Strict Convexity of the Surface Tension for Non-convex Potentials

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Contents

Abstract

We study gradient models on the lattice $\mathbb{Z}^d$ with non-convex interactions. These Gibbs fields (lattice models with continuous spin) emerge in various branches of physics and mathematics. In quantum field theory they appear as massless field theories. Even though our motivation stems from considering vector valued fields as displacements for atoms of crystal structures and the study of the Cauchy-Born rule for these models, our attention here is mostly devoted to interfaces, with the gradient field as an *effective* interface interaction. In this case we prove the strict convexity of the surface tension (interface free energy) for low temperatures and sufficiently small interface tilts using muli-scale (renormalisation group analysis) techniques following the approach of Brydges and coworkers [**Bry09**]. This is a complement to the study of the high temperature regime in [**CDM09**] and it is an extension of Funaki and Spohn's result [**FS97**] valid for strictly convex interactions.

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CHAPTER 1

Introduction

This paper has two related goals.

First, we seek to identify uniform convexity properties for a class of lattice gradient models with non-convex microscopic interactions.

Secondly, we extend the rigorous renormalisation group techniques developed by Brydges and coworkes to models without a discrete rotational symmetry of the interaction. In the presence of symmetry, the set of relevant terms is strongly restricted by the symmetry.

Regarding the first goal, we consider gradient random fields $\{\varphi(x)\}_{x\in\mathcal{L}}$ indexed by a lattice $\mathcal{L}$ with values in $\mathbb{R}^m$, $\varphi(x)\in\mathbb{R}^m$. The term *gradient* is referring to the assumption that the distribution depends only on gradients $\nabla_e\varphi(x)=\varphi(x+e)-\varphi(x)$.

These type of fields are used as effective models of crystal deformation or phase separation. In the former case, where $m=3$ and $\mathcal{L}\subset\mathbb{Z}^3$, the value $\varphi(x)$ plays the role of a displacement of an atom labelled by a site x of a crystal under deformation. Even though the former case is our main motivation, we will restrict our attention here, for simplicity, to the latter case with $m=1$ and $\mathcal{L}=\mathbb{Z}^d$. This is a model describing a phase separation in $\mathbb{R}^{d+1}$ with $\varphi(x)\in\mathbb{R}$ corresponding to the position of the (microscopic) phase separation surface. The model is a reasonably effective approximate description in spite of the fact that it ignores overhangs of separation surface as well as any correlations inside and between the coexisting phases.

The distribution of the interface is given in terms of a Gibbs distribution with nearest neighbour interactions of gradient type, that is, the interaction between neighboring sites $x, x+\mathrm{e}_i$ depends only on the gradient $\nabla_i\varphi(x)=\varphi(x+\mathrm{e}_i)-\varphi(x), i=1,\dots,d$. More precisely, for any finite $\Lambda\subset\mathbb{Z}^d$ we consider the Hamiltonian of the form

$$H_\Lambda(\varphi)=\sum_{x\in\Lambda}\sum_{i=1}^d W(\nabla_i\varphi(x)),$$

where $W\colon\mathbb{R}\to\mathbb{R}$ is a perturbation of a quadratic functions, i.e.

$$W(\eta)=\frac{1}{2}\eta^2+V(\eta)\qquad\text{with some perturbation } V\colon\mathbb{R}\to\mathbb{R}.$$

For a given boundary condition $\psi\in\mathbb{R}^{\partial\Lambda}$, where $\partial\Lambda=\{z\in\mathbb{Z}^d\setminus\Lambda\colon |z-x|=1 \text{ for some } x\in\Lambda\}$, we consider the Gibbs distribution at inverse temperature $\beta>0$ given by

$$\gamma^\psi_{\Lambda,\beta}(\mathrm{d}\varphi)=\frac{1}{Z_\Lambda(\beta,\psi)}\exp\big(-\beta H_\Lambda(\varphi)\big)\prod_{x\in\Lambda}\mathrm{d}\varphi(x)\prod_{x\in\partial\Lambda}\delta_{\psi(x)}(\mathrm{d}\varphi(x)),$$

where the normalisation constant $Z_\Lambda(\beta,\psi)$ is the integral of the density and is called the partition function. One is particularly interested in tilted boundary conditions

$$\psi_u(x) = \langle x, u\rangle, \qquad \text{for some tilt } u \in \mathbb{R}^d.$$

An object of basic relevance in this context is the *surface tension* or *free energy* defined by the limit

$$\sigma_\beta(u) = -\lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{\beta|\Lambda|}\log Z_\Lambda(\beta,\psi_u). \tag{1.1}$$

The surface tension $\sigma_\beta(u)$ can also be seen as the price to pay for tilting a macroscopically flat interface. The existence of the above limit follows from a standard sub-additivity argument.

In the case of a *strictly* convex potential, Funaki and Spohn show in [**FS97**] that σ_β is convex as a function of the tilt. The simplest strictly convex potential is the quadratic one with $V = 0$, which corresponds to a Gaussian model, also called the gradient free field. The convexity of the surface tension plays a crucial role in the derivation of the hydrodynamical limit of the Landau-Ginsburg model in [**FS97**]. Strict convexity of the surface tension for strictly convex W with $0 < c_1 \le W'' \le c_2 < \infty$, was proved in [**DGI00**]. Under the assumption of the bounds of the second derivative of W, a large deviations principle for the rescaled profile with rate function given in terms of the integrated surface tension has been derived in [**DGI00**]. Both papers [**FS97**] and [**DGI00**] use explicitly the conditions on the second derivative of W in their proof. In particular they rely on the Brascamp-Lieb inequality and on the random walk representation of Helffer and Sjöstrand, which requires a strictly convex potential W.

In [**CDM09**] Deuschel *et al* showed the strict convexity of the surface tension for non-convex potentials in the small β (high temperature) regime for potentials of the form

$$W(t) = W_0(t) + g(t),$$

where W_0 is strictly convex as above and where $g \in \mathcal{C}^2(\mathbb{R})$ has a negative bounded second derivative such that $\sqrt{\beta}\|g''\|_{L^1(\mathbb{R})}$ is sufficiently small. These studies have been applied in [**CD12**] to large deviations principle for the profile.

In the present paper, we show the strict convexity of the surface tension for large enough β (low temperatures) and sufficiently small tilt, using multi-scale techniques based on a finite range decomposition of the underlying background Gaussian measure in [**AKM13**].

Note also that, due to the gradient interaction, the Hamiltonian has a continuous symmetry. In particular this implies that no Gibbs measures on $\mathbb{Z}^d$ exist for dimensions $d = 1, 2$ where the field 'delocalises', cf. [**FP81**]. If one considers the corresponding random field of gradients (discrete gradient image of the height field φ) it is clear that its distribution depends on the gradient of the boundary condition of the height field. One can also introduce gradient Gibbs measures in terms of conditional distributions satisfying DLR equations, cf. [**FS97**]. For strictly convex interaction W with bounds on the second derivative, Funaki and Spohn in [**FS97**] proved the existence and uniqueness of an extremal, i.e. ergodic, gradient Gibbs measure for each tilt $u \in \mathbb{R}^d$. In the case of non-convex W, uniqueness of the ergodic gradient component can be violated, for tilt $u = 0$ this has been proved in [**BK07**]. However in this phase transition situation in [**BK07**], the surface tension is not strictly convex at tilt $u = 0$.

The second goal of the present paper is to show in detail how the rigorous renormalisation approach of Brydges and coworkers (see [**BY90**] for early work, [**Bry09**] for a survey and [**BS15a, BS15b, BBS15a, BS15c, BS15d, BBS15b**] for recent developments which go well beyond the gradient models discussed in this paper) can be extended to accommodate a class of models without a discrete rotational symmetry of the interaction.

In accordance with the general renormalization group strategy, the resulting partition function $Z_\Lambda(\beta, \psi_u)$ is obtained by a sequence of "partial integrations" (labelled by an index k). The result of each of them is expressed in terms of two functions: the *"irrelevant" polymers* K_k that are decreasing with each subsequent integration, and the *"relevant" ideal Hamiltonians* H_k—homogeneous quadratic functions of gradients $\nabla\varphi$ parametrized by a fixed finite number of parameters. To fine-tune the procedure so that the final integration yields a result with a straightforward bound we need to assure the smoothness of the procedure with respect to the parameters of a suitably chosen "seed Hamiltonian". However, it turns out that the derivatives with respect to those parameters lead to a loss of regularity of functions K_k and H_k considered as elements in a scale of Banach spaces.

A more detailed summary of the strategy is presented in Chapter 3 where the reader can get an overview of our methods and techniques of the proof. First, however, we will summarize the main claims concerning the convexity of the surface tension $\sigma_\beta(u)$ in Chapter 2. The detailed formulations and proofs are in Chapters 4–8. Miscelaneous technical details are deferred to Appendices.

Various extensions and generalisations of our work are possible.

First, Buchholz has very recently developed a new finite range decomposition for which no loss of regularity occurs in the problem we study [**Buc16**]. However, in the present paper we decided to stick to the usual finite range decomposition and to explain how the loss for regularity can be overcome by a suitable version of the chain rule and the implicit function theorem since we believe that these tools might be useful in other contexts, too.

Secondly, we restrict ourselves to dimensions $d = 2$ and $d = 3$ because in that case there are only two types of linear relevant terms: linear combinations of the first and second discrete derivatives of the field. Our approach can be extended to higher dimensions by including linear terms in higher derivatives of the field. This only requires an extension of the appropriate "homogenisation projection operator" Π_2 used in the definition of quadratic functions H_k (see Chapter 4.3) to relevant polynomials and the corresponding discrete Poincaré type inequalities. In fact, Brydges and Slade [**BS15b**] have recently developed a very general theory which allows one to define the projection onto the relevant polynomials and to prove the necessary estimates.

Thirdly, we focus on scalar valued field even though most our methods carry directly over to the vector valued case which is relevant in elasticity. The discussion of models relevant in elasticity requires, however, also a number of other changes, e.g. the inclusion of non nearest neighbour interactions and the consideration of symmetry under the left action of SO(m) (frame indifference). As a result it is natural to replace our assumption that the microscopic interaction is convex close to its minimum by a more complicated condition. We will thus address the application of our ideas to vector valued fields and models relevant in elasticity in future work.

Fourthly, in this work we focus on the behaviour of the partition function in the large volume limit. As in the work of Bauerschmidt, Brydges and Slade [**BBS15b**] it should be possible to study finer properties, e.g., correlation functions. As a first step in that direction Hilger has recently shown that the scaling limit of the random field becomes a free Gaussian field on the torus (with the renormalised covariance) and that suitably averaged correlation functions converge in the infinite volume limit [**Hil16**].

CHAPTER 2

Setting and Results

2.1. Setup

Let $L > 0$ be a fixed integer. For any integer N we consider the space

$$\boldsymbol{\mathcal{V}}_N = \{\varphi : \mathbb{Z}^d \to \mathbb{R};\ \varphi(x+k) = \varphi(x)\ \forall k \in (L^N\mathbb{Z})^d\}$$

that can be identified with the set of functions on the *torus* $\mathbb{T}_N = \left(\mathbb{Z}/L^N\mathbb{Z}\right)^d$. Using $|x|_\infty = \max_{i=1,\dots,d} |x_i|$ for any $x \in \mathbb{R}^d$ (reserving the notation $|x|$ for the Euclidean norm $\sqrt{\sum x_i^2}$), the torus $\mathbb{T}_N$ may be represented by the lattice cube $\Lambda_N = \{x \in \mathbb{Z}^d \colon |x|_\infty \le \frac{1}{2}(L^N - 1)\}$ of side L^N, once it is equipped with the metric $\rho(x,y) = \inf\{|x-y+k|_\infty : k \in (L^N\mathbb{Z})^d\}$. We view $\boldsymbol{\mathcal{V}}_N$ as a Hilbert space with the scalar product

$$(\varphi, \psi) = \sum_{x\in\mathbb{T}_N} \varphi(x)\psi(x).$$

By $\boldsymbol{\mathcal{X}}_N$ we denote the subspace

$$\boldsymbol{\mathcal{X}}_N = \{\varphi \in \boldsymbol{\mathcal{V}}_N : \sum_{x\in\mathbb{T}_N} \varphi(x) = 0\}, \tag{2.1}$$

of height fields whose sum over the torus is zero. We use λ_N to denote the $(L^{Nd}-1)$-dimensional Hausdorf measure on $\boldsymbol{\mathcal{X}}_N$. We equip the space $\boldsymbol{\mathcal{X}}_N$ with the σ-algebra $\boldsymbol{\mathcal{B}}_{\boldsymbol{\mathcal{X}}_N}$ induced by the Borel σ-algebra with respect to the product topology and use $\mathcal{M}_1(\boldsymbol{\mathcal{X}}_N) = \mathcal{M}_1(\boldsymbol{\mathcal{X}}_N, \boldsymbol{\mathcal{B}}_{\boldsymbol{\mathcal{X}}_N})$ to denote the set of probability measures on $\boldsymbol{\mathcal{X}}_N$, referring to elements in $\mathcal{M}_1(\boldsymbol{\mathcal{X}}_N)$ as to *random gradient fields*.

In this article we study a class of random gradient fields defined (as Gibbs measures) in terms of a non-convex perturbation of a Gaussian gradient field. For a precise definition, we first introduce the *discrete derivatives*

$$\nabla_i\varphi(x) = \varphi(x+\mathrm{e}_i) - \varphi(x),\ \nabla_i^*\varphi(x) = \varphi(x-\mathrm{e}_i) - \varphi(x) \tag{2.2}$$

on $\boldsymbol{\mathcal{V}}_N$. Here, e_i, $i = 1,\dots,d$, are unit coordinate vectors in $\mathbb{R}^d$. Next, let $\mathcal{E}_N(\varphi)$ be the Dirichlet form

$$\mathcal{E}_N(\varphi) = \frac{1}{2}\sum_{x\in\mathbb{T}_N}\sum_{i=1}^d \left(\nabla_i\varphi(x)\right)^2. \tag{2.3}$$

Choosing a function $V\colon \mathbb{R}\to\mathbb{R}$ (satisfying the conditions to be specified later), we consider the Gibbs mesure on the torus corresponding to the Hamiltonian

$$H_N(\varphi) = \mathcal{E}_N(\varphi) + \sum_{x\in\mathbb{T}_N}\sum_{i=1}^d V(\nabla_i\varphi(x)). \tag{2.4}$$

To be able to discuss random fields with a tilt $u = (u_1 \dots, u_d) \in \mathbb{R}^d$, we use the method proposed by Funaki and Spohn [**FS97**] who enforce the tilt on a measure defined on the torus space $\boldsymbol{\mathcal{X}}_N$ by replacing the gradient $\nabla_i\varphi(x)$ in all definitions above by $\nabla_i\varphi(x) - u_i$, $i = 1, \dots, d$, $x \in \mathbb{T}_N$.

Namely, we define the Gibbs mesure on $\mathbb{T}_N$ at inverse temperature β as

$$\gamma^u_{N,\beta}(\mathrm{d}\varphi) = \frac{1}{Z_{N,\beta}(u)} \exp\bigl(-\beta H^u_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi), \tag{2.5}$$

where

$$H^u_N(\varphi) = \mathcal{E}_N(\varphi) + \frac{1}{2}L^{Nd}|u|^2 + \sum_{x\in\mathbb{T}_N}\sum_{i=1}^{d} V(\nabla_i\varphi(x) - u_i) \tag{2.6}$$

(in the last equation we used the fact that substituting $\nabla_i\varphi(x) \mapsto \nabla_i\varphi(x) - u_i$ in $\mathcal{E}_N$, the linear term $\sum_{x\in\mathbb{T}_N}\sum_{i=1}^d u_i\nabla_i\varphi(x)$ vanishes as $\sum_{x\in\mathbb{T}_N}\nabla_i\varphi(x) = 0$ for each $\varphi \in \boldsymbol{\mathcal{V}}_N$ and each $i = 1, \dots, d$). Again, $Z_{N,\beta}(u)$ is the normalizing partition function

$$Z_{N,\beta}(u) = \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta H^u_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi). \tag{2.7}$$

Even though the ultimate goal, in general, is to characterize all limiting gradient Gibbs measures with a fixed mean tilt, and, in particular cases, to prove their unicity, in this paper we will restrict our attention to the discussion of the strict convexity, in u, of the surface tension

$$\sigma_\beta(u) := -\lim_{N\to\infty}\frac{1}{\beta L^{dN}}\log Z_{N,\beta}(u). \tag{2.8}$$

2.2. Main result

To state our main result, we need a condition on smallness of the perturbation V. We will state it in terms of the function $\mathcal{K}_{V,\beta,u} : \mathbb{R}^d \to \mathbb{R}$ associated with the perturbation $V : \mathbb{R} \to \mathbb{R}$ determining the Hamiltonian H^u_N in (2.6) (and with the (inverse) temperature $\beta \geq 0$ and the tilt $u \in \mathbb{R}^d$). Namely, we take

$$\mathcal{K}_{V,\beta,u}(z) = \exp\Bigl\{-\beta\sum_{i=1}^{d} U\bigl(\frac{z_i}{\sqrt{\beta}}, u_i\bigr)\Bigr\} - 1 \tag{2.9}$$

with

$$U(s,t) = V(s-t) - V(-t) - V'(-t)s. \tag{2.10}$$

First, we rewrite the partition function in terms of the function $\mathcal{K}_{V,\beta,u}$. Consider the Gaussian measure ν_β on $\boldsymbol{\mathcal{X}}_N$ corresponding to the Dirichlet form $\beta\mathcal{E}_N(\varphi)$:

$$\nu_\beta(\mathrm{d}\varphi) = \frac{1}{Z^{(0)}_{N,\beta}} \exp\bigl(-\beta\mathcal{E}_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi), \tag{2.11}$$

with

$$Z^{(0)}_{N,\beta} = \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta\mathcal{E}_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi). \tag{2.12}$$

To avoid overloading of the notation, here and in future, we often skip the index referring to N (as above in the case of measure ν_β). Now, the partition function (2.7) is

(2.13)

$$Z_{N,\beta}(u) = Z^{(0)}_{N,\beta} \exp\bigl(-\tfrac{\beta}{2} L^{Nd}|u|^2\bigr) \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta \sum_{x\in\mathbb{T}_N} \sum_{i=1}^{d} V\bigl(\nabla_i\varphi(x) - u_i\bigr)\bigr)\nu_\beta(\mathrm{d}\varphi) =$$

$$= Z^{(0)}_N \exp\bigl(-\beta L^{Nd}(\tfrac{1}{2}|u|^2 + V(u))\bigr) \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta \sum_{x\in\mathbb{T}_N} \sum_{i=1}^{d} U\bigl(\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x), u_i\bigr)\bigr)\nu(\mathrm{d}\varphi),$$

where, denoting $\nu(\mathrm{d}\varphi) = \nu_{\beta=1}(\mathrm{d}\varphi)$ and $Z^{(0)}_N = Z^{(0)}_{N,\beta=1}$, the last equality was obtained by rescaling the field φ by $\frac{1}{\sqrt{\beta}}$, invoking the definition (2.10) and using that $\sum_{x\in\mathbb{T}_N} \nabla_i\varphi(x) = 0$. Expanding the integrand

$$\prod_{x\in\mathbb{T}_N} \Bigl(1 + \exp\bigl\{-\beta \sum_{i=1}^{d} U\bigl(\tfrac{1}{\sqrt{\beta}}\nabla_i\varphi(x), u_i\bigr)\bigr\} - 1\Bigr) \tag{2.14}$$

above and introducing (with a slight abuse of notation), the function

$$\mathcal{K}_{V,\beta,u}(X,\varphi) = \prod_{x\in X} \mathcal{K}_{V,\beta,u}(\nabla\varphi(x)) \tag{2.15}$$

for any subset $X \subset \mathbb{T}_N$, we get

$$Z_{N,\beta}(u) = Z^{(0)}_{N,\beta} \exp\bigl(-\beta L^{Nd}(\tfrac{1}{2}|u|^2 + V(u))\bigr) \int_{\boldsymbol{\mathcal{X}}_N} \sum_{X\subset\mathbb{T}_N} \mathcal{K}_{V,\beta,u}(X,\varphi)\nu(\mathrm{d}\varphi). \tag{2.16}$$

It will be useful to generalize our formulation slightly and, instead of a particular $\mathcal{K}_{V,\beta,u}$ above, to consider for each u a general function $\mathcal{K}_u : \mathbb{R}^d \to \mathbb{R}$ and define

$$\mathcal{Z}_N(u) = \int_{\boldsymbol{\mathcal{X}}_N} \sum_X \mathcal{K}_u(X,\varphi)\nu(\mathrm{d}\varphi) \tag{2.17}$$

with

$$\mathcal{K}_u(X,\varphi) = \prod_{x\in X} \mathcal{K}_u(\nabla\varphi(x)). \tag{2.18}$$

Our main claim is that, under appropriate conditions on the function $u \mapsto \mathcal{K}_u$, the perturbative component of the surface tension,

$$\varsigma(u) := -\lim_{N\to\infty} \frac{1}{L^{dN}} \log \mathcal{Z}_N(u) \tag{2.19}$$

is sufficiently smooth for small u.

Before formulating it in detail, we observe that whenever the claim applies to the case $\mathcal{K}_u = \mathcal{K}_{V,\beta,u}$, the uniform smoothness of $\varsigma(u)$ implies that, for sufficiently large β and small $|u|$, the surface tension $\sigma(u)$ is strictly convex, since, in view of (2.16), we get

$$\sigma_\beta(u) = \tfrac{1}{2}|u|^2 + V(u) + \frac{\varsigma(u)}{\beta} - \lim_{N\to\infty} \frac{1}{\beta L^{dN}} \log Z^{(0)}_{N,\beta} \tag{2.20}$$

The last term is a constant that does not depend on u.

Given any $\zeta > 0$, consider the Banach space $\boldsymbol{E}$ of functions $\mathcal{K} : \mathbb{R}^d \to \mathbb{R}$ with the norm

$$\|\mathcal{K}\|_\zeta = \sup_{z\in\mathbb{R}^d} \sum_{|\boldsymbol{\alpha}|\le r_0} \zeta^{|\boldsymbol{\alpha}|} \big|\partial_z^{\boldsymbol{\alpha}} \mathcal{K}(z)\big| \mathrm{e}^{-\zeta^{-2}|z|^2}. \tag{2.21}$$

Here, the sum is over nonnegative integer multiindices $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}, i = 1, \dots, d$ with $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i \le r_0 \in \mathbb{N}$, and $\partial^{\boldsymbol{\alpha}} = \prod_{i=1}^d \partial_i^{\alpha_i}$. We also use $B_\delta(0) \subset \mathbb{R}^d$ to denote the ball $B_\delta(0) = \{u \mid |u| < \delta\}$.

THEOREM 2.1 (**Strict convexity of the surface tension**). *Let $r_0 \ge 9$. There exist constants $\delta_0 > 0$, $\rho_0 > 0$, $M_0 > 0$, and $\zeta_0 > 0$ such that if the map $\mathbb{R}^d \supset B_\delta(0) \ni u \mapsto \mathcal{K}_u \in \boldsymbol{E}$ is C^3, satisfies the bounds*

$$\|\mathcal{K}_u\|_\zeta \le \rho, \tag{2.22}$$

and

$$\sum_{i=1}^d \Big\|\frac{\partial}{\partial u_i}\mathcal{K}_u\Big\|_\zeta + \sum_{i,j=1}^d \Big\|\frac{\partial^2}{\partial u_i \partial u_j}\mathcal{K}_u\Big\|_\zeta + \sum_{i,j,k=1}^d \Big\|\frac{\partial^3}{\partial u_i \partial u_j \partial u_j}\mathcal{K}_u\Big\|_\zeta \le M \tag{2.23}$$

with $\zeta \ge \zeta_0$, $\rho \le \rho_0$, $\delta \le \delta_0$, $M < M_0$, and $u \in B_\delta(0)$, then the surface tension $\varsigma(u)$ exists with bounds on $\varsigma(u)$, $D\varsigma(u)$, $D^2\varsigma(u)$, and $D^3\varsigma(u)$ depending only on ρ and M uniformly in $u \in B_\delta(0)$.

The proof employs a multi-scale analysis based on ideas going back to the work [**BY90**]. Even though we follow quite closely the approach outlined by Brydges in [**Bry09**], a fair amount of various deviations and generalisations is needed. We believe that this fact and the demands on clarity warrant an independent treatment and the presentation of the proof in full detail.

The reader familiar with [**Bry09**] may, however, find various shortcuts. To facilitate a selective reading, we devote the next Chapter 3 to a presentation of the strategy of the proof, formulating then accurately all main steps of the proof and spelling out all needed extensions of [**Bry09**] in Chapter 4. The proof is then executed in full detail in the remaining chapters.

Before passing to the outline of the proof, we discuss two particular classes of perturbative potentials for which the above theorem applies.

First we verify the assumptions of Theorem 2.1 for a class of perturbations of the form (2.9). This yields a very simple example of a possibly non-convex potential at low temperatures.

PROPOSITION 2.2. *Let $r_0 \in \mathbb{N}$, $\zeta \in (0, \infty)$, $M_0 \ge 1$, and suppose that*

$$V \in C^{r_0+5}(\mathbb{R}), \tag{2.24}$$

$$V(0) = V'(0) = V''(0) = 0, \tag{2.25}$$

$$\|D^k V\|_\infty \le M_0 \quad \textit{for} \quad 2 \le k \le r_0 + 5, \tag{2.26}$$

and

$$V(s) \ge -\tfrac{1}{8}\zeta^{-2}s^2 \quad \textit{for each } s \in \mathbb{R}. \tag{2.27}$$

Then, for any $\rho \in (0,1/2)$, *there exists* $\beta_0 = \beta_0(\zeta,\rho,M_0,r_0)$, $\delta = \delta(\zeta,\rho,M_0,r_0)$, *and* $M(\zeta,M_0,r_0)$ *such that, for any* $\beta \geq \beta_0$, *the map* $\mathbb{R}^d \supset B_\delta(0) \ni u \mapsto \mathcal{K}_{V,\beta,u} \in \boldsymbol{E}$ *is* C^3 *and, for any* $u \in B_\delta(0)$,

$$\|\mathcal{K}_{V,\beta,u}\|_\zeta \leq \rho \tag{2.28}$$

and

$$\sum_{i=1}^d \Big\|\frac{\partial \mathcal{K}_{V,\beta,u}}{\partial u_i}\Big\|_\zeta + \sum_{i,j=1}^d \Big\|\frac{\partial^2 \mathcal{K}_{V,\beta,u}}{\partial u_i \partial u_j}\Big\|_\zeta + \sum_{i,j,k=1}^d \Big\|\frac{\partial^3 \mathcal{K}_{V,\beta,u}}{\partial u_i \partial u_j \partial u_j}\Big\|_\zeta \leq M. \tag{2.29}$$

Moreover, if $r_0 \geq 9$, *there exists* $\bar\beta(M_0)$ *and* $\bar\delta(M_0)$ *such that for all* $\beta \geq \bar\beta_0$, *the function* $\sigma_\beta : B_{\bar\delta}(0) \to \mathbb{R}$ *given in* (2.20) *is* C^3 *and uniformly strictly convex.*

The proof will be given in Section 2.3.

Remark 2.3. (i) Notice that there is no loss of generality in the assumption (2.25). Indeed, the absolute term is just a shift by a constant, the linear term vanishes in view of the condition $\sum_{x\in\mathbb{T}_N} \nabla_i\varphi(x) = 0$, and the quadratic term may be absorbed into the *a priori* quadratic part (2.3).
(ii) The only smallness assumption on V is (2.27). In terms of the full macroscopic potential $W(s) = \frac{1}{2}|s|^2 + V(s)$ it reads

$$W(s) \geq \big(\tfrac{1}{2}W''(0) - \tfrac{1}{8}\zeta^{-2}\big)s^2. \tag{2.30}$$

Of course, the factor $\frac{1}{8}$ can be replaced by any $\theta < 1$. If we could (almost) achieve the optimal value for ζ, $\zeta^{-2} = \frac{1}{2}$, the condition (2.30) would simply say that W is bounded from below by a nondegenerate quadratic function. Due to a number of technical points, however, we need to choose ζ^{-2} rather small to assure the validity of Theorem 2.1. ⋄

Another example is the non-convex potential considered in [**BK07**]. The importance of this case lies in the fact that it is a non-convex potential for which the non-uniqueness of a Gibbs state for a particular temperature and with a particular tilt is actually proven. For the sake of simplicity, the potential considered in [**BK07**] was chosen in a particular form that corresponds to the replacement of $\exp\{-\beta H_N(\varphi)\}$ by

$$\prod_{x\in\mathbb{T}_N}\prod_{i=1}^d\Big[p\exp\Big\{-\frac{1}{2}\big(\nabla_i\varphi(x)\big)^2\Big\} + (1-p)\exp\Big\{-\frac{\kappa}{2}\big(\nabla_i\varphi(x)\big)^2\Big\}\Big] \tag{2.31}$$

(for parameters $\kappa_{\rm O}$ and $\kappa_{\rm D}$ from [**BK07**] we choose $\kappa_{\rm O} = 1$ and $\kappa_{\rm D} = \kappa$). This amounts to replacing $\mathcal{K}_{V,\beta,u}(z) = \exp\{-\beta\sum_{i=1}^d V\big(\frac{z_i}{\sqrt{\beta}} - u_i\big)\} - 1$ by

$$\mathcal{K}_{\kappa,p,u}(z) = \prod_{i=1}^d\Big[p + (1-p)\exp\Big\{\frac{1}{2}(1-\kappa)\big(z_i-u_i\big)^2\Big\}\Big] - 1. \tag{2.32}$$

Indeed, it is enough to observe that (2.31) can be rewritten as

$$\exp\{-\mathcal{E}_N(\varphi)\}\prod_{x\in\mathbb{T}_N}\prod_{i=1}^d\Big[p + (1-p)\exp\Big\{-\frac{1}{2}(1-\kappa)\big(\nabla_i\varphi(x)\big)^2\Big\}\Big]. \tag{2.33}$$

Notice that temperature β is in (2.31) and (2.32) is replaced by the parameter p. The phase transition (non-unicity of Gibbs state with the tilt $u = 0$) mentioned above happens, for κ sufficiently small, for a particular value $p = p_t(\kappa)$. However, this does not prevent the corresponding surface tension to be convex in u (at least for small $|u|$) once p is sufficiently close to 1 (and thus bigger than p_t). This corresponds to the condition of sufficiently large β in the previous Proposition.

Observing that the map $\mathbb{R}^d \ni u \mapsto \mathcal{K}_{\kappa,p,u} \in \boldsymbol{E}$ is clearly analytic for all p, what only needs to be proven to apply Theorem 2.1 is the following claim.

PROPOSITION 2.4. *Let $\kappa \in (0,1)$ be given. There exist $\delta > 0$, $\zeta = \zeta(\delta)$ and M so that so that for any $|u| \le \delta$ one has*

$$\|\mathcal{K}_{\kappa,p,u}\|_\zeta \le \rho \tag{2.34}$$

and

$$\sum_{i=1}^{d}\Big\|\frac{\partial}{\partial u_i}\mathcal{K}_{\kappa,p,u}\Big\|_\zeta + \sum_{i,j=1}^{d}\Big\|\frac{\partial^2}{\partial u_i \partial u_j}\mathcal{K}_{\kappa,p,u}\Big\|_\zeta + \sum_{i,j,k=1}^{d}\Big\|\frac{\partial^3}{\partial u_i \partial u_j \partial u_k}\mathcal{K}_{\kappa,p,u}\Big\|_\zeta \le M \tag{2.35}$$

for any $1-p$ sufficiently small (in dependence on ρ and ζ).

The proof is given below in Section 2.3

2.3. Proofs of the given examples

We collect the outstanding proofs for our two examples above.

PROOF OF PROPOSITION 2.2.

Step 1. Estimate for $\|\mathcal{K}_{V,\beta,u}\|_\zeta$.

This is the key estimate. The main idea is that for z_i small (and also u_i small) we can use the Taylor expansion of $U(\frac{z_i}{\sqrt{\beta}}, u_i)$ in z_i, while for large z_i we rely on the weight $\mathrm{e}^{-\zeta^{-2}|z_i|^2}$ combined with the quadratic lower bound (2.27) on V.

First, let us show that

$$-\beta U(\frac{z_i}{\sqrt{\beta}}, u_i) \le \frac{1}{2}\zeta^{-2} z_i^2 \quad \text{for any} \quad z_i \in \mathbb{R} \quad \text{and any} \quad |u| < \delta, \tag{2.36}$$

whenever $\delta \le \frac{1}{4M_0}\zeta^{-2}$.

Indeed, the Taylor expansion yields

$$\beta\Big|U(\frac{z_i}{\sqrt{\beta}}, u_i)\Big| \le \tfrac{1}{2}\big|V''(s)\big| z_i^2 \tag{2.37}$$

with $|s| \le |u_i| + \big|\frac{z_i}{\sqrt{\beta}}\big|$. Since $V''(0) = 0$ implies that $\big|V''(s)\big| \le M_0|s|$, the right hand side is bounded by $\frac{1}{2}M_0\big(\delta + \big|\frac{z_i}{\sqrt{\beta}}\big|\big) z_i^2$ yielding the claim for $\big|\frac{z_i}{\sqrt{\beta}}\big| \le 3\delta$.

On the other hand, for $\big|\frac{z_i}{\sqrt{\beta}}\big| \ge 3\delta$ we use (2.27) and the observation that $|a| \ge 3|b|$ implies that $(a-b)^2 \le 2a^2$ to get

$$-\beta V(\frac{z_i}{\sqrt{\beta}} - u_i) \le \tfrac{1}{4}\zeta^{-2} z_i^2. \tag{2.38}$$

Moreover, expanding $V'(-u_i)$ around $V'(0) = 0$ up to the order u_i^2, for $|\frac{z_i}{\sqrt{\beta}}| \ge 3\delta$ we get

$$\beta\Big|V'(-u_i)\frac{z_i}{\sqrt{\beta}}\Big| \le \beta\frac{M_0}{2}\delta^2\Big|\frac{z_i}{\sqrt{\beta}}\Big| \le \frac{M_0}{6}\delta z_i^2 \tag{2.39}$$

and, similarly,

$$\beta|V(-u_i)| \le \beta\frac{M_0}{6}\delta^3 \le \frac{M_0}{54}\delta z_i^2, \tag{2.40}$$

yielding the claim since $M_0(\frac{1}{6}+\frac{1}{54})\frac{1}{4M_0} < \frac{1}{4}$.

As a result of (2.36), we are done once $|z|^2 = \sum_{i=1}^d z_i^2 \ge 2\zeta^2 \log\frac{2}{\rho}$. Indeed, under this assumption, we have

$$\left|e^{-\beta U(\frac{z_i}{\sqrt{\beta}},u_i)} - 1\right|e^{-\zeta^{-2}|z|^2} \le \max\left(e^{-\beta U(\frac{z_i}{\sqrt{\beta}},u_i)},1\right)e^{-\zeta^{-2}|z|^2} \le e^{-\frac{1}{2}\zeta^{-2}|z|^2} \le \frac{\rho}{2}. \tag{2.41}$$

Hence, we now focus on the case

$$|z|^2 \le 2\zeta^2\log\frac{2}{\rho}. \tag{2.42}$$

For sufficiently small ρ, set

$$\delta_1 = \frac{\zeta^{-2}}{4M_0}\min\left(1, \frac{\rho}{4\log\frac{2}{\rho}}\right) \le 1 \tag{2.43}$$

and

$$\beta_1 = \frac{2\zeta^2\log\frac{2}{\rho}}{\delta_1^2} \ge 1. \tag{2.44}$$

Then, for $\beta \ge \beta_1$, the relation (2.42) implies that $|z|/\sqrt{\beta} \le |z|/\sqrt{\beta_1} \le \delta_1$ and (2.37) thus for $\delta \le \delta_1$ yields

$$\beta\sum_{i=1}^d \left|U(\frac{z_i}{\sqrt{\beta}},u_i)\right| \le M_0\delta_1|z|^2 \le \frac{\rho}{4}. \tag{2.45}$$

Since $|e^t - 1| \le 2|t|$ for $t \le 1$, we get

$$\left|e^{-\beta\sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}},u_i)} - 1\right| \le \frac{\rho}{2}. \tag{2.46}$$

Together with (2.41) this shows that

$$\sup_{z\in\mathbb{R}^d}\left|e^{-\beta\sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}},u_i)} - 1\right|e^{-\zeta^{-2}|z|^2} \le \frac{\rho}{2} \tag{2.47}$$

as long as $|u| \le \delta \le \delta_1$ and $\beta \ge \beta_1$ with δ_1 and β_1 given by (2.43) and (2.44), respectively.

Step 2. z-derivatives of $\mathcal{K}_{V,\beta,u}$.

We will employ Faà di Bruno's chain rule for higher order derivatives [**Har**] of a function in the form e^f,

$$e^{-f}\partial^{\boldsymbol{\alpha}}e^f = \sum_{\substack{\boldsymbol{\tau}_1,\boldsymbol{\tau}_2,\dots,m_1,m_2,\dots \\ \sum_j m_j\boldsymbol{\tau}_j=\boldsymbol{\alpha}}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\tau}_1!)^{m_1}(\boldsymbol{\tau}_2!)^{m_2}\cdots m_1!m_2!\cdots}\prod_j(\partial^{\boldsymbol{\tau}_j}f)^{m_j}. \tag{2.48}$$

Here, the sum is over distinct partitions $\boldsymbol{\tau}_1,\boldsymbol{\tau}_2,\dots$ of the multiindex $\boldsymbol{\alpha}$ with multiplicities $m_1,m_2,\dots$ (i.e., such that $\sum_j m_j\boldsymbol{\tau}_j = \boldsymbol{\alpha}$) and $\boldsymbol{\tau}! = \tau_1!\dots\tau_d!$ for any multiindex $\boldsymbol{\tau} = (\tau_1,\dots,\tau_d)$.

In our case, we have $f(z) = -\beta\sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}},u_i)$ with

$$\partial_{z_j}f(z) = -\sqrt{\beta}\left(V'(\frac{z_j}{\sqrt{\beta}} - u_j) - V'(-u_j)\right). \tag{2.49}$$

As for the higher derivatives, only the "diagonal" ones, $\partial^k_{z_j} f(z)$, are non-vanishing,

$$\partial^k_{z_j} f(z) = -\frac{1}{\beta^{(k-2)/2}} V^{(k)}(\frac{z_j}{\sqrt{\beta}} - u_j). \tag{2.50}$$

For $|u_i| \le \delta$, we get

$$|\partial^2_{z_j} f(z)| = \big|V''(\frac{z_j}{\sqrt{\beta}} - u_j)\big| \le M_0 \min\big(1, \delta + \big|\frac{z_j}{\sqrt{\beta}}\big|\big) \tag{2.51}$$

and thus, using that $\partial_{z_j} f(0) = 0$, also

$$|\partial_{z_j} f(z)| \le M_0 \min\big(1, \delta + \big|\frac{z_j}{\sqrt{\beta}}\big|\big)|z_j|. \tag{2.52}$$

Moreover, in view of (2.50), we have

$$\sup|\partial^k_{z_j} f(z)| \le \frac{1}{\beta^{(k-2)/2}} M_0 \tag{2.53}$$

for $k \ge 2$. Combining (2.48) with (2.53) and with the particular implication of (2.52),

$$|\partial_{z_j} f(z)| \le M_0 |z|, \tag{2.54}$$

observing that $|z|^r \le 1 + |z|^{r_0}$ whenever $r \le r_0$, and using that $M_0 \ge 1$ and $\beta \ge 1$, we get

$$\Big|\partial^{\boldsymbol{\alpha}} \mathrm{e}^{-\beta \sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}}, u_i)}\Big| \le C(r_0) \mathrm{e}^{-\beta \sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}}, u_i)} M_0^{r_0} (1 + |z|^{r_0}) \tag{2.55}$$

with a suitable constant $C(r_0)$. Using, further, (2.36) and (2.55), we get (note that $\zeta \ge 1$)

$$\zeta^{|\boldsymbol{\alpha}|} \Big|\partial^{\boldsymbol{\alpha}} \mathrm{e}^{-\beta \sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}}, u_i)}\Big| \mathrm{e}^{-\zeta^{-2}|z|^2} \le \xi\, \mathrm{e}^{-\frac{1}{4}\zeta^{-2}|z|^2} \tag{2.56}$$

with

$$\xi = \xi(r_0, h, M_0) = 2C(r_0) \zeta^{2r_0} M_0^{r_0} \big(\tfrac{r_0}{2}\big)^{r_0/2}. \tag{2.57}$$

Here, the factor ξ is a bound on the term $C(r_0)\zeta^{r_0} M_0^{r_0} \mathrm{e}^{-\frac{1}{4}\zeta^{-2}|z|^2}(1+|z|^{r_0})$ obtained with help of the identity $\max_{t>0} \mathrm{e}^{-at^2} t^s = s^s \mathrm{e}^{-s} a^{-s}$ with $t = |z|^2$. As a result, the right hand side of (2.56) is bounded by $\rho/2$ whenever $|z|^2 \ge 4\zeta^2 \log \frac{2\xi}{\rho}$.

For

$$|z|^2 \le 4\zeta^2 \log \frac{2\xi}{\rho} \tag{2.58}$$

we take

$$\delta_2 = \min\big(\delta_1, \frac{\zeta^{-2}}{4M_0 \log \frac{2\xi}{\rho}}\big) \tag{2.59}$$

and

$$\beta_2 = \max\big\{\beta_1, \frac{4\zeta^2 \log \frac{2\xi}{\rho}}{\delta_2^2}\big\}. \tag{2.60}$$

Then, for $\beta \ge \beta_2$ and $|u| \le \delta \le \delta_2$, the bound (2.58) implies that $\frac{|z_i|}{\sqrt{\beta}} \le \delta_2$, yielding, in view of (2.52), the estimate

$$|\partial_{z_j} f(z)| \le 2M_0 \delta_2 |z_j| \tag{2.61}$$

and thus

$$|f(z)| \le \sum_{j=1}^{d} 2M_0\delta_2|z_j|^2 \le 2, \tag{2.62}$$

again in view of (2.58) and the definition of δ_2. Hence, similarly as in (2.56), we get

$$\Big|\zeta^{|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}}\mathrm{e}^{-f(z)}\Big|\mathrm{e}^{-\zeta^{-2}|z|^2} \le 2C(r_0)\zeta^{r_0}\mathrm{e}^2(2M_0)^{r_0}|z|^{r_0}\mathrm{e}^{-\zeta^{-2}|z|^2}\max\big(\delta_2, \tfrac{1}{\sqrt{\beta}}\big) \le \\ \le \overline{C}(r_0, M_0, h)\max\big(\delta_2, \tfrac{1}{\sqrt{\beta}}\big) \tag{2.63}$$

with $\overline{C}(r_0, M_0, h) = 2C(r_0)\zeta^{2r_0}\mathrm{e}^2(2M_0)^{r_0}\big(\frac{r_0}{2}\big)^{\frac{r_0}{2}}$. The factor $\max\big(\delta_2, \frac{1}{\sqrt{\beta}}\big)$ stems from the fact that each first and second derivative of f contributes a factor bounded by $2M_0\delta_2$ (cf. (2.52) and (2.51)), while each higher derivative the factor bounded by $\frac{M_0}{\sqrt{\beta}}$ (cf. (2.53)). Taking now

$$\delta_0 = \min\big(\delta_2, \tfrac{\rho}{\overline{C}(r_0, M_0, h)}\big) \tag{2.64}$$

and

$$\beta_0 = \max\big(\beta_2, \big(\tfrac{\overline{C}(r_0, M_0, h)}{\rho}\big)^2\big), \tag{2.65}$$

we get the sought claim

$$\|\mathcal{K}_{V,\beta,u}\|_{\zeta} \le \rho \tag{2.66}$$

whenever $|u| \le \delta \le \delta_0$ and $\beta \ge \beta_0$.

Step 3. u-derivatives of $\mathcal{K}_{V,\beta,u}$.

The estimates for the u-derivatives of $\mathcal{K}_{V,\beta,u}$ are similar. Indeed,

$$\partial_{u_i}\mathcal{K}_{V,\beta,u} = \mathrm{e}^{f(z)}\partial_i f(z), \tag{2.67}$$

$$\partial_{u_j}\partial_{u_i}\mathcal{K}_{V,\beta,u} = \mathrm{e}^{f(z)}\big(f_j(z)f_i(z) - f_{i,j}(z)\big), \tag{2.68}$$

etc., where

$$f_i(z) = -\beta\sum_{i=1}^{d} U^{(1)}\big(\frac{z_i}{\sqrt{\beta}}, u_i\big), \tag{2.69}$$

$$f_{i,i}(z) = -\beta\sum_{i=1}^{d} U^{(2)}\big(\frac{z_i}{\sqrt{\beta}}, u_i\big), \quad \text{and} \quad f_{i,j}(z) = 0 \text{ if } i \ne j. \tag{2.70}$$

Here, the functions $U^{(\ell)}$ have the same structure as U, but with V replaced by $(-1)^{\ell}\partial^{\ell}V$, e.g.,

$$U^{(1)}(s,t) = V'(s-t) - V'(-t) - V''(-t)s. \tag{2.71}$$

Thus, as in (2.53) and (2.54), we get

$$\beta\sup|\partial_{z_j}^{k}U^{(\ell)}\big(\frac{z_i}{\sqrt{\beta}}, u_i\big)| \le \sup|\partial^{k+\ell}V| \le M_0 \tag{2.72}$$

and

$$\beta|\partial_{z_j}U^{(\ell)}\big(\frac{z_i}{\sqrt{\beta}}, u_i\big)| \le \sup|\partial^{2+\ell}V||z_i| \le M_0|z_i|. \tag{2.73}$$

In addition, we have a new estimate

$$\beta|U^{(\ell)}(\frac{z_i}{\sqrt{\beta}},u_i)| \le \sup|\partial^{2+\ell}V||z_i|^2 \le M_0|z_i|^2. \tag{2.74}$$

Thus, for $|\boldsymbol{\beta}| \in \{1,2,3\}, |\boldsymbol{\alpha}| \in \{0,\dots,r_0\}$,

$$\beta\big|\partial_z^{\boldsymbol{\alpha}}\partial_u^{\boldsymbol{\beta}}\mathrm{e}^{-\beta\sum_{i=1}^d U(\frac{z_i}{\sqrt{\beta}},u_i)}\big| \le C(r_0)\mathrm{e}^{f(z)}M_0^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}(1+|z|^2)^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}. \tag{2.75}$$

Estimate (2.36) yields $|f(z)| \le \frac{1}{2}\zeta^{-2}|z|^2$ if $|u| \le \frac{1}{4M_0}\zeta^{-2}$ (in particular if $|u| < \delta_0$ defined in (2.64)). Then we easily conclude that

$$\left\|\partial^{\boldsymbol{\beta}}\mathcal{K}_{V,\beta,u}\right\|_\zeta \le M(r_0,h,M_0) \text{ for any } |\boldsymbol{\beta}| \in \{1,2,3\}, |u| \le \delta_0, \text{ and } \beta \ge \beta_0, \tag{2.76}$$

with a suitable $M(r_0,h,M_0)$.

Step 4. Uniform convexity of $\sigma(u)$.

