Computer Science and Artificial Intelligence Laboratory
Technical Report

MIT-CSAIL-TR-2012-028 September 7, 2012

A Social-Welfare Optimal Probabilistic Mechanism for Knightian Single-Good Auctions

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Abstract

We provide an optimal probabilistic mechanism for maximizing social welfare in single-good auctions when each player does not know his true valuation for the good, but only a set of valuations that is guaranteed to include his true one.

1 Preliminaries

Finite Focus Adopting a finite perspective, we focus on finite type spaces and on mechanisms that assign finitely many (pure) strategies to the players.

Single-Good Auctions in the Classical and Knightian Settings We study single-good auctions in an incomplete-information, non-Bayesian, private-value setting with quasi-linear utilities.

A type, also called a *valuation*, consists of an integer between 0 and an integral bound, B. The number of players is n, and the true valuation of a player i is θ_i. An outcome ω consists of a pair of profiles, $\omega = (a, P)$, where $a \in \{\perp, 1, \ldots, n\}$ and $P \in \mathbb{R}^n$. We refer to a as the *allocation*: $a = \perp$ signifies that the good is unallocated, and otherwise a is the player who wins the good. Each P_i represents the price paid by player i.

The only knowledge of a player i about the true valuation profile θ consists of a set K_i, i's *approximate valuation*, such that $K_i \ni \theta_i$. A player i is *Knightian* if $K_i \neq \{\theta_i\}$. The profile of approximate valuations is $K \stackrel{\text{def}}{=} K_1 \times \cdots \times K_n$. The auction (or the setting) is Knightian if there exists at least one Knightian player, and classical otherwise. In either case, the social welfare of an allocation A, $\mathrm{SW}(\theta, a)$, is defined to be θ_a if $a \neq \perp$, and 0 otherwise; and the maximum social welfare, $\mathrm{MSW}(\theta)$, is defined to be $\max_i \theta_i$.

For every player i, i's utility function u_i maps a valuation t_i and an outcome $\omega = (a, P)$ to $u_i(t_i, \omega) \stackrel{\text{def}}{=} t_i - P_i$ if $a = i$, and $u_i(t_i, \omega) \stackrel{\text{def}}{=} -P_i$ otherwise.

Incomplete Preferences The only assumption we make about the preferences of a Knightian player i is that i strictly prefers outcome ω to outcome ω' if $u_i(t_i, \omega) \geq u_i(t_i, \omega')$ for all $t_i \in K_i$ and $u_i(t'_i, \omega) > u_i(t'_i, \omega')$ for some $t'_i \in K_i$. (A Knightian player i is free to "complete" his preferences in an arbitrary way, consistent with the above assumption, but a mechanism designer has no clue on how each Knightian player might do that.)

Knowledge Inaccuracy The approximate valuation K_i of a player i is always well defined, but can be very "inaccurate". In particular, if every i has absolutely no clue about his own θ_i, then every K_i coincides with the entire type space of i. In this case no mechanism can be expected to perform well, but intuitively a good mechanism should perform better when the inaccuracy of players' knowledge decreases. Accordingly, we believe that mechanism performance should be defined as a function of such inaccuracy, properly measured. Thus: How should knowledge inaccuracy be measured?

In an auction of a single good, where a valuation essentially consists of a non-negative integer, a simple measure of the knowledge inaccuracy of (a player i with an approximate valuation) K_i would be the its "spread", that is, $\max K_i - \min K_i$, where $\max K_i$ and $\min K_i$ respectively are the maximum and the minimum candidate valuation in K_i. Such a measure, however, is too coarse. For instance, the approximate valuations $K_i = \{10, 20\}$ and $K'_i = \{1010, 1020\}$ would have the same inaccuracy, although K_i is intuitively less accurate than K'_i. Indeed, approximately speaking, in the first case player i "knows that his true valuation is $15 \pm 33\%$", while in the second case he "knows that his true valuation is 1015 $\pm 0.05\%$". We thus choose to take into consideration not only spread but also "magnitude", and define *the inaccuracy of a player i (a set K_i), δ_i*, as follows:

$$\delta_i \stackrel{\text{def}}{=} \frac{\max K_i - \min K_i}{\max K_i + \min K_i} \text{ if } K_i \neq \{0\}, \text{ and } \delta_i \stackrel{\text{def}}{=} 0 \text{ otherwise.}$$

It is immediately seen that δ_i is always in $[0, 1]$ and that it has the following alternative definition: the smallest value r such that K_i is contained in an interval of the form $[x - rx, x + rx]$. Thus, according to our definition each player i knows his own true valuation within a factor δ_i. Indeed, for $K_i = \{10, 20\}$ we have $\delta_i \approx 0.33$; and for $K'_i = \{1010, 1020\}$ we have $\delta_i \approx 0.05$.

A Knightian single-good auction is δ-approximate *if* $\max_i \delta_i \leq \delta$.

For $\delta \in [0, 1]$, we denote by $\mathcal{K}(\delta)$ the set of all δ-approximate valuations K.

Mechanisms In a Knightian auction context, a mechanism M specifies

- a set $S = S_1 \times \cdots \times S_n$, where S_i is the set of pure strategies of player i, and
- a (possibly probabilistic) outcome function, typically denoted by M itself, mapping S to (distributions over) outcomes.

We denote pure strategies by Latin letters, and possibly mixed strategies by Greek ones.

If $\sigma \in \Delta(S_1) \times \cdots \times \Delta(S_n)$, then $M_i^A(\sigma)$ and $M_i^P(\sigma)$ respectively denote the probability that the good is assigned to player i and the expected price paid by i.

Implementation in Dominant and Undominated Strategies In a classical setting of incomplete information there are two main notions of implementation: implementation in dominant strategies and implementation in undominated strategies. The second-price mechanism is efficient according to both notions.

Both notions are naturally extended to the Knightian setting, by properly quantifying also over the members of the players' approximate valuations. Namely, for a player i with approximate valuation K_i, a pure strategy

- $s_i \in S_i$ *Knightian very weakly dominates* $s'_i \in S_i$, if

$$\forall\, \theta_i \in K_i\,,\, \forall\, t_{-i} \in S_{-i} \,:\, \mathbb{E}u_i(\theta_i, M(s_i, t_{-i})) \geq \mathbb{E}u_i(\theta_i, M(s'_i, t_{-i}))\ .$$

  $s_i \in S_i$ is *Knightian very weakly dominant* if for all $s'_i \in S_i$, s_i very weakly dominates s'_i.

- $s_i \in S_i$ is *Knightian (weakly) dominated* by σ_i, in symbols $\sigma_i \overset{\mathrm{w}}{\underset{i,K_i}{\succ}} s_i$, if

$$\begin{aligned}&\forall\, \theta_i \in K_i\,,\, \forall\, t_{-i} \in S_{-i} \,: \mathbb{E}u_i(\theta_i, M(\sigma_i, t_{-i})) \geq \mathbb{E}u_i(\theta_i, M(s_i, t_{-i}))\ \text{, and}\\ &\exists\, \theta_i \in K_i\,,\, \exists\, t_{-i} \in S_{-i} \,: \mathbb{E}u_i(\theta_i, M(\sigma_i, t_{-i})) > \mathbb{E}u_i(\theta_i, M(s_i, t_{-i}))\ .\end{aligned}$$

  $s_i \in S_i$ is *Knightian (weakly) undominated*, $s_i \in \mathsf{UDed}_i(K_i)$, if $\sigma_i \overset{\mathrm{w}}{\underset{i,K_i}{\succ}} s_i$ for no $\sigma_i \in \Delta(S_i)$.

Notice that a Knightian player i does not have a very weakly dominant strategy in the second-price mechanism, and in many other mechanisms where S_i coincides with $\{0, 1, \ldots, B\}$. Nevertheless, dominant-strategy mechanisms continue to be well-defined. Since a variant of the revelation principle still applies, a property is implementable in very weakly dominant strategies if there exists a *(very weakly) dominant-strategy-truthful* mechanism M such that (1) a player i's strategy set S_i consists of bidding *a set of valuations*; (2) bidding K_i is a very weakly dominant strategy for player i; and (3) the property holds for $M(K_1, \cdots, K_n)$.

