IMPEDANCE TRANSFORMATION USING LOSSLESS NETWORKS

by

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ABSTRACT

Two impedances are said to be compatible if one of them can be realized as the input impedance to a two terminal-pair lossless network terminated in the other impedance. A concise set of necessary and sufficient conditions under which two impedances can be compatible is found. Sometimes it is necessary to augment one of the two impedances by inserting a common factor into both its numerator and denominator in order to make it compatible with the second impedance. The conditions under which such a factor exists and methods for finding it are determined.

The conditions derived for positive-real impedances are extended to include stable, active impedances also. A straightforward technique for determining a reactance function from its power series expansion about zero or infinity is found. A technique for the determination of a reactance function from its value specified at a number of complex frequencies is also investigated. The application of these results to the broadbanding problem and cascade synthesis is discussed.

Thesis Supervisor: E. A. Guillemin
Title: Professor of Electrical Communication

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Chapter One

Introduction

A study of the conditions under which one impedance can be transformed into another impedance by a lossless network is worthwhile because of the many applications. For example, it has long been known that maximum power transfer requires a matching of load and generator impedance. Reflectionless transmission on a transmission line requires the load to be matched to the characteristic impedance of the line. Further, a lossless coupling network is usually desirable in both of these cases in order to insure that the signal power is transmitted to the load rather than dissipated in the coupling network.

In problems such as these, the generator impedance or the line's characteristic impedance is usually a resistance and the problem reduces to the transformation of an arbitrary impedance to a resistance. Although this is not always possible when the coupling network is restricted to be lossless, the matching can be approximated closely over a finite bandwidth. This process is called broadbanding and has been studied in detail by Fano[1], Carlin, and LaRosa.[2] Essentially,

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1. Fano, R.M., "Theoretical Limitations on the Broadband Matching of Arbitrary Impedances," Jour. Franklin Inst., 249, Jan. 1950, pp. 57-83, Feb. 1950, pp. 139-154.
2. Carlin, H.J. and LaRosa, R., "Broadband Reflectionless Matching with Minimum Insertion Loss," Proc. Symp. on Modern Network Synthesis, NYC, 1952, pp. 161-178.

the broadbanding process requires that an impedance be chosen which not only approximates a resistance over some frequency range but also is realizable as a two terminal-pair lossless network terminated in the given load impedance. This problem is illustrated in Fig. 1.1 where $Z_0(s)$ is the arbitrary load or terminating impedance and $Z_1(s)$ is the input impedance to the coupling network when it is terminated in $Z_0(s)$.

Many other problems are also of this general form although very different from the broadbanding problem. For example, a filter network which must terminate in some impedance other than a simple resistance (a parallel RC circuit is a common example) has definite restrictions placed on the allowed transfer impedance by the load. From Darlington theory, it is known that the input impedance of such a network is related to this transfer impedance and hence the problem reduces to one of specifying an input impedance which must be realized as a lossless network terminated in a given load impedance.[3] Again Fig. 1.1 is relevant.

Another important problem which fits this form is that of cascade synthesis.[4] In this problem, the impedance $Z_1(s)$ is specified and a lossless two-terminal-pair network must be found which when terminated in another realizable impedance

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3. Guillemin, E.A., "Synthesis of Passive Networks," Wiley, NYC, 1957, p. 446.

4. <u>Ibid</u>, p. 374.

$Z_0(s)$ realizes $Z_1(s)$, the purpose being to ultimately realize $Z_1(s)$ as a lossless network terminated in a resistance.

In all of these problems, one impedance must be transformed into another by means of a lossless network. Two impedances which have the property that one of them can be transformed into the other by means of a realizable lossless network will be said to be compatible. The purpose of this thesis is to find a set of necessary and sufficient conditions under which two arbitrary impedances can be compatible. That is, given the two impedances $Z_1(s)$ and $Z_0(s)$, a test is desired which not only determines whether or not the impedances are compatible, but also gives the characteristics of the lossless coupling network--open circuit impedance parameters, for example--so that synthesis can proceed.

The major work in this area has been that of Fano who studied the broadbanding problem. He derived restrictions on the impedance $Z_1(s)$ imposed by the specified load $Z_0(s)$ in the form of integral relations involving the input reflection factor. This work was extended to the case of lossy coupling networks with minimum insertion loss by Carlin and LaRosa. Their results are very useful in the broadbanding problem but become computationally difficult when the coupling network is at all complex. Moreover, the restrictions derived by Fano are not complete in one sense. Since in the broadbanding problem the input impedance is chosen, the problem of multiplicative factors common to both numerator and denominator need never arise. But if the input impedance is prescribed, the given impedances may not be compatible as given but can be made compatible if the proper common factors are inserted into the impedance $Z_1(s)$. The determination of the conditions under which this can happen and the choice of the common factor when it exists is an important part of the problem which has never been investigated.

The more general problem of transforming one impedance into another has been neglected in the literature except for the work of Ligomenides.[5] He assumes a coupling network of two or three elements and chooses these elements in order to minimize the mean-square error between the desired and actual input impedance at certain prescribed real frequencies. The work does not investigate the conditions under which an exact match is possible and further becomes very difficult to carry through for a coupling network of more than three elements.

Because of the many applications, a concise set of necessary and sufficient conditions under which two impedances can be compatible would be very helpful.

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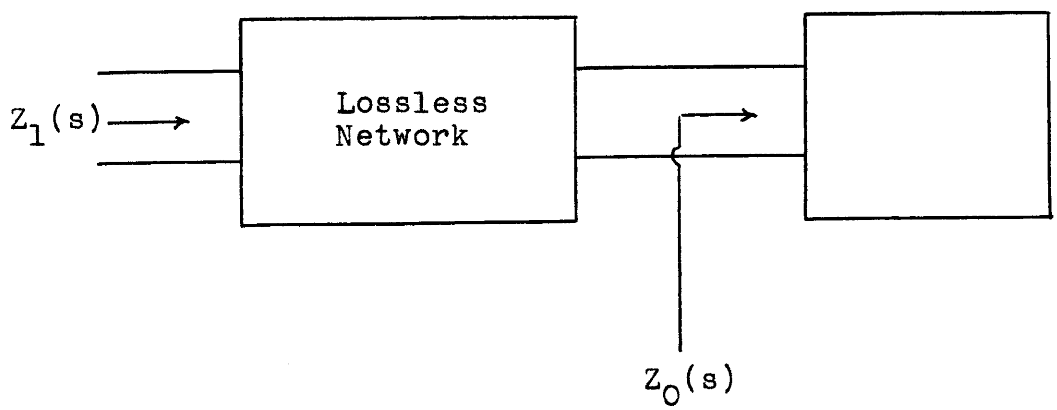
5. Ligomenides, P.A., "A New Design Method for Coupling Networks, With Applications to Broadband Transistor Amplifiers and Antenna Matching," IRE Wescon Conv., 1958, Part 2.

The realization of $Z_1(s)$ shown in Fig. 1.1 may introduce factors into both the numerator and denominator of $Z_1(s)$. That is, if $Z_1(s)$ is specified as the ratio of the polynomials $P_1(s)$ and $Q_1(s)$, the realization may augment each of these polynomials with a factor. Such a factor is hereafter referred to as a common factor. Before treating the general case, the special case of no common factors is first investigated. This restriction implies that the set of open circuit impedance parameters of the lossless coupling network are unique and therefore the characteristics of the network easily studied using conventional techniques such as Darlington theory and cascade-synthesis theory. Although this set will not be unique when common factors are allowed in the impedance $Z_1(s)$, it is unique once a given common factor is inserted provided that no additional common factors are inserted. This permits the results of the special case to be extended to give the conditions under which there exists a common factor which will make the two impedances compatible. The method developed also gives the common factor when it exists.

Once a simple set of necessary and sufficient conditions have been derived, the restriction that the two impedances be RLC realizable can be dropped, so that stable but active impedances can be considered.

Figure 1.1

The Impedance Transformation Problem



Chapter Two

The Special Case of No Common Factors

Consider the problem of determining the necessary and sufficient conditions under which two physically-realizable impedances, $Z_0(s)$ and $Z_1(s)$, are compatible without any common factors inserted into the numerator and denominator of $Z_1(s)$. That is, under what conditions does there exist a physically-realizable, lossless, two-terminal-pair network which, when terminated in $Z_0(s)$, not only has $Z_1(s)$ as its input impedance, but also realizes the correct numerator and denominator polynomials of $Z_1(s)$ without additional factors common to both. For simplicity, let the impedances have no $j\omega$-axis poles, including the origin and infinity. This restriction will be removed later.

There are two approaches to the solution of this problem. The first method is based on the uniqueness property of the Darlington realization procedure (when no common factors are present) and the second on the method of cascade synthesis. Since each method contributes much insight into the problem, a parallel development is worthwhile.

Consider first the Darlington approach.[1] Darlington proved that any physically-realizable driving-point impedance can be

1. Guillemin, op. cit., p. 358.

realized as a lossless two-terminal-pair network terminated in a one-ohm resistor. If no common factors are inserted into the impedance, the open-circuit impedance parameters of the lossless coupling network are unique. This result is used in the following manner. $Z_1(s)$ and $Z_0(s)$ each have a Darlington realization as shown in Fig. 2.1. Here, the z^1_{ij} and z^0_{ij} are the open-circuit impedances of the Darlington lossless network. Moreover, if the impedances $Z_1(s)$ and $Z_0(s)$ are compatible, there must exist a lossless coupling network, N, which when terminated in $Z_0(s)$ yields $Z_1(s)$ as its input impedance (Fig. 2.2). If $Z_0(s)$ is replaced by its Darlington realization as shown in Fig. 2.2, there results two lossless networks, N_1 and the cascade connection of N and N_0, terminated in one-ohm resistances each of which has an input impedance equal to $Z_1(s)$. Moreover, by hypothesis, there are no common factors introduced by these realization methods. Hence the two lossless coupling networks must have the same open-circuit impedance parameters since these are unique.

This reasoning permits the open-circuit impedance parameters of the two lossless coupling networks to be written in terms of the known and unknown impedance parameters of the individual networks. If corresponding impedance parameters are then equated, a set of equations results which can be solved for the open-circuit impedance parameters of the unknown lossless coupling network, N. This reduces the problem to one of studying the realizability of this derived network. The derivation of the open-circuit impedance parameters of this network is most easily carried out by using the ABCD or chain matrices of the various networks.

The chain matrix of a network is merely the matrix relating the input and output quantities. If E_1 and I_1 are the input voltage and current and E_2 and I_2 the output voltage and current (defined in Fig. 2.3), then these variables are related by the chain matrix:

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix}$$

The usefulness of this system of equations lies in the fact that two networks connected in cascade have an overall chain matrix equal to the product in the same order of the chain matrices of the individual networks. Clearly this will be useful in the present problem.

Let M_1, M_0, M_n, and M_{no} be the chain matrices of the networks N_1, N_0, N, and N_{no} respectively, where N_{no} is the cascade-connection of the networks N and N_0 and the other networks are defined in Figs. 2.1 and 2.2. The matrix M_{no} can be written in terms of the matrices M_n and M_0:

$$M_{no} = M_n M_0 \qquad 2.1$$

Since the two lossless networks N_1 and N_{no} must have the same open-circuit impedance parameters, they must also have the same chain matrix. Thus it follows that

$$M_1 = M_{no} = M_n M_0 \qquad 2.2$$

If this matrix equation is solved for the chain matrix of the unknown lossless coupling network N there results

$$M_n = M_1 M_0^{-1} \qquad 2.3$$

where M_0^{-1} denotes the inverse of the matrix M_0 if it exists.

A physical interpretation can be attached to this inverse matrix. Since the product of M_0 and M_0^{-1} must yield the unit matrix, and since the unit matrix represents the set of equations

$$\begin{aligned} E_1 &= E_2 \\ I_1 &= I_2 \end{aligned} \qquad 2.4$$

which are merely the equations governing two wires as shown in Fig. 2.3, it follows that the network N_0, when cascaded with the network realization of the inverse matrix M_0^{-1}, yields just these two wires. If the network N_0 is replaced by its "tee" equivalent for visualization purposes, it is clear that a suitable "tee" equivalent of the inverse network is merely the same network turned end for end with each impedance replaced by its negative. The cascade connection of the networks and the reduction to two wires is shown in Fig. 2.4.

With this realization of the inverse chain matrix, the unknown coupling network N can be replaced by the cascade-connection of the "tee" equivalent circuits of the two networks represented by the chain matrices M_1 and M_0^{-1}. This form is shown in Fig. 2.5. The use of the "tee" equivalents is emphasized only because a useful physical picture results. No restrictions on the generality of the solution are implied. It is shown in Fig. 2.6 that this lossless coupling network yields $Z_1(s)$ when terminated in $Z_0(s)$ and therefore is indeed the correct solution.

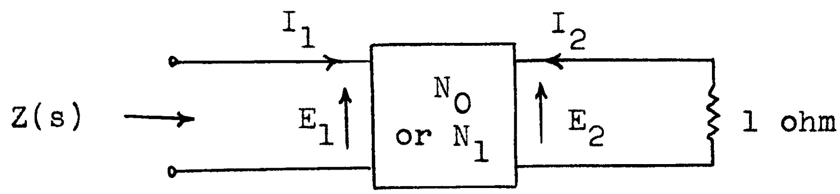
The above reasoning has shown that if $Z_1(s)$ and $Z_0(s)$ are compatible, the network N derived in the above manner is necessarily realizable. On the other hand, if the network N is realizable, $Z_1(s)$ and $Z_0(s)$ are compatible by definition (a physically-realizable network does exist). Hence the condition that N be realizable in order that $Z_0(s)$ and $Z_1(s)$

be compatible is both necessary and sufficient. It should be pointed out that the realizability of this network is not a trivial matter.

Before investigating the realizability of the network N in detail, it is well to digress and rederive the above results from the cascade-synthesis viewpoint. The two techniques together yield the realizability conditions on the network N more easily than either one separately.

Figure 2.1

Darlington Realization of $Z_0(s)$ and $Z_1(s)$

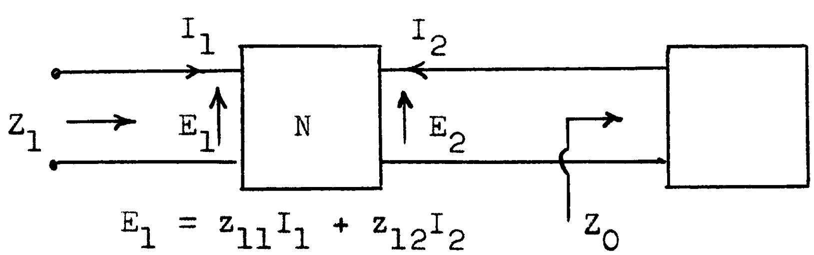


$$E_1 = z_{11}^0 I_1 + z_{12}^0 I_2 \qquad E_1 = z_{11}^1 I_1 + z_{12}^1 I_2$$

$$E_2 = z_{12}^0 I_1 + z_{22}^0 I_2 \qquad E_2 = z_{12}^1 I_1 + z_{22}^1 I_2$$

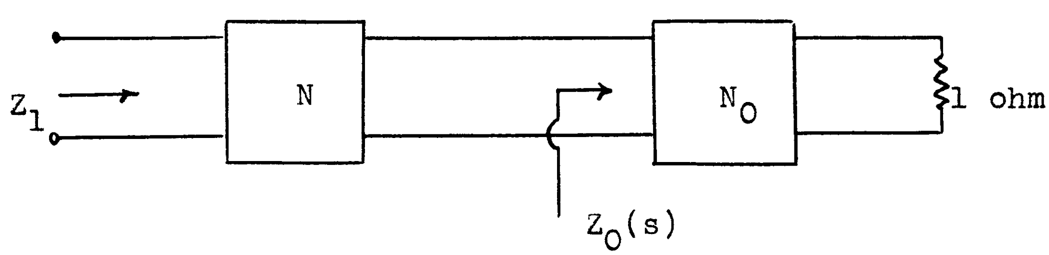
Figure 2.2

The Two Equivalent Realizations of $Z_1(s)$



$$E_1 = z_{11} I_1 + z_{12} I_2$$

$$E_2 = z_{12} I_1 + z_{22} I_2$$



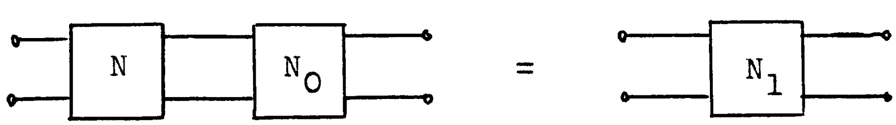
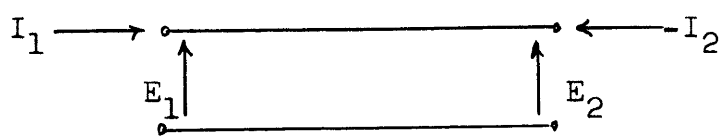


Figure 2.3

Realization of the Unit Chain Matrix



$E_1 = E_2$

$I_1 = I_2$

Figure 2.4

The Cascade Connection of M_0 and M_0^{-1} and Their

Reduction to the Unit Chain Matrix Network

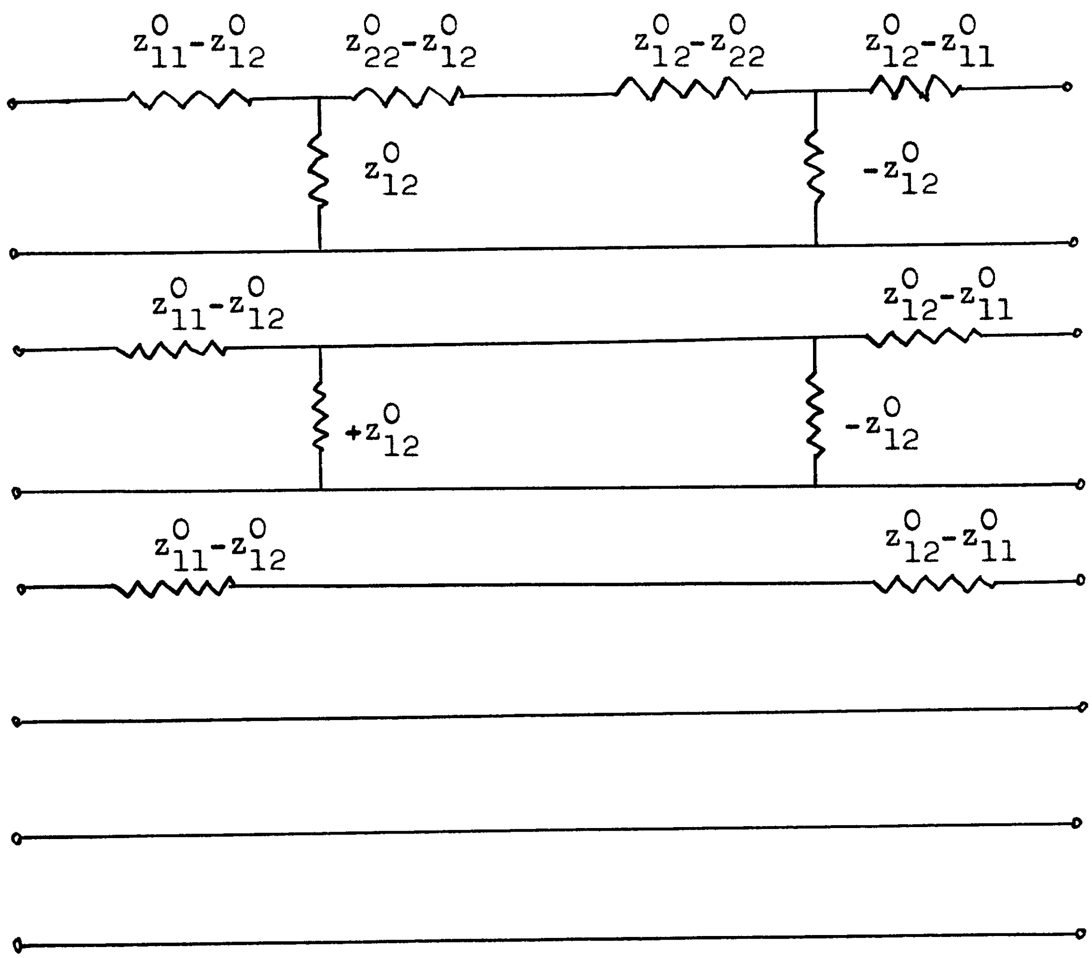


Figure 2.5

Realization of the Network N as the Cascade

Connection of Two Equivalent-Tee Networks

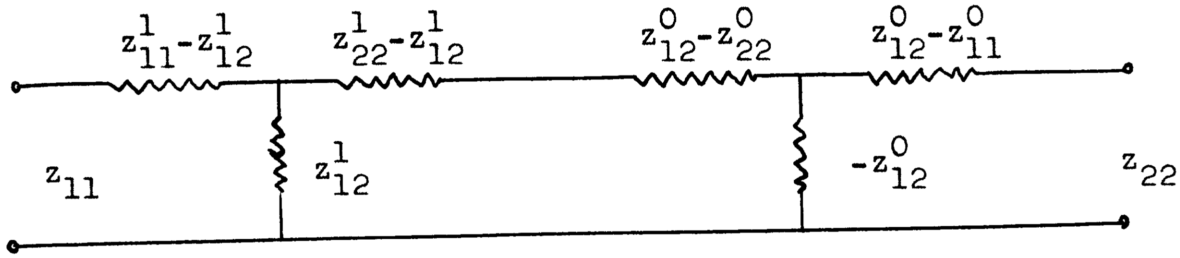
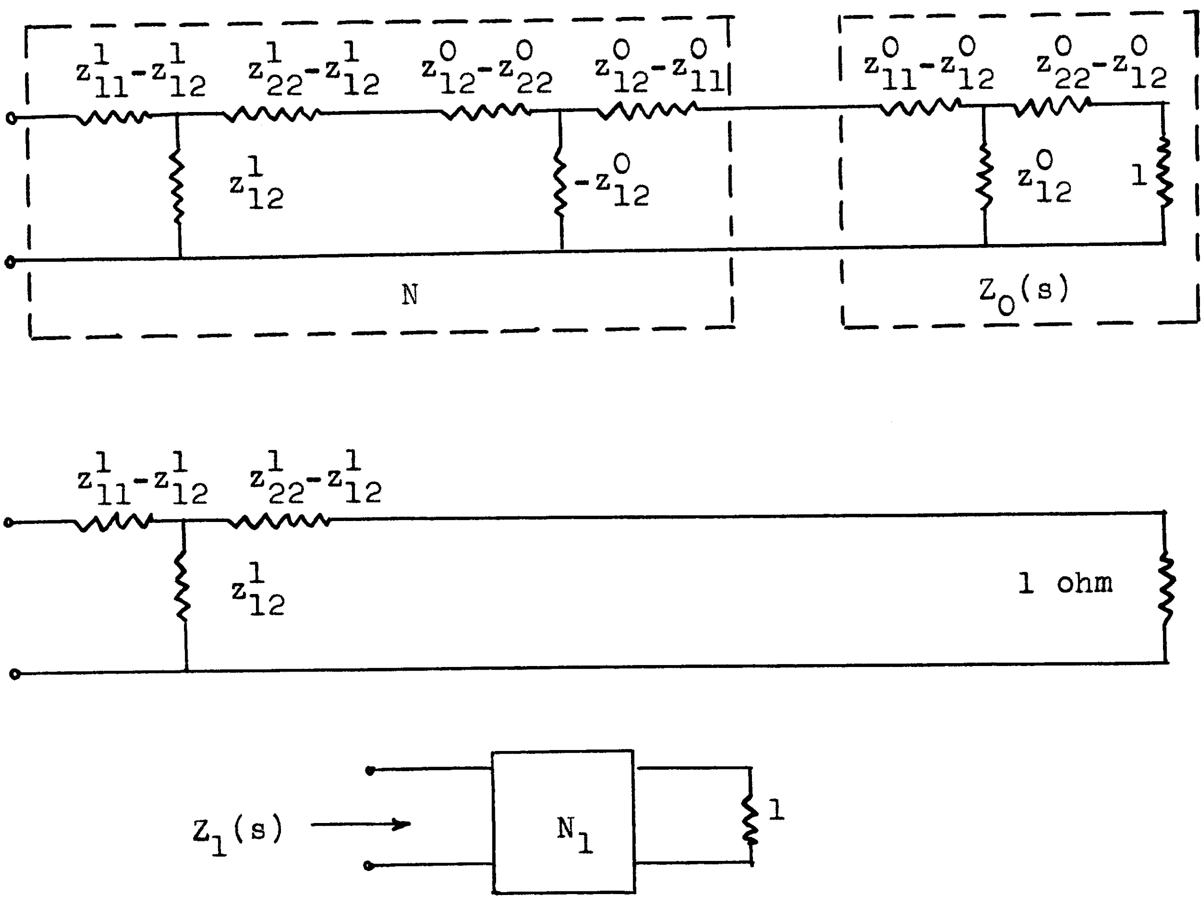