To obtain uniform convexity of $\sigma(u)$, we first fix ρ so small and r_0 and ζ so large that Theorem 2.1 applies. Then for $\beta \ge \beta_0$ and $|u| < \delta_0$ we find that $\varsigma(u)$ is a C^3 function and its first three derivatives in $B_{\delta_0}(0)$ are controlled in terms of ρ and $M = M(r_0,h,M_0)$. In particular,

$$|D^2\varsigma(u)| \le M'(\zeta,M_0,\rho) \text{ if } u \in B_{\delta_0}(0). \tag{2.77}$$

Note that for $|s| \le \frac{1}{4M_0}$, we have $V''(s) \ge -\frac{1}{4}$. Let

$$\bar{\delta}(M_0) = \min\big(\delta_0(\zeta,M_0,\rho,r_0), \frac{1}{4M_0}\big) \tag{2.78}$$

and

$$\bar{\beta}(M_0) = \max\big(\beta_0(\zeta,M_0,\rho,r_0), \frac{1}{4M'(\zeta,M_0,r_0)}\big). \tag{2.79}$$

Then

$$D^2\sigma(u) \ge \mathrm{Id} - \tfrac{1}{4}\mathrm{Id} - \tfrac{1}{4}\mathrm{Id} \ge \tfrac{1}{2}\mathrm{Id} \tag{2.80}$$

for $u \in B_{\bar{\delta}}(0)$ and $\beta \ge \bar{\beta}$.

□

Proof of Proposition 2.4. The proof is similar as the proof of Proposition 2.2.

We will only indicate the main steps. Again, skipping the indices in $\mathcal{K}_{\kappa,p,u}$ and rewriting

$$\mathcal{K}(z) = \prod_{i=1}^d\Big[1 + (1-p)\Big[\exp\Big\{\frac{1}{2}(1-\kappa)\big(z_i-u_i\big)^2\Big\} - 1\Big]\Big] - 1, \tag{2.81}$$

we have

$$0 \le \mathcal{K}(z) \le 2^d(1-p)\exp\Big\{\frac{1}{2}\sum_{i=1}^d\big(z_i-u_i\big)^2\Big\}, \tag{2.82}$$

and, with suitable polynomials $P_{\boldsymbol{\alpha}}(z-u)$, also

$$|\nabla^{\boldsymbol{\alpha}}\mathcal{K}(z)| \le (1-p)P_{\boldsymbol{\alpha}}(z-u)\exp\Big\{\frac{1}{2}\sum_{i=1}^d\big(z_i-u_i\big)^2\Big\}. \tag{2.83}$$

Taking now sufficiently small u and, then, sufficiently large ζ we have

$$\|\mathcal{K}\|_\zeta \leq C(1-p)$$

with the constant C depending on ζ. Similar bounds are valid for the remaining terms in (2.35). $\square$

CHAPTER 3

The Strategy of the Proof

Here we present, in rather broad brush, the main ideas of the proof. Accurate definitions of the needed notions then follow in the succeeding chapter.

As mentioned above, to verify the claim of the theorem, we need to prove that the finite volume perturbative component of the surface tension

$$\varsigma_N(u) := -\frac{1}{L^{dN}} \log \mathcal{Z}_N(u) \tag{3.1}$$

has bounded derivatives uniformly in $N \in \mathbb{N}$.

Here, the partition function $\mathcal{Z}_N(u)$ can be expressed, with a flavour of cluster expansions, in terms of the functions $\mathcal{K}(X,\varphi) = \mathcal{K}_u(X,\varphi)$ as shown in (2.17). However, here comes a difficulty: even though the function $\mathcal{K}(X,\varphi)$ depends only on $\varphi(x)$ with x in the set X and its close neighbourhood and even if for a disjoint union $X = X_1 \cup X_2$ one has $\mathcal{K}(X,\varphi) = \mathcal{K}(X_1,\varphi)\mathcal{K}(X_2,\varphi)$, the Gaussian measure $\nu(\mathrm{d}\varphi)$ with its slowly decaying correlations does not allow to separate the integral of $\mathcal{K}(X,\varphi)$ into a product of integrals with the integrands $\mathcal{K}(X_1,\varphi)$ and $\mathcal{K}(X_2,\varphi)$. This is a non-locality that has to be overcome.

The strategy is to perform the integration in steps corresponding to increasing scales. Before showing what we mean by that, let us make one simple modification. Its importance will be in providing a parameter that will allow us to fine-tune the procedure in such a way that the final integration will eventually yield a result with a straightforward bound.

The parameter in question will be chosen as a *symmetric* $d\times d$*-matrix* $\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Multiplying and dividing the integrand in (2.17) by

$$\exp\Big\{-\tfrac{1}{2}\sum_{x\in\mathbb{T}_N}\sum_{i,j=1}^{d} q_{i,j}\nabla_i\varphi(x)\nabla_j\varphi(x)\Big\} = \exp\Big\{-\tfrac{1}{2}\sum_{x\in\mathbb{T}_N}\langle \boldsymbol{q}\nabla\varphi(x),\nabla\varphi(x)\rangle\Big\} \tag{3.2}$$

and using the definition of the measure ν (by (2.11) with $\beta = 1$), we get

$$\mathcal{Z}_N(u) = \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}}\int_{\boldsymbol{\mathcal{X}}_N} \exp\Big\{-\tfrac{1}{2}\sum_{x\in\mathbb{T}_N}\langle \boldsymbol{q}\nabla\varphi(x),\nabla\varphi(x)\rangle\Big\}\sum_X \mathcal{K}(X,\varphi)\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{3.3}$$

Here, $\mu^{(\boldsymbol{q})}$ is the Gaussian measure on $\boldsymbol{\mathcal{X}}_N$ with the Green function $\mathcal{C}^{(\boldsymbol{q})}$, the inverse of the operator $\mathcal{A}^{(\boldsymbol{q})} = \sum_{i,j=1}^d \big(\delta_{i,j} - q_{i,j}\big)\nabla_i^*\nabla_j$,

$$\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \frac{\exp\big\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\big\}\lambda_N(\mathrm{d}\varphi)}{Z_N^{(\boldsymbol{q})}}, \tag{3.4}$$

with

$$\mathcal{E}_{\boldsymbol{q}}(\varphi) = \tfrac{1}{2}(\mathcal{A}^{(\boldsymbol{q})}\varphi,\varphi) = \tfrac{1}{2}\sum_{x\in\mathbb{T}_N}\sum_{i,j=1}^{d}\big(\delta_{i,j} - q_{i,j}\big)\nabla_i\varphi(x)\nabla_j\varphi(x), \tag{3.5}$$

and

$$Z_N^{(\boldsymbol{q})} = \int_{\boldsymbol{\mathcal{X}}_N} \exp\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\}\lambda_N(\mathrm{d}\varphi). \tag{3.6}$$

Under a suitable assumption about the smallness of $\boldsymbol{q}$ (so that, in particular, the matrix $\mathbf{1}-\boldsymbol{q}$ is positive definite), we will show that the Gaussian measure $\mu^{(\boldsymbol{q})}$ can be decomposed into a convolution $\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \mu_1^{(\boldsymbol{q})} * \cdots * \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi)$ where $\mu_1^{(\boldsymbol{q})}, \ldots, \mu_{N+1}^{(\boldsymbol{q})}$ are Gaussian measures with a particular finite range property. Namely, the covariances $\mathcal{C}_k^{(\boldsymbol{q})}(x)$ of the measures $\mu_k^{(\boldsymbol{q})}$, $k=1,\ldots,N+1$, vanish for $|x| \geq \frac{1}{2}L^k$ with a fixed parameter L with an additional bound on their derivatives with respect to $\boldsymbol{q}$ of the order $L^{-(k-1)(d-1)}$. (See next Chapter for careful definitions and exact formulations; here we concentrate just on the main ideas.)

Now, let us write the integral in (3.3) symbolically as

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H^{(\boldsymbol{q})}} \circ \mathcal{K}^{(\boldsymbol{q})})(\varphi)\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{3.7}$$

Here

$$H^{(\boldsymbol{q})}(X,\varphi) = \tfrac{1}{2}\sum_{x\in X}\sum_{i,j=1}^{d} q_{i,j}\nabla_i\varphi(x)\nabla_j\varphi(x) = \tfrac{1}{2}\sum_{x\in X}\langle \boldsymbol{q}\nabla\varphi(x), \nabla\varphi(x)\rangle, \tag{3.8}$$

the function $\mathcal{K}^{(\boldsymbol{q})}$ is defined as

$$\mathcal{K}^{(\boldsymbol{q})}(X,\varphi) = \exp\Big\{-\tfrac{1}{2}\sum_{x\in X}\langle \boldsymbol{q}\nabla\varphi(x), \nabla\varphi(x)\rangle\Big\}\mathcal{K}(X,\varphi), \tag{3.9}$$

and $\circ$ is the *circle product* notation for the convolutive sum over subsets $X \subset \mathbb{T}_N$,

$$(\mathrm{e}^{-H^{(\boldsymbol{q})}} \circ \mathcal{K}^{(\boldsymbol{q})})(\varphi) = \sum_{X\subset\mathbb{T}_N} \mathrm{e}^{-H^{(\boldsymbol{q})}(\mathbb{T}_N\setminus X,\varphi)}\ \mathcal{K}^{(\boldsymbol{q})}(X,\varphi), \tag{3.10}$$

where we set $H^{(\boldsymbol{q})}(\varnothing,\varphi) = \mathcal{K}^{(\boldsymbol{q})}(\varnothing,\varphi) = 1$.

Replacing $\mu^{(\boldsymbol{q})}$ in (3.7) by the convolution $\mu_1^{(\boldsymbol{q})} * \cdots * \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi)$, we will proceed by integrating first over $\mu_1^{(\boldsymbol{q})}$. It turns out that the form of the integral is conserved. Namely, starting from $H_0^{(\boldsymbol{q})} = H^{(\boldsymbol{q})}$ and $K_0^{(\boldsymbol{q})} = \mathcal{K}^{(\boldsymbol{q})}$, we can define $H_1^{(\boldsymbol{q})}$ and $K_1^{(\boldsymbol{q})}$ so that

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_0^{(\boldsymbol{q})}} \circ K_0^{(\boldsymbol{q})})(\varphi+\xi)\mu_1^{(\boldsymbol{q})}(\mathrm{d}\xi) = (\mathrm{e}^{-H_1^{(\boldsymbol{q})}} \circ K_1^{(\boldsymbol{q})})(\varphi). \tag{3.11}$$

Here, the function $K_1^{(\boldsymbol{q})}(X,\varphi)$ is defined (nonvanishing) only for sets X consisting of L^d-blocks and $H_1^{(\boldsymbol{q})}$ is again a quadratic form like $H_0^{(\boldsymbol{q})}$ but with modified coefficients $q_{i,j}$ and additional linear and constant terms. Recursively, one can define a sequence of pairs $(H_1^{(\boldsymbol{q})}, K_1^{(\boldsymbol{q})}), (H_2^{(\boldsymbol{q})}, K_2^{(\boldsymbol{q})}), \ldots, (H_N^{(\boldsymbol{q})}, K_N^{(\boldsymbol{q})})$ with each $H_k^{(\boldsymbol{q})}$ a quadratic form in $\nabla\varphi$ (plus linear and constant terms) and $K_k^{(\boldsymbol{q})}(X,\varphi)$ defined for sets X consisting of L^{kd}-blocks so that

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_k^{(\boldsymbol{q})}} \circ K_k^{(\boldsymbol{q})})(\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi) = (\mathrm{e}^{-H_{k+1}^{(\boldsymbol{q})}} \circ K_{k+1}^{(\boldsymbol{q})})(\varphi). \tag{3.12}$$

Of course, the difficulty lies in producing correct definitions of consecutive pairs of functions $H_k^{(\boldsymbol{q})}, K_k^{(\boldsymbol{q})}$ so that not only (3.12) is valid, but also that the form of the quadratic function H_k is conserved, the coarse-grained dependence of $K_k^{(\boldsymbol{q})}$ on blocks L^{dk} is maintained, and, most importantly, the size of the perturbation $K_k^{(\boldsymbol{q})}$ in a conveniently chosen norm decreases (the variable $K_k^{(\boldsymbol{q})}$ is *irrelevant* in the language of the renormalisation group theory). See Propositions 4.3-4.6 for an explicit form and properties of the renormalisation transformation $\boldsymbol{T}_k^{(\boldsymbol{q})}\colon (H_k^{(\boldsymbol{q})}, K_k^{(\boldsymbol{q})}) \mapsto (H_{k+1}^{(\boldsymbol{q})}, K_{k+1}^{(\boldsymbol{q})})$.

Using now sequentially the formula (3.12), we eventually get

$$\int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_0^{(\boldsymbol{q})}} \circ K_0^{(\boldsymbol{q})})(\varphi)\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_N^{(\boldsymbol{q})}} \circ K_N^{(\boldsymbol{q})})(\varphi)\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi) \tag{3.13}$$

and thus

$$\mathcal{Z}_N(u) = \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}} \int_{\boldsymbol{\mathcal{X}}_N} (\mathrm{e}^{-H_N^{(\boldsymbol{q})}} \circ K_N^{(\boldsymbol{q})})(\varphi)\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{3.14}$$

At this moment we will invoke an additional feature. Namely, the finite range decomposition can be constructed in such a way that the measures $\mu_1^{(\boldsymbol{q})}, \dots, \mu_{N+1}^{(\boldsymbol{q})}$ depend smoothly on $\boldsymbol{q}$ ([**AKM13**]). As a result it turns out that, in dependence on the original perturbation $\mathcal{K}_u$ (or on V, β, and u in the explicit choice of $\mathcal{K}_u$ as in (2.9)), one can choose the initial value $\boldsymbol{q} = \boldsymbol{q}(\mathcal{K}_u)$ by an implicit function theorem in such a way that $H_N^{(\boldsymbol{q})} = 0$.

However, here we encounter a difficulty stemming from the fact that the action of $T_k^{(\boldsymbol{q})}$, considered on a scale of function spaces, depends on $\boldsymbol{q}$ with certain loss of regularity, see Chapter 6. This leads to a need for employing a suitable version of implicit function theorem as well as a theorem about chain rule for composed maps with loss of regularity (see Appendices D and E for the definitions and proofs).

Also, the "starting" Hamiltonian $H_0^{(\boldsymbol{q})}$ will in general contain, in addition to the quadratic term given by (3.8), also linear and constant terms, i.e., $H_0^{(\boldsymbol{q})}(X,\varphi) = \sum_{x\in X} \mathcal{H}(x,\varphi)$ with

$$\mathcal{H}(x,\varphi) = \lambda + \sum_{i=1}^{d} a_i \nabla\varphi(x) + \sum_{i,j=1}^{d} \boldsymbol{c}_{i,j} \nabla_i\nabla_j\varphi(x) + \frac{1}{2}\sum_{i,j=1}^{d} \boldsymbol{q}_{i,j} \nabla\varphi(x)\nabla_j\varphi(x), \tag{3.15}$$

see (4.91) and (4.17). Note, however, that the constant and linear terms do not lead to a change of the measure $\mu^{(\boldsymbol{q})}$ since by periodicity of φ we have $\sum_{x\in\mathbb{T}_N} \nabla_i\varphi(x) = 0$ and $\sum_{x\in\mathbb{T}_N} \nabla_i\nabla_j\varphi(x) = 0$. For the purpose of this broad outline of the proof we will pretend that we can achieve $H_N^{(\boldsymbol{q})} = 0$ with the choice

$$\lambda = a = \boldsymbol{c} = 0.$$

The general situation will be discussed in Chapter 4.5 below.

Finally, taking into account that the function $K_N^{(\boldsymbol{q})}(X,\cdot)$ is defined only for $X = \Lambda_N$ or $X = \varnothing$, we get

$$\mathcal{Z}_N(u) = \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}} \int_{\boldsymbol{\mathcal{X}}_N} \big(1 + K_N^{(\boldsymbol{q})}(\Lambda_N,\varphi)\big)\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi), \tag{3.16}$$

with $\boldsymbol{q}$ being implicitly dependent on $\mathcal{K} = \mathcal{K}_u$ by the condition that the iteration described above gives $H_N^{(\boldsymbol{q})} = 0$. Note that this formula was derived under the assumption that the constant term λ in the initial perturbation is zero. In general, there is an additional term depending on λ, see (4.95) or (4.110).

Now, to get the sought smoothness with respect to u, we have to evaluate the derivatives with respect to $\boldsymbol{q}$ and show the smooth dependence of implicitly defined $\boldsymbol{q}$ as function of u. The smoothness with respect to $\boldsymbol{q}$ is quite straightforward as the factor $Z_N^{(\boldsymbol{q})}$ can be explicitly computed by Gaussian integration and the derivatives of the integral term can easily be bounded as a consequence of the iterative bounds on $K_N^{(\boldsymbol{q})}$. The smoothness of $\boldsymbol{q}$ as function of u follows by a careful examination of the corresponding implicit function yielding $\boldsymbol{q}$ as function of the initial perturbation $\mathcal{K}_u$ and by smoothness of $\mathcal{K}_u$ as function of u assumed in Theorem 2.1 and proven for the particular classes of potentials considered in Propositions 2.2 and 2.4, see Chapter 4.6.

CHAPTER 4

Detailed Setting of the Main Steps

4.1. Finite range decomposition.

First, we formulate the needed claim about the *finite range decomposition* of the Green function $\mathcal{C}^{(\boldsymbol{q})}$, the inverse of the operator $\mathcal{A}^{(\boldsymbol{q})} = \sum_{i,j=1}^{d} \big(\delta_{i,j} - q_{i,j}\big)\nabla_i^* \nabla_j$ on $\boldsymbol{\mathcal{X}}_N$. We use $\|\boldsymbol{q}\|$ to denote the operator norm of $\boldsymbol{q}$ viewed as operator on $\mathbb{R}^d$ equipped with ℓ_2 metric. Obviously, $\|\boldsymbol{q}\| \le \big(\sum_{i,j} q_{i,j}^2\big)^{1/2}$.

PROPOSITION 4.1. *Let $\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\text{sym}}$ be a symmetric $d\times d$-matrix such that $\|\boldsymbol{q}\| \le \frac{1}{2}$. There exist positive definite operators $\mathcal{C}_k^{(\boldsymbol{q})}$, $k = 1,\dots,N+1$, on $\boldsymbol{\mathcal{X}}_N$ such that*

$$\mathcal{C}^{(\boldsymbol{q})} = \sum_{k=1}^{N+1} \mathcal{C}_k^{(\boldsymbol{q})}. \tag{4.1}$$

The operators $\mathcal{C}_k^{(\boldsymbol{q})}$ commute with translations on $\mathbb{T}_N$. In particular, there exists a function $\mathcal{C}_k^{(\boldsymbol{q})}$ on $\mathbb{T}_N$ such that $\big(\mathcal{C}_k^{(\boldsymbol{q})}\varphi\big)(x) = \sum_{y\in\mathbb{T}_N} \mathcal{C}_k^{(\boldsymbol{q})}(x-y)\varphi(y)$ for each $\varphi \in \boldsymbol{\mathcal{X}}_N$. Moreover,

$$\mathcal{C}_k^{(\boldsymbol{q})}(x) = 0 \ \text{ if } \ |x|_\infty \ge \frac{1}{2}L^k \tag{4.2}$$

and, for each multiindex $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le 3$ and any $a \in \mathbb{N}_0$ there exists a constant $c_{\boldsymbol{\alpha},a}$ such that

$$\sup_{\|\boldsymbol{q}\|\le\frac{1}{2}} |\nabla^{\boldsymbol{\alpha}} D^a \mathcal{C}_k^{(\boldsymbol{q})}(x)(\dot{\boldsymbol{q}},\dots,\dot{\boldsymbol{q}})| \le c_{\boldsymbol{\alpha},a} L^{-(k-1)(d-2+|\boldsymbol{\alpha}|)} L^{\eta(|\boldsymbol{\alpha}|,d)} \|\dot{\boldsymbol{q}}\|^a \tag{4.3}$$

for all $x \in \mathbb{T}_N$ and all $k = 1,\dots,N+1$, with

$$\eta(n,d) = \max\big(\tfrac{1}{4}(d+n-1)^2, d+n+6\big) + 10. \tag{4.4}$$

Here, $\nabla^{\boldsymbol{\alpha}} = \prod_{i=1}^d \nabla_i^{\alpha_i}$ and D is the directional derivative in the direction $\dot{\boldsymbol{q}}$.

The proof can be found in [**AKM13**] which is an extension of ideas in [**BT06**] and [**BGM04**] applied to families of gradient Gaussian measures including vector valued functions. In fact there it is shown that $\mathcal{C}_k^{(\boldsymbol{q})}$ is (real) analytic in $\boldsymbol{q}$ with the natural estimates for all derivatives with respect to $\boldsymbol{q}$.

REMARK 4.2. Since the $\mathcal{C}_k^{(\boldsymbol{q})}$ are translation invariant they are diagonal in the Fourier basis given by $f_p(x) = L^{-dN/2} e^{i\langle p,x\rangle}$ with

$$p \in \widehat{\mathbb{T}}_N = \Big\{p = (p_1,\dots,p_d)\colon p_i \in \big\{-\tfrac{(L^N-1)\pi}{L^N}, -\tfrac{(L^N-3)\pi}{L^N} \dots, 0, \dots, \tfrac{(L^N-1)\pi}{L^N}\big\}\Big\}, \tag{4.5}$$

i.e.,

$$\mathcal{C}_k^{(\boldsymbol{q})} f_p = \widehat{\mathcal{C}}_k^{(\boldsymbol{q})}(p) f_p, \tag{4.6}$$

where the Fourier multiplier $\widehat{\mathcal{C}}_k^{(\boldsymbol{q})}(p)$ is just the discrete Fourier transform of the kernel $\mathcal{C}_k^{(\boldsymbol{q})}$. Equation (4.62) and Lemma 4.3 in [**AKM13**] yield

$$\frac{1}{L^{dN}} \sum_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}} |p|^n \, |D_q^a \widehat{\mathcal{C}}_k^{(\boldsymbol{q})}(p)(\dot{\boldsymbol{q}}, \dots, \dot{\boldsymbol{q}})| \le 2^a a! \, c(n,d) L^{\eta(n,d)} L^{-(k-1)(d+n-2)}. \tag{4.7}$$

This estimate implies (4.3) by the discrete Fourier inversion formula, but it will also be of independent use later. ◇

Now, if a *random field* φ is distributed with respect to the Gaussian measure $\mu^{(\boldsymbol{q})} = \mu_{\mathcal{C}^{(\boldsymbol{q})}}$ on $\boldsymbol{\mathcal{X}}_N$, where the *covariance* $\mathcal{C}^{(\boldsymbol{q})}$ admits a finite range decomposition (4.1), then there exist $N+1$ independent random fields ξ_k, $k = 1, \dots, N+1$, such that each ξ_k is distributed according to the Gaussian measure $\mu_k^{(\boldsymbol{q})} = \mu_{\mathcal{C}_k^{(\boldsymbol{q})}}$ with the covariance $\mathcal{C}_k^{(\boldsymbol{q})}$ and, in distribution,

$$\varphi = \sum_{k=1}^{N+1} \xi_k, \tag{4.8}$$

or,

$$\int_{\boldsymbol{\mathcal{X}}_N} F(\varphi) \mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \mathbb{E}_{N+1} \cdots \mathbb{E}_1 F, \tag{4.9}$$

where $\mathbb{E}_k, k = 1, \dots, N+1$, denote the expectations with respect to the Gaussian measures $\mu_k^{(\boldsymbol{q})}$ and F is taken as a function of $\sum_{k=1}^{N+1} \xi_k$.

Taking into account that operators $\mathcal{C}_k^{(\boldsymbol{q})}$ are of full rank on $\mathcal{X}_N$, standard Gaussian calculus yields an expression in terms of convolutions,

$$\begin{aligned} \int_{\boldsymbol{\mathcal{X}}_N} F(\varphi) \mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) &= \int_{\boldsymbol{\mathcal{X}}_N} F(\varphi) \mu_1^{(\boldsymbol{q})} * \cdots * \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \\ &= \int_{\boldsymbol{\mathcal{X}}_N \times \cdots \times \boldsymbol{\mathcal{X}}_N} F\Big(\sum_{k=1}^{N+1} \xi_k\Big) \mu_1^{(\boldsymbol{q})}(\mathrm{d}\xi_1) \dots \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\xi_{N+1}). \end{aligned} \tag{4.10}$$

Our preferred formulation is to introduce renormalisation maps $\boldsymbol{R}_k^{(\boldsymbol{q})}$ on functions on $\boldsymbol{\mathcal{X}}_N$ by

$$(\boldsymbol{R}_k^{(\boldsymbol{q})} F)(\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} F(\varphi + \xi) \mu_k^{(\boldsymbol{q})}(\mathrm{d}\xi), k = 1, \dots, N. \tag{4.11}$$

Just to be on a firm ground, we can introduce the spaces $M(\boldsymbol{\mathcal{X}}_N)$ of all functions measurable with respect to λ_N on $\boldsymbol{\mathcal{X}}_N$ and view $\boldsymbol{R}_k^{(\boldsymbol{q})}$ as a map $\boldsymbol{R}_k^{(\boldsymbol{q})} \colon \mathcal{U} \subset M(\boldsymbol{\mathcal{X}}_N) \to M(\boldsymbol{\mathcal{X}}_N)$, where

$$\mathcal{U} = \{F \colon \boldsymbol{\mathcal{X}}_N \to \mathbb{R} \colon \text{r.h.s of (4.11) exists and is finite}\}.$$

The integration $\int_{\boldsymbol{\mathcal{X}}_N} F(\varphi) \mu^{(\boldsymbol{q})}(\mathrm{d}\varphi)$ can be viewed, for any $F \in M(\boldsymbol{\mathcal{X}}_N)$, as the consecutive application of maps $\boldsymbol{R}_k^{(\boldsymbol{q})}$ with a final integration with respect to $\mu_{N+1}^{(\boldsymbol{q})}$:

$$\int_{\boldsymbol{\mathcal{X}}_N} F(\varphi) \mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} (\boldsymbol{R}_N^{(\boldsymbol{q})} \dots \boldsymbol{R}_1^{(\boldsymbol{q})} F)(\varphi) \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi). \tag{4.12}$$

Notice that for the operators $\mathcal{C}_N^{(\boldsymbol{q})}$ and $\mathcal{C}_{N+1}^{(\boldsymbol{q})}$ (and the measures $\mu_N^{(\boldsymbol{q})}$ and $\mu_{N+1}^{(\boldsymbol{q})}$) the condition (4.2) is void. However, the suppression condition (4.3) still applies.

4.2. Polymers, polymer functionals, ideal Hamiltonians and norms.

There is a natural hierarchical paving corresponding to the correlation range (4.2) of random fields governed by Gaussian measures μ_k.

Namely, for $k = 0, 1, 2, \ldots, N$, we pave the torus Λ_N by $L^{(N-k)d}$ disjoint cubes of side length L^k. These cubes are all translates (L is odd) of $\{x \in \Lambda_N : |x|_\infty \leq \frac{1}{2}(L^k - 1)\}$ by vectors in $L^k\mathbb{Z}^d$. We call such cubes *k-blocks* or blocks of k-th generation, and use $\mathcal{B}_k$ to denote the set of all k-blocks,

$$\mathcal{B}_k = \mathcal{B}_k(\Lambda_N) = \{B : B \text{ is a } k\text{-block}\}, \quad k = 0, 1, \ldots, N.$$

Single vertices of the lattice are 0-blocks, the starting generation for the renormalisation group transforms, $\mathcal{B}_0 = \Lambda_N$. The only N-block is the torus Λ_N itself, $\mathcal{B}_N = \{\Lambda_N\}$.

A union of k-blocks is called a *k-polymer*. We use $\mathcal{P}_k = \mathcal{P}_k(\Lambda_N)$ to denote the set of all k-polymers in Λ_N and we have $\varnothing \in \mathcal{P}_k$. As N is fixed through the major part of the paper, we often skip Λ_N from the notation as indicated above. Notice that certain ambiguity stems from the fact that every k-polymer is also j-polymer for any $j \leq k$. Nevertheless, we abstain from introducing k-polymer as a pair (X, k) consisting of a set X (union of k-blocks) and a label; the appropriate label will be always clear from the context.

Any subset $X \subset \mathbb{T}_N$ is said to be *connected* if for any $x, y \in X$ there exist a path $x_1 = x, x_2, \ldots, x_n = y$ such that $|x_{i+1} - x_i|_\infty = 1$, $i = 1, \ldots, n-1$. We use $\mathcal{C}(X)$ to denote the *set of connected components* of X. Two connected sets $X, Y \subset \Lambda_N$ are said to be strictly disjoint if their union is not connected. Notice that for any strictly disjoint $X, Y \in \mathcal{P}_k$, we have $\operatorname{dist}(X, Y) > L^k$.

We use $\mathcal{P}_k^{\mathrm{c}}$ to denote the set of all connected k-polymers and we define that $\varnothing \notin \mathcal{P}_k^{\mathrm{c}}$. For a polymer $X \in \mathcal{P}_k$, we use $\mathcal{B}_k(X)$ to denote the set of k-blocks in X and $|X|_k = |\mathcal{B}_k(X)|$ to denote the number of k-blocks in X and $\mathcal{P}_k(X)$ to denote the set of all polymers Y consisting of subsets of blocks from $\mathcal{B}_k(X)$. The set difference $X \setminus Y \in \mathcal{P}_k$ of two polymers $X, Y \in \mathcal{P}_k$ is again a polymer from $\mathcal{P}_k$, $X \setminus Y = \cup_{B \in X, B \notin Y} B$. The *closure* $\overline{X}$ of a polymer $X \in \mathcal{P}_k$ is the smallest polymer $Y \in \mathcal{P}_{k+1}$ of the next generation such that $X \subset Y$.

A polymer $X \in \mathcal{P}_k^{\mathrm{c}}$ is called *small* if $|X|_k \leq 2^d$ and we denote $\mathcal{S}_k = \{X \in \mathcal{P}_k^{\mathrm{c}} : |X|_k \leq 2^d\}$. For any $B \in \mathcal{B}_k$ we define its *small set neighbourhood* B^* to be the cube of the side $(2^{d+1} - 1)L^k$ centered at B. Notice that B^* is the smallest cube for which $B \subset Y$ and $Y \in \mathcal{S}_k$ implies $Y \subset B^*$. For any polymer $X \in \mathcal{P}_k$ we use X^* to denote its *small set neighbourhood*, $X^* = \cup\{B^* : B \in \mathcal{B}_k(X)\}$. Notice that, strictly speaking, the operation of closure $\overline{X}$ and small set neighbourhood X^* should be amended by an index $k+1$ or k indicating the scale from which the relevant blocks are taken. Again we will abstain from cumbersome indexing and avoid ambiguity by clearly stating to which $\mathcal{P}_k$ the considered set X is taken to belong.

Having fixed the parameter N and using a shorthand $\boldsymbol{\mathcal{X}}$ for $\boldsymbol{\mathcal{X}}_N$ in the following, we first introduce the space $M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ of all maps $F : \mathcal{P}_k \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ such that for all $X \in \mathcal{P}_k$ one has $F(X, \cdot) \in M(\boldsymbol{\mathcal{X}})$, the map F is L^k-periodic ($F(\tau_a(X), \tau_a(\varphi)) = F(X, \varphi)$ for any $a \in (L^k\mathbb{Z})^d$, where $\tau_a(B) = B + a$ and $\tau_a(\varphi)(x) = \varphi(x - a)$) and $F(X, \varphi)$ depends only on values of φ on X^* ($\varphi, \psi \in \boldsymbol{\mathcal{X}}$, $\varphi\big|_{X^*} = \psi\big|_{X^*} \implies F(X, \varphi) = F(X, \psi)$ with $\varphi\big|_{X^*}$ denoting the restriction of φ to X^*).

The sets $M(\mathcal{P}_k^{\mathrm{c}},\boldsymbol{\mathcal{X}})$, $M(\mathcal{S}_k,\boldsymbol{\mathcal{X}})$, and $M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ are defined in an analogous way. We also consider the set $M^*(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\supset M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ of the maps $F\colon\mathcal{B}_k\times\boldsymbol{\mathcal{X}}\to\mathbb{R}$ with $F(B,\varphi)$ depending only on values of φ on the extended set $(B^*)^*$.

For functions from $M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ we introduce the *circle product*,

$$F_1,F_2\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}}),\ (F_1\circ F_2)(X,\varphi)=\sum_{Y\subset X}F_1(Y,\varphi)F_2(X\setminus Y,\varphi),\tag{4.13}$$

where we defined $F(\varnothing,\varphi)=:1$. Notice, that the product is defined pointwise in the variable φ. We often skip it and write $(F_1\circ F_2)(X)=\sum_{Y\subset X}F_1(Y)F_2(X\setminus Y)$. Observe that the circle product is commutative and distributive.

For $F\in M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ and $X\in\mathcal{P}_k$, we define

$$F^X(\varphi)=\prod_{B\in\mathcal{B}_k(X)}F(B,\varphi).\tag{4.14}$$

Extending any $F\in M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ to $M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ by taking

$$F(X,\varphi)=F^X(\varphi),\tag{4.15}$$

we get

$$(F_1+F_2)^X=\sum_{Y\subset X}F_1^YF_2^{X\setminus Y}=(F_1\circ F_2)(X)\tag{4.16}$$

directly from the definitions.

For each $x\in\Lambda_N$ we define the functions

$$\mathcal{H}(x,\varphi)=\lambda+\sum_{i=1}^d a_i\nabla\varphi(x)+\sum_{i,j=1}^d\boldsymbol{c}_{i,j}\nabla_i\nabla_j\varphi(x)+\frac12\sum_{i,j=1}^d\boldsymbol{d}_{i,j}\nabla\varphi(x)\nabla_j\varphi(x)\tag{4.17}$$

with coefficients $\lambda\in\mathbb{R},a\in\mathbb{R}^d,\boldsymbol{c}\in\mathbb{R}^{d\times d}$ and $\boldsymbol{d}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

A special role will be played by a subspace $M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\subset M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ of all quadratic functions built from (4.17) of the form

$$H(B,\varphi)=\sum_{x\in B}\mathcal{H}(x,\varphi)=\lambda|B|+\ell(\varphi)+Q(\varphi),\tag{4.18}$$

where

$$\ell(\varphi)=\sum_{x\in B}\Big[\sum_{i=1}^d a_i\,\nabla_i\varphi(x)+\sum_{i,j=1}^d\boldsymbol{c}_{i,j}\,\nabla_i\nabla_j\varphi(x)\Big]\tag{4.19}$$

and

$$Q(\varphi,\varphi)=\frac12\sum_{x\in B}\sum_{i,j=1}^d\boldsymbol{d}_{i,j}\,\nabla_i\varphi(x)\,\nabla_j\varphi(x).\tag{4.20}$$

Sometimes we use the term *ideal Hamiltonians* for functions in $M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$.

Our next aim is to introduce norms $\|\cdot\|_{k,r}$ and $\|\cdot\|_{k+1,r}$ on $M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ (with $r=1,\dots,r_0$, where r_0 is a fixed integer to be chosen later) and a norm $\|\cdot\|_{k,0}$ on $M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$. We begin by introducing, for each $k\in\{0,1,\dots,N\}$ and $X\in\mathcal{P}_k$, two distinct (semi)norms $|\cdot|_{k,X}$ and $|\cdot|_{k+1,X}$ on $\boldsymbol{\mathcal{X}}$. For any $\varphi\in\boldsymbol{\mathcal{X}}$ we define

$$|\varphi|_{k,X}=\max_{1\le s\le 3}\sup_{x\in X^*}\frac1h L^{k\left(\frac{d-2}{2}+s\right)}\left|\nabla^s\varphi(x)\right|\tag{4.21}$$

and

$$|\varphi|_{k+1,X} = \max_{1\le s\le 3} \sup_{x\in X^*} \frac{1}{h} L^{(k+1)\left(\frac{d-2}{2}+s\right)} \left|\nabla^s\varphi(x)\right|, \tag{4.22}$$

where

$$|\nabla^s\varphi(x)|^2 = \sum_{|\boldsymbol{\alpha}|=s} |\nabla^{\boldsymbol{\alpha}}\varphi(x)|^2. \tag{4.23}$$

Next, for any s-linear function S_k on $\boldsymbol{\mathcal{X}}\times\cdots\times\boldsymbol{\mathcal{X}}$, we define

$$|S|^{j,X} = \sup_{|\dot\varphi|_{j,X}\le 1} \left|S_k(\dot\varphi,\dots,\dot\varphi)\right|,\ j=k,k+1, \tag{4.24}$$

and, for any $F\in C^r(\boldsymbol{\mathcal{X}})$, also

$$|F(\varphi)|^{j,X,r} = \sum_{s=0}^{r} \frac{1}{s!} |D^s F(\varphi)|^{j,X}. \tag{4.25}$$

Here, for $s=0$ we take

$$\left|D^0 F(\varphi)\right|^{j,X} = |F(\varphi)|. \tag{4.26}$$

In particular, considering for any $F\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ and any $X\in\mathcal{P}_k$ (and similarly also for any $F\in M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$) the map $F(X):\boldsymbol{\mathcal{X}}\to\mathbb{R}$ defined by $F(X)(\varphi)=F(X,\varphi)$ and its sth derivative $D^sF(X,\varphi)(\dot\varphi,\dots,\dot\varphi)$, we get

$$|F(X,\varphi)|^{j,X,r} = \sum_{s=0}^{r} \frac{1}{s!} \sup_{|\dot\varphi|_{j,X}\le 1} \left|D^sF(X,\varphi)(\dot\varphi,\dots,\dot\varphi)\right|,\ j=k,k+1. \tag{4.27}$$

Now, we are ready to introduce the weighted *strong* norm $|||F(X)|||_{k,X}$ as well as weighted *weak* norm $\|F(X)\|_{k,X,r}$, $r=1,\dots,r_0$ depending on parameters h and ω that will be used for tuning their properties. Introducing the strong weight functions

$$W_k^X(\varphi) = \exp\Big\{\sum_{x\in X} G_{k,x}(\varphi)\Big\} \tag{4.28}$$

with

$$G_{k,x}(\varphi) = \frac{1}{h^2}\big(|\nabla\varphi(x)|^2 + L^{2k}|\nabla^2\varphi(x)|^2 + L^{4k}|\nabla^3\varphi(x)|^2\big), \tag{4.29}$$

we define the weighted strong norm

$$|||F(X)|||_{k,X} = \sup_{\varphi} |F(X,\varphi)|^{k,X,r_0} W_k^{-X}(\varphi) \tag{4.30}$$

with $W_k^{-X}(\varphi) = \big(W_k^X(\varphi)\big)^{-1}$. For $F\in M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$, the norm $|||F(B)|||_{k,B}$ actually does not depend on B in view of periodicity of F, and we use the shorthand $|||F|||_k$.

Further, let $B_x\in\mathcal{B}_k$ be the k-block containing x and let ∂X denote the boundary

$$\partial X = \{y\notin X \mid \exists z\in X \text{ such that } |y-z|=1\} \cup \{y\in X \mid \exists z\notin X \text{ such that } |y-z|=1\} \tag{4.31}$$

(recall that $|\cdot|$ is the Euclidean norm). Introducing the weak weight functions

$$w_k^X(\varphi) = \exp\Big\{\sum_{x\in X} \omega\big(2^d g_{k,x}(\varphi) + G_{k,x}(\varphi)\big) + L^k \sum_{x\in\partial X} G_{k,x}(\varphi)\Big\} \tag{4.32}$$

with $G_{k,x}(\varphi)$ as above and

$$g_{k,x}(\varphi) = \frac{1}{h^2}\sum_{s=2}^{4} L^{(2s-2)k} \sup_{y\in B_x^*} \left|\nabla^s\varphi(y)\right|^2, \tag{4.33}$$

we define the weighted weak norm by

$$\|F(X)\|_{k,X,r} = \sup_{\varphi} \left|F(X,\varphi)\right|^{k,X,r} w_k^{-X}(\varphi),\ \ r = 1,\dots,r_0. \tag{4.34}$$

In addition we also introduce the norm $\|\cdot\|_{k:k+1,X,r}$ that can be viewed as being "halfway between" $\|\cdot\|_{k,X,r}$ and $\|\cdot\|_{k+1,U,r}$ with $U = \overline{X} \in \mathcal{P}_{k+1}$. Namely, we define

$$\|F(X)\|_{k:k+1,X,r} = \sup_{\varphi} \left|F(X,\varphi)\right|^{k+1,X,r} w_{k:k+1}^{-X}(\varphi),\ \ r = 1,\dots,r_0. \tag{4.35}$$

with

$$w_{k:k+1}^X(\varphi) = \exp\Big\{\sum_{x\in X}\big((2^d\omega-1)g_{k:k+1,x}(\varphi) + \omega G_{k,x}(\varphi)\big) + 3L^k\sum_{x\in\partial X} G_{k,x}(\varphi)\Big\}, \tag{4.36}$$

where

$$g_{k:k+1,x}(\varphi) = \frac{1}{h^2}\sum_{s=2}^{4} L^{(2s-2)(k+1)} \sup_{y\in B_x^*} \left|\nabla^s\varphi(y)\right|^2, \tag{4.37}$$

Notice that for the functions $g_{k:k+1,x}$ entering the norm $\|\cdot\|_{k:k+1,X,r}$, we still take $\sup_{y\in B_x^*}$ with k-block B_x. The prefactors $L^{(2s-2)(k+1)}$, however, involve the power $k+1$. Also, the norm $\left|F(X,\varphi)\right|^{k+1,X,r}$ is used, involving $\dot\varphi_{k+1,X}$ in its definition.

For any $r \le r_0$, clearly,

$$\|F(X)\|_{k,X,r} \le |||F(X)|||_{k,X}. \tag{4.38}$$

Inspecting the definitions, it is also easy to show that

$$\|F(X)\|_{k:k+1,X,r} \le \|F(X)\|_{k,X,r} \tag{4.39}$$

once $\omega \ge 2^{d-1}$ (assuring that $2^d\omega(L^2-1) \ge L^2$), and, for any $U \in \mathcal{P}_{k+1} \subset \mathcal{P}_k$ and $F \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}}) \subset M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, also

$$\|F(U)\|_{k+1,U,r} \le \|F(U)\|_{k:k+1,U,r} \le \|F(U)\|_{k,U,r}. \tag{4.40}$$

Next, for any $F \in M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}})$ and a parameter $\mathsf{A} \in \mathbb{R}_+$ we introduce

$$\|F\|_{k,r}^{(\mathsf{A})} = \sup_{X\in\mathcal{P}_k^{\mathrm{c}}} \|F(X)\|_{k,X,r}\Gamma_{k,\mathsf{A}}(X),\ \ r = 1,\dots,r_0, \tag{4.41}$$

where

$$\Gamma_{k,\mathsf{A}}(X) = \begin{cases} \mathsf{A}^{|X|} & \text{if } X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ 1 & \text{if } X \in \mathcal{S}_k. \end{cases} \tag{4.42}$$

Similarly we define also $\|F\|_{k:k+1,r}^{(\mathsf{A})}$. Note that this norm is only defined via functional on connected polymers. Whenever we estimate functionals on arbitrary polymers we simply consider the product over the connected components. Occasionally, when the parameter A is clear from the context, we skip it and write just $\|F\|_{k,r}$ and $\|F\|_{k:k+1,r}$. For $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ we also define

$$\|F\|_{k,r}^{(\mathrm{b})} = \|F(B)\|_{k,B,r}. \tag{4.43}$$

Notice that the right hand side does not depend on B in view of L^k-periodicity of F. Any $F \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ can be restricted to $M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ with $\|F\|_{k,r}^{(\mathrm{b})} \le \|F\|_{k,r}$.

Finally, on the subspace $M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ we define an additional norm $\|\cdot\|_{k,0}$ by taking

$$\|H\|_{k,0} = L^{dk}|\lambda| + L^{\frac{dk}{2}} h \sum_{i=1}^{d} |a_i| + L^{\frac{(d-2)k}{2}} h \sum_{i,j=1}^{d} |\boldsymbol{c}_{i,j}| + \frac{h^2}{2} \sum_{i,j=1}^{d} |\boldsymbol{d}_{i,j}| \tag{4.44}$$

for any $H \in M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ of the form (4.18).

Also, let us stress that the above norms depend on parameters like L, h, and A that are often skipped from the notation. Finally we use the notation

$$\boldsymbol{M}_{k,r} := \{K \in M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}) : \|K\|_{k,r}^{(\mathsf{A})} < \infty\}. \tag{4.45}$$

Sometimes we write $\boldsymbol{M}_r = \boldsymbol{M}_{r,k}$ for brevity. Note that the norms $\|K\|_{k,r}^{(\mathsf{A})} < \infty$ for different $\mathsf{A} > 0$ are equivalent (since there are only finitely many polymers). Thus the definition of $\boldsymbol{M}_{k,r}$ does not depend on A.

4.3. Definition of the renormalisation transformation $\boldsymbol{T}_k : (H_k, K_k) \mapsto (H_{k+1}, K_{k+1})$

Here, we introduce the renormalisation step at a scale k, $k = 0, \dots, N-1$. At each scale k, the interaction will be split between functions H_k and K_k. (Here and in the following we suppress the notation indicating the dependence on $\boldsymbol{q}$, reinstating it only when it will play a crucial role.) The "ideal local Hamiltonian" part H_k is collecting all *relevant* (or marginal) directions under the renormalisation transformation, with all irrelevant ones delegated to the coordinate K_k. There is only limited number of parameters in the relevant coordinate H_k. Being given a pair (H_k, K_k), $H_k \in M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ and $K_k \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, we define a pair (H_{k+1}, K_{k+1}), $H_{k+1} \in M_0(\mathcal{B}_{k+1}, \boldsymbol{\mathcal{X}})$ and $K_{k+1} \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$, so that

$$\boldsymbol{R}_{k+1}(\mathrm{e}^{-H_k} \circ K_k)(\Lambda_N, \varphi) = (\mathrm{e}^{-H_{k+1}} \circ K_{k+1})(\Lambda_N, \varphi) \tag{4.46}$$

with $(\boldsymbol{R}_{k+1}F)(X, \varphi) = \int_{\boldsymbol{\mathcal{X}}} F(X, \varphi + \xi) \mu_{k+1}(\mathrm{d}\xi)$.

As the scale k is fixed in the rest of this chapter, we will skip it and write (H', K') for (H_{k+1}, K_{k+1}), with (4.46) becoming

$$\boldsymbol{R}(\mathrm{e}^{-H} \circ K) = \mathrm{e}^{-H'} \circ K'. \tag{4.47}$$

To define the Hamiltonian H' on the next scale, we first introduce the projection

$$\Pi_2 : M^*(\mathcal{B}, \boldsymbol{\mathcal{X}}) \to M_0(\mathcal{B}, \boldsymbol{\mathcal{X}}) \tag{4.48}$$

as a "homogenization" of the second order Taylor expansion T_2 around zero. Namely, for any $F \in M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ with

$$T_2 F(B, \dot\varphi) = F(B, 0) + DF(B, 0)(\dot\varphi) + \tfrac{1}{2} D^2 F(B, 0)(\dot\varphi, \dot\varphi), \tag{4.49}$$

we define

$$\Pi_2 F(B, \dot\varphi) = F(B, 0) + \ell(\dot\varphi) + Q(\dot\varphi, \dot\varphi) \tag{4.50}$$

so that ℓ is a (unique) linear function of the form (4.19) that agrees with $DF(B, 0)$ on all quadratic functions $\dot\varphi$ on $(B^*)^*$ and Q is a (unique) quadratic function of the form (4.20) that agrees with $\frac{1}{2} D^2 F(B, 0)$ on all affine functions $\dot\varphi$ on $(B^*)^*$. Strictly speaking, we have in mind functions $\dot\varphi \in \boldsymbol{\mathcal{X}}$ such that they are quadratic or affine when restricted to $(B^*)^*$. Since, for $B \in \mathcal{B}_k$, $k \le N-1$, the set $(B^*)^*$ is not wrapped around the torus (as soon as $2^{d+2} \le L$), we do not need to be concerned with a possibility of a contradiction in the assumption of $\dot\varphi \in \boldsymbol{\mathcal{X}}$ having a quadratic or affine restriction to $(B^*)^*$. Clearly, $\Pi_2 F \in M_0(\mathcal{B}, \boldsymbol{\mathcal{X}}) \subset M(\mathcal{B}, \boldsymbol{\mathcal{X}})$ whenever $F \in M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ and $\Pi_2 F = F$ for $F \in M_0(\mathcal{B}, \boldsymbol{\mathcal{X}})$. In particular, we will consider the projection Π_2 on functions $\overline{F} \in M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ of the form

$$\overline{F}(B,\varphi) = \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|} F(X,\varphi) \tag{4.51}$$

for any $F \in M(\mathcal{S}, \boldsymbol{\mathcal{X}})$.

