later arriving demand: the estimated number of bookings increased from 31 to 37 in the second time frame. The higher fares combined with the increase in accepted demand resulted in a \$2100 gain in expected revenues in the second time frame for the uniform distribution, eventually leading to a 0.5% increase in total expected revenues.

The impact of the booking limit on the distribution of the accepted bookings in both time frames is displayed in Figure 3-10. The booking limit greatly reduced the standard deviation of the accepted bookings. However, the load factor is also reduced, as shown in Figure 3-11, which gives an overview of the opposite changes induced by z_1 on both time frames.

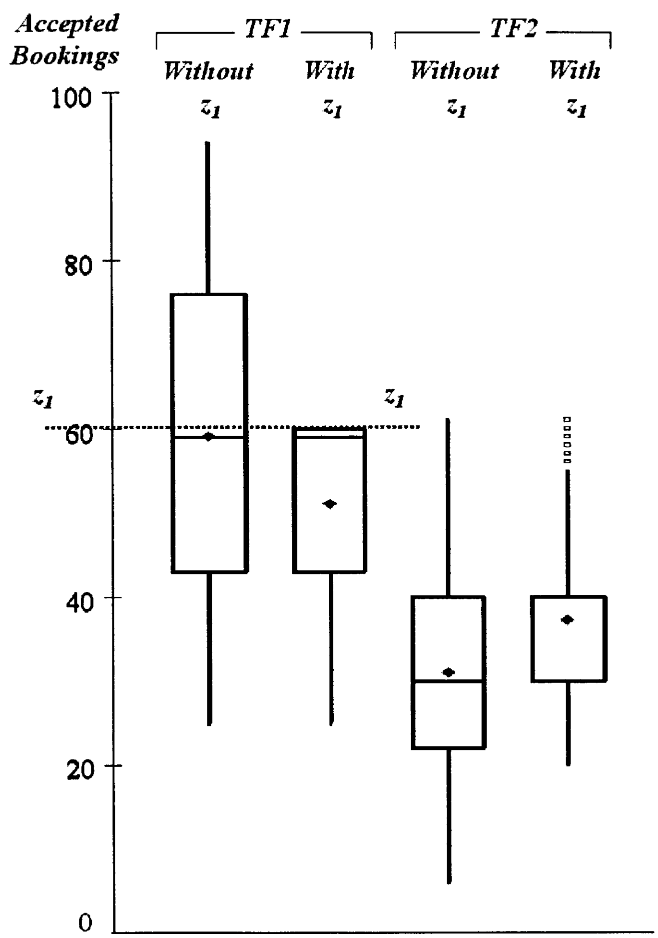


Figure 3-10: Impacts of the booking limit on the accepted bookings in both time frames

	Uniform Demand	*Gaussian Demand*
Revenues without z_1	\$24,856	\$25,107
Revenues with z_1	\$24,970	\$25,292
Change	*0.5%*	*0.7%*

Table 3.6: Impacts of the booking limit on the expected revenues

	TF1	*TF2*	*Total*
Accepted Bookings	51.1	37.2	88.3
Difference with theory	*-14%*	*-8%*	*-11%*
Difference with no z_1	*-14%*	*+20%*	*-2%*
Average Fare (\$)	244	336	283
Revenues (\$)	**12,471**	**12,499**	**24,970**

Table 3.7: Simulation results with the booking limit enforced - uniform distribution

There is trade-off between the increase revenues generated by saving a seat for a later, higher-revenue passenger, and the risk of rejecting a booking for a seat in the first time frame that may not be sold in the end. The 0.5% increase in total revenues may be further enhanced by another well chosen booking limit. The next chapter will focus on determining simultaneously fares and booking limit to optimize the total revenues in a stochastic environment.

3.6.3 Simulations with booking limits on Fare Product 2

In traditional revenue management approaches, all fare products but the highest one have a booking limit. The objective is to ensure that only a maximum number of discounted, or lower, fare products are sold, thus leaving at least a few seats available for the late arriving but high revenue passengers. In our set-up, this would correspond to imposing an additional booking limit on Fare Product 2. Capping the number of bookings that can be accepted for Fare Product 2 in both time frames would ascertain the prolonged availability of Fare Product 1.

	TF1	*TF2*	*Total*
Accepted Bookings	52.2	37.4	89.6
Difference with theory	*-12%*	*-7%*	*-10%*
Difference with no z_1	*-13%*	*+20%*	*-2%*
Average Fare ($)	244	336	282
Revenues ($)	**12,751**	**12,541**	**25,292**

Table 3.8: Simulation results with the booking limit enforced - Gaussian distribution

Our model does not account for this kind of booking limit. Instead we specify a booking limit to protect seats for later time frames. We therefore do take into account the time dimension: our model acknowledges the fact that later arriving passengers probably have a higher willingness-to-pay than early booking passengers. Moreover, we assume, implicitly, that this is likely to be true for both fare products. Furthermore, our model also protects Fare Product 1 versus Fare Product 2, but not in the traditional way: integrating pricing and revenue management allows us to control the passenger mix thanks to the fares. In other words, including the four fare levels as decision variables enabled us to remove the two additional booking limits that would be required to mirror the traditional approach. Nevertheless, we will test the performance of the model when these additional bookings limits are enforced.

Let $z_{2,TF1}$ and $z_{2,TF2}$ be the booking limits for the lower fare product in the first and second time frame, respectively. The optimal fares of the deterministic model imply demands of 36 and 21 for Fare Product 2 in TF1 and TF2. Therefore, in addition to the $z_1 = 60$ booking limit on the total TF1 demand, we impose a booking limit of $z_{2,TF1} = 36$ on the number of accepted Fare Product 2 bookings in the first time frame, and another booking limit of $z_{2,TF2} = 21$ on the number of accepted Fare Product 2 bookings in the second time frame. In this case too, we do not regenerate the underlying demand, in an effort to allow for a more accurate comparison between scenarios. Only the number of accepted bookings changes.

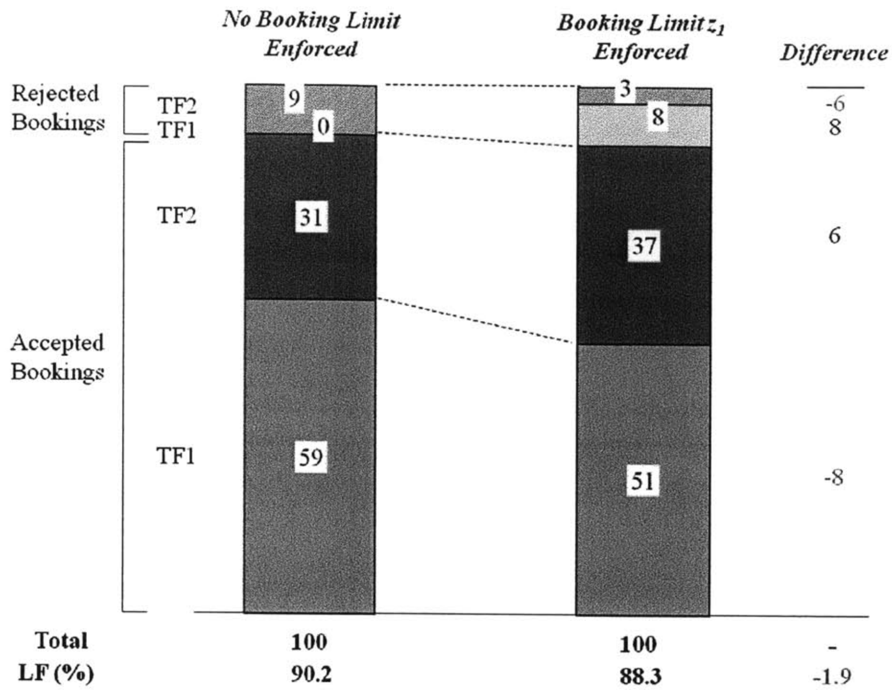


Figure 3-11: Impacts of the booking limit on the type of accepted bookings

The results are summarized in Table 3.9. Enforcing the booking limits on Fare Product 2 in both time frames results in a decrease in the estimated average total revenues from the case where no booking limit is applied. However, the change is very small. It is driven by the decrease in the total number of accepted bookings. With the additional booking limits, the average number of accepted bookings diminished. More Fare Product 2 booking requests were rejected, however the booking limits were not well adapted since this did not result in many additional higher-revenue bookings.

3.6.4 Simulations with a traditional revenue management approach

The last set of simulations is intended to help us compare the deterministic joint optimization approach with a more traditional revenue management method. To do so we used a fixed fare structure combined with a leg-based seat allocation method,

	Uniform Distribution		*Gaussian Distribution*	
	Revenue	***Accepted Bookings***	***Revenue***	***Accepted Bookings***
Simulations without z_1	$24,856	90	$25,107	91
Simulations with z_1	$24,970	88	$25,292	90
Change	*0.5%*	*-2*	*0.7%*	*-1*
Simulations with z_1, $z_{2,TF1}$ and $z_{2,TF2}$	$24,732	87	$25,078	89
Change	*-0.5%*	*-3*	*-0.1%*	*-3*

Table 3.9: Impacts of booking limits on the expected revenues and the accepted bookings

the Expected Marginal Seat Revenue method, also noted EMSRb. In this traditional revenue management approach, the fares and booking limit are considered separately. The fares are not part of the decision process and do not change. It is therefore difficult to propose a straight-forward comparison.

We use the same parameters as those displayed in Table 3.1 to test this traditional revenue management method. In this traditional approach, the fares are assumed to remain fixed throughout the entire selling period and we therefore keep the fares unchanged. We use the formulation of the deterministic joint pricing and seat allocation method to determine the pair of fares that would optimize the revenues. This should mitigate the impact of a poor pricing strategy on the simulated revenues. The optimal fares are $392 and $189. Once the pair of fares is known, it is possible to apply the leg-based seat allocation method, EMSRb, to find the optimal booking limit. As show in Figure 3-12, the deterministic demand for Fare Product *1* and Fare Product *2* is 28 and 72, respectively. Standard deviations of 20 and 12 in $TF1$ and $TF2$ imply standard deviations of 10 and 14 for Fare Product *1* and Fare Product *2*, respectively. The EMSRb booking limit for Fare Product *2* is therefore 71.

The total demand for the two fare products is generated for each time frame, given the new optimal pair of fares. We deduce the demand for each fare product based on

	TF1		*TF2*		*Total*	
	Fare	*Predicted Demand*	*Fare*	*Predicted Demand*	*Average Fare*	*Predicted Demand*
Fare Product 1	392.4	21	392.4	8	473.0	28
Fare Product 2	189.0	32	189.0	40	189.0	72
Total		53		47		100

Figure 3-12: Fares and demands for the traditional revenue management method

the implied probabilities p_1 and p_2. The booking limit of 71 seats is then applied to the demand for Fare Product *2* and the flight capacity limit is applied to the sum of the demands for the two products.

Once again, we use two distinct sets of simulations. In the first set of simulations, the demand is uniformly distributed in both time frames. In a second set of simulations, the demand is normally distributed in both time frames. For both type of demands, we generate 1,000 samples.

	Uniform Demand	*Gaussian Demand*
Accepted Bookings:		
Traditional RM Approach	90.6	90.6
Joint Optimization with z_1	88.3	89.6
Change	*-2.5%*	*-1.1%*
Revenues:		
Traditional RM Approach	\$24,116	\$24,239
Joint Optimization with z_1	\$24,970	\$25,292
Change	*3.5%*	*3.5%*

Table 3.10: Comparison between joint optimization and a traditional revenue management approach

The deterministic joint optimization solution provides a 3.5% increase in revenues

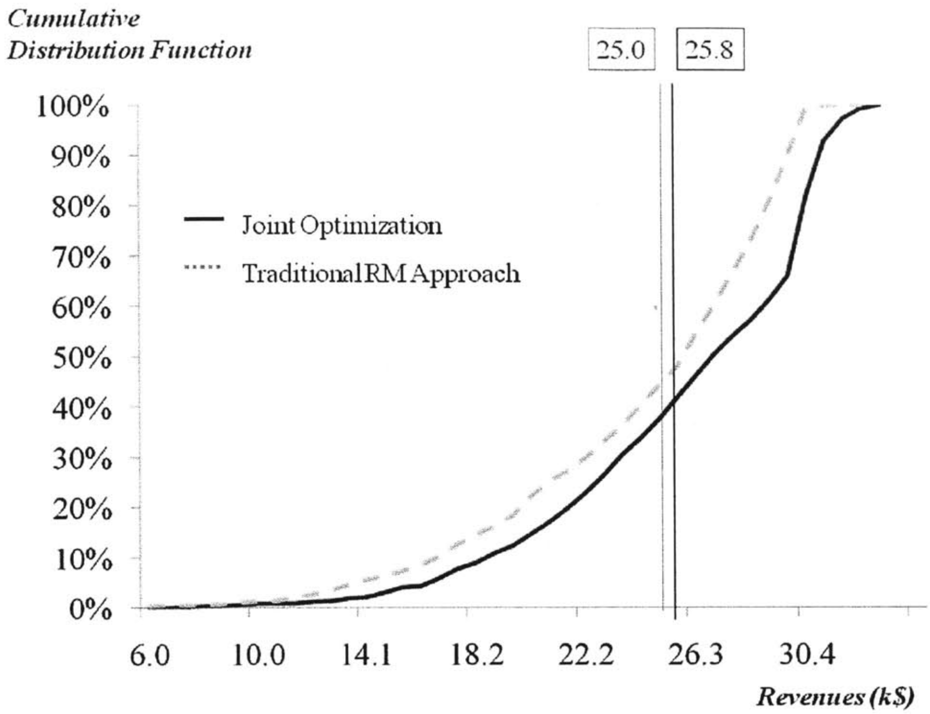


Figure 3-13: Cumulative distribution functions of the revenues, for Gaussian demands

from the traditional approach to revenue management with a single set of fares, as shown in Table 3.10. The pair of fares used for this set of simulations is already partially optimized: we used the deterministic model to find the best pair of fares given the flight capacity. In reality, fares are usually not optimal when traditional revenue management approaches are used and the deterministic model could therefore result in an even higher increase in revenue.

Figure 3-13 shows the cumulative distribution functions of the revenues when the demand in Gaussian. The joint optimization approach can help the airline reduce the risks of lower revenues. The highest revenues are also obtained with the joint optimization approach.

3.7 Summary

In this chapter, we formulated the two-product, two-period joint pricing and seat allocation problem.

We proposed a first approach to solving this problem by first assuming that the demand is deterministic. This assumption enabled us to gain some insights. In the deterministic case, the booking limit was unnecessary, which greatly simplified the objective function.

In the last section, a numerical example was used to illustrate the deterministic approach and run a few sets of simulations in which the demand exhibits some variability. The simulations confirmed the benefits of enforcing the booking limit when the demand is stochastic. By protecting a minimum number of seats from early, low revenues passengers, the booking limit enabled the airline to improve its expected revenues. The simulations showed that even when the demand is uncertain, the proposed approach performed well when compared to a traditional approach to revenue management method. Finally, the proposed approach behaved well under the two types of demand distribution tested. The uniform and Gaussian distributions led to very similar results.

Chapter 4

Stochastic Approach to Joint Pricing and Seat Allocation Optimization

In this chapter, we extend the work presented in Chapter 3 by introducing stochasticity in the demand formulation. The total demand for the fare products is no longer deterministic. While it still depends on the products' prices, it now also encompasses an uncertainty component. This new formulation, in which the demand fluctuates around an average, is intended to be more realistic. The objective of the stochastic model is to simultaneously find the optimal set of fares and booking limit that maximize the expected total revenue.

This chapter is divided into five sections. We begin by reviewing the problem scope. The notations are identical to those used in Chapter 3. In Section 4.2 we state and explain the assumptions regarding the demand formulation. In the following section, we derive the objective function by analysing the effect of demand uncertainty on the total number of accepted bookings, one time frame after the other. The analysis is based on a geometrical analogy. We then go back to the numerical example from the previous chapter to illustrate the new stochastic approach to joint pricing and seat allocation. We compare the results with those obtained with the deterministic approach and the more traditional leg-based revenue management method EMSRb. Finally, we conclude the chapter with a sensitivity analysis and performance analysis.

4.1 Scope of the Problem

The scope of the problem remains the same as in the previous chapter. We place ourselves in a single carrier, single flight, single OD market environment, and the flight has a fixed capacity of C seats.

Again, two fare products are offered. They provide exactly the same in-flight service but are associated with two distinct sets of purchase restrictions and are priced differently. Fare Product *1* represents the more expensive, less restricted, product. The other product, Fare Product *2*, is priced lower, but in return, has additional restrictions and rules. The prices of the two products can change over time.

The booking period is divided into two time frames, noted *TF1* and *TF2*. Bookings start to be accepted at the beginning of the first time frame, *TF1*. The prices of the two products can be modified at the start of each new time period. The price points are decision variables.

Furthermore, the airline can limit the total number of seats to be sold in the first time frame. Any unsold seats at the end of the first time frame is available for booking in the second time frame: the seat inventory is nested.

The notations used in this chapter are the same as the ones used previously.

4.2 Model Assumptions

We make the following assumptions regarding the demand formulation:

1. The demands in the two time frames are independent.

2. The total demand for both products in *TFi* is uncertain, and modelled as a stochastic additive function $n_{total,i} = \mu_i(y_i) + \varepsilon_i$.

3. The expectation of the total demand in *TFi*, denoted μ_i, is a linear function of the lower price: $\mu_i(y_i) = \alpha_i - \beta_i y_i$, with $\alpha_i, \beta_i \geq 0$.

4. The random variable ε_i is uniformly distributed: $\varepsilon_i \sim U[-\sigma_i, \sigma_i]$.

5. The probability that a passenger chooses the less restricted product is $p_i(x_i, y_i) = \dfrac{1}{1 + e^{a_i - b_i y_i + c_i x_i}}$, with $b_i, c_i \geq 0$.

Assumptions 2 and 4 are the two assumptions that were not applied in the previous chapter. Through Assumption 2, we introduce the total demand as a random variable. The mean value of the total demand, given by Assumption 3, corresponds to the deterministic demand of the previous chapter. Assumption 4 implies that the total demand in each time frame is uniformly distributed. In the literature on revenue management, the most commonly used probability distribution is the Gaussian distribution (Weatherford, 1997; Kuyumcu and Garcia Diaz, 2000). However, several reasons, detailed next, led us to chose the uniform distribution over the Gaussian one.

As we shall see in the following section, determining the expected number of accepted bookings, also called the censored demand, over the two time frames is critical to the optimization problem. The probability density function of the sum of two independent random variables is the convolution product of their individual density functions. While we know that the convolution of two unbounded Gaussian probability density functions is a simple Gaussian probability density function, there is no closed-form expression for the convolution of bounded Gaussian distributions. In other words, because the booking limit or the flight capacity truncate the probability distribution function of the demand in the first time frame, the convolution product becomes extremely complex when the distributions are Gaussians. The uniform distribution, on the other hand, enables us derive the probability density function of the sum of two bounded variables.

Nevertheless, we shall note that all the work done with the uniform distribution can be used to model the Gaussian distribution. The Gaussian function can be seen as the limit of a sum of uniform functions, as shown in Figure 4-1, and the convolution product of two sums of functions is the sum of the convolution products of all possible pairs of functions: $\left(\sum_{i=1}^{k_f} f_i\right) \star \left(\sum_{j=1}^{k_g} g_j\right) = \sum_{i=1}^{k_f}\sum_{j=1}^{k_g} (f_i \star g_j)$. The results found for two uniform distributions can be directly applied to every pair of functions $f_i \star g_j$. The findings of this chapter can thus be easily extended and used to approximate the results for Gaussian distributions.

In addition, the uniform distribution does not put as much emphasis on the mean as the Gaussian distribution does. In reality, it might occur that a demand much lower or higher than the mean is more likely than the normal distribution predicts. Modeling the demand with the uniform distribution therefore protects the airline a bit more against lower revenues.