Figure 2.6

Reduction of the Realization of $Z_1(s)$ to its

Darlington Form



Chapter Three

The Cascade Synthesis Approach

In the realization of a driving-point impedance as a lossless two-terminal-pair network terminated in a resistance, it is well known that the zeros of transmission of the lossless network are the zeros of the even part of the impedance and hence the zeros of the transfer impedance.[1,2] It is desirable to realize this lossless network as the cascade connection of a number of two-terminal-pair networks each of which places in evidence one or more of these transmission zeros. The technique for doing this is called cascade synthesis.[3]

The problem is illustrated in Fig. 3.1. Given an impedance $Z_1(s)$ and one or more zeros of its even part, a lossless two terminal-pair network, N, which possesses these transmission zeros, must be extracted from $Z_1(s)$ in such a way as to leave a physically realizable remainder impedance $Z_0(s)$ which does not possess these zeros in its even part and hence is simpler. In other words, a network N must be removed from $Z_1(s)$ so as to leave a remainder impedance $Z_0(s)$ compatible

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1. Guillemin, op. cit., p. 372.

2. The even part of Z(s) is: $\text{Ev } Z(s) = [Z(s) + Z(-s)]/2$. At any frequency at which the $\text{Ev } Z(s) = 0$, Z(s) is an odd function for $Z(-s_0) = -Z(+s_0)$ if $\text{Ev } Z(s_0) = 0$.

3. Hurst, S.R., "New Methods of Transformerless Driving Point Impedance Synthesis," Sc.D. Thesis, MIT, 1955.

with $Z_1(s)$.

The driving-point impedance synthesis problem always has a solution because the remainder impedance is not specified, but is determined by the assumed form of the network N. The problem under consideration is very similiar in that there is an impedance $Z_1(s)$ from which a lossless two-terminal-pair network must be extracted to leave a remainder impedance $Z_0(s)$. The essential difference is that $Z_0(s)$ is completely specified. This causes the problem to be overspecified and leads sometimes to no solution.

Before setting up the problem in the cascade-synthesis scheme, the problem of even-part zeros of $Z_1(s)$ and $Z_0(s)$ must be investigated. To this end, assume $Z_1(s)$ is realizable as a lossless two-terminal-pair network, N, terminated in $Z_0(s)$ as shown in Fig. 3.2. Let z_{11}, z_{12}, and z_{22} be the open-circuit impedance parameters of the lossless coupling network as before. The familiar relation between $Z_1(s)$ and $Z_0(s)$ in terms of these impedances is

$$Z_1(s) = \frac{z_{11}z_{22}-z_{12}^2 + z_{11}Z_0(s)}{z_{22} + Z_0(s)}. \qquad 3.1$$

Since N is a lossless network, each of the z-parameters is an odd function of frequency,[4] s. Multiplying Eq. 3.1 by

$$\frac{z_{22}(-s) + Z_0(-s)}{z_{22}(-s) + Z_0(-s)} \qquad 3.2$$

yields (after simplification):

$$Z_1(s) = z_{11}(s) - \frac{z_{12}^2(s)[z_{22}(-s) + Z_0(-s)]}{[z_{22}(s) + Z_0(s)][z_{22}(-s) + Z_0(-s)]} \qquad 3.3$$

Since the even-parts of $z_{11}(s)$ and $z_{22}(s)$ are zero and since z_{12}^2 is an even function of frequency, the even-part of Eq. 3.3 reduces to

$$\text{Ev } Z_1(s) = [\text{Ev } Z_0(s)]\left[\frac{z_{12}(s)}{z_{22}(s) + Z_0(s)}\right]\left[\frac{z_{12}(-s)}{z_{22}(-s) + Z_0(-s)}\right] \qquad 3.4$$

It is now possible to state and prove a fundamental restriction on the impedance $Z_1(s)$: <u>All zeros of the even-part of $Z_0(s)$ including those at infinity are also zeros of the even-part of $Z_1(s)$ with at least the same multiplicity</u>.

The proof of this statement follows from an examination of the relation between the even parts of $Z_1(s)$ and $Z_0(s)$--Eq. 3.4. Ev $Z_0(s)$ has, in general, three types of zeros: complex zeros which appear with quadrantal symmetry (four quadrant symmetry), even-order zeros on the finite $j\omega$ axis, and zeros at infinity. Consider each type separately.

First suppose the even part of $Z_0(s)$ is zero at a complex frequency $s \mp s_o$ in the right half s-plane (that is, the real part of s_o is positive). In order for Ev $Z_1(s)$ to be nonzero at $s = s_o$, this zero must be cancelled by one of the terms on the right hand side of Eq. 3.4. This zero cannot be cancelled by a pole of $z_{12}(s)$ or $z_{12}(-s)$ since all of these poles must lie on the $j\omega$ axis. Neither can it be cancelled by the term $z_{22}(s) + Z_0(s)$ for this quantity cannot be zero at $s = s_o$ since it is a p.r. impedance and hence can have no right half-plane zeros. The impedance $z_{22}(-s) + Z_0(-s)$ has all of its zeros in the right-half plane and might at first glance appear capable of cancelling the zero of the even-part of $Z_0(s)$. That this cannot be true is seen by writing

$$\begin{aligned} Z_0(-s_o) &= \text{Ev } Z_0(-s_o) + \text{Odd } Z_0(-s_o) \\ &= \text{Ev } Z_0(+s_o) - \text{Odd } Z_0(+s_o). \end{aligned}$$

But the even part of $Z_0(s_o)$ is zero at $s = s_o$ by hypothesis. Hence

$$Z_0(-s_o) = - Z_0(+s_o).$$

This means that

$$z_{22}(-s_o) + Z_0(-s_o) = -[z_{22}(s_o) + Z_0(s_o)] \qquad 3.5$$

Since $z_{22}(s) + Z_0(s)$ is a p.r. impedance and hence cannot be zero in the right half plane, it follows that $z_{22}(-s) + Z_0(-s)$ cannot be zero at $s = s_o$ and the zero of the even part of $Z_0(s)$ at this frequency cannot be cancelled. Because zeros of the even parts of impedances occur with quadrantal symmetry, it follows that no complex zeros of the even part of $Z_0(s)$ can be cancelled, but must also be zeros of the even part of $Z_1(s)$. Moreover, if such an even-part zero is not also an even-part zero of $Z_1(s)$, it cannot be added by merely augmenting numerator and denominator of $Z_1(s)$ with a common factor, for the above proof shows that the even part of $Z_1(s)$ must actually be zero at these frequencies. If any of the complex zeros of the even part of $Z_0(s)$ are not present in the even part of $Z_1(s)$, the two impedances cannot be compatible.

Next consider a zero of the even part of $Z_0(s)$ on the finite $j\omega$ axis. Let $s = j\omega_o$ be such a zero. Since $Z_1(s)$ is p.r., this zero must be of even order. Poles of $z_{12}(s)$ or $z_{12}(-s)$ cannot cancel this zero of the even part of $Z_0(s)$ because the denominator of Eq. 3.4 also has poles at all poles of $z_{12}(s)$ or $z_{12}(-s)$. If the denominator $z_{22} + Z_0$ is zero at $s = j\omega_o$, then $z_{12}(j\omega_o)$ must also be zero or else $Z_1(s)$ would have a pole at $s = j\omega_0$(Eq. 3.1). Since $Z_1(s)$ has no $j\omega$ axis poles by assumption, it follows that the last two terms of Eq. 3.4 must be finite or zero at this frequency and hence cannot cancel the even-part zeros of $Z_0(s)$. This is a reasonable result, for since the even part equals the real part on the $j\omega$ axis, it is clear that $Z_0(s)$ reduces to a reactance function at $s = j\omega_0$. The coupling network is composed entirely of reactances, and therefore $Z_1(j\omega_0)$ must be a reactance, that is, have zero real part.

There remains the possibility of zeros at infinity. From Eq. 3.4 it is clear that a zero of the even part of $Z_0(s)$ at infinity can be cancelled only by poles of the last two terms in this equation. If z_{12} has a pole at infinity, so must z_{22} and therefore the quantity $z_{12}/z_{22} + Z_0$ is finite. If z_{12} is zero at infinity, $z_{22} + Z_0$ can have, at most, a simple zero at infinity and again the quantity $z_{12}/z_{22} + Z_0$ is finite or zero. The impedance z_{12} is a reactance function and must be either zero or infinite at infinity. Hence the above reasoning includes all possible cases. Since the last two terms in Eq. 3.4 cannot be infinite at infinity, zeros of the even part of $Z_0(s)$ cannot cancel and must be zeros of the even part of $Z_1(s)$ also.

This is an important restriction if the impedance $Z_1(s)$ is to be compatible with the impedance $Z_0(s)$. Although necessary, it is not sufficient.

The remaining zeros of the even part of $Z_1(s)$ must be contributed by the lossless coupling network. To investigate this, define[5]

$$Z_1(s) = \frac{m_{11} + n_{11}}{m_{21} + n_{21}} = \frac{P_1(s)}{Q_1(s)}$$

$$Z_0(s) = \frac{m_{10} + n_{10}}{m_{20} + n_{20}} = \frac{P_0(s)}{Q_0(s)}$$

$$\text{Ev } Z_1(s) = \frac{m_{11}m_{21} - n_{11}n_{21}}{Q_1(s)Q_1(-s)} = \frac{A_1(s)}{Q_1(s)Q_1(-s)}$$

$$\text{Ev } Z_0(s) = \frac{m_{10}m_{20} - n_{10}n_{20}}{Q_0(s)Q_0(-s)} = \frac{A_0(s)}{Q_0(s)Q_0(-s)} \qquad 3.6$$

By inspection of the relation between even-parts, Eq. 3.4, it is clear that any zero of the even part of $Z_1(s)$ which is not also an even-part zero of $Z_0(s)$ must be contained in $z_{12}(s)$. The polynomial $A_1(s)$ must contain $A_0(s)$ as a factor and in addition contain the remaining (finite) even-part zeros of $Z_1(s)$. These remaining zeros are realized by a lossless coupling network and therefore must be of even order. If not, $Z_1(s)$ can be augmented as in the Darlington procedure to make them even. Thus the ratio of A_1 to A_0 must be a perfect square (except perhaps for sign). If A_1/A_0 is a perfect square, then the numerator of z_{12} is an even polynomial, say m_{12}, defined by

$$m_{12} = \sqrt{A_1/A_0} \quad . \qquad 3.7$$

If the ratio A_1/A_0 is the negative of a perfect square, then the numerator of z_{12} is an odd polynomial, say n_{12}, defined by

$$n_{12} = \sqrt{-A_1/A_0} \quad . \qquad 3.8$$

These two possibilities lead to two separate cases just as in the Darlington procedure. That is, the open-circuit impedancr parameters of the lossless coupling network are the ratios of even to odd polynomials in Case I and the ratio of odd to even polynomials in Case II. In the Darlington realization procedure, Case A corresponds to a polynomial A which is a perfect square and Case B to a polynomial A which is the negative of a perfect square. In the present problem, Case I corresponds to $Z_1(s)$ and $Z_0(s)$ being the same Darlington case (either both case A or both case B) and case II to the two impedances being opposite Darlington cases (one case A and one case B).

Since the two possibilities are very similiar, the derivation will be carried out in detail only for case I. Case II will be considered later.

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5. The even parts of the polynomials are denoted by the symbol m and the odd parts by the symbol n.

The problem can now be set up within the framework of the cascade-synthesis technique. Define the open-circuit impedance parameters of the lossless coupling network (as before) by

$$z_{11}(s) = m_1/n$$
$$z_{12}(s) = m_{12}/n$$
$$z_{22}(s) = m_2/n. \qquad 3.9$$

The relation between $Z_1(s)$ and $Z_0(s)$ can be written in terms of these impedances:

$$Z_1(s) = z_{11} - \frac{z_{12}^2}{z_{22} + Z_0(s)} \qquad 3.10$$

This can be rearranged to read

$$[z_{11} - Z_1][z_{22} + Z_0] = z_{12}^2 = [m_{12}/n]^2 \qquad 3.11$$

Factor this equation in the usual manner into two equations:

$$z_{11} - Z_1 = \frac{m_{12}^2 Q_0}{nQ_1}$$

$$z_{22} + Z_0 = \frac{Q_1}{nQ_0} . \qquad 3.12$$

The placement of the various factors comes about from the following reasoning. Each equation must have the polynomial $\underline{n}$ in the denominator since the left hand side of each equation has poles at the zeros of this polynomial. Due to Z_1 and Z_0 having poles at the zeros of Q_1 and Q_0 respectively, these polynomials must appear in the denominators of the corresponding equations. But since they do not appear on the right hand side of Eq. 3.11, they must also appear in the numerators of the opposite equations so that the product of the two equations does not contain these polynomials on the right hand side. No extra common factors are permitted. Therefore the only other term to be taken care of is m_{12}^2. Since $z_{22} + Z_0$ is a p.r. function, none of the right half plane zeros of m_{12}^2 can appear in the numerator of the second equation. Thus they must be placed in the numerator of the first equation. At these right half plane zeros, each side of the first equation is zero. But note that the even part of z_{11} is identically zero and the even part of $Z_1(s)$ is zero at these frequencies by hypothesis. Hence the even part of each side of the first equation is zero as well as each side itself being zero. This requires that the odd part be zero also at these frequencies. Since even and odd parts of impedances have zeros with quadrantal symmetry, the first equation must have each side zero at the images of these right half-plane zeros. In other words, the first equation must have zeros at all of the zeros of m_{12} (not only the right half-plane ones), and therefore the entire polynomial m_{12}

must be placed in the numerator of the first equation. The placement of the left half-plane zeros of m_{12} is not arbitrary as has sometimes been thought.[6]

If now each impedance is replaced by its definition in terms of even and odd polynomials (Eqs. 3.6 and 3.9), and the equations separated into even and odd parts, there results,

$$\begin{aligned} m_1 m_{21} - n_{11} n &= m_{20} m_{12}^2 \\ m_1 n_{21} - m_{11} n &= n_{20} m_{12}^2 \\ m_2 m_{20} + n_{10} n &= m_{21} \\ m_2 n_{20} + m_{10} n &= n_{21} \\ A_1/A_0 &= m_{12}^2 \end{aligned} \qquad 3.13$$

If these equations are solved for the open-circuit impedance parameters of the lossless coupling network, N, there results:

$$z_{11} = \frac{m_{11} m_{20} - n_{11} n_{20}}{m_{20} n_{21} - m_{21} n_{20}}$$

$$z_{12} = \frac{m_{12}}{m_{20} n_{21} - m_{21} n_{20}}$$

$$z_{22} = \frac{m_{11} m_{21} - n_{11} n_{21}}{m_{20} n_{21} - m_{21} n_{20}} \qquad 3.14$$

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6. Hurst, op. cit., p.18.

Direct calculation shows that the network shown in Fig. 3.2 realizes these impedances. Comparison of this network with the network derived from Darlington theory (Fig. 2.5) shows that the same network has resulted. Thus the two techniques agree exactly.

Calculations assuming Case II give similiar results with different definitions for the z_{ij} shown in Figure 3.2. In any case, the network is correct if the z_{ij} are interpreted as the z_{ij} determined in the Darlington realization technique.

The solution of Eqs. 3.13 can be written more conveniently as:

$$n = \text{Odd } Q_1(s)Q_0(-s)/A_0$$
$$m_1 = \text{Even } Q_1(s)P_0(-s)/A_0$$
$$m_2 = \text{Even } P_1(s)Q_0(-s)/A_0$$
$$m_{12} = \sqrt{A_1/A_0} \qquad 3.15$$

These results plus the definitions (Eqs. 3.9) lead to the expressions for the overall open-circuit impedance parameters of the network N as shown in Fig. 3.2. Results for Case II are also given there. Although there are four combinations of Darlington cases for $Z_1(s)$ and $Z_o(s)$--two combinations for each impedance--and therefore four different networks N (as defined by the Darlington z_{ij}'s), only two possible sets of overall impedance parameters result, as previously seen.

Again, as in Chapter Two, it is clear that a necessary and sufficient condition that $Z_0(s)$ and $Z_1(s)$ be compatible is that the network N be physically realizable.

Figure 3.1

Illustrating the Cascade-Synthesis Problem

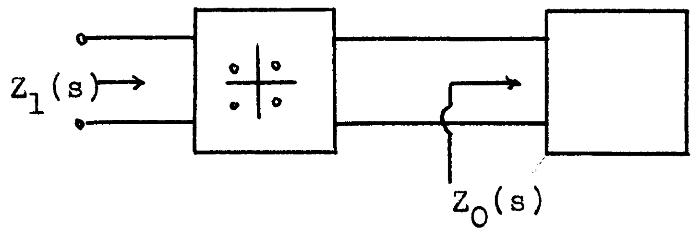
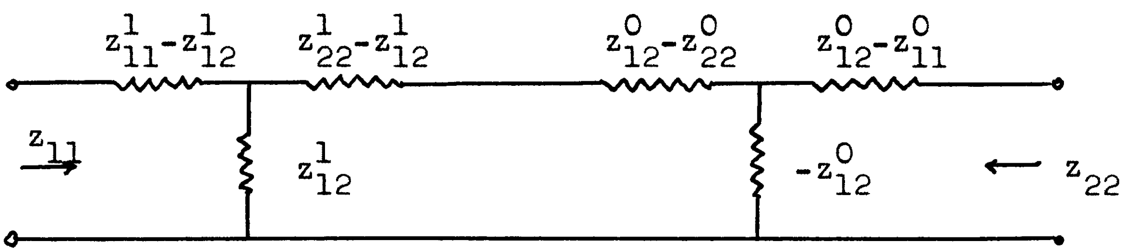


Figure 3.2

Realization of the Lossless Network N



z^1_{ij} = Darlington impedances of Z_1

z^0_{ij} = Darlington impedances of Z_0

<u>Case I</u>

$$z_{11} = \frac{\text{Ev } Q_1(s)P_0(-s)/A_0}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

$$z_{12} = \frac{\sqrt{A_1/A_0}}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

$$z_{22} = \frac{\text{Ev } P_1(s)Q_0(-s)/A_0}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

<u>Case II</u>

$$z_{11} = \frac{\text{Od } Q_1(s)P_0(-s)/A_0}{\text{Ev } Q_1(s)Q_0(-s)/A_0}$$

$$z_{12} = \frac{\sqrt{-A_1/A_0}}{\text{Ev } Q_1(s)Q_0(-s)/A_0}$$

$$z_{22} = \frac{\text{Od } P_1(s)Q_0(-s)/A_0}{\text{Ev } Q_1(s)Q_0(-s)/A_0}$$

Chapter Four

Solution of the Special Case

Both approaches to the problem have resulted in the same necessary and sufficient condition for the compatibility of the two impedances, namely that the network N defined in Fig. 2.5 or 3.2 be physically realizable.