Now we are ready to define the iteration H'. Recalling that $\boldsymbol{R} = \boldsymbol{R}_{k+1}$ is the mapping defined by convolution with μ_{k+1} and starting from $H \in M_0(\mathcal{B}, \boldsymbol{\mathcal{X}})$ and $K \in M(\mathcal{P}, \boldsymbol{\mathcal{X}})$, we define

$$H'(B',\varphi) = \sum_{B\subset B'} \Pi_2\big((\boldsymbol{R}H)(B,\varphi) - \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}(\boldsymbol{R}K)(X,\varphi)\big). \tag{4.52}$$

To define K', we first replace the original variable $H(B,\varphi)$ (or rather $H(B,\varphi+\xi)$ in anticipation of the integration $\boldsymbol{R}$) by $\widetilde{H}(B,\varphi)$, the term in the right hand side sum above,

$$\widetilde{H}(B,\varphi) = \Pi_2\Big((\boldsymbol{R}H)(B,\varphi) - \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}(\boldsymbol{R}K)(X,\varphi)\Big). \tag{4.53}$$

Writing $\tilde{I}(B,\varphi) = \exp\{-\widetilde{H}(B,\varphi)\}$ instead of the original

$$I(B,\varphi+\xi) = \exp\{-H(B,\varphi+\xi)\},$$

and denoting $\tilde{J} = 1 - \tilde{I}$, we introduce

$$\widetilde{K} = \tilde{J} \circ (I-1) \circ K. \tag{4.54}$$

Notice that we are considering here the extension of $\tilde{I}, \tilde{J}$, and I to $M(\mathcal{P}, \boldsymbol{\mathcal{X}})$, resp. $M(\mathcal{P}, \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}})$, according to (4.15). Let us stress that the equation above (and in similar circumstances later) is to be interpreted as an algebraic definition valid pointwise in the variables φ and ξ. It means that $\widetilde{K}$ is actually a function on $\mathcal{P} \times \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}$ defined explicitly by

$$\widetilde{K}(X,\varphi,\xi) = \sum_{\substack{Y,Z\in\mathcal{P}_k(X)\\ Y\cap Z=\varnothing}} \tilde{J}^{X\setminus Y\cup Z}(\varphi)\big(I(\varphi+\xi)-1\big)^Y K(Z,\varphi+\xi). \tag{4.55}$$

Occasionally, we are skipping the polymer variable X but wish to keep the field variables and write, slightly misusing the notation, say, $\widetilde{K}(\varphi,\xi)$ for the mapping $\widetilde{K}(\varphi,\xi) : \mathcal{P} \to \mathbb{R}$ defined by $\widetilde{K}(\varphi,\xi)(X) = \widetilde{K}(X,\varphi,\xi)$. Then the above algebraic equation reads

$$\widetilde{K}(\varphi,\xi) = \tilde{J}(\varphi) \circ \big(I(\varphi+\xi)-1\big) \circ K(\varphi+\xi). \tag{4.56}$$

It is useful to observe that $I - \tilde{I} = (I-1) + \tilde{J}$ yields $I - \tilde{I} = \tilde{J} \circ (I-1)$ and thus $\widetilde{K} = (I - \tilde{I}) \circ K$ suggesting the interpretation of $\widetilde{K}(\varphi,\xi)$ as $K(\varphi+\xi)$ combined with the perturbation $I(\varphi+\xi) - \tilde{I}(\varphi)$.

Now, using $I(\varphi+\xi) = \tilde{I}(\varphi) + \tilde{J}(\varphi) + \big(I(\varphi+\xi) - 1\big)$, we immediately infer that

$$I(\varphi+\xi) = \tilde{I}(\varphi) \circ \tilde{J}(\varphi) \circ \big(I(\varphi+\xi) - 1\big) \tag{4.57}$$

and thus

$$I(\varphi+\xi) \circ K(\varphi+\xi) = \tilde{I}(\varphi) \circ \tilde{J}(\varphi) \circ \big(I-1\big)(\varphi+\xi) \circ K(\varphi+\xi) = \tilde{I}(\varphi) \circ \widetilde{K}(\varphi,\xi). \tag{4.58}$$

As a result,

$$\boldsymbol{R}(I \circ K)(\Lambda_N, \varphi) = (\tilde{I} \circ (\boldsymbol{R}\widetilde{K}))(\Lambda_N, \varphi), \tag{4.59}$$

or, explicitly,

$$\boldsymbol{R}(I \circ K)(\Lambda_N, \varphi) = \sum_{X \in \mathcal{P}(\Lambda_N)} \tilde{I}^{\Lambda_N \setminus X}(\varphi) \int_{\boldsymbol{\mathcal{X}}} \widetilde{K}(X, \varphi, \xi) \mu_{k+1}(\mathrm{d}\xi). \tag{4.60}$$

Here we kept the index $k+1$ at μ_{k+1} to avoid a confusion with the measure $\mu = \mu_1 * \cdots * \mu_{N+1}$.

The function K' on the next scale satisfying (4.47) will be defined by sorting the X-terms according to the next level closure U. While for any $X \in \mathcal{P}(\Lambda_N) \setminus \mathcal{S}(\Lambda_N)$ we attribute the contribution to $K'(U)$ with $U = \overline{X} \in \mathcal{P}(\Lambda_N)'$, for $X \in \mathcal{S}(\Lambda_N)$, we (potentially) split the contribution[1] between several U's. Namely, introducing the factor $\chi(X,U) = \frac{|\{B \in \mathcal{B}(X) \colon \overline{B^*} = U\}|}{|X|}$ for any $X \in \mathcal{S}(\Lambda_N)$ and $\chi(X,U) = \mathbb{1}_{U = \overline{X}}$ for $X \in \mathcal{P}(\Lambda_N) \setminus \mathcal{S}(\Lambda_N)$ (including the case of X consisting of several disjoint components from $\mathcal{S}(\Lambda_N)$), we have

$$(\tilde{I} \circ \widetilde{K})(\Lambda_N, \varphi, \xi) = \sum_{U \in \mathcal{P}'} {I'}^{\Lambda_N \setminus U}(\varphi) \Big[\chi(X,U) \sum_{X \subset U} \tilde{I}^{U \setminus X}(\varphi) \widetilde{K}(X, \varphi, \xi)\Big]. \tag{4.61}$$

Here we used the observation that, for any $X \in \mathcal{S}(\Lambda_N)$ contributing to several U's, we get $\sum_{U \in \mathcal{P}'} \chi(X,U) = 1$ and, also, that $X \subset B^*$ and thus $\overline{X} \subset \overline{B^*}$.

Defining now

$$K'(U, \varphi) = \sum_{X \subset U} \chi(X,U) \tilde{I}^{U \setminus X}(\varphi) \int_{\boldsymbol{\mathcal{X}}} \widetilde{K}(X, \varphi, \xi) \mu_{k+1}(\mathrm{d}\xi) \tag{4.62}$$

for any connected $U \in \mathcal{P}'$, and extending the definition by taking the corresponding product over connected components for a non-connected U, we get

$$\boldsymbol{R}(I \circ K)(\Lambda_N, \varphi) = (I' \circ K')(\Lambda_N, \varphi) \tag{4.63}$$

in view of (4.60) and (4.61).

Notice that if K is L^k-periodic, then K' is obviously L^{k+1}-periodic. Also, the transform conserves the factorisation property of the coordinate K: if K factors on the scale k,

$$X, Y \in \mathcal{P}, \text{ and } X \cap Y = \varnothing, \text{ then } K(X \cup Y, \varphi) = K(X, \varphi) K(Y, \varphi), \tag{4.64}$$

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[1]As will become clear later, the reason for doing so is a need to deal with relevant quadratic terms stemming from K's with $X \in \mathcal{S}$. In anticipation, those terms are already included as the second term in $\widetilde{H}'$ (cf. (4.52)) and the particular way of splitting them among U's leads to the exact cancelations of the corresponding linearized terms. In particular, the linearization of the map $K \to K'$ contains only terms starting with the third order in the Taylor expansion of $K(X, \varphi)$ for X small (cf. (4.83)). Using the fact that only the terms linear in $K(X)$ with $X \in \mathcal{S}$ are relevant in this context, it suffices to introduce a nontrivial χ only for such terms. Our definition is thus a slight simplification of the trick introduced by Brydges [**Bry09**]. We thank Felix Otto and Georg Menz for discussions about this point.

then K' factors on the scale $k+1$.

Indeed, let $X_1, X_2 \in \mathcal{P}$ be such that their closures in $\mathcal{P}'$ are disjoint. Then (assuming that $L > 2^{d+2}$) the range $\frac{1}{2}L^{k+1}$ of the covariance of μ_{k+1} plus twice the possible reach of up to $2^d L^k$ of ${X_1}^*$ and ${X_2}^*$ out of the closures of X_1 and X_2, respectively, does not surpass the minimal distance L^{k+1} of the closure of X_1 from the closure of X_2, and thus

$$(\boldsymbol{R}\widetilde{K})(X_1 \cup X_2, \varphi) = (\boldsymbol{R}\widetilde{K})(X_1, \varphi)(\boldsymbol{R}\widetilde{K})(X_2, \varphi), \tag{4.65}$$

inheriting the property from K, I, and $\tilde{I}$. Now it is easy to observe that this fact actually means that K' factors, as the pairs of sets contributing, according to (4.62), to $K'(U_1, \varphi)$ and $K'(U_2, \varphi)$ with disjoint U_1 and U_2 are necessarily as discussed above.

Let us summarise, reinstating the index k, what we have got.

Proposition 4.3. *Let $k \in \{0, \dots, N-1\}$, $H_k \in M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$, and $K_k \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ be such that it factors. Let $H_{k+1} \in M_0(\mathcal{B}_{k+1}, \boldsymbol{\mathcal{X}})$ be defined by*

$$H_{k+1}(B', \varphi) = \sum_{B \in \mathcal{B}_k(B')} \widetilde{H}_k(B, \varphi), \tag{4.66}$$

where

$$\widetilde{H}_k(B, \varphi) = \Pi_2\Big((\boldsymbol{R}_{k+1} H_k)(B, \varphi) - \sum_{\substack{X \in \mathcal{S}_k \\ X \supset B}} \frac{1}{|X|_k} (\boldsymbol{R}_{k+1} K_k)(X, \varphi)\Big). \tag{4.67}$$

Using $\widetilde{K}_k(\varphi, \xi) = \big(1 - \mathrm{e}^{-\tilde{H}_k(\varphi)}\big) \circ \big(\mathrm{e}^{-H_k(\varphi+\xi)} - 1\big) \circ K_k(\varphi+\xi)$, let $K_{k+1} \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$ be defined by

$$K_{k+1}(U, \varphi) = \sum_{X \in \mathcal{P}_k(U)} \chi(X, U) \exp\Big\{- \sum_{B \in \mathcal{B}_k(U \setminus X)} \widetilde{H}_k(B, \varphi)\Big\} \int_{\boldsymbol{\mathcal{X}}} \widetilde{K}_k(X, \varphi, \xi) \mu_{k+1}(\mathrm{d}\xi) \tag{4.68}$$

for any connected $U \in \mathcal{P}'$, with

$$\chi(X, U) = \begin{cases} \frac{|\{B \in \mathcal{B}_k(X) \colon \overline{B^*} = U\}|}{|X|} & \text{if } X \in \mathcal{S}_k(\Lambda_N), \\ \mathbb{1}_{U = \overline{X}} & \text{if } X \in \mathcal{P}_k(\Lambda_N) \setminus \mathcal{S}_k(\Lambda_N), \end{cases} \tag{4.69}$$

and by the corresponding product over connected components for any non-connected U. Then $K_{k+1} \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$, it factors, and

$$\boldsymbol{R}_{k+1}(\mathrm{e}^{-H_k} \circ K_k)(\Lambda_N, \varphi) = (\mathrm{e}^{-H_{k+1}} \circ K_{k+1})(\Lambda_N, \varphi). \tag{4.70}$$

As a result, introducing

$$\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}) = (H_{k+1}, K_{k+1}) \tag{4.71}$$

with H_{k+1} and K_{k+1} defined by equations (4.66 – 4.68), we get the renormalization map

$$\boldsymbol{T}_k \colon M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}}) \times M(\mathcal{P}_k, \boldsymbol{\mathcal{X}}) \times \mathbb{R}^{d \times d}_{\mathrm{sym}} \to M_0(\mathcal{B}_{k+1}, \boldsymbol{\mathcal{X}}) \times M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}}), \tag{4.72}$$

$k = 0, 1, \dots, N-1$.

4.4. Key properties of the renormalisation transformation

Of course, defining the renormalisation map $\boldsymbol{T}_k$ satisfying (4.70) is only half of our task of the definition of the renormalisation transform. Another part lies in the verification that the choice of coordinates H_k and K_k together with the map $(H_k, K_k) \mapsto (H_{k+1}, K_{k+1})$ indeed isolates relevant and irrelevant variables with correct estimates. Notice that in the definition of $\boldsymbol{T}_k$, we explicitly included the dependence on the matrix $\boldsymbol{q}$. It stems from the dependence of the starting Gaussian measure $\mu = \mu_{\mathcal{C}^{(\boldsymbol{q})}}$ (and of the corresponding generalised Laplacian $\mathcal{A}^{(\boldsymbol{q})}$) on $\boldsymbol{q}$ and it transfers into such a dependence also for the operators $\mathcal{C}_{(k)}^{(\boldsymbol{q})}$ obtained from the finite range decomposition, for the corresponding Green functions $\mathcal{C}_{k,0}^{(\boldsymbol{q})}$ and the measures μ_k, and, eventually, for the operators $\boldsymbol{T}_k$. Even though this dependence often does not appear in our notation, in the following two Propositions, where we state its key properties, we explicitly address this dependence and make it thus explicit also in the notation. For variables H and K we again skip the subscript k and replace $k+1$ by a prime.

It is easy to verify that, for any $\boldsymbol{q}$, the origin $(H, K) = (0, 0)$ is a fixed point of the transformation $\boldsymbol{T}_k$. Further, the H-coordinate of the operator $\boldsymbol{T}_k$ has actually a linear dependence; we can write

$$\boldsymbol{T}_k(H, K, \boldsymbol{q}) = (\boldsymbol{A}_k^{(\boldsymbol{q})} H + \boldsymbol{B}_k^{(\boldsymbol{q})} K, S_k(H, K, \boldsymbol{q})) \tag{4.73}$$

with appropriate linear operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$ and $\boldsymbol{B}_k^{(\boldsymbol{q})}$. While delegating the discussion of the explicit form and the properties of these operators (as well as the linearization of the map S_k) to Proposition 4.7, we begin with the smoothness of the nonlinear part S_k.

The map S_k is given as a composition of several maps and its smoothness will be a consequence of the smoothness of the composing maps. To verify its smoothness we find it useful to introduce a notion differentiability that is rather easy to verify.

Definition 4.4. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be normed linear spaces and $\mathcal{U} \subset \boldsymbol{X}$ be open. We use $C_*^m(\mathcal{U}, \boldsymbol{Y})$ to denote the set of functions $G : \mathcal{U} \to \boldsymbol{Y}$ such that for each $j \leq m$ and $\dot{x} \in X$, the directional derivative

$$D^j f(x, \dot{x}^j) = \frac{\mathrm{d}^j}{\mathrm{d}t^j} G(x + t\dot{x}) \Big|_{t=0} \tag{4.74}$$

at any $x \in \mathcal{U}$ exists and the map $(x, \dot{x}) \in \mathcal{U} \times \boldsymbol{X} \to D^j G(x, \dot{x}^j) \in \boldsymbol{Y}$ is continuous.

The technical reasons for this definition will be apparent later and are explained in great detail in Appendix D. It turns out that this notion is weak only apparently. In particular, for $m \geq 0$ the space $C_*^{m+1}(\mathcal{U}, \boldsymbol{Y})$ is contained in the usual space $C^m(\mathcal{U}, \boldsymbol{Y})$ of Fréchet differentiable functions (with operator norms on multilinear forms from $L_m(\boldsymbol{X}, \boldsymbol{Y})$), see Proposition D.17.

Exploring the smoothness of the nonlinear part S_k of the operator $\boldsymbol{T}_k$, we run into problems stemming from a loss of regularity when deriving S_k with respect to the parameter $\boldsymbol{q}$. For example, it turns out that

$$\|D_1^{j'} D_2^{j''} D_3^{\ell} S_k(H, K, \boldsymbol{q})(\dot{H}^{j'}, \dot{K}^{j''}, \dot{\boldsymbol{q}}^{\ell})\|_{k+1, r-2\ell}^{(\mathsf{A})} \leq C \|\dot{H}\|_0^{j'} (\|\dot{K}\|_{k,r}^{(\mathsf{A})})^{j''} \|\dot{\boldsymbol{q}}\|^{\ell}, \tag{4.75}$$

where the norm $\|\cdot\|_{k+1,r-2\ell}^{(\mathsf{A})}$ in the target space is weaker than the norm $\|\cdot\|_{k,r}^{(\mathsf{A})}$ in the domain space. As a result we are compelled to consider the map S_k with a suitable sequence of normed spaces $\boldsymbol{M} = \boldsymbol{M}_{r_0} \hookrightarrow \boldsymbol{M}_{r_0-2} \hookrightarrow \dots \hookrightarrow \boldsymbol{M}_{r_0-2m}$,

$r_0 > 2m$, defined as the spaces $M_r(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}})$ endowed with the norms $\|\cdot\|_{k,r}^{(\mathsf{A})}$, $r = r_0, r_0-2, \ldots, r_0-2m$, respectively, and the space $\boldsymbol{M}_0$ defined as $M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ with the norm $\|\cdot\|_{k,0}$. Similarly, $\boldsymbol{M}' = \boldsymbol{M}'_{r_0} \hookrightarrow \boldsymbol{M}'_{r_0-2} \hookrightarrow \ldots \hookrightarrow \boldsymbol{M}'_{r_0-2m}$ are defined as $M(\mathcal{P}_{k+1}^{\mathrm{c}}, \boldsymbol{\mathcal{X}})$ with the norms $\|\cdot\|_{r,k+1}^{(\mathsf{A})}$, $r = r_0, r_0-2, \ldots, r_0-2m$. Further, we will use $\widetilde{\boldsymbol{M}}_r$ to denote the closure of $\boldsymbol{M}$ in $\boldsymbol{M}_r$, and similarly for $\widetilde{\boldsymbol{M}}'_r$.

Considering now open subsets $\mathcal{U} \subset \boldsymbol{M}_0 \times \boldsymbol{M}$ and $\mathcal{V} \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$, we will introduce the class of functions that can be described as those $G\colon \mathcal{U} \times \mathcal{V} \to \boldsymbol{M}'$ for which the derivative $D_1^{j'} D_2^{j''} D_3^{\ell} G$ is a continuous map $\mathcal{U} \times \mathcal{V} \times \boldsymbol{M}_0^{j''} \times \widetilde{\boldsymbol{M}}_r^{j'} \times (\mathbb{R}^{d\times d}_{\mathrm{sym}})^{\ell} \to \boldsymbol{M}'_{r-2\ell}$. More formally, we introduce the set $\widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \boldsymbol{M}')$ of maps $G : \mathcal{U} \times \mathcal{V} \to \boldsymbol{M}'$ as follows (see Definition D.24 in a more general setting):

Definition 4.5. Let $r_0, m \in \mathbb{N}$, $r_0 > 2m$. We define $\widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \boldsymbol{M}')$ as the set of all maps $G : \mathcal{U} \times \mathcal{V} \to \boldsymbol{M}'$ such that

(a) $G \in C_*^m(\mathcal{U} \times \mathcal{V}, \boldsymbol{M}'_{r_0-2m})$.

(b) For each $0 \le j' + j'' + \ell \le m$, the function

$$(H, K, \boldsymbol{q}, \dot{H}_1, \ldots, \dot{H}_{j'}, \dot{K}_1, \ldots, \dot{K}_{j''}, \dot{\boldsymbol{q}}_1, \ldots, \dot{\boldsymbol{q}}_\ell) \to \\ \to D_1^{j'} D_2^{j''} D_3^{\ell} G((H, K, \boldsymbol{q}), \dot{\boldsymbol{q}}_1, \ldots, \dot{\boldsymbol{q}}_\ell, \dot{K}_1, \ldots, \dot{K}_{j''}, \dot{H}_1, \ldots, \dot{H}_{j'}),$$

(which is by an implication of the claim (a) (see Theorem D.10) defined as a map $\mathcal{U} \times \mathcal{V} \times \boldsymbol{M}_0^{j'} \times \boldsymbol{M}^{j''} \times (\mathbb{R}^{d\times d}_{\mathrm{sym}})^{\ell} \to \boldsymbol{M}'_{r_0-2m})$ has an extension to a continuous mapping $\mathcal{U} \times \mathcal{V} \times \boldsymbol{M}_0^{j'} \times \widetilde{\boldsymbol{M}}_{r_0-2m+2\ell}^{j''} \times (\mathbb{R}^{d\times d}_{\mathrm{sym}})^{\ell} \to \boldsymbol{M}'_{r_0-2m}$. This extension is also denoted $D_1^{j'} D_2^{j''} D_3^{\ell} G$.

(c) For each $0 \le j' + j'' + \ell \le m$ and $r = r_0, r_0-2, \ldots, r_0-2m+2\ell$, the restriction of $D_1^{j'} D_2^{j''} D_3^{\ell} G$ to $\mathcal{U} \times \mathcal{V} \times \boldsymbol{M}_0^{j'} \times \widetilde{\boldsymbol{M}}_r^{j''} \times (\mathbb{R}^{d\times d}_{\mathrm{sym}})^{\ell}$ (notice that it has been already extended by (b)) has values in $\boldsymbol{M}'_{r-2\ell}$ and is continuous as a mapping between these spaces.

Again, see Appendix D for further context and properties of the notion of smoothness introduced in this way. Contrary to Definition D.24 we abstain from invoking the relevant sequences of normed spaces in the notation as here they are fixed from the context.

In the following we will consider the constants d, ω, and r_0 to be fixed (assuming $d = 2, 3$, $\omega \ge 2(d^2 2^{2d+1} + 1)$ and we will not mention possible dependence of various constants (like L_0, h_0, and A_0 below) on it. For the proof of the results in Chapter 2 $r_0 = 9$ is sufficient, see comment in Remark 4.8).

For fixed values of the parameters L, h, and A in the definition of the norms in Chapter 4.2, let $\mathcal{U}_\rho \subset \boldsymbol{M}_0 \times \boldsymbol{M}_{r_0}$ and $\mathcal{V} \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ be the neighbourhoods of the origin,

$$\mathcal{U}_\rho = \{(H, K) \in \boldsymbol{M}_0 \times \boldsymbol{M}_{r_0} \colon \|H\|_{k,0} < \rho, \|K\|_{k,r_0}^{(\mathsf{A})} < \rho\} \tag{4.76}$$

and

$$\mathcal{V} = \{\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \colon \|\boldsymbol{q}\| < 1/2\}. \tag{4.77}$$

Proposition 4.6 (Smoothness of the nonlinear part S_k). *There exists a constant L_0 and, for any $L \ge L_0$, constants $h_0(L)$ and $\mathsf{A}_0(L)$, and for any $\mathsf{A} \ge \mathsf{A}_0$ a* *constant* $\rho = \rho(\mathsf{A})$ *such that, for any* $k = 0, \dots, N-1$*, any* $L \geq L_0$*,* $h \geq h_0$*, and* $\mathsf{A} \geq \mathsf{A}_0$ *we have*

$$S_k \in \widetilde{C}^m(\mathcal{U}_\rho \times \mathcal{V}, \boldsymbol{M}'), \tag{4.78}$$

and there is a constant $C = C(L, h, \mathsf{A}) > 0$ *such that*

$$\|D_1^{j'} D_2^{j''} D_3^{\ell} S_k(H, K, \boldsymbol{q})(\dot{H}^{j'}, \dot{K}^{j''}, \dot{\boldsymbol{q}}^{\ell})\|_{k+1, r-2\ell}^{(\mathsf{A})} \leq C \|\dot{H}\|_0^{j'} (\|\dot{K}\|_{k,r}^{(\mathsf{A})})^{j''} \|\dot{\boldsymbol{q}}\|^{\ell}, \tag{4.79}$$

for any $(H, K) \in \mathcal{U}_\rho$*,* $\boldsymbol{q} \in \mathcal{V}$*,* $0 \leq j' + j'' + \ell \leq m$*, and* $r = r_0, r_0 - 2, \dots, r_0 - 2m + 2\ell$*.*

The proof will be deferred to Chapter 6, where we will split S_k into a composition of several partial maps and deal with their smoothness separately, isolating in detail the needed restrictions on various constants. Here, instead, we offer a heuristic explanation of the role of the principal constants. The restrictions on L are purely geometric (see Lemma 5.1, Lemma 7.1, Lemma 7.2, Lemma 7.3, Lemma 7.8). In particular, by assuming that $L \geq L_0$ we have $L \geq 2^{d+1}$ implying, for example, that if $B \in \mathcal{B}_k$, then the cube B^* has the side at most L^{k+1} and thus $\overline{B^*} \in \mathcal{S}_{k+1}$. The restrictions on the constant h are more subtle (see Lemma 5.1, Lemma 7.1, Lemma 7.2, Lemma 7.3). Its role is to suppress large fields in the norms $\|F(X)\|_{k,X,r}$ and $|||F(X)|||_{k,X}$ by employing the h-dependent weight factors W_k^X and w_k^X, respectively. When evaluating the norms of the maps $(H, K) \to \widetilde{H}$ (see (4.67)) and $K \to \boldsymbol{R}_{k+1}(K)$, a major part of the coarse grained increase is absorbed into the growth $L^k \to L^{k+1}$ of the corresponding factors in the functions $G_{k,x}$ and $g_{k,x}$ entering the weight factors. However, some surplus remains stemming essentially from the term $L^{\eta(n,d)}$ in the fluctuation bound (4.3) of the finite range decomposition. A suppression of the relevant term is obtained by assuming that $h \geq h_0(L) = h_1 L^{\frac{d^2}{2} + 5d + 16}$ with h_1 depending only on d and ω. Finally, the constant A is responsible for combining the norms $\|\cdot\|_{k,X,r}$ into a single norm $\|\cdot\|_{k,r}^{(\mathsf{A})}$ (see Lemma 6.10 and Lemma 7.2). However, it turns out that the map $K \to \boldsymbol{R}_{k+1}(K)$ leads to acquiring a factor $2^{|X|_k}$ in the norm $\|\cdot\|_{k,X,r}$, yielding an inevitable loss in A in the norm $\|\cdot\|_{k,r}^{(\mathsf{A})}$. Nevertheless, the loss can be recovered when combining the terms in (4.68) while passing to the next scale. Namely, using in the resulting sum stemming from evaluating the norm of (4.68) the geometric bound $|X|_k \geq (1 + \alpha(d))|\overline{X}|_{k+1} - (1 + \alpha(d))2^{d+1}|\mathcal{C}(X)|$ with a constant $\alpha(d) > 0$, we get the original A once we suppose that the map is restricted to sufficiently small domain, e.g. assuming that $\|\boldsymbol{R}_{k+1}(K)\|_{k:k+1,r}^{(\mathsf{A})} \leq \rho(\mathsf{A}) = (2\mathsf{A}^{2^{d+3}})^{-1}$ and taking A sufficiently large depending on L (and d).

The next claim deals with the linearisation of the map $\boldsymbol{T}_k$ at the fixed point $(H, K) = (0, 0)$. For a linear operator $\boldsymbol{L}$ between Banach spaces, we consider here the standard norm $\|\boldsymbol{L}\| = \sup\{\|\boldsymbol{L}(f)\| \colon \|f\| \leq 1\}$, with appropriate norms on the corresponding spaces. Usually we indicate the corresponding norms in an appropriate way, e.g., $\|\boldsymbol{L}\|_{k,r;k+1,0}$ and $\|\boldsymbol{L}\|_{k,r;k+1,r}$, or simply $\|\boldsymbol{L}\|_{r;0}$ and $\|\boldsymbol{L}\|_r$, for a linear mapping $\boldsymbol{L} : \boldsymbol{M}_r \to \boldsymbol{M}'_0$ and $\boldsymbol{L} : \boldsymbol{M}_r \to \boldsymbol{M}'_r$, respectively.

Proposition 4.7 (Linearisation of $\boldsymbol{T}_k$). *The first derivative at* $H = 0$ *and* $K = 0$ *have a triangular form,*

$$D\boldsymbol{T}_k(0, 0, \boldsymbol{q})(\dot{H}, \dot{K}) = \begin{pmatrix} \boldsymbol{A}_k^{(\boldsymbol{q})} & \boldsymbol{B}_k^{(\boldsymbol{q})} \\ \boldsymbol{0} & \boldsymbol{C}_k^{(\boldsymbol{q})} \end{pmatrix} \begin{pmatrix} \dot{H} \\ \dot{K} \end{pmatrix}, \tag{4.80}$$

with

$$(\boldsymbol{A}_k^{(\boldsymbol{q})}\dot H)(B',\varphi)=\sum_{B\in\mathcal{B}(B')}\big[\dot H(B,\varphi)+\sum_{x\in B}\sum_{i,j=1}^{d}\dot d_{i,j}\nabla_i\nabla_j^*\mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)\big], \tag{4.81}$$

$$(\boldsymbol{B}_k^{(\boldsymbol{q})}\dot K)(B',\varphi)=-\sum_{B\in\mathcal{B}(B')}\Pi_2\sum_{\substack{X\in\mathcal{S}\\ X\supset B}}\frac{1}{|X|}\Big(\int_{\boldsymbol{\mathcal{X}}}\dot K(X,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\Big), \tag{4.82}$$

and

$$\begin{aligned}(\boldsymbol{C}_k^{(\boldsymbol{q})}\dot K)(U,\varphi)=\sum_{B:\overline{B^*}=U}\big(1-\Pi_2\big)\sum_{\substack{Y\in\mathcal{S}\\ Y\supset B}}\frac{1}{|Y|}\Big(\int_{\boldsymbol{\mathcal{X}}}\dot K(Y,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\Big)+\\ +\sum_{\substack{X\in\mathcal{P}^c\setminus\mathcal{S}\\ \overline{X}=U}}\int_{\boldsymbol{\mathcal{X}}}\dot K(X,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi).\end{aligned} \tag{4.83}$$

Further, let $\theta\in(1/4,3/4)$ and let L_0 and $h_0=h_0(L)$ be as in Proposition 4.6. There exists a constant $M=M(d)$ and, for any $L\ge L_0$, a constant $\mathsf{A}_0=\mathsf{A}_0(L)$, such that for any $h\ge h_0(L)$ and any $\mathsf{A}\ge\mathsf{A}_0(L)$, the following bounds on the norms of operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$, $\boldsymbol{B}_k^{(\boldsymbol{q})}$, and $\boldsymbol{C}_k^{(\boldsymbol{q})}$ hold independently of N and k and for any $\|\boldsymbol{q}\|\le\frac12$:

$$\|\boldsymbol{C}_k^{(\boldsymbol{q})}\|_r\le\theta,\ \|{\boldsymbol{A}_k^{(\boldsymbol{q})}}^{-1}\|_{r;r}\le\frac{1}{\sqrt{\theta}},\ \text{and}\ \|\boldsymbol{B}_k^{(\boldsymbol{q})}\|_{r;0}\le ML^d, \tag{4.84}$$

$r\ge3$, and for all $\mathsf{A}\ge\mathsf{A}_0$ (note that for the contraction bound for $\boldsymbol{C}^{(\boldsymbol{q})}$ the choice $h\ge h_0$ is sufficient).

Remark 4.8. (i) Notice that as a consequence of Proposition 4.6, the operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$, $\boldsymbol{B}_k^{(\boldsymbol{q})}$, and $\boldsymbol{C}_k^{(\boldsymbol{q})}$ are m-times differentiable with respect to $\boldsymbol{q}$, $\|\boldsymbol{q}\|\le\frac12$, and there exists a finite constant $C=C(h,L)>0$ such that

$$\|\partial_{\boldsymbol{q}}^{\ell}\boldsymbol{A}_k^{(\boldsymbol{q})}\dot H\|_0\le C\|\dot H\|_0,\ \|\partial_{\boldsymbol{q}}^{\ell}\boldsymbol{B}_k^{(\boldsymbol{q})}\dot K\|_0\le C\|\dot K\|_{2\ell+2},\ \|\partial_{\boldsymbol{q}}^{\ell}\boldsymbol{C}_k^{(\boldsymbol{q})}\dot K\|_{r-2\ell}\le C\|\dot K\|_r, \tag{4.85}$$

for any $\ell=1,2,\dots,m$ and any $r\ge2\ell+3$ and $\mathsf{A}\ge\mathsf{A}_0$.
(ii) For the results in Chapter 2 we need $m=3$. Thus $r_0=9$ is sufficient. ◇

Proof of Proposition 4.7. Here, we will only show the validity of the explicit formulas for the operators $\boldsymbol{A}_k^{(\boldsymbol{q})}$, $\boldsymbol{B}_k^{(\boldsymbol{q})}$, and $\boldsymbol{C}_k^{(\boldsymbol{q})}$. The bounds needed for the remaining claims will be proven in Chapter 7.

Starting from (4.66) and (4.67), let us expand the linear and quadratic terms in $\dot H(B,\varphi+\xi)$ into the sum of the terms depending on φ, ξ, and the term proportinal to $\dot Q(\varphi,\xi)$. Observing that the integral with respect to $\mu_{k+1}(\xi)$ of the terms linear in ξ vanishes and that $\Pi_2(\dot H(B,\varphi))=\dot H(B,\varphi)$, we get the expression (4.81) for $\boldsymbol{A}_k^{(\boldsymbol{q})}$ once we notice that $\int_{\boldsymbol{\mathcal{X}}}\dot Q(\xi,\xi)\mu_{k+1}(\mathrm{d}\xi)=\sum_{x\in B}\sum_{i,j=1}^{d}\dot d_{i,j}\nabla_i\nabla_j^*\mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)$.

The formula (4.82) follows directly from the second term on the right hand side of (4.67).

When computing $\boldsymbol{C}_k^{(\boldsymbol{q})}$ we first observe that only linear terms in $\widetilde{K}$ can contribute. Taking $\dot{H} = 0$ and using thus (4.68) with

$$\widetilde{H}(B,\varphi) = -\Pi_2 \sum_{\substack{X\in\mathcal{S}\\ X\supset B}} \frac{1}{|X|}(\boldsymbol{R}\dot{K})(X,\varphi) \tag{4.86}$$

and $\widetilde{K}(\varphi,\xi) = \big(1 - \mathrm{e}^{-\tilde{H}(\varphi)}\big) \circ K(\varphi+\xi)$, we get

$$\begin{aligned}\boldsymbol{C}_k^{(\boldsymbol{q})}(\dot{K})(U,\varphi) = \sum_{Y\in\mathcal{S}} \chi(Y,U) \int_{\boldsymbol{\mathcal{X}}} D\widetilde{K}(0)(\dot{K})(Y,\varphi,\xi)\mu_{k+1}(\mathrm{d}\xi) +\\ + \sum_{\substack{X\in\mathcal{P}^c\setminus\mathcal{S}\\ \overline{X}=U}} \int_{\boldsymbol{\mathcal{X}}} D\widetilde{K}(0)(\dot{K})(X,\varphi,\xi)\mu_{k+1}(\mathrm{d}\xi).\end{aligned} \tag{4.87}$$

Writing $\chi(Y,U) = \sum_{\substack{B\in Y\\ \overline{B^*}=U}} \frac{1}{|Y|}$ and observing that

$$D\widetilde{K}(0)(\dot{K})(B,\varphi,\xi) = \dot{K}(B,\varphi+\xi) - D\mathrm{e}^{-\tilde{H}(0)}(\dot{K})(B,\varphi) \text{ for } Y = B, \tag{4.88}$$

$$D\widetilde{K}(0)(\dot{K})(Y,\varphi,\xi) = \dot{K}(Y,\varphi+\xi) \text{ for } Y \neq B, \tag{4.89}$$

and

$$D\mathrm{e}^{-\tilde{H}(0)}(\dot{K})(B,\varphi) = \Pi_2 \sum_{\substack{Y\in\mathcal{S}\\ Y\supset B}} \frac{1}{|Y|}\big(\boldsymbol{R}\dot{K}\big)(Y,\varphi), \tag{4.90}$$

we get (4.83). □

4.5. Fine tuning of the initial conditions

Our next task is to implement in detail the idea of fine tuning outlined in Chapter 3. More specifically we will choose an initial ideal Hamiltonian (as used in (3.15) and defined in (4.17)),

(4.91)

$$\mathcal{H}(x,\varphi) = \lambda + \sum_{i=1}^{d} a_i \nabla\varphi(x) + \sum_{i,j=1}^{d} \boldsymbol{c}_{i,j}\nabla_i\nabla_j\varphi(x) + \frac{1}{2}\sum_{i,j=1}^{d} \boldsymbol{q}_{i,j}\nabla\varphi(x)\nabla_j\varphi(x)$$

such that the final ideal Hamiltonian vanishes (note that in Chapter 3 we considered only the simplified case $\lambda = a = \boldsymbol{c} = 0$).

Given an initial $\mathcal{K}$ we want to evaluate the integral

$$\mathcal{Z}_N(u) = \int_{\boldsymbol{\mathcal{X}}_N} \prod_{x\in\Lambda} \big(1 + \mathcal{K}(x,\varphi)\big)\,\mu(\mathrm{d}\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} (1\circ\mathcal{K})(\Lambda,\varphi)\,\mu(\mathrm{d}\varphi).$$

Analogously to the calculation in Chapter 3 cf. (3.16) we can rewrite this integral as

$$\begin{aligned}\mathcal{Z}_N(u) &= \int_{\boldsymbol{\mathcal{X}}_N} \mathrm{e}^{\mathcal{H}(\Lambda,\varphi)}\big(\mathrm{e}^{-\mathcal{H}} \circ \mathrm{e}^{-\mathcal{H}}\mathcal{K}\big)(\Lambda,\varphi)\,\mu(\mathrm{d}\varphi)\\ &= \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}}\,\mathrm{e}^{L^{dN}\lambda} \int_{\boldsymbol{\mathcal{X}}_N} \big(\mathrm{e}^{-\mathcal{H}} \circ \mathrm{e}^{-\mathcal{H}}\mathcal{K}\big)(\Lambda,\varphi)\,\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi)\end{aligned} \tag{4.92}$$

where $Z_N^{(\boldsymbol{q})}$ and $Z_N^{(0)}$ are as in Chapter 3. Here we used that $\sum_{x\in\Lambda}\nabla_i\varphi(x) = 0$ and $\sum_{x\in\Lambda}\nabla_i\nabla_j\varphi(x) = 0$ because φ is periodic.

We will now show that for sufficiently small $\mathcal{K}$ there exists an $\mathcal{H} = \mathcal{H}(\mathcal{K})$ such that the second integral in (4.92) deviates from 1 only by an exponential small term and such that the derivatives of this term with respect to $\mathcal{K}$ are also controlled.

To do so we proceed in two steps. We first show that given sufficiently small $\mathcal{K}$ and $\mathcal{H}$ there exists an ideal Hamiltonian $\mathcal{F}_1(\mathcal{K},\mathcal{H}) \in M_0$ and a small 'irrelevant' term $\mathcal{F}_{2N}(\mathcal{K},\mathcal{H}) \in \boldsymbol{M}_{N,r}$ such that

$$\int_{\boldsymbol{\mathcal{X}}_N} \big(\mathrm{e}^{-\mathcal{F}_1(\mathcal{K},\mathcal{H})} \circ \mathrm{e}^{-\mathcal{H}}\mathcal{K}\big)(\Lambda,\varphi)\,\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi) = \int_{\boldsymbol{\mathcal{X}}_N} (1+\mathcal{F}_{2N}(\mathcal{K},\mathcal{H}))\,\mu^{(\boldsymbol{q})}_{N+1}(\mathrm{d}\varphi). \tag{4.93}$$

As a byproduct of this construction we will see that for $\mathcal{K} = 0$ we have $\mathcal{F}_1(0,\mathcal{H}) = 0$ and $\mathcal{F}_{2N}(0,\mathcal{H}) = 0$ for all sufficiently small $\mathcal{H}$. Together with smoothness results for $\mathcal{F}_1$ this implies $D_{\mathcal{H}}\mathcal{F}_1(0,0) = 0$ and the implicit function will guarantee that there exists a unique map $\mathcal{H}$ mapping a neighbourhood of the origin in $\boldsymbol{E}$ to $\boldsymbol{M}_0$ such that

$$\mathcal{F}_1(\mathcal{K},\mathcal{H}(\mathcal{K})) = \mathcal{H}(\mathcal{K}). \tag{4.94}$$

Combining this with (4.93) and (4.92) we get

$$-\log \mathcal{Z}_N(u) = -\log \frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}} - \lambda L^{dN} - \log \int_{\boldsymbol{\mathcal{X}}_N} (1+\mathcal{F}_{2N}(\mathcal{K},\mathcal{H}(\mathcal{K})))\,\mu^{(\boldsymbol{q})}_{N+1}(\mathrm{d}\varphi), \tag{4.95}$$

where

$$\lambda = \pi_0(\mathcal{H}(\mathcal{K})) \quad \text{and} \quad \boldsymbol{q} = \pi_2(\mathcal{H}(\mathcal{K})) \tag{4.96}$$

denote the constant term in $\mathcal{H}(\mathcal{K})$ and the coefficient matrix of the quadratic term, respectively.

We now first explain how to construct the maps $\mathcal{F}_1$ and $\mathcal{F}_{2N}$. We rewrite the entire cascade of maps $\boldsymbol{T}_k$ in terms of a single map on a suitably defined Banach space. First, we introduce the Banach spaces

$$\boldsymbol{Y}_r = \big\{\boldsymbol{y} = (H_0, H_1, K_1, \ldots, H_{N-1}, K_{N-1}, K_N)\colon H_k \in \boldsymbol{M}_{k,0}, K_k \in \boldsymbol{M}_{k,r}\big\} \tag{4.97}$$

with the norms

$$\|\boldsymbol{y}\|_{\boldsymbol{Y}_r} = \max_{k\in\{0,\ldots,N-1\}} \frac{1}{\eta^k}\|H_k\|_{k,0} \vee \max_{k\in\{1,\ldots,N\}} \frac{\alpha}{\eta^k}\|K_k\|_{k,r} \tag{4.98}$$

for $r = 1,\ldots,r_0$ and with parameters $\eta \in (0,1)$ and $\alpha \geq 1$ to be chosen later. Here, to avoid ambiguity, we reinstated index k also in the notation for normed spaces; we write $\boldsymbol{M}_{k,0}$ and $\boldsymbol{M}_{k,r}$ instead of $\boldsymbol{M}_0$ and $\boldsymbol{M}_r$ used previously. Notice that the terms K_0 and H_N are not present in $\boldsymbol{y} \in \boldsymbol{Y}_r$; while the latter is put to be 0, the former is singled out as an initial condition for a separate treatment. Also, notice that $\|\boldsymbol{y}\|_{\boldsymbol{Y}_r} \leq \|\boldsymbol{y}\|_{\boldsymbol{Y}_{r+1}}$ and thus $\boldsymbol{Y}_{r+1} \hookrightarrow \boldsymbol{Y}_r$.

Taking into account the dependence of $\boldsymbol{T}_k$ on $\boldsymbol{q}$ (the matrix in the quadratic term of $\mathcal{H}$) and on the initial perturbation $\mathcal{K} \in \boldsymbol{E}$ (see (2.21)) we define the map

$$\boldsymbol{\mathcal{T}}\colon \boldsymbol{Y}_r \times \boldsymbol{E} \times M_0 \to \boldsymbol{Y}_r \tag{4.99}$$

by

$$\boldsymbol{\mathcal{T}}(\boldsymbol{y},\mathcal{K},\mathcal{H}) = \overline{\boldsymbol{y}}. \tag{4.100}$$

Here, $\overline{\boldsymbol{y}}$ is given by recursive equations,

$$\begin{aligned} \overline{H}_k &= \boldsymbol{A}_k^{-1}\big(H_{k+1} - \boldsymbol{B}_k K_k\big),\\ \overline{K}_{k+1} &= S_k(H_k, K_k, \boldsymbol{q}) = \boldsymbol{C}_k K_k + S_k(H_k, K_k, \boldsymbol{q}) - D_2 S_k((0,0,\boldsymbol{q}), K_k). \end{aligned} \tag{4.101}$$

for $k = 0, \dots, N-1$. Here $\boldsymbol{C}_k K_k = D_2 S_k((0,0,\boldsymbol{q}), K_k)$ and $S_k(H_k, K_k, \boldsymbol{q}) - D_2 S_k((0,0,\boldsymbol{q}), K_k)$ is the nonlinear part of the map S_k. In addition, we set $H_N = 0$ and define $K_0 \in M(\mathcal{P}_0, \boldsymbol{\mathcal{X}})$ by $K_0 = \mathrm{e}^{-\mathcal{H}}\mathcal{K}$, i.e., by

$$K_0(X,\varphi) := \prod_{x\in X}\big(\exp(-\mathcal{H}(x,\varphi))\mathcal{K}(\nabla\varphi(x))\big) \tag{4.102}$$

with $\mathcal{K} \in \boldsymbol{E}$ and $\mathcal{H} \in M_0$.

Observe now that, for a given $\mathcal{K}$ and $\mathcal{H}$, the $2N$-tuple $\boldsymbol{y}$ is a fixed point of $\boldsymbol{\mathcal{T}}$, i.e., $\boldsymbol{\mathcal{T}}(\boldsymbol{y}, \mathcal{K}, \mathcal{H}) = \boldsymbol{y}$ if and only if

$$\boldsymbol{T}_k(H_k, K_k, \boldsymbol{q}) = (H_{k+1}, K_{k+1}), k = 0, \dots, N-1, \tag{4.103}$$

with $K_0 = \mathrm{e}^{-\mathcal{H}}\mathcal{K}$ and $H_N = 0$. Our task thus is to find a map $\mathcal{F}$ from a neighbourhood of origin in $\boldsymbol{E} \times M_0$ to $\boldsymbol{Y}_r$ so that

$$\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K},\mathcal{H}), \mathcal{K}, \mathcal{H}) = \mathcal{F}(\mathcal{K},\mathcal{H}). \tag{4.104}$$

This can be done with help of the Implicit Function Theorem E.1 using the bounds from Propositions 4.7 and 4.6 to verify its hypothesis. In Proposition 8.1, we will summarize the smoothness properties of the obtained fixed point map $\mathcal{F}$. Note that for $\mathcal{K} = 0$ the vector $\boldsymbol{y} = 0$ is a fixed point for every $\mathcal{H}$. Thus

$$\mathcal{F}(0,\mathcal{H}) = 0. \tag{4.105}$$

Taking now for $\mathcal{F}_1$ and $\mathcal{F}_{2N}$ the first and last component of $\mathcal{F}$, corresponding to H_0 and K_N, the equality (4.93) readily follows from the definition of $\mathcal{F}$.