A mechanism M implements a property in (weakly) undominated strategies if the property holds for $M(s)$ for all $s \in \mathsf{UDed}(K)$, where $\mathsf{UDed}(K) \overset{\text{def}}{=} \mathsf{UDed}_1(K_1) \times \cdots \times \mathsf{UDed}_n(K_n)$.

Social-welfare Performance Seeking very robust guarantees, we define the social-welfare performance of a mechanism M in an "*ex post* manner", by taking the worst case over all sets of approximate valuation profiles and all possible true valuations. That is,

In a δ-approximate auction, the social-welfare performance of a very weakly dominant-strategy-truthful mechanism M is

$$\min_{K \in \mathcal{K}(\delta), \theta \in K} \frac{\mathrm{SW}(\theta, M(K_1, \ldots, K_n))}{\mathrm{MSW}(\theta)}\ ,$$

and that of an undominated-strategy mechanism M is

$$\min_{K \in \mathcal{K}(\delta), \theta \in K, s \in \mathsf{UDed}(K)} \frac{\mathrm{SW}(\theta, M(s))}{\mathrm{MSW}(\theta)}\ .$$

2 Our Prior Results

In our prior work (CMZ12), we prove two results for Knightian auctions of a single-good: a "positive" result and a "negative" one. Namely,

Theorem ((CMZ12))**.** *Let M be the second-price mechanism with any deterministic tie-breaking rule. Then, for all $\delta \in [0,1]$, all valuation bounds B, all profiles K of δ-approximate valuations in $\{0,\ldots,B\}$, all true valuation profiles $\theta \in K$, and all strategy profiles $v \in \mathsf{UDed}(K)$:*

$$\mathrm{SW}\big(\theta, M(v)\big) \geq \left(\frac{1-\delta}{1+\delta}\right)^2 \mathrm{MSW}(\theta) - 2\,\frac{1-\delta}{1+\delta} \ .$$

Theorem ((CMZ12))**.**

(a) *Let M be a* deterministic *mechanism for n players with valuation bound B. If $B \geq \frac{1+\delta}{\delta}$, then there exist a profile K of δ-approximate valuations, a true valuation profile $\theta \in K$, and a strategy profile $s \in \mathsf{UDed}(K)$ such that:*

$$\mathrm{SW}\big(\theta, M(s)\big) \leq \left(\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{3}{B}\right) \mathrm{MSW}(\theta) \ .$$

(b) *Let M be a possibly* probabilistic *mechanism for n players with valuation bound B. If $B \geq \frac{1+\delta}{\delta}$, then there exists a profile K of δ-approximate valuations, a true valuation profile $\theta \in K$, and a strategy profile $s \in \mathsf{UDed}(K)$ such that:*

$$\mathrm{SW}\big(\theta, M(s)\big) \leq \left(\left(\frac{1-\delta}{1+\delta}\right)^2 + \frac{3}{B} + \frac{4\delta}{(1+\delta)^2 n}\right) \mathrm{MSW}(\theta) \ .$$

Our two results show that there is a potential gap between the performance of the optimal deterministic and probabilistic mechanisms. Thus, can the performance of the second-price mechanism in the Knightian setting be improved by a probabilistic mechanism? We prove that the answer is yes, and we actually explicitly construct an optimal probabilistic mechanism, provided that the inaccuracy parameter δ (but not the actual profile of δ-approximate valuations!) is known to the mechanism designer.[1]

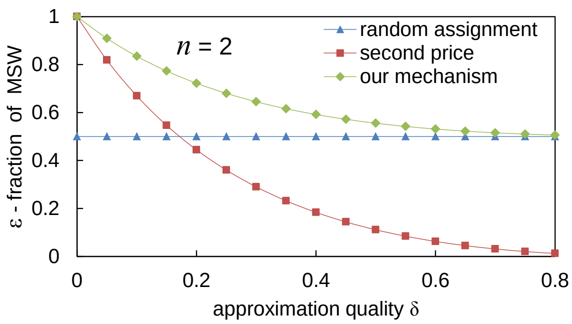
3 Statement and Discussion of Our Result

Theorem 1. *$\forall n$, $\forall \delta \in (0,1)$, and $\forall B$, there exists a mechanism $M^{(\delta)}_{\mathsf{opt}}$ such that for every profile K of δ-approximate valuations in $\{0,1,\ldots,B\}$, every true-valuation profile $\theta \in K$, and every undominated strategy profile $v \in \mathsf{UDed}(K)$:*

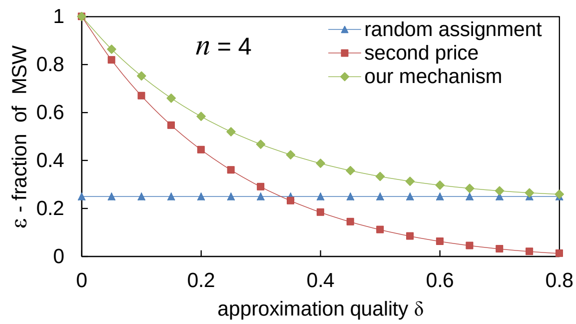
$$\mathbb{E}\Big[\mathrm{SW}\big(\theta, M^{(\delta)}_{\mathsf{opt}}(v)\big)\Big] \geq \left(\frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4\delta}{n(1+\delta)^2}\right) \mathrm{MSW}(\theta) \ .$$

[1] Actually one can show similar results when the designer knows an upper bound δ' sufficiently close to δ.

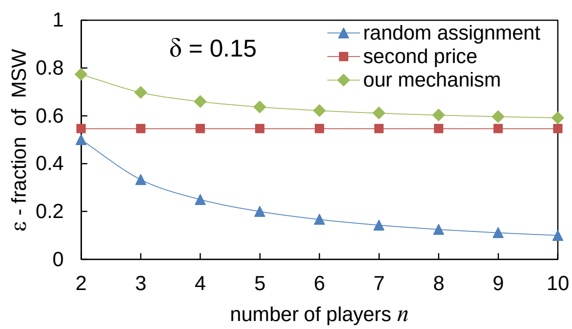
Practicality. The proof of Theorem 1 is very technically hard in this paper. Nonetheless, we would like to emphasize that $M_{\text{opt}}^{(\delta)}$ is very *practically* played, as it requires almost no computation from the players, and a very small amount of computation from the mechanism. In addition, its performance is *practically* preferable to that of the second-price mechanism. For instance, when $\delta = 0.5$, $M_{\text{opt}}^{(\delta)}$ guarantees a social welfare that is at least **five** times higher than that of the second-price mechanism when there are 2 players, and at least **three** times higher when there are 4 players.



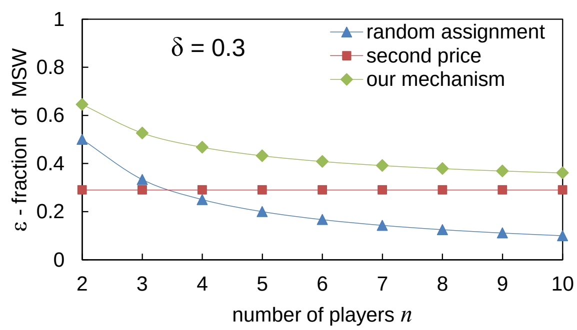
(a) With $n = 2$ players, the second-price mechanism performs *worse* than randomly assigning the good for $\delta > 0.18$.



(b) With $n = 4$ players, the second-price mechanism performs *worse* than randomly assigning the good for $\delta > 0.34$.



(c) With $\delta = 0.15$, the second-price mechanism always performs better than randomly assigning the good.



(d) With $\delta = 0.3$, the second-price mechanism performs *worse* than randomly assigning the good for $n = 2, 3$.

Figure 1: Performance of our optimal mechanism

Performance Diagrams In Figure 1 we compare the social welfare guarantees of:

- randomly assigning the good ($\varepsilon = \frac{1}{n}$),
- the second-price mechanism ($\varepsilon = \frac{(1-\delta)^2}{(1+\delta)^2}$, see (CMZ12)), and
- our optimal mechanism ($\varepsilon = \frac{(1-\delta)^2}{(1+\delta)^2} + \frac{4\delta}{n(1+\delta)^2}$, see Theorem 1).