The realizability conditions for reactive networks such as this are well known but will be summarized.[1] Let z_{11}, z_{12}, and z_{22} be the open-circuit impedance parameters of a lossless network. The network is realizable if and only if

1. z_{11} and z_{22} are each physically realizable reactance functions.
2. z_{12} has only simple poles, each of which is a pole of both z_{11} and z_{22}.
3. at each pole of z_{12}, the residue must be real and satisfy the residue condition $k_{11}k_{22}-k_{12}^2$ positive or zero (k_{ij} is the residue of z_{ij}).

The realizability conditions can be greatly simplified for the network under consideration. Inspection of Eqs. 3.14 shows that the three open-circuit impedance parameters have the same finite poles. Since none of these poles are poles of $Z_1(s)$--for $Z_1(s)$ is assumed to have no $j\omega$ axis poles--the residue condition must be satisfied with the equal sign.

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1. Guillemin, _op_. _cit_., p. 218.

That is, at each pole, $k_{11}k_{22}-k_{12}^2 = 0$. Poles whose residues satisfy this condition are said to be compact. Inspection of Eq. 3.4, the relation between the even parts of $Z_1(s)$ and $Z_0(s)$, shows that $z_{12}(s)$ can not have a higher order pole at infinity than does $z_{22}(s)$ because this would cause the even part of $Z_1(s)$ to have a lower order zero at infinity than does the even part of $Z_0(s)$. Again since by assumption $Z_1(s)$ has no pole at infinity, any pole of the z_{ij} at infinity must also be compact.

Because of these things, the realizability conditions on the network N reduce to: <u>The network N is physically realizable if and only if the impedance $z_{22}(s)$ is a physically realizable reactance function</u>. This follows from the following reasoning. If $z_{22}(s)$ is p.r., at each pole the residue k_{22} is real and positive. Since $k_{11}k_{22}-k_{12}^2 = 0$, k_{11} must also be positive and real. This is also true for any pole at infinity. Hence if $z_{22}(s)$ is p.r., then so is z_{11} and since the residue condition is satisfied at all poles, so is the network defined by the set z_{11}, z_{12}, z_{22}.

The realizability conditions on $z_{22}(s)$ can most easily be derived by considering the cascade-synthesis development of Chapter Three. Consider Eq. 3.12 (again, Case I is assumed):

$$z_{22} + Z_0 = \frac{Q_1}{nQ_0} \qquad 3.12$$

The impedance $z_{22}(s)$ is a reactance function and thus has zero even part. Hence, the even part of $Z_0(s)$ must equal the even part of Q_1/nQ_0. Using the same notation as in Eqs. 3.6, there results

$$\mathrm{Ev}\ Z_0 = \frac{A_0}{Q_0(s)Q_0(-s)}$$

$$= \frac{\mathrm{Odd}\ Q_1(s)Q_0(-s)}{nQ_0(s)Q_0(-s)} \qquad 4.1$$

or

$$\mathrm{Odd}\ Q_1(s)Q_0(-s) = nA_0. \qquad 4.2$$

This is a major restriction on the permissible locations of the poles of the two impedances $Z_1(s)$ and $Z_0(s)$. This is not a trivial requirement, nor is it automatically fulfilled. It is a very stringent and special requirement, and places many limitations on the types of impedances which can be compatible with a given impedance.

This requirement, Eq. 4.2, can also be seen from the equivalent network (Fig. 2.5) very easily. Consider the impedance z_{22} as defined by this network. At a right half-plane or $j\omega$-axis zero, s_o, of the even part of $Z_0(s)$, the impedance $z_{12}^0 = 0$. Hence if $z_{22}^1 - z_{22}^0$ is non-zero at $s = s_o$, z_{22} behaves like $-z_{11}^0$ in the vicinity of this frequency. But z_{11}^0 is a p.r. function, meaning that in the vicinity of this frequency, z_{22} is behaving like a negative-real function, that is, z_{22} has a negative real-part in the vicinity of $s = s_o$. Therefore there are points in the right half-plane for which $z_{22}(s)$ has a negative real-part. It would then be impossible for z_{22} to be p.r. or the network N physically realizable. If N is to be physically realizable, it is necessary for $z^1_{22} - z^0_{22}$ to be zero at the frequency $s = s_o$ in such a way that

$$\frac{[z^0_{12}]^2}{z^1_{22} - z^0_{22}} \neq 0.$$

If this quantity is written in terms of polynomials, there results

$$\frac{[z^0_{12}]^2}{z^1_{22} - z^0_{22}} = \frac{n_{21}A_0}{n_{20}[m_{20}n_{21}-m_{21}n_{20}]} \qquad 4.3$$

$$= \frac{n_{21}}{n_{20}} \left[\frac{A_0}{\text{Odd } Q_1(s)Q_0(-s)} \right] \qquad 4.4$$

If this quantity is to be non-zero when $A_0 = 0$, it is clear that the quantity Odd $Q_1(s)Q_0(-s)$ must contain A_0 as a factor. This is the same requirement as in Eq. 4.2. Physically, this requires that $z^1_{22} = z^0_{22}$ at any zero of the even part of $Z_0(s)$, a very special condition.

The above condition, together with the requirement that the even part of $Z_1(s)$ contain all of the zeros of the even part of $Z_0(s)$, constitute the major restrictions on compatible impedances.

But these conditions, although necessary in order that $z_{22}(s)$ be p.r., are not sufficient. The p.r. character of z_{22} can be investigated by considering the impedance $Q_1(s)/nQ_0(s)$, which is related to z_{22} by Eq. 3.12:

$$z_{22} = \frac{Q_1}{nQ_0} - Z_0 \qquad 4.5$$

A necessary and sufficient condition that z_{22} be p.r. is that the impedance Q_1/nQ_0 be p.r. The necessity is easy to show. If z_{22} is p.r., Q_1/nQ_0 is p.r. since the sum of p.r. functions is p.r. The sufficiency is also straightforward. Consider z_{22} as defined in Eq. 4.5. Since Q_1/nQ_0 and Z_0 are both p.r., z_{22} has no right half-plane poles. On the $j\omega$ axis, the real part of z_{22} is identically zero (since the odd part of $Q_1(s)$ times $Q_0(-s)$ is equal to nA_0 by the previous restriction, the real part of $Q_1/nQ_0 = \text{Re } Z_0$). The remaining condition of the p.r. test is that $j\omega$-axis poles of z_{22} be simple with real, positive residues. But Z_0 has no $j\omega$-axis poles by hypothesis. Hence any $j\omega$-axis poles of z_{22} are contributed by Q_1/nQ_0 alone, and these assuredly have real, positive residues since Q_1/nQ_0 is assumed p.r. Thus the sufficiency is shown.

The coupling network, N, will be physically realizable if and only if Q_1/nQ_0 is p.r. and the two restrictions previously derived are satisfied. But the pr character of Q_1/nQ_0 is easy to determine. It is not necessary to test Q_1/nQ_0 (which has $j\omega$-axis poles and is therefore difficult to handle) but is sufficient to test the simpler function, nQ_0/Q_1.

The impedance nQ_0/Q_1 has no right half-plane poles (the zeros of the polynomial $Q_1(s)$ are the poles of $Z_1(s)$ which is p.r.). The real part along the $j\omega$ axis is

$$\mathrm{Re}\,\frac{nQ_0}{Q_1} = \frac{n\ \mathrm{Odd}\ Q_0(s)Q_1(-s)}{Q_1(s)Q_1(-s)}\ , \quad s = j\omega. \qquad 4.6$$

Making the substitution Odd $Q_1(s)Q_0(-s) = nA_0$ gives

$$\mathrm{Re}\,\frac{nQ_0}{Q_1} = \frac{-n^2A_0}{Q_1(s)Q_1(-s)}\ , \quad s = j\omega. \qquad 4.7$$

Eq. 4.7 is always positive since A_0, $-n^2(j\omega)$, and $Q_1(j\omega)Q_1(-j\omega)$ are always positive. Hence, the only thing left to test to determine whether or not nQ_0/Q_1 is positive-real or not is whether any $j\omega$-axis poles are simple with real, positive residues. But $Q_1(s)$ has no $j\omega$-axis zeros by assumption. The only possible $j\omega$-axis poles of nQ_0/Q_1 must therefore lie at infinity. Therefore the only way nQ_0/Q_1 can fail being p.r. is that it have either a multiple order pole at infinity or else a simple pole at infinity with a negative residue. Thus the remaining necessary condition on $Z_1(s)$ and $Z_0(s)$ in order that they be compatible is that nQ_0/Q_1 be realizable at infinity, that is, have at most a simple pole with a real, positive residue.

The three conditions can be simply stated:

1. The even part of $Z_1(s)$ must contain all of the zeros of the even part of $Z_0(s)$, including those at infinity, with at least the same multiplicity. The remaining zeros must be of even order (Z_1 can be augmented if this later condition is not satisfied, but not if the former condition is not satisfied).

2. The odd part of $Q_1(s)Q_0(-s)$ must contain A_0 as a factor (A_0 contains all of the finite zeros of the even part of $Z_0(s)$).

3. The quantity nQ_0/Q_1 must have, at most, a simple pole at infinity with a positive residue. The quantity n is defined by

$$n = \text{Odd } Q_1(s)Q_0(-s)/A_0.$$

These conditions were derived assuming Case I. The results for Case II are similiar. They are:

1. The even part of $Z_1(s)$ must contain all of the even-part zeros of $Z_0(s)$, including those at infinity, with at least the same multiplicity. The remaining zeros must be of even order (Z_1 can be augmented if this later condition is not satisfied, but not if the former condition is not fulfilled).

2. The <u>even</u> part of $Q_1(s)Q_0(-s)$ must contain A_0 as a factor.

3. The quantity mQ_0/Q_1 must have, at most, a simple pole at infinity with a positive residue. The polynomial <u>m</u> is defined by

$$m = \text{Ev } Q_1(s)Q_0(-s)/A_0.$$

It has been shown that the conditions listed above are both necessary and sufficient for the impedance $z_{22}(s)$ to be positive-real and hence physically realizable. It has also been shown that the realizability of $z_{22}(s)$ guarantees the realizability of the network, N, shown in Fig. 3.2. In turn, the network N has the property that when it is terminated in $Z_0(s)$, its input impedance is $Z_1(s)$. Hence it follows that the three derived conditions are both necessary and sufficient conditions that $Z_1(s)$ and $Z_0(s)$ be compatible without any extra common factors augmenting $Z_1(s)$.

Given two impedances, it is a simple matter to test them to determine whether or not they are compatible without common factors. The tests do not involve laborious work, equation solving, etc., but merely require one polynomial to be divided by another to determine whether or not the latter is a factor of the former. The third condition (nQ_0/Q_1 realizable at infinity) merely requires that the order of the pole at infinity be calculated and the sign of its residue observed. This can be done by inspection. The tests have the added advantage that, if the two impedances are compatible, the polynomials necessary for the calculation of the open-circuit impedance parameters of the coupling network are available at the end of the test. Hence synthesis can proceed without a great deal of additional calculation. These results are summarized in Table 4.1.

It is worthwhile to consider several examples.

Example 1

$$Z_0 = \frac{P_0}{Q_0} = \frac{6s^2 + 5s + s}{s^2 + 3s + 2}$$

$$Z_1 = \frac{P_1}{Q_1} = \frac{6s^4 + 6s^3 + 12s^2 + 7s + 3}{6s^5 + 6s^4 + 18s^3 + 13s^2 + 9s + 2}$$

First, the even-parts are computed:

$$A_0 = \text{Ev } P_0(s)Q_0(-s) = 6s^4 + 6$$
$$A_1 = \text{Ev } P_1(s)Q_1(-s) = 6s^4 + 6$$

Since $A_0 = A_1$, the first condition is satisfied. This is a Case I problem (A_1/A_0 is a perfect square). Next the second condition is examined:

$$\frac{\text{Odd } Q_1(s)Q_0(-s)}{A_0} = s^3 + 2s = n$$

Condition II is satisfied since the remainder is a polynomial. Moreover, this polynomial is $\underline{n}$, the denominator of the open-circuit impedance functions of the lossless coupling network.

The third condition is that nQ_0/Q_1 must be realizable at infinity. As $\underline{s}$ approaches infinity, nQ_0/Q_1 approaches $s^3(s^2)/6s^5$ which is 1/6. This is realizable and therefore the third condition is also satisfied and the two impedances are compatible. If the coupling network is desired, it may be computed from table 4.1.

$$m_2 = \text{Ev } Q_1(s)P_0(-s)/A_0 = s^2 + 1$$
$$m_{12} = \sqrt{A_1/A_0} = 1$$
$$m_1 = \text{Ev } P_1(s)Q_0(-s) = s^2 + 1.$$

Hence

$$z_{11} = \frac{m_1}{n} = \frac{s^2 + 1}{s^3 + 2s}$$

$$z_{12} = \frac{m_{12}}{n} = \frac{1}{s^3 + 2s}$$

$$z_{22} = \frac{m_2}{n} = \frac{s^2 + 1}{s^3 + 2s}$$

Since $Z_1(s)$ has no $j\omega$-axis poles, these impedances form a compact set. In addition, all of the zeros of z_{12} lie at infinity. Hence the network is realized merely by expanding z_{22} into a Cauer form about infinity. The network is shown in Fig. 4.1.

Example 2

$$Z_0 = \frac{6s^2 + 5s + 3}{s^2 + 3s + 2}$$

$$Z_1 = \frac{6s^5 + 6s^4 + 12s^3 + 7s^2 + 3s}{s^5 + 15s^4 + 13s^3 + 15s^2 + 7s + 3}$$

Calculation of the even-parts yields,

$$A_0 = 6s^4 + 6$$
$$A_1 = -s^6(6s^4 + 6).$$

Clearly A_0 is contained as a factor in A_1. Moreover, since $A_1/A_0 = -s^6$, this is a Case II situation.

Condition two yields:

$$m = \frac{\text{Ev } Q_1(s)Q_0(-s)}{A_0} = 2s^2 + 1.$$

Condition three yields: As s approaches infinity

mQ_0/Q_1 approaches $2/s$.

This is realizable and hence all three conditions are satisfied and the two impedances are compatible. Using the Case II formulae for the open-circuit impedances yields

$$z_{11} = z_{22} = \frac{s^3 + s}{2s^2 + 1}$$

$$z_{12} = \frac{s^3}{2s^2 + 1}$$

Again the set is compact. All of the transmission zeros are at the origin. Hence the network is realized by expanding z_{22} into a Cauer form about the origin. The circuit is shown in Fig. 4.2.

Example 3

$$Z_0 = \frac{102s^2 + 170s + 119}{14s^2 + 17s + 12}$$

$$Z_1 = \frac{3s^2 + 2s + 1}{s^3 + s^2 + 2s + 1}$$

Calculation of the even-parts yields

$$A_0 = 1428(s^4 + 1)$$
$$A_1 = s^4 + 1.$$

Condition I is satisfied and the impedances are Case I types.

The second condition yields

$$\frac{\text{Odd } Q_1(s)Q_0(-s)}{A_0} = \text{not a polynomial.}$$

The two impedances are not compatible because condition two is not satisfied. But suppose $Z_1(s)$ is augmented with the polynomial $(s + 1)^2$, that is, both the numerator and denominator of $Z_1(s)$ are multiplied by this polynomial. This doesn't change $Z_1(s)$ but does affect the compatibility.

After augmentation, there results

$$Z_1 = \frac{3s^4 + 8s^3 + 8s^2 + 4s + 1}{s^5 + 3s^4 + 5s^3 + 6s^2 + 4s + 1}$$

The even parts are now:

$$A_0 = 1428(s^4 + 1)$$
$$A_1 = (s^4 + 1)(1 - s^2)^2$$

A_0 is contained in A_1 and A_1/A_0 is a perfect square. Condition I is still satisfied and the impedances belong to Case I. Condition II now becomes:

$$\frac{\text{Odd } Q_1(s)Q_0(-s)}{A_0} = \frac{14s^3 + 31s}{1428} = n.$$

Condition II is now satisfied. Condition III becomes:

$$nQ_o/Q_1 \text{ approaches a constant.}$$

Hence Condition III is satisfied and the two impedances are therefore compatible. Calculation of the open-circuit impedance parameters of the coupling network yields:

$$z_{11} = \frac{42s^2 + 12}{14s^3 + 31s}$$

$$z_{12} = \frac{1428}{14s^3 + 31s}$$

$$z_{22} = \frac{136s^2 + 119}{14s^3 + 31s} .$$

This last example is interesting in that two apparently noncompatible impedances were converted to compatible ones merely by augmenting $Z_1(s)$ by the proper common factor (a perfect square). The price for this is an increase in complexity, for the numerator and denominator of $Z_1(s)$ are increased by four in order. However this is more than compensated for because now a solution exists whereas before none existed.

Thus, it is necessary to investigate this possibility and also to remove the restriction of no $j\omega$-axis poles in the impedances.

Table 4.1

Summary of Testing Procedure

$$Z_0 = \frac{P_0}{Q_0} \qquad \text{Ev } Z_0 = \frac{A_0}{Q_0(s)Q_0(-s)}$$

$$Z_1 = \frac{P_1}{Q_1} \qquad \text{Ev } Z_1 = \frac{A_1}{Q_1(s)Q_1(-s)}$$

Case I

1. A_1 must contain A_0 as a factor.
2. A_1/A_0 must be a perfect square (if not, augment Z_1)
3. $n = \text{Odd } Q_1(s)Q_0(-s)/A_0$ must be a polynomial.
4. nQ_0/Q_1 must be realizable at infinity.

Case II

1. A_1 must contain A_0 as a factor.
2. $-A_1/A_0$ must be a perfect square (if not, augment Z_1)
3. $m = \text{Ev } Q_1(s)Q_0(-s)/A_0$ must be a polynomial.
4. mQ_0/Q_1 must be realizable at infinity.

The Network N

Case I:

$$z_{11} = \frac{\text{Ev } Q_1(s)P_0(-s)/A_0}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

$$z_{12} = \frac{\sqrt{A_1/A_0}}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

$$z_{22} = \frac{\text{Ev } P_1(s)Q_0(-s)/A_0}{\text{Od } Q_1(s)Q_0(-s)/A_0}$$

Case II:

$$z_{11} = \frac{\text{Od } Q_1(s)P_0(-s)/A_0}{\text{Ev } Q_1(s)Q_0(-s)/A_0}$$

$$z_{12} = \frac{\sqrt{-A_1/A_0}}{\text{Ev } Q_1(s)Q_0(-s)/A_0}$$

$$z_{22} = \frac{\text{Od } P_1(s)Q_0(-s)/A_0}{\text{Ev } Q_1(s)Q_0(-s)/A_o}$$

Figure 4.1

Realization for Example 1

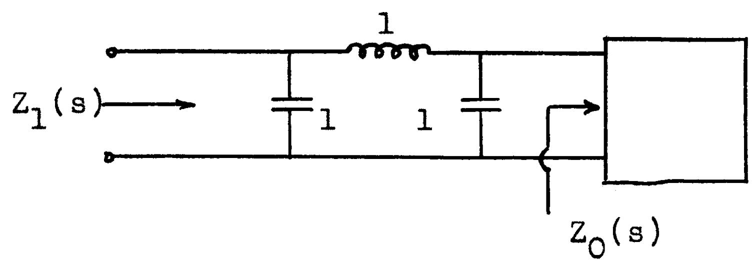
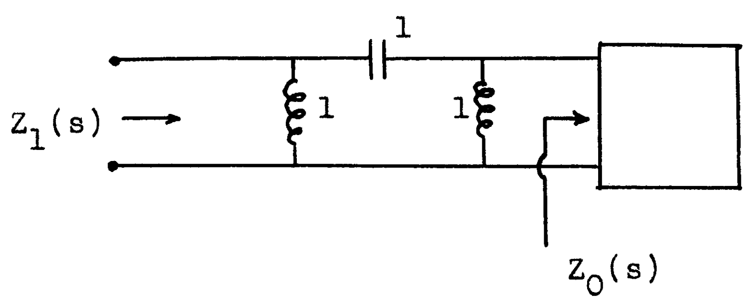


Figure 4.2

Realization for Example 2



Chapter Five

The Common Factor Problem

The set of conditions under which two impedances can be compatible which has been derived is a sufficient but not necessary set. This is due to the restriction that the two impedances be compatible without factors common to both numerator and denominator of $Z_1(s)$. That augmenting with common factors can help was seen in an example at the end of the last chapter. If such augmentation is allowed in the realization--and this doesn't affect the impedances--the compatibility conditions can be greatly relaxed.

The effect of augmenting factors is easily seen. Consider first the Darlington realization technique. To be specific, consider the impedance

$$Z(s) = \frac{s + 1}{s^2 + s + 1} \quad .$$

This impedance has an even part whose numerator is unity, and therefore is a Case A impedance. The open-circuit impedance parameters of the lossless coupling network are given by

$$z_{11} = 1/s$$
$$z_{12} = 1/s$$
$$z_{22} = [s^2 + 1]/s$$

The network realization involves only two elements and is shown in Fig. 5.1. Now consider the same impedance with both numerator and denominator multiplied by a common factor. This factor must be a perfect square so that the numerator of the even part remains a perfect square. Let the common factor be $(s + 1)^2$. The impedance is then given by

$$Z(s) = \frac{(s + 1)^3}{s^4 + 3s^3 + 4s^2 + 3s + 1} .$$

The open-circuit impedance parameters of the lossless coupling network are given by

$$z_{11} = \frac{3s^2 + 1}{3s^3 + 3s}$$

$$z_{22} = \frac{s^4 + 4s^2 + 1}{3s^3 + 3s}$$

$$z_{12} = \frac{1 - s^2}{3s^3 + 3s} .$$

The realization of this network is shown in Fig. 5.2. Notice that completely different impedance parameters and networks result. In fact, the second lossless network has a transmission zero at a positive-real frequency and hence does not exist as an unbalanced network without mutual coupling whereas the first network which does not have this transmission zero does exist in this form. Yet both of these networks when terminated in a one-ohm resistor realize the

the same impedance Z(s). The effect of the common factor is not trivial. It changes the lossless coupling network appreciably.