Now we can easily construct the map $\mathcal{H}$. The condition (4.105) and the differentiability of $\mathcal{F}$ (see Proposition 8.1) imply that

$$D_{\mathcal{H}}\mathcal{F}_1(0,0) = 0. \tag{4.106}$$

Thus we can apply the implicit function theorem in the space C^m_* to get the following result.

THEOREM 4.9. *Let $2m+3 \le r_0$. There exist constants $\rho_1, \rho_2 > 0$, and a parameter $\zeta > 0$ in the definition of the norm on the space $\boldsymbol{E}$ introduced in* (2.21) *such that there exists a C^m_*-map $\mathcal{H}\colon B_{\boldsymbol{E}}(\rho_1) \to B_{\boldsymbol{M}_0}(\rho_2)$ satisfying the fixed point equations*

$$\mathcal{F}_1(\mathcal{K}, \mathcal{H}(\mathcal{K}))) = \mathcal{H}(\mathcal{K}) \tag{4.107}$$

and

$$\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K},\mathcal{H}(\mathcal{K})), \mathcal{K}, \mathcal{H}(\mathcal{K})) = \mathcal{F}(\mathcal{K},\mathcal{H}(\mathcal{K})) \tag{4.108}$$

for all $\mathcal{K} \in B_{\boldsymbol{E}}(\rho_1)$. Moreover, the C^m_*- norm of the map $\mathcal{H}$ is bounded uniformly in N. We may choose $\rho_2 < \frac{1}{4}h^2$. Then in view of (4.44) *the matrix $\boldsymbol{q} = \pi_2 \circ \mathcal{H}(\mathcal{K})$ of the quadratic part of $\mathcal{H}(\mathcal{K})$ satisfies $|\boldsymbol{q}| < \frac{1}{2}$.*

4.6. Proof of strict convexity—Theorem 2.1

We are following the strategy outlined in Chapter 3, but we now consider the full ideal Hamiltonian $\mathcal{H}$ in (4.91) and not just the quadratic part. To prove the strict convexity of the surface tension $\sigma_\beta(u)$, we need to prove that its perturbative component $\varsigma(u)$ is smooth in the tilt u. This amounts to obtaining a uniform bound (in $N \in \mathbb{N}$) on the approximation

$$\varsigma_N(u) := -\frac{1}{L^{dN}}\log \mathcal{Z}_N(u) \tag{4.109}$$

with $\mathcal{Z}_N(u)$ defined in (2.17). In view of the equality (4.95), applied with $\mathcal{K} = \mathcal{K}_u$, we have

$$\begin{aligned} \varsigma_N(u) &= -\frac{1}{L^{dN}} \log\Big(\frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}}\Big) - \lambda \\ &+ \frac{1}{L^{dN}} \log\Big(\int_{\boldsymbol{\mathcal{X}}_N} \Big(1 + \mathcal{F}_{2N}(\mathcal{K}_u, \mathcal{H}(\mathcal{K}_u))(\Lambda_N, \varphi)\Big) \mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi)\Big), \end{aligned} \tag{4.110}$$

where, as in (4.96),

$$\lambda = \pi_0(\mathcal{H}(\mathcal{K}_u)) \quad \text{and} \quad \boldsymbol{q} = \pi_2(\mathcal{H}(\mathcal{K}_u)) \tag{4.111}$$

denote the constant term in $\mathcal{H}(\mathcal{K}_u)$ and the coefficient matrix of the qudratic term, respectively.

The proof of strict convexity thus consists of the following three steps.

Step 1: Choose all needed constants according to Propositions 4.6 and 4.7. In particular, we choose (with a fixed d) the constants L, h, A, $\bar\rho = \bar\rho(\mathsf{A})$, and a constant C, so that the claims from Propositions 4.6 and 4.7 (i.e., differentiability and uniform smoothness of the renormalization maps $\boldsymbol{T}_k$ as well as the contractivity of the linearisation) are valid for any $(H, K, \boldsymbol{q}) \in \mathcal{U}_\rho$ (in particular, $\|\boldsymbol{q}\| \le \frac{1}{2}$).

Step 2: Apply Theorem 4.9 to get the existence and smoothness properties of the map $\mathcal{H} : B_{\boldsymbol{E}}(\rho_1) \to B_{\boldsymbol{M}_0}(\rho_2)$.

Step 3: Finally, address the dependence of $\mathcal{K}_u$ on the tilt u: according to the assumptions of Theorem 2.1 we have a C^3 tilt map τ, $u \mapsto \tau(u) = \mathcal{K}_u$. Choosing δ sufficiently small, we have $\tau(B_\delta(0)) \subset B_{\boldsymbol{E}}(\rho) \subset \boldsymbol{E}$.

Having this in mind, we show that the right hand side of (4.110) is three times continuously differentiable in u with bounded derivatives, by analysing each of the three terms separately.

The first term on the right hand side of (4.110) can easily be computed as

$$-\log\Big(\frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}}\Big) = \tfrac{1}{2} \log\det\big(\mathcal{A}^{(\boldsymbol{q})} \mathcal{C}^{(0)}\big). \tag{4.112}$$

Consider the dual torus

$$\widehat{\mathbb{T}}_N = \Big\{ p = (p_1, \dots, p_d) \colon p_i \in \big\{ -\tfrac{(L^N-1)\pi}{L^N}, -\tfrac{(L^N-3)\pi}{L^N}, \dots, \tfrac{(L^N-1)\pi}{L^N} \big\}, i = 1, \dots, d \Big\}, \tag{4.113}$$

and the functions $f_p(x) = \mathrm{e}^{i\langle p, x\rangle}$. The family $\big\{ |\Lambda_N|^{-1/2} f_p \big\}_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}}$ is an orthonormal basis of $\boldsymbol{\mathcal{V}}_N$. The eigenvalues of $\mathcal{A}^{(\boldsymbol{q})}$ are

$$\sigma(p) = \langle q^{(p)}, (\mathbb{1} + \boldsymbol{q}) q^{(p)} \rangle = \sum_{l,j=1}^{d} q_l^{(p)} \big(\delta_{l,j} + \boldsymbol{q}_{l,j}\big) q_j^{(p)}, \ p \in \widehat{\mathbb{T}}_N \tag{4.114}$$

with $q_j^{(p)} = \mathrm{e}^{ip_j} - 1$, $j = 1, \dots, d$. Note that $q_l^{(p)} q_j^{(p)} \approx p_l p_j$. The eigenvalues for $\mathcal{A}^{(0)}$ and $\mathcal{C}^{(0)}$ are $\langle q^{(p)}, q^{(p)} \rangle \approx \|p\|^2$ and $\langle q^{(p)}, q^{(p)} \rangle^{-1} \approx \|p\|^{-2}, p \in \widehat{\mathbb{T}}_N$, respectively. We get

$$\log\det\big(\mathcal{A}^{(\boldsymbol{q})} \mathcal{C}^{(0)}\big) = \operatorname{Tr}\log\big(\mathbb{1} + \mathcal{A}^{(\boldsymbol{q})} \mathcal{C}^{(0)}\big) = \sum_{p \in \widehat{\mathbb{T}}_N \setminus \{0\}} \log\Big(1 + \frac{\langle q^{(p)}, \boldsymbol{q} q^{(p)} \rangle}{\langle q^{(p)}, q^{(p)} \rangle}\Big). \tag{4.115}$$

Since the sum over the torus has $L^{dN}-1$ terms it follows that

$$-\frac{1}{L^{dN}}\log\Big(\frac{Z_N^{(\boldsymbol{q})}}{Z_N^{(0)}}\Big)$$

is a smooth function of $\boldsymbol{q}$ with derivatives bounded uniformly in N. Thus

$$u\mapsto -\frac{1}{L^{dN}}\log\Big(\frac{Z_N^{\pi_2(\mathcal{H}(\mathcal{K}_u))}}{Z_N^{(0)}}\Big)$$

is a C_*^3 mapping with uniformly bounded derivatives. Note that the chain rule initially states that this map is C_*^3, but $\mathbb{R}^d$ being a finite dimensional vector space it is actually a C^3 mapping according to Proposition D.17.

As regards the second term we know from Theorem 4.9 and the chain rule that $u\mapsto\mathcal{H}(\mathcal{K}_u)$ is C_*^3. Thus the map $u\mapsto\lambda=\pi_0(\mathcal{H}(\mathcal{K}_u))$ is C_*^3 and hence C^3 because the map is defined a neighbourhood in the finite dimensional space $\mathbb{R}^d$.

Regarding the last term

$$\log\Big(\int_{\boldsymbol{\mathcal{X}}_N}\Big(1+\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))(\Lambda_N,\varphi)\Big)\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi)\Big)$$

we first note that for a positive function G the k-th derivative of $\log G$ is a polynomial in $\frac{1}{G}$ and the first k derivatives of G. Since $\mu_{N+1}^{(\boldsymbol{q})}$ is a probability measure, it suffices to show that

$$\Big|\int_{\boldsymbol{\mathcal{X}}_N}\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))(\Lambda_N,\varphi)\,\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi)\Big|\le\frac{1}{2}. \tag{4.116}$$

and to estimate the derivatives of the integral. We thus need to estimate

$$T(u):=\int_{\boldsymbol{\mathcal{X}}_N}\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))(\Lambda_N,\varphi)\,\mu_{N+1}^{(\boldsymbol{q})}(\mathrm{d}\varphi),\quad\text{where }\boldsymbol{q}=\pi_2(\mathcal{H}(\mathcal{K}_u)), \tag{4.117}$$

and its derivatives with respect to u. The integral in (4.117) is exactly the application of the renormalisation map R_1, defined in (6.16), evaluated at zero:

$$T(u)=(R_1^{(\boldsymbol{q})}P)(\Lambda_N,0)\quad\text{where }P=\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))\text{ and }\boldsymbol{q}=\pi_2(\mathcal{H}(\mathcal{K}_u)).$$

Thus we can apply the estimates for R_1 stated in Lemma 6.5 and in Lemma 5.1 (iv). We introduce the notation

$$\widetilde{R}_1(K,\mathcal{H}):=(R_1^{(\boldsymbol{q})}K)(\Lambda_N,0)=R_1(K,q)(\Lambda_N,0).$$

It will later be convenient to view $\tilde{R}_1$ as a function of K and $\mathcal{H}$ even thus it depends on $\mathcal{H}$ only through $\boldsymbol{q}=\pi_2(\mathcal{H})$. We get

$$T(u)=\widetilde{R}_1\Big(\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u)),\mathcal{H}(\mathcal{K}_u)\Big).$$

Now by Lemma 5.1 (iv) (note that there is only one N-block), Proposition 8.1, the definition (4.98) of the norm on $\mathcal{F}$, Theorem 4.9 and the assumptions on $\mathcal{K}_u$ in Theorem 2.1 we get

$$|T(u)|\le\|\mathcal{F}_{2N}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))\|\le 2\frac{\eta^N}{\alpha}\|\mathcal{F}(\mathcal{K}_u,\mathcal{H}(\mathcal{K}_u))\|_{\boldsymbol{Y}_0}\le C\frac{\eta^N}{\alpha}.$$

Thus (4.116) holds if N is large enough (note that α and C are independent of N).

To verify the differentiability of T we recall the notation

$$(F\diamond G)(x,\mathcal{H})=F(G(x,\mathcal{H}),\mathcal{H})$$

to rewrite $T(u)$ as

$$T(u) = \big(\widetilde{R}_1 \diamond \mathcal{F}_{2N}\big)(\mathcal{K}_u, \mathcal{H}(\mathcal{K}_u))$$

Now by Proposition 8.1 we have $\mathcal{F}_{2N} \in \widetilde{C}^m(B_{\boldsymbol{X}\times\boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2), \boldsymbol{Y})$ with bounds on the derivatives which are independent of N. Here $\boldsymbol{Y} = \boldsymbol{Y}_{r_0} \hookrightarrow \boldsymbol{Y}_{r_0-2} \hookrightarrow \ldots \hookrightarrow \boldsymbol{Y}_{r_0-2m}$ and in the domain we use the trivial scale $\boldsymbol{X}_m = \ldots = \boldsymbol{X}_0 = \boldsymbol{E}$.

By Lemma 6.5 we have $\widetilde{R}_1 \in \widetilde{C}^m(\boldsymbol{Y} \times B_{\widehat{\rho}_2}, \mathbb{R})$ (as long as $\widehat{\rho}_2 < \frac{1}{4}h^2$), again with bounds on the derivatives which are independent of N. Thus the chain rule with loss of regularity, Theorem D.29, shows that $\widetilde{R}_1 \diamond \mathcal{F}_{2N} \in \widetilde{C}^m(B_{\boldsymbol{X}\times M_0}(\widehat{\rho}_1, \widehat{\rho}_2), \mathbb{R})$ with uniformly bounded derivatives. Since the scale $\boldsymbol{X}_m = \ldots = \boldsymbol{X}_0 = \boldsymbol{E}$ is trivial (and since the target is just $\mathbb{R}$) this implies that $\widetilde{R}_1 \diamond \mathcal{F}_{2N} \in C^m_*(B_{\boldsymbol{X}\times M_0}(\widehat{\rho}_1, \widehat{\rho}_2), \mathbb{R})$ Together with the regularity of $\mathcal{H}$ (see Theorem 4.9) and the assumptions on $\mathcal{K}_u$ in Theorem 2.1 we get $T \in C^3_*(B(\delta_0))$ with uniformly bounded derivatives. Since $B(\delta_0) \subset \mathbb{R}^d$ by Proposition D.17 this is the same as $T \in C^3(B(\delta_0))$. $\square$

CHAPTER 5

Properties of the Norms

As a preparation for the proof of Propositions 4.7 and 4.6, we first address the factorisation properties of the norms defined in Chapter 4.2 and prove a bound on the integration map $\boldsymbol{R}_k$ defined in (4.11). Recalling that the norms $\|\cdot\|_{k,X,r}$ depend on parameters L, h, and ω, we summarise their properties in the following lemma. Using $\eta(n,d)$ defined by (4.4), we introduce $\kappa(d) := \frac{1}{2}\big(d + \eta(2\lfloor\frac{d+2}{2}\rfloor + 8, d)\big)$ with $\lfloor t\rfloor$ denoting the integer value of t. Notice that $\kappa(d) \le d^2/2 + 5d + 16$.

LEMMA 5.1. *Let $\omega \ge 1 + 18\sqrt{2}$, $N \in \mathbb{N}$, $N \ge 1$, and $L \in \mathbb{N}$ odd, $L \ge 3$. Given $k \in \{0,\dots,N-1\}$, let $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ factor (at the scale k), and let $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$. Then, the norms $\|\cdot\|_{k,X,r}, \|\cdot\|_{k:k+1,X,r}, r \in \{1,\dots,r_0\}$, and $|||\cdot|||_{k,X}, X \in \mathcal{P}_k$, satisfy the following conditions:*

- *(i)* $\|K(X)\|_{k,X,r} \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k,Y,r}$ *and* $\|K(X)\|_{k:k+1,X,r} \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k:k+1,Y,r}$,
- *(iia)* $\|F^X K(Y)\|_{k,X\cup Y,r} \le \|K(Y)\|_{k,Y,r} |||F|||_k^{|X|_k}$ *as well as*
- *(iib)* $\|F^X K(Y)\|_{k:k+1,X\cup Y,r} \le \|K(Y)\|_{k:k+1,Y,r} |||F|||_k^{|X|_k}$ *for $X, Y \in \mathcal{P}_k$ disjoint,*
- *(iii)* $|||\mathbb{1}(B)|||_{k,B} = 1$ *for $B \in \mathcal{B}_k$,*
- *(iv) There exists a constant $h_1 = h_1(d,\omega)$ depending only on the dimension d and value of the parameter ω, such that for any $h \ge L^{\kappa(d)} h_1$ and $X \in \mathcal{P}_k$, we have* $\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r} \le 2^{|X|_k} \|K(X)\|_{k,X,r}$.

PROOF.
(i) Notice first that for any $F_1, F_2 \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$ and any (not necessarily disjoint) $X_1, X_2 \in \mathcal{P}_k$, we have

$$|F_1(X_1)(\varphi)F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r} \le |F_1(X_1)(\varphi)|^{k,X_1,r} |F_2(X_2)(\varphi)|^{k,X_2,r}. \tag{5.1}$$

Indeed, using the definition of the norm $|\cdot|^{k,X,r}$ and fact that a Taylor expansion of a product is the product of Taylor expansions, we have

$$|F_1(X_1)(\varphi)F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r} \le |F_1(X_1)(\varphi)|^{k,X_1\cup X_2,r} |F_2(X_2)(\varphi)|^{k,X_1\cup X_2,r}. \tag{5.2}$$

Observing now that for any $\dot\varphi \in \boldsymbol{\mathcal{X}}_N$ we have $|\dot\varphi|_{k,X_1} \le |\dot\varphi|_{k,X_1\cup X_2}$, we get

$$\sup_{|\dot\varphi|_{k,X_1\cup X_2}\le 1} |D^s F_1(X_1)(\varphi)(\dot\varphi,\dots,\dot\varphi)| \le \sup_{|\dot\varphi|_{k,X_1}\le 1} |D^s F_1(X_1)(\varphi)(\dot\varphi,\dots,\dot\varphi)|, \tag{5.3}$$

implying

$$|F_1(X_1)(\varphi)|^{k,X_1\cup X_2,r} \le |F_1(X_1)(\varphi)|^{k,X_1,r} \tag{5.4}$$

and similarly for F_2, yielding thus (5.1).

Iterating (5.1) we can use it for $K(X,\varphi)=\prod_{Y\in\mathcal{C}(X)}K(Y)(\varphi)$, yielding

$$|K(X,\varphi)|^{k,X,r}\le\prod_{Y\in\mathcal{C}(X)}|K(Y)(\varphi)|^{k,Y,r}\tag{5.5}$$

and, similarly,

$$|K(X,\varphi)|^{k+1,X,r}\le\prod_{Y\in\mathcal{C}(X)}|K(Y)(\varphi)|^{k+1,Y,r}\tag{5.6}$$

To conclude, it then suffices to observe that

$$w_k^X(\varphi)=\prod_{Y\in\mathcal{C}(X)}w_k^Y(\varphi)\quad\text{and}\quad w_{k:k+1}^X(\varphi)=\prod_{Y\in\mathcal{C}(X)}w_{k:k+1}^Y(\varphi).\tag{5.7}$$

Here, in both cases, we use the fact that the partition $X=\cup_{Y\in\mathcal{C}(X)}Y$ splits both X and its boundary ∂X into disjoint components: $Y_1,Y_2\in\mathcal{C}(X)$, $Y_1\neq Y_2$ implies that $\operatorname{dist}(Y_1,Y_2)>L^k$ and thus $Y_1\cap Y_2=\varnothing$, $\partial Y_1\cap\partial Y_2=\varnothing$, and $\partial X=\cup_{Y\in\mathcal{C}(X)}\partial Y$.
(iia) Using (iterated) (5.1) for $\prod_{B\in\mathcal{B}_k(X)}F(B)(\varphi)K(Y)(\varphi)$, we have

$$\big|\big(F^XK(Y)\big)(\varphi)\big|^{k,X\cup Y,r}\le\prod_{B\in\mathcal{B}_k(X)}|F(B)(\varphi)|^{k,B,r}|K(Y)(\varphi)|^{k,Y,r}.\tag{5.8}$$

Bounding the right hand side by

$$\prod_{B\in\mathcal{B}_k(X)}|||F(B)|||_{k,B}\|K(Y)\|_{k,Y,r}\prod_{B\in\mathcal{B}_k(X)}W_k^B(\varphi)w_k^Y(\varphi),\tag{5.9}$$

we get (ii) once we verify that

$$\prod_{B\in\mathcal{B}_k(X)}W_k^B(\varphi)w_k^Y(\varphi)\le w_k^{X\cup Y}(\varphi).\tag{5.10}$$

Inserting the definitions of the strong and weak weight functions, (5.10) is satisfied once

$$L^k\sum_{x\in\partial Y}G_{k,x}(\varphi)\le\sum_{x\in X}\big(2^d\omega g_{k,x}(\varphi)+(\omega-1)G_{k,x}(\varphi)\big)+L^k\sum_{x\in\partial(X\cup Y)}G_{k,x}(\varphi).\tag{5.11}$$

To verify this, it suffices to notice that each $y\in\partial Y\setminus\partial(X\cup Y)$ is necessarily contained in ∂B for some $B\in\mathcal{B}_k(X)$ (a block on the boundary of X touching Y). Thus, it suffices to show that for each such B one has

$$L^k\sum_{x\in\partial B}G_{k,x}(\varphi)\le\sum_{x\in B}\big(2^d\omega g_{k,x}(\varphi)+(\omega-1)G_{k,x}(\varphi)\big).\tag{5.12}$$

Indeed, applying Proposition B.5 (a), we have

$$\begin{aligned}&h^2L^k\sum_{x\in\partial B}G_{k,x}(\varphi)\le\\&\le 2\mathfrak{c}\Big(\sum_{x\in B}|\nabla\varphi(x)|^2+L^{2k}\sum_{x\in U_1(B)}|\nabla^2\varphi(x)|^2\Big)+L^k\sum_{x\in\partial B}\sum_{s=2}^{3}L^{(2s-2)k}|\nabla^s\varphi(x)|^2\le\\&\le h^2 2\mathfrak{c}\sum_{x\in B}G_{k,x}(\varphi)+h^2 2\mathfrak{c}L^k\sum_{z\in\partial B}g_{k,z}(\varphi),\end{aligned}\tag{5.13}$$

where z is any point $z \in B$. Observing that the size of the set ∂B is at most $(L^k+2)^d - (L^k-2)^d \le 2^d L^{(d-1)k}$ once $2 \le L$, we get the seeked bound once

$$2\mathfrak{c} \le \omega - 1. \tag{5.14}$$

Observing that $\mathfrak{c} < 3\sqrt{2}$, this condition is satisfied with our choice of ω.
(iib) The proof is similar, with (5.11) replaced by

$$\begin{aligned} 3L^k \sum_{x\in\partial Y} G_{k,x}(\varphi) \le \sum_{x\in X} &\big((2^d\omega - 1) g_{k:k+1,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big) \\ &+ 3L^k \sum_{x\in\partial(X\cup Y)} G_{k,x}(\varphi) \end{aligned} \tag{5.15}$$

that, in its turn, needs (5.12) in a slightly stronger version,

$$3L^k \sum_{x\in\partial B} G_{k,x}(\varphi) \le \sum_{x\in B} \big((2^d\omega - 1) g_{k:k+1,x}(\varphi) + (\omega-1) G_{k,x}(\varphi)\big). \tag{5.16}$$

This is satisfied once

$$6\mathfrak{c} \le \omega - 1. \tag{5.17}$$

(iii) follows immediately from the definition.
(iv) Since convolution commutes with differentiation we have

$$D^s \int K(\varphi+\xi)\mu_{k+1}(\mathrm{d}\xi) = \int D^s K(\varphi+\xi)\mu_{k+1}(\mathrm{d}\xi). \tag{5.18}$$

For a vector $(A_0, A_1, \dots, A_r)$ consisting of $A_0 \in \mathbb{R}$ and multilinear symmetric maps $A_s\colon \boldsymbol{\mathcal{X}}^{\otimes s} \to \mathbb{R}, s \in \mathbb{N}$, we consider the norm

$$|(A_0, \dots, A_r)| := \sum_{s=0}^r \frac{1}{s!} |A_s|^{k+1,X} \tag{5.19}$$

with $|A_s|^{k+1,X}$ defined by (4.24). Then

$$|K(\varphi), DK(\varphi), \dots, D^r K(\varphi))| = |K(\varphi)|^{k+1,X,r}.$$

Now fix φ and apply Jensen's inequality to map $\xi \mapsto (K(\varphi+\xi), \dots, D^r K(\varphi+\xi))$. This yields

$$\Big| \int K(\varphi+\xi)\mu_{k+1}(\mathrm{d}\xi) \Big|^{k+1,X,r} = \int |K(\varphi+\xi)|^{k+1,X,r} \mu_{k+1}(\mathrm{d}\xi). \tag{5.20}$$

Since

$$|\dot\varphi|_{k,X} \le L^{-\frac{d}{2}} |\dot\varphi|_{k+1,X}, \tag{5.21}$$

we also have

$$|K(X,\varphi+\xi)|^{k+1,X,r} \le |K(X,\varphi+\xi)|^{k,X}. \tag{5.22}$$

As a result,

$$\|(\boldsymbol{R}_{k+1}K)(X)\|_{k:k+1,X,r} \le \sup_\varphi \int |K(X,\varphi+\xi)|^{k,X,r} \mu_{k+1}(\mathrm{d}\xi) w_{k:k+1}^{-X}(\varphi). \tag{5.23}$$

Estimating the integrand $|K(X,\varphi+\xi)|^{k,X,r}$ from above by

$$\|K(X)\|_{k,X,r} w_k^X(\varphi+\xi),$$

the proof of the needed bound amounts to showing that

$$\int_{\boldsymbol{\mathcal{X}}_N} w_k^X(\varphi+\xi)\mu_{k+1}(\mathrm{d}\xi) \le 2^{|X|} w_{k:k+1}^X(\varphi). \tag{5.24}$$

As this result will be used also later in different circumstances, we state it as a separate Lemma.

Lemma 5.2. *Let $\omega \ge 1+6\sqrt{2}$. There exists a constant $h_1 = h_1(d,\omega)$ such that for any $N \ge 1$, L odd, $L \ge 5$, $h \ge L^{\kappa(d)} h_1$, $k \in \{0,\dots,N-1\}$, $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, and any $X \in \mathcal{P}_k$, we have*

$$\int_{\boldsymbol{\mathcal{X}}_N} w_k^X(\varphi+\xi)\mu_{k+1}(\mathrm{d}\xi) \le 2^{|X|_k} w_{k:k+1}^X(\varphi). \tag{5.25}$$

Proof. We will prove the bound (5.25) in three steps:

Step 1. Expanding the terms $(\nabla\varphi(x)+\nabla\xi(x))^2$ in $\sum_{x\in X} G_{k,x}(\varphi+\xi)$ and using the Cauchy's inequality $(a+b)^2 \le 2a^2+2b^2$ for the remaining terms (those that are preceded by a power in L that allows to absorb the resulting prefactors while passing to the next scale), we have

$$\begin{aligned} h^2\sum_{x\in X} G_{k,x}(\varphi+\xi) \le \sum_{x\in X}\big(|\nabla\varphi(x)|^2+|\nabla\xi(x)|^2\big) + 2\Big|\sum_{x\in X}\nabla\varphi(x)\nabla\xi(x)\Big| + \\ + 2\sum_{x\in X}\Big(L^{2k}|\nabla^2\varphi(x)|^2 + L^{2k}|\nabla^2\xi(x)|^2 + L^{4k}|\nabla^3\varphi(x)|^2 + L^{4k}|\nabla^3\xi(x)|^2\Big). \end{aligned} \tag{5.26}$$

For the remaining terms occurring in $w_k^X(\varphi+\xi)$, we simply write (again by Cauchy's inequality)

$$g_{k,x}(\varphi+\xi) \le 2g_{k,x}(\varphi) + 2g_{k,x}(\xi) \tag{5.27}$$

and

$$L^k G_{k,x}(\varphi+\xi) \le 2L^k G_{k,x}(\varphi) + 2L^k G_{k,x}(\xi). \tag{5.28}$$

Step 2. In view of Proposition B.6, we bound the mixed term $2\big|\sum_{x\in X}\nabla\varphi(x)\nabla\xi(x)\big|$ by

$$L^{2k}\sum_{x\in X\cup\partial^- X}|\nabla^2\varphi(x)|^2 + L^k\sum_{x\in\partial^- X}|\nabla\varphi(x)|^2 + \frac{1+\mathfrak{c}d}{L^{2k}}\sum_{x\in X\cup\partial^- X}\xi(x)^2 + \mathfrak{c}\sum_{x\in X}|\nabla\xi(x)|^2. \tag{5.29}$$

The sum over X in the first term above will be estimated by the regulator $g_{k:k+1,x}(\varphi)$ of the next generation. Namely, combining, for any $x \in X$, its terms with the corresponding φ-terms on the second line in (5.26), we have

$$\begin{aligned} 3L^{2k}|\nabla^2\varphi(x)|^2 + 2L^{4k}|\nabla^3\varphi(x)|^2 \le \\ \le 3L^{-2}L^{2(k+1)}|\nabla^2\varphi(x)|^2 + 2L^{-4}L^{4(k+1)}|\nabla^3\varphi(x)|^2 \le 3L^{-2}h^2 g_{k:k+1,x}(\varphi), \end{aligned} \tag{5.30}$$

where we are assuming that

$$2L^{-2} \le 3. \tag{5.31}$$

The remaining sum over $\partial^- X \setminus X$, together with the second term in (5.29), will be absorbed into the sum $\sum_{x\in\partial X} G_{k,x}(\varphi)$. Collecting now all the φ-terms in $\log w_k(\varphi+\xi)$ with expanded mixed term, we get the bound

$$\sum_{x\in X} 2^{d+1}\omega g_{k,x}(\varphi) + \sum_{x\in X}\omega G_{k,x}(\varphi) + 3\omega L^{-2}\sum_{x\in X} g_{k:k+1,x}(\varphi) + 3L^k \sum_{x\in\partial X} G_{k,x}(\varphi). \tag{5.32}$$

This is bounded by

$$\log w^X_{k:k+1}(\varphi) = \sum_{x\in X}\big((2^d\omega-1)g_{k:k+1,x}(\varphi) + \omega G_{k,x}(\varphi)\big) + 3L^k\sum_{x\in\partial X} G_{k,x}(\varphi) \tag{5.33}$$

once

$$(3+2^{d+1})\omega \le (2^d\omega - 1)L^2. \tag{5.34}$$

This condition, including also (5.31), are satisfied once $L \ge 5$.

Turning now to the ξ-terms in $h^2\log w_k(\varphi+\xi)$ with expanded mixed term, we get the bound

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) + \sum_{x\in X}\omega\big((1+\mathfrak{c})|\nabla\xi(x)|^2 + 2L^{2k}|\nabla^2\xi(x)|^2 + 2L^{4k}|\nabla^3\xi(x)|^2\big) + \\ + \omega(1+\mathfrak{c}d)L^{-2k}\sum_{x\in X\cup\partial^- X}\xi(x)^2 + 2L^k\sum_{x\in\partial X} h^2 G_{k,x}(\xi). \tag{5.35}$$

Bounding the last term with the help of Proposition B.5, we get

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) + \sum_{x\in U_1(X)}\big(\omega(1+\mathfrak{c}d)L^{-2k}\xi(x)^2 + (\omega(1+\mathfrak{c})+4\mathfrak{c})|\nabla\xi(x)|^2 + \\ + (2\omega+8\mathfrak{c})L^{2k}|\nabla^2\xi(x)|^2 + (2\omega+8\mathfrak{c})L^{4k}|\nabla^3\xi(x)|^2 + 4\mathfrak{c}L^{6k}|\nabla^4\xi(x)|^2\big). \tag{5.36}$$

Finally, the term $g_{k,x}(\xi)$ containing l_∞-norm of $\nabla^s\xi$, $s=2,3,4$, is bounded with the help of the Sobolev inequality from Proposition A.1. Taking B^* for the B_n with $n=(2^{d+1}-1)L^k$, we get

$$\|\nabla^s\xi\|^2_{l_\infty(B^*)} \le \mathfrak{C}^2(2^{d+1}-1)^2\frac{1}{L^{kd}}\sum_{l=0}^{\widetilde{M}} L^{2lk}\sum_{x\in B^*}|\nabla^l\nabla^s\xi|^2(x), \tag{5.37}$$

where $\widetilde{M} = \lfloor\frac{d+2}{2}\rfloor$ is the integer value of $\frac{d+2}{2}$ and in computing the pre-factor we took into account that $2\lfloor\frac{d+2}{2}\rfloor - d \le 2$. Notice that the constant $\mathfrak{C}$ depends (also through $\widetilde{M}$) only on the dimension d. As a result, we are getting

$$\sum_{x\in X} h^2 2^{d+1}\omega g_{k,x}(\xi) \le \\ \le 2^{d+1}\omega\sum_{x\in X}\sum_{s=2}^{4} L^{(2s-2)k}\mathfrak{C}^2(2^{d+1}-1)^2\frac{1}{L^{kd}}\sum_{l=0}^{M} L^{2lk}\sum_{y\in B^*_x}|\nabla^l\nabla^s\xi|^2(x) \le \\ \le 2^{d+1}\omega 2^{d+1}\mathfrak{C}^2(2^{d+1}-1)^{d+2}3L^{-2k}\sum_{l=2}^{M+4} L^{2lk}\sum_{y\in X^*}|\nabla^l\xi|^2(x), \tag{5.38}$$

where in the last inequality we took into account that each point $y \in X^*$ may accur in B_x^* for at most $(2^{d+1}-1)^d L^{dk}$ points $x \in X$.

Summarising, under the conditions (5.31), (5.34), we have

$$w_k^X(\varphi+\xi) \le w_{k:k+1}^X(\varphi) \exp\Big(h^{-2}\frac{\overline{C}}{L^{2k}} \sum_{x\in X^*} \sum_{l=0}^{M+4} L^{2lk} |\nabla^l \xi(x)|^2\Big) \tag{5.39}$$

with the constant

$$\overline{C} = \max\{\omega(1+\mathfrak{c}d), \omega(1+\mathfrak{c})+4\mathfrak{c}, 2(\omega+8\mathfrak{c}) + 32^{d+1}\omega\mathfrak{C}^2(2^{d+1}-1)^{d+2}\} \tag{5.40}$$

that depends, afters ω is chosen, only on the dimension d.

Step 3. We first bound the term in ξ in (5.39) by a smooth Gaussian and then bound the remaining integral. Let η_{X^*} be a smooth cut-off function such that $\operatorname{supp} \eta_{X^*} \subset (X^*)^*, \eta_{X^*} = 1$ on X^*, and

$$\big|\nabla^l \eta_{X^*}\big| \le \Theta L^{-lk}. \tag{5.41}$$

Then the bound in (5.39) implies taht

$$w_k^X(\varphi+\xi) \le w_{k:k+1}^X(\varphi) \exp\big(\frac{1}{2}\varkappa(\mathcal{B}_k\xi,\xi)\big), \tag{5.42}$$

where $\varkappa = 2\overline{C}h^{-2}$ and

$$(\mathcal{B}_k\xi,\xi) = \frac{1}{L^{2k}} \sum_{x\in\Lambda_N} \sum_{l=0}^{M+4} L^{2lk} \big|\eta_{X^*}(x)(\nabla^l\xi)(x)\big|^2. \tag{5.43}$$

Explicitly,

$$\mathcal{B}_k = \mathcal{B}_k^{(0)} + \sum_{l=1}^{M+4} \mathcal{B}_k^{(l)} \tag{5.44}$$

with

$$\mathcal{B}_k^{(l)}\xi = \frac{1}{L^{2k}}(\nabla^l)^*\eta_{X^*}^2\nabla^l\xi,\ l=1,\dots,\widetilde{M}+4, \quad\text{and}\quad \mathcal{B}_k^{(0)}\xi = \frac{1}{L^{2k}}\Pi(\eta_{X^*}^2\xi), \tag{5.45}$$

where $\Pi\colon \boldsymbol{\mathcal{V}}_N \to \boldsymbol{\mathcal{X}}_N$ is the projection $(\Pi\varphi)(x) = \varphi(x) - \frac{1}{|\Lambda_N|}\sum_{y\in\Lambda_N}\varphi(y)$ (for $l \ge 1$ the projection is not needed since $(1, \nabla_i^*\varphi) = (\nabla_i 1, \varphi) = 0$).

It remains only to show that

$$\int_{\boldsymbol{\mathcal{X}}_N} \exp\big(\frac{1}{2}\varkappa(\mathcal{B}_k\xi,\xi)\big)\mu_{k+1}(\mathrm{d}\xi) \le 2^{|X|}.$$

A formal Gaussian calculation with respect to the measure μ_{k+1} with the covariance operator $\mathcal{C}_{k+1}$ yields

$$\begin{aligned}\int_{\boldsymbol{\mathcal{X}}_N} \exp\big(\frac{1}{2}\varkappa(\mathcal{B}_k\xi,\xi)\big)\mu_{k+1}(\mathrm{d}\xi) &= \Big(\frac{\det(\mathcal{C}_{k+1}^{-1} - \varkappa\mathcal{B}_k)}{\det(\mathcal{C}_{k+1}^{-1})}\Big)^{-\frac{1}{2}} \\ &= \det\Big(\mathsf{I} - \varkappa\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}\Big)^{-\frac{1}{2}}.\end{aligned} \tag{5.46}$$

To justify this calculation we will derive a bound on the spectrum $\sigma(\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}})$ in the following lemma.

Lemma 5.3. *Using the shorthand $\eta(d) := \eta(2\lfloor\frac{d+2}{2}\rfloor + 8, d) = 2\kappa(d) - d$, we have:*

(i) The operators $\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}$ *are symmetric and positive definite.*

There exist constants M_0 *and* M_1 *that depend only on the dimension* d *such that for any* N *and any* $k = 1,\ldots,N$,

(ii) $\sup\sigma(\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}) \leq M_0 L^{d+\eta(d)}$ *and*

(iii) $\mathrm{Tr}\Big(\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}\Big) \leq M_1|X|_k L^{\eta(d)}$.

Postponing momentarily the proof of the Lemma, we first observe that $\varkappa < \frac{1}{2M_0L^{d+\eta(d)}}$ with $h \geq L^{\kappa(d)}4\overline{C}M_0$, and thus the eigenvalues λ_j, $j = 1,\ldots,L^{Nd}-1$ of $\varkappa\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}$ lie between 0 and $\frac{1}{2}$. The formal Gaussian calculation is then justified and

(5.47)
$$\log\det\Big(\mathsf{I}-\varkappa\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}\Big) \geq \sum_i \log(1-\lambda_i) \geq \sum_i -2\lambda_i = -2\mathrm{Tr}\Big(\varkappa\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}\Big)$$
$$\geq -2M_1L^{\eta(d)}\varkappa|X|_k = -4\overline{C}M_1L^{\eta(d)}h^{-2}|X|_k.$$

Hence

$$\det\Big(\mathsf{I}-\varkappa\mathcal{C}_{k+1}^{\frac{1}{2}}\mathcal{B}_k\mathcal{C}_{k+1}^{\frac{1}{2}}\Big)^{-\frac{1}{2}} \leq \mathrm{e}^{\frac{2\overline{C}M_1|X|_k}{h^2}L^{\eta(d)}} \leq \mathrm{e}^{\frac{2\overline{C}M_1|X|_k}{h_1^2}L^{-d}} \tag{5.48}$$

and the Lemma 5.2 follows with

$$h_1(d,\omega)^2 \geq 4\overline{C}\max\big(M_0, \tfrac{M_1}{5^d 2\log 2}\big). \tag{5.49}$$

□

Proof of Lemma 5.3.

The claim (i) follows from definitions.

The estimate (ii) follows from the estimate

$$\|\mathcal{B}_k\mathcal{C}_{k+1}\xi\|_2 \leq M_0L^{d+\eta(d)}\|\xi\|_2 \text{ for all } \xi\in\boldsymbol{\mathcal{X}}_N. \tag{5.50}$$

For $\mathcal{B}_k^{(0)}$, we first observe that

$$L^{2k}\|\mathcal{B}_k^{(0)}\xi\|_2 = \|\Pi(\eta_{X^*})^2\xi\|_2 \leq \|(\eta_{X^*})^2\xi\|_2 \leq \|\xi\|_2. \tag{5.51}$$

In view of Proposition 4.1, the operator $\mathcal{C}_{k+1}$ acts by convolution with respect to the function $\mathcal{C}_{k+1}$. With the bounds (5.51), (4.2), (4.3), and $c_{\max} = \max_{|\boldsymbol{\alpha}|\leq 2(M+4)} c_{\boldsymbol{\alpha},0}$, we have (recall that $\eta(0,d) \leq \eta(2\lfloor\frac{d+2}{2}\rfloor+8,d) = \eta(d)$))

(5.52)
$$\|\mathcal{B}_k^{(0)}\mathcal{C}_{k+1}\xi\|_2 \leq L^{-2k}\|\mathcal{C}_{k+1}\xi\|_2 \leq L^{-2k}\sum_{z\in\Lambda_N}|\mathcal{C}_{k+1}(z)|\|\xi\|_2 \leq c_{\max}L^{d+\eta(d)}\|\xi\|_2.$$

For $\mathcal{B}_k^{(l)}$ we use the discrete product rule

$$\nabla_i(fg) = \nabla_i f\mathsf{S}_i g + \mathsf{S}_i f\nabla_i g, \tag{5.53}$$

where

$$(\mathsf{S}_i f)(x) := \tfrac{1}{2}f(x) + \tfrac{1}{2}f(x+\mathrm{e}_i). \tag{5.54}$$

The operations S_i commute with all discrete derivatives. Using multiindex notation

$$\nabla^{\boldsymbol{\alpha}} := \prod_{i=1}^d \nabla_i^{\alpha_i} \quad\text{and}\quad \mathsf{S}^{\boldsymbol{\alpha}} := \prod_{i=1}^d \mathsf{S}_i^{\alpha_i}, \tag{5.55}$$

we get the Leibniz rule

$$\nabla^{\boldsymbol{\gamma}}(fg) = \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \big(\mathsf{S}^{\boldsymbol{\alpha}} \nabla^{\boldsymbol{\beta}} f\big)\big(\mathsf{S}^{\boldsymbol{\beta}} \nabla^{\boldsymbol{\alpha}} g\big), \tag{5.56}$$

with suitable constants $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Thus

$$\mathcal{B}_k^{(l)} \mathcal{C}_{k+1}\xi = L^{(2l-2)k} \sum_{|\boldsymbol{\gamma}|=l} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \mathsf{S}^{\boldsymbol{\alpha}} (\nabla^{\boldsymbol{\beta}})^* (\eta_{X^*})^2 \mathsf{S}^{\boldsymbol{\beta}} (\nabla^{\boldsymbol{\alpha}})^* \nabla^{\boldsymbol{\gamma}} \mathcal{C}_{k+1}\xi. \tag{5.57}$$

Notice that $\|\mathsf{S}^{\beta}\| = 1$ (with the operator norm induced by l^2 norms on $\mathcal{V}_N$). Further, using (5.41), (4.23), and again (5.56), we have

$$\big|(\nabla^{\boldsymbol{\beta}})^* (\eta_{X^*})^2\big| \le \Theta^2 C_{\max} L^{-k|\boldsymbol{\beta}|} \tag{5.58}$$

with

$$C_{\max} = \sum_{\substack{\boldsymbol{\alpha},\boldsymbol{\beta} \\ |\boldsymbol{\alpha}+\boldsymbol{\beta}|\le M+4}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}}. \tag{5.59}$$

As a result we get, recalling that $l \le \widetilde{M}+4$, where $\widetilde{M} = \lfloor \frac{d+2}{2} \rfloor$, and that

$$\eta(2(\widetilde{M}+4), d) = \eta(d),$$

$$\begin{aligned} \|\mathcal{B}_k^{(l)} \mathcal{C}_{k+1}\| &\le \\ \le L^{(2l-2)k} \sum_{|\boldsymbol{\gamma}|=l} \sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \Theta^2 C_{\max} L^{-k|\boldsymbol{\beta}|} L^{(k+1)d} c_{\max} L^{-k(d-2+|\boldsymbol{\alpha}|+l)} L^{\eta(d)} &\le \\ \le \Theta^2 C_{\max}^2 c_{\max} L^{d+\eta(d)}. \end{aligned} \tag{5.60}$$

This completes the proof of (ii) with $M_0 = \Theta^2 C_{\max}^2 c_{\max}$.

To prove the estimate (iii), we first observe that $\mathcal{C}_k \mathbb{1}_{\Lambda_N} = 0$. Hence $\mathcal{B}_k \mathcal{C}_k$ can be viewed as an operator from $\mathcal{V}_N$ (instead of $\boldsymbol{\mathcal{X}}_N$) to $\mathcal{V}_N$ with the same trace. To compute the trace of $\mathcal{B}_k \mathcal{C}_{k+1}$ we now use the orthonormal basis given by the unit coordinate vectors

$$\mathrm{e}_x(z) = \begin{cases} 1\,, z = x, \\ 0\,, z \ne x. \end{cases} \tag{5.61}$$

According to (5.57), for $l \ge 1$ we get

$$\big|(\mathrm{e}_x, \mathcal{B}_k^{(l)} \mathcal{C}_{k+1} \mathrm{e}_x)\big| = 0 \quad \text{whenever} \quad x \notin (X^*)^*. \tag{5.62}$$

For $x \in (X^*)^*$ we use (5.57) and the bound

$$\sup_z \big|(\nabla^{\boldsymbol{\alpha}})^* \nabla^{\boldsymbol{\gamma}} \mathcal{C}_{k+1}(z)\big| \le c_{\max} L^{-k(d-2+|\boldsymbol{\alpha}|+|\boldsymbol{\gamma}|)} L^{\eta(d)} \tag{5.63}$$

to conclude that

$$\big|(\mathrm{e}_x, \mathcal{B}_k^{(l)} \mathcal{C}_{k+1} \mathrm{e}_x)\big| \le \Theta^2 C_{\max}^2 c_{\max} L^{-kd+\eta(d)} \tag{5.64}$$

and

$$\mathrm{Tr} \mathcal{B}_k^{(l)} \mathcal{C}_{k+1} = \sum_{x \in \Lambda_N} (\mathrm{e}_x, \mathcal{B}_k^{(l)} \mathcal{C}_{k+1} \mathrm{e}_x) \le \Theta^2 C_{\max}^2 c_{\max} 2^{d+2} L^{\eta(d)} |X|_k. \tag{5.65}$$

For $\mathcal{B}_k^{(0)}$, we explicitly express the projection, $\Pi \mathrm{e}_x = \mathrm{e}_x - \mathbb{1}_{\Lambda_N} \frac{1}{|\Lambda_N|}$, yielding

$$L^{2k}(\mathrm{e}_x, \mathcal{B}_k^{(0)} \mathcal{C}_{k+1} \mathrm{e}_x) = (\Pi \mathrm{e}_x, \eta_{X^*}^2 \mathcal{C}_{k+1} \mathrm{e}_x) = \\ = (\mathrm{e}_x, \eta_{X^*}^2 \mathcal{C}_{k+1} \mathrm{e}_x) - (\mathbb{1}_{\Lambda_N} \frac{1}{|\Lambda_N|}, \eta_{X^*}^2 \mathcal{C}_{k+1} \mathrm{e}_x) \\ = \eta_{X^*}^2(x) \mathcal{C}_{k+1}(0) - \frac{1}{|\Lambda_N|} (\mathbb{1}_{\Lambda_N}, \eta_{X^*}^2 \mathcal{C}_{k+1} \mathrm{e}_x). \tag{5.66}$$

Therefore

$$\mathrm{Tr} \mathcal{B}_k^{(0)} \mathcal{C}_{k+1} = \sum_{x \in \Lambda_N} (\mathrm{e}_x, \mathcal{B}_k^{(0)} \mathcal{C}_{k+1} \mathrm{e}_x) = \\ = L^{-2k} \Big(\sum_{x \in \Lambda_N} \eta_{X^*}^2(x) \Big) \mathcal{C}_{k+1}(0) - \frac{1}{|\Lambda_N|} (\mathbb{1}_{\Lambda_N}, \eta_{X^*}^2 \mathcal{C}_{k+1} \mathbb{1}_{\Lambda_N}) \le \\ \le L^{-2k} c_{\max} L^{-k(d-2)} L^{\eta(d)} \sum_{x \in (X^*)^*} 1 \le c_{\max} 2^{d+2} L^{\eta(d)} |X|_k. \tag{5.67}$$

Thus

$$\mathrm{Tr}\Big(\mathcal{B}_k \mathcal{C}_{k+1} \Big) \le C \big| \big(X^*\big)^* \big|_k \le (M+5) \Theta^2 C_{\max}^2 c_{\max} 2^{d+2} L^{\eta(d)} |X|_k.$$

We get the claim (iii) with $M_1 = (\widetilde{M} + 5) \Theta^2 C_{\max}^2 c_{\max} 2^{d+2}$. □

Remark 5.4. Notice that, with the particular values of M_0 and M_1 given above, we can choose h_1 fulfilling (5.49) by taking

$$h_1^2 = \overline{C} (\widetilde{M} + 5) M_0. \tag{5.68}$$

◇

CHAPTER 6

Smoothness

We prove Proposition 4.6 asserting the smoothness of the renormalisation map

$$S : \mathcal{U} \times B_{\frac{1}{2}} \subset (M_0(\mathcal{B}, \boldsymbol{\mathcal{X}}) \times M(\mathcal{P}^{\mathrm{c}}, \boldsymbol{\mathcal{X}})) \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to M((\mathcal{P}')^{\mathrm{c}}, \boldsymbol{\mathcal{X}}) \tag{6.1}$$

on a suitable scale of functions spaces. Here, $\mathcal{B} = \mathcal{B}_k$, $\mathcal{P} = \mathcal{P}_k$, and $\mathcal{P}' = \mathcal{P}_{k+1}$ with k fixed. (Later, when the dependence of the map S on k will be crucial, we will use the notation S_k instead of S.) Let us recall the explicit formula (4.68) for $K_{k+1} = K' = S(H, K, \boldsymbol{q})$,

$$K'(U, \varphi) = \sum_{X \in \mathcal{P}(U)} \chi(X, U) \widetilde{I}^{U \setminus X}(\varphi) \int_{\boldsymbol{\mathcal{X}}} \big(\widetilde{J}(\varphi) \circ P(\varphi + \xi)\big)(X) \mu_{k+1}(\mathrm{d}\xi) \tag{6.2}$$

with $\widetilde{I} = \mathrm{e}^{-\widetilde{H}}$, $\widetilde{J} = 1 - \widetilde{I}$, $P = (I-1) \circ K$, and $I = \mathrm{e}^{-H}$.