4.3 Objective Function

The total revenues generated by the sale of the two products over the course of the two time frames is given by:

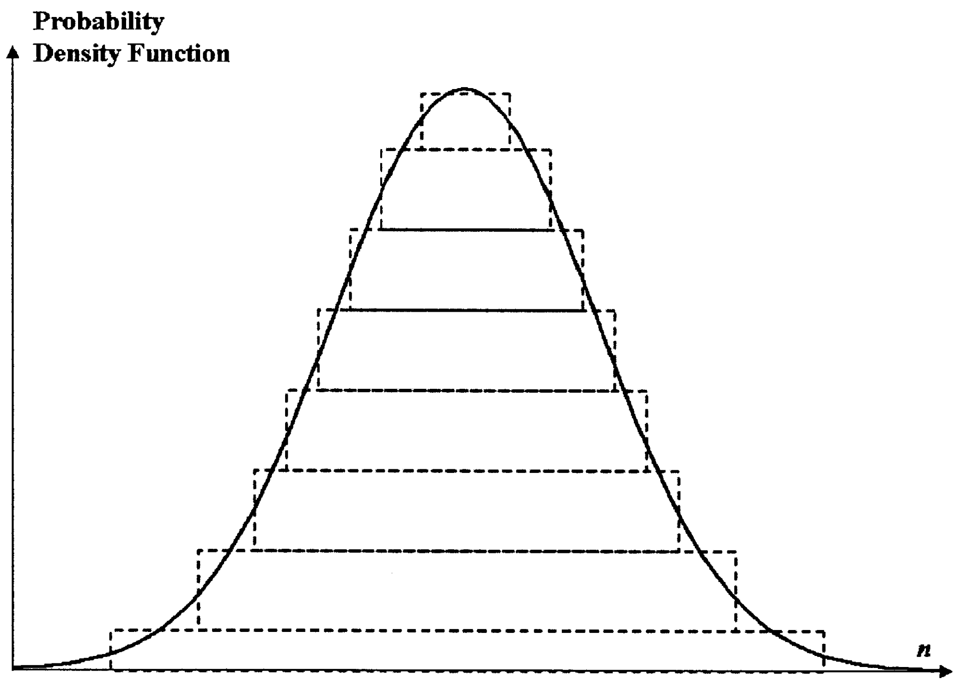


Figure 4-1: Gaussian as the limit of a sum of uniform functions

$$R_{total} = R_1 + R_2$$

$$\text{with } R_1 = \begin{cases} n_{total,1}\left[p_1 x_1 + (1-p_1)\, y_1\right], & \text{if } n_{total,1} < z_1; \\ z_1 \left[p_1 x_1 + (1-p_1)\, y_1\right], & \text{otherwise.} \end{cases}$$

$$R_2 = \begin{cases} n_{total,2}\left[p_2 x_2 + (1-p_2)\, y_2\right], & \text{if } n_{total,2} < C - min\left(z_1, n_{total,1}\right); \\ \left[C - min\left(z_1, n_{total,1}\right)\right]\left[p_2 x_2 + (1-p_2)\, y_2\right], & \text{otherwise.} \end{cases}$$

The two underlying demands $n_{total,1}$ and $n_{total,2}$ are uniformly distributed. The objective function for the stochastic model is therefore the expected value of the total revenue:

$$\overline{R}_{total} = \int \min\left(n_{total,1}, z_1\right)\left[p_1 x_1 + \left(1 - p_1\right) y_1\right] f_1\left(n_{total,1}\right) dn_{total,1}$$
$$+ \iint \min\left(n_{total,2}, C - min\left(n_{total,1}, z_1\right)\right)\left[p_2 x_2 + \left(1 - p_2\right) y_2\right] f_1\left(n_{total,1}\right) f_2\left(n_{total,2}\right) dn_{total,1} dn_{total,2}$$

The expected revenue from the first time frame only depends on few variables. Once again, we will analyse the revenue functions for the two time frames separately, starting with the simpler first time frame revenue.

Time Frame 1

In this section, we will drop the index 1 to simplify the notation. The revenue function is:

$$R\left(x, y, z\right) = \begin{cases} n_{total} p x + n_{total}\left(1 - p\right) y, & \text{if } n_{total} < z; \\ z p x + z\left(1 - p\right) y, & \text{otherwise.} \end{cases}$$

The expected revenue in $TF1$ is given by the following equation:

$$\overline{R}\left(x, y, z\right) = \int_{-\infty}^{\infty} \min\left(n_{total}, z\right)\left[p x + \left(1 - p\right) y\right] f\left(n_{total}\right) dn_{total}$$

For $z \in [\mu - \sigma; \mu + \sigma]$ and $z \leq C$, we have:

$$\begin{aligned}
\overline{R}(x,y,z) &= \int_{-\infty}^{z} n_{total}\left[px + (1-p)y\right] f(n_{total})\, dn_{total} \\
&\quad + \int_{z}^{\infty} z\left[px + (1-p)y\right] f(n_{total})\, dn_{total} \\
&= \left[px + (1-p)y\right]\left[\int_{-\infty}^{z} n_{total} f(n_{total})\, dn_{total} + \int_{z}^{\infty} z f(n_{total})\, dn_{total}\right] \\
&= \left[px + (1-p)y\right]\left[\int_{\mu-\sigma}^{z} \frac{n_{total}}{2\sigma}\, dn_{total} + \int_{z}^{\mu+\sigma} \frac{z}{2\sigma}\, dn_{total}\right] \\
&= \left[px + (1-p)y\right]\left[\frac{z+\mu}{2} - \frac{(\mu - z)^2}{4\sigma} - \frac{\sigma}{4}\right]
\end{aligned}$$

We can introduce the expected value of the censored demand, noted $\overline{n}_{accepted}$:

$$\overline{n}_{accepted} = \begin{cases} \mu, & \text{if } z \geq \mu + \sigma; \\ \dfrac{z+\mu}{2} - \dfrac{(\mu - z)^2}{4\sigma} - \dfrac{\sigma}{4}, & \text{if } z \in [\mu - \sigma; \mu + \sigma]; \\ z, & \text{if } z \leq \mu - \sigma. \end{cases}$$

The expected revenue in $TF1$ can then be rewritten as:

$$\overline{R}(x,y,z) = \left[px + (1-p)y\right]\overline{n}_{accepted}.$$

It is important to note the difference between the underlying demand and the actually observed number of bookings or censored demand.

The underlying demand is the demand that would be observed if the airline could accept all the booking requests it received. However, in reality, they are physical constraints that bound the total number of bookings that can be accepted. Once the flight capacity or the booking limits are reached, the airline has to reject booking requests. Therefore, the total number of accepted bookings, which is the total censored demand, is in most cases different from and lower than the underlying total demand.

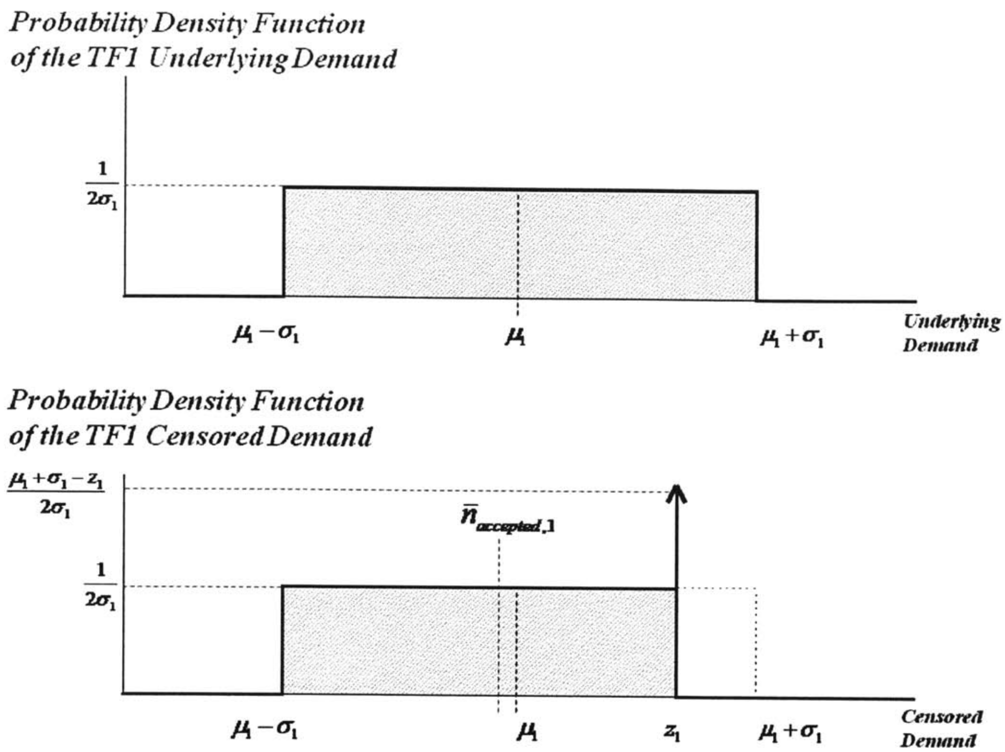


Figure 4-2: Underlying and constrained, also called censored, demands in TF1

The revenues generated by the sale of the two products depends on the number of accepted bookings. Therefore, the expected revenues are a function of the expected censored demand, noted $\overline{n}_{accepted,i}$. The probability density function of the censored demand in *TF1* depends on the booking limit imposed by the airline, as shown in Figure 4-2. The expected censored demand is a function of $\mu_i\,(y_i)$, z and σ_i. The expected value of the underlying total demand, μ_i, and the booking limit z_1 are lower and upper bounds of the expected censored total demand.

Time Frame 2

The maximum number of bookings that can be accepted in the second time frame depends on remaining capacity at the end of the first time frame, i.e. $C - min\,(n_{total,1}, z_1)$, as outlined in the following equation:

$$R_2 = \begin{cases} n_{total,2}\left[p_2 x_2 + (1-p_2)\, y_2\right], & \text{if } n_{total,2} < C - min\left(z_1, n_{total,1}\right); \\ \left[C - min\left(z_1, n_{total,1}\right)\right]\left[p_2 x_2 + (1-p_2)\, y_2\right], & \text{otherwise.} \end{cases}$$

The second time frame's revenue function not only depends upon the flight capacity, but also the upon variables form the first time frames. The flight capacity could have been, for the second time frame, the equivalent of the booking limit from the first time frame. However, the seat supply for the two time frames is nested: the physical constraint embodied by the flight capacity does not solely apply to bookings of a single time frame but to the combined bookings of both time frames. The combined censored demands of the two time frames have to verify $\overline{n}_{accepted,1} + \overline{n}_{accepted,2} \leq C$. The first time's booking limit and the flight capacity affect the number of bookings that can be accepted in the two time frames, as shown in Figure 4-3.

The expected value of the revenues in *TF2* is a function of the expected censored demand in *TF2*, noted $\overline{n}_{accepted,2}$.

$$\overline{R_2} = \left[p_2 x_2 + (1-p_2)\, y_2\right] \overline{n}_{accepted,2}$$

.

One way to determine the expected value of the censored *TF2* demand would consist of first evaluating the expected value of the sum of the two censored demands $\overline{n}_{accepted,1} + \overline{n}_{accepted,2}$, and then subtracting the known $\overline{n}_{accepted,1}$. This would involve a cumbersome convolution of a censored and an uncensored probability density function. There is, however, a more straight-forward and intuitive method to obtain $\overline{n}_{accepted,2}$.

Indeed, the expected value of the total censored *TF2* demand can be found geo-

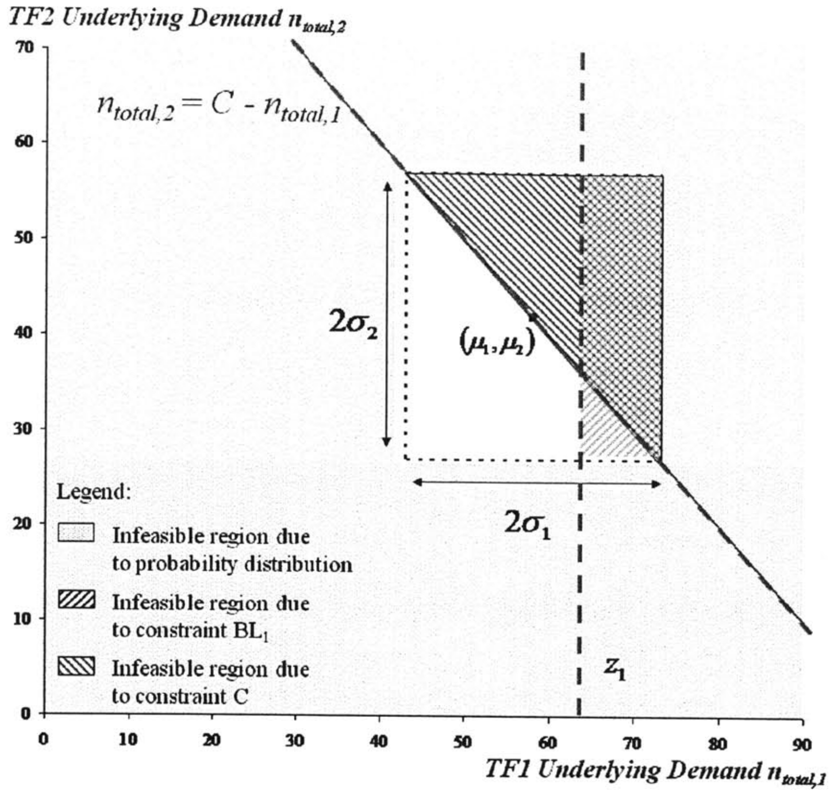


Figure 4-3: Constraints on the bookings in the two time frames

metrically. The probability density functions of the underlying, underlying, demands in *TF1* and *TF2* are fairly simple. The region of possible values for $n_{total,1}$ and $n_{total,2}$ can be divided into four smaller regions by the constraints z_1 and C. Those four regions are shown in Figure 6-1.

Region I is not affected by either constraint. All the points in this region are equally likely to be drawn.

In Region II, the bookings in *TF1* were lower than the booking limit and are therefore equally likely to be drawn. They are not affect by either the booking limit or the capacity constraint. However, the sum of booking requests in *TF1* and *TF2* are greater than C. The capacity constraint therefore applies to *TF2*. In this region, the demand in *TF2* is censored to $C - n_{total,1}$.

In the other two regions, Region III and Region IV, the booking limit had to be enforced in the first time frame. However, the booking limit and capacity constraint

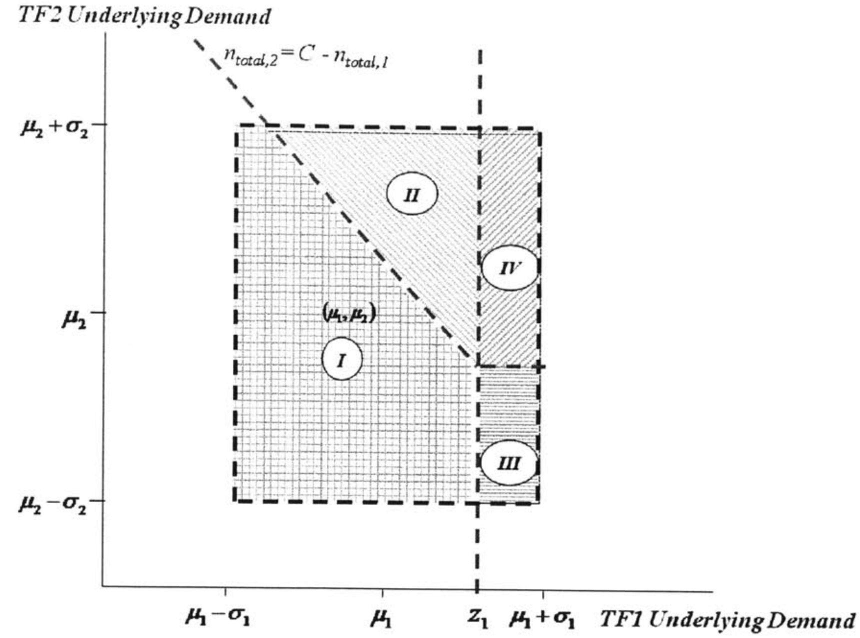


Figure 4-4: Divide and conquer

are not enforced simultaneously. The booking limit $z1$ is applied first and will affect the number of accepted bookings in *TF1* only. The capacity constraint is applied later, but its impact on the number of accepted *TF2* bookings depends on the number of bookings accepted in *TF1*.

In Region III, the booking limit z_1 applies to the first time frame's bookings: the *TF1* demand is censored and exactly z_1 bookings are accepted. The underlying *TF2* demand of this third region is between $\mu_2 - \sigma_2$ and $C - z_1$. Therefore, when the booking limit is enforced in the first time frame, the sum of bookings for *TF1* and underlying demand for *TF2* is lower than the capacity. The *TF2* demand is not capped.

In Region IV, both constraints apply. The booking limit z_1 was enforced in the first time frame and the accepted number of bookings in this region is exactly equal to z_1. Then, in the second time frame, the underlying demand is too high again and the accpeted number of bookings is exactly equal to $C - z_1$. there are more than z_1 booking requests in *TF1* and more than $C - z_1$ requests in *TF2*.

For each of one these four regions, we can easily find the ordinate of the barycentre, and therefore deduct $\overline{n}_{accepted,2}$.

We assume that the condition $Cap - z_1 \leq \mu_2 + \sigma_2$ is always satisfied. In order words, we assume that the fares are such that the flight capacity can be reached. If this is not the case, the total demand is much lower than the capacity and there is little meed for seat allocation optimization. There are three cases to consider, depending on the position of the flight capacity constraint with respect to the underlying *TF2* demand. We shall first consider the case for which: $\mu_1 - \sigma_1 \leq C - \mu_2 - \sigma_2 \leq z_1$. This is the case depicted by Figure 6-1.

We shall first consider Regions I & II. We shall compare the ordinates of these two regions in the two following cases: without and with the capacity constraint. When the flight capacity is not enforced, the ordinate of the combined regions is simply μ_2. Let $Y_{I,II}$ be the ordinate of the barycenter of these two regions when the capacity constraint is applied. The two regions' characteristics of interest are listed in the two tables, Table 4.1 and Table 4.2.

Region	***Barycenter's Ordinate***	***Area***
I	unknown	$2\sigma_2(z_1 - \mu_1 + \sigma_1) - \dfrac{(\mu_2 + \sigma_2 - C + z_1)^2}{2}$
II	$\dfrac{C - z_1 + 2\mu_2 + 2\sigma_2}{3}$	$\dfrac{(\mu_2 + \sigma_2 - C + z_1)^2}{2}$
Total	μ_2	$2\sigma_2(z_1 - \mu_1 + \sigma_1)$

Table 4.1: Underlying demands in Regions I and II

By substitution, we have $Y_{I,II} = \mu_2 - \dfrac{(\mu_2 + \sigma_2 - C + z_1)^3}{12\sigma_2(z_1 - \mu_1 + \sigma_1)}$.

Similarly, we can define the ordinates of the Regions III and IV's barycentres, as shown in Table 4.3. The ordinate of the two regions' barycenter is given by $Y_{III,IV} = \dfrac{C - z_1 + \mu_2}{2} - \dfrac{(\mu_2 - C + z_1)^2}{4\sigma_2} - \dfrac{\sigma_2}{4}$.

Region	*Barycenter's Ordinate*	*Area*
I	unknown	$2\sigma_2(z_1-\mu_1+\sigma_1)-\frac{(\mu_2+\sigma_2-C+z_1)^2}{2}$
II	$\frac{2C-2z_1+\mu_2+\sigma_2}{3}$	$\frac{(\mu_2+\sigma_2-C+z_1)^2}{2}$
Total	$Y_{I,II}$	$2\sigma_2(z_1-\mu_1+\sigma_1)$

Table 4.2: Censored demands in Regions I and II

Region	*Barycenter's Ordinate*	*Area*
III	$\frac{1}{2}(C-z_1+\mu_2-\sigma_2)$	$(C-z_1-\mu_2+\sigma_2)(\mu_1+\sigma_1-z_1)$
IV	$C-z_1$	$(\mu_2+\sigma_2-C+z_1)(\mu_1+\sigma_1-z_1)$
Total	$Y_{III,IV}$	$2\sigma_2(\mu_1+\sigma_1-z_1)$

Table 4.3: Censored demands in Regions III and IV

As, $2\sigma_1\overline{n}_{accepted,2}=Y_{I,II}(z_1-\mu_1+\sigma_1)+Y_{III,IV}(\mu_1+\sigma_1-z_1)$, we have, for $\mu_1-\sigma_1\leq C-\mu_2-\sigma_2\leq z_1$:

$$\overline{n}_{accepted,2}=\left[\mu_2-\frac{(\mu_2+\sigma_2-C+z_1)^3}{12\sigma_2(z_1-\mu_1+\sigma_1)}\right]\frac{(z_1-\mu_1+\sigma_1)}{2\sigma_1}+\left[\frac{C-z_1+\mu_2}{2}-\frac{(\mu_2-C+z_1)^2}{4\sigma_2}-\frac{\sigma_2}{4}\right]\frac{\mu_1+\sigma_1-z_1}{2\sigma_1}.$$

We can use the same geometrical approach to determine the expected censored demand in the two remaining cases. For $\mu_1-\sigma_1\leq z_1\leq C-\mu_2-\sigma_2$, we have $\overline{n}_{accepted,2}=\mu_2$. For $C-\mu_2-\sigma_2\leq\mu_1-\sigma_1\leq z_1$, the expected censored demand is

$$\overline{n}_{accepted,2}=\left[\mu_2-\frac{z_1^2-(\mu_1-\sigma_1)^2}{12\sigma_2}-\frac{(\mu_2+\sigma_2+\mu_1-\sigma_1-C)^2}{4\sigma_2}\right]\frac{(z_1-\mu_1+\sigma_1)}{2\sigma_1}+\left[\frac{C-z_1+\mu_2}{2}-\frac{(\mu_2-C+z_1)^2}{4\sigma_2}-\frac{\sigma_2}{4}\right]\frac{\mu_1+\sigma_1-z_1}{2\sigma_1}.$$

Figure 4-5 is an illustration of the expected censored demand in $TF2$ as a function

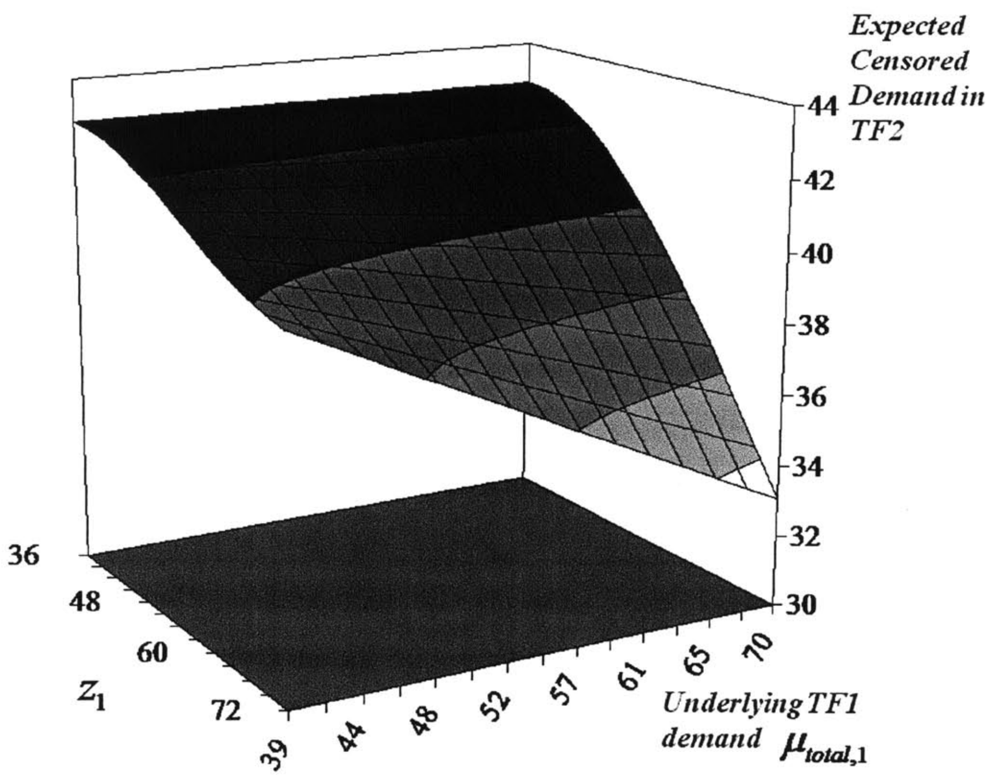


Figure 4-5: The expected censored demand in TF2 as a function of the TF1 demand and booking limit

of the first time frame's demand and booking limit.