Now consider the case of compatible impedances. It has been shown that the coupling network, N, can be represented by the cascade connection of the Darlington lossless network associated with $Z_1(s)$ and the negative of the transposed Darlington network associated with $Z_0(s)$ as far as its terminal behaviour is concerned (see Fig. 2.5). If this network is not realizable, the two impedances are not compatible without common factors being added. With the previous example in mind, the possibility suggests itself that one or both of these lossless networks might be modified by introducing common factors into the impedances $Z_1(s)$ and/or $Z_0(s)$ in such a way as to make the network N physically realizable. The problem is to determine when this is possible, and how to carry out the mechanics of finding suitable common factors.

To this end, consider a physical interpretation of the common factor in the Darlington synthesis. If the impedance (before augmenting) is given by

$$Z(s) = \frac{m_1 + n_1}{m_2 + n_2} \qquad 5.1$$

the Darlington impedances are given by (assuming Case A)

$$\begin{aligned} z_{11} &= m_1/n_2 \\ z_{22} &= m_2/n_2 \\ z_{12} &= \sqrt{m_1m_2-n_1n_2}\;/n_2 \\ z &= n_1/n_2. \end{aligned} \qquad 5.2$$

The impedance Z(s) can be written in terms of these as

$$Z(s) = \frac{\frac{n_1}{n_2} + \frac{m_1}{n_2}R}{\frac{m_2}{n_2} + R} \qquad 5.3$$

where R is the one-ohm terminating resistance. If the one-ohm resistance is replaced by a constant resistance lattice whose input impedance is one ohm (Fig. 5.3), R is replaced by

$$R = \frac{(m_a + n_a)^2}{(m_a + n_a)^2} \qquad 5.4$$

where the impedances of the lattice arms is given by

$$\begin{aligned} z_a &= m_a/n_a \\ z_b &= n_a/m_a \end{aligned} \qquad 5.5$$

or vice versa.

Replacing R in Eq. 5.3 by its definition in Eq. 5.4 gives after clearing fractions

$$Z(s) = \frac{(m_1 + n_1)(m_a + n_a)^2}{(m_2 + n_2)(m_a + n_a)^2} . \qquad 5.6$$

Hence, the common factor in the impedance Z(s) may be interpreted as the replacement of the one-ohm terminating resistance by a one-ohm constant resistance lattice.[1]

With this in mind, consider the effect of common factors in the impedances $Z_0(s)$ and $Z_1(s)$ which are to be compatible. If a common factor is introduced into $Z_0(s)$ but not $Z_1(s)$, the even part of $Z_0(s)$ will have that factor in its numerator. Hence by Condition I of table 4.1 this factor must also be multiplied into $Z_1(s)$ if the impedances are to be compatible. The effect of augmentation of both $Z_0(s)$ and $Z_1(s)$ with common factors is best illustrated by treating the factor as resulting from a constant resistance lattice termination.

Introduction of the common factor into the numerator and denominator of both $Z_0(s)$ and $Z_1(s)$ modifies the lossless networks by the connection of the lattice network at the output terminals as shown in Fig. 5.4. If now these lossless networks are connected so as to yield the coupling network N (N_1 connected in cascade with the transposed network N_0 with all impedances replaced by their negatives in N_0) the network of Fig. 5.5 results. The two lattice networks now reduce to the trivial network shown in Fig. 5.6 and thus the network N degenerates to the same network it was before the common factors were introduced. Hence if N was not physically realizable without common factors, it is still not physically realizable after common factors are added to both impedances.

Next consider the remaining alternative, the augmentation of $Z_1(s)$ but not $Z_0(s)$. Again realizing this common factor by means of a lattice, the Darlington network of $Z_1(s)$ is modified by the addition of the lattice but the Darlington network of $Z_0(s)$ is not changed. Connecting these networks so as to realize the coupling network N gives the result shown in Fig. 5.7. This single lattice does not disappear in this case and can affect the realizability of N as the following reasoning shows. The network N is realizable if and only if the impedance $z_{22}(s)$ is physically realizable. This impedance is given by

$$z_{22}(s) = \frac{\dfrac{n_{10}}{n_{20}} - \dfrac{m_{10}}{n_{20}} z_x}{-\dfrac{m_{20}}{n_{20}} + z_x} \qquad 5.7$$

where $z_x = m_{21}/n_{21}$ when no common factors have been in-

1. This was pointed out by Fano, op. cit.

serted and is

$$z_x = \frac{1 + \frac{m_{21}}{n_{21}}\left[\frac{m_a^2 + n_a^2}{2m_a n_a} \right]}{\frac{m_{21}}{n_{21}} + \frac{m_a^2 + n_a^2}{2m_a n_a}} \qquad 5.8$$

when the common factor $(m_a + n_a)^2$ has been inserted into the numerator and denominator of $Z_1(s)$. This expression for z_x is recognized as merely a linear fractional transformation of the impedance m_{21}/n_{21}. Note that the only part of the network N_1 which affects the realizability of N is the output impedance and this changes radically with the insertion of common factors into $Z_1(s)$. Hence if Z_0 and Z_1 are not compatible as they stand, perhaps a Hurwitz polynomial can be found which will transform m_{21}/n_{21} into an impedance which will make $z_{22}(s)$ a p.r. impedance.

In order to arrive at a set of necessary and sufficient conditions on the impedances $Z_0(s)$ and $Z_1(s)$ in order for them to be compatible, this common factor problem must be resolved. The problem is best approached by noting that after the common factor, $P_a^2(s)$ has been found and inserted into the impedance $Z_1(s)$, this modified impedance together with $Z_0(s)$ must satisfy the conditions for compatibility without additional common factors. Consider Case I impedances. Of the three necessary and sufficient conditions, Condition I cannot be affected by the addition of common factors. That is, if the even part of $Z_1(s)$ does not contain all of the zeros of the even part of $Z_0(s)$ with at least the same multiplicity, the impedances are not compatible and cannot be made compatible. But Conditions II and III can be modified by the addition of common factors. When these factors have been added to the polynomials $P_1(s)$ and $Q_1(s)$, the compatibility conditions become:

$$\text{Condition II:} \quad \text{Odd } P_a^2(s)Q_1(s)Q_0(-s) = nA_0$$
$$\text{Condition III:} \quad nQ_0/P_a^2Q_1 \text{ realizable at infinity.} \tag{5.9}$$

The problem now is to reduce these conditions on the polynomials $Q_1(s)$ and $Q_0(s)$ to conditions on the polynomial P_a.

This problem appears related to the real-part sufficiency problem. However attempts to solve it in this manner prove fruitless because both sides of Eq. 5.9 have unknown polynomials in contrast with the usual situation where only one side has an unknown polynomial. Also the unknown polynomial must be a perfect square.

For these reasons, another approach appears necessary. In order to control the Hurwitz character of the polynomial $P_a = m_a + n_a$, it is desirable to reduce Conditions II and III to conditions on the impedance $z_a = m_a/n_a$ which is easily tested for p.r. character. Then the polynomial $P_a(s)$ is found by adding together the numerator and denominator of $z_a(s)$. Since this polynomial is multiplied into both the numerator and denominator of the impedance $Z_1(s)$, the constant multiplier of $P_a(s)$ is arbitrary.

Consider first Condition III (Eq. 5.9) which reads

$$Y = \frac{nQ_0(s)}{P_a^2(s)Q_1(s)}$$

must be realizable at infinity. This admittance Y can be manipulated into a more interesting form by replacing n by its definition (Eq. 5.9):

$$Y = \frac{\text{Odd } P_a^2(s)Q_1(s)Q_0(-s)}{A_0} \; \frac{Q_0(s)}{P_a^2(s)Q_1(s)} \qquad 5.10$$

Multiply numerator and denominator of this expression by $Q_0(-s)$ and write it as

$$Y = \frac{\text{Odd } P_a^2(s)Q_1(s)Q_0(-s)}{P_a^2(s)Q_1(s)Q_0(-s)} \; \frac{Q_0(s)Q_0(-s)}{A_0} \qquad 5.11$$

Now $P_a^2(s)Q_1(s)Q_0(-s)$ can be written as the sum of its even and odd parts. Hence Y can be written

$$Y = \frac{Q_0(s)Q_0(-s)/A_0}{1 + W(s)} \qquad 5.12$$

where
$$W(s) = \frac{\text{Ev } P_a^2(s)Q_1(s)Q_0(-s)}{\text{Odd } P_a^2(s)Q_1(s)Q_0(-s)} \qquad 5.13$$

Note that the function in the numerator of Y is the reciprocal of the even part of $Z_0(s)$ and is completely known. The quantity W(s) is an odd function of frequency. Condition III requires that this admittance Y(s) be realizable at infinity, that is, have at most a simple pole with a positive residue.

If the even part of $Z_0(s)$ has a t^{th} order zero at infinity, then as s approaches infinity, the even part of $Z_0(s)$ approaches Ks^{-t} where K is a constant. Since for $s = j\omega$ the even-part equals the real -part, this function must be positive for $s = j\omega$. If t/2 is an even number, $(j\omega)^t = \omega^t$ and K must be positive. If t/2 is odd, $(j\omega)^t = -\omega^t$ and K must be a negative number. Hence the reciprocal of the even part of $Z_0(s)$ has a t^{th} order pole at infinity with a positive coefficient if t/2 is even and a negative coefficient if t/2 is odd.

If t = 0 so that the real part of $Z_0(s)$ has a non-zero value at infinity, Condition III is automatically satisfied since Y(s) can not have a pole at infinity. Hence for t = 0, there are no restrictions on the quantity W(s) and therefore there are also no restrictions on the common factor $P_a^2(s)$ due to Condition III.

If t = 2, 4, 6, ..., then W(s) must have a pole of order $t \pm 1$ at infinity so that Y(s) will have only a simple pole at infinity [since W(s) is an odd function of frequency, it cannot have an even order pole at infinity]. In addition, if W(s) has a pole of order t - 1 at infinity, its coefficient must be the same sign as that of the even part of $Z_0(s)$ so that Y(s) will have a positive residue in its simple pole.

Thus Condition III has reduced to a condition on the quantity W(s) which involves the common factor $P_a^2(s)$. This can further be reduced by expanding W(s):

$$W(s) = \frac{\text{Ev } P_a^2 Q_1(s)Q_0(-s)}{\text{Odd } P_a^2 Q_1(s)Q_0(-s)}$$

$$= \frac{[\text{Ev } P_a^2][\text{Ev } Q_1(s)Q_0(-s)] + [\text{Odd } P_a^2][\text{Odd } Q_1(s)Q_0(-s)]}{[\text{Ev } P_a^2][\text{Odd } Q_1(s)Q_0(-s)] + [\text{Odd } P_a^2][\text{Ev } Q_1(s)Q_0(-s)]} \qquad 5.14$$

Next define

$$z_{eo}(s) = \frac{\text{Ev } Q_1(s)Q_0(-s)}{\text{Odd } Q_1(s)Q_0(-s)} \qquad 5.15$$

The polynomial $P_a(s)$ can be written as the sum of its even and odd parts as $P_a = m_a + n_a$. Hence

$$\frac{\text{Ev } P_a^2(s)}{\text{Odd } P_a^2(s)} = \frac{m_a^2 + n_a^2}{2m_a n_a} = \frac{1}{2}\left[\frac{m_a}{n_a} + \frac{n_a}{m_a}\right] \qquad 5.16$$

Let $z_a(s)$ be defined as either m_a/n_a or n_a/m_a, whichever has a pole at infinity. With these definitions, W(s) becomes

$$W(s) = \frac{z_a^2 z_{eo} + z_{eo} + 2z_a}{z_a^2 + 2z_a z_{eo} + 1} \qquad 5.17$$

Notice that $z_{eo}(s)$ is completely known and that W(s) involves only this known impedance and the arbitrary impedance $z_a(s)$. The requirements on $z_a(s)$ in order that W(s) have a $t \pm 1$ order pole at infinity can now be found.

Since $z_{eo}(s)$ is not a p.r. function, it can have any order pole at infinity. Consider the three possibilities.

1. Suppose $z_{eo}(s)$ is zero at infinity. Then since $z_a(s)$ has a simple pole at infinity, the numerator of Eq. 5.17 has at most a simple pole at infinity. The denominator has a second order pole at infinity. Hence W(s) is zero at infinity and Y(s) cannot be realizable.

2. Suppose $z_{eo}(s)$ has a multiple order pole at infinity or else a simple pole with a positive residue. If the order of this pole is n, then the numerator of W(s) has an (n + 2) order pole at infinity and the denominator an n + 1 order pole at infinity. Hence W(s) has a simple pole at infinity. Moreover the residue of this pole is positive. A first order pole of W(s) means that the even part of $Z_0(s)$ must have a second order zero at infinity so that Y(s) can have only a simple pole. However, for t = 2, t/2 is odd and the coefficients of the pole of the reciprocal of Ev $Z_0(s)$ and the pole of W(s) are opposite in sign. Hence, Y(s) cannot be realizable.

3. The only remaining possibility is that $z_{eo}(s)$ have a simple pole with a negative residue at infinity. In thiscase, the numerator of W(s) has a third order pole at infinity with a negative residue. The denominator can have either a second order pole at infinity or else any order zero since $z_a(s)$ can be chosen so that $z_a^2 + 2z_a z_{eo} + 1$ has a multiple order zero at infinity.

Clearly, if the even part of $Z_0(s)$ has a t^{th} order zero at infinity, the quantity $z_a^2 + 2z_a z_{eo} + 1$ must have a t - 2 or t - 4 order zero at infinity. If this quantity is expanded into a power series about infinity, the result must be of the form

$$z_a^2 + 2z_a z_{eo} + 1 = \rho_{-2}s^2 + \rho_0 + \rho_2 s^{-2} + \ldots\ldots\ldots +$$

$$+ \rho_{t-4}s^{-(t-4)} + \rho_{t-2}s^{-(t-2)} + \ldots\ldots\ldots \qquad 5.18$$

and the ρ_k must all be zero for $k = -2, 0, 2, \ldots, t-6$. Hence Eq. 5.18 becomes

$$z_a^2 + 2z_a z_{eo} + 1 = \rho_{t-4}s^{-(t-4)} + \rho_{t-2}s^{-(t-2)} + \ldots\ldots \qquad 5.19$$

If t/2 is even, the coefficient ρ_{t-4} must be negative or zero. If t/2 is odd, ρ_{t-4} must be positive or zero [when ρ_{t-4} is not zero, Y(s) has a simple pole at infinity and the sign of ρ_{t-4} must be such that the residue of the pole of Y(s) is positive].

Thus the condition that Y(s) be realizable at infinity has reduced to the requirement that the first few terms of the power series expansion about infinity of $z_a^2 + 2z_a z_{eo} + 1$ must be zero. This requirement can finally be reduced to conditions on the impedance $z_a(s)$ in the following manner.

Let $z_a(s)$ and $z_{eo}(s)$ have power series expansions about infinity of the form

$$z_a(s) = \beta_0 s + \beta_1 s^{-1} + \beta_3 s^{-3} + \ldots\ldots$$
$$z_{eo}(s) = \alpha_0 s + \alpha_1 s^{-1} + \alpha_3 s^{-3} + \ldots\ldots \qquad 5.20$$

If these definitions are placed into Eq. 5.19, the following equations result:

$$\begin{aligned}
\rho_{-2} &= \beta_0[\beta_0 + 2\alpha_0] \\
\rho_0 &= \beta_1[2\beta_0 + 2\alpha_0] + 2\beta_0\alpha_1 + 1 \\
\rho_2 &= \beta_3[2\beta_0 + 2\alpha_0] + \beta_1^2 + 2[\alpha_3\beta_0 + \alpha_1\beta_1] \\
\rho_4 &= \beta_5[2\beta_0 + 2\alpha_0] + 2[\beta_1\beta_3 + \beta_0\alpha_5 + \beta_1\alpha_3 + \beta_3\alpha_1] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad 5.21
\end{aligned}$$

Note that in these equations, the ρ's are zero and all of the α_k are known. The first equation involves β_0 alone, the second β_0 and β_1, the third β_0, β_1, and β_3, etc. The n^{th} equation involves the first n of the β_k. Hence this is not a simultaneous set of equations but the set can be solved one at a time for each of the β_k. Setting ρ_{-2}, ρ_0,....., ρ_{t-6} equal to zero gives values for β_0, β_1, β_3, ..., β_{t-5}. The value of ρ_{t-4} is arbitrary except that it must be positive if t/2 is odd and negative if t/2 is even. The value of β_{t-3} can be determined in terms of ρ_{t-4}.

Hence Condition III reduces to the requirement that the first t/2 terms of the power series expansion about infinity of $z_a(s)$ assume certain specific values.

In Appendix I it is shown that a physically realizable reactance function is uniquely determined by its power series expansion about infinity. To be more explicit, Appendix I shows that the first n elements of the Cauer network having series inductances and shunt capacitances are uniquely related to the first n terms of the power series expansion of the reactance about infinity. Moreover, these relations are not simultaneous equations, but may be solved one at a time for the elements of the Cauer network. If only n terms of the power series expansion are specified, only the first n elements of the Cauer network are specified. Any arbitrary value may be assumed for the remaining elements in the network without changing the first n terms of the power series expansion about infinity.

Using these results, the procedure for calculating the impedance $z_a(s)$ is very straightforward. The first t/2 terms of the power series expansion of $z_a(s)$ about infinity are calculated from Eqs. 5.21. Using the method of Appendix I, the first t/2 elements of the Cauer network for $z_a(s)$ are calculated. The remaining elements of the network are arbitrary and thus can be selected so that Condition II is also satisfied.

Thus, to summarize, the requirement that $nQ_0/P_a^2Q_1$ be realizable at infinity reduces to a realizability condition on the impedance $z_a(s)$ computed by the above scheme. If and only if $z_a(s)$ is p.r. can this condition be fulfilled. The scheme is computationally simple, as an example will show, and leads to an element by element determination of $z_a(s)$. If any element comes out negative, the test stops and the impedances $Z_0(s)$ and $Z_1(s)$ cannot be compatible. If all elements come out positive, Condition III is fulfilled.

The remaining Condition II, that the odd part of $P_a^2 Q_1(s)Q_0(-s)$ contain A_0 as a factor, has not yet been considered. If $A_0 = 1$ so that the even part of $Z_0(s)$ has all of its zeros at infinity, then this second condition is automatically satisfied and the satisfaction of Condition III as above is both necessary and sufficient. When A_0 is not a constant, the arbitrary elements in the network realization of $z_a(s)$ must be chosen so as to satisfy Condition II. Before proceeding further, an example will be given in order to show the simplicity of the procedure derived.

Example 1

$$\text{Ev } Z_0(s) = \frac{1}{Q_0(s)Q_0(-s)}$$

$$\text{Ev } Z_1(s) = \frac{1}{Q_1(s)Q_1(-s)}$$

where

$$Q_0(s) = s^3 + s^2 + 2s + 1$$

$$Q_1(s) = s^5 + 6s^4 + 13s^3 + 14s^2 + 9s + 2$$

First compute the polynomial $Q_1(s)Q_0(-s)$:

$$Q_1(s)Q_0(-s) = [-s^8 - 9s^6 - 15s^4 - 2s^2 + 2] + [-5s^7 - 12s^5 - 8s^3 + 5s].$$

These are Case I impedances. Condition I is satisfied by inspection. The impedances are not compatible without common factors because as s approaches infinity, nQ_0/Q_1 approaches $-5s^5$ which is not realizable [note that Condition II is automatically satisfied since $A_0 = 1$].

To determine whether or not they are compatible with a common factor inserted into $Z_1(s)$, the impedance $z_{eo}(s)$ is formed from the polynomial $Q_1(s)Q_0(-s)$ calculated above:

$$z_{eo}(s) = \frac{-s^8 - 9s^6 - 15s^4 - 2s^2 + 2}{-5s^7 - 12s^5 - 8s^3 + 5s}$$

The even part of $Z_0(s)$ has a sixth order zero at infinity. Hence $W(s)$ must have either a fifth or seventh order pole at infinity. But $z_{eo}(s)$ has a simple pole at infinity with a positive residue and therefore there exists no common factor $P_a^2(s)$ which can produce a multiple order pole in $W(s)$. Hence the impedances are not compatible at all. That is, there exists no lossless, realizable coupling network which will transform $Z_0(s)$ into $Z_1(s)$.

Example 2

Define the even-parts of the two impedances as in Example 1 except that the polynomials $Q_0(s)$ and $Q_1(s)$ are given by

$$Q_0(s) = 83s^3 + 664s^2 + 2053s + 2646$$
$$Q_1(s) = s^5 + 6s^4 + 13s^3 + 14s^2 + 9s + 2.$$

First calculate $Q_1(s)Q_0(-s)$ and form $z_{eo}(s)$, the ratio of the even and odd parts of this polynomial:

$$z_{eo}(s) = \frac{-83s^8 + 852s^6 - 2264s^4 + 56849s^2 + 5292}{166s^7 - 2202s^5 + 68950s^3 + 24708s}$$

By inspection, nQ_0/Q_1 is not realizable at infinity. Thus the impedances are not compatible without common factors. To determine whether or not a common factor exists, consider $z_{eo}(s)$. This has the required simple pole at infinity with a negative residue. Proceed by expanding $z_{eo}(s)$ about infinity:

$$z_{eo}(s) = -\frac{s}{2} - \frac{3}{2s} + \frac{14454}{83s^3} + \ldots\ldots\ldots$$

Since $t = 6$ [the even part of $Z_0(s)$ has a sixth order zero at infinity], it is necessary to set ρ_{-2} and ρ_0 equal to zero. Since $t/2$ is odd, ρ_2 must be chosen positive or zero. Eqs. 5.21 become:

$$\alpha_0 = -1/2; \quad \alpha_1 = -3/2; \quad \alpha_3 = 14454/83.$$

$$0 = \beta_0 - 1; \quad \beta_0 = 1$$

$$\beta_1 = \frac{-2\beta_0\alpha_1 - 1}{2\beta_0 + 2\alpha_0} = 2$$

The third equation becomes

$$\rho_2 = \beta_3 + 28659/83$$

or

$$\beta_3 = \rho_2 - 28659/83.$$

Since ρ_2 must be positive or zero, β_3 can be chosen between the limits of -28659/83 and infinity.