In Figure 1a and Figure 1b we compare ε versus δ, and in Figure 1c and Figure 1d we compare ε versus n. The green data, our mechanism, is always better (at times significantly) than the other two mechanisms.

4 Proof of Theorem 1

In this section we explicitly construct and analyze the desired mechanism $M_{\mathsf{opt}}^{(\delta)}$. This process is not going to be trivial, so we first provide a concise representation of our mechanism in Section 4.1 independent of our proof. Next, we show a generic lemma for constructing undominated-strategy mechanisms in Knightian setting in Section 4.2. The details of our proof are then broken into Section 4.3, Section 4.4 and Section 4.5, where we provide intuitions as needed.

4.1 Our Optimal Mechanism $M_{\mathsf{opt}}^{(\delta)}$

In this section we provide a concise description to our optimal mechanism in Theorem 1 for a single-good auction with n players, valuation bound B and approximation accuracy δ. We first construct the following allocation function:

Definition 4.1. *For every $\delta \in (0,1)$, and let $D_\delta \stackrel{\text{def}}{=} \left(\frac{1+\delta}{1-\delta}\right)^2 - 1 > 0$. We define the function $f^{(\delta)} \colon [0,B]^{[n]} \to [0,1]^{[n]}$ as follows:*

- *for every $z = (z_1, \dots, z_n) \in [0,B]^{[n]}$ such that $z_1 \geq z_2 \geq \cdots \geq z_n$, let $n^* \in \{1, 2, \dots, n\}$ be the index in $[n]$ (which exists and is unique) such that*

$$z_1 \geq \cdots \geq z_{n^*} > \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta} \geq z_{n^*+1} \geq \cdots \geq z_n \enspace ,$$

  Then set

$$f_i^{(\delta)}(z) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} \cdot \frac{z_i(n^*+D_\delta) - \sum_{j=1}^{n^*} z_j}{z_i D_\delta}, & \text{if } i \leq n^*, \\ 0, & \text{if } i > n^*; \end{cases}$$

- *for other z, define $f^{(\delta)}$ by extending it symmetrically.*

The code for our mechanism $M_{\mathsf{opt}}^{(\delta)}$ is:

Code for outcome function of $M_{\mathsf{opt}}^{(\delta)}$

public parameter: $\delta \in (0,1)$

inputs: $v_1, \dots, v_n \in \{0, 1, \dots, B\}$

output: (i, P), where $i \in [n] \cup \{\perp\}$ is the winning player and $P \in \mathbb{R}^{[n]}$ is the price profile

pseudocode:

1. Draw r uniformly at random in $[0,1]$.
2. (Define $f_0^{(\delta)} \stackrel{\text{def}}{=} 0$.)
3. If there exists $i \in [n]$ such that $\sum_{j=0}^{i-1} f_i^{(\delta)}(v) < r \leq \sum_{j=0}^{i} f_i^{(\delta)}(v)$:
   - Compute $P_i \stackrel{\text{def}}{=} v_i - \frac{\int_0^{v_i} f_i^{(\delta)}(z, v_{-i})\, dz}{f_i^{(\delta)}(v)}$, and $P_j = 0$ for $j \neq i$, and output (i, P).
4. Otherwise, output $(\perp, (0, \dots, 0))$. (No player is assigned the good.)

We note that our mechanism can be tweaked to make sure that the good is always assigned to some player. But the proof is more involved than it already is, and we leave it to a future version of this paper.

4.2 The Distinguishable Monotonicity Lemma

To prove that a given social welfare performance is guaranteed in undominated strategies, as it is needed for Theorem 1, we are happy to work with a suitable class of restricted mechanisms, using only very special strategies and allocation functions. But what should "suitable" mean?

On one hand, these restrictions should suffice for proving Theorem 1. On the other hand, they should ensure that the undominated strategies corresponding to a given approximate-valuation set can be characterized in a way that is both *conceptually simple* and *easy to work with*.

Specifically, we consider mechanisms whose strategies consist of possible valuations, namely the set $\{0, \ldots, B\}$, and whose allocation functions (over $\{0, 1, \ldots, B\}^{[n]}$) are restrictions of integrable functions (over $[0, B]^{[n]}$) satisfying a suitable monotonicity property. A simple lemma, the *Distinguishable Monotonicity Lema*, will then guarantee that for this type of mechanisms:

> *the set of undominated strategies of a player i with approximate valuation K_i is contained in* $\{\min K_i, \min K_i + 1, \ldots, \max K_i\}$.

We believe that this simple property will be useful beyond our immediate need to prove Theorem 1. Note that:

- Our setting is still discrete: continuous domains are only tools for proving the lemma.
- The Distinguishable Monotonicity Lemma, when specialized to the case where players know their valuations exactly, is a strengthening of a classical lemma that characterizes those mechanisms that are (very-weakly-)dominant-strategy-truthful in single-good auctions.
- The Distinguishable Monotonicity Lemma actually applies to *all* single-parameter domains, not just single-good auctions (the same way that the classical lemma does).

4.2.1 Details

Before we describe our lemma, let us recall a traditional way to define auction mechanisms from suitable allocation functions.

Definition 4.2. *If $f : [0, B]^{[n]} \to [0, 1]^{[n]}$ is an integrable[2] allocation function, then we denote by M_f the mechanism where the strategy space $S = \{0, 1, \ldots, B\}^{[n]}$ and, on input bid profile $v \in S$,*

- *with probability $f_i(v)$ the good is assigned to player i, and*
- *if player i wins, he pays $P_i = v_i - \frac{\int_0^{v_i} f_i(z, v_{-i})\, dz}{f_i(v_i, v_{-i})}$ (and all other players pay $P_j = 0$ for $j \neq i$.)*

[2]Specifically, we require that, for each v_{-i}, the function $f_i(z, v_{-i})$ is integrable with respect to z on $[0, B]$.

Remark 4.3.
- M_f is deterministic if and only if $f(\{0,1,\dots,B\}) \subseteq \{0,1\}^{[n]}$.
- For all player i and bid profile v, the expected price $M_i^P(v) = v_i \cdot f_i(v_i, v_{-i}) - \int_0^{v_i} f_i(z, v_{-i})\, dz$.
- We stress that M_f continues to have discrete strategy space $S = \{0,1,\dots,B\}$, as the analysis over a continuous domain for f is only a tool for proving the lemma.
- Recall that an allocation function f is monotonic if each f_i is non-decreasing in the bid of player i, for any fixed choice of bids of all other players. In the exactly-valuation world, the class of mechanisms M_f's when f is both integrable and monotonic gives a full characterization to all (very-weakly-)dominant-strategy-truthful mechanisms in single-good auctions.

Now, we want to slightly strengthen this notion of monotonicity.

Definition 4.4. *Let $f : [0,B]^{[n]} \to [0,1]^{[n]}$ be a allocation function. For $d \in \{1,2\}$, we say that f is d-**distinguishably monotonic** (d-DM, for short) if f is integrable, monotonic, and satisfying the following "distinguishability" condition:*

$$\forall\, i \in [n]\,,\ \forall v_i, v_i' \in S_i \ \text{s.t.}\ v_i \le v_i' - d,\ \exists\, v_{-i} \in S_{-i} \quad \int_{v_i}^{v_i'} \big(f_i(z, v_{-i}) - f_i(v_i, v_{-i})\big)\, dz > 0 \ .$$

If f is d-DM, we say that M_f is d-DM.

Distinguishable monotonicity is certainly an additional requirement to monotonicity, but actually quite mild. Indeed, the second-price mechanism is 2-DM and, if ties are broken at random, even 1-DM (see Example 4.5). Yet, in our approximate-valuation world, this mild additional requirement is quite useful for "controlling" the *undominated strategies* of a mechanism, and thus for engineering implementations of desirable social choice functions in undominated strategies.