It will be useful to split the map S into a composition of a series of maps and to deal with them one by one. To this end, we first recall the notation for relevant normed spaces. In Section 4.4 we have already introduced the sequence of normed spaces $\boldsymbol{M} = \boldsymbol{M}_{r_0} \hookrightarrow \boldsymbol{M}_{r_0-2} \hookrightarrow \ldots \hookrightarrow \boldsymbol{M}_{r_0-2m}$, defined as $\boldsymbol{M}_r = \{K \in M(\mathcal{P}^{\mathrm{c}}, \boldsymbol{\mathcal{X}}) : \|K\|^{(\mathsf{A})}_{k,r} < \infty\}$ and equipped with the norm $\|\cdot\|^{(\mathsf{A})}_{k,r}$, $r = r_0, r_0-2, \ldots, r_0-2m$, the space $\boldsymbol{M}_0 = (M(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), \|\cdot\|_{k,0})$, and the sequence of spaces $\boldsymbol{M}' = \boldsymbol{M}'_{r_0} \hookrightarrow \boldsymbol{M}'_{r_0-2} \hookrightarrow \ldots \hookrightarrow \boldsymbol{M}'_{r_0-2m}$ with $\boldsymbol{M}'_r = \{K \in M(\mathcal{P}^{\mathrm{c}}_{k+1}, \boldsymbol{\mathcal{X}}), \|K\|^{(\mathsf{A})}_{r,k+1} < \infty\}$, equipped with the norm $\|\cdot\|^{(\mathsf{A})}_{k+1,r}$, $r = r_0, r_0 - 2, \ldots, r_0 - 2m$. We also introduce the space $\boldsymbol{M}_{|||} = \{F \in M(\mathcal{B}, \boldsymbol{\mathcal{X}}), |||F|||_k < \infty\}$.

One difficulty is that convolution with the measure μ_{k+1} does not preserve the factorization in connected k-polymers.[1] More precisely, if

$$K(X, \varphi) = \prod_{Y \in \mathcal{C}(X)} K(Y, \varphi)$$

and if

$$RK(X, \varphi) := \int_{\boldsymbol{\mathcal{X}}} K(X, \varphi + \xi) \mu_{k+1}(\mathrm{d}\xi),$$

then in general

$$RK(X, \varphi) \neq \prod_{Y \in \mathcal{C}(X)} RK(Y, \varphi)$$

because the support of the covariance $\mathcal{C}_{k+1}$ has range bounded by $L^{k+1}/2$ but not by $L^k/2$. Thus we cannot only consider functionals defined for connected k-polymers but we need to consider functionals which involve all k-polymers and we define

$$\widehat{\boldsymbol{M}}_r := \{K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}}), \|K\|^{(\mathsf{A},\mathsf{B})}_{k,r} < \infty\}, \tag{6.3}$$

[1] We are grateful to S. Buchholz for pointing this out and for suggesting the use of the norm $\|\cdot\|^{(\mathsf{A},\mathsf{B})}_{k,r}$.

$$\widehat{\boldsymbol{M}}_{:,r} := \{K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}}), \|K\|^{(\mathsf{A},\mathsf{B})}_{k:k+1,r} < \infty\}, \tag{6.4}$$

where

$$\|K\|^{(\mathsf{A},\mathsf{B})}_{k,r} := \sup_{X \in \mathcal{P}_k \setminus \varnothing} \Gamma_{\mathsf{A}}(X) \mathsf{B}^{|\mathcal{C}(X)|} \|K(X)\|_{k,X,r} \tag{6.5}$$

with

$$\Gamma_{\mathsf{A}}(X) := \prod_{Y \in \mathcal{C}(X)} \Gamma_{\mathsf{A}}(Y) \quad \text{for } X \in \mathcal{P} \setminus \varnothing \tag{6.6}$$

and where $\|\cdot\|^{(\mathsf{A},\mathsf{B})}_{k:k+1,r}$ is defined in the same way using $\|K(X)\|_{k:k+1,X,r}$. Note that the definition of the spaces does not depend on the weights $\mathsf{A} > 0$ and $\mathsf{B} > 0$ since there are only finitely many polymers.

The map S will be rewritten as a composition of several partial maps: The exponential map,

$$E\colon \boldsymbol{M}_0 \to \boldsymbol{M}_{|||} \text{ defined by} \tag{6.7}$$

$$E(\widetilde{H}) = \exp\{-\widetilde{H}\} = \widetilde{I}, \tag{6.8}$$

three polynomial maps,

$$P_1\colon \boldsymbol{M}_{|||} \times \boldsymbol{M}_{|||} \times \widehat{\boldsymbol{M}}_{:,r_0} \to \boldsymbol{M}'_{r_0} \text{ defined by} \tag{6.9}$$

$$P_1(\widetilde{I}, \widetilde{J}, \widetilde{P})(U, \varphi) = \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing}} \chi(X_1 \cup X_2, U) \widetilde{I}^{U \setminus (X_1 \cup X_2)}(\varphi) \widetilde{J}^{X_1}(\varphi) \widetilde{P}(X_2, \varphi), \tag{6.10}$$

$$P_2\colon \boldsymbol{M}_{|||} \times \boldsymbol{M}_r \to \boldsymbol{M}_r \text{ defined by} \tag{6.11}$$

$$P_2(I, K) = (I - 1) \circ K, \tag{6.12}$$

$$P_3\colon \boldsymbol{M}_r \to \widehat{\boldsymbol{M}}_r, \tag{6.13}$$

$$(P_3 K)(X, \varphi) = \prod_{Y \in \mathcal{C}(X)} K(Y, \varphi) \tag{6.14}$$

and, finally, two linear renormalisation maps that are the source of loss of regularity,

$$R_1\colon \widehat{\boldsymbol{M}}_{r_0} \times B_{\frac{1}{2}} \to \widehat{\boldsymbol{M}}_{:,r_0} \text{ defined by} \tag{6.15}$$

$$R_1(P, \boldsymbol{q})(X, \varphi) = (\boldsymbol{R}^{(\boldsymbol{q})} P)(X, \varphi) = \int_{\boldsymbol{\mathcal{X}}} P(X, \varphi + \xi) \mu^{(\boldsymbol{q})}_{k+1}(\mathrm{d}\xi), \quad X \in \mathcal{P}, \tag{6.16}$$

$$R_2\colon \boldsymbol{M}_0 \times \boldsymbol{M}_{r_0} \times B_{\frac{1}{2}} \to \boldsymbol{M}_0 \text{ defined by} \tag{6.17}$$

$$R_2(H, K, \boldsymbol{q})(B, \varphi) = \Pi_2\Big((\boldsymbol{R}^{(\boldsymbol{q})} H)(B, \varphi) - \sum_{\substack{X \in \mathcal{S} \\ X \supset B}} \tfrac{1}{|X|} (\boldsymbol{R}^{(\boldsymbol{q})} K)(X, \varphi) \Big), \tag{6.18}$$

where we write $B_{\frac{1}{2}} = \big\{ \boldsymbol{q} \in \mathbb{R}^{d \times d}_{\mathrm{sym}} : \|q\| < \frac{1}{2} \big\}$.

In terms of these maps we have

$$S(H, K, \boldsymbol{q}) = P_1\big(E(R_2(H, K, \boldsymbol{q})),\, 1 - E(R_2(H, K, \boldsymbol{q})),\, R_1(P_3(P_2(E(H), K)), \boldsymbol{q}) \big). \tag{6.19}$$

Notice that the norms on the corresponding spaces are chosen in a natural way, with the exception of the space $M(\mathcal{P}, \boldsymbol{\mathcal{X}})$ in the role of the domain space of the map P_1 as well as the target space of the map R_1, that comes equipped with the norm $\|\cdot\|^{(\mathsf{A},\mathsf{B})}_{k:k+1,r_0}$. This is driven by the bound (iv) from Lemma 5.1 that makes the norm $\|K(X,\cdot)\|_{k:k+1,r}$ natural for the map R_1. The additional weight $\mathsf{B}^{|\mathcal{C}(X)|}$ in the norms of $\widehat{\boldsymbol{M}}_r$ and $\widehat{\boldsymbol{M}}_{:,r}$ plays an important role in the estimates for the map P_1 and is a substitute for the fact that we no longer deal with maps which factor in connected k-polymers. More precisely if K factors we can use the bound (i) from Lemma 5.1 to conclude that

$$\|K(X)\|_{k,X,r} \leq \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_{k,Y,r} \leq \Gamma_{\mathsf{A}}(X)^{-1}\left[\|K\|^{(\mathsf{A})}_{k,r}\right]^{|\mathcal{C}(X)|}.$$

This provides additional smallness if $\|K\|^{(\mathsf{A})}_{k,r}$ is small and the number of connected components $|\mathcal{C}(X)|$ is large. If K does not factor we can use the bound

$$\|K(X)\|_{k,X,r} \leq \Gamma_{\mathsf{A}}(X)^{-1}\mathsf{B}^{-|\mathcal{C}(X)|}\|K\|^{(\mathsf{A},\mathsf{B})}_{k,r}$$

instead to get a good decay for a large number of components.

The dependence on the parameters A and B in the definition of the weak norms (4.41) and in the norm (6.5) plays an important role here, we thus incorporate it explicitly into the notation and write, e.g., $\|\cdot\|^{(\mathsf{A})}_{k,r}$. Note that for a fixed N (where L^N is the system size) the norms $\|\cdot\|^{(\mathsf{A})}_{k,r}$ and $\|\cdot\|^{(\mathsf{A},\mathsf{B})}_{k,r}$ are equivalent for all $\mathsf{A} > 0$ and $\mathsf{B} > 0$ (because there are only finitely many polymers), but the constant in the equivalences depend strongly on N. Since we are interested in bounds on the derivatives which are independent of N a careful choice of the parameters A and B is crucial.

In the following sections we will show that all maps introduced above belong to the class $\widetilde{C}^m(\boldsymbol{X}\times B_{\frac{1}{2}},\boldsymbol{Y})$, introduced in Appendix D, for suitable scales of spaces $\boldsymbol{X} = \boldsymbol{X}_m \hookrightarrow \ldots \hookrightarrow \boldsymbol{X}_0$ and $\boldsymbol{Y} = \boldsymbol{Y}_m \hookrightarrow \ldots \hookrightarrow \boldsymbol{Y}_0$. Finally we will use the chain rule in the $\widetilde{C}^m$ spaces to show that the same regularity for the composed map S, see Section 6.7. In fact the maps above actually possess arbitrarily many Fréchet derivatives (or are even real-analytic) but the setting of the $\widetilde{C}^m$ spaces is setting which naturally goes with the estimates that are independent of N (where L^N is the system size).

Let us first discuss the partial maps one by one, starting from the most interior one in the composition (6.19).

6.1. Immersion $E\colon \boldsymbol{M}_0 \to \boldsymbol{M}_{|||}$

While the norm $\|H\|_{k,0}$ is expressed directly in terms of the co-ordinates $\lambda, a, \boldsymbol{c}, \boldsymbol{d}$ of the ideal Hamiltonian $H \in \boldsymbol{M}_0$, the terms involving $E(H)(B,\varphi) = \mathrm{e}^{-H(B,\varphi)}$ will be evaluated with the help of the norm $|||\cdot|||_k$. Considering thus the map $E\colon \boldsymbol{M}_0 \to \boldsymbol{M}_{|||}$, we have:

Lemma 6.1. *We have* $|||H|||_k \leq 5\|H\|_{k,0}$ *for any* $H \in \boldsymbol{M}_0$. *Moreover, there exist constants* $\delta = \delta(r_0)$ *and* $C = C(r_0)$ *so that* E *is smooth on* $B_\delta = \{H \in \boldsymbol{M}_0 \colon \|H\|_{k,0} < \delta\}$ *with uniformly bounded derivatives,*

$$|||D^j E(H)(\dot H,\ldots,\dot H)|||_k \leq C\|\dot H\|^j_{k,0}, \qquad j\leq m. \tag{6.20}$$

In particular we have

$$|||E(H) - 1|||_k \leq C\|H\|_{k,0}. \tag{6.21}$$

REMARK 6.2. The definition of norm $|||\cdot|||_k$ involves the parameter r_0 (see (4.30)) but the statement does not depend on r_0. $\diamond$

PROOF. Let $H \in \boldsymbol{M}_0$ and $B \in \mathcal{B}$. First, we estimate $|||H(B,\cdot)|||_{k,B}$ by $\|H\|_{k,0}$. In view of the definitions (4.30) and (4.25), we need to compute the norms

$$|D^p H(B,\varphi)|^{k,B}, \quad p = 0,1,2,$$

(the higher derivatives vanish as H is a quadratic function).

Starting with $p = 0$ and recalling the definitions (4.18)–(4.20), we get

$$|D^0 H(B,\varphi)|^{k,B} = |H(B,\varphi)| \le |\lambda| L^{dk} + L^{\frac{dk}{2}} \sum_{i=1}^{d} |a_i| \big(\sum_{x\in B} |\nabla\varphi(x)|^2 \big)^{1/2} + \tag{6.22}$$
$$+ L^{\frac{dk}{2}} \sum_{i,j=1}^{d} |\boldsymbol{c}_{i,j}| \big(\sum_{x\in B} |\nabla^2\varphi(x)|^2 \big)^{1/2} + \sum_{x\in B} |\nabla\varphi(x)|^2 \tfrac{1}{2} \sum_{i,j=1}^{d} |\boldsymbol{d}_{i,j}|.$$

Here, when evaluating the term $\sum_{x\in B}\sum_{i=1}^d |a_i||\nabla_i\varphi(x)|$, we first apply the Cauchy-Schwarz inequality in $\mathbb{R}^d$ and using the bound $|a| = \big(\sum_{i=1}^d |a_i|^2\big)^{1/2} \le \sum_{i=1}^d |a_i| = |a|_1$, we then employ the Cauchy-Schwarz inequality for the second time on the sum $\sum_{x\in B} 1\cdot|\nabla\varphi(x)|$ with $|\nabla\varphi(x)|^2 = \sum_{i=1}^d |\nabla_i\varphi(x)|^2$. Similarly we treat the next term with $|\nabla^2\varphi(x)|^2 = \sum_{i,j=1}^d |\nabla_i\nabla_j\varphi(x)|^2$. In the last term we just use the bound $\big|\frac{1}{2}\sum_{i,j=1}^d \boldsymbol{d}_{i,j}\nabla_i\varphi(x)\nabla_j\varphi(x)\big| \le \frac{1}{2}\|\boldsymbol{d}\||\nabla\varphi(x)|^2$ and then evaluate the operator norm, $\|\boldsymbol{d}\| \le (\sum_{i,j+1}^d \boldsymbol{d}_{i,j}^2)^{1/2} \le \sum_{i,j=1}^d |\boldsymbol{d}_{i,j}|$.

Hence,

$$\begin{aligned} &\big|H(B,\varphi)\big| \le \\ &\le \|H\|_{k,0}\Big(1 + \frac{1}{h}\big(\sum_{x\in B}|\nabla\varphi(x)|^2\big)^{1/2} + \frac{1}{h}L^k\big(\sum_{x\in B}|\nabla^2\varphi(x)|^2\big)^{1/2} + \frac{1}{h^2}\sum_{x\in B}|\nabla\varphi(x)|^2\Big) \\ &\le 2\|H\|_{k,0}\Big(1 + \frac{1}{h^2}\sum_{x\in B}\big(|\nabla\varphi(x)|^2 + L^{2k}|\nabla^2\varphi(x)|^2\big)\Big) \le 2\|H\|_{k,0}\Big(1 + \log W^B(\varphi)\Big), \end{aligned} \tag{6.23}$$

where we took into account the definition (4.28) of the weight function $W^B(\varphi) = W_k^B(\varphi)$.

Similarly, taking into account that $DH(B,\varphi)(\dot\varphi) = \ell(\dot\varphi) + 2Q(\varphi,\dot\varphi)$, we get

$$\begin{aligned} &|DH(B,\varphi)|^{k,B} = \sup_{|\dot\varphi|_{k,B}\le 1} |\ell(\dot\varphi) + 2Q(\varphi,\dot\varphi)| \le \\ &\le \sup\{|\ell(\dot\varphi)| + |2Q(\varphi,\dot\varphi)| : \sup_{x\in B^*}|\nabla\dot\varphi(x)| \le hL^{-\frac{kd}{2}} \text{ and } \sup_{x\in B^*}|\nabla^2\dot\varphi(x)| \le hL^{-\frac{kd}{2}-k}\} \\ &\le hL^{-\frac{kd}{2}}\big\{L^{kd}\sum_{i=1}^{d}|a_i| + L^{kd}L^{-k}\sum_{i,j=1}^{d}|\boldsymbol{c}_{i,j}| + \sum_{i,j=1}^{d}|\boldsymbol{d}_{i,j}|\sum_{x\in B}|(\nabla\varphi)(x))|\big\} \le \\ &\le \|H\|_{k,0}\big(1 + \frac{1}{h}\big(\sum_{x\in B}|\nabla\varphi(x)|^2\big)^{1/2}\big) \le 2\|H\|_{k,0}\big(1 + \log W^B(\varphi)\big) \end{aligned} \tag{6.24}$$

and

$$|D^2 H(B,\varphi)|^{k,B} \le 2h^2 L^{-dk} L^{dk} \sum_{i,j=1}^{d} |\boldsymbol{d}_{i,j}| \le 2\|H\|_{k,0}. \tag{6.25}$$

Recalling that $D^3 H(B,\varphi)(\dot\psi,\dot\psi,\dot\psi) = 0$, we finally get

(6.26)
$$|||H|||_k = |||H(B,\cdot)|||_{k,B} \le 5 \sup_\varphi W_k^{-B}(\varphi)\|H\|_{k,0}(1+\log W^B(\varphi)) \le 5\|H\|_{k,0}.$$

To get $|||E(H)|||_k$, we need to compute the norms $|D^p E(H)(B,\varphi)|^{k,B}$, $p = 0,\dots,r_0$. Using again Faà di Bruno's chain rule for higher order derivatives and the bounds (6.23), (6.24), and (6.25), we get

$$|D^p E(H)(B,\varphi)|^{k,B} \le B_{r_0} \mathrm{e}^{-H(B,\varphi)} \Big(1 + 2\|H\|_{k,0}(1+\log W^B(\varphi))\Big)^p \tag{6.27}$$

with the constant $B_{r_0} \le r_0^{r_0}$ bounding the number of partitions of the set $\{1,\dots,p\}$. Hence,

(6.28)
$$\begin{aligned}|||E(H)|||_k &\le B_{r_0} \sup_\varphi \mathrm{e}^{-H(B,\varphi)} W^{-B}(\varphi) \sum_{p=0}^{r_0} \big(1+2\|H\|_{k,0}(1+\log W^B(\varphi))\big)^p \le \\ &\le B_{r_0} \sum_{p=0}^{r_0} \sup_\varphi \mathrm{e}^{2\|H\|_{k,0}(1+\log W^B(\varphi))} W^{-B}(\varphi) \mathrm{e}^{2p\|H\|_{k,0}(1+\log W^B(\varphi))} \le \\ &\le (r_0+1)B_{r_0} \mathrm{e}^{2(1+r_0)\|H\|_{k,0}} \sup_\varphi \mathrm{e}^{2\|H\|_{k,0}(1+r_0)\log W^B(\varphi)} W^{-B}(\varphi) < e(r_0+1)B_{r_0}\end{aligned}$$

once $\|H\|_{k,0}$ is sufficiently small to assure that $2\|H\|_{k,0}(1+r_0) \le 1$ (we took into account that $W^B(\varphi) \ge 1$).

Computing the derivative of the exponent $E(H)$ as a composed function, we get $DE(H)(\dot H)(B,\varphi) = E(H)(B,\varphi)\dot H(B,\varphi)$. Using, similarly as when proving (5.1), the fact that a Taylor expansion of a product is the product of Taylor expansions, we get

$$|DE(H)(\dot H)(B,\varphi)|^{k,B,r_0} \le |E(H)(B,\varphi)|^{k,B,r_0} |\dot H(B,\varphi)|^{k,B,r_0}. \tag{6.29}$$

Applying now (6.27) and (6.23)–(6.25), we get

(6.30)
$$\begin{aligned}&|DE(H)(\dot H)(B,\varphi)|^{k,B,r_0} \le \\ &\le \mathrm{e}^{-H(B,\varphi)}(r_0+1)\big(1+2\|H\|_{k,0}(1+\log W^B(\varphi))\big)^{r_0} 5\,\|\dot H\|_{k,0}(1+\log W^B(\varphi))\end{aligned}$$

yielding

(6.31)
$$\begin{aligned}&|||DE(H)(\dot H)|||_k \\ &\le \sup_\varphi \mathrm{e}^{-H(B,\varphi)}(r_0+1)W^{-B}(\varphi)\mathrm{e}^{2r_0\|H\|_{k,0}(1+\log W^B(\varphi))}\, 10\,\|\dot H\|_{k,0} \mathrm{e}^{\frac12 \log W^B(\varphi)} \\ &\le 10\,\mathrm{e}(r_0+1)\|\dot H\|_{k,0}\end{aligned}$$

if $4r_0\|H\|_{k,0} \le 1$. Similarly, we get the bounds for higher derivatives. Formally the estimate (6.21) follows from (6.20) and the identity

$$E(H) - 1 = \int_0^1 DE(tH)(H)\,\mathrm{d}t.$$

□

6.2. The map P_2

LEMMA 6.3. *Consider the map $P_2\colon \boldsymbol{M}_{|||} \times \boldsymbol{M}_r \to \boldsymbol{M}_r$ defined in (6.12), restricted to $B_{\rho_1}(1) \times B_{\rho_2} \subset \boldsymbol{M}_{|||} \times \boldsymbol{M}_r$ with the balls $B_{\rho_1}(1) = \{I\colon |||I-1|||_k < \rho_1\}$ and $B_{\rho_2} = \{K\colon \|K\|_{k,r}^{(\mathsf{A})} < \rho_2\}$ and the target space $\boldsymbol{M}_r$ equipped with the norm $\|\cdot\|_{k,r}^{(\mathsf{A}/2)}$. For any $\mathsf{A} \geq 2$ and ρ_1, ρ_2 such that*

$$\rho_1 < (2\mathsf{A})^{-1}, \text{ and } \rho_2 < (2\mathsf{A}^{2^d})^{-1}, \tag{6.32}$$

the map P_2 restricted to $B_{\rho_1} \times B_{\rho_2}$ is smooth and satisfies the bound

$$\begin{aligned} &\frac{1}{j_1! j_2!} \big\| \big(D_1^{j_1} D_2^{j_2} P_2\big)(I,K)(\dot I, \dots, \dot I, \dot K, \dots, \dot K)\big\|_{k,r}^{(\mathsf{A}/2)} \\ &\qquad \leq (2\mathsf{A})^{j_1} \big(2\mathsf{A}^{2^d}\big)^{j_2} |||\dot I|||_k^{j_1} \big(\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^{j_2} \end{aligned} \tag{6.33}$$

for any $j_1, j_2 \in \mathbb{N}$. In particular,

$$\|P_2(I,K)\|_{k,r}^{(\mathsf{A}/2)} \leq 2\mathsf{A}|||I-1|||_k + 2\mathsf{A}^{2^d} \|K\|_{k,r}^{(\mathsf{A})}. \tag{6.34}$$

PROOF. Recall that

$$\big((I-1)\circ K\big)(X) = \sum_{Y \in \mathcal{P}(X)} (I-1)^{X\setminus Y} K(Y), \ X \in \mathcal{P}^{\mathrm{c}}, \tag{6.35}$$

with $(I-1)^{X\setminus Y} = \prod_{B \in \mathcal{B}(X\setminus Y)} \big(I(B)-1\big)$ and $K(Y) = \prod_{Z \in \mathcal{C}(Y)} K(Z)$, where $\mathcal{C}(Y)$ denotes the set of components of $Y \in \mathcal{P}$.

Hence,

$$\begin{aligned} &\frac{1}{j_1! j_2!} \big(D_1^{j_1} D_2^{j_2} \big((I-1)\circ K\big)(X)(\dot I, \dots, \dot I, \dot K, \dots, \dot K) = \\ &= \sum_{\substack{Y \in \mathcal{P}(X), Y_1 \in \mathcal{P}(X\setminus Y), |Y_1| = j_1 \\ \mathcal{J} \subset \mathcal{C}(Y), |\mathcal{J}| = j_2}} (I-1)^{(X\setminus Y)\setminus Y_1} \dot I^{Y_1} \prod_{Z \in \mathcal{C}(Y)\setminus \mathcal{J}} K(Z) \prod_{Z \in \mathcal{J}} \dot K(Z). \end{aligned} \tag{6.36}$$

Further, recall that, by definition of the norm $\|K\|_{k,r}^{(\mathsf{A})}$, we have

$$\|K(Z)\|_{k,Z,r} \leq \Gamma_A(Z)^{-1} \|K\|_{k,r}^{(\mathsf{A})} \text{ for any } Z \in \mathcal{P}_k^{\mathrm{c}}.$$

Notice also that

$$\mathsf{A}^{|Z|-2^d} \leq \max(1, \mathsf{A}^{|Z|-2^d}) \leq \Gamma_{\mathsf{A}}(Z) \leq \mathsf{A}^{|Z|} \tag{6.37}$$

for any $\mathsf{A} \geq 1$ and any $Z \in \mathcal{P}^{\mathrm{c}}$. Using the bounds (iia) and (i) from Lemma 5.1, assumptions (6.32), as well as the lower bound on $\Gamma_{\mathsf{A}}(Z)$ above and the fact that the number of terms in the sum is bounded by $2^{|X|}$, we get

$$\begin{aligned} &\|P_2(I,K)(X)\|_{k,X,r} \leq \\ &\leq \sum_{Y \in \mathcal{P}(X)} |||I-1|||_k^{|X\setminus Y|} \big(\|K\|_{k,r}^{(\mathsf{A})}\big)^{|\mathcal{C}(Y)|} \mathsf{A}^{2^d |\mathcal{C}(Y)|} \mathsf{A}^{-|Y|} \leq \mathsf{A}^{-|X|} 2^{|X|} = \big(\tfrac{\mathsf{A}}{2}\big)^{-|X|}, \end{aligned} \tag{6.38}$$

cf. [**Bry09**, Lemma 6.3]. Similarly, using that $\binom{n}{j} \le 2^n$, we get the claim

(6.39)
$$\begin{aligned}
&\frac{1}{j_1!j_2!}\|(D_1^{j_1}D_2^{j_2}P_2)(I,K)(X)(\dot I,\dots,\dot I,\dot K,\dots,\dot K)\|_{k,X,r}\\
&\quad\le \sum_{Y\in\mathcal{P}(X)} \tbinom{|X\setminus Y|}{j_1}|||I-1|||_k^{|X\setminus Y|-j_1}|||\dot I|||_k^{j_1}\tbinom{|\mathcal{C}(Y)|}{j_2}\\
&\qquad\times\big(\|K\|_{k,r}^{(\mathsf{A})}\big)^{|\mathcal{C}(Y)|-j_2}\big(\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^{j_2}\mathsf{A}^{2^d\mathcal{C}(Y)}\mathsf{A}^{-|Y|}\\
&\quad\le \sum_{Y\in\mathcal{P}(X)} 2^{|X\setminus Y|}(2\mathsf{A})^{-(|X\setminus Y|-j_1)}|||\dot I|||_k^{j_1}2^{|\mathcal{C}(Y)|}(2\mathsf{A}^{2^d})^{-(|\mathcal{C}(Y)|-j_2)}\big(\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^{j_2}\\
&\qquad\times \mathsf{A}^{2^d\mathcal{C}(Y)}\mathsf{A}^{-|Y|}=\\
&\quad= \sum_{Y\in\mathcal{P}(X)} 2^{j_1}\mathsf{A}^{-(|X\setminus Y|-j_1)}|||\dot I|||_k^{j_1}2^{j_2}\mathsf{A}^{j_2 2^d}\big(\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^{j_2}A^{-|Y|}\\
&\quad\le 2^{|X|}(2\mathsf{A})^{j_1}|||\dot I|||_k^{j_1}(2\mathsf{A}^{2^d})^{j_2}\big(\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^{j_2}\mathsf{A}^{-|X|}.
\end{aligned}$$

Finally, (6.34) follows from the fact that $P_2(1,0)=0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}P_2(1+t(I-1),tK) &= D_1P_2(1+t(I-1),tK)(I-1)\\ &\quad + D_2P_2(1+t(I-1),tK)K.\end{aligned} \tag{6.40}$$

□

6.3. The map P_3

LEMMA 6.4. *Let* $\mathsf{A}\ge 1$, $\mathsf{B}\ge 1$. *Consider the map* $P_3\colon \boldsymbol{M}_r \to \widehat{\boldsymbol{M}}_r$ *defined by*

$$(P_3K)(X) = \prod_{Y\in\mathcal{C}(X)} K(Y). \tag{6.41}$$

restricted to $B_\rho = \{K\in\boldsymbol{M}_r : \|K\|_{k,r}^{(\mathsf{A})} < \rho\}$ *and the target space* $\widehat{\boldsymbol{M}}_r$ *equipped with the norm* $\|\cdot\|_{k,r}^{(\mathsf{A},\mathsf{B})}$. *For any*

$$\rho \le (2\mathsf{B})^{-1} \tag{6.42}$$

the map P_3 *restricted to* B_ρ *is smooth and satisfies the bound*

$$\frac{1}{j!}\big\|\big(D_1^jP_3)(K)(\dot K,\dots,\dot K)\big\|_{k,r}^{(\mathsf{A},\mathsf{B})} \le \big(2\mathsf{B}\|\dot K\|_{k,r}^{(\mathsf{A})}\big)^j \tag{6.43}$$

for any $j_1,j_2\in\mathbb{N}$.

PROOF. The proof is similar to, but simpler than, the proof of Lemma 6.3. We have

$$\frac{1}{j!}D^jP_3(K)(X)(\dot K,\dots,\dot K) = \sum_{\mathcal{J}\subset\mathcal{C}(X),|\mathcal{J}|=j}\ \prod_{Z\in\mathcal{C}(X)\setminus\mathcal{J}} K(Z)\prod_{Z\in\mathcal{J}}\dot K(Z). \tag{6.44}$$

Thus using the estimate $\binom{|\mathcal{C}(X)|}{j} \le 2^{|\mathcal{C}(X)|}$ and the identity $\Gamma_{\mathsf{A}}(X) = \prod_{Z\in\mathcal{C}(X)} \Gamma_{\mathsf{A}}(Z)$ and arguing as in the proof of Lemma 6.3 we get

(6.45)

$$\mathsf{B}^{|\mathcal{C}(X)|}\Gamma_{\mathsf{A}}(X)\frac{1}{j!}\|D^j P_3(K)(X)(\dot K,\dots,\dot K)\|_{k,X,r} \le (2\mathsf{B})^{|\mathcal{C}(X)|}\left(\|K\|_{k,r}^{(\mathsf{A})}\right)^{|\mathcal{C}(X)|-j} \times \times \left(\|\dot K\|_{k,r}^{(\mathsf{A})}\right)^{j}.$$

Since $2\mathsf{B}\|K\|_{k,r}^{(\mathsf{A})} \le 2\mathsf{B}\rho \le 1$ it follows that

$$\frac{1}{j!}\|D^j P_3(K)(X)(\dot K,\dots,\dot K)\|_{k,r}^{(\mathsf{A},\mathsf{B})} \le \left(2\mathsf{B}\|\dot K\|_{k,r}^{(\mathsf{A})}\right)^{j} \tag{6.46}$$

and this finishes the proof. □

6.4. The map R_1

LEMMA 6.5. *Let $m\in\mathbb{N}$, $2m\le r_0$, and for any $n=0,1,\dots,m$, let $\boldsymbol{X}_n$ denote the space $\widehat{\boldsymbol{M}}_{r_0-2m+2n}$ equipped with the norm $\|\cdot\|_{\boldsymbol{X}_n} = \|\cdot\|_{k,r_0-2m+2n}^{(\mathsf{A},\mathsf{B})}$ and $\boldsymbol{Y}_n$ the space $\widehat{M}_{:,r_0-2m+2n}$ equipped with the norm $\|\cdot\|_{\boldsymbol{Y}_n} = \|\cdot\|_{k:k+1,r_0-2m+2n}^{(\mathsf{A}/2,\mathsf{B}/2^{2^d})}$. Further, let $B_{\frac12} = \{\boldsymbol{q}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\colon \|\boldsymbol{q}\| < \frac12\}$. Consider the map $R_1\colon \boldsymbol{X}\times B_{\frac12}\to\boldsymbol{Y}$ defined in (6.16) with $\boldsymbol{X}=\boldsymbol{X}_m=\widehat{\boldsymbol{M}}_{r_0}$ and $\boldsymbol{Y}=\boldsymbol{Y}_m=\widehat{\boldsymbol{M}}_{:,r_0}$. There exists a constant $C=C(r_0,d)$ such that for any $h\ge L^{\kappa(d)}h_1$ with $h_1=h_1(d,\omega)$ and $\kappa(d)$ as in Lemma 5.1 (iv) (see (5.68)), $\mathsf{A}\ge 2$, and any $r=1,\dots,r_0$, we have*

$$R_1 \in \widetilde{C}^m(\boldsymbol{X}\times B_{\frac12},\boldsymbol{Y}). \tag{6.47}$$

Moreover the constants in the estimates of the relevant derivatives are independent of k and N. More precisely for $0\le\ell\le m, 0\le n\le m-\ell$, there are $C(n,d)>0$ such that

$$\|D_2^\ell R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}}^\ell)\|_{\boldsymbol{Y}_n} \le C(n,d)\|P\|_{\boldsymbol{X}_{n+\ell}}\|\dot{\boldsymbol{q}}\|^\ell, \tag{6.48}$$

$$\|D_1D_2^\ell R_1(P,\boldsymbol{q},\dot P,\dot{\boldsymbol{q}}^\ell)\|_{\boldsymbol{Y}_n} \le C(n,d)\|\dot P\|_{\boldsymbol{X}_{n+\ell}}\|\dot{\boldsymbol{q}}\|^\ell, \tag{6.49}$$

$$D_1^2D_2^\ell R_1(P,\boldsymbol{q},\dot P^2,\dot{\boldsymbol{q}}^\ell) = 0. \tag{6.50}$$

REMARK 6.6. (i) Note that (6.49) follows from (6.48) since R_1 is linear in the first argument, whereas (6.50) is trivial.
(ii) The proof below actually shows that

$$\|D_2^\ell R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}}^\ell)(X)\|_{k:k+1,X,n} \le C(n,d)2^{|X|}\|P(X)\|_{k,X,n+\ell}\|\dot{\boldsymbol{q}}\|^\ell \tag{6.51}$$

The estimate (6.48) then follows by the choice of weights $\mathsf{A}/2$ and $\mathsf{B}/2^{2^d}$ on the target space, see Step 2 of the proof.
(iv) It follows from Step 1 in the proof, the bound

$$\|\boldsymbol{R}^{(\boldsymbol{q})}P(X)\|_{k:k+1,X,r} \le 2^{|X|_k}\|P(X)\|_{k,X,r}$$

in Step 2 of the proof and the linearity of R_1 in the first argument that R_1 is actually a real-analytic map from $\boldsymbol{X}_r\times B_{\frac12}$ to $\boldsymbol{Y}_r$ without any loss of regularity. The bounds on the corresponding derivatives depend, however, on the system size N and the level k, while the bounds stated in Lemma 6.5 do not. ◇

PROOF. Recall from (6.16) that

$$R_1(P,\boldsymbol{q})(X,\varphi) = (\boldsymbol{R}^{(\boldsymbol{q})}P)(X,\varphi) = \int_{\boldsymbol{\mathcal{X}}} P(X,\varphi+\xi)\,\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi).$$

The fact that R_1 maps $\boldsymbol{M}_m \times B_{\frac{1}{2}}$ to $\boldsymbol{Y}_m$ follows from Lemma 5.1(iv). Note that R_1 is linear in P. Thus by Lemma D.31 it suffices to show that

(i) For each $P \in \boldsymbol{X}_m$ and $0 \le \ell \le m$ the map $\boldsymbol{q} \mapsto \boldsymbol{R}^{(\boldsymbol{q})}P$ is in $C^\ell_*(B_{\frac{1}{2}};\boldsymbol{Y}_{m-\ell})$.

(ii) For each $\boldsymbol{q}_0 \in B_{\frac{1}{2}}$ there exist $\delta, C > 0$ such that

$$\|D^\ell_{\boldsymbol{q}}\boldsymbol{R}^{(\boldsymbol{q})}(P,\boldsymbol{q}),\dot{\boldsymbol{q}}^\ell)\|_{\boldsymbol{Y}_n} \le C\|P\|_{\boldsymbol{X}_{n+\ell}}\|\dot{\boldsymbol{q}}\|^\ell$$

for any $0 \le \ell \le m, 0 \le n \le m-\ell$, and for all $(P,\boldsymbol{q},\dot{\boldsymbol{q}}) \in \boldsymbol{X}_m \times B_\delta(\boldsymbol{q}_0) \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$.

We split the proof of (i) and (ii) into seven steps below. Note that the required constant C will be given as the maximum of all constants in (6.48) and (6.49). We first show (i) in step 1 below. Indeed we even show that $\boldsymbol{q} \mapsto \boldsymbol{R}^{(\boldsymbol{q})}P$ is real-analytic with values in $\boldsymbol{Y}_m \subset \boldsymbol{Y}_{m-\ell}$.

Step 1: Assume that $P \in \boldsymbol{M}_r = (M_r(\mathcal{P}^{\mathrm{c}}_k,\boldsymbol{\mathcal{X}}),\|\cdot\|^{(\mathsf{A})}_{k,r})$ for some $r \in \{r_0,\dots,r_0-2m\}$. Then the map

$$\boldsymbol{q} \mapsto R_1(P,\boldsymbol{q})$$

is real-analytic from $B_{\frac{1}{2}}$ to $\boldsymbol{M}_{:,r} = (M_r(\mathcal{P}^{\mathrm{c}}_k,\boldsymbol{\mathcal{X}}),\|\cdot\|^{(\mathsf{A}/2)}_{k:k+1,r})$. First it suffices to show the result for $r=0$, since differentiation with respect to φ commutes with $\boldsymbol{R}^{(\boldsymbol{q})}$. Secondly it suffices to consider a fixed polymer X, since there are only finitely many polymers. Thus we need to show the following: If

$$\|P(X)\|_{k,X,0} = \sup_\xi \frac{|P(X,\xi)|}{w^X_k(\xi)} < \infty,$$

then the map

$$B_{\frac{1}{2}} \ni \boldsymbol{q} \mapsto \int_{\boldsymbol{\mathcal{X}}} P(X,\cdot+\xi)\,\mu_{\mathcal{C}^{(\boldsymbol{q})}_{k+1}}(\mathrm{d}\xi)$$

is real-analytic with values in the space of continuous functions F of the field with the weighted norm

$$\|F\|_{k:k+1,X,0} = \sup_\varphi \frac{|F(\varphi)|}{w^X_{k:k+1}(\varphi)}.$$

This follows from Gaussian calculus (see Lemma C.1), Lemma 5.3 and the properties of the finite range decomposition, see Proposition 4.1. To see this recall (5.42), i.e.,

$$w^X_k(\varphi+\xi) \le w^X_{k:k+1}(\varphi)\mathrm{e}^{\frac{1}{2}\varkappa(\mathcal{B}_k\xi,\xi)},$$

where $\varkappa = 2\overline{C}h^{-2}$ and $\mathcal{B}_k$ is given by (5.43). If h_1 and $\kappa(d)$ are chosen as in Lemma 5.1 and $h \ge L^{\kappa(d)}h_1$ then it follows from Lemma 5.3 that for $\boldsymbol{q} \in B_{\frac{1}{2}}$ and $\mathcal{C}_{k+1} = \mathcal{C}^{(\boldsymbol{q})}_{k+1}$ we have

$$0 \le \mathcal{C}^{1/2}_{k+1}\varkappa\mathcal{B}_k\mathcal{C}^{1/2}_{k+1} \le \frac{1}{2}\mathrm{Id} \text{ and hence } \mathcal{C}^{-1}_{k+1} > \varkappa\mathcal{B}_k, \tag{6.52}$$

i.e.,

$$B_{\frac{1}{2}} \ni \boldsymbol{q} \mapsto \mathcal{U}_k,$$

where we define

$$\mathcal{U}_k := \{\mathcal{C} \in \mathrm{Sym}^{(+)}(\boldsymbol{\mathcal{X}})\colon \mathcal{C}^{-1} > \varkappa\mathcal{B}_k\}.$$

By Lemma C.1 the map

$$\mathcal{C} \mapsto \int_{\boldsymbol{\mathcal{X}}} P(\cdot + \xi)\, \mu_{\mathcal{C}}(\mathrm{d}\xi)$$

is real-analytic from $\mathcal{U}_k$ to the desired space. Finally, by Proposition 4.1 and (6.52) the map $\boldsymbol{q} \mapsto \mathcal{C}_{k+1}^{(\boldsymbol{q})}$ is real-analytic from $B_{\frac{1}{2}}$ to $\mathcal{U}_k$.

Hence $\boldsymbol{q} \mapsto R_1(P, \boldsymbol{q})$ is real-analytic from $B_{\frac{1}{2}}$ to the space $\boldsymbol{M}_{:,r}$, and thus (i) is proven.

In the remaining steps we are going to prove (ii). In step 2 we show the bounds for $\ell = 0$ followed by the bound for $\ell = 1$ in step 3 to step 6. The bounds for higher derivatives are then finally settled in step 7.