The assumption that the demand follows a uniform distribution allowed us to determine the exact expression of the censored demand. The geometrical approach would not have been possible with a Gaussian distribution. Instead, one would have had to resort to simulations to estimate the number of accepted bookings. The use of this simple distribution will be particularly useful as we move to a multiple-time frame optimization problem. As we shall seen in the next chapter, it is possible to extend the geometrical approach to additional time periods.

Total Expected Revenues

The objective function is the expected total revenues generated by the sale of the two products over the course of the two time frames:

$$\overline{R}_{total} = \overline{n}_{accepted,1}\left[x_1 p_1 + (1-p_1)\, y_1\right] + \overline{n}_{accepted,2}\left[x_2 p_2 + (1-p_2)\, y_2\right]$$

with $\overline{n}_{accepted,1} = \dfrac{z_1+\mu_1}{2} - \dfrac{(\mu_1 - z_1)^2}{4\sigma_1} - \dfrac{\sigma_1}{4}$,

and, for $\mu_1 - \sigma_1 \le C - \mu_2 - \sigma_2 \le z_1$:

$$\begin{aligned}\overline{n}_{accepted,2} = {} & \left[\mu_2 - \frac{(\mu_2+\sigma_2-C+z_1)^3}{12\sigma_2\,(z_1-\mu_1+\sigma_1)}\right]\frac{(z_1-\mu_1+\sigma_1)}{2\sigma_1} \\ & + \left[\frac{C-z_1+\mu_2}{2} - \frac{(\mu_2-C+z_1)^2}{4\sigma_2} - \frac{\sigma_2}{4}\right]\frac{\mu_1+\sigma_1-z_1}{2\sigma_1};\end{aligned}$$

for $\mu_1 - \sigma_1 \le z_1 \le C - \mu_2 - \sigma_2$: $\overline{n}_{accepted,2} = \mu_2$;

for $C - \mu_2 - \sigma_2 \le \mu_1 - \sigma_1 \le z_1$:

$$\overline{n}_{accepted,2} = \left[\mu_2 - \frac{z_1^2 - (\mu_1 - \sigma_1)^2}{12\sigma_2} - \frac{(\mu_2 + \sigma_2 + \mu_1 - \sigma_1 - C)^2}{4\sigma_2}\right] \frac{(z_1 - \mu_1 + \sigma_1)}{2\sigma_1}$$
$$+ \left[\frac{C - z_1 + \mu_2}{2} - \frac{(\mu_2 - C + z_1)^2}{4\sigma_2} - \frac{\sigma_2}{4}\right] \frac{\mu_1 + \sigma_1 - z_1}{2\sigma_1}.$$

In this model, we have, by construction, $\overline{n}_{accepted,1} \leq z_1$ and $\sum_{i=1}^{2} \overline{n}_{accepted,i} \leq C$. The two booking constraints z_1 and C are included in the objective function.

The objective function is a non-linear, neither concave nor convex function. A non-linear maximization technique, such as the Powell's algorithm can be used to determine the optimal set of fares and booking limit. The deterministic optimal solution, derived from the deterministic problem, can be used as a starting point.

4.4 Numerical Results

In this section, we illustrate the stochastic joint pricing and seat allocation approach with a numerical example. This example will help us understand the advantages of the stochastic model over the deterministic one. Furthermore, simulations drawn from this example will enable us to test the model under different assumptions and compare the joint approach to a more traditional revenue management approach.

The assumed parameters for the demands in the two time frames considered are given in Table 4.4. The demand and portion parameters are identical to the parameters used in Chapter 3. However, for the stochastic approach, we have two additional parameters which describe the standard deviation associated to the total demand in each time frame: s_1 and s_2. The relationship between the uniform distribution's standard deviation s and distance between upper bound and mean, noted σ in the problem formulation, is given by $\sigma = s\sqrt{3}$. The flight capacity is 100 seats.

	TF1	***TF2***
Total Demand $n_{total,t}$	$\alpha_1 = 135$ $\beta_1 = 0.44$	$\alpha_2 = 85$ $\beta_2 = 0.20$
Probability p_t	$a_1 = 0.864$ $b_1 = -0.020$ $c_1 = 0.009$	$a_2 = -0.038$ $b_2 = -0.016$ $c_2 = 0.008$
Standard deviation	$s_1 = 20$	$s_2 = 12$

Table 4.4: Parameters for the demand functions

The optimal stochastic solution to this joint pricing and seat allocation problem is obtained with Matlab. The maximum total revenue is reached with the fares and booking limit of Table 4.5. The stochastic optimal fares are higher than the deterministic fares, and the increase is larger in the first time frame. These higher fares should lower the demand, yet the optimal booking limit is also higher than the implied deterministic booking limit.

	TF1	***TF2***
Fare Product 1	$x_1^* = \$383.4$	$x_2^* = \$482.8$
Fare Product 2	$y_1^* = \$196.5$	$y_2^* = \$236.9$
Booking limit	$z_1^* = 73$	

Table 4.5: Stochastic optimal solution to the joint pricing and seat allocation problem

Based on the assumed input parameters, we deduce the total number of accepted bookings, or censored demand, and revenues implied by these fares and booking limit, as shown in Table 4.6. The optimal fares and booking limit determined with the stochastic approach result in a total expected censored demand of 82.6, or an average load factor of 82.6%. The expected total revenue is \$25.7k. This forecasted expected revenue is about 9% lower than the \$28.1k revenue predicted with the deterministic model. However, the previous chapter's simulations revealed that, in practice, the

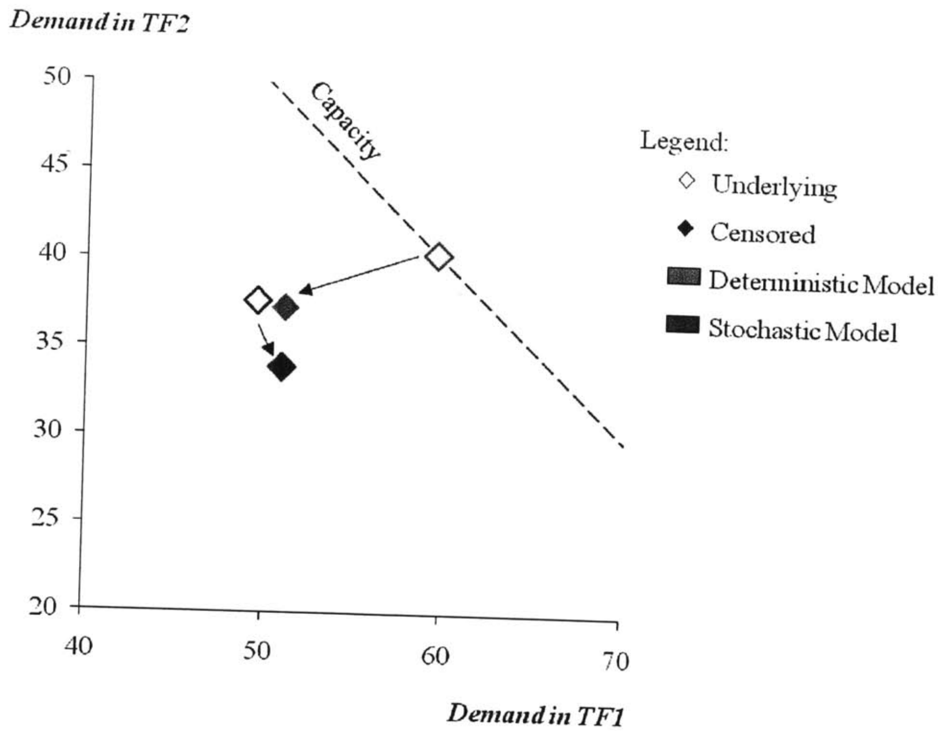


Figure 4-6: Difference between the two time frames' underlying and censored demand

average revenue generated by the deterministic optimal solution in a stochastic environment only averages $25.0k.

	TF1	***TF2***	***Total***
Average fare ($)	279.8	355.8	
Underlying Demand	$\mu^*_{total,1}$ =49.5	$\mu^*_{total,2}$ =37.6	μ^*_{total} =87.1
Accepted Bookings	$\overline{n}^*_{total,1}$ =48.7	$\overline{n}^*_{total,2}$ =33.9	$\overline{n}^*_{total}$ =82.6
Revenues (k$)	$\overline{R}^*_1 = \mathbf{13.6}$	$\overline{R}^*_2 = \mathbf{12.1}$	$\overline{R}^*_{total} = \mathbf{25.7}$

Table 4.6: Implied censored demand and revenues

Figure 4-6 shows the underlying and censored demands for the two models, deterministic and stochastic. In the case of the deterministic model, the underlying demand corresponding to the optimal solution matches the flight capacity exactly. The simulations revealed that the demand variability, the booking limit and the flight capacity all constrain the demand to about 88 instead. The stochastic model,

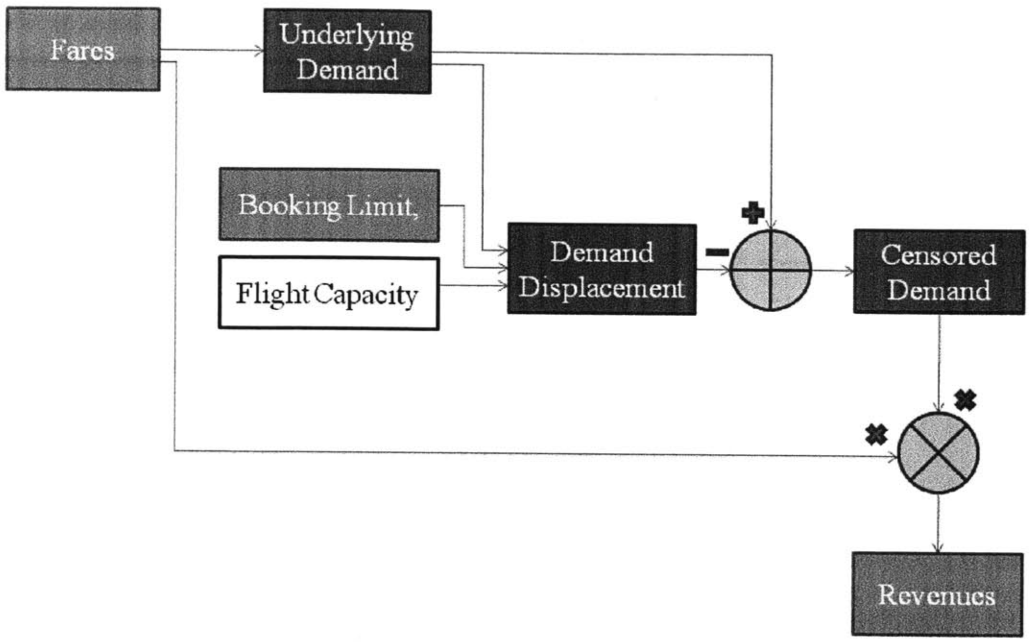


Figure 4-7: Impact of fares and booking limit on the revenues

which takes into account uncertainty and constraints, recommends fares which lower the total underlying demand altogether. The displacement between underlying and censored demands is reduced. As shown in Figure 4-6, the censored demand obtained with the stochastic solution should not be too far off the actual total number of bookings observed with the deterministic solution in practice. Yet, since the fares are optimized in light of the constraints and their impact on the observed demand, the average revenues provided by the stochastic model should ultimately be higher.

Figure 4-7 is a schematic overview of the relationship between the problem's inputs, the fares and booking limit, in green in the graph, the different types of demands, in blue, and the revenue. Fares stimulate the underlying demand, and their effects therefore cascade down to the censored demand as well. The booking limit and the flight capacity constrain the underlying demand. The amplitude of a constraint's impact on the underlying demand can be seen as inversely proportional to their distance. This impact, labelled as "demand displacement" in Figure 4-7, is subtracted from the underlying demand, yielding the censored demand. Eventually, the fares intervene again directly, since the revenues is the product of the censored demand and an average fare.

Increasing the fares lowers the underlying demand, which lowers the impact of the flight capacity. The censored demand thus undergoes minimal changes. However, because the fares, the other revenue component, are higher, the total revenues increase. The booking limit and fares, adjusted simultaneously, can increase the expected total revenues of the airline overtime.

4.5 Simulations

We shall now run several sets of simulations in which the demand is stochastic and analyse the results of the stochastic model.

4.5.1 Simulations with the stochastic optimal set of fares and booking limit

In the first set of simulations, we use the fares and booking limit corresponding to the stochastic optimal solution (see Table 4.5). The booking limits do not affect the probability that a passenger chooses the higher fare product.

As was the case in Chapter 3, the total demand for the two fare products is generated at the beginning of each time period. We run two distinct types of simulations. In the first type of simulations, the demand is uniformly distributed in both time frames. In a second type of simulations, the demand is normally distributed in both time frames. The demands for the two time frames are independent. For both type of demands, we generate 1,000 samples. We use the parameters displayed in Table

4.4, to model the expected demands. The standard deviations of the demands in $TF1$ and $TF2$ are 20 and 12, respectively, as outlined in Table 4.4. For the normal distribution, the demand is truncated in order to prevent any instance of negative demand. Finally, the flight capacity does not affect the probability that a passenger chooses the higher fare product.

These simulations should, in a first time, enable us to ascertain the accuracy of the predicted revenues. The simulations will also help us compare the performance of the stochastic model when the demand is not uniformly distributed but normally distributed.

Table 4.7 summarizes the findings. The simulations results match the values predicted by the stochastic model in the uniform distribution case. The results are higher than expected when the demand is normally distributed.

	Uniform Demand	***Gaussian Demand***
$TF1$ Accepted Bookings	48.5	48.1
$TF2$ Accepted Bookings	34.2	35.0
Load factor	82.8%	83.1%
Diff. w/ Deterministic	*-5.5*	*-6.5*
Total Revenues (k$)	**25.8**	**25.9**
Rel. Diff. w/ Deterministic	*2.5%*	*2.5%*

Table 4.7: Average censored demands and revenues

The estimated expected censored demands are very close to the predicted censored demands for both demand distribution types. In the case of the uniform distribution, the relative difference between the simulated censored demand and the predicted expected censored demand is -0.3% and 1.2% for $TF1$ and $TF2$, respectively.

The stochastic model is based on the uniform distribution and the relative differences between the theoretical and simulated results with a Gaussian distribution are therefore slightly larger for the Gaussian distribution. We should note that the estimated average demands are higher for the Gaussian distribution. The Gaussian distribution gives more weight to the mean of the underlying demand. Thus, the expected censored demand for this distribution type lies between the uniform distributions' $\overline{n}_{accepted}$ and μ. We can use the $\overline{n}_{accepted}$ computed for the uniform distribution as a lower bound for the expected censored demand in the case of a Gaussian distribution.

For the uniform demand distribution, the average total revenue is only 0.4% higher than the predicted revenue. For the Gaussian distribution, the expected revenues are 1.0% higher than predicted, due to the higher average censored demand.

The set of fares and booking limit determined with the stochastic model led to 2.5% increase in revenues from the deterministic model. The average load factor is however 5-6 points lower than it used to be with the deterministic solution.

Figure 4-8 shows the histogram of the revenues for the deterministic and stochastic model when the demand is uniformly distributed. The revenues obtained with the stochastic optimal solution span over a larger range of values. The minimum revenue observed with the stochastic solution is $11.0k instead of $13.8k for the deterministic fares and booking limit. However, revenues are higher than $28.5k in 43% of the cases with the stochastic approach, versus 7% only in the case of the deterministic approach. The stochastic model can provide a 2.5% increase in revenues on average.

4.5.2 Simulations with re-optimized $TF2$ fares

While the stochastic optimal solution provides a 2.5% increase in revenues, it also lowers the load factor by about 7 points. By modifying the $TF2$ fares at the end of the first time frame, given the observed $TF1$ demand, we may be able to adjust the demand to better match the remaining capacity.

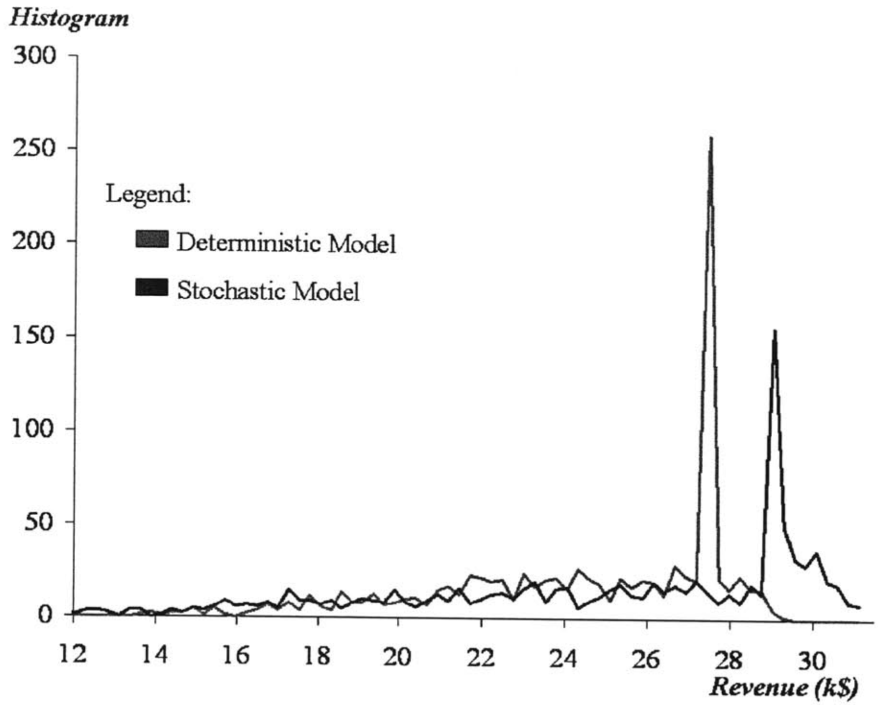


Figure 4-8: Revenue histogram for the uniform demand distribution

Previously, a very low underlying $TF1$ demand would leave much more seats available for the second time frame than anticipated. Yet, the $TF2$ fares, optimal for the average demand, are ill-suited to accommodate the "extra" capacity. As we re-adjust the $TF2$ at the end of the second time frame, we can lower the optimal x_2 and y_2 to foster the demand and fill-in the remaining capacity, while maintaining the revenues to an even level. This should help us improve the load factor without sacrificing the revenues.

In this second test, we allow for the re-optimization of the fares at the end of the first time frame. At the end of $TF1$, the remaining capacity is computed as $C - min\left(n_{total,1}, z_1\right)$. The stochastic approach can then be applied to the second time frame only, with the flight capacity set equal to the remaining capacity. The newly determined set of optimal fares for $TF2$ is then used to generate the demand. We keep the same generated demand for the first time frame, in order to compare this set of simulations with the previous "static" one.

	Uniform Demand	*Gaussian Demand*
$TF2$ average fare:		
Static	\$355.8	\$355.8
Re-optimized	\$335.7	\$337.1
$TF2$ average accepted bookings:		
Static	34.3	35.0
Re-optimized	36.5	37.0
Average load factor:		
Static	82.8%	83.1%
Re-optimized	85.0%	85.1%
Changes in LF	*2.1*	*2.0*
Average total revenues:		
Static	\$25.8k	\$25.9k
Re-optimized	\$25.8k	\$25.9k
Change in revenues	*0.1%*	*0.0%*

Table 4.8: Impacts of the re-optimization on the expected revenues

For both types of demand distribution, the adjustment of the $TF2$ fares at the end of the first time frame results in very little changes in the revenues but improves the load factor, as shown in Table 4.8. The average re-optimised $TF2$ fare is about \$20 lower than initial optimal average of \$355.8, fostering the demand. The $TF2$ censored demand is about 2 points higher. This resulted in an increased load factor, averaging 85%. As anticipated, the decrease in average fare combined to the increase in the number of accepted bookings ultimately lead to very small changes in the revenues: +0.1% for the uniform distribution and 0.0% change for the Gaussian distribution.