Example 4.5 (Second-Price Mechanism)**.** Recall that the *second-price mechanism* is a direct mechanism that assigns the good to the highest bidder at a price equal to the second-highest bid: it is a pair $M_{\mathsf{2P}} = (S_{\mathsf{2P}}, M_{\mathsf{2P}})$ with $S_{\mathsf{2P}} = \{0,1,\dots,B\}^{[n]}$ and

$M_{\mathsf{2P}}(v) \stackrel{\text{def}}{=}$

1. Assign the good to the highest bidder: $i^* \stackrel{\text{def}}{=} \arg\max_{i\in[n]} v_i$.
2. Charge the highest bidder the second price: $P_{i^*} \stackrel{\text{def}}{=} \max_{\substack{i\in[n]\\ i\neq i^*}} v_i$.

   (And everyone else pays nothing.)

In the language of allocation functions, M_{2P} can be represented as a mechanism M_f, where f is 2-DM; also, if we require that the ties (for the highest bidder) are broken at random (by giving a positive, but not necessarily equal, probability to every highest bidder), then f is 1-DM. So let us prove these two facts:

Proof. At a high level, the allocation function f for M_{2P} is almost unique, except for those input bids that contain ties. Now take an arbitrary second-price mechanism M_{2P} with a specific tie breaking rule. For each player $i \in [n]$, we define f_i as follows: for every bid sub-profile $z_{-i} \in [0,B]^{[n]-\{i\}}$, letting $x^* \stackrel{\text{def}}{=} \max_{j\neq i} z_j$,

- for every $x < x^*$, define $f_i(x, z_{-i}) \stackrel{\text{def}}{=} 0$,
- for every $x > x^*$, define $f_i(x, z_{-i}) \stackrel{\text{def}}{=} 1$,
- for $x = x^*$ then there is a tie, in which case
  - if z_{-i} is a valid integer bid in $\{0, 1, \dots, B\}^{[n]-\{i\}}$, then define $f_i(x^*, z_{-i})$ to be the winning probability according to how M_{2P} breaks the tie,[3] and
  - if z_{-i} is not a valid integer bid in $\{0, 1, \dots, B\}^{[n]-\{i\}}$, then define $f_i(x^*, z_{-i})$ arbitrarily (say 0 for example).[4]

One can verify that the mechanism M_f according to the definition above is exactly the given M_{2P}; this is because the two coincide on the allocation probabilities for all integer points $v \in \{0, 1, \dots, B\}^{[n]}$, and the price (recall the integral in Definition 4.2) is exactly the winning probability multiplied by the second highest bid.

It is clear that f is monotonic. Moreover, for any $v_i, v_i' \in \{0, 1, \dots, B\}$ such that $v_i < v_i' - 1$, let everyone else bid some integer x in the open interval (v_i, v_i') by setting $v_{-i} = \{x, x, \dots, x\}$. By construction, $f_i(v_i, v_{-i}) = 0$ and every (not necessarily integer) $z \in (x, v_i']$ satisfies $f(z, v_{-i}) = 1$; but this establishes that f is 2-DM:

$$\int_{v_i}^{v_i'} \big(f_i(z, v_{-i}) - f_i(v_i, v_{-i})\big)\, dz \geq (v_i' - x)(1 - 0) \geq 1 > 0 \ .$$

If instead we tweak M_{2P} to break ties at random by giving a positive (but not necessarily equal) probability to every highest bidder, then, for any $v_i, v_i' \in \{0, 1, \dots, B\}$ such that $v_i < v_i'$, let everyone else bid some $x = v_i$ by setting $v_{-i} = \{x, x, \dots, x\}$. We have that for every $z \in (x, v_i']$, $f(z, v_{-i}) = 1$, but $f(v_i, v_{-i}) < 1$ (since every highest bidder is awarded the good with positive probability); but this establishes that f is 1-DM:

$$\int_{v_i}^{v_i'} \big(f_i(z, v_{-i}) - f_i(v_i, v_{-i})\big)\, dz \geq (v_i' - x) \cdot (1 - f(v_i, v_{-i})) > 0 \ ,$$

as desired. $\square$

Lemma 4.6 (Distinguishable Monotonicity Lemma). *If f is a d-DM allocation function, then M_f is such that, for any player i and δ-approximate-valuation profile K,*

$$\begin{aligned} &\mathsf{UDed}_i(K_i) \subseteq \{\min K_i, \dots, \max K_i\} \ \text{ if } d = 1, \text{ and} \\ &\mathsf{UDed}_i(K_i) \subseteq \{\min K_i - 1, \dots, \max K_i + 1\} \quad \text{if } d = 2. \end{aligned}$$

(Above, $\min K_i$ and $\max K_i$ respectively denote the minimum and maximum integers in K_i.)

Proof. For every $i \in N$, let $v_i^{\perp} \stackrel{\text{def}}{=} \min K_i$ and $v_i^{\top} \stackrel{\text{def}}{=} \max K_i$. Then, to establish our lemma it suffices to prove that, $\forall i \in N$ and $\forall d \in \{1, 2\}$, the following four properties hold:

[3] If in M_{2P} player i receives the good with probability 1, then we set $f_i(x^*, z_{-i}) \stackrel{\text{def}}{=} 1$; if player i receives the good with probability 0.2, then $f_i(x^*, z_{-i}) \stackrel{\text{def}}{=} 0.2$, and so on.

[4] Indeed, M_f will never be invoked on an input with more than one non-integer points. It invokes integer points for calculating allocation probabilities, and one non-integer points for calculating the price.

1. $v_i^\perp$ *very-weakly dominates every* $v_i \le v_i^\perp - d$.
2. $v_i^\top$ *very-weakly dominates every* $v_i \ge v_i^\top + d$.
3. *There is a strategy sub-profile* v_{-i} *for which* $v_i^\perp$ *is strictly better than every* $v_i \le v_i^\perp - d$.
4. *There is a strategy sub-profile* v_{-i} *for which* $v_i^\top$ *is strictly better than every* $v_i \ge v_i^\top + d$.

Proof of Property 1. Fix any (pure) strategy sup-profile $v_{-i} \in S_{-i}$ for the other players and any possible true valuation $\theta_i \in K_i$. Letting $v^\perp = (v_i^\perp, v_{-i})$ and $v = (v_i, v_{-i})$, we prove that

$$\begin{aligned}
&\mathbb{E}\Big[u_i\big(\theta_i, M(v^\perp)\big)\Big] - \mathbb{E}\Big[u_i\big(\theta_i, M(v)\big)\Big] \\
&= \big(f_i(v^\perp) - f_i(v)\big) \cdot \theta_i - \big(M_i^P(v^\perp) - M_i^P(v)\big) \\
&= \big(f_i(v^\perp) - f_i(v)\big) \cdot \theta_i - \left(v_i^\perp \cdot f_i(v^\perp) - \int_0^{v_i^\perp} f_i(z, v_{-i})\, dz - v_i \cdot f_i(v) + \int_0^{v_i} f_i(z, v_{-i})\, dz\right) \\
&= \big(f_i(v^\perp) - f_i(v)\big) \cdot (\theta_i - v_i^\perp) + \int_{v_i}^{v_i^\perp} \big(f_i(z, v_{-i}) - f_i(v)\big)\, dz \ .
\end{aligned}$$

Now note that, since $\theta_i \in K_i$, $\theta_i - v_i^\perp = \theta_i - \min K_i \ge 0$; moreover, by the monotonicity of f, whenever $z \ge v_i$, it holds that $f_i(z, v_{-i}) \ge f_i(v)$. We deduce that $\mathbb{E}u_i\big(\theta_i, M(v^\perp)\big) \ge \mathbb{E}u_i\big(\theta_i, M(v)\big)$. We conclude that $v_i^\perp$ very-weakly dominates v_i.

Proof of Property 2. Analogous to that of Property 1 and omitted.

Proof of Property 3. Due to the d-distinguishable monotonicity of M, $v_i \le v_i^\perp - d$ implies the existence of a strategy sub-profile v_{-i} making $\int_{v_i}^{v_i^\perp} \big(f_i(z, v_{-i}) - f_i(v)\big)\, dz$ strictly positive. For such v_{-i}, therefore, playing $v_i^\perp$ is strictly better than v_i.