Next determine the elements of the impedance $z_a(s)$. From the relations given in Appendix I, the first three elements are

$$L_1 = 1$$
$$C_2 = 1/2$$
$$L_3 = -4/\beta_3.$$

In order for L_3 to be positive or zero, β_3 must be chosen between zero and -28659/83. Clearly it is most advantageous to choose it equal to zero so that L_3 is infinite. If this is done, the impedance $z_a(s)$ reduces to two elements and is given by

$$z_a(s) = [s^2 + 2]/s$$

Hence the polynomial $P_a(s)$ is given by

$$P_a(s) = s^2 + s + 2,$$

and the augmenting polynomial $P_a^2(s)$ is given by

$$P_a^2(s) = s^4 + 2s^3 + 5s^2 + 4s + 4.$$

Since $z_a(s)$ is realizable, it follows that if $Z_1(s)$ is augmented by multiplying both numerator and denominator by the polynomial $P_a^2(s)$, then $Z_1(s)$ and $Z_0(s)$ are compatible with no further common factors inserted in $Z_1(s)$. That is, $Z_0(s)$ and the augmented $Z_1(s)$ satisfy the conditions of Table 4.1. Hence the equations of Table 4.1 may be used to calculate the open-circuit impedance parameters of the lossless coupling network. It is unnecessary to test these impedances for realizability, for the p.r. character of the impedance $z_a(s)$ guarantees that these are realizable and therefore that the network N is realizable. Notice that Condition II of Eq. 5.9 is automatically satisfied in this example since $A_0 = 1$. Thus for this special case, the compatibility tests become very simple.

Additional polynomials $P_a^2(s)$ can be found by merely adding additional elements onto the given impedance since these additional terms in the Cauer form do not affect the first two terms of the series expansion about infinity. Any value for the element L_3 can be assumed within the legitimate range. Notice in this procedure that it is never necessary to choose more than one parameter (β_3 in the present example), and this parameter is always simply chosen by forcing the last element of the network to be zero or infinite.

Notice also that it is not necessary to calculate all of the β_k before solving for the network elements. As each β is determined, the corresponding element can be determined. For example, first calculate β_0 and determine L_1. Next calculate β_1 and C_2, etc. If at any step an element comes out negative, the process stops. No solution exists. If the process proceeds to a satisfactory conclusion, the work has not been wasted for it is necessary to perform this work to calculate the polynomial $P_a(s)$ which is needed in order to proceed with the synthesis. Hence the technique has the important advantage that all of the work done in testing the impedances for compatibility is useful in the synthesis of the lossless coupling network.

All of this work assumed that $Z_0(s)$ and $Z_1(s)$ were Case I impedances. If instead they are Case II impedances, the only change in the procedure is that instead of working with $z_{eo}(s)$, the above procedure uses $z_{oe}(s)$, that is, the ratio of the odd part to the even part of $Q_1(s)Q_0(-s)$. Hence if t is greater than zero, $z_{oe}(s)$ must have a simple pole at infinity with a negative residue etc. The α_k are found by expanding $z_{oe}(s)$ about infinity rather than $z_{eo}(s)$ as for Case I impedances. The difference is thus very small. Notice, however, that the same polynomials cannot be used with both Case I and Case II impedances since if $z_{eo}(s)$ has a pole at infinity, $z_{oe}(s)$ must have a zero at infinity and vice versa.

An additional nicety of these results is that Conditions II and III separate, that is, may be applied one at a time. This simplifies calculation of the common factor $P_a^2(s)$.

Figure 5.1

Realization of $Z(s) = \dfrac{s + 1}{s^2 + s + 1}$

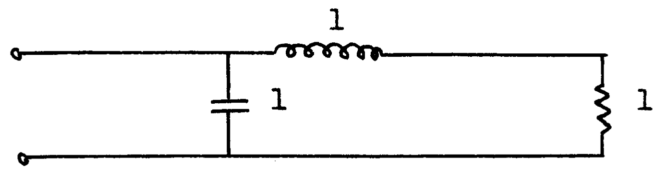


Figure 5.2

Alternate Realization of $Z(s) = \dfrac{s + 1}{s^2 + s + 1}$

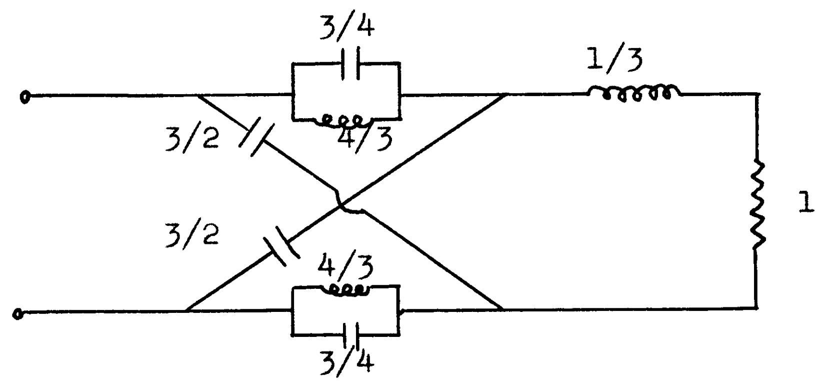


Figure 5.3

The Constant Resistance Lattice

$$\frac{[m_a + n_a]^2}{[m_a + n_a]^2} = 1$$

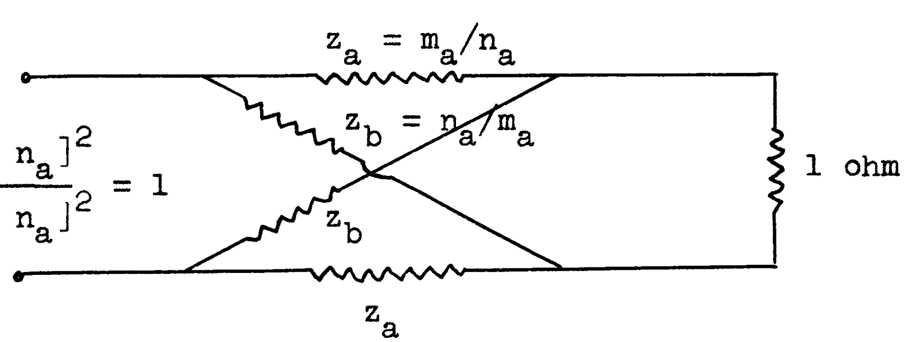


Figure 5.4

Effect of Common Factors on Darlington Realizations

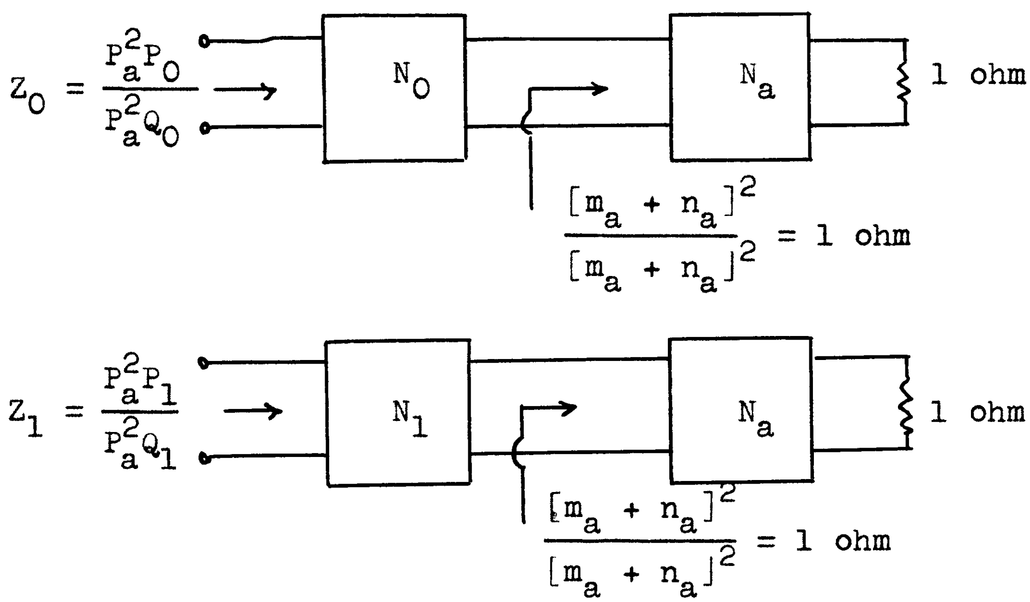


Figure 5.5

The Lossless Coupling Network N

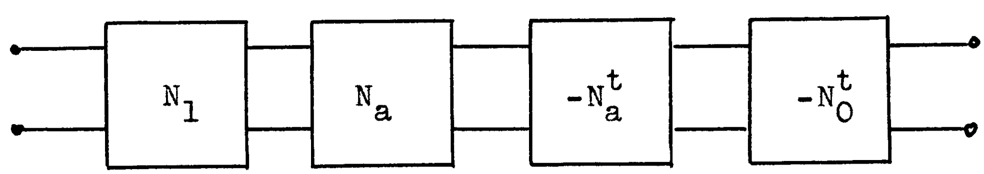


Figure 5.6

Reduction of the Cascaded Lattices

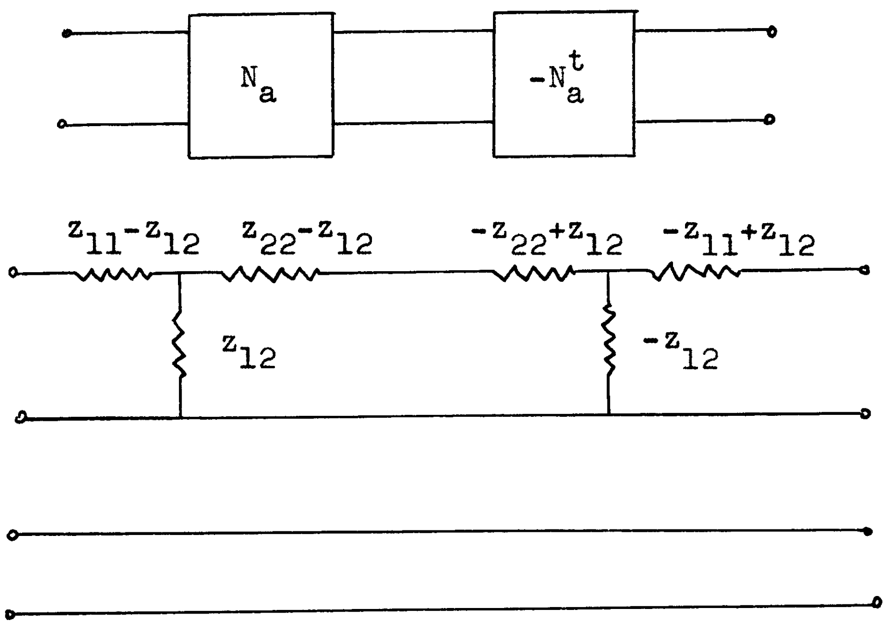
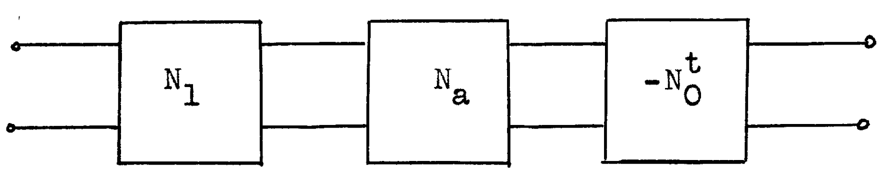


Figure 5.7

The Lossless Coupling Network N when $Z_1(s)$ Has A Common Factor But $Z_0(s)$ Does Not



Chapter Six

Completion of the Common Factor Problem

Of the three necessary and sufficient conditions on two compatible impedances, the first condition which restricts the even-part zeros of $Z_1(s)$ cannot be fulfilled by augmenting with common factors if it is not fulfilled by the original impedances. The third condition--that the admittance nQ_0/Q_1 be realizable at infinity--determines the first t/2 elements of the Cauer network realization of the impedance $z_a(s)$ where t is the order of the zero of the even part of $Z_0(s)$ at infinity. The remaining elements of $z_a(s)$ must be chosen to fulfill Condition II: the odd part of the polynomial $P_a^2Q_1(s)Q_0(-s)$ contains A_0 as a factor [Case I impedances are still being considered]. In order to reduce this condition to restrictions on the impedance $z_a(s)$, Condition II is manipulated as follows.

$$nA_0 = \text{Odd } P_a^2(s)Q_1(s)Q_0(-s) \qquad 6.1$$

$$nA_0 = [\text{Ev } P_a^2][\text{Odd } Q_1(s)Q_0(-s)] + [\text{Odd } P_a^2][\text{Ev } Q_1(s)Q_0(-s)] \qquad 6.2$$

Making use of the definitions for $z_a(s)$ and $z_{eo}(s)$--Eqs. 5.15 and 5.16--this condition reduces to

$$z_a^2 + 2z_a z_{eo} + 1 = \frac{2nA_0 z_a}{[\text{Odd } Q_1(s)Q_0(-s)][\text{Odd } P_a^2(s)]} \qquad 6.3$$

Consider first zeros of A_0 which lie at the origin. Since $z_{eo}(s)$ is formed by taking the ratio of the even to odd parts of the product $Q_1(s)Q_0(-s)$, it is clear that the even part of $Q_1(s)Q_0(-s)$ is not zero for $s = 0$ and therefore that $z_{eo}(s)$ has a pole at the origin with the same order as the zero of the polynomial Odd $Q_1(s)Q_0(-s)$. Examination of Eq. 6.3 shows that the only way it can be satisfied when A_0 has zeros at $s = 0$ is if $z_a(s)$ has a pole at $s = 0$ and $z_{eo}(s)$ has a simple pole with a negative residue at $s = 0$. In this case, the left-hand side of the equation can have a multiple order zero at $s = 0$. In general, if A_0 has a k^{th} order zero at the origin, $z_a^2 + 2z_a z_{eo} + 1$ must have a $k - 2$ order zero at $s = 0$.

Define the power series expansions about the origin for z_a, z_{eo}, and $z_a^2 + 2z_a z_{eo} + 1$ by

$$\begin{aligned} z_a(s) &= \beta_0 s^{-1} + \beta_1 s + \beta_3 s^3 + \beta_5 s^5 + \ldots\ldots \\ z_{eo}(s) &= \alpha_0 s^{-1} + \alpha_1 s + \alpha_3 s^3 + \alpha_5 s^5 + \ldots\ldots \\ z_a^2 + 2z_a z_{eo} + 1 &= \rho_{-2} s^{-2} + \rho_0 + \rho_2 s^2 + \rho_4 s^4 + \ldots\ldots \end{aligned} \qquad 6.4$$

The following equations result from these definitions and the requirement that Eq. 6.3 have a k-2 order zero at the origin:

$$\rho_{-2} = \beta_0[\beta_0 + 2\alpha_0]$$
$$\rho_0 = \beta_1[2\beta_0 + 2\alpha_0] + 2\beta_0\alpha_1 + 1$$
$$\rho_2 = \beta_3[2\beta_0 + 2\alpha_0] + \beta_1^2 + 2[\beta_0\alpha_3 + \beta_1\alpha_1]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad 6.5$$

Note that these equations are exactly the same as the set 5.21 derived for coefficients of the power series of $z_a(s)$ about infinity. Their solution is just as simple. If A_0 has a k^{th} order zero at the origin, ρ_{-2}, ρ_0, ..., ρ_{k-4} must all be zero. This gives k/2 equations which can be solved one at a time for β_0, β_1, ..., β_{k-3}, the first k/2 terms in the power series expansion about the origin of $z_a(s)$.

Part of the network $z_a(s)$ has already been specified by Condition III (assuming that the even part of Z_0 has some even-part zeros at infinity). Let the open-circuit impedance parameters of this part of the network be z_{11}, z_{12}, and z_{22}. Let the remainder impedance be $z_x(s)$ as shown in Fig. 6.1. The relation between z_a, z_x, z_{11}, z_{12}, and z_{22} is given by

$$[z_{11} - z_a][z_{22} + z_x] = z_{12}^2 \qquad 6.6$$

Since the open-circuit impedance parameters z_{11}, z_{12}, and z_{22} are known, the first k/2 terms in the power series expansion of $z_x(s)$ about the origin can easily be found.

Appendix II shows that by a simple frequency transformation, the problem of determining an impedance from its power series expansion about the origin is easily reduced to the determination of a reactance from its power series expansion about infinity. Hence, knowing the first k/2 terms in the power series expansion about the origin of $z_x(s)$ means that the first k/2 elements of the Cauer network for $z_x(s)$ may easily be found. No simultaneous equations need be solved. The elements are calculated one at a time, as before. Again these elements must be positive if the impedances are to be compatible because the Cauer network is cannonic and if $z_x(s)$ exists at all, it must exist in this form.

In the satisfaction of Condition III, the last element determined may be arbitrary, that is, may be chosen within some prescribed limits. In this case, this element is neglected while calculating the impedance $z_x(s)$ but after this impedance is determined, the arbitrary elements which can be added to $z_x(s)$ are selected so as to make the last term of the power series expansion about infinity of $z_a(s)$ correct. For example, if the last element determined by Condition III is an inductor which must be within certain limits, the requirement is that $z_x(s)$ have a pole at infinity with residue within these limits. This is accomplished by choosing the arbitrary elements at the output of $z_x(s)$ properly. If no permissible choice of this remainder will give the proper size pole of $z_x(s)$ at infinity, then again no solution exists and the original impedances cannot be compatible.

Thus, to summarize, if the even part of $Z_0(s)$ has a t^{th} order zero at infinity and a k^{th} order zero at the origin, the first $(k + t)/2$ elements of the network realization of $z_a(s)$ are determined. The form of the resulting network is shown in Fig. 6.2. Here again, any arbitrary reactance which does not change the power series expansion of $z_a(s)$ about either zero or infinity may be added to the output of this network, leaving part of $z_a(s)$ arbitrary so that the remainder of Condition II may be satisfied. An example is in order here.

Example 1

$$\text{Ev } Z_0 = \frac{s^4}{Q_0(s)Q_0(-s)}$$

$$\text{Ev } Z_1 = \frac{s^4}{Q_1(s)Q_1(-s)}$$

$$Q_1(s) = s^4 + 2s^3 + 3s^2 + 2s + 1$$
$$Q_0(s) = s^4 + 4.00s^3 + 7.88s^2 + 5.49s + 1.83$$

First the impedances are checked to determine whether or not they are compatible without common factors. Condition I is seen by inspection to be satisfied since each impedance has a fourth order zero at zero and infinity. Next the product $Q_1(s)Q_0(-s)$ is formed and written as $z_{eo}(s)$:

$$z_{eo}(s) \quad \frac{s^8 + 2.88s^6 + 7.51s^4 + 2.39s^2 + 1.83}{-2s^7 + 0.28s^5 - 1.02s^3 - 1.82s}$$

By inspection of the denominator of $z_{eo}(s)$, $s^4 = A_0$ is not contained as a factor and therefore the impedances are not compatible without common factors. Since the even part of $Z_0(s)$ has zeros at both the origin and infinity, $z_{eo}(s)$ must have simple poles at zero and infinity with negative residues. This condition is fulfilled and therefore a common factor may exist.

First Condition III is fulfilled by selecting the first t/2 elements of the Cauer network for $z_a(s)$. In this case, t = 4, so the first two elements will be determined. Expanding $z_{eo}(s)$ about infinity gives:

$$z_{eo}(s) = -\frac{s}{2} - 1.51\ s^{-1} + \ldots\ldots\ldots$$

Using Eqs. 5.21, these equations result:

$$\rho_{-2} = 0 = \beta_0 - 1$$
$$\rho_0 \ = \beta_1 + 2[-1.51] + 1$$

Hence

$$\beta_0 = 1$$

$$\beta_1 = \rho_0 + 2.02$$

Since $t/2 = 2$ is even, ρ_0 must be negative or zero. Using the formulae of Appendix I, the first two elements of the network realization of $z_a(s)$ are found to be

$$L_1 = 1 \text{ henry}$$

$$1/C_2 = 2.02 + \rho_0.$$

Since ρ_0 must be negative or zero, C_2 must be chosen between the limits of 1/2.02 and infinity. The choice for C_2 is delayed until the remainder of the network is determined so that it may be chosen to minimize the number of elements and hence minimize the size of $P_a^2(s)$. The portion of $z_a(s)$ so determined is shown in Fig. 6.3.