As shown in Figure 4-9, the probability of revenues higher than \$30.0k greatly increases when the $TF2$ fares are re-optimized at the end of the first time period. Re-optimizing the fares lowers the risk of low revenues.

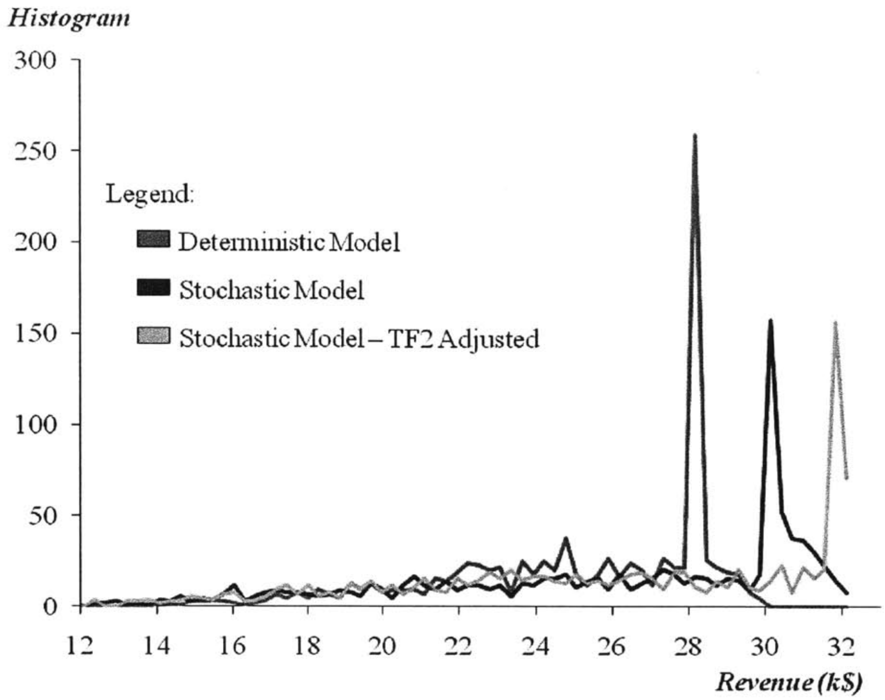


Figure 4-9: Revenue histogram for the uniform distribution when TF2 fares are adjusted

4.5.3 Simulations with booking limits on Fare Product 2

The stochastic model we developed includes a single booking limit. This booking limit protect seats for the later time frames versus the earlier one, rather than for a higher fare product versus a lower fare product. In practice, airlines would impose a booking limit on the lower fare product as well, as discussed in the previous chapter. We will take advantage of our numerical example to test this other type of booking limits on our model.

Let $z_{2,TF1}$ and $z_{2,TF2}$ be the booking limits for the lower fare product in the first and second time frame, respectively. These limits are not part of the stochastic model's output. We shall therefore test several levels for each one of them. The tested values for $z_{2,TF1}$ are between a slightly lower value than the underlying expected Fare Product 2 demand in TF1, given the optimal TF1 fare y_1^*, and the maximum demand for Fare Product 2 in TF1, which is defined by optimal TF1 booking limit z_1^*. For the second time frame, we test values between 17, which is slightly lower than the expected underlying demand for Fare Product 2 in TF2 and 45. In this case too, we do not regenerate the underlying demand in neither time frames, in an effort to allow for a more accurate comparison between scenarios.

The results are summarized in Table 4.9 and represented on Figure 4-10. The additional booking limits led to a -3.5% to +0.02% change in revenues. Very low TF2 booking limits on Fare Product 2 systematically result in a decrease in revenues. The lower $z_{2,TF2}$, the larger the decrease. Larger values of this same booking limit do not, however, affect the revenues. The change in revenues seems to be mostly driven by $z_{2,TF1}$.

For $z_{2,TF1} = 31$, which is about 15% higher than the expected number of booking requests for Fare Product 2 in the first time frame, we observe a slight increase in revenues (+0.02%) when $z_{2,TF2}$ is large enough. All other combinations of booking limits led to a decrease in revenues.

Several factors may account for these results. Our optimization model does not include booking limits on Fare Product 2. Therefore, enforcing such booking limits with the stochastic optimal set of fares and z_1^* is unlikely to be optimal and provide a very large increase in revenues in our simulations. Additionally, the objective of limits on lower fare products is to protect seats for passengers with a higher willingness-to-pay who usually arrive late in the booking process. The time dimension of this strategy is already taken into account in our model. By dividing the selling period into two time frames and imposing a booking limit on the first time frame, we ensure that seats are saved for the late, high-revenue passengers of TF2.

The fares, which are now decision variables, also enable us to control the demands for the two fare products in each time frame. These demands are assumed to be dependent on the two products' fares and the stochastic model's output therefore implicitly optimizes the mix of Fare Product 1 and 2 passengers. The potential revenue impact of additional booking limits on Fare Product 2 is lessen.

Furthermore, our model does not make any assumptions on the arrival order of Fare Product 1 and 2 passengers within each time frame. Contrary to Weatherford (1997) for example, we do not assume that all lower-revenue passengers arrive first and higher-revenue passengers last. In this type of unrealistic scenario, enforcing a booking limit on Fare Product 2 bookings would probably lead to large increase in revenues.

Lastly, in our model, the probability that a passengers buys the higher fare product is deterministic. A booking limit of Fare Product 2 may lead to a larger increase in revenue if we had large variations in the passenger mix.

		$z_{2,TF2}$							
		17	21	25	29	33	37	41	45
$z_{2,TF1}$	41	-2.7	-1.8	-1.3	-0.1	-0.5	-0.5	-0.5	-0.5
	37	-2.4	-1.5	-1.0	-0.0	-0.2	-0.2	-0.2	-0.2
	34	-1.6	-1.3	-0.8	-0.2	-0.1	-0.1	-0.1	-0.1
	31	-1.5	-1.1	-0.8	-0.6	+0.0	+0.0	+0.0	+0.0
	27	-2.3	-1.3	-0.9	-0.5	-0.1	-0.1	-0.1	-0.1
	24	-3.5	-1.8	-1.3	-0.2	-0.6	-0.6	-0.6	-0.6

Table 4.9: Changes in revenues due to the implementation of Fare Product2's booking limits

4.5.4 Simulations with a traditional revenue management approach

This set of simulations is intended to help us compare the stochastic joint optimization model with a traditional revenue management approach. To do so we used, as we did in Chapter 3, a fixed fare structure combined with a leg-based seat allocation

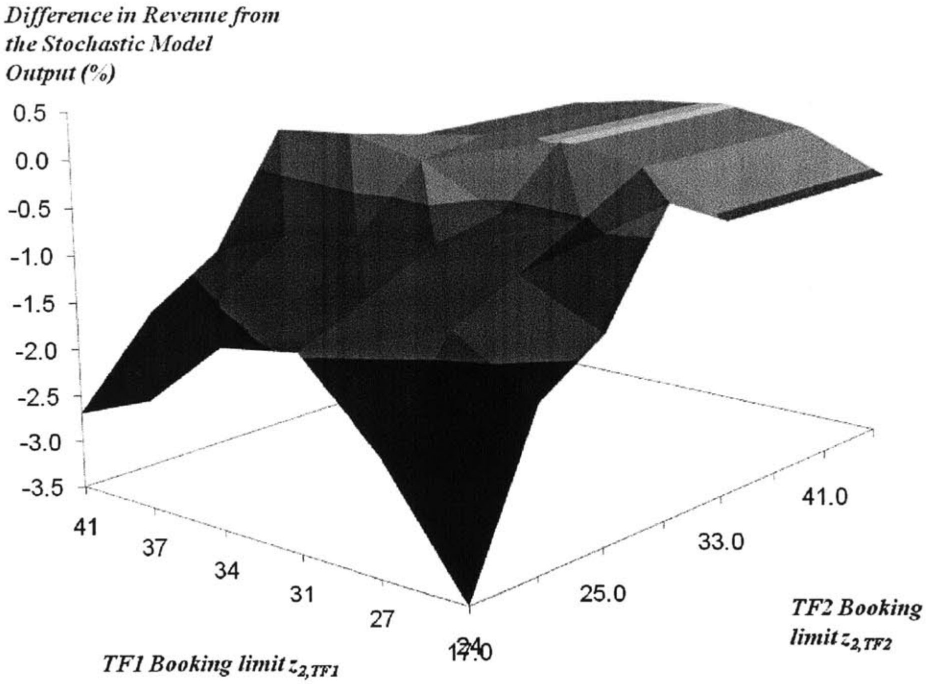


Figure 4-10: Decrease in revenues due to the implementation of Fare Product2's booking limits

method, the Expected Marginal Seat Revenue method, also noted EMSRb. For the joint approach, we do not re-adjust the $TF2$ fares.

We use the same parameters as those displayed in Table 4.4 . In the traditional approach, the fares are assumed to remain fixed throughout the entire selling period. We use the stochastic joint optimization approach to find the optimal pair of fares for EMSRb. In this formulation, the booking limit of the first time frame is equal to the flight capacity. The optimal pair of fares for $TF1$ and $TF2$ is : \$428 and \$211. As shown in Figure 4-11, the predicted censored demand for Fare Product *1* and Fare Product *2*, given the fares, is 24 and 62, respectively. The associated standard deviations are 11 and 13. The implied EMSRb booking limit for Fare Product *2* is 76.

The total demand for the two fare products is generated for each time frame, given the optimal pair of fares. We deduce the demand for each fare product based on the implied probabilities p_1 and p_2. The booking limit of 76 seats is then applied

	TF1		TF2		Total	
	Fare	*Predicted Constrained Demand*	*Fare*	*Predicted Constrained Demand*	*Average Fare*	*Predicted Constrained Demand*
Fare Product 1	428	18	428	6	428	24
Fare Product 2	211	25	211	37	211	62
Total		43		43		86

Figure 4-11: Fares and demands for the traditional revenue management method

to demand for Fare Product *2* and the flight capacity limit is applied to the sum of the demands for the two products.

Once again, we use two distinct sets of simulations. In the first set of simulations, the demand is uniformly distributed in both time frames. In a second set of simulations, the demand is normally distributed in both time frames. For both type of demands, we generate 1,000 samples.

	Uniform Demand	***Gaussian Demand***
Accepted Bookings:		
Traditional RM Approach	82.0	82.0
Stochastic Joint Optimization	82.8	83.1
Difference	*1.0%*	*1.4%*
Revenues:		
Traditional RM Approach	\$24.9k	\$25.0k
Stochastic Joint Optimization	\$25.8k	\$25.9k
Difference	*3.4%*	*3.9%*

Table 4.10: Comparison between stochastic joint optimization and a traditional revenue management approach

The comparison is summarised in Table 4.10. The two methods lead to similar load factors. The load factor for the joint approach is only one point higher than

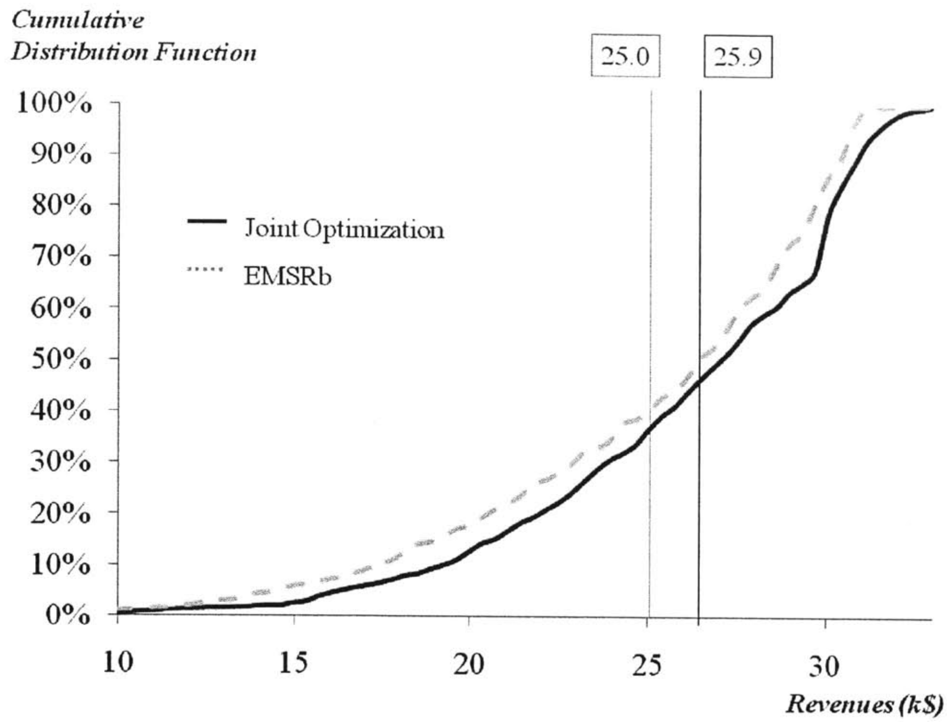


Figure 4-12: Cumulative distribution functions of the revenues, for Gaussian demands

the traditional revenue management method's load factor. However, even with the Gaussian distribution, which is the assumed distribution for the calculations in EMSRb, the stochastic joint optimization approach provides a 3.9% increase in revenues. Figure 4-12 shows the cumulative distribution functions of the revenues when the demand in Gaussian.

The joint optimization approach can help the airline reduce the risks of lower revenues. The best revenues are also obtained with the joint optimization approach.

4.6 Sensitivity to Forecasting Errors

There are three sets of input parameters to the stochastic joint optimization model: the total demand parameters, the probability parameters, and the demand uncertainty. Different estimates of these parameters will change the output of the model, the optimal set of fares and booking limit. In this section, we build on the previous numerical example to discuss the impact of the input parameters on the output of the stochastic model. We then run simulations in which the demand generated follows the original, true, input values, as displayed in Table 4.4, but where the fares and booking limits are the newly found stochastic outputs. This allows us assess the impact of forecasting errors on the final revenues.

For each cases of forecasting error, we then find the optimal solutions corresponding to the deterministic model and the fixed fare structure,traditional leg-based revenue management method. Simulations are then run again to determine the average revenues obtained with these new sets of fares and booking limit, allowing a comparison with the stochastic model.

Estimates of the Standard Deviations

The true parameters for the demand and the probability that a passenger chooses Fare Product *2* are unchanged and are outlined in Table 4.4. The true standard deviations remain the same: $s_1 = 20$ and $s_2 = 12$.

For the purpose of this discussion, we first assume that the true value of the demand uncertainty is unknown and assume different values. Based on the assumed s_1 and s_2, we use the stochastic model to determine the optimal set of fares and booking limits. This allows us to appreciate how a change in the uncertainty affects the model's output. Furthermore, we run simulations in which the demand generated follows the true parameters but the fares and booking limit imposed are those corresponding to the assumed demand uncertainty. In order words, the generated demand has a standard deviation of $s_1 = 20$ and $s_2 = 12$, but the assumed standard deviations used to determine the optimal solution are different.

First, we assume different levels of uncertainty in the first time frame, keeping s_2 unchanged. We then change the level of uncertainty in the second time frame, keeping s_1 to its true value of 20. The different values tested, as well as the corresponding optimal set of fares and booking limits are shown in Table 4.11 and Table 4.12. The two figures, Figure 4-13 and Figure 4-14, give a graphical overview of the impact of the change in uncertainty on the five output components. A small difference in s_1 only leads to, at most, a \$3 variation in the optimal fares, and barely affects the booking limit. However, if the uncertainty is largely underestimated, and halved for example, the variation in the optimal fares increases, and the suggested booking limit is much lower. The impact on the fares of the two time frames have a similar amplitude.

As the demand uncertainty increases in the second time frame, the first time frame's fares increase steadily, while the second time frame's fares decrease. As the uncertainty in $TF2$ increases, the stochastic model warrants lower fares to ensure the $TF2$ demand is high enough in the worst case scenario. Simultaneously, to improve the expected revenue overall, the model suggests a higher booking limit for the first time frame.

	Assumed s_1	***Assumed*** s_2	x_1^*	y_1^*	z_1^*	x_2^*	y_2^*	***Difference in Revenue (%)***
a/	**10**	12	378	193	69	481	236	-1.28
b/	**15**	12	381	195	73	482	236	-0.54
c/	**18**	12	383	196	73	482	236	-1.32
d/	**23**	12	384	197	73	483	237	-1.35
e/	**25**	12	384	197	73	484	238	-2.45
f/	**30**	12	384	196	73	485	239	-0.15

Table 4.11: Sensitivity analysis to uncertainty in the first time frame

The optimal set of fares and booking limit deduced with the assumed standard deviations are used to run simulations. The generated demand, however, follows the true parameters. For each combination of standard deviations, 1,000 samples were used to find the average revenue. The results were compared to the average revenue obtained when the inputs used for the stochastic model are the true parameters. The relative differences in revenues are summarized in Table 4.11 and Table 4.12. An

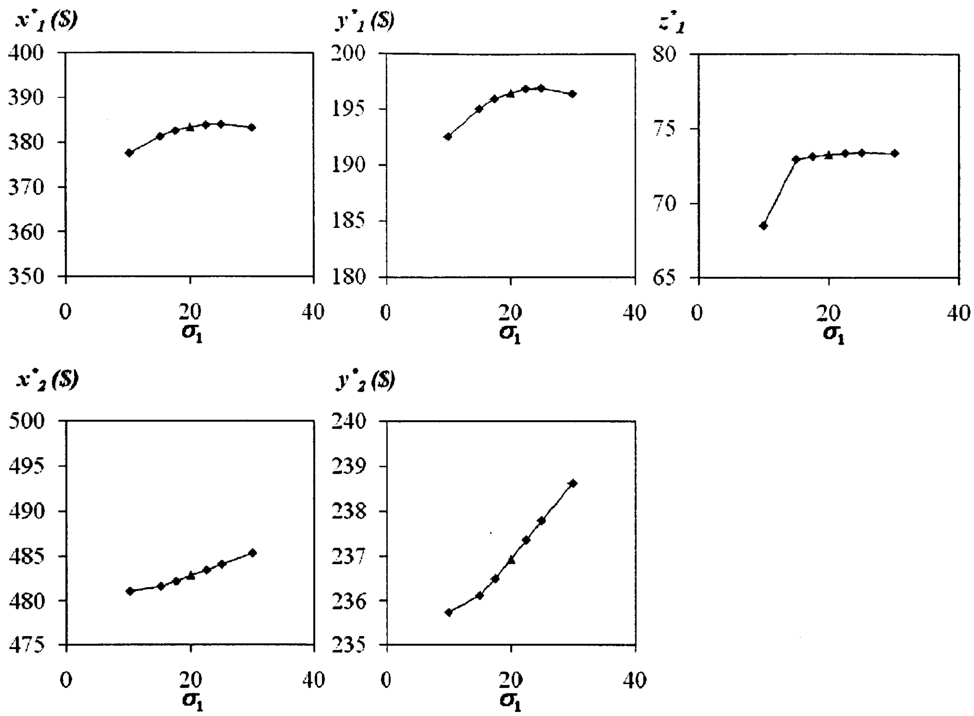


Figure 4-13: Changes in the optimal solution, given uncertainty in the first time frame

inadequate estimate of the demand uncertainty led to 0.1 to 2.0% decrease in revenues. As shown in Figure 4-15, a large difference between the true and assumed $TF2$ standard deviations led to the larger decrease in revenues.