Step 2: Bounds on $\boldsymbol{R}^{(\boldsymbol{q})}$. By Lemma 5.1(iv) we have for all $\boldsymbol{q} \in B_{\frac{1}{2}}$ the following estimate

$$\|\boldsymbol{R}^{(\boldsymbol{q})} P(X)\|_{k:k+1,X,r} \leq 2^{|X|_k} \|P(X)\|_{k,X,r}.$$

For connected polymers Y we have

$$2^{|Y|}\Gamma_{\mathsf{A}/2}(Y) \leq 2^{2^d}\Gamma_{\mathsf{A}}(Y). \tag{6.53}$$

Thus for general polymer X we get

$$2^{|X|}\Gamma_{\mathsf{A}/2}(X) \leq 2^{2^d|\mathcal{C}(X)|}\Gamma_{\mathsf{A}}(X). \tag{6.54}$$

and thus

$$2^{|X|}\left(\mathsf{B}/2^{2^d}\right)^{|\mathcal{C}(X)|}\Gamma_{\mathsf{A}/2}(X) \leq \mathsf{B}^{|\mathcal{C}(X)|}\Gamma_{\mathsf{A}}(X). \tag{6.55}$$

Therefore

$$\|\boldsymbol{R}^{(\boldsymbol{q})} P\|_{k:k+1,r}^{(\mathsf{A}/2,\mathsf{B}/2^{2^d})} \leq \|P\|_{k,r}^{(\mathsf{A},\mathsf{B})},$$

and hence with $r = r_0 - 2m + 2n$ we obtain

$$\|R_1(P,\boldsymbol{q})\|_{\boldsymbol{Y}_n} = \|\boldsymbol{R}^{(\boldsymbol{q})} P\|_{\boldsymbol{Y}_n} \leq \|P\|_{\boldsymbol{X}_n}, \quad \text{for all } \boldsymbol{q} \in B_{\frac{1}{2}}. \tag{6.56}$$

Step 3: Bounds for $D_2R_1(P, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Let $\boldsymbol{q} \in B_{\frac{1}{2}}$ and $\|\dot{\boldsymbol{q}}\| = 1$ and write $\boldsymbol{\gamma}(t) = \boldsymbol{q} + t\dot{\boldsymbol{q}}$ in the following. By Lemma C.2 and (C.23) we have

$$\begin{aligned} D_2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}})(X,\varphi) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_{\boldsymbol{\mathcal{X}}} P(X,\varphi+\xi)\, \mu_{\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}}(\mathrm{d}\xi) \\ &= \int_{\boldsymbol{\mathcal{X}}} A_{\dot{\mathcal{C}}_{k+1}} P(X,\varphi+\xi)\, \mu_{\mathcal{C}_{k+1}^{(\boldsymbol{q})}}(\mathrm{d}\xi) = (\boldsymbol{R}^{(\boldsymbol{q})} A_{\dot{\mathcal{C}}_{k+1}} P)(X,\varphi) \end{aligned}$$

with

$$\dot{\mathcal{C}}_{k+1} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}$$

and where the functional $A_{\dot{\mathcal{C}}_{k+1}}$ is defined as

$$A_{\dot{\mathcal{C}}_{k+1}} P(X,\xi) = \sum_{i,j=1}^{L^{dN}-1} D^2P(X,\xi,e_i,e_j)(\dot{\mathcal{C}}_{k+1})_{i,j},$$

where $\{e_j\}_{j=1}^{L^{dN}-1}$ is any orthonormal basis of $\boldsymbol{\mathcal{X}}$ and $(\dot{\mathcal{C}}_{k+1})_{i,j} = (\dot{\mathcal{C}}_{k+1}e_i, e_j)$. By Step 2 we obtain the following bound for the derivative with respect to $\boldsymbol{q}$, for $0 \leq n \leq m-1$,

$$\|D_2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}})\|_{\boldsymbol{Y}_n} \leq \|A_{\dot{\mathcal{C}}_{k+1}} P\|_{\boldsymbol{X}_n}. \tag{6.57}$$

Step 4: Estimate for $\|A_{\dot{\mathcal{C}}_{k+1}} P\|$. We now express and estimate the functional $A_{\dot{\mathcal{C}}_{k+1}} P$ using the orthonormal Fourier basis $\{f_p\}_{p\in\widehat{\mathbb{T}}_N}$ of the (complexified space) $\boldsymbol{\mathcal{X}}$ given by

$$f_p(x) = \frac{\mathrm{e}^{i\langle p,x\rangle}}{L^{dN/2}}, \quad p \in \widehat{\mathbb{T}}_N, x \in \Lambda_N. \tag{6.58}$$

We denote by $\widehat{\dot{\mathcal{C}}_{k+1}}(p)$ the Fourier multiplier of $\dot{\mathcal{C}}_{k+1}$. Now $\mathcal{C}_{k+1}^{(\boldsymbol{q})}$ and hence $\dot{\mathcal{C}}_{k+1}$ are diagonal in the Fourier basis and

$$\dot{\mathcal{C}}_{k+1} f_p = \widehat{\dot{\mathcal{C}}_{k+1}}(p) f_p \quad \text{ with } \widehat{\dot{\mathcal{C}}_{k+1}}(p) \in \mathbb{R}.$$

Thus by (C.13)

$$\begin{aligned} A_{\dot{\mathcal{C}}_{k+1}} P(X,\xi) = \sum_{p\in\widehat{\mathbb{T}}_N} D^2 P(X,\xi,\dot{\mathcal{C}}_{k+1} f_p, \overline{f}_p) \\ \sum_{p\in\widehat{\mathbb{T}}_N} D^2 P(X,\xi,f_p,\overline{f}_p)\widehat{\dot{\mathcal{C}}_{k+1}}(p). \end{aligned}$$

We claim that
(6.59)

$$|A_{\dot{\mathcal{C}}_{k+1}} D^2 P(X,\xi)\dot{\mathcal{C}}_{k+1})|^{k,X,r-2} \le r(r-1)|P(X,\xi)|^{k,X,r} \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} |f_p|^2_{k,X} |\widehat{\dot{\mathcal{C}}_{k+1}}|(p)\,.$$

whenever $\dot{\mathcal{C}}_{k+1}$ is diagonal in the Fourier basis. In particular we now show that there exists a $C(n,d) > 0$ such that for $0 \le n \le m-1$ the following estimate holds,

$$\|A_{\dot{\mathcal{C}}_{k+1}} P\|_{\boldsymbol{X}_n} \le C(n)\|P\|_{\boldsymbol{X}_{n+1}} \sum_{p\in\widehat{\mathbb{T}}_N} |f_p|^2 \widehat{\dot{\mathcal{C}}_{k+1}}(p). \tag{6.60}$$

Indeed, using the fact that $\widehat{\dot{\mathcal{C}}_{k+1}}(p)$ is real and the definition of the trace we have

$$\begin{aligned} G(X,\xi) :=& \mathrm{Tr}\big(D^2 P(X,\xi)\dot{\mathcal{C}}_{k+1})\big) = A_{\dot{\mathcal{C}}_{k+1}} P(X,\xi) \\ =& \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} \big(f_p, D^2 P(X,\xi) f_p\big)\widehat{\dot{\mathcal{C}}_{k+1}}(p) \\ =& \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} \big(\mathrm{Re}\,(f_p), D^2 P(X,\xi)\mathrm{Re}\,(f_p)\big)\widehat{\dot{\mathcal{C}}_{k+1}}(p) \\ &+ \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} \big(\mathrm{Im}\,(f_p), D^2 P(X,\xi)\mathrm{Im}\,(f_p)\big)\widehat{\dot{\mathcal{C}}_{k+1}}(p). \end{aligned} \tag{6.61}$$

By a standard symmetrisation argument we have

$$|D^\sigma P(X,\xi)(\dot{\varphi}_1,\ldots,\dot{\varphi}_\sigma)| \le \frac{\sigma^\sigma}{\sigma!}|D^\sigma P(X,\xi)|^{k,X} \prod_{i=1}^{\sigma} |\dot{\varphi}_i|_{k,X}. \tag{6.62}$$

Set

$$M := \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}} |f_p|^2_{k,X} |\widehat{\dot{\mathcal{C}}_{k+1}}|(p)\,. \tag{6.63}$$

Then for all $\dot\varphi$ with $|\dot\varphi|_{k,X}\le 1$ we have

$$
\begin{aligned}
&|D^sG(X,\xi)(\dot\varphi,\dots,\dot\varphi)|\le\\
&\le \sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}|D^{s+2}P(X,\xi)(\dot\varphi,\dots,\dot\varphi;\mathrm{Re}\,(f_p),\mathrm{Re}\,(f_p))||\widehat{\dot{\mathcal{C}}_{k+1}}|(p)\\
&+\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}|D^{s+2}P(X,\xi)(\dot\varphi,\dots,\dot\varphi;\mathrm{Im}\,(f_p),\mathrm{Im}\,(f_p))||\widehat{\dot{\mathcal{C}}_{k+1}}|(p)\\
&\le 2\frac{(s+2)^{s+2}}{(s+2)!}|D^{s+2}P(X,\xi)|^{k,X,r}|\dot\varphi|_{k,X}^sM.
\end{aligned}
\tag{6.64}
$$

Hence $|D^sG(X,\xi)|^{k,X}\le C(r_0)M|D^{s+2}P(X,\xi,X)|^{k,X}$, for all $s\le r_0-2$ and $C(r_0)=2\frac{r_0^{r_0}}{r_0!}$. This yields

$$
\begin{aligned}
&|G(X,\xi)|^{k,X,r-2}\le\\
&\le M\sum_{s=0}^{r-2}\frac{1}{s!}|D^{s+2}P(X,\xi)|^{k,X}\le r(r-1)C(r_0)M\sum_{s=0}^{r-2}\frac{1}{(s+2)!}|D^{s+2}P(X,\xi)|^{k,X}\le\\
&\le r(r-1)C(r_0)M|P(X,\xi)|^{k,X,r}
\end{aligned}
\tag{6.65}
$$

and hence the assertion (6.60). Note that in the proof we only used the fact that $\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}$ is diagonal in the Fourier basis. Hence the same computation yields the corresponding result for the higher derivatives

$$
\begin{aligned}
&|\mathrm{Tr}(D^2P(X,\xi)\frac{\mathrm{d}^j}{\mathrm{d}t^j}\mathcal{C}_{k+1}^{\boldsymbol{\gamma}(t)})|^{k,X,r-2}\le\\
&\le r(r-1)C(r_0)|P(X,\xi)|^{k,X,r}\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}|f_p|_{k,X}^2\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\widehat{\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}}(p)\right|.
\end{aligned}
\tag{6.66}
$$

Step 5: Estimate for the term (6.63) **involving the Fourier multiplier.** Let

$$\boldsymbol{\gamma}(t)=\boldsymbol{q}+t\dot{\boldsymbol{q}}\qquad\text{with } \boldsymbol{q}\in B_{\frac12}\text{ and }\|\dot{\boldsymbol{q}}\|=1.$$

We claim that, with our choice of h, there exists $C=C(n,d)>0$ such that

$$\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}|f_p|_{k,X}^2\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\widehat{\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}}(p)\right|\le Cj!. \tag{6.67}$$

To see this note first that by the definition of the $|\cdot|_{k,X}$ norm

$$|f_p|_{k,X}\le\frac{1}{h}\frac{1}{L^{Nd/2}}L^{kd/2}\max(|p|,L^k|p|^2,L^{2k}|p|^3)\,. \tag{6.68}$$

The estimate (4.7) in Remark 4.2 can be rewritten as

$$\sum_{p\in\widehat{\mathbb{T}}_N\setminus\{0\}}|p|^n\left|\frac{\mathrm{d}^j}{\mathrm{d}t^j}\widehat{\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}}(p)\right|\le C2^jj!\,L^{\eta(n,d)+n+d-2}L^{-k(n+d-2)}L^{dN}, \tag{6.69}$$

where $\eta(n,d)=\max(\frac14(d+n-1)^2,d+n+6)+10$. Applying this estimate with $n=2,4$ and 6 and using the monotonicity of $\eta(n,d)$ in n, we need a bound on $\eta(6,d)+4+d$. It turns out that $\eta(6,d)+4+d\le 2\kappa(d)$ whenever $d\ge 2$. Indeed, this amounts to showing that $\eta(6,d)+4 \leq \eta(12,d)$ (with $2\lfloor \frac{d+2}{2}\rfloor + 8 = 12$ for $d=2$). Using this and assuming that $h_1 \geq 1$, we can conclude that

$$h^{-2}L^{\eta(n,d)+n+d-2} \leq 1 \tag{6.70}$$

for $n = 2,4,6$, implying thus (6.67).

Step 6: Estimate for $D_2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}})$. It follows from Step 3, (6.60) with $\dot{\mathcal{C}}_{k+1} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}$, and Step 5 with $j=1$ for any $0 \leq n \leq m-1$ that there exists $C(n,d) > 0$ such that

$$\|D_2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}})\|_{\boldsymbol{Y}_n} \leq \|A_{\dot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_n} \leq C(n,d)\|P\|_{\boldsymbol{X}_{n+1}}. \tag{6.71}$$

Step 7: Bounds for the higher derivatives $D_2^{\ell}R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}}^{\ell})$. These bounds follow from Gaussian calculus in Lemma C.4, the chain rule and the estimates for $\frac{\mathrm{d}^j}{\mathrm{d}t^j}\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}$ (see step 5). We consider first the case $\ell = 2$. As in (C.1) in appendix C we set

$$H(\mathcal{C})(\cdot) = \int_{\boldsymbol{\mathcal{X}}} P(X,\cdot+\xi)\,\mu_{\mathcal{C}}(\mathrm{d}\xi),$$

respectively,

$$\widetilde{h}(t)(\cdot) = \int_{\boldsymbol{\mathcal{X}}} P(X,\cdot+\xi)\,\mu_{\mathcal{C}_{k+1}^{\boldsymbol{\gamma}(t)}}(\mathrm{d}\xi).$$

By Lemma C.4 and (C.24) we obtain

$$\begin{aligned} D_2^2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}})(X,\varphi) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}R_1(P,\boldsymbol{\gamma}(t))(X,\varphi) = D^2H(\mathcal{C}_{k+1},\dot{\mathcal{C}}_{k+1},\dot{\mathcal{C}}_{k+1}) \\ &+ DH(\mathcal{C},\ddot{\mathcal{C}}_{k+1}) = R_1(A^2_{\dot{\mathcal{C}}_{k+1}}P,\boldsymbol{q})(X,\varphi) + R_1(A_{\ddot{\mathcal{C}}_{k+1}}P,\boldsymbol{q})(X,\varphi) \end{aligned}$$

where we use that

$$\dot{\mathcal{C}}_{k+1} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))} \quad \text{ and } \ddot{\mathcal{C}}_{k+1} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))}.$$

By step 2 we have the estimate

$$\|D_2^2R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}},\dot{\boldsymbol{q}})\|_{\boldsymbol{X}_n} \leq \big(\|A^2_{\dot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_n} + \|A_{\ddot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_n}\big).$$

Now step 4 and step 5 yield the following bound, for $0 \leq n \leq m-2$,

$$\|A_{\ddot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_n} \leq C(n)\|P\|_{\boldsymbol{X}_{n+1}} \leq C(n)\|P\|_{\boldsymbol{X}_{n+2}}.$$

Applying now the steps 4 and 5 twice we get that

$$\|A^2_{\dot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_n} \leq C(n)\|A_{\dot{\mathcal{C}}_{k+1}}P\|_{\boldsymbol{X}_{n+1}} \leq C(n)\|P\|_{\boldsymbol{X}_{n+2}},$$

and thus the required estimate for the second derivative $D_2^2R_1$. For general $\ell \geq 2$ it follows from Lemma C.4 and the chain rule that

$$D_2^{\ell}R_1(P,\boldsymbol{q},\dot{\boldsymbol{q}}^{\ell})$$

is a linear combination of terms of the form

$$R_1(A_{\dot{\mathcal{C}}_1}\cdots A_{\dot{\mathcal{C}}_{\kappa}}P,\boldsymbol{q})$$

where

$$\dot{\mathcal{C}}_i := \frac{\mathrm{d}^{j_i}}{\mathrm{d}t^{j_i}}\Big|_{t=0}\mathcal{C}_{k+1}^{(\boldsymbol{\gamma}(t))} \quad \text{ with } \sum_{i=1}^{\kappa} j_i = \ell.$$

Thus the desired estimate follows from step 2 and a κ-fold application of (6.60) and step 5. $\square$

6.5. The map R_2

LEMMA 6.7. *Let $m \in \mathbb{N}$, $2m+2 \le r_0$. For $n = 0,,\ldots,m$ let $\boldsymbol{Z}_n$ denote the space $\boldsymbol{M}_{r_0-2m+2n}$ equipped with the norm $\|\cdot\|_{\boldsymbol{Z}_n} = \|\cdot\|^{(\mathsf{A})}_{k,r_0-2m+2n}$. Let $\boldsymbol{X}_n = \boldsymbol{M}_0 \times \boldsymbol{Z}_n$, $\boldsymbol{Y}_n = \boldsymbol{M}_0$ (for all n) and $B_{\frac{1}{2}} = \{\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \colon \|\boldsymbol{q}\| < \frac{1}{2}\}$. Consider the map $R_2 \colon \boldsymbol{M}_0 \times \boldsymbol{X} \times B_{\frac{1}{2}} \to \boldsymbol{Y}$, defined in (6.18) with $\boldsymbol{X} = \boldsymbol{X}_m = \boldsymbol{M}_0 \times \boldsymbol{M}_{r_0}$ and $\boldsymbol{Y} = \boldsymbol{Y}_m = \boldsymbol{M}_0$. There exists a constant $C = C(d)$ such that for any $h \ge L^{\kappa(d)} h_1$ with $h_1 = h_1(d,\omega)$ and $\kappa(d)$ as in Lemma 5.1 (iv), $\mathsf{A} \ge 1$, we have*

$$R_2 \in \widetilde{C}^m(\boldsymbol{X} \times B_{\frac{1}{2}}, \boldsymbol{Y}). \tag{6.72}$$

Moreover for any $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ with $|\boldsymbol{q}| < \frac{1}{2}$ and $|\dot{\boldsymbol{q}}| \le 1$, and any $\ell \le m$, we have

$$\begin{aligned} &\|D_1^j D_2^n D_3^\ell R_2(H,K,\boldsymbol{q})(\dot{H},\dot{K},\dot{\boldsymbol{q}},\ldots,\dot{\boldsymbol{q}})\|_{k,0} \\ &\qquad \le C \begin{cases} \|H\|_{k,0} + \|K\|_{\boldsymbol{Z}_\ell} & \text{if } j=0, n=0, \\ \|\dot{H}\|_{k,0} & \text{if } j=1, n=0, \\ \|\dot{K}\|_{\boldsymbol{Z}_\ell} & \text{if } j=0, n=1; \end{cases} \end{aligned} \tag{6.73}$$

and

$$D_1^j D_2^m D_3^\ell R_2(H,K,\boldsymbol{q}) = 0 \quad \text{if } j+m \ge 2. \tag{6.74}$$

REMARK 6.8. It follows from Remark 6.6 and Lemma 6.9 below that the map R_2 is actually a real analytic map from $\boldsymbol{M}_0 \times \boldsymbol{M}_2$ to $\boldsymbol{M}_0$. $\diamond$

First, we estimate the main component of R_2, namely the map Π_2.

LEMMA 6.9. *Let $B \in \mathcal{B}_k$, $X \in \mathcal{S}_k$ with $X \supset B$, and let $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$. Then*

$$\|\Pi_2 K(X,\cdot)\|_{k,0} \le [2^d(d^{\frac{3}{2}} + d) + d^{\frac{1}{2}}]\, |K(X,0)|^{k,X,2}. \tag{6.75}$$

Note that since $X \in \mathcal{S}_k$ we have $X \subset B^*$ and thus the maps $\varphi \mapsto K(X,\varphi)$ can be viewed as an element of $M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ on which the projection Π_2 was defined.

PROOF. Let $H = \Pi_2 K(X,\cdot)$. By definition we have $H(B,\dot{\varphi}) = L^{dk}\lambda + \ell(\dot{\varphi}) + Q(\dot{\varphi},\dot{\varphi})$, where

$$\ell(\dot{\varphi}) = \sum_{x\in B}\sum_{i=1}^d a_i \nabla_i \dot{\varphi} + \boldsymbol{c}_{i,j} \nabla_i \nabla_j \dot{\varphi}(x) \tag{6.76}$$

$$Q(\dot{\varphi},\dot{\varphi}) = \frac{1}{2}\sum_{x\in B}\sum_{i,j=1}^d \boldsymbol{d}_{i,j} \nabla_i \dot{\varphi}(x) \nabla_j \dot{\varphi}(x) \tag{6.77}$$

and

$$L^{dk}\lambda = K(X,0) \tag{6.78}$$

$$\ell(\dot{\varphi}) = DK(X,0)(\dot{\varphi}) \quad \forall \dot{\varphi} \text{ quadratic + affine in } (B^*)^* \tag{6.79}$$

$$Q(\dot{\varphi},\dot{\varphi}) = \frac{1}{2} D^2 K(X,0)(\dot{\varphi},\dot{\varphi}) \quad \forall \dot{\varphi} \text{ affine in } (B^*)^* \tag{6.80}$$

To estimate $\boldsymbol{d}_{i,j}$ and a_i we consider functions $\dot\varphi$ which are linear on $((B^*)^*)^*$

$$\dot\varphi = \sum_{i=1}^d \eta_i \pi_i, \tag{6.81}$$

where $\eta = (\eta_i)_{i=1,\dots,d} \in \mathbb{R}^d$, and π_i is the co-ordinate projection $\pi_i(x) = x_i$ for $x \in \mathbb{Z}^d$. Then for $x \in (B^*)^*$ we have $\nabla_i \dot\varphi(x) = \eta_i$ and $\partial^\alpha \dot\varphi(x) = 0$ if $|\alpha| = 2$ or $|\alpha| = 3$. Hence,

$$\begin{aligned} L^{dk} \Big|\frac{1}{2}\sum_{i,j=1}^d \boldsymbol{d}_{i,j}\eta_i\eta_j\Big| = |Q(\dot\varphi)| = \Big|\frac{1}{2}D^2K(X,0)(\dot\varphi,\dot\varphi)\Big| \le \frac{1}{2}\big|D^2K(X,0)\big|^{k,X} |\dot\varphi|^2_{k,X} \\ = \frac{1}{2}\big|D^2K(X,0)\big|^{k,X} h^{-2}\sum_{i=1}^d |\eta_i|^2 L^{dk}. \end{aligned} \tag{6.82}$$

This yields $\max_{|\eta|_2=1}|\frac{1}{2}\sum_{i,j=1}^d \boldsymbol{d}_{i,j}\eta_i\eta_j| \le \frac{1}{2}h^{-2}\big|D^2K(X,0)\big|^{k,X}$ and thus

$$\sum_{i,j=1}^d |\boldsymbol{d}_{i,j}| \le d\left(\sum_{i,j=1}^d |\boldsymbol{d}_{i,j}|^2\right)^{\frac{1}{2}} \le d^{\frac{3}{2}}\left(\lambda_{\max}(\boldsymbol{d}^2)\right)^{1/2} \le \frac{1}{2}d^{\frac{3}{2}}h^{-2}\big|D^2K(X,0)\big|^{k,X}. \tag{6.83}$$

Similarly, we have

$$L^{dk}\sum_{i=1}^d a_i\eta_i = \ell(\dot\varphi) = DK(X,0)(\dot\varphi) \le \big|DK(X,0)\big|^{k,X} h^{-1}\left(\sum_{i=1}^d |\eta_i|^2\right)^{\frac{1}{2}} L^{\frac{dk}{2}}. \tag{6.84}$$

The choice $\eta_i = a_i$ yields

$$\sum_{i=1}^d |a_i| \le d^{\frac{1}{2}}\left(\sum_{i=1}^d |a_i|^2\right)^{\frac{1}{2}} \le d^{\frac{1}{2}}h^{-1}L^{-\frac{dk}{2}}\big|DK(X,0)\big|^{k,X}. \tag{6.85}$$

For the evaluation of the second derivative we use a test function which satisfies

$$\dot\varphi(x) = \frac{1}{2}\sum_{i,j=1}^d \eta_{i,j}(x-\overline{x})_i(x-\overline{x})_j \quad \forall x \in ((B^*)^*)^*, \tag{6.86}$$

where $\overline{x} = \frac{1}{|B|}\sum_{x\in B} x$ and $\eta_{i,j} = \eta_{j,i}$. Then, for any $x \in (B^*)^*$,

$$\begin{aligned} \nabla_j\dot\varphi(x) &= \sum_{i=1}^d \eta_{i,j}(x-\overline{x})_i, \\ \nabla_i\nabla_j\dot\varphi(x) &= \eta_{i,j}, \text{ and} \\ \nabla^\alpha\dot\varphi(x) &= 0 \quad \text{ for } |\alpha| = 3. \end{aligned} \tag{6.87}$$

Now $|(x-\overline{x})_i| \le \frac{2^{d+1}-1}{2} L^k \le 2^d L^k$ for any $x \in (B^*)^*$ and thus $|\nabla_j \dot\varphi(x)| \le d^{\frac12} (\sum_{i=1}^d |\eta_{i,j}|^2)^{\frac12} 2^d L^k$ which yields

$$|\dot\varphi|_{k,B} \le \frac{1}{h}\big(2^d d^{\frac12} L^k L^{\frac{kd}{2}} (\sum_{i,j=1}^d |\eta_{i,j}|^2)^{\frac12} + |L^{k(\frac{d}{2}+1)}\big)(\sum_{i,j=1}^d |\eta_{i,j}|^2)^{\frac12} \le \tag{6.88}$$

$$\le (2^d d^{\frac12}+1) h^{-1} L^{k(\frac{d}{2}+1)} (\sum_{i,j=1}^d |\eta_{i,j}|^2)^{\frac12}.$$

Note that $\sum_{x\in B} \eta_{i,j} (x-\overline{x})_i a_i$ vanishes in view of the definition of $\overline{x}$. Hence

$$\sum_{i,j=1}^d L^{dk} \eta_{i,j} \boldsymbol{c}_{i,j} = \tag{6.89}$$

$$= \ell(\dot\varphi) \le |DK(X,0)|^{k,X} |\dot\varphi|_{k,X} \le (2^d d^{\frac12}+1) h^{-1} L^{k(\frac{d}{2}+1)} (\sum_{i,j=1}^d |\eta_{i,j}|^2)^{\frac12} |DK(X,0)|^{k,X}$$

Taking $\eta_{i,j} = \boldsymbol{c}_{i,j}$ we get

$$\sum_{i,j=1}^d |\boldsymbol{c}_{i,j}| \le d \left(\sum_{i,j=1}^d |\boldsymbol{c}_{i,j}|^2 \right)^{\frac12} \le (2^d d^{\frac32} + d) h^{-1} L^{-(\frac{d}{2}-1)k} |DK(X,0)|^{k,X}. \tag{6.90}$$

This yields the assertion with

$$C(d) = \max(1, d^{\frac12} + 2^d (d^{\frac32}+d), d^{\frac32}) = d^{\frac12} + 2^d (d^{\frac32}+d). \tag{6.91}$$

□

Proof of Lemma 6.7. We first note that $R_2(H,K,\boldsymbol{q}) = R_{2,a}^{(\boldsymbol{q})} H + R_{2,b}^{(\boldsymbol{q})} K$ where $R_{2,a}^{(\boldsymbol{q})}$ and $R_{2,b}^{(\boldsymbol{q})}$ are linear maps. Thus (6.74) is obvious. To prove the remaining statements we can consider the maps $H \mapsto R_{2,a}^{(\boldsymbol{q})} H$ and $K \mapsto R_{2,b}^{(\boldsymbol{q})} K$ separately. We will establish the relevant estimates for the directional derivatives $t \mapsto R_{2,a}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}$ and $t \mapsto R_{2,b}^{(\boldsymbol{q}+t\dot{\boldsymbol{q}})}$. The assertion on the existence and continuity of the total derivatives then follows as in the proof of Lemma 6.5, using in particular the continuity of the map $\boldsymbol{q} \mapsto \boldsymbol{R}^{(\boldsymbol{q})}$. We first consider the map

$$R_{2,a}^{(\boldsymbol{q})} H := \Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} H \tag{6.92}$$

which acts on ideal Hamiltonians. The integral of an odd functions against $\mu_{k+1}^{(\boldsymbol{q})}$ is zero and

$$\int_{\boldsymbol{\mathcal{X}}} Q(\xi,\xi)\, \mu_{k+1}^{(\boldsymbol{q})} = L^{dk} \frac12 \sum_{i,j} \boldsymbol{d}_{i,j} \nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0) \tag{6.93}$$

(cf. (4.81)). Thus $\boldsymbol{R}^{(\boldsymbol{q})} H$ is again an ideal Hamiltonian and the action of $R_2^{(a)}$ in the coordinates $(\lambda, a, \boldsymbol{c}, \boldsymbol{d})$ for H is simply

$$(\lambda, a, \boldsymbol{c}, \boldsymbol{d}) \mapsto (\lambda + \sum_{i,j} \boldsymbol{d}_{i,j} \nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0), a, \boldsymbol{c}, \boldsymbol{d}) \tag{6.94}$$

By (4.3) we have $|\nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)| \le C(d) L^{\eta(2,d)} L^{-dk}$ and thus

$$\|R_{2,a}^{(\boldsymbol{q})} H\| \le (1 + C(d) h^{-2} L^{\eta(2,d)}) \|H\|_{k,0} \le C(d) \|H\|_{k,0}, \tag{6.95}$$

where we used the lower bound on h in the assumption of the lemma. The estimates for $D_q^\ell R_{2,a}^{(\boldsymbol{q})} H$ follow in the same way from (4.3) since $h^2 \geq L^{\kappa(d)} \geq L^{\eta(8,d)}$.

Now let $X \in \mathcal{S} : k$ with $X \supset B$ and let $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$. We will estimate

$$\Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(X, \cdot) \tag{6.96}$$

and its derivatives with respect to $\boldsymbol{q}$. The operator $R_{2,b}^{(\boldsymbol{q})}$ is obtained by taking a sum over all such X (for a fixed block B) with weight $\frac{1}{|X|_k}$. Since there are at most $(3^d - 1)^{2^d}$ such polymers X is suffices to estimate (6.96).

By Lemma 6.9 and Lemma 5.1 (iv) we have

$$\begin{aligned} \frac{1}{C(d)} \|\Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(X, \cdot)\|_{k,0} &\leq |\boldsymbol{R}^{(\boldsymbol{q})} K(X, 0)|^{k,X,2} \\ \leq \int_{\boldsymbol{\mathcal{X}}} |K(X, \xi)|^{k,X,2} \mu_{k+1}^{(\boldsymbol{q})}(d\xi) &\leq 2^{|X|_k} \|K(X)\|_{k,X,2} \leq 2^{2^d} \|K(X)\|_{k,X,2} \\ &\leq 2^{2^d} \|K\|_{k,2}^{(\mathsf{A})}. \end{aligned} \tag{6.97}$$

The derivatives with respect to $\boldsymbol{q}$ are estimated using Gaussian calculus and the estimates used in the proof of Lemma 6.5. Let $\|\boldsymbol{q}\| < \frac{1}{2}$ and $\|\dot{\boldsymbol{q}}\| = 1$, and consider the curve $\boldsymbol{\gamma}(t) = \boldsymbol{q} + t\dot{\boldsymbol{q}}$ on a sufficiently small interval $(-a, a)$. Let

$$G(X, \varphi) := \operatorname{Tr}\big[D^2 K(X, \varphi) \dot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big]. \tag{6.98}$$

Then (see Appendic C)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\boldsymbol{R}^{(\boldsymbol{\gamma}(t))} K)(X, \varphi) = (\boldsymbol{R}^{(\boldsymbol{q})} G)(X, \varphi) \tag{6.99}$$

Now by (6.66) and (6.67) as well as the assumption on h we have

$$|G(X, \varphi)|^{k,X,2} \leq C |K(X, \varphi)|^{k,X,4}.$$

Using again Lemma 6.9 and Lemma 5.1 (iv) we get

$$\begin{aligned} \frac{1}{C(d)} \|D_q \Pi_2 \boldsymbol{R}^{(\boldsymbol{q})} K(X, \cdot)(\dot{\boldsymbol{q}})\|_{k,0} &= \frac{1}{C(d)} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Pi_2 \boldsymbol{R}^{(\boldsymbol{\gamma}(t))} K(X, \cdot) \right\|_{k,0} \\ \leq |(\boldsymbol{R}^{(\boldsymbol{q})} G)(X, 0)|^{k,X,2} \leq 2^{2^d} \|G(X)\|_{k,X,2} &\leq C 2^{2^d} \|K(X)\|_{k,X,4} \leq C 2^{2^d} \|K\|_{k,4}^{(\mathsf{A})}. \end{aligned} \tag{6.100}$$

The higher derivatives with respect to t are estimated in a similar way using the functions

$$G_2(X, \varphi) := \operatorname{Tr}\big[D^2 K(X, \varphi) \ddot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big], \quad G_3(X, \varphi) := \operatorname{Tr}\big[D^2 G(X, \varphi) \dot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big], \tag{6.101}$$

$$G_4(X, \varphi) := \operatorname{Tr}\big[D^2 K(X, \varphi) \dddot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big], \quad G_5(X, \varphi) := \operatorname{Tr}\big[D^2 G(X, \varphi) \ddot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big], \tag{6.102}$$

$$G_6(X, \varphi) := \operatorname{Tr}\big[D^2 G_3(X, \varphi) \dot{\mathcal{C}}_{k+1}^{(\boldsymbol{q})}\big]. \tag{6.103}$$

and the estimates (see (6.66) and (6.67))

$$|G_2(X, \xi)|^{k,X,2} + |G_4(X, \xi)|^{k,X,2} \leq C |K(X, \xi)|^{k,X,4}, \tag{6.104}$$

$$|G_3(X, \xi)|^{k,X,2} + |G_5(X, \xi)|^{k,X,2} \leq C |G(X, \xi)|^{k,X,4} \leq C |K(X, \xi)|^{k,X,6}, \tag{6.105}$$

$$|G_6(X, \xi)|^{k,X,2} \leq C |G_3(X, \xi)|^{k,X,4} \leq C |K(X, \xi)|^{k,X,8}. \tag{6.106}$$

□

6.6. The map P_1

LEMMA 6.10. *Consider the map*

$$P_1 \colon \boldsymbol{M}_{|||} \times \boldsymbol{M}_{|||} \times \widehat{\boldsymbol{M}}_{:,r} \to \boldsymbol{M}'_r$$

defined in (6.10)*, restricted to* $B_{\rho_1}(1) \times B_{\rho_2} \times \widehat{\boldsymbol{M}}_{:,r} \subset \boldsymbol{M}_{|||} \times \boldsymbol{M}_{|||} \times \widehat{\boldsymbol{M}}_{:,r}$ *with the balls* $B_{\rho_1}(1)$ *and* B_{ρ_2} *defined in terms of respective norms* $|||\cdot|||_k$*, i.e.,* $B_{\rho_1}(1) = \{\widetilde{I} \in \boldsymbol{M}_{|||} \colon |||\widetilde{I} - 1|||_k < \rho_1\}$ *and* $B_{\rho_2} = \{\widetilde{J} \in \boldsymbol{M}_{|||} \colon |||\widetilde{J}|||_k < \rho_2\}$*, and the target space* $\boldsymbol{M}'_r$ *equipped with the norm* $\|\cdot\|^{(\mathsf{A})}_{k+1,r}$*. There exists* $\mathsf{A}_0 = \mathsf{A}_0(L,d)$ *such that for any* $\mathsf{A} \geq \mathsf{A}_0$ *and* ρ_1, ρ_2*, and* $\tilde{B}$ *such that*

$$\rho_1 \leq 1/2, \ \ \rho_2 < (2\mathsf{A}^{1+2^{d+2}})^{-1} \ \ \text{and} \ \ \tilde{\mathsf{B}} \geq \mathsf{A}^{2^{d+3}} \tag{6.107}$$

the map P_1 *is smooth and, for any* $j_1, j_2 \in \mathbb{N}$*, satisfies the bounds*

$$\begin{aligned} \tfrac{1}{j_1!}\tfrac{1}{j_2!}\|D_1^{j_1} D_2^{j_2} P_1(\widetilde{I}, \widetilde{J}, \widetilde{P})(\dot{\widetilde{I}}, \dots, \dot{\widetilde{I}}, \dot{\widetilde{J}}, \dots, \dot{\widetilde{J}})\|^{(\mathsf{A})}_{k+1,r} \leq \\ \leq |||\dot{\widetilde{I}}|||_k^{j_1} \left(\mathsf{A}^{1+2^{d+2}} |||\dot{\widetilde{J}}|||_k\right)^{j_2} \max\left(\|\widetilde{P}\|^{(\mathsf{A}/4,\tilde{\mathsf{B}})}_{k:k+1,r}, 1\right), \end{aligned} \tag{6.108}$$

$$\begin{aligned} \tfrac{1}{j_1!}\tfrac{1}{j_2!}\|D_1^{j_1} D_2^{j_2} D_3 P_1(\widetilde{I}, \widetilde{J}, \widetilde{P})(\dot{\widetilde{I}}, \dots, \dot{\widetilde{I}}, \dot{\widetilde{J}}, \dots, \dot{\widetilde{J}}, \dot{\widetilde{P}})\|^{(\mathsf{A})}_{k+1,r} \leq \\ \leq |||\dot{\widetilde{I}}|||_k^{j_1} \left(\mathsf{A}^{1+2^{d+2}} |||\dot{\widetilde{J}}|||_k\right)^{j_2} \|\dot{\widetilde{P}}\|^{(\mathsf{A}/4,\tilde{\mathsf{B}})}_{k:k+1,r}, \end{aligned} \tag{6.109}$$

$$D_1^{j_1} D_2^{j_2} D_3^{j_3} P_1 = 0 \quad \text{for } j_3 \geq 2. \tag{6.110}$$

PROOF. Since P_1 is affine in the last argument, (6.110) is obvious and (6.109) follows from (6.108). Indeed since $\widetilde{P}(\varnothing) \equiv 1$ the map P_1 can be written as

$$P_1(\widetilde{I}, \widetilde{J}, \widetilde{P}) = P_1^0(\widetilde{I}, \widetilde{J}) + P_1^1(\widetilde{I}, \widetilde{J}, \widetilde{P}) \tag{6.111}$$

with

$$P_1^0(\widetilde{I}, \widetilde{J})(U) = \sum_{X_1 \in \mathcal{P}(U)} \chi(X_1, U) \widetilde{I}^{U \setminus X_1} \widetilde{J}^{X_1}, \tag{6.112}$$

$$P_1^1(\widetilde{I}, \widetilde{J}, \widetilde{P}) = \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing, X_2 \neq \varnothing}} \chi(X_1 \cup X_2, U) \widetilde{I}^{U \setminus (X_1 \cup X_2)} \widetilde{J}^{X_1} \widetilde{P}(X_2) \tag{6.113}$$

Since P_1^1 is linear in P we have

$$D_3 P_1(\widetilde{I}, \widetilde{J}, \widetilde{P})(\dot{\widetilde{P}}) = P_1^1(\widetilde{I}, \widetilde{J}, \dot{\widetilde{P}}) = \lim_{\lambda \to \infty} \tfrac{1}{\lambda} P_1(\widetilde{I}, \widetilde{J}, \lambda \widetilde{P}) \tag{6.114}$$

and an analogous identity holds for $\frac{1}{j_1!}\frac{1}{j_2!} D_1^{j_1} D_2^{j_2} D_3 P_1$. Thus (6.109) follows from (6.108).

To prove (6.108) we first consider the case $j_1 = j_2 = 0$. Pick $U \in \mathcal{P}^{\mathrm{c}}_{k+1}$. Taking into account that

$$\|F(U)\|_{k+1,U,r} \leq \|F(U)\|_{k:k+1,U,r},$$

and applying Lemma 5.1 (iib) we get

$$(6.115)\ \|P_1^1(\widetilde{I},\widetilde{J},\widetilde{P})(U)\|_{k+1,U,r} \le$$
$$\le \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing, X_2\neq\varnothing}} \chi(X_1\cup X_2,U)|||\widetilde{I}|||_k^{|U\setminus(X_1\cup X_2)|}|||\widetilde{J}|||_k^{|X_1|}\,\|\widetilde{P}(X_2)\|_{k:k+1,X_2,r}$$
$$\le \sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing}} \chi(X_1\cup X_2,U)\,2^{|U\setminus(X_1\cup X_2)|}\mathsf{A}^{-(1+2^{d+2})|X_1|}\|\widetilde{P}\|_{k:k+1,r}^{(\mathsf{A}/4,\tilde{\mathsf{B}})}\Gamma_{\mathsf{A}/4}(X_2)^{-1}\tilde{\mathsf{B}}^{-|\mathcal{C}(X_2)|}$$

Now

$$\Gamma_{\mathsf{A}/4}(X_2) \ge \left(\tfrac{\mathsf{A}}{4}\right)^{|X_2|-2^d|\mathcal{C}(X_2)|} \tag{6.116}$$

and using that $\tilde{\mathsf{B}} \ge \mathsf{A}^{2^{d+3}}$ and $2^{d+3}-2^d \ge 2^{d+2}$ we get

$$(6.117)\ \|P_1^1(\widetilde{I},\widetilde{J},\widetilde{P})(U)\|_{k+1,U,r} \le$$
$$\le 4^{|U|}\sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing, X_2\neq\varnothing}} \chi(X_1\cup X_2,U)\mathsf{A}^{-(1+2^{d+2})|X_1|-|X_2|-2^{d+2}|\mathcal{C}(X_2)|)}\ \|\widetilde{P}\|_{k:k+1,r}^{(\mathsf{A}/4,\tilde{\mathsf{B}})}.$$

Now, we will rely on the combinatorial Lemma 6.16 from [**Bry09**] stated in (F.2) in Lemma F.1,

$$|X|_k \ge (1+\alpha(d))|\overline{X}|_{k+1}-(1+\alpha(d))2^{d+1}|\mathcal{C}(X)| \ \text{ with }\ \alpha(d)=\frac{1}{(1+2^d)(1+6^d)}. \tag{6.118}$$

Applying this inequality with $X=X_1\cup X_2$ and using the trivial estimate $\mathcal{C}(X_1\cup X_2)\le |X_1|+\mathcal{C}(X_2)$, we get

$$(1+2^{d+2})|X_1|_k+|X_2|_k+2^{d+2}|\mathcal{C}(X_2)| \ge (1+\alpha(d))|\overline{X_1\cup X_2}|_{k+1} \tag{6.119}$$

and thus

$$\begin{aligned}&\|P_1^1(\widetilde{I},\widetilde{J},\widetilde{P})(U)\|_{k+1,U,r}\\ &\quad\le 4^{|U|_k}\sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing, X_2\neq\varnothing}} \chi(X_1\cup X_2,U)\mathsf{A}^{-(1+\alpha(d))|\overline{X_1\cup X_2}|_{k+1}}\ \|\widetilde{P}\|_{k:k+1,r}^{(\mathsf{A}/4,\tilde{\mathsf{B}})}.\end{aligned} \tag{6.120}$$

Similarly we obtain for P_1^0

$$\begin{aligned}\|P_1^0(\widetilde{I},\widetilde{J})(U)\|_{k+1,U,r} &\le \sum_{X_1\in\mathcal{P}(U)}\chi(X_1,U)|||\widetilde{I}|||_k^{|U\setminus X_1|}|||\widetilde{J}|||_k^{|X_1|}\\ &\le 2^{|U|}\sum_{X_1\in\mathcal{P}(U)}\chi(X_1,U)\mathsf{A}^{-(1+2^{d+2})|X_1|}\end{aligned} \tag{6.121}$$

Since $\alpha(d)\le 1\le 2^{d+2}$ and since $|X_1|_k \ge |\overline{X_1}|_{k+1}$ it is easy to combine the estimates for P_1^1 and P_1^0. To prove (6.108) for $j_1=j_2=0$ it thus suffices to show that

$$\Gamma_{\mathsf{A}}(U)\ 4^{|U|_k}\sum_{\substack{X_1,X_2\in\mathcal{P}(U)\\ X_1\cap X_2=\varnothing}}\chi(X_1\cup X_2,U)\mathsf{A}^{-(1+\alpha(d))|\overline{X_1\cup X_2}|_{k+1}}\le 1. \tag{6.122}$$

for any $U\in\mathcal{P}_{k+1}^{\mathrm{c}}$ once

$$\mathsf{A}\ge \mathsf{A}_0(L,d) = (12)^{L^d(1+2^d)(1+6^d)}. \tag{6.123}$$

If $|U|_{k+1} \le 2^d$ then $\Gamma_{\mathsf{A}}(U) = 1$ and we use $|U|_k = L^d |U|_{k+1}$ as well as the fact that the sum in (6.122) has at most $3^{|U|_k} \le 3^{L^d 2^d}$ terms, each contributing at most $\mathsf{A}^{-1} \le \mathsf{A}^{-2^d \alpha(d)}$ to bound the left hand side of (6.122) by

$$4^{(2L)^d} 3^{(2L)^d} \mathsf{A}^{-1} \le \left((12)^{L^d} \mathsf{A}^{-\alpha(d)} \right)^{2^d} \le 1. \tag{6.124}$$

For $|U|_{k+1} > 2^d$, there is no $B \in \mathcal{P}_k$ such that $U = \overline{B^*}$ and as a result $X_1 \cup X_2$ is not small and $U = \overline{X_1 \cup X_2}$ (cf. definition (4.69) of $\chi(X_1 \cup X_2, U)$). Hence, using again that the number of terms in the sum is bounded by $3^{|U|_k}$, we can bound the left hand side of (6.122) by

$$\begin{aligned} &\mathsf{A}^{|U|_{k+1}} 4^{L^d |U|_{k+1}} \mathsf{A}^{-(1+\alpha(d))|U|_{k+1}} \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing}} \chi(X_1 \cup X_2, U) \\ &\le (12)^{L^d |U|_{k+1}} \mathsf{A}^{-\alpha(d)|U|_{k+1}} \le 1 \end{aligned} \tag{6.125}$$

once $(12)^{L^d} \mathsf{A}^{-\alpha(d)} \le 1$.

For the derivatives

$$\begin{aligned} &\tfrac{1}{j_1!} \tfrac{1}{j_2!} D_1^{j_1} D_2^{j_2} P_1^1(\widetilde{I}, \widetilde{J}, \widetilde{P})(U)(\dot{\widetilde{I}}, \dots, \dot{\widetilde{I}}, \dot{\widetilde{J}}, \dots, \dot{\widetilde{J}}) \\ &= \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing, X_2 \ne \varnothing}} \chi(X_1 \cup X_2, U) \sum_{\substack{Y_1 \in \mathcal{P}(U \setminus (X_1 \cup X_2)), |Y_1| = j_1 \\ Y_2 \in \mathcal{P}(X_1), |Y_2| = j_2}} \widetilde{I}^{(U \setminus (X_1 \cup X_2)) \setminus Y_1} (\dot{\widetilde{I}})^{Y_1} \widetilde{J}^{X_1 \setminus Y_2} (\dot{\widetilde{J}})^{Y_2} \widetilde{P}(X_2) \end{aligned} \tag{6.126}$$

we proceed as above in (6.115) and (6.117) to get

$$\begin{aligned} &\tfrac{1}{j_1!} \tfrac{1}{j_2!} \| D_1^{j_1} D_2^{j_2} P_1^1(\widetilde{I}, \widetilde{J}, \widetilde{P})(U)(\dot{\widetilde{I}}, \dots, \dot{\widetilde{I}}, \dot{\widetilde{J}}, \dots, \dot{\widetilde{J}}) \|_{k+1, U, r} \le \\ &\le \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing, X_2 \ne \varnothing}} \chi(X_1 \cup X_2, U) \binom{|U \setminus (X_1 \cup X_2)|}{j_1} |||\widetilde{I}|||_k^{|U \setminus (X_1 \cup X_2)| - j_1} \binom{|X_1|}{j_2} \times \\ &\qquad \times |||\widetilde{J}|||_k^{|X_1| - j_2} \| P(X_2) \|_{k:k+1, X_2, r} \, |||\dot{\widetilde{I}}|||_k^{j_1} \, |||\dot{\widetilde{J}}|||_k^{j_2} \le \\ &\le \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing, X_2 \ne \varnothing}} \chi(X_1 \cup X_2, U) 2^{|U \setminus (X_1 \cup X_2)|} 2^{|U \setminus (X_1 \cup X_2)| - j_1} 2^{|X_1|} \times \\ &(2\mathsf{A}^{1+2^{d+2}})^{-|X_1| + j_2} \left(\tfrac{\mathsf{A}}{4}\right)^{-|X_2| + 2^d |\mathcal{C}(X_2)|} \mathsf{A}^{-2^{d+3} |\mathcal{C}(X_2)|} \, \|\widetilde{P}\|_{k:k+1, r}^{(\mathsf{A}/4, \tilde{\mathsf{B}})} \, |||\dot{\widetilde{I}}|||_k^{j_1} \, |||\dot{\widetilde{J}}|||_k^{j_2} \le \\ &\le \|\widetilde{P}\|_{k:k+1, r}^{(\mathsf{A}/4, \tilde{\mathsf{B}})} \, |||\dot{\widetilde{I}}|||_k^{j_1} \left(\mathsf{A}^{1+2^{d+2}} |||\dot{\widetilde{J}}|||_k \right)^{j_2} \times \\ &\quad \times 4^{|U|} \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing, X_2 \ne \varnothing}} \chi(X_1 \cup X_2, U) \mathsf{A}^{-(1+2^{d+2})|X_1| - |X_2| - 2^{d+2} |\mathcal{C}(X_2)|)}. \end{aligned} \tag{6.127}$$

Similarly we get

$$\frac{1}{j_1!}\frac{1}{j_2!}\|D_1^{j_1}D_2^{j_2}P_1^0(\widetilde{I},\widetilde{J})(U)(\dot{\widetilde{I}},\ldots,\dot{\widetilde{I}},\dot{\widetilde{J}},\ldots,\dot{\widetilde{J}})\|_{k+1,U,r} \leq \tag{6.128}$$
$$\leq \sum_{X_1\in\mathcal{P}(U)} \chi(X_1,U)2^{|U\setminus X_1|}2^{|U\setminus X_1)|-j_1}2^{|X_1|}\,(2\mathsf{A}^{1+2^{d+2}})^{-|X_1|+j_2}\,|||\dot{\widetilde{I}}|||_k^{j_1}\,|||\dot{\widetilde{J}}|||_k^{j_2}$$
$$\leq |||\dot{\widetilde{I}}|||_k^{j_1}\left(\mathsf{A}^{1+2^{d+2}}|||\dot{\widetilde{J}}|||_k\right)^{j_2}4^{|U|}\sum_{X_1\in\mathcal{P}(U)}\chi(X_1,U)\mathsf{A}^{-(1+2^{d+2})|X_1|}$$

Now (6.108) follows as in the case $j_1 = j_2 = 0$ by using (6.119) and (6.122) as well as the obvious estimates $\alpha(d) \leq 1 \leq 2^{d+2}$ and $|X_1|_k \geq |\overline{X_1}|_{k+1}$.