Next Condition II must be satisfied. Since all of the zeros of A_0 lie at the origin, the method described above will give the complete solution for the impedance $z_a(s)$ and hence the entire polynomial $P_a(s)$. The polynomial A_0 has a fourth order zero at $s = 0$, requiring that ρ_{-2} and ρ_0 be set equal to zero in Eq. 6.5. The power series expansion about the origin of $z_{eo}(s)$ has only a first order pole at required and is given by

$$z_{eo}(s) = -s^{-1} - 0.750s + \ldots\ldots$$

Therefore Eqs. 6.5 become:

$$0 = \beta_0 + 2\alpha_0 = \beta_0 - 2$$

$$\beta_0 = 2$$

$$0 = \beta_1[2\beta_0 + 2\alpha_0] + 2\beta_0\alpha_1 + 1$$

$$0 = 2\beta_1 - 2.000$$

$$\beta_1 = 1.000$$

Thus the power series expansion of $z_a(s)$ about the origin is given by

$$z_a(s) = 2s^{-1} + 1.000s + \ldots\ldots\ldots$$

Only the first element of z_a has been specified, the series one-henry inductance. Ignore the capacitance C_2 which is specified with certain limits, calculate the elements which determine the behaviour of $z_a(s)$ at the origin and then choose arbitrary elements to satisfy the range of C_2 required. From Fig. 6.3, the impedance $z_x(s)$ is given by (in series form):

$$z_x(s) = z_a(s) - s = 2s^{-1} + 0$$

The network realization of $z_x(s)$ is obviously a 1/2 farad capacitor. Hence $z_a(s)$ becomes the series connection of a one-henry inductance and a one-half farad capacitor. Next $z_x(s)$ is checked so that the input capacity lies within the limits required by Condition III. Since it does, the problèm is complete. The result is

$$z_a(s) = s + 2/s$$
$$P_a(s) = s^2 + s + 2$$
$$P_a^2(s) = s^4 + 2s^3 + 5s^2 + 4s + 4$$

If $Z_1(s)$ is augmented by multiplying numerator and denominator by $P_a^2(s)$, the impedances must be compatible without further common factors. To check this, the following calculations are performed:

$$P_a^2(s)Q_1(s) = s^8 + 4s^7 + 12s^6 + 22s^5 + 32s^4 + 32s^3 + 25s^2 + 12s + 4.$$

$$\text{Odd } P_a^2(s)Q_1(s)Q_0(-s) = [18.8s^3 + 28.76s][s^4].$$

By inspection, this odd part contains A_0 as a factor. Moreover, the quantity n is given by

$$n = 18.8s^3 + 28.76s$$

Condition III becomes (as $\underline{s}$ approaches infinity):

$$nQ_0/P_a^2Q_1 \text{ approaches } 18.8s^7/s^8 \ .$$

Since this is zero, Condition III is also satisfied and thus the impedances are proved compatible. Note that it was not necessary to make these checks since the realizability of $z_a(s)$ is both necessary and sufficient for compatibility.

When the first $(t + k)/2$ elements of the network realization of $z_a(s)$ have been found from Conditions II and III, there still remains the problem of satisfying Condition II when A_0 has zeros not at the origin. The meaning of this condition is best seen from Eq. 6.3. At each zero of A_0, the quantity $z_a^2 + 2z_az_{eo} + 1$ must be zero. If the zero is of the second order, the derivative of this function must also be zero etc. Hence Condition II can be written

$$z_a^2 + 2z_az_{eo} + 1 = 0 \text{ for each zero of } A_0(s) \qquad 6.7$$

Because A_0 is an even function of $\underline{s}$, all of its zeros must appear with quadrantal symmetry. Denote by s_k the k^{th} complex zero of A_0 which appears in the first quadrant of the s-plane. Then A_0 has zeros at $s = \pm s_k$ and $s = \pm \bar{s}_k$, where $\bar{s}_k$ is the conjugate of s_k. The requirement that

$z_a^2 + 2z_a z_{eo} + 1$ be zero at $s = s_k$ is sufficient to insure that it is also zero at $s = -s_k$ and $s = \pm \bar{s}_k$ since it is an even function of frequency. Thus only 1/4 of the zeros of A_0 need be considered. Similiarly, if A_0 has zeros on the $j\omega$ axis, they must be of even order and therefore it is necessary to make $z_a^2 + 2z_a z_{eo} + 1$ and its derivative equal to zero only at one of these zeros, for the complex conjugate zero is taken care of automatically.

To reduce Condition II to conditions on $z_a(s)$, merely solve Eq. 6.7 for the value of $z_a(s)$ at each of the zerosof A_0 which lie in the first quadrant. In the case of multiple order zeros, the value of the derivative of $z_a(s)$ must also be found. In either case, Condition II reduces to the requirement that $z_a(s)$ be a physically realizable reactance function which assumes certain complex values at certain complex frequencies.

These values of $z_a(s)$ are given by

$$z_a(s_k) = - z_{eo} \pm \sqrt{z_{eo}^2 - 1} \qquad 6.8$$

The proper sign is chosen so that $z_a(s_k)$ will have a positive real part because $z_a(s)$ must be p.r.

Knowing the value of $z_a(s)$ at these complex frequencies allows the value of the remainder impedance $z_x(s)$ to be found. Again let z_{11}, z_{12}, and z_{22} be the open-circuit impedance parameters of the lossless network determining $z_a(s)$ which has already been found. Let $z_x(s)$ again be the arbitrary remainder reactance. Then these are related by

$$[z_{11} - z_a][z_{22} + z_x] = z_{12}^2 \qquad 6.9$$

This equation is easily solved for the values of $z_x(s)$ at each s_k. Hence Condition II reduces to the requirement that $z_x(s)$ assume certain specified complex values at certain specified complex frequencies.

The determination of a reactance function from its value specified at a finite number of discrete frequencies (complex) has been studied and the results given in Appendix III. There it is shown how a simple reactance may be found when it exists. This technique proceeds as follows. First a set of linear simultaneous algebraic equations are solved for the reactance function of smallest size which satisfies the given conditions. This reactance is tested for p.r. character. If it is p.r., the problem is solved. This reactance is attached to the output terminals of the network realization of $z_a(s)$, $z_a(s)$ calculated, $P_a(s)$ calculated, and finally the desired augmenting polynomial $P_a^2(s)$ calculated.

If this reactance, $z_x(s)$, is not p.r., there is a possibility that a more complex function can be found which not only will satisfy the restrictions but also be p.r. A simple test is described in Appendix III which tells whether or not such a reactance exists and also gives some indication of how complex it must be. If it exists, a trial and error method for finding it is described. Once it has been found, assuming it exists, the problem is completed for the polynomial $P_a^2(s)$ is easily found. Again a simple example is in order.

Example 2

$$Z_0(s) = \frac{102s^2 + 170s + 119}{14s^2 + 17s + 12}$$

$$Z_1(s) = \frac{3s^2 + 2s + 1}{s^3 + s^2 + 2s + 1}$$

Calculating the even parts of these impedances yields the numerators

$$A_0 = 1428(s^4 + 1)$$
$$A_1 = s^4 + 1.$$

Condition I is satisfied since $Z_0(s)$ has no even-part zeros at zero or infinity and the polynomial A_0 is a factor of A_1. These are Case I impedances.

The polynomial $Q_1(s)Q_0(-s) = [145s^5 + 23s^3 + 7s] + [-3s^4 -8s^2 + 12]$. The odd part of this polynomial does not contain $(s^4 + 1)$ as a factor and therefore the two impedances are not compatible without common factors being added.

Since the even part of $Z_0(s)$ has no zeros at infinity, Condition III is automatically satisfied. Because A_0 has no zeros at the origin, only the zero in the first quadrant must be considered. The requirement on $z_a(s)$ in order that Condition II be satisfied is:

$$z_a = - z_{eo} \pm \sqrt{z_{eo}^2 - 1} \quad \text{at } s = [1 + j]/\sqrt{2} \ .$$

Calculation of $z_{eo}(s)$ at this frequency from the polynomial $Q_1(s)Q_0(s)$ already determined gives

$$z_{eo} = -1/\sqrt{2} \quad \text{at } s = [1 + j]/\sqrt{2} \ .$$

In this case, z_{eo} happens to be pure real. Next z_a is determined at this frequency to be

$$z_a = [1 \pm j]/\sqrt{2} \quad \text{at } s = [1 + j]/\sqrt{2} \ .$$

Choose the plus sign for convenience. For this simple result, the methods of Appendix III are not necessary. Clearly a suitable $z_a(s)$ is given by

$$z_a(s) = s$$

From the impedance $z_a(s)$, the polynomial $P_a(s)$ is found by adding together the numerator and denominator. The result is

$$P_a(s) = s + 1.$$

The augmenting polynomial is the square of this polynomial. Augmenting $Z_1(s)$ yields the impedance discussed in example 3 of Chapter 4. There it is shown to be compatible with the given $Z_0(s)$.

Since from the network realization of $z_a(s)$ a common factor $P_a^2(s)$ is to be calculated, the simplest network is desired for $z_a(s)$ because this will lead to the smallest order common factor in $Z_1(s)$ and therefore the simplest coupling network. A problem of economy may arise in this procedure if Condition II is not satisfied by the simplest reactance network. That is, as shown in Appendix III, a reactance function which assumes certain specified values at certain specified complex frequencies is not guaranteed to be simple. If this impedance turns out to be very complex, the order of the common factor is very high. This is important because each additional element in $z_a(s)$ increases the order of the common factor by two. In any practical problem, there will be a limit on the allowed size of the coupling network, necessitating the introduction of a fourth compatibility condition that the common factor not exceed a certain size.

The size of the polynomial $P_a^2(s)$ is therefore of interest. It has been shown that if the even part of $Z_0(s)$ has a t^{th} order zero at infinity and a k^{th} order zero at the origin, the network $z_a(s)$ will have $(k + t)/2$ elements. If the even part has n zeros not at zero or infinity, then, since only n/4 zeros need be considered and since for each zero two quantities are specified, n/2 elements are necessary. The form of this network realization is shown in Fig. 6.4. It is clear that one element in $z_a(s)$ is necessary for each pair of zeros of the even part.Hence the order of the polynomial $P_a(s)$ is the same as the order of the denominator polynomial $Q_0(s)$ and the order of the augmenting factor P_a^2 is twice this size. Of course it may be larger or smaller in any particular instance, but in general it will be this size.

Again it must be noted that this procedure was derived assuming Case I impedances. If instead the impedances belong to Case II, the whole procedure must be carried out using $z_{eo}^{-1}(s)$, that is, the reciprocal of $z_{eo}(s)$ which is denoted $z_{oe}(s)$. Except for this change, all results are the same for the two cases.

The whole procedure for testing two impedances for compatibility is summarized below.

Summary

Using Table 4.1, test the given impedances to determine whether or not they are compatible without common factors being added to $Z_1(s)$. If so, the test is complete and the open-circuit impedance parameters of the coupling network may be calculated from the equations of Table 4.1.

If the impedances are not compatible without common factors, determine the first t/2 terms in the series expansion about infinity of $z_{eo}(s)$ where t is the order of the zero of the even part of $Z_0(s)$ at infinity. From these terms, determine the first t/2 terms in the power series expansion about infinity of $z_a(s)$ and solve for the first t/2 elements of the network realization for $z_a(s)$. These elements must be positive.

Solve for the first k/2 terms in the power series expansion about the origin of $z_{eo}(s)$ where k is the order of the zero of the even part of $Z_0(s)$ at the origin. From these terms, determine the first k/2 terms in the power series expansion about the origin of $z_a(s)$. Using these terms and the part of $z_a(s)$ already found, determine the first k/2 elements in the remainder network, $z_x(s)$. These elements must be positive.

Solve for the required value of $z_a(s)$ at each of the first quadrant zeros of the even part of Z_0 (and the appropriate number of derivatives if these zeros are multiple order zeros). Using these values and the part of the network already determined, solve for the required value of the remainder impedance (and its derivatives if necessary) at these frequencies and determine a p.r. reactance function which assumes these complex values at these frequencies.

Calculate the impedance $z_a(s)$ as a ratio of polynomials, add together its numerator and denominator and square this resulting polynomial to form the required common factor $P_a^2(s)$. Multiply both the numerator and denominator of $Z_1(s)$ by this polynomial. The resulting $Z_1(s)$ is now compatible with the original $Z_0(s)$ and the equations given in Table 4.1 can be used to calculate the open-circuit impedance parameters of the lossless coupling network.

Figure 6.1

The Portion of the Network Realization of

$z_a(s)$ Determined by Condition III

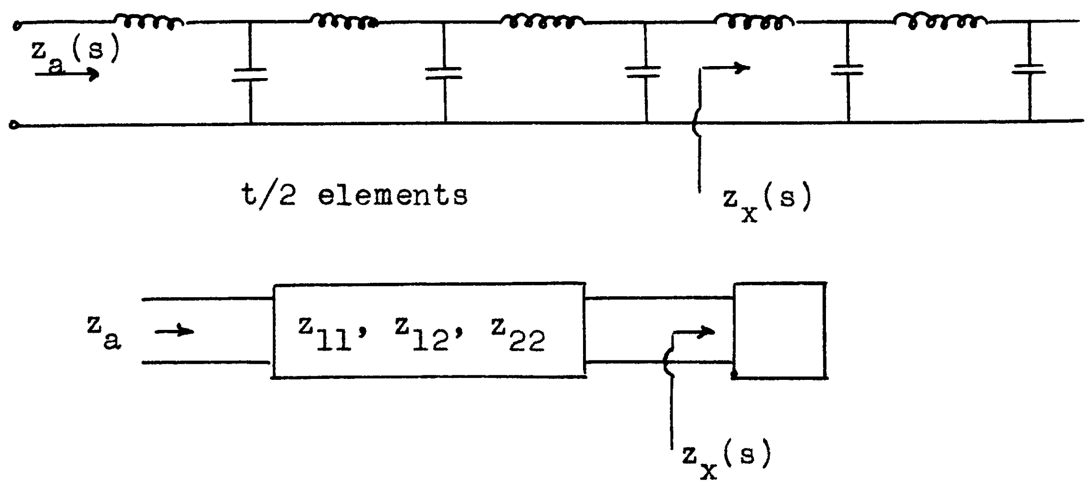


Figure 6.2

The Part of the Network Realization for $z_a(s)$ Determined

By the Zeros of the Even Part at Zero and Infinity

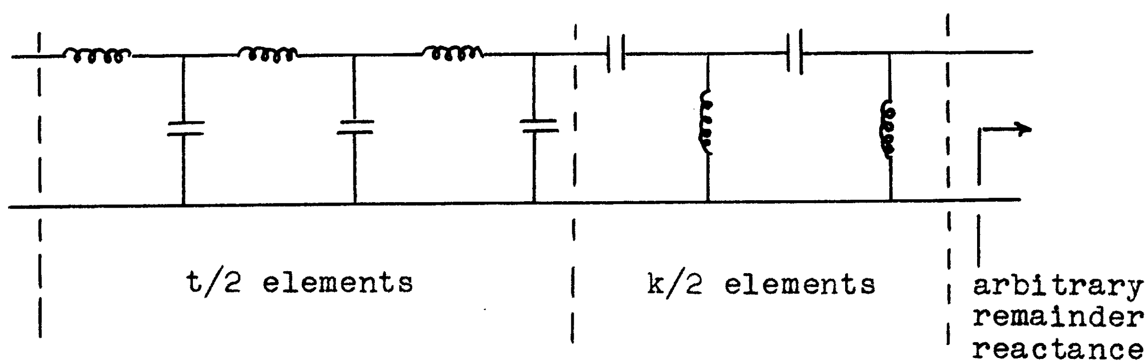


Figure 6.3

Realization for $z_a(s)$ in Example One

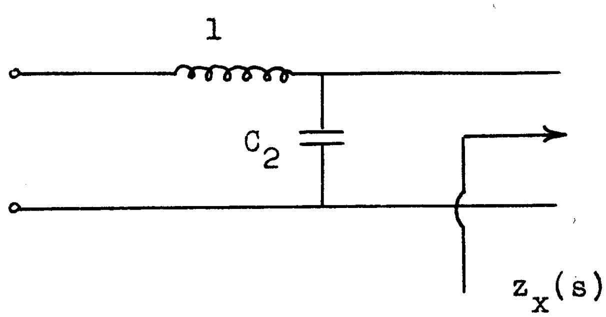
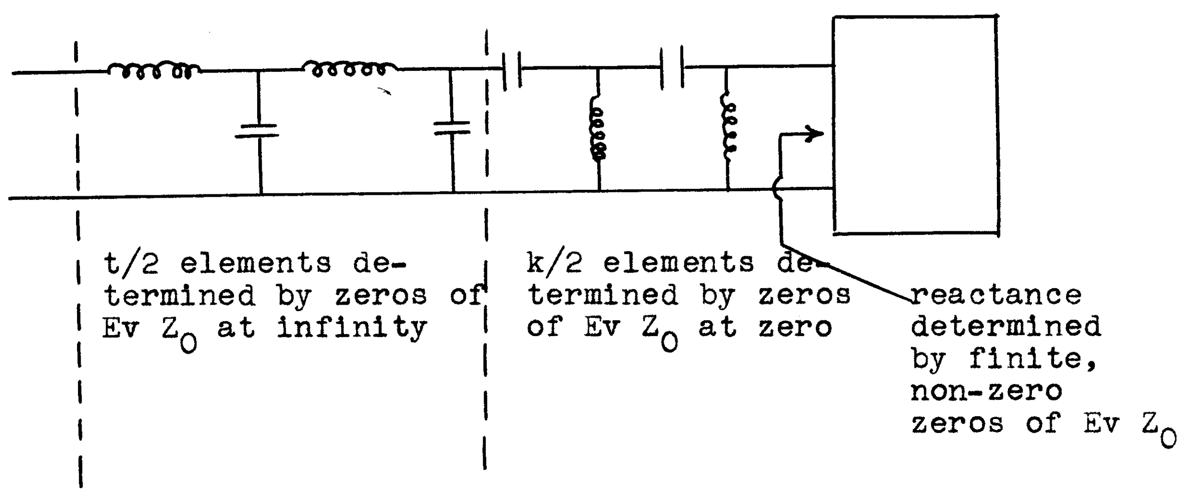


Figure 6.4

General Form of Network Realization of $z_a(s)$



Chapter Seven

Extension of the Solution

A Uniqueness Theorem: The open-circuit impedance parameters of the lossless coupling network are not unique in general because the polynomial $P_a^2(s)$ is not unique. It is important to understand the limitations on the use of common factors.

Limitations on the use of common factors can be stated in the form of a "quasi" uniqueness theorem. This theorem states that if $Z_1(s)$ and $Z_0(s)$ are compatible as they stand--that is, without augmenting $Z_1(s)$ with common factors--the open-circuit impedance parameters of the lossless coupling network are unique. This is equivalent to saying that there exists no Hurwitz polynomial which can be used to augment $Z_1(s)$ and leave the resulting impedances compatible.

The proof of this theorem is very simple. First suppose that the even part of $Z_0(s)$ has at least a second order zero at infinity. In order to satisfy Condition III, W(s) must have a multiple order pole at infinity (see Eqs. 5.15 and 5.19). Since $Z_1(s)$ and $Z_0(s)$ are compatible without common factors, $z_{eo}(s)$ must have a multiple order pole at infinity [W(s) reduces to $z_{eo}(s)$ when no common factor is used]. But from the discussion in Chapter Five it is clear that with a common factor added to numerator and denominator of $Z_1(s)$ the quantity $W(s)$ cannot have a multiple order pole at infinity unless $z_{eo}(s)$ has a simple pole at infinity with a negative residue. Hence the theorem is true when the even part of $Z_0(s)$ has at least a second order zero at infinity.

If the even part of $Z_0(s)$ has no zeros at infinity, it follows that all of the zeros must be finite. Consider Condition II which is expanded in Eq. 6.2. Since the impedances are compatible without common factors, the odd part of $Q_1(s)Q_0(-s)$ is zero at each zero of A_0. From Eq. 6.2 it follows that at each such zero,

$$[\text{Odd } P_a^2][\text{Ev } Q_1(s)Q_0(-s)] = 0.$$

Since Odd $P_a^2(s)$ must have only simple zeros located on the $j\omega$ axis, the even part of $Q_1(s)Q_0(-s)$ and therefore $Q_1(s)Q_0(-s)$ itself must be zero at each zero of A_0. At any right half-plane zero of A_0, $Q_0(-s)$ must be zero for $Q_1(s)$ is a Hurwitz polynomial. But this means that the even part of $Z_0(s)$ is not zero here for both numerator and denominator are zero. Hence if the even part of Z_0 has any right half-plane zeros, the theorem is proved.

If all of the zeros of the even part lie on the $j\omega$ axis, $Q_1(s)$ must have all these zeros since such zeros must be of even order and Odd $P_a^2(s)$ can have only simple zeros. This means, in turn, that $Z_1(s)$ has jω-axis poles at each zero of the even part of Z_0. Even if Z_1 is allowed to have jω-axis poles, this is a trivial case. If $Z_1(s)$ cannot have jω-axis poles, it follows that the theorem is proved.

The significance of this result is twofold. If the two impedances are compatible without common factors, they cannot be compatible if common factors are added. Thus it is impossible to introduce common factors in order to get a more desirable set of open-circuit impedance parameters for the coupling network. If the impedances are compatible with common factors, then after the common factor has been inserted into $Z_1(s)$, no further common factors may be inserted. However, in general there are an infinite number of different augmenting factors which will make the impedances compatible if there is one such factor.

<u>The Problem of Poles on the Real Frequency Axis:</u> The restriction assumed throughout this work, that there are no jω-axis poles in the two impedances, can be removed by slightly complicating the necessary and sufficient conditions derived previously.

Consider first Condition I, that the impedance $Z_1(s)$ have all of the even-part zeros of the impedance $Z_0(s)$ with at least the same multiplicity. Examination of Eq. 3.4 shows that this restriction must still hold when the two impedances have jω-axis poles except in the special case where Ev Z_0 is zero on the jω axis and $Z_1(s)$ has a pole at the same frequency. This exception occurs because the impedance $z_{22}(s) + Z_0(s)$ can have a jω-axis zero which then cancels the even-part zero and also produces a pole of $Z_1(s)$ [see Eq. 3.1]. Despite this exception, the numerator of the even part of $Z_1(s)$, A_1, must still contain this zero so that A_1 still contains A_0 as a factor.

For the above reason, the problem may be set up within the framework of the cascade-synthesis method just as before. However, Conditions II and III are no longer sufficient although they are still necessary.

That they are not sufficient follows from examination of Eqs. 3.12. For example, if $Z_1(s)$ has a jω-axis pole, nQ_0/Q_1 may have a jω-axis pole and its residue must be tested in addition to the other p.r. tests. Also, if $Z_0(s)$ has a jω-axis pole, the residue of Q_1/nQ_0 may be smaller than the residue of Z_0 at this pole, meaning that $z_{22}(s)$ must have this pole with a negative residue. Thus if $Z_0(s)$ has a jω-axis pole, this pole may appear in $z_{22}(s)$ and its residue must be tested. Similiarly, if $Z_1(s)$ has a jω-axis pole, $m_{12}^2Q_0/nQ_1$ may have that pole with a large negative residue. Then examination of Eq. 3.12 shows that z_{11} will have this pole with a negative residue. In other words, the presence of jω-axis poles in one or both impedances creates degenerate situations which must be handled separately.