	Assumed s_1	***Assumed*** s_2	x_1^*	y_1^*	z_1^*	x_2^*	y_2^*	***Difference in Revenue (%)***
a/	20	**4**	375	191	66	488	241	-1.05
b/	20	**7**	378	193	69	486	239	-1.22
c/	20	**10**	381	195	71	484	238	-2.15
d/	20	**14**	385	198	75	482	236	-0.83
e/	20	**17**	387	199	78	480	237	-0.53
f/	20	**20**	389	200	82	478	234	-1.10

Table 4.12: Sensitivity analysis to uncertainty in the second time frame

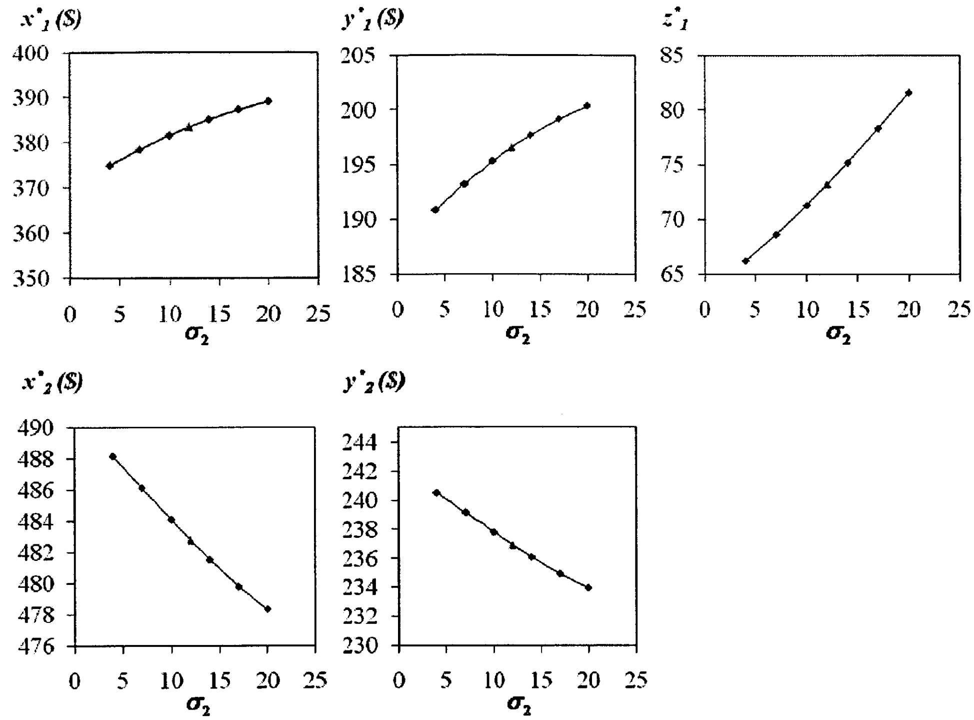


Figure 4-14: Changes in the optimal solution, given uncertainty in the second time frame

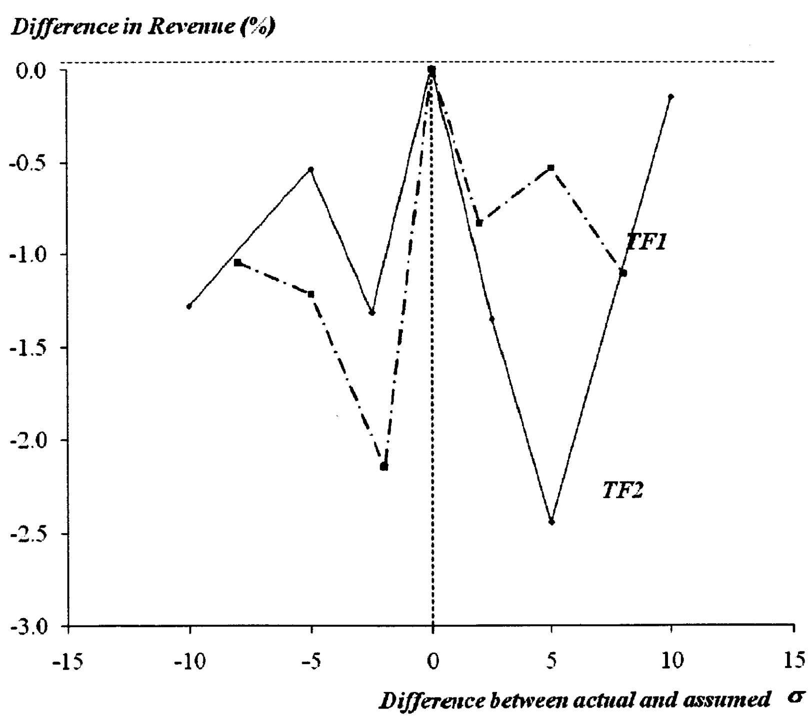


Figure 4-15: Changes in revenues due to uncertainty forecasting errors

Estimates of the Demand and Probability Parameters

We now assume that we know the true value of the standard deviations, but ignore the exact value of the other input parameters, one at a time. Based on guesses of α, β, a, b or c, we use the stochastic model to determine the optimal set of fares and booking limits. This should allows us to assess how critical forecasting is to the model's output. Furthermore, we run simulations in which the demand generated follows the true parameters but the fares and booking limit imposed are those corresponding to the assumed demand uncertainty. The resulting revenue is compared to the average revenue obtained when the inputs used for the stochastic model are the true parameters.

The relative differences in revenues are summarized in Table 4.13 and Table 4.14. A 10% error in the estimate of the one of the two time frames' demand parameter can lead to a 1.4 to 3.8% decrease in revenues. Over-estimating the parameter α seems to have the largest detrimental effect on the revenues. The fares are too high for the actual underlying demand.

An inadequate estimate of the probability function seems to have a smaller impact on the revenues. As shown in Table 4.14, the decrease in revenue varies between 0.4 and 2.2%.

Comparison with the deterministic model and the more traditional approach to revenue management method

We now compare the impact of forecasting errors on the revenues obtained with the stochastic solution, and ,on the one hand, the deterministic solution and, on the other hand, a traditional revenue management approach. For these simulations too, the demand generated follows the true parameters. The numerical results are summarized in Tables 4.15, 4.16, 4.17 and 4.18.

	Relative Diff. with True Parameter	Value	x_1^*	y_1^*	z_1^*	x_2^*	y_2^*	Diff. in Revenue (%)
α_1	+10%	149	418	220	78	488	241	-2.7
	-10%	122	350	174	69	477	233	-1.5
α_2	+10%	94	389	200	68	524	264	-3.8
	-10%	77	377	192	79	442	209	-1.4
β_1	+10%	0.48	358	179	70	482	236	-1.9
	-10%	0.39	415	218	77	484	238	-2.0
β_2	+10%	0.22	382	195	76	450	215	-1.4
	-10%	0.18	385	198	70	523	264	-2.0

Table 4.13: Sensitivity analysis to the demand input parameters

The expected revenues from the deterministic solution or traditional revenue management approach's solution are systematically lower than the stochastic solution. As shown in Tables 4.15 and 4.16, the stochastic model provides at least a 1.5% increase in revenues from the other two approaches when the uncertainty input parameter is off.

4.7 Performance Analysis

In this section, we extend the numerical study to analyze the performance of the stochastic model under different levels of demand and flight capacity. For each one of the input variables, two new levels are tested, as shown in Table 4.19. The inputs are changed one at a time.

For each combination of input variables, we determine the optimal set of fares and booking limit for the deterministic model, the stochastic model and the more traditional revenue management approach. We then run numerical simulations to determine the average revenue for each case considered, given the new demand, flight

	Relative Diff. with True Parameter	Value	x_1^*	y_1^*	z_1^*	x_2^*	y_2^*	Diff. in Revenue (%)
a_1	+10%	0.950	380	197	73	482	237	-1.3
	-10%	0.778	387	196	74	483	237	-2.2
a_2	+10%	-0.042	383	197	73	483	237	-0.4
	-10%	-0.034	383	197	73	483	237	-0.5
b_1	+10%	0.022	409	198	76	483	237	-1.3
	-10%	0.018	368	196	72	483	237	-0.8
b_2	+10%	0.018	384	197	71	512	239	-0.3
	-10%	0.014	383	196	75	458	235	-1.6
c_1	+10%	0.010	365	196	72	483	237	-2.1
	-10%	0.008	422	197	76	483	237	-0.6
c_2	+10%	0.009	382	196	75	445	236	-1.2
	-10%	0.007	384	197	71	520	238	-1.8

Table 4.14: Sensitivity analysis to the probability input parameters

capacity and deduced set of optimal fares and booking limit.

The stochastic approach to joint pricing and seat allocation performs consistently well, regardless of the input parameters.

Table 4.20 shows the percentage difference in average revenue between the stochastic model and the deterministic model for each one of the 27 input combinations of demand uncertainty and flight capacity. The stochastic model provides a 0.3 to 6.0% increase in revenue.

Table 4.21 summarizes the percentage difference in average revenue between the stochastic model and the traditional approach to revenue management, with a fixed fares tructure and the EMSRb seat allocation method. The stochastic joint optimization consistently outperforms the traditional approach by providing a 0.3 to

	Assumed s_1	s_2	*Diff. in revenues (%) from* deterministic solution	traditional RM solution
a/	**10**	12	2.93	3.18
b/	**15**	12	3.70	3.98
c/	**18**	12	2.89	1.38
d/	**23**	12	2.85	1.91
e/	**25**	12	1.71	1.45
f/	**30**	12	4.11	3.91

Table 4.15: Revenue comparison with a forecasting error in the first time's standard deviation

	Assumed s_1	s_2	*Diff. in revenues (%) from* deterministic solution	traditional RM solution
a/	20	**4**	3.17	7.18
b/	20	**7**	2.99	3.38
c/	20	**10**	2.02	2.54
d/	20	**14**	3.39	2.40
e/	20	**17**	3.71	4.83
f/	20	**20**	3.11	3.24

Table 4.16: Revenue comparison with a forecasting error in the second time's standard deviation

7.0% increase in revenue.

As the demand parameters and the flight capacity change, the expected revenues from the deterministic solution or traditional revenue management approach's solution are systematically lower than the stochastic solution. The numerical results are summarized in Tables 4.22 to 4.25.

4.8 Summary

In this chapter, we propose a stochastic approach to solving the joint pricing and seat allocation optimization problem. The demand is assumed to be a uniformly dis-

	Relative Diff. with True Parameter	*Value*	*Diff. in revenues (%) from deterministic solution*	*traditional RM solution*
α_1	+10%	149	0.3	2.1
	-10%	122	7.9	4.8
α_2	+10%	94	0.5	0.8
	-10%	77	5.6	3.1
β_1	+10%	0.48	4.5	3.5
	-10%	0.39	0.1	2.4
β_2	+10%	0.22	4.0	3.1
	-10%	0.18	2.5	3.9

Table 4.17: Revenue comparison with a forecasting error in the demand parameter

tributed random variable. Its mean remains a linear function of the lower fare. We use a geometrical analogy to determined the censored demand of each time frame and express the new objective revenue function. The stochastic approach relies on both the fares and the booking limit to maximize the total revenues generated by the two fare products over the two time periods considered.

The same numerical example as the one of Chapter 3 is used to illustrate this stochastic approach and run a few sets of simulations. The simulations confirmed the benefits of accounting for the demand uncertainty and the constraints imposed on the demand in the problem formulation. The simulations showed that the proposed approach performs well when compared to a traditional revenue management approach, with fixed fares and a leg-based seat allocation method. Finally, the proposed approach behaved well under the two types of demand distribution tested. The uniform and Gaussian distributions led to fairly similar results.

	Relative Diff. with True Parameter	Value	Diff. in revenues (%) from deterministic solution	traditional RM solution
a_1	+10%	0.950	1.4	1.5
	-10%	0.778	2.4	2.0
a_2	+10%	-0.042	2.9	3.2
	-10%	-0.034	3.5	4.2
b_1	+10%	0.022	4.3	2.7
	-10%	0.018	2.3	2.1
b_2	+10%	0.018	3.7	4.7
	-10%	0.014	2.5	3.4
c_1	+10%	0.010	2.1	1.1
	-10%	0.008	4.1	3.8
c_2	+10%	0.009	2.5	3.5
	-10%	0.007	1.9	2.2

Table 4.18: Revenue comparison with a forecasting error in the probability parameter

Capacity C	90	100	110
σ_1	10	20	30
σ_2	7	12	17
α_i	-10%		+10%
β_i	-10%		+10%
a_i	-10%		+10%
b_i	-10%		+10%
c_i	-10%		+10%

Table 4.19: Levels of demand and flight capacity

σ_1	σ_2	Capacity 90	100	110
10	7	0.8	2.1	3.3
10	12	2.1	2.9	3.8
10	17	2.0	3.5	5.8
20	7	0.7	0.8	4.5
20	12	2.1	3.5	3.2
20	17	2.4	3.9	5.6
30	7	0.3	2.5	4.1
30	12	1.3	1.9	5.2
30	17	3.6	3.2	6.1

Table 4.20: Increase in revenue between stochastic and deterministic joint optimization (%)

σ_1	σ_2	Capacity 90	100	110
10	7	2.5	3.0	2.5
10	12	3.5	2.1	2.4
10	17	1.8	4.1	2.9
20	7	3.1	2.5	3.9
20	12	4.5	3.3	0.3
20	17	4.0	3.1	3.2
30	7	6.5	5.7	2.7
30	12	5.9	2.6	3.9
30	17	7.0	4.7	4.6

Table 4.21: Increase in revenue between stochastic joint optimization and the traditional revenue management approach (%)

	TF1	*TF2*	Capacity 90	100	110
α_i	**148.5**	85.0	2.2	2.7	3.2
	121.5	85.0	1.4	3.5	4.7
	135.0	**93.5**	2.0	1.7	2.7
	135.0	**76.5**	3.5	3.3	5.1
β_i	**0.479**	0.200	2.5	3.8	4.9
	0.392	0.200	0.9	2.4	4.8
	0.435	**0.220**	10.4	3.6	5.8
	0.435	**0.180**	0.6	2.9	4.6

Table 4.22: Increase in revenue between stochastic and deterministic joint optimization (%)

	TF1	*TF2*	Capacity 90	100	110
α_i	**148.5**	85.0	4.2	2.6	1.4
	121.5	85.0	5.4	5.0	5.4
	135.0	**93.5**	6.2	3.8	3.8
	135.0	**76.5**	3.3	0.7	0.9
β_i	**0.479**	0.200	6.0	3.4	4.8
	0.392	0.200	1.7	2.2	0.7
	0.435	**0.220**	10.9	1.1	1.6
	0.435	**0.180**	4.6	4.8	5.3

Table 4.23: Increase in revenue between stochastic joint optimization and the traditional revenue management approach (%)

	TF1	*TF2*	Capacity 90	100	110
a_i	**0.950**	-0.038	2.5	4.2	4.3
	0.778	-0.038	1.5	3.2	5.6
	0.864	**-0.042**	0.6	2.0	3.7
	0.864	**-0.034**	2.4	2.2	4.7
b_i	**0.022**	0.016	3.0	6.7	1.0
	0.018	0.016	2.6	2.9	6.6
	0.020	**0.018**	1.2	4.0	0.7
	0.020	**0.014**	3.9	2.8	2.8
c_i	**0.010**	0.008	1.0	2.1	4.7
	0.008	0.008	2.7	4.1	4.4
	0.009	**0.009**	1.2	2.5	5.1
	0.009	**0.007**	2.5	2.6	4.9

Table 4.24: Increase in revenue between stochastic and deterministic joint optimization (%)

	TF1	*TF2*	Capacity 90	100	110
a_i	**0.950**	-0.038	4.2	4.3	2.0
	0.778	-0.038	2.4	2.5	4.6
	0.864	**-0.042**	3.1	2.7	1.1
	0.864	**-0.034**	6.5	2.7	1.6
b_i	**0.022**	0.016	5.4	4.6	1.7
	0.018	0.016	3.6	3.7	17.0
	0.020	**0.018**	3.4	5.3	0.1
	0.020	**0.014**	3.2	2.5	1.4
c_i	**0.010**	0.008	3.0	3.6	3.6
	0.008	0.008	6.4	5.6	3.0
	0.009	**0.009**	4.8	4.1	2.8
	0.009	**0.007**	4.4	4.1	2.5

Table 4.25: Increase in revenue between stochastic joint optimization and the traditional revenue management approach (%)

Chapter 5

Heuristics for the multiple-period problem

The proposed stochastic model for the two-product, two-time frame joint pricing and seat allocation optimization problem provides a significant increase in revenue from the deterministic model or the tested traditional revenue management method given a traditional, yet optimized, fare structure. The natural next step to improve the model consists in extending it to additional time frames. Dividing the selling horizon into more but smaller time frames would allow an increased number of changes in the fares, which should, to a certain point, help match more closely the changing characteristics of the passengers over the booking process. Ultimately, solving for the multiple-time frame optimization problem should further enhance the revenues.

Extending the deterministic model to additional time frames does not present any problem. The stochastic model, however, will not scale up as easily. With each time frame, we add not only another uncertainty parameter and but also three decision variables. Furthermore, as seen in the previous chapter, the censored demand of a time frame is dependent on the demand that materialized in the previous time frame. As we increase the number of time intervals, the demands become very quickly more intricate.

In this chapter, we describe how the stochastic model can be extended to additional time frames. In the first section, we go over the assumptions and introduce new notations. We then present a recursive approach to determine each time frame's censored demand. As shown in an example, the problem complexity increases very rapidly and we therefore propose alternative heuristics to overcome the curse of dimension in Section 5.3. Then, we use a numerical example to illustrate the heuristics and compare their performance with the stochastic and deterministic models.

5.1 Notations and assumptions

The scope of the problem remains the same as in the previous chapter, with a single carrier, a single flight, a single OD market environment, and a fixed capacity of C seats.

Again, two fare products are offered, Fare Product *1* and Fare Product *2*. The prices of the two products can change at the start of each of the k time frames constituting the booking period.

The airline can limit the total number of seats to be sold by the end of each time frame. All the unsold seats are available for booking in the later time frames.

The notations and assumptions of this chapter are similar to the ones used previously.

- x_i is the price of Fare Product *1* in TFi
- y_i is the price of Fare Product *2* in TFi. We impose that for all i, $y_i \leq x_i$.
- z_i is the booking limit corresponding to TFi. We have $z_k = C$.
- $n_{total,i}$ is the combined underlying demand for Fare Product *1* and *2* in TFi, and $n_{total,i} = \mu_i(y_i) + \varepsilon_i$.

- The expectation of the total demand in TFi, denoted μ_i, is a linear function of the lower price: $\mu_i(y_i) = \alpha_i - \beta_i y_i$, with $\alpha_i, \beta_i \geq 0$.

- The random variable ε_i is uniformly distributed: $\varepsilon_i \sim U[-\sigma_i, \sigma_i]$.

- The minimum and maximum values that the underlying demand $n_{total,i}$ can take are noted $\mu_{min,i}$ and $\mu_{max,i}$.

- $n_{accepted,i}$ is the total number of accepted bookings in TFi, also called the censored demand, when all booking limits are enforced. For example, $n_{accepted,2} = min\left[z_2 - min\left(z_1, n_{total,1}\right), n_{total,2}\right]$. Furthermore, we define $n_{accepted,0}$ as the null function.

- $\overline{n}_{accepted,i}$ is the average total censored demand for both products in TFi, when all booking limits are enforced.

- f_i is the probability density function of the sum of the censored demands $\sum_{j=1}^{i} \overline{n}_{total,j}$ of $TF1$ to TFi.

- u_i is the probability density function of the total underlying demand in TFi.

- p_i is the probability that a random passenger chooses Fare Product *1* in TFi.

- R_i is the total revenues generated by the combined sale of the two fare products in TFi. $\overline{R}_i$ represent the expected revenues.

- R_{total} is the total revenues generated by the sale of the two fare products over the entire booking period. $\overline{R}_{total}$ is the total expected revenues.

5.2 Expected total revenues and the expected censored demands

Expected total revenues

The objective function is the expected value of the total revenues generated by the sale of the two products over the course of the k time frames considered. The total revenue function is given by:

$$R_{total} = \sum_{i=1}^{k} R_i$$

with, $\forall i \in [1, \ldots, k]$,

$$R_i = \begin{cases} n_{total,i} \left[p_i x_i + (1 - p_i)\, y_i\right], & \text{if } n_{total,i} < z_i - \sum_{t=1}^{i-1} n_{accepted,t}; \\ \left(z_i - n_{accepted,i-1}\right) \left[p_i x_i + (1 - p_i)\, y_i\right], & \text{otherwise.} \end{cases}$$

The expected total revenue function for two time frames can be generalized to k time frames:

$$\overline{R}_{total} = \sum_{i=1}^{k} \overline{n}_{accepted,i} \left[x_i p_i + (1 - p_i)\, y_i\right]$$

The challenge consists in expressing the censored demand for each time frame, given the demand and booking limits of all the prior time frames.

Determining the expected censored demands

The rather straight-forward geometrical approach used in the previous chapter to find the expected values of the censored $TF2$ demand can be extended to three or more dimensions, with some modifications. For all TFi with $i \geq 2$, we shall still be able to define four possible regions and analyse them one by one to determine the TFi censored demand. However, we will have to introduce convolution products to express the probability density function of the sum of the previous time frames' censored demands. This step was not necessary in Chapter 4 because the probability function of the censored $TF1$ demand was simply a truncated uniform distribution. However, the probability density function of the sum of $TF1$ and $TF2$ censored demands is altogether more complex, even though we assumed the underlying demands are uniform distributions. It is thanks to this simplified assumption that we can continue deriving the expression of the censored demands for more time frames.

A generalized methodology to find the censored demand for all time frames is described below. It is a recurring method.

<u>Initialization - *TF1*</u>

1. Define the probability density function f_1 of the censored total demand in *TF1.*

2. Deduct the expected censored total demand $\overline{n}_{accepted,1}$. From Chapter 4, we have:

$$\overline{n}_{accepted,1} = \frac{z_1 + \mu_1}{2} - \frac{(\mu_1 - z_1)^2}{4\sigma_1} - \frac{\sigma_1}{4}$$

<u>Recurrence - *TFk* for $k \geq 2$</u>

1. Find the probability density function f_{k-1} of the sum $\sum_{i=1}^{k-1} \overline{n}_{accepted,i}$ of all the previous time frames' censored total demand.

   If $k \geq 3$, then:

$$f_{k-1}(x) = \begin{cases} f_{k-2} \star u_{k-1}(x) & \text{if } x \in \left[\sum_{i=1}^{k-1} \mu_i, z_{k-1}\right[, \\ \int_{z_{k-1}}^{\infty} f_{k-2} \star u_{k-1}(t)\, dt & \text{if } x = z_{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

   If $k = 2$, then f_1 is simply the function found in the first step of the initialization process.