Proof of Property 4. Analogous to that of Property 3 and omitted. □

4.3 A Very Restricted Search

In order to leverage our Distinguishably Monotonicity Lemma 4.6, it is natural for us to search for $M^{(\delta)}_{\mathsf{opt}}$ among 1-DM mechanisms. Let us now distill an additional requirement for the underlying allocation function of such mechanisms that suffices for our goals. We shall do so in terms of the following positive quantity D_δ: for all $\delta \in (0, 1)$,

$$D_\delta \stackrel{\text{def}}{=} \left(\frac{1+\delta}{1-\delta}\right)^2 - 1.$$

Definition 4.7. *We say that a allocation function f is* δ**-good** *if it is* 1*-DM and:*

$$\forall\, i \in [n],\ \forall\, v \in \{0, 1, \ldots, B\}^{[n]}, \quad \sum_{j=1}^{n} f_j(v) v_j + D_\delta \cdot f_i(v) v_i \ge \frac{1}{n} \cdot v_i(n + D_\delta) \ . \tag{4.1}$$

The reason why the additional requirement is sufficient is easily understood:

Lemma 4.8. *If f is δ-good, then M_f satisfies that such that for every δ-approximate-valuation profile K, every strategy profile $s \in \mathsf{UDed}(K)$ and every true-valuation profile $\theta \in K$:*

$$\mathbb{E}\Big[\mathrm{SW}\big(\theta, M_f(v)\big)\Big] \ge \left(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2}\right) \mathrm{MSW}(\theta) \ .$$

Proof. Let K be an arbitrarily chosen δ-approximate-valuation profile. Then, because in any allocation the social welfare coincides with the welfare of a given player, to prove our lemma it suffices to prove that

$$\forall \theta \in K, \quad \forall v \in \mathsf{UDed}(K), \quad \forall i \in [n], \quad \sum_{j=1}^{n} \theta_j f_j(v) \geq \left(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \right) \theta_i \ . \tag{4.2}$$

For every $i \in [n]$, let $x_i \in \mathbb{R}$ be such that $K_i \subseteq \delta[x_i]$, and let $\delta[x] = \delta[x_1] \times \cdots \times \delta[x_n]$. Then, $\theta \in K$ and the Distinguishable Monotonicity Lemma respectively imply

$$(1-\delta)x_i \leq \theta_i \leq (1+\delta)x_i \quad \text{and} \quad (1-\delta)x_i \leq \min K_i \leq v_i \leq \max K_i \leq (1+\delta)x_i.$$

Combining these two chains of inequalities yields

$$\frac{1-\delta}{1+\delta} v_i \leq \theta_i \leq \frac{1+\delta}{1-\delta} v_i \ . \tag{4.3}$$

Let us now argue that Eq. 4.2 holds by arbitrarily fixing v and i and showing that it is impossible to construct a “bad” θ so as to violate Eq. 4.2.

In trying to construct a “bad” θ, it suffices to choose θ_j (for $j \neq i$) to be as small as possible, since θ_j only appears on the left-hand side with a positive coefficient. For θ_i, however, we may want to choose it as large as possible if $f_i(v) \geq \big(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2}\big)$, or as small as possible otherwise. So there are two extreme θ's.

Considering these extreme choices, we conclude that no θ contradicts Eq. 4.2 if:

$$\sum_{j=1}^{n} \Big(\frac{1-\delta}{1+\delta}\Big) v_j f_j(v) \geq \left(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \right) \Big(\frac{1-\delta}{1+\delta}\Big) v_i \ , \text{ and}$$

$$\sum_{j=1}^{n} \Big(\frac{1-\delta}{1+\delta}\Big) v_j f_j(v) + \Big(\frac{1+\delta}{1-\delta} - \frac{1-\delta}{1+\delta}\Big) v_i f_i(v) \geq \left(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2} \right) \Big(\frac{1+\delta}{1-\delta}\Big) v_i \ .$$

Simplifying the above equations, Eq. 4.2 holds if both the following inequalities hold:

$$\sum_{j=1}^{n} v_j f_j(v) \geq \frac{n + D_\delta}{n} \cdot \frac{1}{D_\delta + 1} \cdot v_i \ , \tag{4.4}$$

$$\sum_{j=1}^{n} v_j f_j(v) + D_\delta \cdot v_i f_i(v) \geq \frac{n + D_\delta}{n} v_i \ . \tag{4.5}$$

Note that Eq. 4.5 holds because it is implied by the hypothesis that f is δ-good; note also that Eq. 4.4 holds because it is implied by Eq. 4.5. Indeed, since $\frac{1}{D_\delta+1} = \big(\frac{1-\delta}{1+\delta}\big)^2 < 1$ for all $\delta \in (0,1)$,

$$\sum_{j=1}^{n} v_j f_j(v) \geq \frac{1}{D_\delta + 1} \left(\sum_{j=1}^{n} v_j f_j(v) + D_\delta v_i f_i(v) \right) \geq \frac{1}{D_\delta + 1} \frac{n + D_\delta}{n} v_i \ .$$

Thus both Eq. 4.2 and our lemma hold. $\square$

4.4 Our Allocation Function

In light of our last lemma, all is left is to find a suitable δ-good allocation function f.

Some intuition. If the players' bids are not "clustered", then f should clearly give a much higher probability mass to the highest bids, as lower bids are less likely to come from players with high true valuations. However, when the highest bids are close to each other, it is hard for f to "infer" from them who the player with the highest true valuation really is — after all, we are in an approximate-valuation model. The intelligent thing for f to do in such a case is to assign the good to a randomly chosen high-bidding player. To achieve optimality, however, one must be much more careful in allocating probability mass, and some complexities should be expected.

Since the mechanism $M_{\mathsf{opt}}^{(\delta)}$ of Theorem 1 is allowed to depend on the approximation accuracy δ, we construct its allocation function, $f^{(\delta)}$, depending on it. Our proposed $f^{(\delta)}$ derives from the players' bids a threshold, and probabilistically chooses the winning player only among those bids lying above the threshold. We now explain the rationale for these choices.

Recall that, to be δ-good, a allocation function $f : [0,B]^{[n]} \to [0,1]^{[n]}$ should satisfy Eq. 4.1, that is:

$$\forall\, i \in [n],\ \forall\, v \in \{0,1,\ldots,B\}^{[n]}, \quad \sum_{j=1}^{n} f_j(v)v_j + D_\delta \cdot f_i(v)v_i \geq \frac{1}{n} \cdot v_i(n+D_\delta) \ .$$

A reasonable guess to "solve for f" is to restrict our attention to *symmetric* functions. The most natural candidate is simply

$$\forall z \in [0,B]^{[n]}, \quad f_i(z) = \frac{1}{n} \cdot \frac{z_i(n+D_\delta) - \sum_{j=1}^{n} z_j}{z_i D_\delta} \ .$$

One could verify that the function f, in addition to being symmetric, sums up to 1, is 1-DM, and satisfies the desired condition Eq. 4.1. (In fact, as we shall see, the above candidate f coincides with our proposed $f^{(\delta)}$ when no threshold is introduced.) We would be done, except for one crucial fact: f sometimes takes negative values!

We therefore need to "patch" the guessed function f by forcing non-negativity while maintaining the other required properties, and this is exactly where the idea of a threshold, winners, and losers comes in. Roughly, only players with sufficiently low reported valuations are at risk of a "negative probability" and, because are most likely to have low true valuations, we remove them from the auction altogether. To preserve the other properties, though, we need to re-weight the function, thereby obtaining Eq. 4.6. Thus, at high level, we simply keep removing players until all of the players are given non-negative probability (by virtue of being in the auction or having been thrown out). A similar idea previously appeared in (CLS+11).

While the introduction of a threshold fixes the "negativity problem", it introduces additional complexities. (For example, even the simple task of verifying monotonicity, where the bids of all players but i are fixed, becomes non-trivial. Indeed, the number of winners n^* varies as the bid of player i increases, and thus the definition of $f^{(\delta)}$ varies too.)

Let us now proceed more formally. Recall that $D_\delta \stackrel{\text{def}}{=} \left(\frac{1+\delta}{1-\delta}\right)^2 - 1 > 0$.