For any combination of jω-axis poles in the two impedances $Z_0(s)$ and $Z_1(s)$, a necessary and sufficient set of conditions can be written as:

1. Conditions I, II, and III must hold as before with the exception of the possible special case under Condition I described above.

2. At any jω-axis poles of $Z_1(s)$ which are also poles of nQ_0/Q_1, the residue must be tested to insure that it is positive real.

3. The open-circuit impedance parameters must be found from Eqs. 3.12 rather than the formulae derived in Table 4.1. That is,

$$z_{11} = \frac{m_{12}^2 Q_0}{nQ_1} + Z_1$$

$$z_{22} = \frac{Q_1}{nQ_0} - Z_0$$

   This is to insure that jω-axis poles of Z_1 and Z_0 which may be poles of z_{11} and z_{22} but not z_{12} are included.

4. Poles of z_{11}, z_{12}, and z_{22} which are also poles of Z_1 must be tested so that all residue conditions are satisfied.

Thus the conditions are essentially the same as when no $j\omega$-axis poles are present, except that after the open-circuit impedance parameters of the lossless coupling network have been found, some residues must be tested.

In the most common case, these extra tests can be obviated. If $Z_1(s)$ has a $j\omega$-axis pole at a frequency at which the real part of $Z_0(s)$ is not zero and Z_0 does not have a pole at this frequency also, this pole may be removed from $Z_1(s)$ and the remainder tested with $Z_0(s)$ for compatibility. This follows because not only does $Q_1(s)$ have this zero, but also $n(s)$. Hence z_{22} and z_{12} do not have this pole but z_{11} does have the pole and its residue must be the same as that of $Z_1(s)$ which is necessarily positive. Hence it may be removed without affecting the realizability of the remainder of the lossless coupling network.

The Case of Active Impedances: Another restriction which may be removed from this problem is that the impedances $Z_0(s)$ and $Z_1(s)$ be positive-real, that is, RLC realizable. To remove this restriction, consider two impedances which have no right half-plane or $j\omega$-axis poles. These impedances may have a negative real-part on the $j\omega$ axis but will not be permitted to have any pole at infinity. In addition, if the impedances have a simple zero at infinity, they will be assumed to vanish like a positive capacitance.

With these restrictions in mind, each of the steps previously taken can be repeated to determine any additional conditions on the impedances. First is the derivation of Condition I regarding the zeros of the even parts of the impedances. Examination of Eq. 3.4, the relation between even parts, shows that, just as before, zeros of the even part of $Z_0(s)$ cannot be cancelled by the other terms of the equation [when the even part of Z_0 is zero, $z_{22} + Z_0$ is an odd function of frequency, meaning that if it is zero in the left half-plane, it is also zero in the right half-plane and therefore a right half-plane pole of $Z_1(s)$]. A possible exception occurs if z_{12} has a simple zero at some frequency and $z_{22} + Z_0$ has a second order zero at the same real frequency. In this case, $z_{12}/z_{22} + Z_0$ has a simple pole whereas $z_{12}^2/z_{22} + Z_0$ has no pole. Thus Condition I no longer necessarily holds in the case of active impedances as it did in the case of passive impedances. Clearly the same exception can occur at a frequency not on the $j\omega$ axis since the impedance $z_{22} + Z_0$ needn't be p.r. and therefore can have right half-plane zeros. Thus if the even part of Z_1 does not contain the zeros of the even part of Z_0, the impedances may still be compatible. In this case, it is necessary to augment $Z_1(s)$ with common factors so that again the numerator of the even part of $Z_1(s)$ contains A_0 as a factor with the remainder either a perfect square or the negative of a perfect square.

After this is done, the problem may be set up in terms of the cascade-synthesis method. Assuming no $j\omega$-axis poles in either impedance, the set z_{11}, z_{12}, and z_{22} defined by Eq. 3.12 are realizable if and only if the impedance $z_{22}(s)$ is p.r.

Since $z_{22}(s)$ is to be a p.r. reactance function, it must be an odd function of frequency. Taking the even parts of both sides of Eq. 3.12 produces the same result as Condition II, namely

$$\text{Odd } Q_1(s)Q_0(-s) = nA_0.$$

$Z_0(s)$ fails being p.r. only because its real part goes negative on the $j\omega$ axis. Hence there exists a finite positive constant, R, such that $R + Z_0$ is a p.r. impedance. If R is added to both sides of Eq. 3.12, it is clear that z_{22} is p.r. if and only if the impedance

$$R + \frac{Q_1}{nQ_0}$$

is p.r.

It is possible to show that $Q_1/nQ_0 + R_2$ is p.r. if $nQ_0/Q_1 + R_1$ is p.r. To this end, consider

$$Y = \frac{1}{\frac{Q_1}{nQ_0} + R_2} = \frac{nQ_0}{Q_1 + nQ_0R_2} \qquad 7.1$$

To test this for p.r. character, first test for right half-plane poles. The denominator is $Q_1 + nQ_0R_2$ and can be written

$$R_2[nQ_0 + \frac{Q_1}{R_2}].$$

Since $nQ_0/Q_1 + R_1$ is p.r. by hypothesis, it follows that the denominator of Eq. 7.1 can have no right half-plane zeros if $1/R_2$ is greater than or equal to R_1.

To test the real part on the $j\omega$ axis, form the even part of $Y(s)$ and use the condition Odd $Q_1(s)Q_0(-s) = nA_0$ to get

$$\text{Ev } Y(s) = \frac{-n^2[A_0 + R_2Q_0(s)Q_0(-s)]}{[Q_1(s) + n(s)Q_0(s)R_2][Q_1(-s)+n(-s)Q_0(-s)R_2]} \qquad 7.2$$

This will be positive on the $j\omega$ axis if and only if

$$A_0 + R_2|Q_0|^2 \geq 0 \quad \text{for all } \omega. \qquad 7.3$$

This inequality is fulfilled automatically whenever A_0 is positive. Let the most negative value of $A_0/|Q_0|^2$ be $-R_3$ where R_3 is a positive number and let this occur at frequency $\omega = \omega_0$. Then the inequality holds for all ω provided that R_2 is greater than or equal to R_3:

$$R_2 \geq R_3 \qquad 7.4$$

The last test for p.r. character involves $j\omega$-axis poles. Since Q_1 has no $j\omega$-axis zeros by assumption, the only possible $j\omega$-axis pole occurs at infinity. Hence Condition III becomes Q_1/nQ_0 have at most a simple pole at infinity with a positive residue. This is exactly the same condition as for p.r. impedances.

It is necessary to show that the two inequalities -- $\frac{1}{R_2} \geq R_1$ and $R_2 \geq R_3$--can be simultaneously fulfilled, that is, that a value for R_2 does indeed exist. Thus it is necessary to show that R_3 is less than or equal to $1/R_1$. This can be done in the following manner.

By hypothesis, $nQ_0/Q_1 + R_1$ is p.r. Hence its real part must be positive on the $j\omega$ axis. This real part can be written with the aid of Condition II ($s = j\omega$):

$$\text{Re}\left[R_1 + \frac{nQ_0}{Q_1}\right] = R_1 + \frac{-[\text{Odd } Q_1(s)Q_0(-s)]^2}{A_0 Q_1(s)Q_1(-s)} \qquad 7.5$$

Multiplying numerator and denominator by $Q_0(s)Q_0(-s)$ and replacing $Q_1(j\omega)$ by $R + jX$ gives

$$\frac{X^2 \, |Q_0|^2}{A_0[R^2 + X^2]} + R_1 \geq 0 \text{ for all } \omega. \qquad 7.6$$

At $\omega = \omega_o$, this inequality becomes

$$\frac{X^2}{-R_3[R^2 + X^2]} + R_1 \geq 0 \qquad 7.7$$

or finally

$$\frac{1}{R_1} \leq R_3[1 + (R/X)^2] \qquad 7.8$$

Since $(R/X)^2$ is positive, it is clear that the inequality holds and therefore a value for R_2 does exist.

Thus it has been proved that $z_{22}(s)$ is p.r. if and only if nQ_0/Q_1 is realizable at infinity as before. These results are rather surprising since for passive impedances Conditions I, II, and III must hold whereas for active impedances, only Conditions II and III must hold. The simplicity of the result is certainly not expected.

Although Condition I need not be fulfilled, in general, for active impedances, the condition that the polynomial A_1/A_0

be a perfect square is now very important. This is so because in the case of active impedances with no right half-plane poles, the real part goes negative only because the polynomial A_1 or A_0 has odd order jω-axis zeros. Hence if A_1/A_0 is a polynomial with any odd order jω-axis zeros, the impedance $Z_1(s)$ cannot be augmented and therefore the impedances cannot be compatible. This is obviously physically for $Z_1(s)$ is to be realized as a lossless network terminated in $Z_0(s)$. From power considerations, whenever the real part of $Z_0(s)$ is positive, so must the real part of $Z_1(s)$ and when the real part of $Z_0(s)$ is negative, so must the real part of $Z_0(s)$. It follows that the even parts of the two impedances must have coincident jω-axis odd-order zeros.

If the two impedances satisfy the three conditions, the open-circuit impedance parameters of the lossless coupling network are calculated in the same way as for passive impedances. If the two impedances are not compatible as they stand, a common factor may be added. The calculation proceeds in exactly the same way as for passive impedances since the common factor must be the square of a Hurwitz polynomial. Hence, the more general problem of compatible active impedances works out almost exactly the same as for passive impedances.

Chapter Eight

Conclusions

The result of this thesis has been the derivation of a concise set of necessary and sufficient conditions under which two impedances can be compatible. Augmenting factors, when they exist, are found by an element-by-element calculation of the network realization of $z_a(s)$ which is the ratio of the even to odd parts of the common factor $P_a(s)$. This facilitates testing $P_a(s)$ for Hurwitz character since this is guaranteed if all of the elements of $z_a(s)$ are positive. Once the two given impedances have been proved compatible, simple formulae for the open-circuit impedances of the lossless coupling network are available from which the coupling network can be determined.

In order to determine the conditions under which augmenting factors exist, the problem of calculating a reactance function from its power series expansion about zero or infinity was solved. The result is a simple technique for an element-by-element calculation of the Cauer form of the impedance.

The problem of determining a reactance function which assumes certain prescribed complex values at specified complex frequencies has also been investigated and a technique developed to determine whether or not a simple solution exists and also a trial and error method for finding the solution when it is known to exist.

All of this work is also applicable on the admittance basis if every letter z is changed to the letter y, that is, if Z_1 and Z_0 are replaced by Y_1 and Y_0 respectively and open-circuit impedance parameters replaced by short-circuit admittance parameters. This follows because the two formulations are exact duals of one another.

The conditions under which two stable but active impedances are compatible has also been determined by extending the results derived for p.r. impedances. The results for active impedances are almost exactly the same in form as for p.r. impedances.

As pointed out in the introduction, the importance of compatible impedances lies in the many applications. The solution, therefore, must be simple enough to apply to these problems. The simple set of necessary and sufficient conditions derived in Chapter Four satisfy this requirement completely. Since in a synthesis problem common factors can usually be ignored, the simple form of Conditions I, II, and III should provide much insight into the applications discussed.

For example, consider the broadbanding problem. Suppose the load impedance is $Z_0(s)$ as usual and the numerator of its even part is A_0. Since A_0 is an even polynomial, it can be factored into the product of two polynomials R(s) and R(-s) where R(s) is a Hurwitz polynomial. It is well known that an impedance (minimum reactive) can be defined by its even part. Therefore, define the impedance $Z_1(s)$ by its even part:

$$\text{Ev } Z_1 = \frac{A_1}{Q_1(s)Q_1(-s)} \; . \qquad 8.1$$

It is clear from Condition I [A_1 must contain A_0 as a factor] that the smallest possible impedance $Z_1(s)$ is one for which $A_1 = A_0$. Assuming that the even part of Z_0 has no zeros at infinity, it is permissible to define

$$\text{Ev } Z_1 = \frac{A_0}{R(s)R(-s)} = 1 \qquad 8.2$$

If the even part of Z_0 has some zeros at infinity, a larger polynomial R(s) must be selected so that the even part of $Z_1(s)$ will have the same number of zeros at infinity. This might be done by multiplying R(s) by a factor of appropriate size with coefficients to be determined.

Although this impedance $Z_1(s)$ can not be compatible with Z_0, Conditions II and III show that the only thing preventing compatibility is the denominator of $Z_1(s)$, the polynomial $R(s)$. If $R(s)$ is replaced by another polynomial $R_1(s)$ which not only satisfies Conditions II and III but also approximates the polynomial $R(s)$ over some given frequency interval, then the impedance $Z_1(s)$ defined by

$$\text{Ev } Z_1(s) = \frac{A_0}{R_1(s)R_1(-s)} \qquad 8.3$$

is compatible with $Z_0(s)$ and also approximates a resistance over the given frequency range. Note that the broadbanding problem has reduced to the choosing of only one polynomial, certainly not an impossible job.

Conditions II and III will not leave much freedom in the choice of $R_1(s)$. This difficulty can be alleviated somewhat by multiplying both numerator and denominator of the even part of $Z_1(s)$ by a factor $P_a^2(s)$, an arbitrary common factor. Then the broadbanding process reduces to the selection of a polynomial $R_1(s)$ which satisfies Conditions II and III and also approximates the polynomial $P_a^2(s)R(s)$ over some frequency interval. The arbitrary coefficients of $P_a^2(s)$ should make the choice of $R_1(s)$ easier and allow a better solution to the broadbanding problem.

This approach to the broadbanding problem is attractive because the work done is restricted to choosing a polynomial. Once this is done, the open-circuit impedance parameters of the lossless coupling network are easily found by the relations in Table 4.1. Hence despite the fact that the resulting coupling network may be very large, the approximation problem should be relatively simple. This is in marked contrast with the method of Ligomenides, for example, which becomes very difficult when networks of more than three elements are considered.

The problem of limitations on allowable transfer functions for lossless networks terminated in a specified impedance is also an interesting and practical area which has not been fully investigated. In simple cases, this reduces directly to the compatibility requirement on two impedances. However this is not true in general. The results of this thesis should provide an approach to this problem.

Appendix One

On the Determination of a Reactance Function From Its Series Expansion About Infinity

Any physically realizable reactance function may be realized by its cannonical Cauer form with series inductances and shunt capacitances.[1] If the inductances are labeled L_1, L_3, L_5, etc., and the capacitances C_2, C_4, C_6, etc. as shown in Fig. A1.1, the input impedance, z(s), may be written as an infinite series about infinity. The first few terms are

$$z(s) = sL_1 + (sC_2)^{-1} - s^{-3}(L_3C_2^2)^{-1} + s^{-5}[(L_3^2C_2^2C_4)^{-1} + (L_3^2C_2^3)^{-1}] - s^{-7}[(C_2^2L_3^2C_4^2L_5)^{-1} + (C_2^2L_3^2C_4^2)^{-1} + (C_2^3L_3^2C_4)^{-1} + (C_2^4L_3^3)^{-1}] + \ldots\ldots \qquad \text{A1.1}$$

This expansion is verified by taking a network with five elements, calculating the input impedance and expanding it into a series. The number of terms in each of the coefficients of the higher order terms increases very quickly, making it unfeasible to write them out explicitly in terms of the various elements. Ignoring this problem for the moment, several things suggest themselves.

1. Guillemin, *op. cit.*, p. 87.

For example, if the terms are numbered 1, 2, ... with the k^{th} term denoted by t_k, it is clear from an examination of the above series that the k^{th} term involves the first k elements but not any higher numbered elements. That is, each term of the series introduces an element not in the previous terms, and introduces only one such element. This stops of course when k = b, the number of elements in the given reactance.

Moreover, the element e_k [e_k is L_k if k is odd, and is C_k if k is even] appears as $1/e_k$ in the first term in which it appears, and not to some higher power. Also, in case the term t_k is made up of the sum of several quantities, e_k appears in only one of these quantities. Hence the term t_k can be written

$$t_k = A + B/e_k \qquad \text{A1.2}$$

where A and B are functions of elements e_{k-1}, e_{k-2}, ..., L_3, and C_2 but are not functions of e_k or any higher numbered elements. These statements are true for at least the first five terms of the series by inspection of Eq. A1.1 and are now proved in general.

That element e_{k+1} appears for the first time in term t_{k+1} can be proved by induction since it is known to be true for the first five terms. Hence, assume it is true for element e_k and prove that this implies that it is necessarily true for element e_{k+1} also.

Write the impedance z(s) in its continued fraction expansion about infinity

$$z(s) = sL_1 + \cfrac{1}{sC_2 + \cfrac{1}{\ddots\, se_k + \cfrac{1}{se_{k+1}}}} \qquad \text{A1.3}$$

This expression for the impedance of the network with k+1 elements can be rewritten as a similiar expression involving only k elements by defining

$$e_k' = e_k[1 + \frac{1}{e_k e_{k+1} s^2}] \qquad \text{A1.4}$$

By assumption, the k^{th} term of the expansion of z(s) about infinity is given by

$$t_k = [A + B/e_k]/s^{2k-3} \qquad \text{A1.5}$$

where A and B are functions of elements C_2, L_3, ..., e_{k-1}. Substituting for e_k from Eq. A1.4 gives

$$t_k = [A + \frac{B}{e_k(1 + \frac{1}{e_k e_{k+1} s^2})}]/s^{2k-3} \qquad \text{A1.6}$$

Expanding about infinity gives

$$t_k = [A + B/e_k]/s^{2k-3} - \frac{B}{e_k^2 e_{k+1} s^{2k-1}} + \dots \qquad \text{A1.7}$$

Hence, the element e_{k+1} appears with s raised to a power two greater than that in the term t_k and therefore must appear in term t_{k+1}. Moreover, it appears linearily in that term as shown above. Thus the theorem is proved by induction.

Inspection of this proof and the original equation shows also that the coefficients of e_k in the first term in which it appears is given by the square of the products of all the previous elements, C_2, L_3, ..., e_{k-1}. That is, in term t_k, the element e_k appears in the term

$$\frac{1}{e_k(C_2 L_3 C_4 \dots\dots e_{k-2} e_{k-1})^2} \qquad \text{A1.8}$$

Thus if an impedance z(s) is specified by its expansion about infinity and if it has five or fewer elements, the determination of the impedance is straightforward.

If z(s) is specified as

$$z(s) = a_1 s + a_2 s^{-1} + a_3 s^{-3} + \ldots\ldots., \qquad \text{A1.9}$$

then the following equations may be written:

$$
\begin{aligned}
a_1 &= L_1 \\
a_2 &= 1/C_2 \\
a_3 &= -1/L_3 C_2^2 \\
a_4 &= 1/L_3^2 C_2^2 C_4 + 1/L_3^2 C_2^3 \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\qquad \text{A1.10}
$$

As many equations may be written as terms in Eq. A1.1 are known. Notice that these are not simultaneous equations, but are solvable in order. That is, from the first equation, L_1 is determined, from the second equation, C_2 determined, from the third equation and the value of C_2 already calculated, L_3 determined, etc. Thus no simultaneous equations need be solved. The determination of the elements is an easy calculation. If more than five elements are needed, this procedure becomes computationally difficult even though there are no theoretical problems because of the difficulty of writing out explicitly the higher numbered terms as functions of the elements.

If more than five elements are necessary, another procedure is available. Suppose element e_{k+1} and all higher numbered elements are removed from the network (let element e_n become infinite for n = k+1, k+2, ...). Then the series expansion of the resulting impedance z'(s) has its first k terms the same as those in the expansion of z(s) since none of the higher numbered elements appear in these terms. But the $(k+1)^{th}$ term is different. Denote this term by t'_{k+1}. The relation between t_{k+1} and t'_{k+1} is given by

$$t_{k+1} = t'_{k+1} + \frac{1}{e_{k+1}(e_k e_{k-1} \ldots L_3 C_2)^2} . \qquad \text{A1.11}$$

Although t'_{k+1} is a very complicated function of elements e_k, e_{k-1}, ..., L_3, and C_2, it can be determined numerically in a straightforward manner. The impedance z'(s) is found from the known elements and expanded into an infinite series about infinity. The $(k+1)^{th}$ term is t'_{k+1}. This allows Eq. A1.11 to be used to solve for the element e_{k+1}. Then the process can be repeated again.

The procedure for determining the reactance from its series expansion may be summarized as follows.

1. Determine L_1, C_2, L_3, C_4, and L_5 from Eqs. A1.10.
2. Let the impedance corresponding to these elements be z_5. Calculate z_5 and expand it about infinity. The coefficient of the sixth term is t'_6. Calculate C_6 from the relation

$$t_6 = t'_6 + [C_6(L_5 C_4 L_3 C_2)^2]^{-1}.$$

3. Calculate the impedance z_6 which corresponds to the six known elements. Expand it into a series and find the seventh term, t_7'. Calculate L_7 from the relation

$$t_7 = t_7' + [L_7(C_6L_5C_4L_3C_2)^2]^{-1}.$$

4. Continue this process until an element comes out infinite. This must be the end of the network or else the series does not represent a physically realizable reactance function.

This problem is similiar to one which Fano investigated.[2] He expanded the reflection factor of a lossless network terminated in a resistance about infinity and showed that the network could be found from the series expansion. His technique, however, was limited to a four or five element network because it required that the terms in the series be written explicitly as functions of the elements in the network, and this becomes impossible to carry out when there are many elements in the network.