2. Define the four possible regions, as shown in Figure 5-1. The x-axis is the sum of the previous time frames' censored demand, $\sum_{j=1}^{k-1} \overline{n}_{accepted,j}$. The y-axis represents the underlying demand for the time frame of interest, TFk. The two constraints considered are z_{k-1} and z_k.

   Region I is not affected by either constraint.

   In Region II, the sum of the censored demand from *TF1* to *TFk-1* is lower than the booking limit z_{k-1}. The other constraint, z_k, nonetheless, applies to the sum of bookings from *TF1* to *TFk*.

   In Region III, the booking limit z_{k-1} applies to sum of the accepted bookings from *TF1* to *TFk-1*: the sum is censored and only z_{k-1} bookings are accepted. The sum of the censored demand from *TF1* to *TFk* is, however, lower than the capacity C. The *TFk* demand is therefore not capped.

   In Region IV, both constraints apply: the number of accepted bookings from the first k-1 time frames is equal to z_{k-1} and the bookings in the k^{th} time frame are censored to $z_k - z_{k-1}$.

   For each of one these four regions, we can find the ordinate of the barycentre, and then deduct $\overline{n}_{accepted,k}$.

3. Find the ordinate of Region II's barycentre when there are no constraints.

$$g(x) = \begin{cases} \int_x^{\infty} f_{k-1}(t)\, dt\,, & \text{if } x \in [z_2 - n_2 - \sigma_2; z_1] \\ 0, & \text{otherwise} \end{cases}$$

   Deduct the area of Region II: $Area = \int_{z_2 - z_1}^{n_2 + \sigma 2} g(y)\, dy$. Normalize function g, and note it G. Obtain the coordinate of the region's barycentre, $Y_{noconstraints} = \int_{z_2 - z_1}^{n_2 + \sigma 2} yG(y)\, dy$.

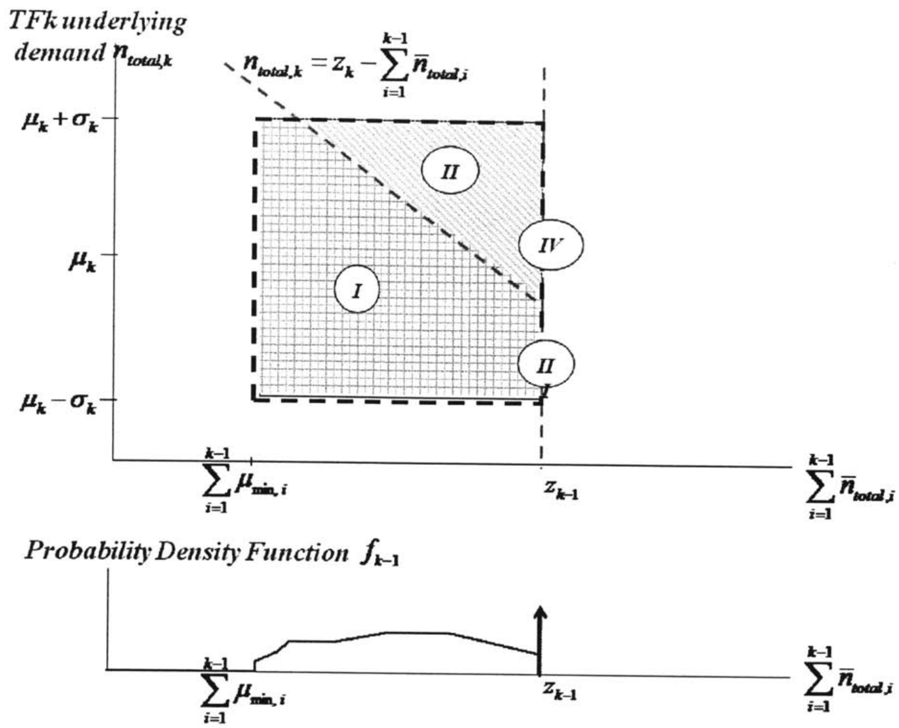


Figure 5-1: Constraints on the bookings in the two time frames

4. Find the ordinate of Region II's barycentre when the constraint z_k is applied.

$$h(x) = \begin{cases} (x - z_3) f_{k-1}(x), & \text{if } x \in [z_2 - n_2 - \sigma_2; z_1] \\ 0, & \text{otherwise} \end{cases}$$

Normalize this function and note it H. Deduct the coordinate of the region's barycentre, $Y_{constraints} = \int_{z_2 - z_1}^{n_2 + \sigma 2} yH(y)\, dy$.

5. Substitute to find $Y_{I,II}$, the ordinate of Regions I and II's combined barycentre:

$$Y_{I,II} = \mu_k + (Y_{constraints} - Y_{noconstraints}) \frac{Area}{(z_{k-1} - min)\, 2\sigma_k}$$

6. Determine the ordinate of Region III's barycentre.

$$Y_{III} = \frac{1}{2}(z_k - z_{k-1} - \mu_{min,k})$$

The length of this region is $(z_k - z_{k-1} - \mu_{min,k})$.

7. Determine the ordinate of Region IV's barycentre. In this region, the z_k booking limit applies. Therefore, all the TFk bookings are capped to $z_k - z_{k-1}$, and $Y_{IV} = z_k - z_{k-1}$. The length of this region is $2\sigma_k - (z_k - z_{k-1} - \mu_{min,k})$.

8. Deduct the ordinate of Regions III and IV's combined barycentre:

$$Y_{III,IV} = Y_{III}\frac{2\sigma_k - (z_k - z_{k-1} - \mu_{min,k})}{2\sigma_k} + Y_{III}\frac{(z_k - z_{k-1} - \mu_{min,k})}{2\sigma_k}$$

9. Deduct $\overline{n}_{accepted,k}$ from $Y_{I,II}$ and $Y_{III,IV}$.

The function f_i becomes more complex with each iteration. The initial advantage presented by simplicity of the uniform distribution soon disappears as we go from one truncated convolution product to another. Accordingly, the analysis of Region II, steps 3 and 4 of the recurrence, becomes fairly complicated. The resulting $Y_{I,II}$, the ordinate of Regions I and II's combined barycentre, grows larger with each iteration, involving an increasingly large number of variables. As the following example shows, the approach is very quickly impractical. Heuristics may thus be more suitable to the multiple-time frame joint pricing and seat allocation optimization problem.

Example with three time frames

The proposed methodology gives us the expected censored demand in *TF3*:

$$\overline{n}_{accepted,3} = (1 - \nu)\,Y_{I,II} + \nu Y_{III,IV}$$

$$\text{with}\begin{cases} \nu &= \dfrac{z_1 - z_2 + \mu_{max,2}}{4\sigma_1\sigma_2} \cdot \dfrac{2\mu_{max,1} + \mu_{max,2} - z_2 - z_1}{2} \\ Y_{I,II} &= \mu_3 - \dfrac{\left(\mu_{max,3} - z_3 + z_2\right)^3}{16\sigma_3\left(z_2 - \mu_{min,1} - \mu_{min,2}\right)^2} \cdot \dfrac{4\sum\left(\mu_{max,i}\right) + 3\left(z_3 - z_2\right) + \mu_{max,3}}{3\sum\left(\mu_{max,i}\right) + 2\left(z_3 - z_2\right) + \mu_{max,3}} \\ Y_{III,IV} &= \dfrac{z_3 - z_2 + \mu_3}{2} - \dfrac{\left(\mu_3 - z_3 + z_2\right)^2}{4\sigma_3} - \frac{\sigma_3}{4} \end{cases}$$

The expression for $Y_{I,II}$ is already very complex. On the other hand, $Y_{III,IV}$'s form is not altered as we move from one time frame to the other. The main difficulty does arise from the increasingly complex probability density function f_i of the sum of the censored demands.

5.3 Heuristics for Cases with more than Two Time Frames

In light of the fact that the probability density function f_i of the sum of the censored demands is the main source of complexity, we derive three heuristics to the joint pricing and seat allocation optimization. Each heuristic tackles the problem in a different way.

With the first heuristic, we altogether bypass the complexity induced by f_i by simply assuming that the censored demands are independent. Convolutions are no longer necessary, and the recurrence simply becomes the same as the initialization.

With the second heuristic, we also focus on the first step of the recurrence. However, the change is more subtle. We smooth out f_i by imposing that it be a uniform distribution. This greatly simplifies the subsequent steps 3 and 4 of the recurrence.

With the third heuristic, we acknowledge the simplicity of the problem formulation when the number of time periods is kept to two and use dichotomy to revert to this case systematically.

Heuristic One - Non-nested Inventory

The nested characteristic of our model is at the root of the increased complexity of the objective function. The expected censored demands are all interrelated. Indeed, the booking limit z_i applies not only to the demand in TFi but also to the sum of the censored demands from all the anterior time frames. As a consequence, the censored demand for the i^{th} time frame is a function of its own and all the previous time frame's booking limits z, expected demand μ, and demand uncertainty σ. The expression does not scale up.

Relaxing the nested assumption and imposing a partitioned inventory rule instead could greatly simplify the analysis. In the first proposed heuristic, the time frames are no longer nested: if the demand during a time frame is lower than the booking limit, the remaining unsold seats are considered lost and are not available for booking in subsequent time frames. Under this rule, the objective function is:

$$R_{total} = \sum_{i=1}^{k} R_i$$
$$\text{with, } \forall i \in [2, \ldots, k],$$
$$R_i = \begin{cases} n_{total,i}\left[p_i x_i + (1-p_i)\, y_i\right], & \text{if } n_{total,i} < z_i; \\ z_i\left[p_i x_i + (1-p_i)\, y_i\right], & \text{otherwise.} \end{cases}$$

We then have $\sum_{i=1}^{k} z_i = C$. The revenue function R_i is identical to the previous chapter's $TF1$ revenue function. Therefore, the expected revenue function for all time frames is given by the simple equation:

$$\overline{R}_{total} = \sum_{i=1}^{k} \left[p_i x_i + (1-p_i)\, y_i\right] \overline{n}_{accepted,i}$$

$$\text{with,}\forall i,\quad \overline{n}_{accepted,i} = \begin{cases} \mu_i, & \text{if } z_i \geq \mu_i + \sigma_i; \\ \dfrac{z_i+\mu_i}{2} - \dfrac{(\mu_i - z_i)^2}{4\sigma_i} - \dfrac{\sigma_i}{4}, & \text{if } z_i \in [\mu_i - \sigma_i; \mu_i + \sigma_i]; \\ z_i, & \text{otherwise.} \end{cases}$$

The objective function is greatly simplified.

We shall now analyze how this simplification affects the outcome of the optimization. The simplicity of the heuristic is due to the fact that we ignore a portion of possible bookings, as shown in red in Figure 5-3. The extra revenue generated when unsold seats from previous time frames are available to ulterior time frames is not taken into account by the new problem formulation. If the later time frames' average fare is higher than the earlier time frames' average fare, then the optimization process will either tend to increase the booking limit for the later time frame or tend to decrease the fares in an attempt to increase the average censored demand. However, since $\sum_{i=1}^{k} z_i = C$, this will have the consequence of lowering the earlier booking limit or increasing the fares. Thus, this heuristic will be more conservative than the stochastic model in the first time frames. The heuristic will recommend a lower booking limit early on to protect more seats for the later time frames. Furthermore, the optimal fares set by the heuristic are likely to be higher than those proposed by the stochastic model. As a result, the total accepted demand will not be as high. However, since the fares are likely to be higher than with the stochastic model, it is difficult to assess how the revenues will decrease.

When implementing this heuristic, the booking limits should ideally be "re-nested". The first time frame's booking limit z_1 remains unchanged, but for TFi with $i > 1$ the booking limit should be set to $\sum_{k=1}^{i} z_k$. By doing so, we mitigate the revenue loss by ensuring that unsold seats from previous time frames become available to ulterior

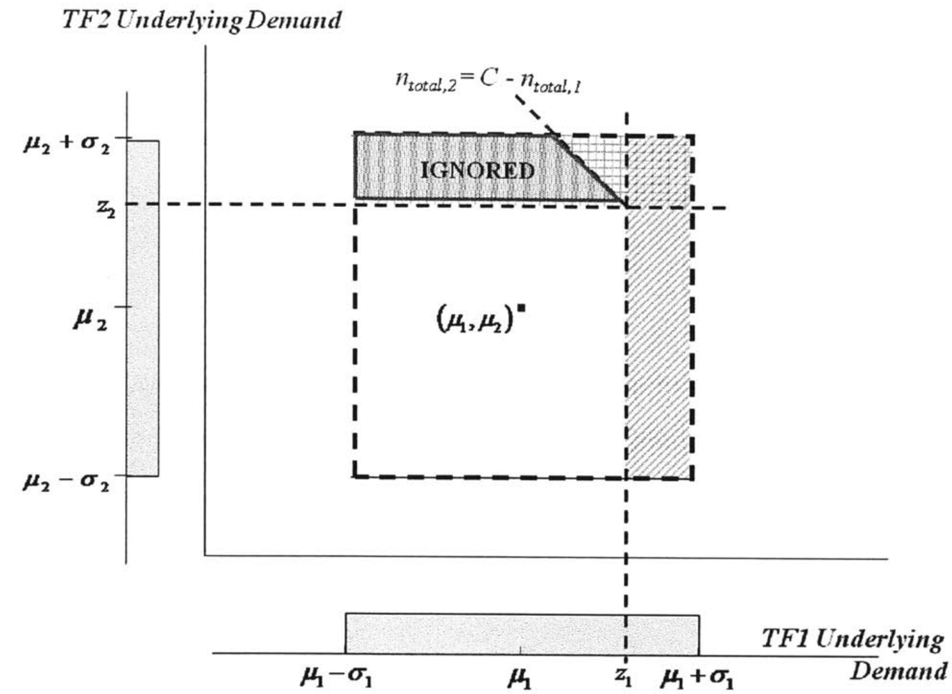


Figure 5-2: Possible bookings ignored by the first heuristic

time frames.

Heuristic Two - Simplified Probability Density Function

With the first heuristic, we simplify the analysis by assuming the demands for the different time frames are independent of each other. This approach is conservative and ignores some of the potential revenues. Another alternative would consist in simplifying the probability density function of the sum of the censored demand while keeping the nested structure of the demand.

Much of the difficulty arises from the increasing complexity of the function f_i, the probability density function the sum of the censored demands $\sum_{j=1}^{i} \overline{n}_{accepted,j}$ of $TF1$ to TFi. As described in the first step of the recurrence, in Section 5.2, the function is the convolution product of the previous time frame's f_{i-1} and a uniform distribution. The complexity of convolution product increases with the index of the time frame, and reflects in the following steps of the recurrence. Therefore, one alternative would

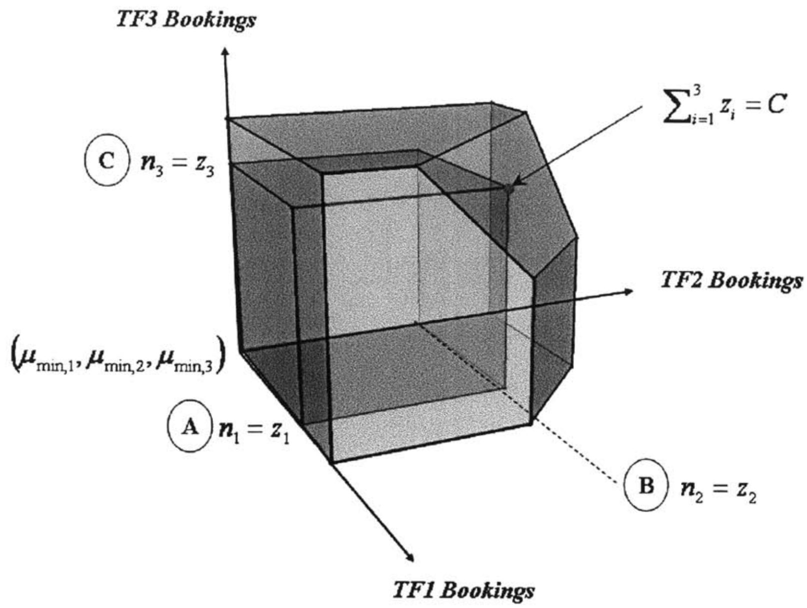


Figure 5-3: Possible bookings ignored by the first heuristic

consist in simplifying this function.

For example, the convolution product can be replaced by a new uniform distribution: For $k \geq 2$,

$$f_{k-1}(x) = \begin{cases} \dfrac{1}{\prod_{i=1}^{k-1} \sigma_i} & \text{if } x \in \left[\sum_{i=1}^{k-1} \mu_{min,i}, z_{k-1}\right[, \\ \dfrac{\sum_{i=1}^{k-1} \mu_{max,k} - z_{k-1}}{\prod_{i=1}^{k-1} \sigma_i} & \text{if } x = z_{k-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 5-4 illustrates the change in the probability density function this imposes.

This change enables us to draw insights from the two period example. The new probability density function f_{i-1} has the form of f_2. Therefore, we can use the results found for the censored demand for $TF2$ in the two-time period problem and generalize

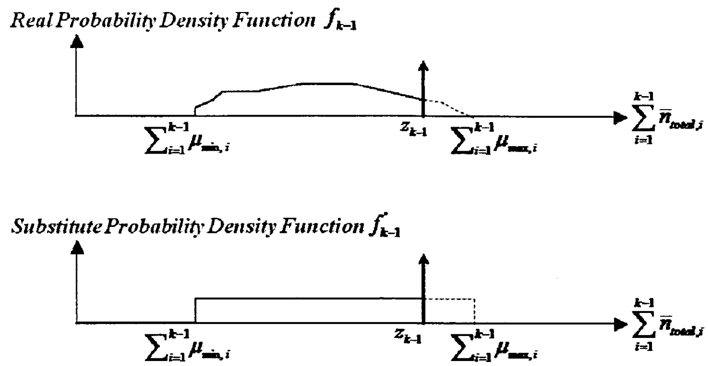


Figure 5-4: Change in the probability density function

it. The expected revenues are therefore:

$$\overline{R}_{total} = \sum_{i=1}^{k} \overline{n}_{accepted,i} \left[x_i p_i + (1 - p_i)\, y_i \right]$$

with, for $k = 1$, $\quad \overline{n}_{accepted,1} = \dfrac{z + \mu}{2} - \dfrac{(\mu - z)^2}{4\sigma} - \dfrac{\sigma}{4}$

and, for $k \geq 2$,

$$\overline{n}_{accepted,k} = \left[\mu_k - \frac{(\mu_k + \sigma_k - z_k + z_{k-1})^3}{12\sigma_k\,(z_{k-1} - \mu_{equi} + \sigma_{equi})} \right] \frac{(z_{k-1} - \mu_{equi} + \sigma_{equi})}{2\sigma_{equi}}$$
$$+ \left[\frac{z_k - z_{k-1} + \mu_k}{2} - \frac{(\mu_k - z_k + z_{k-1})^2}{4\sigma_k} - \frac{\sigma_k}{4} \right] \frac{\mu_{equi} + \sigma_{equi} - z_{k-1}}{2\sigma_{equi}}$$

where $\mu_{equi} = \sum_{i=1}^{k-1} \mu_k,$ and $\sigma_{equi} = \sum_{i=1}^{k-1} \sigma_k.$

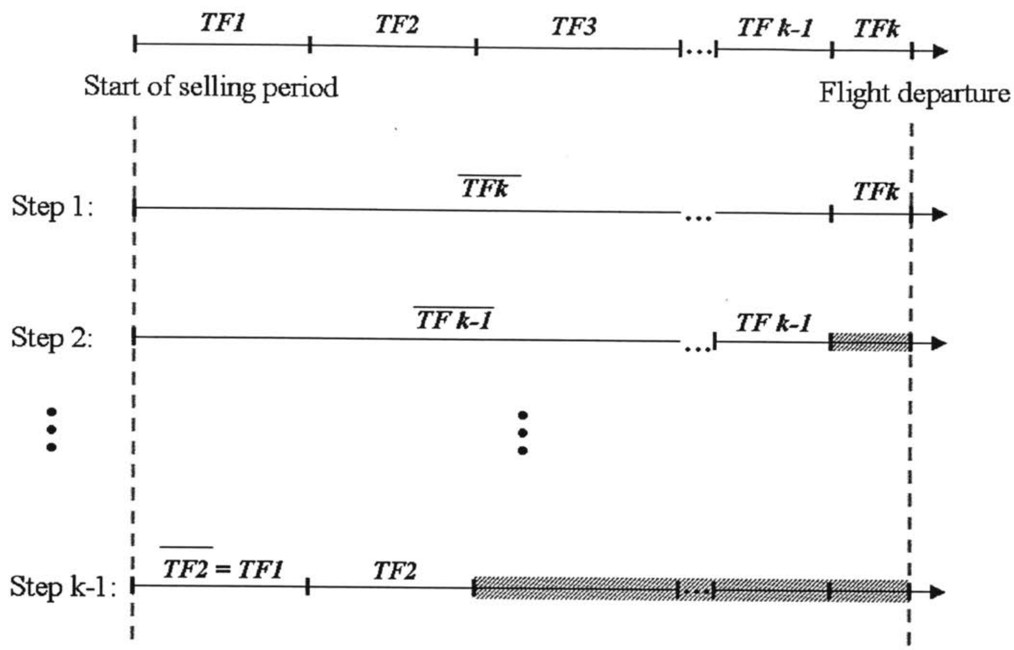
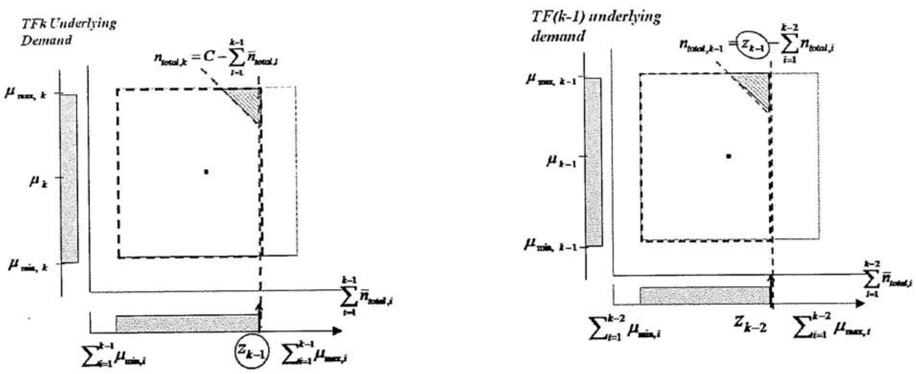


Figure 5-5: Dichotomy

Heuristic Three - Dichotomy

The two-time period problem formulation was fairly simple. We therefore suggest the use of dichotomy to unravel the process and revert to the two-time period case systematically.