Definition 4.9. *For every $\delta \in (0,1)$, define the function $f^{(\delta)}\colon [0,B]^{[n]} \to [0,1]^{[n]}$ as follows: for every $i \in [n]$ and every $z = (z_1, \ldots, z_n) \in [0,B]^{[n]}$*

- *if $z_1 \geq z_2 \geq \cdots \geq z_n$, then*

$$f_i^{(\delta)}(z) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} \cdot \frac{z_i(n^*+D_\delta) - \sum_{j=1}^{n^*} z_j}{z_i D_\delta}, & \text{if } i \leq n^*, \\ 0, & \text{if } i > n^*; \end{cases} \quad , \tag{4.6}$$

  where $n^* \in \{1, 2, \ldots, n\}$ is the index in $[n]$ (whose existence and uniqueness will be proved shortly) such that

$$z_1 \geq \cdots \geq z_{n^*} > \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta} \geq z_{n^*+1} \geq \cdots \geq z_n \ . \tag{4.7}$$

- *else, $f_i^{(\delta)}(z) \stackrel{\text{def}}{=} f_{\pi(i)}^{(\delta)}(z_{\pi(1)}, \ldots, z_{\pi(n)})$ where π is any permutation of the players such that $z_{\pi(1)} \geq \cdots \geq z_{\pi(n)}$ (i.e., we define $f_i^{(\delta)}$ by extending it symmetrically).*

We call $\frac{\sum_{j=1}^{n^*} z_j}{n^*+D_\delta}$ the **threshold**, *players $1, \ldots, n^*$ the* **winners**, *and players $n^*+1, \ldots, n$ the* **losers**.

Lemma 4.10. *$f^{(\delta)}$ is a well-defined allocation function.*

Proof. We first prove that n^* exists and is unique, and begin with the existence proof.

Assume, without loss of generality, that $z_1 \geq z_2 \geq \cdots \geq z_n$. Note that there exists an index n' in $[n]$ such that

$$\forall\, i > n', \quad z_i \leq \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \ .$$

Indeed, Eq. 4.8 vacuously holds for $n' = n$. Now take n'' to be the least such index. Accordingly,

$$\forall\, i > n'', \quad z_i \leq \frac{\sum_{j=1}^{n''} z_j}{n'' + D_\delta} \ . \tag{4.8}$$

Next we claim that

$$\forall\, i \leq n'', \quad z_i > \frac{\sum_{j=1}^{n''} z_j}{n'' + D_\delta} \ . \tag{4.9}$$

To prove Eq. 4.9, it suffices to consider $i = n''$ because z is non-increasing. Indeed, by the minimality of n'' we know that ("$n'' - 1$ does not work", that is) there exists some $j \geq n''$ such that

$$z_{n''} \geq z_j > \frac{\sum_{j=1}^{n''-1} z_j}{n'' - 1 + D_\delta} \ ,$$

which, after rearranging, is equivalent to $z_{n''} > \frac{\sum_{j=1}^{n''} z_j}{n''+D_\delta}$ as desired.

At last, combining Eq. 4.8 and Eq. 4.9, and choosing $n^* = n''$, Eq. 4.7 is satisfied.

Next, we prove that n^* is unique. Suppose by way of contradiction that there exist two integers $n^\perp$ and $n^\top$, with $n^\perp < n^\top$ both satisfying Eq. 4.7. Now define

$$S^\perp \stackrel{\text{def}}{=} \sum_{j=1}^{n^\perp} z_j, \quad S^\top \stackrel{\text{def}}{=} \sum_{j=1}^{n^\top} z_j, \quad S^\Delta \stackrel{\text{def}}{=} S^\top - S^\perp, \quad \text{and } n^\Delta \stackrel{\text{def}}{=} n^\top - n^\perp \ .$$

By invoking Eq. 4.7 with $n^\top$ and $n^\perp$, we deduce that for $i \in \{n^\perp + 1, \ldots, n^\top\}$,

$$\frac{S^\perp}{n^\perp + D_\delta} \geq z_i > \frac{S^\top}{n^\top + D_\delta} = \frac{S^\perp + S^\Delta}{n^\perp + n^\Delta + D_\delta} \ .$$

Averaging over all z_i for $i \in \{n^\perp + 1, \ldots, n^\top\}$, we get

$$\frac{S^\perp}{n^\perp + D_\delta} \geq \frac{S^\Delta}{n^\Delta} > \frac{S^\perp + S^\Delta}{n^\perp + n^\Delta + D_\delta} \ . \tag{4.10}$$

Let us now show that the second inequality of Eq. 4.10 contradicts the first inequality Eq. 4.10:

$$\begin{aligned} \frac{S^\Delta}{n^\Delta} > \frac{S^\perp + S^\Delta}{n^\perp + n^\Delta + D_\delta} &\Leftrightarrow (n^\perp + n^\Delta + D_\delta) S^\Delta > n^\Delta (S^\perp + S^\Delta) \\ &\Leftrightarrow (n^\perp + D_\delta) S^\Delta > n^\Delta S^\perp \Leftrightarrow \frac{S^\Delta}{n^\Delta} > \frac{S^\perp}{(n^\perp + D_\delta)} \ . \end{aligned} \tag{4.11}$$

The contradiction establishes the uniqueness of n^*.

We are left to prove that (a) $f_i^{(\delta)}(z) \geq 0$ for every i and z, and (b) $\sum_i f_i^{(\delta)}(z) \leq 1$ for every z. (Indeed, the last two properties imply that $f_i^{(\delta)}(z) \leq 1$.)

Assume, again without loss of generality, that $z_1 \geq z_2 \geq \cdots \geq z_n$. Eq. 4.7 tells us that $z_i(n^* + D_\delta) - \sum_{j=1}^{n^*} z_j \geq 0$ for each $i \leq n^*$, so (a) follows immediately. As for (b),

$$\begin{aligned} \sum_{i=1}^{n} f_i^{(\delta)}(z) &= \frac{1}{n} \cdot \frac{n + D_\delta}{n^* + D_\delta} \cdot \sum_{i=1}^{n^*} \frac{z_i(n^* + D_\delta) - \sum_{j=1}^{n^*} z_j}{z_i D_\delta} \\ &= \frac{1}{n} \cdot \frac{n + D_\delta}{(n^* + D_\delta) D_\delta} \cdot \left(n^*(n^* + D_\delta) - \sum_{i=1}^{n^*} \sum_{j=1}^{n^*} \frac{z_j}{z_i} \right) \\ &\leq \frac{1}{n} \cdot \frac{n + D_\delta}{(n^* + D_\delta) D_\delta} \cdot (n^*(n^* + D_\delta) - n^* n^*) = \frac{n + D_\delta}{n} \cdot \frac{n^*}{n^* + D_\delta} \leq 1 \ . \end{aligned}$$

□

Lemma 4.11. *$f^{(\delta)}$ is monotonic.*

Proof. By symmetry it suffices to show that $f^{(\delta)}$ is monotonic with respect to the n-th coordinate. Without loss of generality, assume $z_1 \geq z_2 \geq \cdots \geq z_{n-1}$. We need to prove that for any $z_n^\perp$ and $z_n^\top$ with $0 \leq z_n^\perp < z_n^\top \leq B$,

$$f_n^{(\delta)}(z_{-n}, z_n^\perp) \leq f_n^{(\delta)}(z_{-n}, z_n^\top) \ . \tag{4.12}$$

We will prove Eq. 4.12 in three steps.