□

6.7. Proof of Proposition 4.6

Proposition 4.6 now follows from the estimates on the maps $E, P_1, R_1, R_2,\ P_2$ and P_3 and the chain rule, Theorem D.29, in connection with Remark D.30 which provides uniform control of the relevant derivatives. For the convenience of the reader we spell out the details. We first write S as a composition of five maps $\boldsymbol{F}_1,\ldots,\boldsymbol{F}_5$ and describe the scales of Banach spaces $\boldsymbol{X}^{(i)}, i=1,\ldots,5$, on which these maps are defined. Then we recursively identify neighbourhoods $\boldsymbol{U}^{(i)}\subset\boldsymbol{X}^{(i)}$ such that

$$\boldsymbol{F}_i \in \widetilde{C}^m(\boldsymbol{U}^{(i)}\times B_{\frac{1}{2}}), \quad i=1,\ldots,5,$$

and verify that $\boldsymbol{F}_i(\boldsymbol{U}^{(i)}\times B_{\frac{1}{2}})\subset\boldsymbol{U}^{(i-1)}$ for $i\geq 2$ and that each map $\boldsymbol{F}_i$ satisfies the assumptions of the chain rule Theorem D.29. Recall the definitions in Appendix D and denote by $\diamond$ the composition defined by

$$\big(\boldsymbol{F}\diamond\boldsymbol{G}\big)(\boldsymbol{x},\boldsymbol{p}) := \boldsymbol{F}(\boldsymbol{G}(\boldsymbol{x},\boldsymbol{p}),\boldsymbol{p}). \tag{6.129}$$

Define

$$\tilde{\mathsf{B}} = \mathsf{A}^{2^{d+3}}, \quad \mathsf{B} = 2^{2^d}\tilde{\mathsf{B}}. \tag{6.130}$$

In the following we will always assume

$$r_0 \geq 2m+2. \tag{6.131}$$

We also assume that

$$\mathsf{A} \geq \mathsf{A}_0(L,d) \tag{6.132}$$

where $\mathsf{A}_0(L,d)$ is the quantity in Lemma 6.10 and

$$h \geq L^{\kappa(d)}h_1 \quad \text{with} \quad h_1 = h_1(d,\omega) \tag{6.133}$$

and $\kappa(d)$ as in Lemma 5.1 (iv) (see (5.68)).

Note that

$$S = \boldsymbol{F}_1\diamond\boldsymbol{F}_2\diamond\boldsymbol{F}_3\diamond\boldsymbol{F}_4\diamond\boldsymbol{F}_5, \tag{6.134}$$

where the maps $\boldsymbol{F}_i, i=1,\ldots,5$, and the scales of Banach spaces are given by

$$\boldsymbol{F}_1\colon \boldsymbol{X}^{(1)}\times B_{\frac{1}{2}}\to\boldsymbol{X}^{(0)}, \quad \boldsymbol{F}_1(K_1,K_2,K_3,\boldsymbol{q}) = P_1(K_1,K_2,K_3), \tag{6.135}$$

with

$$\begin{aligned} \boldsymbol{X}^{(1)}_n &= \boldsymbol{M}^2_{|||} \times (\widehat{\boldsymbol{M}}_{:,r_0-2m+2n}, \|\cdot\|^{(\mathsf{A}/4,\tilde{\mathsf{B}})}_{k:k+1,r_0-2m+2n}) \\ \boldsymbol{X}^{(0)}_n &= (\boldsymbol{M}'_{r_0-2m+2n}, \|\cdot\|^{(\mathsf{A})}_{k+1,r_0-2m+2n}), \\ B_{\frac{1}{2}} &= \{\boldsymbol{q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \colon \|\boldsymbol{q}\| < \frac{1}{2}\}; \end{aligned} \tag{6.136}$$

and

$$\boldsymbol{F}_2 \colon \boldsymbol{X}^{(2)} \times B_{\frac{1}{2}} \to \boldsymbol{X}^{(1)}, \quad \boldsymbol{F}_2(H, K, \boldsymbol{q}) := (E(H), 1 - E(H), R_1(K, \boldsymbol{q})), \tag{6.137}$$

with

$$\boldsymbol{X}^{(2)}_n = (\boldsymbol{M}_0, \|\cdot\|_{k,0}) \times (\widehat{\boldsymbol{M}}_{r_0-2m+2n}, \|\cdot\|^{(\mathsf{A}/2,\mathsf{B})}_{k,r_0-2m+2n}); \tag{6.138}$$

and

$$\boldsymbol{F}_3 \colon \boldsymbol{X}^{(3)} \to \boldsymbol{X}^{(2)}, \quad \boldsymbol{F}_3(H, K) := (H, P_3(K)), \tag{6.139}$$

with

$$\boldsymbol{X}^{(3)}_n = (\boldsymbol{M}_0, \|\cdot\|_{k,0}) \times (\boldsymbol{M}_{r_0-2m+2n}, \|\cdot\|^{(\mathsf{A}/2)}_{k,r_0-2m+2n}) \tag{6.140}$$

$$\boldsymbol{F}_4 \colon \boldsymbol{X}^{(4)} \times B_{\frac{1}{2}} \to \boldsymbol{X}^{(3)}, \quad \boldsymbol{F}_4(H, \widetilde{K}, K, \boldsymbol{q}) := (R_2(H, K, \boldsymbol{q}), P_2(\widetilde{K}, K)), \tag{6.141}$$

with

$$\boldsymbol{X}^{(4)}_n = (\boldsymbol{M}_0, \|\cdot\|_{k,0}) \times \boldsymbol{M}_{|||} \times (\boldsymbol{M}_{r_0-2m+2n}, \|\cdot\|^{(\mathsf{A})}_{k,r}) \tag{6.142}$$

and

$$\boldsymbol{F}_5 \colon \boldsymbol{X}^{(5)} \times B_{\frac{1}{2}} \to \boldsymbol{X}^{(4)}, \quad \boldsymbol{F}_5(H, K) := (H, E(H), K), \tag{6.143}$$

with

$$\boldsymbol{X}^{(5)}_n = (\boldsymbol{M}_0, \|\cdot\|_{k,0}) \times (\boldsymbol{M}_{r_0-2m+2n}, \|\cdot\|^{(\mathsf{A})}_{k,r_0-2m+2n}). \tag{6.144}$$

Let

$$\begin{aligned} &\boldsymbol{U}^{(1)} = B_{\rho_1}(1) \times B_{\rho_2} \times \widehat{\boldsymbol{M}}_{:,r_0} \subset \boldsymbol{X}^{(1)}_m \ \text{ with} \\ &\rho_1 \le \frac{1}{2},\ \rho_2 < \big(2\mathsf{A}^{1+2^{d+2}}\big)^{-1}. \end{aligned} \tag{6.145}$$

Then by Lemma 6.10 we have

$$\boldsymbol{F}_1 \in \widetilde{C}^m(\boldsymbol{U}^{(1)} \times B_{\frac{1}{2}}, \boldsymbol{X}^{(0)}), \tag{6.146}$$

and the derivatives of $\boldsymbol{F}_1$ satisfy the assumptions of the chain rule, Theorem D.29. Let $C_{6.1}$ denote the constant in (6.21) in Lemma 6.1 (we may assume that $C_{6.1} \ge 1$) and let

$$\rho_3 = \frac{1}{C_{6.1}} \min\{\rho_1, \rho_2\} = \frac{\rho_2}{C_{6.1}}. \tag{6.147}$$

Then $H \in B_{\rho_3}$ implies that $E(H) - 1 \in B_{\rho_1} \cap B_{\rho_2} \subset \boldsymbol{M}^2_{|||}$. Thus the choice

$$\boldsymbol{U}^{(2)} := B_{\rho_3} \times \widehat{\boldsymbol{M}}_{r_0}$$

yields

$$\boldsymbol{F}_2(\boldsymbol{U}^{(2)} \times B_{\frac{1}{2}}) \subset \boldsymbol{U}^{(1)}. \tag{6.148}$$

Moreover by Lemma 6.1 and Lemma 6.5 the map $\boldsymbol{F}_2 \colon \boldsymbol{U}^{(2)} \times B_{\frac{1}{2}} \to \boldsymbol{X}_m^{(1)}$ satisfies the assumptions of the chain rule, Theorem D.29.

Let

$$\rho_4 := (2\mathsf{B})^{-1}, \quad \boldsymbol{U}^{(3)} = B_{\rho_3} \times B_{\rho_4}$$

Then

$$\boldsymbol{F}_3(\boldsymbol{U}^{(3)} \times B_{\frac{1}{2}}) \subset \boldsymbol{U}^{(2)} \tag{6.149}$$

and by Lemma 6.4 the map F_3 is a smooth map on $\boldsymbol{U}^{(3)}$ and on $\boldsymbol{U}^{(3)}$ satisfies the assumptions of the chain rule Theorem D.29. Note that we are applying Lemma D.32 for those maps which do not depend on $\boldsymbol{q}$ like F_1, F_2 and F_5.

We have $\rho_4 \leq 1$. Let $C_{6.7}$ be the constant in Lemma 6.7 and let

$$\rho_5 = \frac{\rho_3}{2C_{6.7}}, \; \rho_6 = \frac{\rho_4}{4\mathsf{A}}, \; \rho_7 = \min\left\{\frac{\rho_3}{2C_{6.7}}, \frac{\rho_4}{4\mathsf{A}^{2^d}}\right\}. \tag{6.150}$$

Then it follows from (6.34) in Lemma 6.3 and Lemma 6.7 (with $r_1 = r_0$) that

$$\boldsymbol{F}_4(B_{\rho_5} \times B_{\rho_6}(1) \times B_{\rho_7} \times B_{\frac{1}{2}}) \subset B_{\rho_3} \times B_{\rho_4} = \boldsymbol{U}^{(3)}. \tag{6.151}$$

Set $\boldsymbol{U}^{(4)} := B_{\rho_5} \times B_{\rho_6}(1) \times B_{\rho_7}$. Then $\boldsymbol{F}^{(4)} \colon \boldsymbol{U}^{(4)} \times B_{\frac{1}{2}} \to \boldsymbol{X}_m^{(3)}$ satisfies the assumptions of the chain rule.

Finally set

$$\rho_8 = \frac{\rho_6}{C_{6.1}}, \; \rho_9 = \rho_7, \; \text{ and } \boldsymbol{U}^{(5)} = B_{\rho_8} \times B_{\rho_9}. \tag{6.152}$$

Then $\boldsymbol{F}_5(\boldsymbol{U}^{(5)} \times B_{\frac{1}{2}}) \subset \boldsymbol{U}^{(4)}$ and $\boldsymbol{F}_5 \colon \boldsymbol{U}^{(5)} \times B_{\frac{1}{2}} \to \boldsymbol{X}_m^{(4)}$ satisfies the assumptions of the chain rule. Now an application of the chain rule, Theorem D.29, shows that the conclusions of Proposition 4.6 hold with $\rho = \min\{\rho_8, \rho_9\}$.

□

CHAPTER 7

Linearization of the Renormalization Map

Here we prove Proposition 4.7 summarizing the properties of the linearization (4.80) of the maps $\boldsymbol{T}_k$ at the fixed point $(H_k, K_k) = (0, 0)$ guaranteeing that H_k and K_k are the relevant and irrelevant variables, respectively. First, we prove the contraction property of the operator $\boldsymbol{C}^{(\boldsymbol{q})}$ in Section 7.2. We finish the proof of Proposition 4.7 in Section 7.2 with the bounds on the operators ${\boldsymbol{A}^{(\boldsymbol{q})}}^{-1}$ and $\boldsymbol{B}^{(\boldsymbol{q})}$.

7.1. Contractivity of operator $\boldsymbol{C}^{(\boldsymbol{q})}$

LEMMA 7.1. *Let $\theta \in (\frac{1}{4}, \frac{3}{4})$ and $\omega \geq 2(d^2 2^{2d+1} + 1)$. Consider the constant $h_1 = h_1(d, \omega)$ and $\kappa(d)$ chosen from Lemma 5.1 and let $L \geq 2^d + 1$, $h \geq L^{\kappa(d)} h_1(d, \omega)$. There exists $\mathsf{A}_0 = \mathsf{A}_0(d, L)$ such that*

$$\|\boldsymbol{C}^{(\boldsymbol{q})}\|_r^{(\mathsf{A})} = \sup_{\|K\|_{k,r}^{(\mathsf{A})} \leq 1} \|\boldsymbol{C}^{(\boldsymbol{q})} K\|_{k+1,r}^{(\mathsf{A})} \leq \theta. \tag{7.1}$$

for any $\|\boldsymbol{q}\| \leq \frac{1}{2}$, any $k = 1, \dots, N$, $r = 1, \dots, r_0$, and any $\mathsf{A} \geq \mathsf{A}_0$.

PROOF. Let us begin by evaluating the large set term: the last term on the right hand side of (4.83).

LEMMA 7.2. *Let $L \geq 2^d + 1$ and $\omega \geq 18\sqrt{2} + 1$. Whenever $h \geq L^{\kappa(d)} h_1$, and A such that $2\mathsf{A}^{-\frac{2\alpha}{1+2\alpha}} \leq \frac{1}{8}\delta(d, L)$ with α from Lemma F.1 and $\delta(d, L)$ from Lemma F.2, then*

$$\|F\|_{k+1,r}^{(\mathsf{A})} \leq \tfrac{\theta}{2} \|K\|_{k,r}^{(\mathsf{A})} \tag{7.2}$$

for any $K \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$. Here, the function $F \in M(\mathcal{P}_{k+1}, \boldsymbol{\mathcal{X}})$ is defined by

$$F(U, \varphi) = \sum_{\substack{X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ \overline{X} = U}} \int_{\boldsymbol{\mathcal{X}}} K(X, \varphi + \xi) \mu_{k+1}(\mathrm{d}\xi). \tag{7.3}$$

PROOF. Considering, for any $X \subset U$, the function $(\boldsymbol{R}_{k+1} K)(X, \varphi)$ and its norm $|(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,U,r}$ as defined by (4.25), we have

$$\sup_{\varphi} |(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,U,r} w_{k+1}^{-U} \leq \sup_{\varphi} |(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,X,r} w_{k:k+1}^{-X}. \tag{7.4}$$

To see it, we just notice that, as in (5.4) in the proof of Lemma 5.1, one has

$$|(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,U,r} \leq |(\boldsymbol{R}_{k+1} K)(X, \varphi)|^{k+1,X,r} \tag{7.5}$$

and that

$$w_{k+1}^{-U}(\varphi) \leq w_{k:k+1}^{-X}. \tag{7.6}$$

The last inequality amounts to

$$\sum_{x\in X}\big((2^d\omega-1)g_{k:k+1,x}(\varphi)+\omega G_{k,x}(\varphi)\big)+3L^k\sum_{x\in\partial X}G_{k,x}(\varphi)\le \tag{7.7}$$
$$\le\sum_{x\in U}\omega\big(2^d g_{k+1,x}(\varphi)+G_{k+1,x}(\varphi)\big)+L^{k+1}\sum_{x\in\partial U}G_{k+1,x}(\varphi).$$

This is clearly valid since $g_{k:k+1,x}(\varphi)\le g_{k+1,x}(\varphi)$, $G_{k,x}(\varphi)\le G_{k+1,x}(\varphi)$, and any $x\in\partial X\setminus\partial U$ is necessarily contained in ∂B for some $B\in\mathcal{B}_k(U\setminus X)$ and, in view of (5.16), for each such B one has

$$3L^k\sum_{x\in\partial B}G_{k,x}(\varphi)\le\sum_{x\in B}\omega\big(2^d g_{k+1,x}(\varphi)+G_{k+1,x}(\varphi)\big) \tag{7.8}$$

once $\omega\ge 6c+1$.

Combining now (7.4) with the bound from Lemma 5.1 (iv), we get

$$\Gamma_{k+1,A}(U)\|F(U)\|_{k+1,U,r}\le \mathsf{A}^{|U|_{k+1}}\sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{S}_k\\ \overline{X}=U}}2^{|X|_k}\|K(X)\|_{k,X,r}\le \tag{7.9}$$
$$\le\|K\|_{k,r}^{(\mathsf{A})}\,\mathsf{A}^{|U|_{k+1}}\sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{S}_k\\ \overline{X}=U}}\big(\tfrac{\mathsf{A}}{2}\big)^{-|X|_k}\le\|K\|_{k,r}^{(\mathsf{A})}\sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{S}_k\\ \overline{X}=U}}\big(2\mathsf{A}^{-\frac{2\alpha}{1+2\alpha}}\big)^{|X|_k}\le\tfrac{\theta}{2}\|K\|_{k,r}^{(\mathsf{A})}.$$

Here, in the last two inequalities, we first used $|X|_k\ge(1+2\alpha(d))|\overline{X}|_{k+1}$ for any X contributing to the sum (see [**Bry09**, Lemma 6.15]; (F.1) in Lemma F.1) and then applied Lemma F.2 assuming that $2\mathsf{A}^{-\frac{2\alpha}{1+2\alpha}}\le\frac{\theta}{2}\delta(d,L)$. □

Turning to the first term on the right hand side of (4.83), we have:

LEMMA 7.3. *Let* $L\ge 7$, $\omega\ge 2(d^2 2^{2d+1}+1)$, $h\ge L^{\kappa(d)}h_1$, *and* $K\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ *with* $G\in M(\mathcal{P}_{k+1},\boldsymbol{\mathcal{X}})$ *defined by*

$$G(U,\varphi)=\sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}}\big(1-\Pi_2\big)\sum_{\substack{X\in\mathcal{S}_k,\\ X\supset B}}\frac{1}{|X|_k}(\boldsymbol{R}_{k+1}K)(X,\varphi). \tag{7.10}$$

Then

$$\|G\|_{k+1,r}^{(\mathsf{A})}\le 2^{d+2^d}(3^d-1)^{2^d}\big(5L^{-\frac{d}{2}}+2^{d+3}L^{\frac{d}{2}-2}+9L^{-1}\big)\|K\|_{k,r}^{(\mathsf{A})} \tag{7.11}$$

for any $\mathsf{A}>1$.

REMARK 7.4. Notice that (7.11) is used later only for $d\le 3$. Our method can be extended also to include higher dimension when employing additional higher order terms to estimate the projection of the second Taylor polynomial. ◇

PROOF. Notice first that the sum vanishes unless $U\in\mathcal{S}_{k+1}$ and, necessarily, for any contributing X, one has $X\subset U$ and $X^*\subset U^*$. As a result, the norms in (7.11) contain only the contributions of small sets and do not depend on A according to the definition of the factor $\Gamma_{j,\mathsf{A}}(X)$, $j=k,k+1$. Considering $R\in M^*(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$ defined by $R(B,\varphi)=\sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}}\frac{1}{|X|_k}(\boldsymbol{R}_{k+1}K)(X,\varphi)$ and replacing the operator $1-\Pi_2$ by $(1-T_2)+(T_2-\Pi_2)$, we split $G(U,\varphi)$ into two terms,

$$G_1(U,\varphi)=\sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}}(1-T_2)R(B,\varphi) \tag{7.12}$$

and

$$G_2(U,\varphi) = \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} (T_2-\Pi_2)R(B,\varphi), \tag{7.13}$$

and evaluate them separately in Lemma 7.6 and Lemma 7.7.

First, however, considering the norm $|F(X,\varphi)|^{j,X,r}$, $j=k,k+1$, as defined in (4.27) for any $F\in\mathcal{M}(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$ with $X\in\mathcal{P}_k$ and $\varphi\in\boldsymbol{\mathcal{X}}$, we prove the following.

LEMMA 7.5. *Let* $F\in\mathcal{M}(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$, $X\in\mathcal{P}_k$, $r=1,\dots,r_0$, *and* $j=k,k+1$. *Then*

$$|F(X,\varphi)-T_2F(X,\varphi)|^{j,X,r} \le (1+|\varphi|_{j,X})^3 \sup_{t\in(0,1)} \sum_{s=3}^{r} \frac{1}{s!}|D^sF(X,t\varphi)|^{j,X}. \tag{7.14}$$

PROOF. Cf. [**Bry09**, Lemma 6.8]. Introducing the shorthands

$$f(\varphi)=(1-T_2)F(X,\varphi)$$

and

$$f_s(\varphi)=D^sF(X,\varphi)(\dot\varphi,\dots,\dot\varphi)$$

for any $s\ge 1$, we express the terms contributing to the left hand side of (7.14) with the help of the integral form of the Taylor polynomial remainder,

$$f(\varphi)=\int_0^1 \frac{(1-t)^2}{2}D^3F(X,t\varphi)(\varphi,\varphi,\varphi)\,\mathrm{d}t, \tag{7.15}$$

$$\begin{aligned} Df(\varphi)(\dot\varphi) = f_1(\varphi)-f_1(0)-Df_1(0)(\varphi) &= \int_0^1 (1-t)D^2f_1(t\varphi)(\varphi,\varphi)\,\mathrm{d}t = \\ &= \int_0^1 (1-t)D^3F(X,t\varphi)(\dot\varphi,\varphi,\varphi)\,\mathrm{d}t, \end{aligned} \tag{7.16}$$

$$\begin{aligned} \frac{1}{2}D^2f(\varphi)(\dot\varphi,\dot\varphi) &= \frac{1}{2}\big(f_2(\varphi)-f_2(0)\big) = \\ &= \frac{1}{2}\int_0^1 Df_2(t\varphi)(\varphi)\,\mathrm{d}t = \int_0^1 D^3F(X,t\varphi)(\dot\varphi,\dot\varphi,\varphi)\,\mathrm{d}t, \end{aligned} \tag{7.17}$$

and, for $s\ge 3$,

$$\frac{1}{s!}D^sf(\varphi)(\dot\varphi,\dots,\dot\varphi) = \frac{1}{s!}D^sF(X,\varphi)(\dot\varphi,\dots,\dot\varphi). \tag{7.18}$$

Summing all the right hand sides above and using the bound

$$|D^{s+m}F(X,t\varphi)(\dot\varphi,\dots,\dot\varphi,\varphi,\dots,\varphi)| \le |D^{s+m}F(X,t\varphi)|^{j,X}|\dot\varphi|_{j,X}^s|\varphi|_{j,X}^m, \tag{7.19}$$

as well as the fact that

$$|\varphi|_{j,X}^3\int_0^1 \frac{(1-t)^2}{2}\,\mathrm{d}t + |\varphi|_{j,X}^2\int_0^1(1-t)\,\mathrm{d}t + \frac{1}{2}|\varphi|_{j,X} + \frac{1}{3!} = \frac{1}{3!}(1+|\varphi|_{j,X})^3, \tag{7.20}$$

we get the seeked result. ∎

LEMMA 7.6. *Let $K \in \mathcal{M}(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$, $X \in \mathcal{S}_k$, $B \in \mathcal{B}_k(X)$, and $U = \overline{B^*}$, and assume that $L \geq 7$, $\omega \geq 2(d^2 2^{2d+1} + 1)$, and $h \geq L^{\kappa(d)} h_1$. Then*

(7.21)
$$\sup_{\varphi} |(\boldsymbol{R}_{k+1}K)(X,\varphi) - T_2(\boldsymbol{R}_{k+1}K)(X,\varphi)|^{k+1,X,r} w_{k+1}^{-U}(\varphi) \leq 5L^{-\frac{3d}{2}} 2^{|X|_k} \|K(X)\|_{k,X,r}.$$

For G_1 defined in (7.12) *we have*

$$\|G_1(U)\|_{k+1,U,r} \leq 5\, 2^{d+2^d} (3^d - 1)^{2^d} L^{-\frac{d}{2}} \|K\|_{k,r}^{(\mathsf{A})}. \tag{7.22}$$

PROOF. Lemma 7.5 yields

$$|(\boldsymbol{R}_{k+1}K)(X,\varphi) - T_2(\boldsymbol{R}_{k+1}K)(X,\varphi)|^{k+1,X,r} \leq \tag{7.23}$$
$$\leq (1 + |\varphi|_{k+1,X})^3 \sup_{t\in(0,1)} \sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X,t\varphi)|^{k+1,X}$$

for any $\varphi \in \boldsymbol{\mathcal{X}}$. Interchanging differentiation and integration, we get

$$\sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X,t\varphi)|^{k+1,X} \leq \tag{7.24}$$
$$\leq \sum_{s=3}^{r} \frac{1}{s!} \sup_{\dot\varphi\neq 0} \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi) \Big| \frac{D^s K(X, t\varphi+\xi)(\dot\varphi,\ldots,\dot\varphi)}{|\dot\varphi|_{k+1,X}^s} \Big| =$$
$$= \sum_{s=3}^{r} \frac{1}{s!} \sup_{\dot\varphi\neq 0} \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi) \Big| \frac{D^s K(X, t\varphi+\xi)(\dot\varphi,\ldots,\dot\varphi)}{|\dot\varphi|_{k,X}^s} \frac{|\dot\varphi|_{k,X}^s}{|\dot\varphi|_{k+1,X}^s} \Big| \leq$$
$$\leq L^{-\frac{3d}{2}} \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\, |K(X, t\varphi+\xi)|^{k,X,r}.$$

In the last inequality we used the bound (5.21). Next, we apply

$$|K(X,t\varphi+\xi)|^{k,X,r} \leq \|K(X)\|_{k,X,r} w_k^X(t\varphi+\xi)$$

and (5.25), to get

(7.25)
$$\sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X,t\varphi)|^{k+1,X} \leq 2^{|X|_k} L^{-\frac{3d}{2}} \|K(X)\|_{k,X,r} \frac{w_{k:k+1}^X(\varphi)}{w_{k+1}^U(\varphi)} w_{k+1}^U(\varphi).$$

Here we also used the fact that $w_{k:k+1}^X(t\varphi)$ is monotone in t.

Bounding $(1 + |\varphi|_{k+1,X})^3$ with the help of

$$(1+u)^3 \leq 5\mathrm{e}^{u^2} \tag{7.26}$$

(proven by showing that $\min_{u\geq 0} \frac{\mathrm{e}^{u^2}}{(1+u)^3} \geq \frac{1}{5}$), we would like to show that

$$|\varphi|_{k+1,X}^2 \leq \log \frac{w_{k+1}^U(\varphi)}{w_{k:k+1}^X(\varphi)}. \tag{7.27}$$

Notice, first, that

(7.28)
$$\log \frac{w^U_{k+1}(\varphi)}{w^X_{k:k+1}(\varphi)} \ge \sum_{x\in U\setminus X}\big((2^d\omega-1)g_{k+1,x}(\varphi)+\omega G_{k+1,x}(\varphi)\big)+\sum_{x\in U} g_{k:k+1,x}(\varphi)+ \\ + L^k(L-3)\sum_{x\in\partial U} G_{k+1,x}(\varphi) - 3L^k \sum_{x\in\partial X\setminus\partial U} G_{k,x}(\varphi) \ge \\ \ge \sum_{x\in U\setminus X}(2^d\omega-1)g_{k+1,x}(\varphi)+L^k(L-3)\sum_{x\in\partial U} G_{k+1,x}(\varphi).$$

To verify the last inequality, we show that

$$3L^k \sum_{x\in\partial X\setminus\partial U} G_{k,x}(\varphi) \le \sum_{x\in U} g_{k:k+1,x}(\varphi) + \sum_{x\in U\setminus X}\omega G_{k+1,x}(\varphi) \tag{7.29}$$

in analogy with (5.15). Indeed, arguing that any $x\in\partial X\setminus\partial U$ is contained in ∂B for $B\in\mathcal{B}_k(U\setminus X)$, and applying again Proposition B.5 (a), we have

(7.30)
$$h^2L^k\sum_{x\in\partial B} G_{k,x}(\varphi)\le \\ \le 2c\Big(\sum_{x\in B}|\nabla\varphi(x)|^2+L^{2k}\sum_{x\in U_1(B)}|\nabla^2\varphi(x)|^2\Big)+L^k\sum_{x\in\partial B}\sum_{s=2}^3 L^{(2s-2)k}|\nabla^s\varphi(x)|^2\le \\ \le h^2 2\mathfrak{c}\sum_{x\in B} G_{k,x}(\varphi)+h^2 2\mathfrak{c}L^k\sum_{x\in\partial B} L^{-2}g_{k:k+1,z}(\varphi),$$

where z is any point $z\in B$. Using $|\partial B|\le 2^dL^{(d-1)k}$, we get the seeked bound once $\omega\ge 18\sqrt{2}$ and $L\ge 5$ (when $6\mathfrak{c}\le\omega$ and $6\mathfrak{c}L^{-2}\le 1$).

In view of (7.28) and using that $|\varphi|^2_{k+1,X}\le|\varphi|^2_{k+1,U}$, it suffices to show that

$$|\varphi|^2_{k+1,U}\le\sum_{x\in U\setminus X}(2^d\omega-1)g_{k+1,x}(\varphi)+L^k(L-3)\sum_{x\in\partial U} G_{k+1,x}(\varphi). \tag{7.31}$$

Clearly,

$$h^2|\varphi|^2_{k+1,U}\le\sum_{1\le s\le 3} L^{(k+1)(d-2+2s)}\max_{x\in U^*}|\nabla^s\varphi(x)|^2 \tag{7.32}$$

Applying Lemma B.7, we get

(7.33)
$$L^{(k+1)d}\max_{x\in U^*}|\nabla\varphi(x)|^2\le\frac{2L^{(k+1)d}}{|\partial U|}\sum_{x\in\partial U}|\nabla\varphi(x)|^2+2L^{(k+1)d}(\mathrm{diam}U^*)^2\max_{x\in U^*}|\nabla^2\varphi(x)|^2.$$

Using that $|\partial U|\ge 2dL^{(k+1)(d-1)}$, the first term above is covered by the second term on the right hand side of (7.31) once $L\ge 7$,

$$\frac{2L^{(k+1)d}}{|\partial U|}\le\frac{2L^{(k+1)d}}{2dL^{(k+1)(d-1)}}=\frac{1}{d}L^{k+1}\le L^k(L-3). \tag{7.34}$$

Taking into account that $\mathrm{diam}U^*\le d2^dL^{k+1}$ (here we use the fact that U is necessarily contained in a block of the side $2L^{k+1}$), the second term is bounded by $d^2 2^{2d+1}L^{(k+1)(d+2)}\max_{x\in U^*}|\nabla^2\varphi(x)|^2$ and will be treated together with the remaining terms $\max_{x\in U^*}|\nabla^s\varphi(x)|^2$, $s=2,3$, contained in $|\varphi|^2_{k+1,U}$.

Using the fact that the number of $(k+1)$-blocks in U is at most 2^d, we get

$$\max_{x\in U^*}|\nabla^s\varphi(x)|^2 \le 2^d \sum_{B\in\mathcal{B}_{k+1}(U)} \max_{x\in B^*}|\nabla^s\varphi(x)|^2. \tag{7.35}$$

This yields

$$(d^2 2^{2d+1} L^{(k+1)(d+2)} + L^{(k+1)(d+2)}) \max_{x\in U^*}|\nabla^2\varphi(x)|^2 \le \\ \le 2^d(d^2 2^{2d+1}+1)L^{(k+1)(d+2)} \sum_{B\in\mathcal{B}_{k+1}(U)} \max_{x\in B^*}|\nabla^2\varphi(x)|^2. \tag{7.36}$$

and

$$L^{(k+1)(d+4)} \max_{x\in U^*}|\nabla^3\varphi(x)|^2 \le 2^d L^{(k+1)(d+4)} \sum_{B\in\mathcal{B}_{k+1}(U)} \max_{x\in B^*}|\nabla^3\varphi(x)|^2. \tag{7.37}$$

Each of the terms on the right hand sides will be bounded by the corresponding term in

$$h^2 \sum_{x\in B\setminus X} (2^d\omega - 1) g_{k+1,x}(\varphi) = (2^d\omega-1) \sum_{x\in B\setminus X} \sum_{s=2}^{4} L^{(2s-2)(k+1)} \sup_{y\in B_x^*} |\nabla^s\varphi(y)|^2, \tag{7.38}$$

Indeed, observing that $g_{k+1,x}(\varphi)$ is constant over each $(k+1)$-block $B\subset U$, and the volume of $B\setminus X$ is at least $L^{kd}(L^d - 2^d) = L^{(k+1)d}(1-(\frac{2}{L})^d)$ since the number of k-blocks in X is at most 2^d, while B consists of L^d of them, we need

$$2^d(d^2 2^{2d+1}+1)L^{(k+1)(d+2)} \le (2^d\omega-1)L^{(k+1)d}(1-(\tfrac{2}{L})^d)L^{2(k+1)} \tag{7.39}$$

and

$$2^d L^{(k+1)(d+4)} \le (2^d\omega-1)L^{(k+1)d}(1-(\tfrac{2}{L})^d)L^{4(k+1)}. \tag{7.40}$$

These conditions are satisfied once $\omega \ge 2(d^2 2^{2d+1}+1)$.

In summary, combining (7.25), (7.26), and (7.27), we have

$$(1+|\varphi|_{k+1,X})^3 \sum_{s=3}^{r} \frac{1}{s!} |D^s(\boldsymbol{R}_{k+1}K)(X,t\varphi)|^{k+1,X} \le \\ \le 5L^{-\frac{3d}{2}} 2^{|X|_k} \|K(X)\|_{k,X,r}\, w_{k+1}^U(\varphi). \tag{7.41}$$

for any $\varphi\in\boldsymbol{\mathcal{X}}$ and any $t\in(0,1)$, finishing thus the proof of the inequality (7.21).

To prove the bound (7.22), we use that $|\mathcal{B}_k(U)|\le(2L)^d$ and the obvious bound $|\{X\in\mathcal{S}_k \mid X\supset B\}| \le (3^d-1)^{2^d}$, to get

$$\|G_1(U)\|_{k+1,U,r} \le 5\,L^{-\frac{3d}{2}} \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} 2^{|X|_k} \|K(X)\|_{k,X,r} \le \\ \le 5\,L^{-\frac{3d}{2}}(2L)^d(3^d-1)^{2^d}\|K\|_{k,r}^{(\mathsf{A})} 2^{2^d} \le 5\,2^{d+2^d}(3^d-1)^{2^d} L^{-\frac{d}{2}} \|K\|_{k,r}^{(\mathsf{A})}. \tag{7.42}$$

□

LEMMA 7.7. *Let* $K \in \mathcal{M}(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$, $U = \overline{B^*}$, *and assume that* $L \geq 7$ *and* $\omega \geq 2(d^2 2^{2d+1} + 1)$. *For* G_2 *defined in* (7.13) *we have*

$$\|G_2(U)\|_{k+1,U,r} \leq \leq 2^{2^d+d+1}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{\frac{d}{2}-2} + (8L^{-1}+2L^{-2})\big)\|K\|_{k,r}. \tag{7.43}$$

Recall that $G_2(U,\varphi) = \sum_{\substack{B\in\mathcal{B}_k(U)\\ \overline{B^*}=U}} (T_2 - \Pi_2)R(B,\varphi)$ with $R \in M^*(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$ defined by $R(B,\varphi) = \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}(\boldsymbol{R}_{k+1}K)(X,\varphi)$. The polynomial $\Pi_2 R(B,\varphi) = \lambda|B| + \ell(\varphi) + Q(\varphi,\varphi)$ is characterised by taking a unique linear function $\ell(\varphi)$ of the form (4.19), $\ell(\varphi) = \sum_{x\in(B^*)^*}\big[\sum_{i=1}^d a_i\,\nabla_i\varphi(x) + \sum_{i,j=1}^d \boldsymbol{c}_{i,j}\,\nabla_i\nabla_j\varphi(x)\big]$, that agrees with $DR(B,0)(\varphi)$ on all quadratic functions φ on $(B^*)^*$ and a unique quadratic function $Q(\varphi,\varphi)$ of the form (4.20), $Q(\varphi,\varphi) = \sum_{x\in(B^*)^*}\sum_{i,j=1}^d \boldsymbol{d}_{i,j}\,\nabla_i\varphi(x)\,\nabla_j\varphi(x)$, that agrees with $\frac{1}{2}D^2R(B,0)(\varphi,\varphi)$ on all affine functions φ on $(B^*)^*$.

In view of the definition of the map $\boldsymbol{R}_{k+1}$ we can write

$$R(B,\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\, R_\xi(B,\varphi)$$

with

$$R_\xi(B,\varphi) = \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} K(X,\xi+\varphi).$$

Observing that

$$D(\boldsymbol{R}_{k+1}K)(X,0)(\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\, DK(X,\xi)(\varphi),$$
$$D^2(\boldsymbol{R}_{k+1}K)(X,0)(\varphi,\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\, D^2K(X,\xi)(\varphi,\varphi),$$

and introducing, similarly as above, $\Pi_2 R_\xi(B,\varphi) = \lambda_\xi|B| + \ell_\xi(\varphi) + Q_\xi(\varphi,\varphi)$, the unicity implies that $\ell(\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\,\ell_\xi(\varphi)$ and $Q(\varphi,\varphi) = \int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\,Q_\xi(\varphi,\varphi)$.

Given that $G_2(B,\varphi) = (T_2 - \Pi_2)R(B,\varphi)$ is a polynomial of second order, we have $|G_2(B,\varphi)|^{k+1,U,r} = |G_2(B,\varphi)|^{k+1,U,2}$. In a preparation for the evaluation of this norm, we first evaluate separately the absolute value of the linear and quadratic terms $P_1(\varphi)$ and $P_2(\varphi)$ in $G_2(B,\varphi)$.

Observing that for any affine function φ_1 and any quadratic function φ_2 on $(B^*)^*$ we have $P_1(\varphi - \varphi_1 - \varphi_2) = P_1(\varphi)$, we get

$$\begin{aligned} \big|P_1(\varphi)\big| &= \Big|\int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi)\,\big(DR_\xi(B,0)(\varphi-\varphi_1-\varphi_2) - \ell_\xi(\varphi-\varphi_1-\varphi_2)\big)\Big| \leq \\ &\leq (2^{d+2}-1)\sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}\|K(X)\|_{k,X,r}|\varphi-\varphi_1-\varphi_2|_{k,B^*}\int_{\boldsymbol{\mathcal{X}}}\mu_{k+1}(\mathrm{d}\xi)w_k^X(\xi) \leq \\ &\leq 2^{2^d}(3^d-1)^{2^d}(2^{d+2}-1)\|K\|_{k,r}|\varphi-\varphi_1-\varphi_2|_{k,B^*}. \end{aligned} \tag{7.44}$$

Here, we first used the inequalities

$$|\ell_\xi(\varphi)| \leq (2^{d+2}-2)\sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}|K(X,\xi)|^{k,X,r}|\varphi|_{k,B^*} \tag{7.45}$$

and

$$|DR_\xi(B,0)(\varphi)| \le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} |K(X,\xi)|^{k,X,r} |\varphi|_{k,X} \tag{7.46}$$

combined with the bounds $|K(X,\xi)|^{k,X,r} \le \|K(X)\|_{k,X,r} w_k^X(\xi)$ and $|\varphi|_{k,X} \le |\varphi|_{k,B^*}$, and then the bounds $\int_{\boldsymbol{\mathcal{X}}} \mu_{k+1}(\mathrm{d}\xi) w_k^X(\xi) \le 2^{|X|_k}$, and, as in (7.42), $|\{X \in \mathcal{S}_k \mid X \supset B\}| \le (3^d-1)^{2^d}$. To verify (7.45), we first observe that $\ell_\xi(\varphi) = \sum_{i=1}^d a_i(\xi)\, s_i + \sum_{i,j=1}^d \boldsymbol{c}_{i,j}(\xi)\, t_{i,j}$ where $s_i = s_i(\varphi) = \sum_{x\in(B^*)^*} \nabla_i\varphi(x)$ and $t_{i,j} = t_{i,j}(\varphi) = \sum_{x\in(B^*)^*} \nabla_i\nabla_j\varphi(x)$. The same values of "average slopes" $\mathbf{s} = \{s_i\}$ and $\mathbf{t} = \{t_{i,j}\}$ are obtained with the quadratic function

(7.47)

$$\varphi_{\mathbf{s},\mathbf{t}}(x) = L^{-dk}(2^{d+2}-3)^{-d} \sum_i \big(s_i - \sum_j (t_{i,j}+t_{j,i})\overline{x_j}\big) x_i + L^{-dk}(2^{d+2}-3)^{-d} \sum_{i,j} t_{i,j} x_i x_j,$$

where $\overline{x_j} = L^{-dk}(2^{d+2}-3)^{-d} \sum_{y\in B} y_j$ (notice that $(B^*)^*$ contains $(2^{d+2}-3)^d$ k-blocks). Further, observe that

$$\begin{aligned} h|\varphi_{\mathbf{s},\mathbf{t}}|_{k,X} &= \max\Big(L^{\frac{dk}{2}} \max_{x\in X^*} |\nabla\varphi_{\mathbf{s},\mathbf{t}}(x)|, L^{\frac{dk}{2}+k} \max_{x\in X^*} |\nabla^2\varphi_{\mathbf{s},\mathbf{t}}(x)|\Big) \le \\ &\le L^{-\frac{dk}{2}}(2^{d+2}-3)^{-d} \max\Big(|\mathbf{s}| + 2|\mathbf{t}|\tfrac{1}{2}L^k(2^{d+2}-3), L^k|\mathbf{t}|\Big) = \\ &= L^{-\frac{dk}{2}}(2^{d+2}-3)^{-d}|\mathbf{s}| + L^{-\frac{dk}{2}+k}(2^{d+2}-3)^{-d+1}|\mathbf{t}| \le (1+2^{d+2}-3)h|\varphi|_{k,B^*}. \end{aligned} \tag{7.48}$$

Here, the last inequality, valid for any φ such that $s_i(\varphi) = s_i$ and $t_{i,j}(\varphi) = t_{i,j}$, is implied by obvious bounds $\max_{x\in(B^*)^*}|\nabla_i\varphi(x)| \ge L^{-dk}(2^{d+2}-3)^{-d}|s_i|$ and $\max_{x\in(B^*)^*}|\nabla_i\nabla_j\varphi(x)| \ge L^{-dk}(2^{d+2}-3)^{-d}|t_{i,j}|$.

Now, for the quadratic function $\varphi_{\mathbf{s},\mathbf{t}}$ we have $\ell_\xi(\varphi_{\mathbf{s},\mathbf{t}}) = DR_\xi(B,0)(\varphi_{\mathbf{s},\mathbf{t}})$. As a result,

$$\begin{aligned} |\ell_\xi(\varphi)| &= |\ell_\xi(\varphi_{\mathbf{s},\mathbf{t}})| \le \\ &\le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} |DK(X,\xi)(\varphi_{\mathbf{s},\mathbf{t}})| \le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} |K(X,\xi)|^{k,X,r} |\varphi_{\mathbf{s},\mathbf{t}}|_{k,X} \le \\ &\le (2^{d+2}-2) \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k} |K(X,\xi)|^{k,X,r} |\varphi|_{k,B^*}. \end{aligned} \tag{7.49}$$

Here, the last inequality, valid for any φ such that $s_i(\varphi) = s_i$ and $t_{i,j}(\varphi) = t_{i,j}$, is implied by obvious bounds $\max_{x\in(B^*)^*}|\nabla_i\varphi(x)| \ge L^{-dk}(2^{d+2}-3)^{-d}|s_i|$ and $\max_{x\in(B^*)^*}|\nabla_i\nabla_j\varphi(x)| \ge L^{-dk}(2^{d+2}-3)^{-d}|t_{i,j}|$.

Choosing now, for any fixed φ, the functions φ_1 and φ_2 as an optimal approximation in accordance with the Poincaré inequalities,

$$\inf_{\varphi_1 \text{ affine}} |\varphi-\varphi_1|_{k,B^*} \le \frac{1}{h} L^{k(\frac{d}{2}+1)} \sup_{x\in(B^*)^*} |\nabla^2\varphi(x)| \le L^{-(\frac{d}{2}+1)} |\varphi|_{k+1,B^*} \tag{7.50}$$

and

(7.51)

$$\inf_{\substack{\varphi_1 \text{ affine,}\\ \varphi_2 \text{ quadratic}}} |\varphi-\varphi_1-\varphi_2|_{k,B^*} \le \frac{1}{h} L^{k(\frac{d}{2}+2)} \sup_{x\in(B^*)^*} |\nabla^3\varphi(x)| \le L^{-(\frac{d}{2}+2)} |\varphi|_{k+1,B^*},$$

we get

$$\big|P_1(\varphi)\big| \le L^{-(\frac{d}{2}+2)} 2^{2^d}(3^d-1)^{2^d}(2^{d+2}-1)\|K\|_{k,r}|\varphi|_{k+1,B^*}. \tag{7.52}$$

Similarly for the quadratic part. First, we prove the bound

$$|P_2(\varphi,\varphi)| \le 2^{2^d+1}(3^d-1)^{2^d}\|K\|_{k,r}|\varphi|^2_{k,B^*}. \tag{7.53}$$

While deriving it, the bound (7.45) is replaced by

$$|Q_\xi(\varphi,\varphi)| \le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}|K(X,\xi)|^{k,X,r}|\varphi|^2_{k,B^*}. \tag{7.54}$$

For its proof we consider the linear function

$$\varphi_{\mathbf{s}}(x) = L^{-dk}(2^{d+2}-3)^{-d}\sum_i s_i x_i \tag{7.55}$$

with the slope $s_i = s_i(\varphi)$ and

$$\begin{aligned}(7.56)\quad & h|\varphi_{\mathbf{s}}|_{k,X} = \\ &= L^{\frac{dk}{2}}\max_{x\in X^*}|\nabla\varphi_{\mathbf{s}}(x)| \le L^{-\frac{dk}{2}}(2^{d+2}-3)^{-d}|\mathbf{s}| = L^{-\frac{dk}{2}}(2^{d+2}-3)^{-d}|\mathbf{s}| \le h|\varphi|_{k,B^*}\end{aligned}$$

yielding

$$\begin{aligned}(7.57)\quad |Q_\xi(\varphi,\varphi)| = |Q_\xi(\varphi_{\mathbf{s}},\varphi_{\mathbf{s}})| &\le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}|\tfrac12 D^2K(X,\xi)(\varphi_{\mathbf{s}},\varphi_{\mathbf{s}})| \le \\ \le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}|K(X,\xi)|^{k,X,r}|\varphi_{\mathbf{s}}|^2_{k,X} &\le \sum_{\substack{X\in\mathcal{S}_k\\ X\supset B}} \frac{1}{|X|_k}|K(X,\xi)|^{k,X,r}|\varphi|^2_{k,B^*}.\end{aligned}$$

Validity of (7.53) for all φ, implies $|P_2(\varphi,\psi)| \le 2^{2^d+2}(3^d-1)^{2^d}\|K\|_{k,r}|\varphi|_{k,B^*}|\psi|_{k,B^*}$ for all φ and ψ. Taking now into account that $P_2(\varphi_1,\varphi_1)=0$ for any affine function φ_1, we rewrite $P_2(\varphi,\varphi) = 2P_2(\varphi,\varphi-\varphi_1) - P_2(\varphi-\varphi_1,\varphi-\varphi_1)$ to get

$$|P_2(\varphi,\varphi)| \le 2^{2^d+1}(3^d-1)^{2^d}\|K\|^{(\mathsf{A})}_{k,r}|\varphi-\varphi_1|_{k,B^*}(4|\varphi|_{k,B^*}+|\varphi-\varphi_1|_{k,B^*}). \tag{7.58}$$

Applying further (7.50), we get

$$|P_2(\varphi,\varphi)| \le \big(4L^{-(d+1)}+L^{-(d+2)}\big)2^{2^d+1}(3^d-1)^{2^d}\|K\|^{(\mathsf{A})}_{k,r}|\varphi|^2_{k+1,B^*}. \tag{7.59}$$

Finally, combining (7.52) and (7.59), we get

$$\begin{aligned}(7.60)\quad & |\big(T_2-\Pi_2\big)R(B,\varphi)| \le \\ &\le 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)}+(8L^{-(d+1)}+2L^{-(d+2)})|\varphi|_{k+1,B^*}\big)|\varphi|_{k+1,B^*}\|K\|^{(\mathsf{A})}_{k,r}.\end{aligned}$$

For the first and second the derivatives, we first notice that

$$D\big(P_1(\varphi)+P_2(\varphi,\varphi)\big)(\dot\varphi) = P_1(\dot\varphi)+2P_2(\varphi,\dot\varphi) \tag{7.61}$$

and

$$D^2\big(P_1(\varphi)+P_2(\varphi,\varphi)\big)(\dot\varphi,\dot\varphi) = 2P_2(\dot\varphi,\dot\varphi) \tag{7.62}$$

yielding with the help of (7.52) and (7.59)

$$\big|D\big(P_1(\varphi)+P_2(\varphi,\varphi)\big)\big|^{k+1,B^*} \leq \tag{7.63}$$
$$\leq 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)}+(16L^{-(d+1)}+4L^{-(d+2)})|\varphi|_{k+1,B^*}\big)\|K\|^{(\mathsf{A})}_{k,r}$$

and, using again (7.59),

$$\big|D^2\big(P_1(\varphi)+P_2(\varphi,\varphi)\big)\big|^{k+1,B^*} \leq 2^{2^d}(3^d-1)^{2^d}(8L^{-(d+1)}+2L^{-(d+2)})\|K\|^{(\mathsf{A})}_{k,r}. \tag{7.64}$$

Combining last two inequalities with (7.60), we get

$$\big|\big(T_2-\Pi_2\big)R(B,\varphi)\big|^{k+1,B^*,r} \leq 2^{2^d}(3^d-1)^{2^d}\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)}+ \tag{7.65}$$
$$+(8L^{-(d+1)}+2L^{-(d+2)})(1+|\varphi|_{k+1,B^*})\big)(1+|\varphi|_{k+1,B^*})\|K\|^{(\mathsf{A})}_{k,r}.$$

With $(1+u)^2 \leq 2\mathrm{e}^{u^2}$ and (7.27), we get

$$\|G_2(U)\|_{k+1,U,r} \leq \tag{7.66}$$
$$\leq 2^{2^d+1}(3^d-1)^{2^d}(2L)^d\big((2^{d+2}-1)L^{-(\frac{d}{2}+2)}+(8L^{-(d+1)}+2L^{-(d+2)})\big)\|K\|^{(\mathsf{A})}_{k,r}$$

yielding the sought bound. □

The proof of Lemma 7.1 is the finished by combining the claims of Lemma 7.2 and Lemma 7.3. □

7.2. Bounds on the operators $\boldsymbol{A}^{(\boldsymbol{q})^{-1}}$ and $\boldsymbol{B}^{(\boldsymbol{q})}$

The bounds on operators $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ are rather straightforward.