We first divide the selling horizon into two time frames, *TF1* and *TF2*. We determine the set of fares and booking limit z_1 that result in the maximum total expected revenues from those time frames, based the two time period joint optimization approach described earlier in this chapter. We then further divide *TF1* into two new intervals. The joint optimization approach can be applied again to those two new intervals, once the total number of seats available to the demands from those time frames is set to z_1. We repeat this step $k-2$ times, until all the time frames are covered.



(a) First Step (b) Second Step

Figure 5-6: First steps of the heuristic

5.4 Performance Analysis

As we have in the previous chapters, we use a numerical example to illustrate the approaches newly introduced and assess their various performance. The numerical example in this chapter will have three time frames.

We expand the results from Chapter 3 to find the deterministic model's optimal solution to this multiple-time frame joint pricing and seat allocation problem. The revenues obtained with the deterministic solution will be used as a benchmark. The optimal solutions of the other four approaches, the stochastic model and its three heuristics, will be compared to this first model's output and simulations will then be run to assess the performance of each approach.

The assumed parameters for the demands in the three time frames considered are given in Table 5.1. The flight capacity is 100 seats. All else being held constant, the total demand decreases as the time frame considered is closer to the departure date. On the other hand, the passengers become less price sensitive. The standard deviation decreases with the total demand. At the same time, all else being equal, people seem to be more inclined to choose the higher fare product as we get closer to departure, even if there is not much difference between the last two time frames. All these are characteristics observed when analyzing real booking data.

	TF1	*TF2*	*TF3*
Total Demand $n_{total,t}$	$\alpha_1 = 110$ $\beta_1 = 0.35$	$\alpha_2 = 66$ $\beta_2 = 0.19$	$\alpha_3 = 44$ $\beta_3 = 0.10$
Probability p_t	$a_1 = 1.023$ $b_1 = 0.022$ $c_1 = 0.001$	$a_2 = 0.512$ $b_2 = 0.019$ $c_2 = 0.009$	$a_3 = -0.022$ $b_3 = 0.016$ $c_3 = 0.008$
Standard deviation	$s_1 = 16$	$s_2 = 10$	$s_3 = 7$

Table 5.1: Parameters for the demand functions

Deterministic Model

All the results from Chapter 3 can be extended to the multiple-time frame problem. The booking limits z_i are all redundant variables. The optimization problem is given below:

$$\text{Maximize } R = \sum n_{total,i} p_i x_i + n_{total,i} (1 - p_i) y_i$$
$$\text{Subject to } \sum \alpha_i - \beta_i y_i - C \leq 0$$

The optimal deterministic solution to this joint pricing and seat allocation problem is obtained with Matlab. The maximum total revenue is reached with the fares and booking limit outlined in Table 5.2. In the simulations we will set the booking limits as follows: $z_1 = \alpha_1 - \beta_1 y_1^*$ and $z_2 = \sum \alpha_i - \beta_i y_i^*$.

Based on the assumed parameters, we can deduce the censored demand and revenues implied by those fares and nested booking limits.

	TF1	*TF2*	*TF3*
Fare Product 1	$x_1^* = \$336.8$	$x_2^* = \$376.1$	$x_3^* = \$484.5$
Fare Product 2	$y_1^* = \$174.4$	$y_2^* = \$185.7$	$y_3^* = \$239.2$
Implied booking limit	$z_1^* = 49$	$z_2^* = 79$	

Table 5.2: Optimal solution for the deterministic model

	TF1	*TF2*	*TF3*	*Total*
Average fare ($)	237	265	361	
Underlying Demand	48.6	30.4	21.0	100.0

Table 5.3: Fares and demand implied by the deterministic model's output

Stochastic Model

The censored demand for the first two time frames are detailed in the previous chapter. We will use the equations derived from the recurring methodology proposed in Section 5.2 to determine the expected censored demand for the third and last time frame. This enables us to find the stochastic model's exact optimal solution for this three-time frame problem. Table 5.4 shows this optimal solution.

The stochastic model's optimal fares are higher than the deterministic model's fares in all three time frames. This is consistent with the findings of Chapter 4: the fares are used as a means of lowering the underlying demand and lessen the impact of the flight capacity on the censored demand. However, the gaps closes-in as we get to the last time frame. The load factor will subsequently be much lower than when the deterministic output is implemented. However, because seats are protected for the later arriving but higher revenue passengers, and because the fares are higher on average, the revenues should be greater overall. From Chapter 4's numerical example, we can expected an increase in revenues around 2-3% .

Based on the assumed parameters, we can deduce the censored demand and rev-

	TF1	*TF2*	*TF3*
Fare Product 1	$x_1^* = \$368.7$	$x_2^* = \$401.3$	$x_3^* = \$498.9$
Fare Product 2	$y_1^* = \$195.7$	$y_2^* = \$202.7$	$y_3^* = \$248.8$
Implied booking limit	$z_1^* = 65$	$z_2^* = 87$	

Table 5.4: Optimal solution for the stochastic model

	TF1	*TF2*	*TF3*	*Total*
Average fare ($)	269	290	375	
Underlying Demand	41.1	27.1	20.1	88.3
Accepted Bookings	40.8	25.5	18.1	84.4
Revenues (k$)	$\overline{R}_1^* = \mathbf{11.0}$	$\overline{R}_2^* = \mathbf{7.4}$	$\overline{R}_3^* = \mathbf{6.8}$	$\overline{R}_{total}^* = \mathbf{25.2}$

Table 5.5: Fares and demands implied by the stochastic model's output

enues implied by those fares and booking limit, summarized in Table 5.5. The optimal fares and booking limit determined with the stochastic approach should result in a total expected censored demand of 84.6, or an average load factor of 84.6%. The expected total revenue is $25.2k.

Heuristic One

With the first heuristic, we assume that the inventory is partitioned instead of being nested across time frames and we solve for the corresponding joint optimal solution. The optimal set of fares and booking limits is shown in Table 5.6.

The first time frame's fares are very close to the stochastic model's $TF1$ fares. However, the other ones are higher than for the stochastic model and thus much higher than they were in the deterministic case. The booking limits on the other hand are very low. At most, we will have accepted $z_1 + z_2 = 76$ passengers by the end of the second time frame, versus 87 in the stochastic case. The load factor corresponding to

	TF1	*TF2*	*TF3*
Fare Product 1	$x_1^* = \$369.0$	$x_2^* = \$408.9$	$x_3^* = \$519.3$
Fare Product 2	$y_1^* = \$195.8$	$y_2^* = \$207.7$	$y_3^* = \$262.3$
Implied booking limit	$z_1^* = 46$	$z_2^* = 76$	

Table 5.6: Optimal solution for the non-nested heuristic

	TF1	*TF2*	*TF3*	*Total*
Average fare ($)	269	297	396	
Underlying Demand	41.1	26.1	18.8	86.0

Table 5.7: Fares and demands implied by the non-nested heuristic's output

the first heuristic is expected to be very low.

Heuristic Two

For the second heuristic, we revert back to the nested inventory structure but simplify the probability density function. The optimal solution for this second heuristic is shown in Table 5.8.

	TF1	*TF2*	*TF3*
Fare Product 1	$x_1^* = \$369.9$	$x_2^* = \$396.3$	$x_3^* = \$503.7$
Fare Product 2	$y_1^* = \$196.4$	$y_2^* = \$199.3$	$y_3^* = \$252.0$
Implied booking limit	$z_1^* = 69$	$z_2^* = 82$	

Table 5.8: Optimal solution for the second heuristic

	TF1	*TF2*	*TF3*	*Total*
Average fare ($)	270	285	380	
Underlying Demand	40.9	27.7	19.8	88.4

Table 5.9: Fares and demands implied by the second heuristic's output

These fares are very closed to those determined by the stochastic model. There is at most a $5 difference. The second booking limit, however, is a bit lower than it were with the stochastic model.

Heuristic Three

The third heuristic is based on dichotomy. The optimal solution is outlined in Table 5.10.

The other two heuristics mostly had higher fares than the stochastic model. This last heuristic, on the contrary, seems to warrant much lower fares, especially early on. In the three time frames, the fares are between the deterministic and stochastic optimal fares and this is combined to the highest booking limits so far: 74 and 91 for the first and second time frame, respectively. The lower fares and high booking limits ought to result in a higher number of accepted bookings in the first two time frames. Nevertheless, since it is also accompanied by a lower average fare, the impact on the total revenues is unsure. In the last time frame, there should be fewer accepted bookings due to the limited number of remaining seats. The $TF3$ revenue may very be low.

	TF1	***TF2***	***TF3***
Fare Product 1	$x_1^* = \$352.0$	$x_2^* = \$381.8$	$x_3^* = \$491.9$
Fare Product 2	$y_1^* = \$184.6$	$y_2^* = \$189.5$	$y_3^* = \$244.7$
Implied booking limit	$z_1^* = 74$	$z_2^* = 91$	

Table 5.10: Optimal solution for the dichotomy heuristic

	TF1	***TF2***	***TF3***	***Total***
Average fare ($)	252	270	369	
Underlying Demand	45.0	29.6	20.5	95.1

Table 5.11: Fares and demands implied by the dichotomy heuristic's output

Simulations with the optimal set of fares and booking limit

We now compare the performance of those five different approaches by implementing the proposed fares and booking limits and simulating the demand.

In this chapter too, the total demand for the two fare products is a random variable generated at the beginning of each time period. We will again test two possible probability density functions: a uniform and a Gaussian probability density function. In the first set of simulations, the demand is uniformly distributed in all three time frames. In a second set of simulations, the demand is normally distributed in all three time frames. The demands for the three time frames are independent. For both type of demands, we generate 1,000 samples. The mean total demand is a linear function of the lower fare, y_i, and we use the same parameters, displayed in Table 5.1, to model these expected values. The standard deviations of the demands in $TF1$, $TF2$ and $TF3$ are 16, 10 and 7, respectively, which corresponds to about a third of the deterministic demand for each time frame. For the normal distribution, the demand is truncated in order to prevent any instance of negative demand.

The booking constraints or the flight capacity do not affect the probability that a passenger chooses the higher fare product. We run five scenarios for each type of probability density function:

1. simulations with the deterministic optimal fares and booking limits;
2. simulations with the stochastic optimal fares and booking limits;
3. simulations with the first, non-nested inventory, heuristic's optimal fares and booking limits;
4. simulations with the optimal fares and booking limits of the second heuristic, with the simplified demand probability density function;
5. and, finally, simulations with the optimal fares and booking limits of the thrid and last heuristic, based on dichotomy.

Tables 5.12 and 5.13 summarize the findings. The estimated average censored demands are very close to the predicted censored demands for the stochastic model. In the case of the uniform distribution, the relative difference between the simulated censored demand and the predicted average censored demand is -1.2%, -0.1% and 1.0% for $TF1$, $TF2$ and $TF3$, respectively. The final load factor is 84.6% and the total revenues are as expected equal to $25.1k. This represents a 2.8% increase in revenues from the deterministic case. Nevertheless, as observed in the previous chapters, the load factor is also lower with the stochastic fares and booking limits than the with the deterministic solution.

The stochastic model is based on the uniform distribution and the relative differences between the theoretical and simulated results for the three time frames are therefore slightly larger for the Gaussian distribution. Since, the Gaussian distribution gives more weight to the mean of the underlying demand, the expected censored demand for this distribution type lies between the uniform distributions' $\overline{n}_{total}$ and μ and the estimated average censored demands are thus higher for the Gaussian distribution. The revenues are consequently larger with the Gaussian distribution than the uniform one.

	Deterministic Model	***Stochastic Model***	***Heuristic One***	***Heuristic Two***	***Heuristic Three***
$\overline{n}_{accepted,1}$	42.2	40.3	36.2	40.4	45.3
$\overline{n}_{accepted,2}$	27.2	25.5	24.9	24.6	26.6
$\overline{n}_{accepted,3}$	19.9	18.3	18.6	18.4	16.6
Load factor	89.3%	84.1%	79.6%	83.5%	88.4%
$\overline{R}_{total}$ (k$)	**24.4**	**25.1**	**24.5**	**24.9**	**24.7**
Change in rev. from deterministic		*2.8%*	*0.4%*	*2.2%*	*1.3%*

Table 5.12: Average censored demands and revenues, for a uniform demand distribution

	Deterministic Model	*Stochastic Model*	*Heuristic One*	*Heuristic Two*	*Heuristic Three*
$\overline{n}_{accepted,1}$	42.5	40.5	37.1	40.0	44.6
$\overline{n}_{accepted,2}$	27.6	26.1	25.1	25.7	28.2
$\overline{n}_{accepted,3}$	19.7	18.1	18.3	18.1	16.8
Load factor	89.8%	84.8%	80.6%	83.8%	89.6%
$\overline{R}_{total}$ (k\$)	**24.5**	**25.3**	**24.7**	**25.0**	**25.1**
Change in rev. from deterministic		*3.1%*	*0.9%*	*2.1%*	*2.4%*

Table 5.13: Average censored demands and revenues, for a Gaussian demand distribution

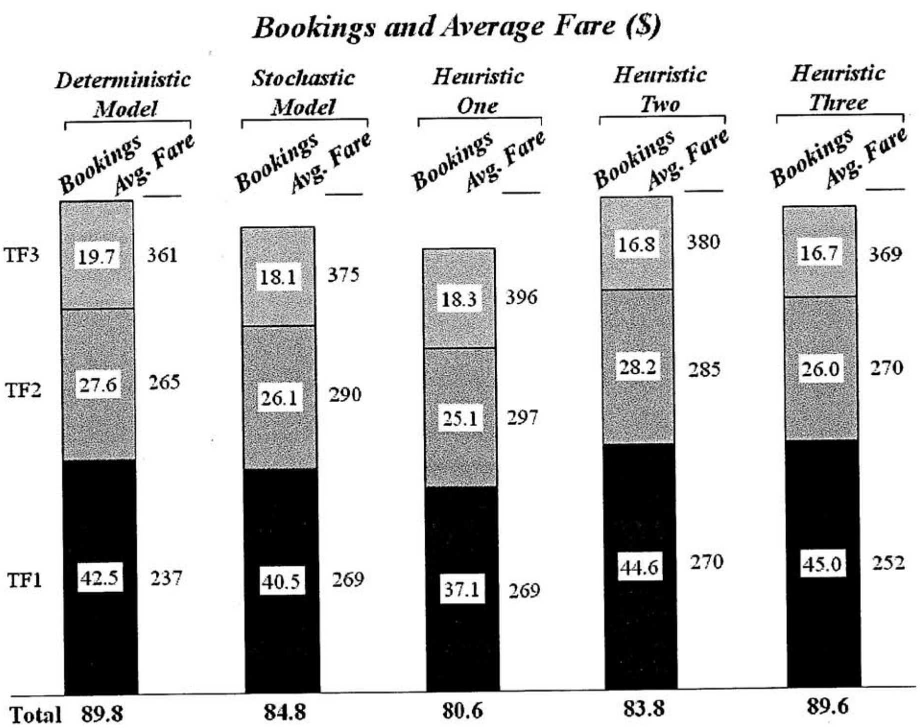


Figure 5-7: Accepted demand by time frame

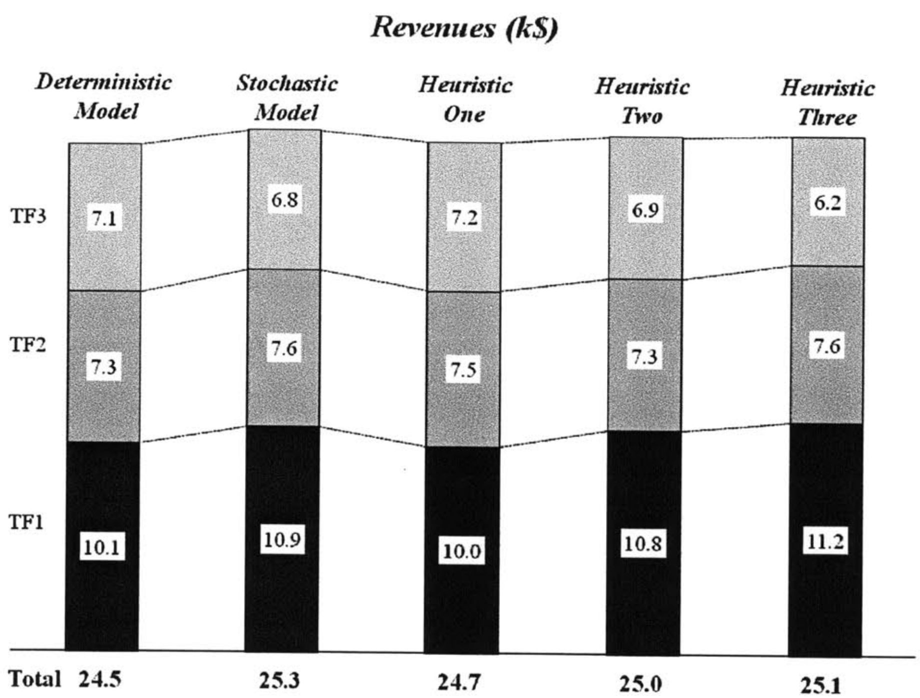


Figure 5-8: Revenue by time frame

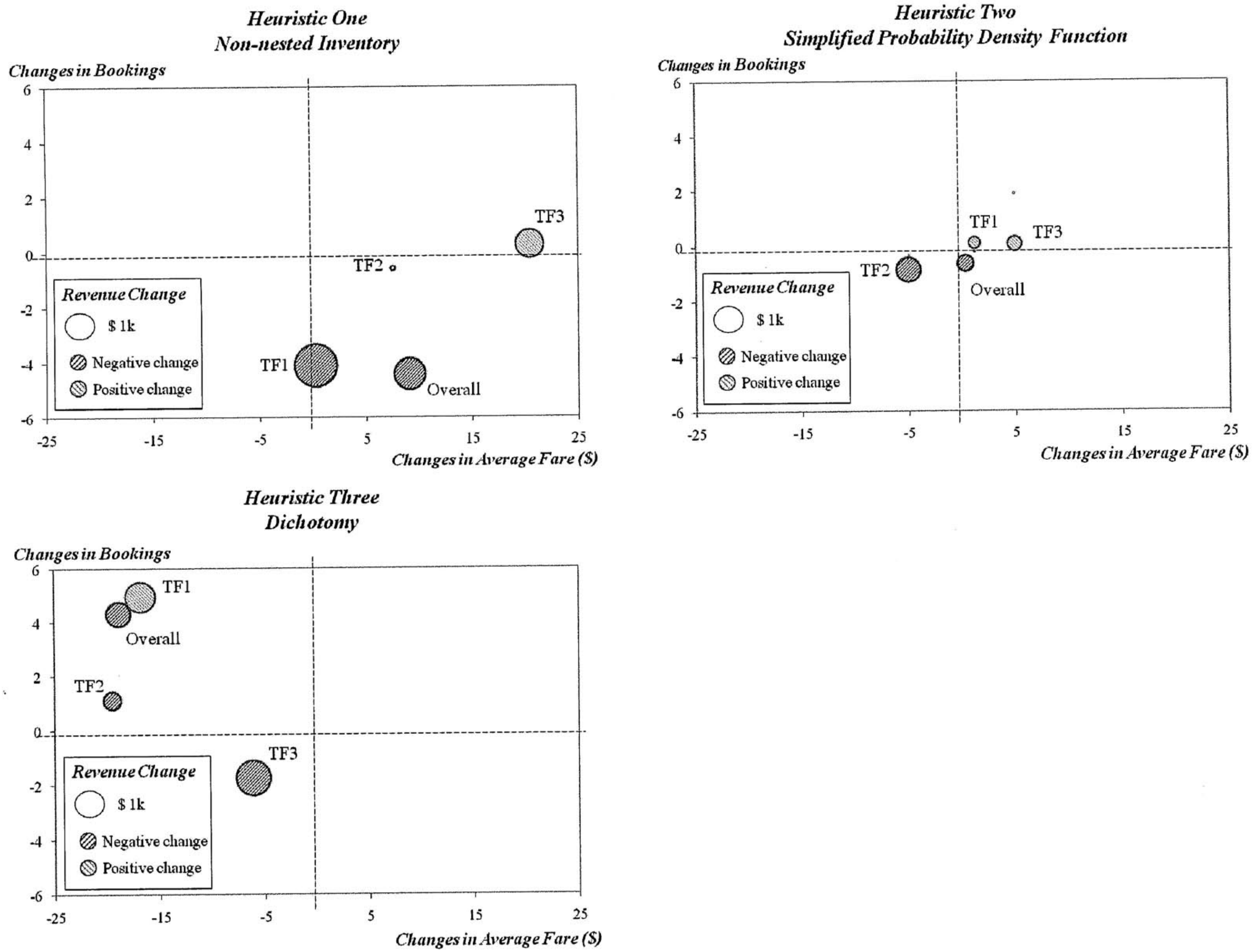


Figure 5-9: Differences between the stochastic model and the heuristics - uniform distribution