- **Step 1.** Letting n' be the number of winners in a game where only the first $n-1$ players are bidding z_{-n}, we first prove that:

$$z_n \leq \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \longrightarrow f_n^{(\delta)}(z_{-n}, z_n) = 0 \text{ (i.e., } n \text{ is a loser)} \tag{4.13}$$

$$z_n > \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \longrightarrow f_n^{(\delta)}(z_{-n}, z_n) > 0 \text{ (i.e., } n \text{ is a winner)} \tag{4.14}$$

  To show Eq. 4.13, recall that, in the game with only the first $n-1$ players bidding z_{-n}, we have n' winners satisfying,

$$\forall i \in \{1, 2, \ldots, n'\},\ z_i > \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta}; \quad \forall i \in \{n'+1, \ldots, n-1\},\ z_i \leq \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \ .$$

  Then imagine that player n comes with bid z_n that is at most $\frac{\sum_{j=1}^{n'} z_j}{n'+D_\delta}$. In this new game, because the threshold does not change, the set of winners continues to be $\{1, 2, \ldots, n'\}$ and therefore n must be a loser. Indeed,

$$\forall i \in \{1, 2, \ldots, n'\}, z_i > \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta}; \quad \forall i \in \{n'+1, \ldots, n\}, z_i \leq \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \ .$$

  To show Eq. 4.14, we actually prove its contrapositive: namely,

$$f_n^{(\delta)}(z_{-n}, z_n) = 0 \text{ (i.e. } n \text{ is a loser)} \longrightarrow z \leq \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta} \ .$$

  Let n^* be the number of winners when $f_n^{(\delta)}(z_{-n}, z_n) = 0$, that is, in the game where there are n players, the bid profile is z, and player n is a loser; then,

$$\forall i \in \{1, 2, \ldots, n^*\}, z_i > \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta}; \quad \forall i \in \{n^*+1, \ldots, n\}, z_i \leq \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta} \ .$$

  The above also implies the following, where player n has been removed:

$$\forall i \in \{1, 2, \ldots, n^*\}, z_i > \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta}; \quad \forall i \in \{n^*+1, \ldots, n-1\}, z_i \leq \frac{\sum_{j=1}^{n^*} z_j}{n^* + D_\delta} \ .$$

  This means, n^* is also the number of winners for the $(n-1)$-player game, i.e., $n^* = n'$. This gives $z_n \leq \frac{\sum_{j=1}^{n^*} z_j}{n^*+D_\delta} = \frac{\sum_{j=1}^{n'} z_j}{n'+D_\delta}$. □

Because of Step 1, we only need to show Eq. 4.12 for $z_n^\perp$ and $z_n^\top$ satisfying $z_n^\top > z_n^\perp > \frac{\sum_{j=1}^{n'} z_j}{n'+D_\delta}$. Notice that in such a case, player n is always a winner. Therefore, let $\{1, \ldots, n^\perp, n\}$ and $\{1, \ldots, n^\top, n\}$ be the winners when the bid profiles are $(z_{-n}, z_n^\perp)$ and $(z_{-n}, z_n^\top)$ respectively.

- **Step 2.** We now prove that
$$n^{\perp} \geq n^{\top} \ . \tag{4.15}$$
Assume by way of contradiction that $n^{\perp} < n^{\top}$ and. As in Lemma 4.10, set $n^{\Delta} \stackrel{\text{def}}{=} n^{\top} - n^{\perp}$, $S^{\perp} \stackrel{\text{def}}{=} \sum_{j=1}^{n^{\perp}} z_j$ and $S^{\top} \stackrel{\text{def}}{=} \sum_{j=1}^{n^{\top}} z_j = S^{\perp} + S^{\Delta}$. Then each player i, with $n^{\perp} \leq i < n^{\top}$, is a loser when the bid profile is $(z_{-n}, z_n^{\perp})$ while a winner when the bid profile is $(z_{-n}, z_n^{\top})$; in particular,
$$\frac{S^{\perp} + z_n^{\perp}}{n^{\perp} + 1 + D_\delta} \geq z_i > \frac{S^{\top} + z_n^{\top}}{n^{\top} + 1 + D_\delta} = \frac{S^{\perp} + S^{\Delta} + z_n^{\top}}{n^{\perp} + n^{\Delta} + 1 + D_\delta} \ .$$
Averaging over all $n^{\perp} \leq i < n^{\top}$ we get:
$$\frac{S^{\perp} + z_n^{\perp}}{n^{\perp} + 1 + D_\delta} \geq \frac{S^{\Delta}}{n^{\Delta}} > \frac{S^{\perp} + S^{\Delta} + z_n^{\top}}{n^{\perp} + n^{\Delta} + 1 + D_\delta}$$
but this is already a contradiction, since the right hand side is equivalent to (using a similar technique as Eq. 4.11):
$$\frac{S^{\Delta}}{n^{\Delta}} > \frac{S^{\perp} + S^{\Delta} + z_n^{\top}}{n^{\perp} + n^{\Delta} + 1 + D_\delta} \Leftrightarrow \frac{S^{\Delta}}{n^{\Delta}} > \frac{S^{\perp} + z_n^{\top}}{n^{\perp} + 1 + D_\delta} \ ,$$
which actually contradicts the left hand side, as $z_n^{\top} > z_n^{\perp}$. Therefore, $n^{\perp} \geq n^{\top}$. □

We now use the fact that $n^{\perp} \geq n^{\top}$ to obtain Eq. 4.12 for such $z_n^{\perp}$ and $z_n^{\top}$ satisfying $z_n^{\top} > z_n^{\perp} > \frac{\sum_{j=1}^{n'} z_j}{n' + D_\delta}$.

- **Step 3.** We now prove Eq. 4.12.

If $n^{\perp} = n^{\top}$, then for both $(z_{-n}, z_n^{\top})$ and $(z_{-n}, z_n^{\perp})$, the set of winners is $\{1, 2, \ldots, n^{\perp}, n\}$. Let $n^* = n^{\perp} + 1 = n^{\top} + 1$ be the number of winners and we get
$$\begin{aligned} f_n^{(\delta)}(z_{-n}, z_n^{\perp}) &= \frac{1}{n} \cdot \frac{n + D_\delta}{n^* + D_\delta} \cdot \frac{z_n^{\perp}(n^* + D_\delta) - \sum_{j=1}^{n^*-1} z_j - z_n^{\perp}}{z_n^{\perp} D_\delta} \\ &\leq \frac{1}{n} \cdot \frac{n + D_\delta}{n^* + D_\delta} \cdot \frac{z_n^{\top}(n^* + D_\delta) - \sum_{j=1}^{n^*-1} z_j - z_n^{\top}}{z_n^{\top} D_\delta} = f_n^{(\delta)}(z_{-n}, z_n^{\top}) \ . \end{aligned}$$

If $n^{\perp} > n^{\top}$, let $n^{\perp} = n^{\top} + n^{\Delta}$, $S^{\top} = \sum_{j=1}^{n^{\top}} z_j$ and $S^{\perp} = \sum_{j=1}^{n^{\perp}} z_j = S^{\top} + S^{\Delta}$ as before. Then we average over all z_i for $n^{\top} < i \leq n^{\perp}$ and get:
$$\frac{S^{\Delta}}{n^{\Delta}} > \frac{S^{\perp} + z_n^{\perp}}{n^{\perp} + 1 + D_\delta} = \frac{S^{\top} + S^{\Delta} + z_n^{\perp}}{n^{\top} + n^{\Delta} + 1 + D_\delta} \ . \tag{4.16}$$
But this is equivalent to (again using the same technique as Eq. 4.11)
$$\frac{S^{\Delta}}{n^{\Delta}} > \frac{S^{\top} + z_n^{\perp}}{n^{\top} + 1 + D_\delta} \ . \tag{4.17}$$

Letting $C_1 = \frac{n+D_\delta}{n}$, we now do the final calculation:

$$
\begin{aligned}
& f_n^{(\delta)}(z_{-n}, z_n^\top) - f_n^{(\delta)}(z_{-n}, z_n^\perp) \\
&= C_1 \cdot \Big(\frac{z_n^\top(n^\top + 1 + D_\delta) - S^\top - z_n^\top}{(n^\top + 1 + D_\delta) z_n^\top} - \frac{z_n^\perp(n^\perp + 1 + D_\delta) - S^\perp - z_n^\perp}{(n^\perp + 1 + D_\delta) z_n^\perp} \Big) \\
&= C_1 \cdot \Big(\frac{S^\perp + z_n^\perp}{(n^\perp + 1 + D_\delta) z_n^\perp} - \frac{S^\top + z_n^\top}{(n^\top + 1 + D_\delta) z_n^\top} \Big) \\
&= C_2 \cdot \Big((S^\perp + z_n^\perp)(n^\top + 1 + D_\delta) z_n^\top - (S^\top + z_n^\top)(n^\perp + 1 + D_\delta) z_n^\perp \Big) \\
&= C_2 \cdot \Big((S^\top + S^\Delta + z_n^\perp)(n^\top + 1 + D_\delta) z_n^\top - (S^\top + z_n^\top)(n^\top + n^\Delta + 1 + D_\delta) z_n^\perp \Big) \\
&= C_2 \cdot \Big(S^\top(n^\top + 1 + D_\delta)(z_n^\top - z_n^\perp) + S^\Delta(n^\top + 1 + D_\delta) z_n^\top - n^\Delta(S^\top + z_n^\top) z_n^\perp \Big) \\
&\geq C_2 \cdot \Big(S^\top(n^\top + 1 + D_\delta)(z_n^\top - z_n^\perp) + S^\Delta(n^\top + 1 + D_\delta) z_n^\top - n^\Delta(S^\top + z_n^\perp) z_n^\top \Big) \geq 0
\end{aligned}
$$