As an example of this process, consider the impedance z(s) defined by

$$z(s) = 2s + s^{-1} - s^{-3} + 2s^{-5} - 5s^{-7} + 14s^{-9} - 42s^{-11} +$$

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2. Fano, <u>op. cit.</u>

$$+ 132s^{-13} - 428s^{-15} + 1416s^{-17} + \ldots\ldots$$

Equations A1.10 yield the first five elements from the first five terms:

$$L_1 = 2 \qquad C_2 = 1 \qquad L_3 = 1 \qquad C_4 = 1 \qquad L_5 = 1$$

Since no further explicit coefficients are known, the extended procedure must be used. Impedance z_5 is calculated from these five elements to be

$$z_5 = 2s + \frac{s^3 + 2s}{s^4 + 3s^2 + 1}$$
$$= 2s + s^{-1} - s^{-3} + 2s^{-5} - 5s^{-7} + 13s^{-9} + \ldots$$

The sixth term is $t_6' = 13$. Hence (since the product of the elements C_2 through L_5 is unity) element C_6 is given by

$$1/C_6 = t_6 - 13 = 14 - 13 = 1.$$

Using this value for C_6, impedance z_6 is calculated to be

$$z_6 = 2s + \frac{s^4 + 3s^2 + 1}{s^5 + 4s^3 + 3s}$$
$$= 2s + s^{-1} - s^{-3} + 2s^{-5} - 5s^{-7} + 14s^{-9} - 41s^{-11} + \ldots\ldots$$

Here $t_7' = 41$ and $t_7 = 42$. Again the product of the elements

C_2 through L_5 is unity and C_6 is unity. Hence

$$L_7 = 1.$$

This process is continued and yields the eighth element to be

$$C_8 = 1.$$

The impedance z_8 is calculated and expanded into a series. The result is exactly the same as the original series. Hence element L_9 is infinite as are all higher numbered elements. The impedance z_8 is then the desired impedance and is guaranteed p.r. because all of its elements are positive.

The only difficult part of these computations is the calculation of the impedances z_6, z_7, etc., and even these are simple to carry out if the standard ladder techniques are used. Hence a straightforward technique for constructing an impedance from its expansion about infinity has been found.

Appendix Two

On the Determination of a Reactance Function From Its Power Series Expansion About the Origin

If the power series expansion about the origin of a reactance is specified in the form

$$z(s) = b_1 s^{-1} + b_2 s + b_3 s^3 + b_4 s^5 + \ldots\ldots. \qquad \text{A2.1}$$

the simple frequency transformation obtained by replacing s by 1/s changes this series into the form

$$z(s) = b_1 s + b_2 s^{-1} + b_3 s^{-3} + b_4 s^{-5} + \ldots. \qquad \text{A2.2}$$

which is recognized as the power series expansion about infinity of a reactance. If the original series represented a p.r. function, so must the modified series since a p.r. function of a p.r. function is itself p.r. Thus using the results of Appendix I, the elements in the network realization of $z(s^{-1})$ are found and then the reverse transformation applied. An inductance of L henries is replaced by a capacitance $C = 1/L$ and a capacitor C is replaced by an inductance $L = 1/C$.

The determination of the reactance may be done directly without the use of the frequency transformation in a manner similiar to that used in Appendix I. The network realization must be of the form shown in Fig. A2.1.

If the impedance of this network is expanded into a series about the origin, the first few terms are

$$z(s) = 1/sC_1 + sL_2 - s^3L_2^2C_3 + s^5[L_2^3C_3^2 + L_2^2C_3^2L_4] + \ldots \quad A2.3$$

Again it is noticed that the k^{th} term must be of the form

$$t_k = t_k' + e_k[e_{k-1}e_{k-3}\ldots L_4C_3L_2]^2 \qquad A2.4$$

where t_k' is a function of L_2, C_3, ..., e_{k-1} only.

Hence the procedure is to determine the first few elements from the explicit formulae given above. Then with element k set equal to zero, the impedance of the network is determined from the known values of elements C_1,....., e_{k-1} and expanded about the origin to determine term t_k'. Then element k is calculated from Eq. A2.4 and the procedure repeated until all elements have been found.

At any step in the solution, a negative element means z(s) is not realizable and hence the calculation stops. A quick rejection test exists. A necessary condition on the coefficients is that the pole at the origin must have a positive coefficient. The remaining coefficients must alternate in sign with the first one positive.

Appendix III

On the Determination of a Reactance Function Whose Value Is Specified at a Number of Complex Frequencies

It is of some interest to investigate the problem of determining a reactance function from its value specified at a number of complex frequencies. That is, if $s_1, s_2, \ldots, s_n$ are complex frequencies and if $z(s)$ is the desired reactance function, then the following data are specified:

$$
\begin{aligned}
z(s_1) &= r_1 + jx_1 \\
z(s_2) &= r_2 + jx_2 \\
&\ldots\ldots\ldots\ldots\ldots \\
z(s_n) &= r_n + jx_n
\end{aligned}
\qquad \text{A3.1}
$$

The problem is to find a physically realizable reactance function $z(s)$ which assumes these complex values at the given frequencies.

A similiar problem arises in an extension of the Miyata procedure for the synthesis of RLC driving-point impedances.[1] There it is necessary to determine a reactance function which assumes prescribed values at real frequencies.[2] The solution proposed by E.A. Guillemin makes use of real-part

1. Guillemin, op. cit., pp. 412-431.

2. Neitzert, Carl, "The Synthesis of a Two-Terminal Non-Dissipative Network For a Finite Band of Frequencies," Sc. D. Thesis, MIT, 1936.

sufficiency techniques and may lead to non-realizable impedances.

In this Miyata procedure, the data is presented not as values of the reactance at certain real frequencies, but rather in the form of a p.r. reactance function. That is, it is required that a reactance function z(s) be found which is equal to the negative of another reactance function at certain prescribed real frequencies. Thus the Miyata problem differs from the present problem in that no such reactance function is available.

If in the Miyata problem a non p.r. function results, it is not clear whether or not a more complicated p.r. function exists. In the application contemplated for the present problem, it is necessary to know not only whether or not a solution exists, but also how complex the resulting reactance function must be. Hence it is desirable to formulate a technique for going from the given frequency data directly to a reactance function and a method for predicting the complexity of this reactance function. Moreover, the method must not be restricted to the case of real frequencies only.

Consider the number of independent pieces of data specified. If n values of the reactance function are specified, there are 2n pieces of data since at each frequency the reactance has both a real and imaginary part. Hence the smallest reactance function which can satisfy this data is one with 2n poles (this is clear from consideration of the Cauer realization of a reactance function which produces one element for each pole and hence one element for each independent piece of specified information). Thus if a polynomial $P_{xo}(s)$ is defined as

$$\begin{aligned} P_{xo}(s) &= m_{xo} + n_{xo} \\ &= a_{2n}s^{2n} + a_{2n-1}s^{2n-1} + \ldots + a_1 s + 1 \end{aligned} \qquad \text{A3.2}$$

then a reactance function $z_{xo}(s)$ can be defined as the ratio of even to odd parts of this polynomial:

$$z_{xo}(s) = m_{xo}/n_{xo} \qquad \text{A3.3}$$

If this reactance is now evaluated at each complex frequency and the given value for $z_{xo}(s)$ used, a set of complex linear equations result. If these n equations are separated into 2n equations by separating the real and imaginary parts of each equation, there result a set of 2n linear algebraic equations in the 2n unknown coefficients $a_{2n}, \ldots\ldots, a_2, a_1$. Since there are as many equations as unknowns, a solution will usually exist. If an infinite number of solutions exist, this means that a smaller reactance function can be used to satisfy the given data. In any case, the reactance function may be found and therefore the polynomial $P_{xo}(s)$ also.

In addition to specifying the reactance at a given frequency, its derivative may also be specified. In this case, a linear equation also results. This may be seen by multiplying out Eq. A3.3 to get

$$n_{xo}(s)z_{xo}(s) = m_{xo}(s) \tag{A3.4}$$

Taking the derivative of both sides of this equation gives

$$n'_{xo}(s)z_{xo}(s) + n_{xo}(s)z'_{xo}(s) = m'_{xo}(s) \tag{A3.5}$$

Evaluating this expression at the given frequency and using the specified value of z(s) and z'(s) gives a linear equation in the unknown coefficients. It follows that derivatives of any order may be specified and a linear set of equations will still result.

Solving these simultaneous equations gives the coefficients of the polynomial $P_{xo}(s)$. If this polynomial is Hurwitz, the problem is solved for then $z_{xo}(s)$ is p.r. However there is no guarantee that is will be Hurwitz except in the case where only one frequency is specified.

There is no lack of generality in assuming all of the specified frequencies to be in the first quadrant of the s-plane or on the $j\omega$ axis since a reactance function is an odd function of frequency. A necessary condition on the given data is that each of the real parts be positive and the argument associated with each of the specified values of the reactance be less than the argument of the frequency variable, s, for z(s) must be a p.r. function. If this condition is fulfilled, a p.r. impedance necessarily exists which assumes these values at these frequencies. However a reactance function--which is a special case of an impedance function--need not exist. Moreover, the p.r. criterion says nothing at all about the complexity of the impedance assuming it exists at all.

From the above argument, it is clear that the assumption of minimum size reactance functions will often lead to a non p.r. function. Hence a method of determining whether or not a simple reactance function exists and a method for finding it are necessary.

One approach is as follows. Let the reactance function just found be $z_{xo}(s)$ and the desired reactance function $z_x(s)$. Then the desired result is

$$z_{xo}(s) = z_x(s) \text{ at } s = s_1, \ldots, s_n. \qquad \text{A3.6}$$

In terms of the even and odd parts of the corresponding polynomials this becomes

$$\frac{m_{xo}}{n_{xo}} = \frac{m_x}{n_x} \text{ at } s = s_1,\ldots\ldots,s_n. \qquad \text{A3.7}$$

Notice that the original data has now been put into the form in which Guillemin's data is presented. The only differance between the two problems is that $s_1,\ldots\ldots,s_n$ may be complex and z_{xo} is not p.r.

This requirement (Eq. A3.7) can be rewritten as

$$\frac{m_{xo}n_x}{n_{xo}m_x} = 1 \text{ at } s = s_1,\ldots\ldots s_n. \qquad \text{A3.8}$$

If now a constant K is placed on the left side, the problem reduces to the requirement that the equation be satisfied at each specified frequency with the constant K equal to unity.

$$K\frac{m_{xo}n_x}{n_{xo}m_x} = 1 \text{ at } s = s_k \text{ and } K = 1. \qquad \text{A3.9}$$

If now K is varied from zero to infinity and the locus of solutions to the equation plotted, the result is the familiar root locus plot.[3] If $z_x(s) = n_x/m_x$ rather than

3. Truxal, J.G., "Control System Synthesis," McGraw-Hill, NYC, 1955, Ch. 4.

m_x/n_x, the polynomials m_x and n_x are merely interchanged in Eq. A3.9.

However this problem is much simpler than the one usually encountered in the study of Feedback Systems for here all of the polynomials are even or odd and hence the loci will have four quadrant symmetry. Moreover, the polynomials m_x and n_x come from a physically realizable reactance function and hence all of their zeros must lie on the $j\omega$ axis and must separate one another (poles and zeros of $z_x(s)$ must alternate on the $j\omega$ axis).

To draw the loci, the zeros of m_{xo} and n_{xo} are determined approximately--that is, their approximate location in the s-plane is found. It is not necessary to know them exactly, but only whether they lie on the real axis or in the complex plane or on the $j\omega$ axis. Then the alternating poles and zeros of $z_x(s)$ are sketched on the $j\omega$ axis. There will be very little choice how these are entered. If $z_{xo}(s)$ has no poles or zeros on the $j\omega$ axis, there is no choice at all. Notice that the exact locations are not of interest yet. From experience with root loci plots, it is known that the general form of the loci does not change as poles and zeros move around but changes only when the relative placement is modified.

Once the loci have been sketched--and this is easily done--the requirement is that the loci pass through the frequencies $s_1, \ldots\ldots, s_n$ when K = 1. The value of K is not important for this can be changed by moving the locations of the jω axis poles and zeros of $z_x(s)$. What is important is that the loci pass through or in the vicinity of the desired frequencies. For example, if s_1 lies in the complex plane and no loci ever go into the complex plane for any choice of $z_x(s)$, it is clear that no matter how complex a function is assumed, no p.r. function exists to satisfy the given data.

From examination of the root locus plot, it is possible to tell at a glance whether or not a solution will exist and also approximately how complex an impedance $z_x(s)$ is necessary. This is the use of the plot, to tell the approximate order of complexity of the solution. If the solution appears to exist, a general polynomial $P_x(s)$ with coefficients to be determined is assumed. Then the requirement is equivalent to

$$\text{Odd } P_x(s)P_{xo}(-s) = nA_o \qquad \text{A3.10}$$

where A_O is the even polynomial containing the required roots $s_1, \ldots\ldots, s_n$ (and their images and conjugates), with each root appearing to the required order. The quantity n in Eq. A3.10 is merely the leftover part of the left-hand-side of the equation and has no physical significance. Note that

A_0 is the same polynomial which occurs in the numerator of the even part of $Z_0(s)$ except that any zeros at the origin are removed. If $z_x(s) = n_x/m_x$, Eq. A3.10 becomes

$$\text{Ev } P_x(s)P_{xo}(-s) = mA_0 \qquad \text{A3.11}$$

Multiplying out Eq. A3.10 gives a number of equations relating the unknown coefficients of $P_x(s)$ and n.[4] The coefficients of n are of no interest so that the set of equations is solved for the coefficients of $P_x(s)$ in terms of the coefficients of n (there are more unknowns than equations). Then these coefficients of n are chosen so as to leave P_x a Hurwitz polynomial. This last procedure requires some trial and error, but since a solution is known to exist, success is assured. Notice that multiple order zeros of A_0--implying that $z_x(s)$ and some derivatives were specified in Eqs. A3.1--do not affect this procedure at all. The method is best illustrated by examples.

Example 1

Let a reactance be defined at the two frequencies listed:

$$z(s) = 3\sqrt{2}/10[3 + j] \quad \text{at } s = (1 + j)/\sqrt{2}$$

$$z(s) = 10/27 \quad \text{at } s = 2$$

4. This method is similiar to the real-part sufficiency method used by Guillemin in reference 1.

Assume a third order polynomial $P_{xo}(s) = a_3s^3 + a_2s^2 + a_1s + 1$. Then $z_{xo}(s)$ is the ratio of, say, the odd part to the even part. Evaluating this impedance at the two given frequencies gives the following three equations:

$$2a_1 - 4a_3 = 10/3$$
$$4a_1 - (10/3)a_2 + 2a_3 = 0$$
$$2a_1 - (40/27)a_2 + 8a_3 = 10/27$$

These equations have the unique solution

$$a_3 = 0 \qquad a_2 = 2 \qquad a_1 = 5/3.$$

Hence the reactance $z_{xo}(s)$ is given by

$$z_{xo}(s) = \frac{(5/3)s}{2s^2 + 1}$$

This reactance is seen by inspection to be p.r. and hence the solution is complete.

Example 2

Define the impedance as in example 1 except that at the second frequency, $s = 2$, let the impedance be $z(s) = 2$. The equations corresponding to the assumed third order reactance function are given by

$$2a_1 - 4a_3 = 10/3$$
$$4a_1 - (10/3)a_2 + 2a_3 = 0$$
$$2a_1 - 8a_2 + 8a_3 = 2$$

Their solution is

$$a_3 = -11/9 \qquad a_2 = -15/9 \qquad a_1 = -7/9$$

Hence $z_{xo}(s)$ is given by

$$z_{xo}(s) = \frac{11s^3 + 7s}{15s^2 - 9}$$

This is seen to be non p.r. The pole-zero plot of this impedance and the root-locus plots for the two possibilities $z_x(s) = m_x/n_x$ and $z_x(s) = n_x/m_x$ are shown in Fig. A3.1. Examination of these loci shows that it is impossible to find a p.r. reactance function which will assume the required values. This is so because although the loci pass through the required two frequencies, the same locus passes through both frequencies and certainly can't do this for a single value of gain K. Notice also that the addition of more poles and zeros to the reactance $z_x(s)$ doesn't change the general shape of the root locus plot. Thus no matter how complex a $z_x(s)$ may be assumed, there will be no solution to the problem.

Suppose the specified frequencies are such that the polynomial A_0 is given by

$$A_0 = (s^4 + 1)(s^4 + s^2 + 4)$$

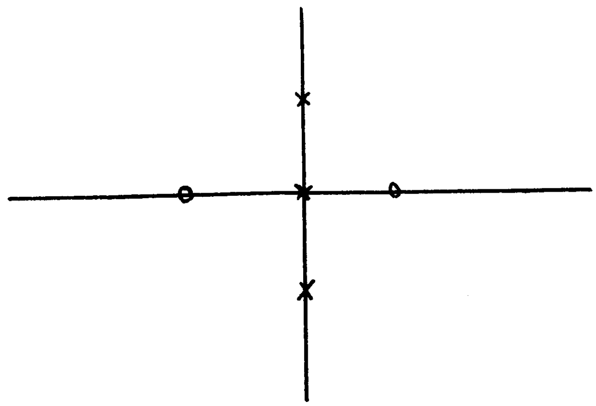
and $z_{xo}(s)$ has been found to be

$$z_{xo}(s) = \frac{(s^2 + 3)(s^2 + 4)(s^2 + 5)}{s(s^2 + 1)(s^2 + 2)}$$

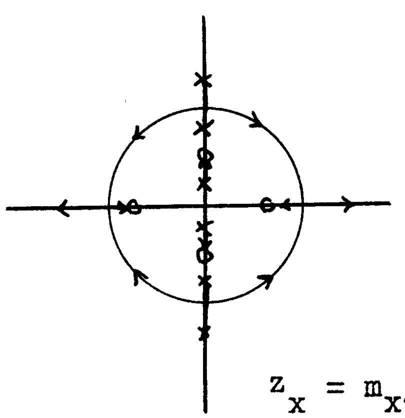
The pole-zero plot of this impedance is shown in Fig. A3.2. The impedance is obviously not p.r. If $z_x = m_x/n_x$, the root locus plot is shown in Fig. A3.2 for one choice of placement of the zeros and poles of z_x. Note that two different loci pass through the complex plane in the vicinity of the required frequencies. It seems plausible therefore that a solution does exist. In this case, an eighth order polynomial will probably suffice (P_{xo} is sixth order so P_x must be at least seventh order) since the loci must not be severely distorted in order to pass through the required frequencies.

Figure A3.1

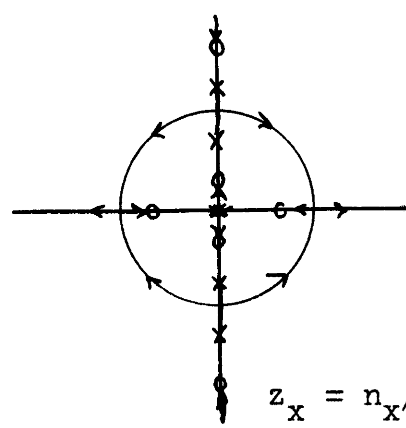
Root Locus Plot for Example Two



Pole-zero plot of $z_{xo}(s)$



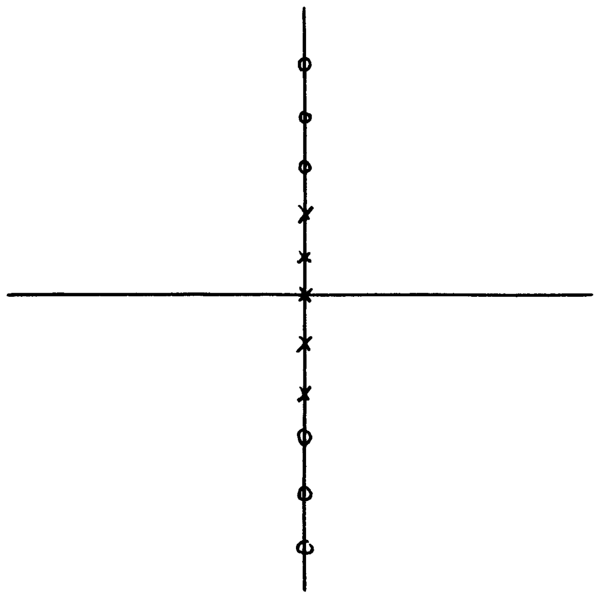
$z_x = m_x/n_x$



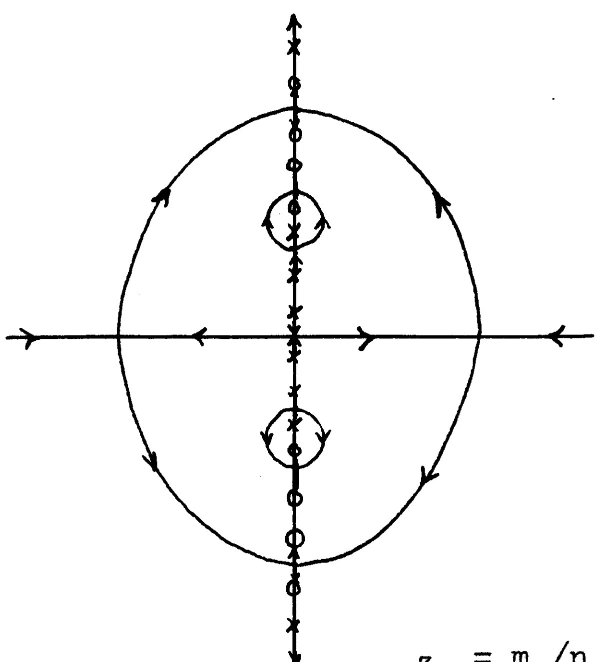
$z_x = n_x/m_x$

Figure A3.2

Root Locus Plot for Example Three



Pole-zero plot of $z_{xo}(s)$



$z_x = m_x/n_x$

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Biography

James D. Schoeffler was born February 9, 1933 in Cleveland, Ohio. He began studying Electrical Engineering at Case Institute of Technology in 1951 and received the B.S. degree in 1955. He returned to Case in 1955 as a Graduate Assistant and in 1956 as a Teaching Fellow, receiving the M.S. degree in 1957.

From 1957 to 1960 he attended the Massachusetts Institute of Technology as a General Electric Fellow for one year, a Teaching Assistant the next year, and an instructor the last year.