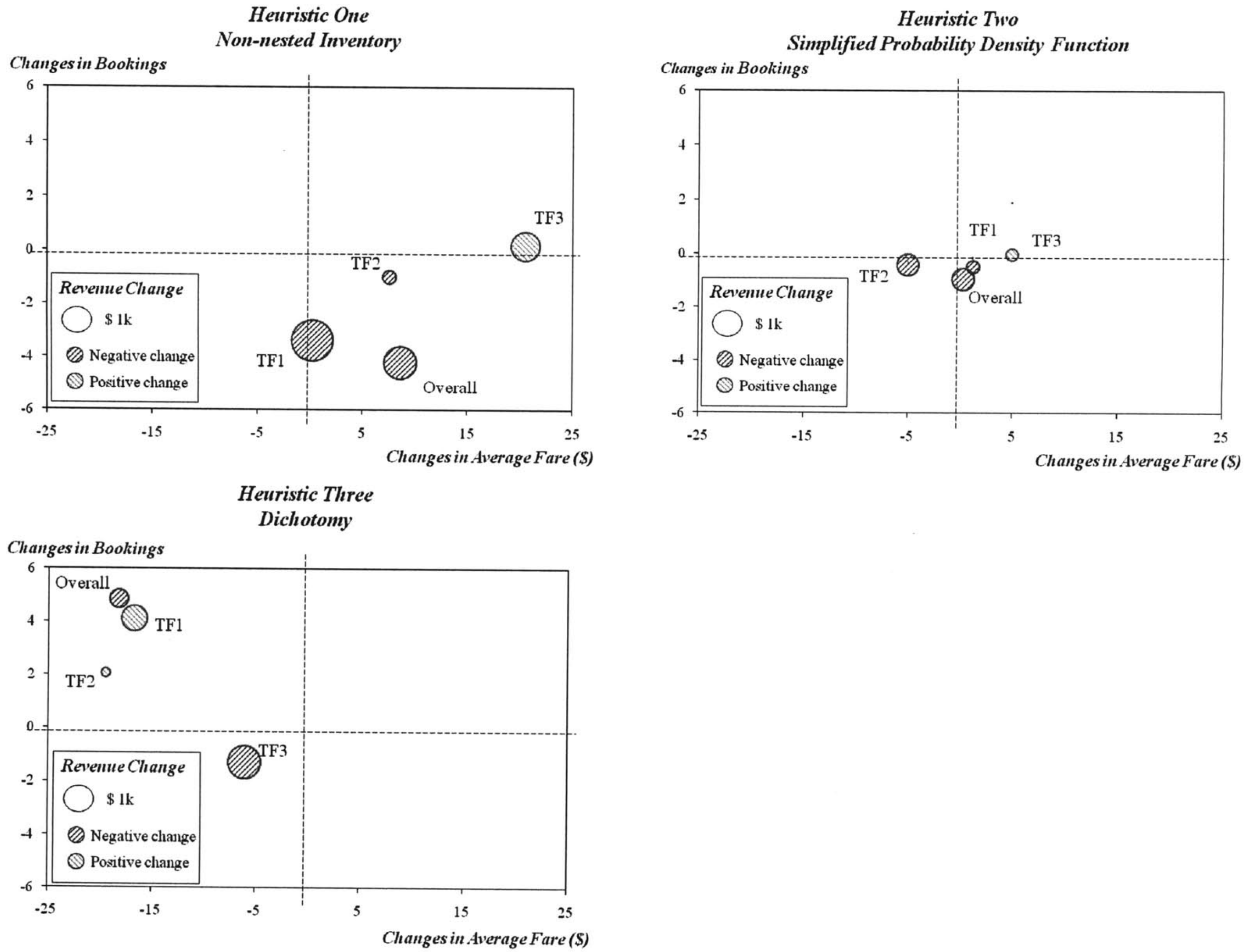


Figure 5-10: Differences between the stochastic model and the heuristics - Gaussian distribution

The highest revenues are obtained with the stochastic solution and the lowest with the deterministic one. The first heuristic, with the partitioned inventory, not only leads to the lowest revenues of all heuristics, but also to the lowest load factor overall. This heuristic only provides a 0.4 - 0.9% increase in revenues from the deterministic revenues, and the associated total number of bookings is in the low 80's, which is about 9 points lower than the 89% deterministic load factor.

The second heuristic, with the simplified demand probability density function, provides a steady 2.1 - 2.2% increase in revenues. For this numerical example, this heuristic is the best of all three when the demand is uniformly distributed. With a load factor only 0.6 points lower than the 84.1% of the stochastic load factor, and a small 0.6% decrease in revenues from the stochastic revenues, this heuristic seems to be a good approach to the multiple-time period joint optimization problem, when the actual demand is uniformly distributed, as assumed by the heuristic. As shown on Figures 5-9, the changes in fares or bookings are minimal with this heuristic. Consequently, the impact on the revenue is small.

However, it should also be noted that when the demand is normally distributed, the best heuristic is the third one. Indeed, while it still outperforms the deterministic solution or the first heuristic's solution, the second heuristic's revenues and load factor are lower than the third's. With a load factor in the high 80's, second only to the deterministic load factor, the third heuristic also exhibit the second best revenues overall: $25.1k, a 2.4% increase from the base case. The third heuristic's load factor is also very high when the demand is uniformly distributed. As anticipated, the low fares and high booking limits warranted by the third heuristic fostered the demand in the first two time frames. The bubble chart of Figure 5-9 shows how this combination of factors generated a large increase in revenues in $TF1$, mitigating the overall loss and largely increasing the overall number of accepted bookings.

The last two heuristics perform well. Heuristic Two, which assumed a simplified probability density function for later time frames, exhibited a consistent 2.1 - 2.2% revenue increase from the deterministic case. This heuristic's optimal solution is close to the stochastic optimal solution and changes in fares, bookings and revenues are minimal. More importantly, the changes in revenues are more balanced across time frames. With this heuristic, one can expect good revenues from the last time frame, which usually is the time frame in which high revenue passengers materialized. The third heuristic's good performance seems to rely mainly on the large number of bookings from the first time frame. This heuristic, based on dichotomy, is in a sense less risky by putting the emphasis on $TF1$ rather than $TF3$. In the numerical example considered, this third heuristic seems to be the best under the assumption that the demand is normally distributed. To generalize this finding, we would have to run more tests. However, one possible explanation to the fact that the second heuristic is not as good with a normal distribution is that, by construction, this heuristic relies heavily on the uniform distribution. It therefore systematically underestimates the number of accepted bookings.

5.5 Summary

In this chapter, we consider the multiple-time frame joint pricing and seat allocation optimization problem. A generalized methodology is outlined to determine the expression of all the censored demands, and therefore derive the objective revenue function. However, the probability density function of the sum of censored demands becomes very quickly overly complex. The censored demand for a third time frame already involves an important number of variables and is difficult to handle. In light of the increasingly complexity arising from the probability density function, we derive three heuristics to the joint pricing and seat allocation optimization problem.

A numerical example is used to illustrate the performance of these three heuristics. To compare the results with those obtained with the deterministic and stochastic approaches, we limit ourselves to a three-time frame numerical example. The stochastic solution yields the best revenues overall, and all three heuristics outperformed the deterministic model. In particular, two of the three heuristics provided very good revenues.

Chapter 6

Conclusions

We conclude the dissertation by summarizing the results and contributions of our research. We also discuss implementation challenges of the optimization methods. We then propose possible extensions of our models and suggest future research directions.

6.1 Research findings and contributions

Pricing and revenue management are two essential levers to maximize the sales of an airline's seat inventory and increase revenues. The two processes are complementary and interrelated. Both share the ultimate goal of maximizing the expected revenues of the airline and both affect the consumer's choice set. By setting fares and travel constraints, pricing defines the global set of options that could be available to a passenger. Booking limits, on the other hand, may render one or more of these options unavailable at the time of booking and therefore restrict the actual choice set of a passenger. The two processes have, nevertheless, traditionally been studied and even practised separately. For decades, researchers have considered them as two distinct optimization problems.

Researchers started addressing the issue of airline joint pricing and seat allocation optimization in the late 1990's. The papers published since then have either addressed the multiple-product single-time-period problem or the single-product, multiple-time-period problem. Some studies put a particular emphasis on joint optimization within a network, or in the presence of a competitor. Few studies proposed a simultaneous optimization of pricing and seat allocation. Most of them relied on an iterative approach, optimizing pricing first and then turning to seat allocation.

This research complements the existing body of work by addressing the multiple-product, multiple time frame joint pricing and seat allocation optimization problem. We placed ourselves in a single carrier, single OD market environment to remove any network or competitive effects. We first considered two time frames only. Two fare products are offered. The demands for the products are mutually dependent. The objective is to maximize the revenue generated by the sale of the two products by simultaneously determining the optimal fares and booking limit for the two products over the course of the two time periods.

We started with the simple deterministic case. The demand, a function of the products' prices, is assumed to be deterministic. This allowed us to show that the booking limit on the first time period's bookings is, in the deterministic case, a redundant variable. The number of decision variables is thus reduced to four, which simplifies the optimization problem. The deterministic model offers a rapid solution to the problem and is used as a benchmark to test more elaborate approaches.

We tested the optimal solution of the deterministic model in a stochastic environment, with simulations in which the demand generated is a random variable. The simulation results show that the deterministic model can provide a 3% increase in revenues over a traditional revenue management approach, even when the demand is uncertain. Furthermore, the simulations confirm the benefits of enforcing the booking limit when the demand is stochastic. By protecting a minimum number of seats from being sold to early, low-revenue passengers, the booking limit enables the airline to improve its expected revenues.

In Chapter 4, we introduced stochasticity in the model by assuming that, for each time frame, the total demand for the two fare products is a uniformly distributed random variable. The mean of the underlying demand is still a function of the lower fare. The presence of a limit on the total number of bookings that can be accepted in a time frame can increase the expected revenues. It thus becomes critical to understand the impact of the booking limit and the capacity on the accepted number of bookings, the censored demand.

We chose the uniform probability density function over the more commonly used Gaussian distribution after realizing that the latter distribution is at the root of the greatest difficulties encountered by researchers working on joint optimization. The probability density function of the sum of two independent random variables is the convolution product of their individual density functions. While the convolution of two unbounded Gaussian probability density functions is a simple Gaussian probability density function, there is no closed-form expression for the convolution of bounded Gaussian distributions. This is not the case for the uniform distribution. Furthermore, the uniform distribution can ultimately be used to model the Gaussian distribution, since the Gaussian function can be seen as the limit of a sum of uniform functions.

The uniform distribution presents the advantage of allowing us to derive the closed-form solution to the problem, through a geometrical analogy. The impact of the booking limit and the flight capacity on the underlying demands of the two time frames can be shown on a two-dimension graph, as displayed in Figure 6-1. The geometrical analysis of the two constraints' impact on the underlying demand allows us to derive the closed-form expression of the two time frames' censored demand, and by extension the objective revenue function.

Simulations confirm the benefits of accounting for the demand uncertainty and the constraints imposed on the demand in the problem formulation. The simulations

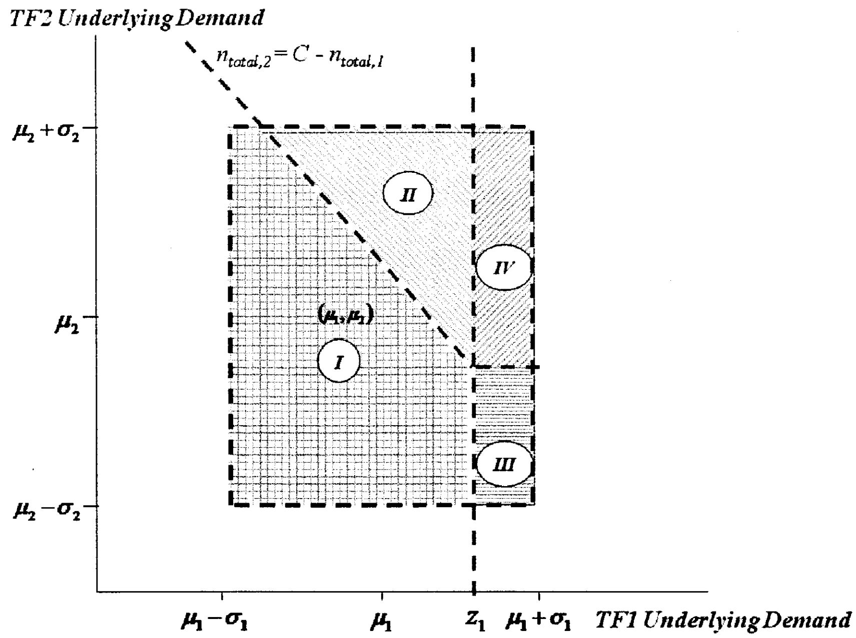


Figure 6-1: Divide and conquer

show that the stochastic model can provide a substantial 2% revenue increase over the deterministic model or a 3-4% increase from a more traditional revenue management approach with near-optimal fixed fares. We also showed that the approach behaves well in an stochastic environment where the demand is normally distributed.

Finally, the stochastic optimization model was extended to account for additional time frames. A recursion was proposed to derive the exact form of the censored demand for all the time frames considered. However, the approach does not scale up very well, as illustrated by an example. The compounded effects of successive booking limits, the increasing number of parameters and variables, all greatly increase the complexity of the censored demands. We therefore further analyzed the characteristics of our model to identify the origins of the recursion's complexity. Based on our understanding of the features that can be simplified, we proposed three different heuristics.

A first heuristic, which assumes that the capacity allocated to one time frame is not available to any other time frame was suggested. In order words, we assumed that the demands for the different time frames are independent of each other. We showed that this approach is attractive for its facilitated implementation but ignores an important revenue source. The extra revenue that can be generated when unsold seats from previous time frames are available to later time frames is not taken into account by this first heuristic.

The two other heuristics keep the nested structure of the inventory over the multiple time frames intact. We focused on simplifying the probability density functions of the demand in the different time frames. In the second heuristic, we assumed that the density function of the sum of the censored demands is uniform. Much of the recursion difficulty comes from the increasing complexity of this probability density function. Determining the exact expression of the convolution products of bounded uniform distributions is possible, but becomes rapidly difficult as more and more variables are introduced. We proposed to not compute the exact expression and approximate it instead by a uniform distribution, with a similar mean and standard deviation.

Lastly, we used dichotomy and the results from the initial stochastic model to derive the third heuristic. The two-time period problem formulation in Chapter 4 is fairly simple. We therefore suggested the use of dichotomy to unravel the multiple-time period process and revert to the two-time period case systematically.

A numerical example was used to illustrate the performance of these three heuristics. To compare the results with those obtained with the deterministic and stochastic approaches, we limited ourselves to a three-time frame numerical example. Although the stochastic solution yields the best revenues overall, all three heuristics outperform the deterministic model. In particular, the last two heuristics perform very when compared to the stochastic and deterministic models. The second heuristics results in a revenue increase of about 2% from the deterministic solution, which represents 60% of the potential increase from the stochastic solution. The third heuristic performed very well under the assumption that the demand is normally distributed, with a 2.4% increase in revenues from the deterministic solution. This increase corresponds to about 75% of the stochastic solution's potential increase.

6.2 Implementation challenges

The models developed in this dissertation can enable an airline to effectively merge the practice of pricing and revenue management. There seems to be a trend in this direction among major airlines and our methods can be a starting point for a few appropriate markets.

The proposed approaches have the potential to improve the revenue of an airline by 2-3%. However, the models do not account for any network or competitive effects. The joint pricing and seat allocation heuristics could therefore first be implemented in markets where the airline has very few connecting passengers, to mitigate any network effects. Most importantly, the airline should also pick markets in which it has not only a large market share, but also pricing power. If the airline has a large market share, it should have a more comprehensive understanding of the underlying demand characteristics. Furthermore, estimating the relationship between a market's total demand and the fares requires the ability to change the market's lowest fare and observe how this change affects the demand. It is therefore crucial to have the pricing power for the origin-destination markets in question.

Once the origin-destination markets are chosen, the airline will have to calibrate the model's demand functions. Historical data may not be fit for regressions. For example, the price variations over a period of time could be too small. The airline may then have to run small-scale pricing experiments to unveil the characteristics of the underlying total demand and estimate the model's parameters. Booking limits imposed by the airline during the booking period may constrain the observed demand.

To get to the underlying demand, one possibility would be to use the equations developed in this dissertation and reverse engineer them. Based on the observed number of bookings, the booking limits, and the fares, one could use the Chapter 4's formulae to run regressions and find the corresponding underlying demand characteristics.

When total demand and probability coefficients are estimated for all time frames, the airline can run one of the heuristics to jointly optimize pricing and seat allocation. We believe that Heuristic Two, with simplified probability density functions, and Heuristic Three, based on dichotomy, are the most effective of the three heuristics.

Implementing the heuristic's optimal set of fares and booking limit may be not be straight-forward. Filing the optimal set of fares with third party vendors may require some flexibility and advance notice. Some airlines with innovative fare structures may have already experienced similar difficulties with global distribution systems. Nevertheless, the models can relatively easily be implemented on an airline's own website. Depending on the revenue management system in place, the airline users may have to overwrite the system's own booking limits to ensure that the heuristic's optimal solution is enforced on the origin-destination markets of interest. The airline may also decide to impose additional booking limits on the lower fare products, as discussed in Section 4.5.3, to mitigate the risk of an abnormally low buy-up rate during the booking process.

Our models for joint pricing and seat allocation optimization can easily be first used as a joint pricing and revenue management guideline by many airlines. Finding the optimal corresponding set of fares and booking limits can provide insightful pricing recommendations very quickly and efficiently, even with current revenue management environments.

6.3 Future research directions

The research presented in this dissertation can be expanded. We propose several research directions:

1. **Modify the total demand function to account for both fares**

   In the current demand formulation, the total demand is a function of the lower fare only. We assume that changing Fare Product 1's fare does not alter the OD market's total underlying demand. We showed that this assumption holds for OD markets and fare structures such that the number of consumers that will only consider Fare Product 1 is relatively small compared to the total demand and that these consumers are relatively price inelastic. However, there are markets in which this assumption is not be applicable. It would then be appropriate to include the second fare x_i in the total demand function as well. The geometrical analogy could be used to find the impact of booking limits on the combined demands of the two groups.

2. **Extend the model to more fare products**

   The research could be extended to account for more than two fare products. We focused on improving the model by extending it to account for additional time frames. Dividing the selling horizon into more but smaller time frames, which airlines do in reality, helps match more closely the changing characteristics of passengers over the booking process. However, most airline also offer multiple fare products. To introduce additional products in our model, the demand formulation would have to be adapted. The total demand function is likely to be mostly dependent on a few number of fares, and may therefore not require much changes if the first suggested research direction is undertaken first. The probability that a passenger will choose of the multiple products will however have to be adapted. The logit model may no longer be appropriate and the interactions between the fare products will have to be further analysed.

3. **Include network effects**

The stochastic model could be extended to take into account network effects. If we consider a network of flights, then the demand for different fare products for different origin-destination markets will be competing for the same resources, the flight capacity. In short, the sum of the demands for the different origin-destination markets will be capped by the flight's total capacity. Prior work on multiple fare products could be useful if we assume that some the fare products are for local passengers while the rest is designed for connecting passengers. The geometrical analogy could be used again to find the impact of booking limits on the different types of demands.

4. **Introduce a competitor**

   This research could also serve as a mean of introducing a competitor. We could imagine having two airlines, Airline 1 and its competitor, Airline 2. To simplify the first analysis we could assume that each airline only offers one fare product, and that the pricing strategy of Airline 2 is known and stable. The capacity of each airline is also known. Airline 1 offers Fare Product 1 and Airline 2 offers Fare Product 2, but the average fare for Fare Product 2 over the booking period can be anticipated. Then the objective would be to find the set of fares and booking limits that would maximize Airline 1's revenues, given Airline 2's pricing strategy and the two airlines' capacities. To successfully implement this model, an airline will have to have very good forecasting capabilities.

In this dissertation, we developed a stochastic model to determine simultaneously the set of fares and the first time period's booking limit that maximize the revenue generated by the sale of two fare products over two time periods. We further proposed three heuristics to tackle the multiple-time period joint pricing and seat allocation problem. The performance of the model and the heuristics in the few numerical examples used are very promising. The proposed approaches have the potential to improve the revenue of an airline by 2-4%. This model should enable an airline to effectively merge the practice of pricing and revenue management. There seems to be a trend in this direction among major airlines and this method can be a starting point in a few appropriate markets.

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