Here the last inequality has used $z_n^\top - z_n^\perp \geq 0$ and $S^\Delta(n^\top + 1 + D_\delta) - n^\Delta(S^\top + z_n^\perp) > 0$ (by Eq. 4.17). □

This finishes the proof that $f^{(\delta)}$ is monotonic. □

Lemma 4.12. *$f^{(\delta)}$ is 1-distinguishably monotonic.*

Proof. We already know from Lemma 4.11 that $f^{(\delta)}$ is monotonic. Also, the integrability of $f^{(\delta)}$ is obvious, because $f^{(\delta)}$ is piecewise continuous, and there are at most n pieces, as the number of winners decreases when z_n increases (recall Eq. 4.15). We are therefore left to prove the "distinguishability condition".

Fix a player $i \in [n]$ and two distinct valuations $v_i, v_i' \in \{0, 1, \ldots, B\}$, and assume that $v_i < v_i'$. Define $v_{-i} \stackrel{\text{def}}{=} (v_i, v_i, \ldots, v_i)$, then:

- $f(v_i, v_{-i}) = \frac{1}{n}$ since there are n winners, all bidding the same valuation.
- $f(z, v_{-i}) = \frac{1}{n D_\delta}(D_\delta + n - 1 - \frac{v_i}{z}(n-1)) > \frac{1}{n}$, when $v_i < z \leq (1 + D_\delta) v_i$.

Here the upper bound $z \leq (1 + D_\delta) v_i$ is to make sure that the number of winners is still n on input (z, v_{-i}). Notice that $f(z, v_{-i})$ is a function that is *strictly* increasing when z increases in such range, and therefore

$$\int_{v_i}^{v_i'} \big(f_i(z, v_{-i}) - f_i(v_i, v_{-i})\big)\, dz \geq \int_{v_i}^{\min\{v_i', (1+D_\delta) v_i\}} \big(f_i(z, v_{-i}) - f_i(v_i, v_{-i})\big)\, dz > 0 \ ,$$

as desired. □

Lemma 4.13. *$f^{(\delta)}$ is δ-good.*

Proof. We already know from Lemma 4.12 that $f^{(\delta)}$ is 1-DM. Therefore, in order to prove that $f^{(\delta)}$ is δ-good, we only need to show that Eq. 4.1 holds. We will actually prove that Eq. 4.1 holds not only for the discrete cube $\{0, 1, \ldots, B\}^{[n]}$ but also in the continuous cube $[0, B]^{[n]}$.

Without loss of generality, assume $z_1 \geq z_2 \geq \cdots \geq z_n$. We first observe that:

$$\sum_{i=1}^{n} f_i^{(\delta)}(z)z_i = \sum_{i=1}^{n^*} f_i^{(\delta)}(z)z_i = \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} \cdot \sum_{i=1}^{n^*} \frac{z_i(n^*+D_\delta) - \sum_{j=1}^{n^*} z_j}{D_\delta}$$
$$= \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} \cdot \left(\sum_{i=1}^{n^*} z_i\right) \ .$$

For each player k with $k > n^*$, because he is a loser, we have,

$$\sum_{j=1}^{n} f_j^{(\delta)}(z)z_j + D_\delta \cdot f_k^{(\delta)}(z)z_k = \sum_{i=1}^{n} f_i^{(\delta)}(z)z_i = \frac{n+D_\delta}{n} \cdot \frac{\sum_{i=1}^{n^*} z_i}{n^*+D_\delta} \geq \frac{n+D_\delta}{n} \cdot z_k$$

satisfying Eq. 4.1, where the last inequality is due to $k > n^*$ and Eq. 4.7.

For each winner i (i.e., with $i \leq n^*$), we have

$$\sum_{j=1}^{n} f_j^{(\delta)}(z)z_j + D_\delta \cdot f_i^{(\delta)}(z)z_i = \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} \cdot \left(\sum_{i=1}^{n^*} z_i\right) + D_\delta \cdot f_i^{(\delta)}(z)z_i$$
$$= \frac{1}{n} \cdot \frac{n+D_\delta}{n^*+D_\delta} z_i(n^*+D_\delta) = \frac{1}{n} \cdot z_i(n+D_\delta)$$

again satisfying Eq. 4.1. □

4.5 Our Mechanism $M_{\mathsf{opt}}^{(\delta)}$

Theorem 1. *$\forall n$, $\forall \delta \in (0,1)$, and $\forall B$, there exists a mechanism $M_{\mathsf{opt}}^{(\delta)}$ such that for every δ-approximate-valuation profile K, every true-valuation profile $\theta \in K$, and every strategy profile $v \in \mathsf{UDed}(K)$:*

$$\mathbb{E}\Big[\mathrm{SW}\big(\theta, M_{\mathsf{opt}}^{(\delta)}(v)\big)\Big] \geq \left(\frac{(1-\delta)^2 + \frac{4\delta}{n}}{(1+\delta)^2}\right) \mathrm{MSW}(\theta) \ .$$

Proof. By Lemma 4.13, the function $f^{(\delta)}$ from Definition 4.9 is a (well-defined) allocation function that is also δ-good. Therefore, invoking Lemma 4.8, the mechanism $M_{\mathsf{opt}}^{(\delta)} \stackrel{\text{def}}{=} M_{f^{(\delta)}}$ yields the target social welfare. ■

Finally, we note that $M_{\mathsf{opt}}^{(\delta)}$ can be implemented efficiently (just like the second-price mechanism):

Claim 4.14. *The outcome function M of $M_{\mathsf{opt}}^{(\delta)}$ is efficiently computable.*

Proof. It suffices to show that both the allocation function $M^A = f^{(\delta)}\big|_{\{0,1,\ldots,B\}^{[n]}}$ and expected price function M^P are efficiently computable over $\{0,1,\ldots,B\}^{[n]}$.

First, $f^{(\delta)}$ is efficiently computable for trivial reasons: the number of winners n^* is between 1 and n and can be determined in linear time.

However, M^P is efficiently computable for a more involved reason. Without loss of generality, we show how to compute the expected price for player n as a function of v_n, i.e.,

$$M_n^P(v_{-n}, v_n) = f_n^{(\delta)}(v_{-n}, v_n) \cdot v_n - \int_0^{v_n} f_n^{(\delta)}(v_{-n}, z)\, dz \ .$$

Indeed, when v_{-n} is fixed, $f_n^{(\delta)}$ is a function piece-wisely defined according with respect to v_n, since different values of v_n may result in different numbers of winners n^*. Assume without loss of generality that $v_1 \geq v_2 \geq \cdots \geq v_{n-1}$, and let n' be the number of winners when player n is absent.

When $v_n \leq \frac{\sum_{j=1}^{n'} v_j}{n'+D_\delta}$, the proof of the monotonicity of $f^{(\delta)}$ implies that $f_n^{(\delta)} = 0$, so that integral below this line is zero.

When $v_n > \frac{\sum_{j=1}^{n'} v_j}{n'+D_\delta}$, one can again see from the proof of the monotonicity of $f^{(\delta)}$ that n^* is non-increasing as a function of v_n. Therefore, $f_n^{(\delta)}$ contains at most n different pieces and, for each piece with n^* fixed, $f_n^{(\delta)}(v_{-n}, v_n) = a + b/v_n$ is a function that is symbolically intergrable. Therefore, the only question is how to calculate the pieces for $f_n^{(\delta)}$.

This is again not hard, by using a simple line sweep method. One can start from $v_n = \frac{\sum_{j=1}^{n'} v_j}{n'+D_\delta}$ and move v_n upwards. At any moment, one can calculate the earliest time that Eq. 4.7 is violated, and claim that another piece of $f_n^{(\delta)}$ is found. □

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