LEMMA 7.8. *Let $\theta \in (\frac{1}{4},\frac{3}{4})$ and $\omega \geq 2(d^2 2^{2d+1}+1)$. Consider the constant $h_1 = h_1(d,\omega)$, $\mathsf{\kappa}(d)$, $\mathsf{A}_0 = \mathsf{A}_0(d,L)$ as chosen from Lemma 7.1. Then there exists $L_0(d)$ such that*

$$\|\boldsymbol{A}^{(\boldsymbol{q})^{-1}}\|_{0;0} \leq \frac{1}{\sqrt{\theta}} \tag{7.67}$$

and there exists $M = M(d)$ such that

$$\|\boldsymbol{B}^{(\boldsymbol{q})}\|_{r;0} \leq ML^d \tag{7.68}$$

for any $\|\boldsymbol{q}\| \leq \frac{1}{2}$, any $N \in \mathbb{N}$, $k = 1,\dots,N$, $r = 1,\dots,r_0$, and any $L \geq L_0$, $h \geq L^{\mathsf{\kappa}} h_1$, and $\mathsf{A} \geq \mathsf{A}_0$.

PROOF.

When expressed in the coordinates $\dot\lambda, \dot a, \dot{\boldsymbol{c}}, \dot{\boldsymbol{d}}$ of $\dot H$, the linear map $\boldsymbol{A}$ according to (4.81) keeps $\dot a, \dot{\boldsymbol{c}}$, and $\dot{\boldsymbol{d}}$ unchanged and only shifts $\dot\lambda$ by

$$\frac{1}{2}\sum_{x\in B}\sum_{i,j=1}^{d} \dot d_{i,j}\nabla_i\nabla_j^*\mathcal{C}^{(\boldsymbol{q})}_{k+1}(0).$$

Hence, $\boldsymbol{A}^{-1}$ only makes the opposite shift and thus

$$\begin{aligned}(7.69)\quad \|\boldsymbol{A}^{-1}\dot{H}\|_{k,0} &= \\ &= L^{dk}|\dot{\lambda}| + L^{\frac{dk}{2}} h \sum_{i=1}^{d} |\dot{a}_i| + L^{\frac{(d-2)}{2}k} h \sum_{i,j=1}^{d} |\dot{\boldsymbol{c}}_{i,j}| + \frac{h^2}{2} \sum_{i,j=1}^{d} |\dot{\boldsymbol{d}}_{i,j}| \\ &\quad + \frac{L^{dk}}{2} \sum_{i,j=1}^{d} |\dot{\boldsymbol{d}}_{i,j}| \big|\nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)\big|.\end{aligned}$$

Using

$$\frac{1}{2} \sum_{i,j=1}^{d} |\dot{\boldsymbol{d}}_{i,j}| \le \frac{1}{h^2} \|\dot{H}\|_{k,0}, \tag{7.70}$$

we get

$$\|\boldsymbol{A}^{-1}\dot{H}\|_{k,0} \le (1 + c_{2,0} L^{\eta(d)} h^{-2}) \|\dot{H}\|_{k+1,0}$$

using that $\max_{i,j=1}^{d} \big|\nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)\big| \le c_{2,0} L^{-kd} L^{\eta(d)}$ according to Proposition 4.1. Given that $h^2 \ge L^{2\kappa(d)} = L^{\eta(d)+d}$ we can get

$$1 + c_{2,0} L^{\eta(d)} h^{-2} \le 1 + c_{2,0} L^{-d} \le \theta^{-1/2} \tag{7.71}$$

once $L > \big(\frac{2c_{2,0}}{\log 4}\big)^{1/d}$.

For the second bound, using Lemma 6.9, the first inequality of (4.40) and Lemma 5.1(iv),

$$\begin{aligned}(7.72)\quad \|\boldsymbol{B}K\|_{k+1,0} &\le \sum_{B \in \mathcal{B}_k(B')} \Big\| \Pi_2 \sum_{\substack{X \in \mathcal{S}_k, \\ X \supset B}} \frac{1}{|X|_k} (\boldsymbol{R}_{k+1} K)(X) \Big\|_{k+1,0} \le \\ &\le \sum_{B \in \mathcal{B}_k(B')} C \sum_{\substack{X \in \mathcal{S}_k, \\ X \supset B}} \frac{1}{|X|_k} \|(\boldsymbol{R}_{k+1} K)(X)\|_{k:k+1,X,r} \\ &\le \sum_{B \in \mathcal{B}_k(B')} \sum_{\substack{X \in \mathcal{S}_k, \\ X \supset B}} \frac{C 2^{|X|_k}}{|X|_k} \|K(X)\|_{k,X,r} \le \\ &\le \sum_{B \in \mathcal{B}_k(B')} \sum_{\substack{X \in \mathcal{S}_k, \\ X \supset B}} \frac{C 2^{|X|_k}}{|X|_k} \|K_k\|_k^{(\mathsf{A})} \le L^d M \|K_k\|_k^{(\mathsf{A})},\end{aligned}$$

for any $B' \in \mathcal{B}_{k+1}$. Here the factor L^d comes from the number of blocks $B \in \mathcal{B}_k(B')$ and we included into $M = M(d)$ the constant $C = C(d)$ as well as the bound on the number of short polymers containing a fixed block. □

Lemma 7.1 in conjunction with the estimates above give the estimates (4.84) in Proposition 4.7.

Proof of Remark 4.8.

The smoothness of the operators with respect to the fine tuning parameter $\boldsymbol{q}$ follows for $\boldsymbol{B}^{(\boldsymbol{q})}$ and $\boldsymbol{C}^{(\boldsymbol{q})}$ with the corresponding bounds in Chapter 6 and for $\boldsymbol{A}^{(\boldsymbol{q})}$ from the regularity of the finite range decomposition (4.3), i.e., (4.85) follows with $C = C(d, h, L, \omega) > 0$ and $r \ge 2\ell + 3$ and all $\|\boldsymbol{q}\| \le \frac{1}{2}$. □

CHAPTER 8

Fine Tuning of the Initial Conditions

Finally, we address the fine tuning Theorem 4.9. First, in Section 8.1, we prove the smoothness of the map $\mathcal{F}$ assigning a fixed point of the renormalisation map $\boldsymbol{\mathcal{T}}$ to initial values $\mathcal{H}$ and $\mathcal{K}$. Then we can specify the map $\mathscr{H}$ that chooses the initial ideal Hamiltonian $\mathcal{H}$ in a self-consistent way so that it is reproduced in the first component H_0 of $\mathcal{F}$. Its properties summarized in Theorem 4.9 are proven in Section 8.2.

8.1. Properties of the map $\mathcal{F}$

Considering the space $\boldsymbol{E}$ with the norm $\|\cdot\|_\zeta$ with $\zeta > 0$ as defined in (2.21) and the Banach space $\boldsymbol{Y}_r$ introduced in (4.97) and (4.98), we find a map $\mathcal{F}$ from a neighbourhood of origin in $\boldsymbol{E} \times \boldsymbol{M}_0$ (with a shorthand $\boldsymbol{M}_0 = M_0(\mathcal{B}_0, \boldsymbol{\mathcal{X}})$) to $\boldsymbol{Y}_r$ so that $\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K},\mathcal{H}),\mathcal{K},\mathcal{H}) = \mathcal{F}(\mathcal{K},\mathcal{H})$ with the following smoothness properties.

Proposition 8.1. *Let $d = 2,3$, $\omega \geq 2(d^2 2^{2d+1} + 1)$, $r_0 \geq 9$, and $2m + 2 \leq r_0$ be fixed and let L_0, $h_0(L)$, $\mathsf{A}_0(L)$, $M > 0$ (see (4.98)), and $\theta \in (1/4, 3/4)$ be the constants from Propositions 4.6 and 4.7. Then there exist constants $\alpha = \alpha(M,\theta) \geq 1$ and $\eta = \eta(\theta) \in (0,1)$ determining the norm of the spaces $\boldsymbol{Y}_r$, $r = r_0, r_0 - 2, \dots, r_0 - 2m$ and, for any $L \geq L_0$, $h \geq h_0(L)$, and $\mathsf{A} \geq \mathsf{A}_0(L)$, a constant $\zeta = \zeta(h)$ determining the norm $\|\cdot\|_\zeta$ on $\boldsymbol{E}$ and constants $\widehat{\rho}, \widehat{\rho}_1, \widehat{\rho}_2 > 0$ so that there exists a unique function $\mathcal{F}\colon B_{\boldsymbol{E}\times\boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2) \to B_{\boldsymbol{Y}_{r_0}}(\widehat{\rho})$ solving the equation $\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K},\mathcal{H}),\mathcal{K},\mathcal{H}) = \mathcal{F}(\mathcal{K},\mathcal{H})$ (see (4.104)). Moreover,*

$$\mathcal{F} \in \widetilde{C}^m(B_{\boldsymbol{E}\times\boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2), \boldsymbol{Y}) \tag{8.1}$$

with bounds on derivatives that are uniform in N, i.e., there is $\widehat{C}$ such that

$$\|D^j_{\mathcal{K}} D^\ell_{\mathcal{H}} \mathcal{F}(\mathcal{K},\mathcal{H})(\dot{\mathcal{K}},\dots,\dot{\mathcal{K}},\dot{\mathcal{H}},\dots,\dot{\mathcal{H}})\|_{\boldsymbol{Y}_{r_0-2\ell}} \leq \widehat{C}\|\dot{\mathcal{H}}\|_0^\ell \|\dot{\mathcal{K}}\|_\zeta^j, \tag{8.2}$$

for all $(\mathcal{K},\mathcal{H}) \in B_{\boldsymbol{E}\times\boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2)$ and all $\ell, j \in \mathbb{N}_0$ with $\ell + j \leq n \leq m$.

The proof of Proposition 8.1 is based on Theorem E.1 applied in conjunction with Propositions 4.6 and 4.7. Here, the map $\boldsymbol{\mathcal{T}} : \boldsymbol{Y} \times \boldsymbol{E} \times M_0 \to \boldsymbol{Y}$ plays the role of the map F and the sequence of spaces $\boldsymbol{Y} = \boldsymbol{Y}_{r_0} \hookrightarrow \boldsymbol{Y}_{r_0-2} \hookrightarrow \dots \hookrightarrow \boldsymbol{Y}_{r_0-2m}$, $2m < r_0$, the role of the sequence $\boldsymbol{X}_n$, $n = m, m-1, \dots, 0$. Using $\mathcal{O}_\rho := B_{\boldsymbol{Y}}(\rho)$, $\mathcal{W}_\rho := B_{\boldsymbol{E}}(\rho) = \{\mathcal{K} \in \boldsymbol{E} : \|\mathcal{K}\|_\zeta \leq \rho\}$, and $\mathcal{V}_\rho := \{\mathcal{H} \in \boldsymbol{M}_0 : \|\mathcal{H}\|_0 \leq \rho\}$, we just have to verify the assumptions of Theorem E.1, that is we need to prove the following claim.

Lemma 8.2. *Let L, h, and A be constants as in Proposition 8.1 and let $\theta \in (1/4, 3/4)$ and $M > 0$ be the constants from Proposition 4.7. Then there exist parameters α and η of the norms in $\boldsymbol{Y}_r$ depending only on θ and M, constants $\rho > 0$, and ζ depending on h and A, so that:*

(i) $\boldsymbol{\mathcal{T}} \in \widetilde{C}^m(\mathcal{O}_\rho \times \mathcal{W}_\rho \times \mathcal{V}_\rho, \boldsymbol{Y})$ *with the bounds on corresponding derivatives that are uniform in* N*,*
(ii) $\boldsymbol{\mathcal{T}}(0,0,\mathcal{H}) = 0$ *for all* $\mathcal{H} \in \mathcal{V}_\rho$*, and*
(iii) $\left\| D_1\boldsymbol{\mathcal{T}}(\boldsymbol{y},0,\mathcal{H})\big|_{\boldsymbol{y}=0} \right\|_{\mathcal{L}(\boldsymbol{Y}_r,\boldsymbol{Y}_r)} \leq \theta$ *for all* $\mathcal{H} \in \mathcal{V}_\rho$ *and* $r = r_0, r_0-2, \ldots, r_0-2m$*.*

Proof. Let us recall the definition of the map $\boldsymbol{\mathcal{T}}$. The $2N$ coordinates of the image

$$\boldsymbol{\mathcal{T}}(\boldsymbol{y},\mathcal{K},\mathcal{H}) = \overline{\boldsymbol{y}} = (\overline{H}_0, \overline{H}_1, \overline{K}_1, \ldots, \overline{H}_{N-1}, \overline{K}_{N-1}, \overline{K}_N) \tag{8.3}$$

are defined by

$$\begin{aligned} \overline{H}_k &= \big(\boldsymbol{A}_k^{(\mathcal{H})}\big)^{-1}\big(H_{k+1} - \boldsymbol{B}_k^{(\mathcal{H})}K_k\big) \quad \text{and} \\ \overline{K}_{k+1} &= S_k(H_k, K_k, \mathcal{H}), \end{aligned} \tag{8.4}$$

where we set $H_N = 0$ and

$$K_0(X,\varphi) := \exp\Big\{-\sum_{x\in X}\mathcal{H}(x,\varphi)\Big\}\prod_{x\in X}\mathcal{K}(\nabla\varphi(x)) \tag{8.5}$$

with $\mathcal{K} \in \boldsymbol{E}$. Notice that $\boldsymbol{A}_k^{\mathcal{H}}, \boldsymbol{B}_k^{\mathcal{H}}$, and $S_k(H_k, K_k, \mathcal{H})$ depend on $\mathcal{H}$ only through the coefficient of its quadratic term $\boldsymbol{q} = \boldsymbol{q}(\mathcal{H})$. We will also use a shorthand

$$K_0(X,\varphi) =: K_0^{(\mathcal{K},\mathcal{H})}(X,\varphi) = \prod_{x\in X}\mathcal{K}_0^{(\mathcal{K},\mathcal{H})}(x,\varphi) \tag{8.6}$$

with

$$\mathcal{K}_0^{(\mathcal{K},\mathcal{H})}(x,\varphi) = \exp\big\{-\mathcal{H}(x,\varphi)\big\}\mathcal{K}(\nabla\varphi(x)). \tag{8.7}$$

Here we explicitly invoke the dependence of the map S_k on k in contradistinction to Chapter 6, where the index k was omitted. Notice that the only two coordinates of $\overline{\boldsymbol{y}}$ that depend on $\mathcal{K}$ (through K_0) are $\overline{H}_0 = \big(\boldsymbol{A}_0^{(\mathcal{H})}\big)^{-1}\big(H_1 - \boldsymbol{B}_0^{(\mathcal{H})}K_0\big)$ and $\overline{K}_1 = S_0(H_0, K_0, \mathcal{H})$.

(i) The fact that $\boldsymbol{\mathcal{T}} \in \widetilde{C}^m(\mathcal{O}_\rho \times \mathcal{W}_\rho \times \mathcal{V}_\rho, \boldsymbol{Y})$ follows from Propositions 4.6 and 4.7. We will treat separately the coordinates $\overline{K}_{k+1}$, $k = 1, 2, \ldots, N-1$, the coordinates $\overline{H}_k$, $k = 1, 2, \ldots, N-1$, and finally, the coordinates $\overline{H}_0$ and $\overline{K}_1$ that depend on $\mathcal{K}$.

Reinstating the dependence on k, we denote more explicitly the sequence of normed spaces $\boldsymbol{M}_{k,r} = \{M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}) : \|\cdot\|_{k,r}^{(\mathsf{A})} < \infty\}$, $r = r_0, r_0-2, \ldots, r_0-2m$, as well as $\boldsymbol{M}_{k,0} = (M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), \|\cdot\|_{k,0})$. Then the claim of Proposition 4.6 is that the maping $S_k : \mathcal{U}_{k,\rho} \times \mathcal{V}_{1/2} \to \boldsymbol{M}_{k+1} = \boldsymbol{M}_{k+1,r_0}$ belongs to $\widetilde{C}^m(\mathcal{U}_{k,\rho} \times \mathcal{V}_{1/2}, \boldsymbol{M}_{k+1})$ for all $k = 1, 2, \ldots, N-1$. Here,

$$\mathcal{U}_{k,\rho} = \{(H,K) \in \boldsymbol{M}_{k,0} \times \boldsymbol{M}_{k,r_0} : \|H\|_{k,0} < \rho, \|K\|_{k,r_0}^{(\mathsf{A})} < \rho\}$$

For the coordinates $\overline{H}_k$, $k = 1, 2, \ldots, N-1$, we first observe that the defining map $\overline{H}_k = (\boldsymbol{A}_k^{(\mathcal{H})})^{-1}\big(H_{k+1} - \boldsymbol{B}_k^{(\mathcal{H})}K_k\big)$ is linear in H_{k+1} and K_k and that it does not depend on $\mathcal{K}$. Consider thus the map

$$G : (\boldsymbol{y},\mathcal{H}) \mapsto (\boldsymbol{A}_k^{(\mathcal{H})})^{-1}\big(H_{k+1} - \boldsymbol{B}_k^{(\mathcal{H})}K_k\big) \tag{8.8}$$

and verify that $G \in \widetilde{C}^m(\boldsymbol{Y} \times \mathcal{V}_\rho, \boldsymbol{M}_{k,0})$.

First, we will address the smoothness of the term $\boldsymbol{B}_k^{(\mathcal{H})} K_k$. Comparing the formula (4.82) with (6.18), we see that

$$\boldsymbol{B}_k^{(\mathcal{H})} K_k(B', \varphi) = - \sum_{B \in \mathcal{B}(B')} R_2(0, K_k, \boldsymbol{q}(\mathcal{H})), \tag{8.9}$$

obtaining the needed smoothness relying on the fact that $R_2 \in \widetilde{C}^m(\mathcal{U}_{k,\rho} \times \mathcal{V}_\rho, \boldsymbol{M}_{k,0})$ (see Lemma 6.7) and the fact that the projection $\mathcal{H} \mapsto \boldsymbol{q}(\mathcal{H})$ is a linear mapping.

Denoting $H = H_{k+1} - \boldsymbol{B}_k^{(\mathcal{H})} K_k \in \boldsymbol{M}_{k+1,0}$ and rewriting it in terms of the coordinates $\lambda, a, \boldsymbol{c}, \boldsymbol{d}$ we see that the linear operator $(\boldsymbol{A}_k^{(\mathcal{H})})^{-1}$ only shifts the coordinate λ by

$$-\frac{1}{2} \sum_{x \in B} \sum_{i,j=1}^{d} \boldsymbol{d}_{i,j} \nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q}(\mathcal{H}))}(0), \tag{8.10}$$

keeping the other coordinates unchanged (cf. the proof of Lemma 7.8). The derivatives of this shift can be estimated by finite range decomposition bound (4.3) yielding

$$\sup_{\|\mathcal{H}\|_0 \le \frac{1}{2}} \left| (D^\ell \nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q}(\mathcal{H}))})(0)(\dot{\mathcal{H}}, \ldots, \dot{\mathcal{H}}) \right| \le c_{2,\ell} L^{-kd} L^{\eta(2,d)} \|\dot{\mathcal{H}}\|_0^\ell \tag{8.11}$$

where we used that

$$\frac{1}{2} \sum_{i,j=1}^{d} |\boldsymbol{d}_{i,j}| \le \frac{1}{h^2} \|H\|_{k+1,0} \tag{8.12}$$

according to (4.44). Hence

$$\begin{aligned} \|D^\ell((\boldsymbol{A}_k^{(\boldsymbol{q}(\mathcal{H}))})^{-1} H)(\dot{\mathcal{H}}, \ldots, \dot{\mathcal{H}})\|_{k,0} &= \|D^\ell G(\boldsymbol{y}, \mathcal{H})(\dot{\mathcal{H}}, \ldots, \dot{\mathcal{H}})\|_{k,0} \\ &\le c_{2,\ell} L^{\eta(2,d)} h^{-2} \|H\|_{k+1,0} \|\dot{\mathcal{H}}\|_0^\ell, \end{aligned} \tag{8.13}$$

for $\|\mathcal{H}\|_0 \le \frac{1}{2}$ and $\boldsymbol{y} \in \boldsymbol{Y}$. Actually, in [**AKM13**] it is shown that $\nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)$ is analytic in $\boldsymbol{q}$.

Finally, we consider the coordinates $\overline{H}_0$ and $\overline{K}_1$. Their derivatives with respect to $\mathcal{K}$ have to be evaluated by composing the derivatives of $\overline{H}_0$ and $\overline{K}_1$ with respect to K_0 with the derivatives of K_0 with respect to $\mathcal{K}$. We first deal with the coordinate $\overline{K}_1$ which can be viewed as a composition of maps

$$F : \boldsymbol{M}_0 \times \boldsymbol{E} \times \boldsymbol{M}_0 \to \boldsymbol{M}_{0,0} \times \boldsymbol{M}_{0,r_0} \text{ and } S_0 : (\boldsymbol{M}_0 \times \boldsymbol{M}_{0,r_0}) \times \boldsymbol{M}_0 \to \boldsymbol{M}_{1,r_0}. \tag{8.14}$$

Indeed, with

$$F(H_0, \mathcal{K}, \mathcal{H}) = (H_0, K_0^{(\mathcal{K}, \mathcal{H})}) \tag{8.15}$$

we get

$$\overline{K}_1 = S_0 \diamond F, \quad \text{i.e.,} \quad \overline{K}_1(H_0, \mathcal{K}, \mathcal{H}) = S_0(F(H_0, \mathcal{K}, \mathcal{H}), \mathcal{H}). \tag{8.16}$$

Here, $K_0^{(\mathcal{K}, \mathcal{H})}$ is the polymer defined in (8.6), where we explicitly denoted the dependence on $\mathcal{K}$ and $\mathcal{H}$.

Now, we apply the Chain Rule according to Theorem D.29 jointly with Remark D.30 providing bounds on derivatives that are uniform in N. The needed condition $S_0 \in \widetilde{C}^m(\mathcal{U}_{0,\rho} \times \mathcal{V}_{1/2}, \boldsymbol{M}_1)$ is just the corresponding claim (4.78) from Proposition 4.6. For the map F, there is no grading on the domain space $\boldsymbol{M}_0 \times \boldsymbol{E} \times \boldsymbol{M}_0$, and we will actually show that $F \in C^m_*(\mathcal{U}_{0,\rho} \times \mathcal{W}_\rho \times \mathcal{V}_\rho, \boldsymbol{M}_0 \times \boldsymbol{M}_{0,r_0})$. Indeed, choosing a suitable parameter ζ and ρ, both depending on h, we will prove that the derivative $D^j D^\ell K_0^{(\mathcal{K},\mathcal{H})}(\dot{\mathcal{K}}^j, \dot{\mathcal{H}}^\ell)$ exists and

$$\left\| D^j D^\ell K_0^{(\mathcal{K},\mathcal{H})}(\dot{\mathcal{K}}^j, \dot{\mathcal{H}}^\ell) \right\|_{0,r} \le C_1 \|\dot{\mathcal{K}}\|_\zeta^j \|\dot{\mathcal{H}}\|_0^\ell \tag{8.17}$$

for any $j, \ell \le m+1$ with $C_1 = C_1(h, \mathsf{A}, m)$, and thus also

$$\lim_{(\mathcal{K}',\mathcal{H}') \to (\mathcal{K},\mathcal{H})} \left\| D^j D^\ell K_0^{(\mathcal{K},\mathcal{H})}(\dot{\mathcal{K}}^j, \dot{\mathcal{H}}^\ell) - D^j D^\ell K_0^{(\mathcal{K},\mathcal{H})}(\dot{\mathcal{K}}^j, \dot{\mathcal{H}}^\ell) \right\|_{0,r} = 0 \tag{8.18}$$

for any $j, \ell \le m$ and any $(H_0, \mathcal{K}, \mathcal{H}) \in \mathcal{U}_{0,\rho} \times \mathcal{W}_\rho \times \mathcal{V}_\rho$.

Indeed, in view of the product form in (8.5) and (8.6), we first have

$$D^\ell K_0(X,\varphi)(\dot{\mathcal{H}}, \dots, \dot{\mathcal{H}}) = \sum_{\substack{k \in \mathbb{N}_0^X : \\ \sum_{x\in X} k_x = \ell}} \frac{(-1)^\ell \ell!}{\prod_{x\in X} k_x!} \prod_{x \in X} \Big(\dot{\mathcal{H}}(x,\varphi)^{k_x} \mathrm{e}^{-\mathcal{H}(x,\varphi)} \mathcal{K}(\nabla\varphi(x)) \Big), \tag{8.19}$$

and thus

$$\begin{aligned} D^j D^\ell K_0^{(\mathcal{K},\mathcal{H})}(\dot{\mathcal{K}}^j, \dot{\mathcal{H}}^\ell) = &\sum_{\substack{k \in \mathbb{N}_0^X : \\ \sum_{x\in X} k_x = \ell}} \sum_{\substack{Y \subset X \\ |Y| = j}} \frac{(-1)^\ell \ell!}{\prod_{x\in X} k_x!} \prod_{x \in X} \Big(\dot{\mathcal{H}}(x,\varphi)^{k_x} \mathrm{e}^{-\mathcal{H}(x,\varphi)} \Big) \\ &\times \prod_{y\in Y} \dot{\mathcal{K}}(\nabla\varphi(y)) \prod_{y \in X\setminus Y} \mathcal{K}(\nabla\varphi(y)) \\ = &\sum_{\substack{k \in \mathbb{N}_0^X : \\ \sum_{x\in X} k_x = \ell}} \sum_{\substack{Y \subset X \\ |Y| = j}} \frac{(-1)^\ell \ell!}{\prod_{x\in X} k_x!} \prod_{x \in X} \Big(\dot{\mathcal{H}}(x,\varphi)^{k_x} \Big) \prod_{x\in Y} \dot{\mathcal{K}}_0^{(\mathcal{K},\mathcal{H})}(x,\varphi) \prod_{x \in X\setminus Y} \mathcal{K}_0^{(\mathcal{K},\mathcal{H})}(x,\varphi). \end{aligned} \tag{8.20}$$

Here, we use the shorthand $\dot{\mathcal{K}}_0^{(\mathcal{K},\mathcal{H})}(x,\varphi) = \exp\{-\mathcal{H}(x,\varphi)\} \dot{\mathcal{K}}(\nabla\varphi(x))$. Observing that, in the case $k = 0$, the unit blocks are actually single sites, $\mathcal{B}_k(\Lambda_N) = \Lambda_N$, we can apply the claim (iia) of Lemma 5.1 to get

$$\begin{aligned} &\Big\| \prod_{y\in Y} \dot{\mathcal{K}}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi) \prod_{y\in X\setminus Y} \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi) \Big\|_{0,X,r} \\ &\qquad \le \prod_{y\in Y} |||\dot{\mathcal{K}}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}|||_{0,\{y\}} \prod_{y\in X\setminus Y} |||\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}|||_{0,\{y\}}. \end{aligned} \tag{8.21}$$

Here we introduced the shorthands

$$\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi) = -\dot{\mathcal{H}}(y,\varphi)^{k_y} \mathcal{K}_0^{(\mathcal{K},\mathcal{H})}(y,\varphi) \tag{8.22}$$

and

$$\dot{\mathcal{K}}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi) = -\dot{\mathcal{H}}(y,\varphi)^{k_y} \dot{\mathcal{K}}_0^{(\mathcal{K},\mathcal{H})}(y,\varphi). \tag{8.23}$$

Further, using definitions (4.30) and (4.27),

$$|||\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y y)}|||_{0,\{y\}} = \sup_\varphi |\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi)|_{0,\{y\},r_0} \exp\{-G_{0,y}(\varphi)\} \tag{8.24}$$

with the weight function $G_{0,y}(\varphi)$ defined in (4.29) and

$$|\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi)|_{0,\{y\},r_0} = \sum_{r=0}^{r_0} \frac{1}{r!} \sup_{|\dot\varphi|_{0,\{y\}}\le 1} \big|D^r \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi)(\dot\varphi,\dots,\dot\varphi)\big|. \tag{8.25}$$

Using the definition (4.21), we can bound

$$|\dot\varphi|_{0,\{y\}} = \max_{1\le s\le 3} \sup_{w\in\{y\}^*} \frac{1}{h}\big|\nabla^s\dot\varphi(w)\big| \ge \max\big(\frac{1}{h}|\nabla\dot\varphi(y)|, \frac{1}{h}|\nabla^2\dot\varphi(y)|\big). \tag{8.26}$$

Now

(8.27)

$$\sup_{|\dot\varphi|_{0,\{y\}}\le 1} \big|D^r \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi)(\dot\varphi,\dots,\dot\varphi)\big| \le \sup_{|\dot\varphi|_{0,\{y\}}\le 1} \left| \frac{\mathrm{d}^r \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi+t\dot\varphi)}{\mathrm{d}t^r}\Big|_{t=0}\right|.$$

Defining $v=\nabla\varphi(y),\ w=\nabla^2\varphi(y)$, and $z=\frac{1}{h}\big(|v|^2+|w|^2\big)^1/2$ we notice that

$$\frac{\mathrm{d}^r \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi+t\dot\varphi)}{\mathrm{d}t^r}$$

is a sum of terms of the form

(8.28)

$$(\dot\lambda+\dot a v+\tfrac12\langle\dot{\boldsymbol q}v,v\rangle+\dot c w)^{i_0}(\dot a\dot v+\langle\dot{\boldsymbol q}v,\dot v\rangle+\dot c\dot w)^{i_1}\langle\dot{\boldsymbol q}\dot v,\dot v\rangle^{i_2}(a\dot v+\langle\boldsymbol q v,\dot v\rangle+z\dot w)^{j_1}\langle\boldsymbol q\dot v,\dot v\rangle^{j_2}\times$$
$$\times\exp\{-(\lambda+av+\tfrac12\langle\boldsymbol q v,v\rangle+cw)\}\frac{\mathrm{d}^s\mathcal{K}(v+t\dot v)}{\mathrm{d}t^s}\Big|_{t=0}$$

such that $i_0+i_1+i_2=k_y$ and $i_1+2i_2+j_1+2j_2+s=r$. Using the definition of the norm $\|\mathcal{H}\|_0$ and the fact that $\frac{1}{h}\max(|\dot v|,|\dot w|)\le|\dot\varphi|_{0,\{y\}}\le 1$, the absolute value of the prefactor above can be bounded by

$$2^{i_1+i_2+j_1+j_2}\|\dot{\mathcal{H}}\|_0^{j_1+j_2}\big(1+z\big)^{2i_0+i_1+j_1}$$

Now assume that

$$\|\mathcal{H}\|_0 \le \widetilde\rho \le 1. \tag{8.29}$$

Since $k_y\le m+1$ and $j_1\le m+1$ we have

$$(1+z)^{2i_0+i_1+j_1} \le (1+z)^{4(m+1)} \le \big(1+\frac{16(m+1)}{\widetilde\rho}\big)^{2(m+1)}\exp\{\widetilde\rho z^2\}. \tag{8.30}$$

In the last inequality we used that for $a>0$, $z\ge 0$,

$$(1+z)^a \le \big(1+\frac{2a}{\widetilde\rho}\big)^{a/2}\exp\{\widetilde\rho z^2\} \tag{8.31}$$

To see this observe that for $a>0$ the maximum of the function

$$t\mapsto(1+t)^a\exp\{-\widetilde\rho t^2\} \tag{8.32}$$

for $t\ge 0$ is attained at

$$t=\bar t=\frac12\big(\sqrt{1+\frac{2a}{\widetilde\rho}}-1\big)$$

and is bounded by

$$(1+\bar t)^a \le (1+2\bar t)^a = \big(1+\frac{2a}{\widetilde\rho}\big)^{a/2}.$$

As a result, there exists a constant $\overline{C}(r_0)$ so that for $|\dot\varphi| \le 1$ and hence $|\dot v| \le h$, we have

$$\begin{aligned}\left|\frac{\mathrm{d}^r \mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(\varphi+t\dot\varphi)}{\mathrm{d}t^r}\Big|_{t=0}\right| &\le \overline{C}(r_0)\big(1+\tfrac{16(m+1)}{\widetilde\rho}\big)^{2(m+1)}\|\dot{\mathcal{H}}\|_0^{k_y}\times\\ &\times\exp\{\widetilde\rho|z|^2\}\Bigg(\sum_{s=0}^{r_0}\left|\frac{\mathrm{d}^s\mathcal{K}(v+t\dot v)}{\mathrm{d}t^s}\Big|_{t=0}\right|\Bigg)\\ &\le \overline{C}(r_0)\big(1+\tfrac{16(m+1)}{\widetilde\rho}\big)^{2(m+1)}\|\dot{\mathcal{H}}\|_0^{k_y}\exp\{\widetilde\rho z^2\}\sum_{|\alpha|\le r_0} h^{|\alpha|}|\partial_v^\alpha \mathcal{K}(v)|\end{aligned}\tag{8.33}$$

for any $\|\mathcal{H}\|_0 \le \widetilde\rho$, and any $r \le r_0$. Finally, choosing

$$\zeta \ge h\tag{8.34}$$

and taking into account that

$$G_{0,y}(\varphi) \ge \frac{1}{h^2}|\nabla\varphi(y)|^2 + \frac{1}{h^2}|\nabla^2\varphi(y)|^2 = z^2\tag{8.35}$$

and the definition (2.21) of the norm $\|\mathcal{K}\|_\zeta$ and using $|v| \le hz$ we get

$$|||\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}|||_{0,\{y\}} \le \widetilde{C}\|\dot{\mathcal{H}}\|_0^{k_y}\sup_{z\ge 0}\Big(\exp\{(\widetilde\rho-1)z^2\}\exp\{\zeta^{-2}h^2z^2\}\|\mathcal{K}\|_\zeta\Big)\tag{8.36}$$

with

$$\widetilde{C} = \widetilde{C}(r_0,m,h,\widetilde\rho) = \overline{C}(r_0)\big(1+\frac{16(m+1)}{\widetilde\rho}\big)^{2(m+1)}.\tag{8.37}$$

The same estimate holds for $\dot{\mathcal{K}}^{(\mathcal{H},\dot{\mathcal{H}},k_y)}$ if we replace $\|\mathcal{K}\|_\zeta$ on the right hand side by $\|\dot{\mathcal{K}}\|_\zeta$. The exponential term can be controlled if for given h we choose ζ and $\widetilde\rho$ such that

$$\frac{h^2}{\zeta^2} + \widetilde\rho \le 1.\tag{8.38}$$

In particular we may take

$$\widetilde\rho = \frac{1}{2} \quad\text{and}\quad \zeta = \sqrt{2}h.\tag{8.39}$$

Note that (8.38) implies (8.34) and (8.29).

Summarising, we get,

$$\begin{aligned}\Big\|\prod_{y\in Y}\dot{\mathcal{K}}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(\nabla\varphi(y))\prod_{y\in X\setminus Y}\mathcal{K}_0^{(\mathcal{H},\dot{\mathcal{H}},k_y)}(y,\varphi)\Big\|_{0,X,r} &\le\\ \le \widetilde{C}^{|X|}\|\dot{\mathcal{H}}\|_0^\ell\|\dot{\mathcal{K}}\|_\zeta^j\|\mathcal{K}\|_\zeta^{|X|-j}.\end{aligned}\tag{8.40}$$

Since $\ell \le m+1$ the sum in (8.20) over $k \in \mathbb{N}_0^X$ with $\sum_{x\in X} k_x = \ell$ involves at most $(m+2)^{|X|}$ terms. The sum over Y involves at most $2^{|X|}$ terms. The counting terms with the factorial in (8.20) are bounded by $(m+1)!$. Thus (8.20) and (8.40) give

$$\begin{aligned}&\|D_1^j D_2^\ell K_0(X,\mathcal{K},\mathcal{H},\dot{\mathcal{K}},\ldots,\dot{\mathcal{K}},\dot{\mathcal{H}},\ldots,\dot{\mathcal{H}})\|_{0,r}\\ &\qquad\le (m+1)!(2(m+2))^{|X|}\widetilde{C}^{|X|}\|\mathcal{K}\|_\zeta^{X-j}\|\dot{\mathcal{K}}\|_\zeta^j\|\dot{\mathcal{H}}\|_0^\ell.\end{aligned}\tag{8.41}$$

Thus with $\zeta = \sqrt{2}h$ we have for all $\mathcal{K} \in B_{\boldsymbol{E}}(\rho_1)$ with

$$\rho_1 = \rho_1(\mathsf{A}) = \big(2(m+2)(m+1)!\mathsf{A}\widetilde{C}\big)^{-1} \tag{8.42}$$

and all $\mathcal{H} \in B_{\boldsymbol{M}_0}(\widetilde{\rho})$ with $\widetilde{\rho} = \frac{1}{2}$,

$$\begin{aligned} &\Gamma_{\mathsf{A}}(X)\|D_1^j D_2^\ell K_0(X, \mathcal{K}, \mathcal{H}, \dot{\mathcal{K}}, \dots, \dot{\mathcal{K}}, \dot{\mathcal{H}}, \dots, \dot{\mathcal{H}})\|_{0,r} \\ &\qquad \le C_1 \|\dot{\mathcal{K}}\|_\zeta^j \|\mathcal{H}\|_0^\ell \end{aligned} \tag{8.43}$$

with

$$C_1 = C_1(\mathsf{A}, m) = \big((m+1)!(2(m+2)\widetilde{C}\mathsf{A}\big)^{m+1}. \tag{8.44}$$

Finally, for the coordinate $\overline{H}_0 = (\boldsymbol{A}^{(\mathcal{H})})_0^{-1}\big(H_1 - \boldsymbol{B}_0^{(\mathcal{H})} K_0\big)$, we can again apply the Chain Rule according to Theorem D.29. The image coordinate $\overline{H}_0$ is obtained as a composition of maps

$$F : \boldsymbol{M}_{1,0} \times \boldsymbol{E} \times \boldsymbol{M}_0 \to \boldsymbol{M}_{1,0} \times \boldsymbol{M}_{0,r_0} \text{ and } G : (\boldsymbol{M}_{1,0} \times \boldsymbol{M}_{0,r_0}) \times \boldsymbol{M}_0 \to \boldsymbol{M}_{0,r_0} \tag{8.45}$$

with

$$F(H_1, \mathcal{K}, \mathcal{H}) = (H_1, K_0^{(\mathcal{K},\mathcal{H})}) \text{ and } G((H_1, K_0), \mathcal{H}) = (\boldsymbol{A}_0^{(\mathcal{H})})^{-1}\big(H_1 - \boldsymbol{B}_0^{(\mathcal{H})} K_0\big) \tag{8.46}$$

yielding $\overline{H}_0 = G \diamond F$. Both needed conditions, $G \in \widetilde{C}^m(\boldsymbol{Y} \times \mathcal{V}_\rho, \boldsymbol{M}_{0,r_0})$ as well as $F \in C_*^m(\mathcal{U}_{1,\rho} \times \mathcal{W}_\rho \times \mathcal{V}_\rho, \boldsymbol{M}_{1,0} \times \boldsymbol{M}_{0,r_0})$ have been already proven.

(ii) This is an immediate consequence of the definition of the map $\boldsymbol{\mathcal{T}}$ and the fact that $S(0, 0, \mathcal{H}) = 0$ (cf. (4.62)).

(iii) Using that $K_0 = 0$ for $\mathcal{K} = 0$ and that $\frac{\partial S_k}{\partial H_k}(0, 0, \mathcal{H}) = \frac{\partial S_k}{\partial K_k}(0, 0, \mathcal{H}) = 0$, we can compute the derivatives of $\overline{\boldsymbol{y}} = \boldsymbol{\mathcal{T}}(\boldsymbol{y}, 0, \mathcal{H})$ at $\mathcal{H} = 0$:

$$\begin{aligned} \frac{\partial \overline{H}_k}{\partial H_j} &= \begin{cases} \boldsymbol{A}_k^{-1} & \text{if } j = k+1,\ j = 0, \dots, N-2 \\ 0 & \text{otherwise,} \end{cases} \\ \frac{\partial \overline{H}_k}{\partial K_j} &= \begin{cases} -\boldsymbol{A}_k^{-1}\boldsymbol{B}_k & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{8.47}$$

and

$$\begin{aligned} &\frac{\partial \overline{K}_{k+1}}{\partial H_j} = 0, \\ &\frac{\partial \overline{K}_{k+1}}{\partial K_j} = \begin{cases} \boldsymbol{C}_k & \text{if } j = k \neq 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{8.48}$$

for $k, j = 0, \dots, N-1$.

Consider now a vector $\boldsymbol{y} \in \boldsymbol{Y}_r$ with $\|\boldsymbol{y}\|_{\boldsymbol{Y}_r} \le 1$ and its image $\overline{\boldsymbol{y}}$ under the map $\frac{\partial \boldsymbol{\mathcal{T}}(\boldsymbol{y},0,\mathcal{H})}{\partial \boldsymbol{y}}\Big|_{\boldsymbol{y}=0}$,

$$\overline{\boldsymbol{y}} = \frac{\partial \boldsymbol{\mathcal{T}}(\boldsymbol{y}, 0, \mathcal{H})}{\partial \boldsymbol{y}}\Big|_{\boldsymbol{y}=0} \boldsymbol{y}. \tag{8.49}$$

Since $\|\boldsymbol{y}\|_{\boldsymbol{Y}_r} \le 1$, we have $\|H_k^{(\boldsymbol{y})}\|_{k,0} \le \eta^k$, $k = 0, \dots, N-1$, and $\|K_k^{(\boldsymbol{y})}\|_{k,r} \le \frac{\eta^k}{\alpha}$, $k = 1, \dots, N$, for the coordinates $H_k^{(\boldsymbol{y})}, K_k^{(\boldsymbol{y})}$ of the vector $\boldsymbol{y}$. Using $H_k^{(\overline{y})}, K_k^{(\overline{y})}$, for the coordinates of the image $\overline{y}$, we get

$$\|H_0^{(\overline{\boldsymbol{y}})}\|_{k,0} \le \|\boldsymbol{A}_0^{-1}\|\eta;$$

$$\|H_k^{(\overline{\boldsymbol{y}})}\|_{k,0} \le \|\boldsymbol{A}_k^{-1}\|\eta^{k+1} + \|\boldsymbol{A}_k^{-1}\|\|B_k\|\frac{\eta^k}{\alpha} \le \frac{\eta^k}{\sqrt{\theta}}(\eta + \frac{M}{\alpha}), k = 1, \dots, N-2;$$

$$\|H_{N-1}^{(\overline{\boldsymbol{y}})}\|_{N-1,0} \le \|\boldsymbol{A}_{N-1}^{-1}\|\|\boldsymbol{B}_{N-1}\|\frac{\eta^{N-1}}{\alpha} \le \frac{\eta^{N-1}M}{\alpha\sqrt{\theta}};$$

$$\|K_1^{(\overline{\boldsymbol{y}})}\|_{k,r} = 0;$$

$$\|K_k^{(\overline{\boldsymbol{y}})}\|_{k,r} \le \|\boldsymbol{C}_{k-1}\|\frac{\eta^k}{\alpha} \le \theta\frac{\eta^k}{\alpha}, k = 2, \dots, N.$$

As a result,

$$\|\overline{\boldsymbol{y}}\|_{\boldsymbol{Y}_r} \le \big(\frac{1}{\sqrt{\theta}}(\eta + \frac{M}{\alpha})\big) \vee \frac{\theta}{\eta}.$$

It suffices to choose the parameters η and α so that $\eta + M/\alpha \le \theta^{1/2}$ $(\theta < \eta < \theta^{1/2})$, yielding

$$\Big\|\frac{\partial \boldsymbol{\mathcal{T}}(\boldsymbol{y}, 0, \mathcal{H})}{\partial \boldsymbol{y}}\Big|_{\boldsymbol{y}=0}\Big\|_{\mathcal{L}(\boldsymbol{Z}_s, \boldsymbol{Z}_s)} \le \theta < 1, \quad s = r_0, r_0 - 2, \dots, r_0 - 6. \tag{8.50}$$

□

Proof of Proposition 8.1. Having thus, in Lemma 8.2, verified the assumptions (E.1)-(E.4) of Theorem E.1 for the map $\boldsymbol{\mathcal{T}}$ in the role of F, there exist constants $\widehat{\rho}_1$, $\widehat{\rho}_2$, and $\widehat{\rho}$ depending (through ρ in Lemma 8.2) on h and A and $\widehat{C}$, depending (through $C = C(L, h, \mathsf{A})$ in Proposition 4.6) on L, h, and A, and the map

$$\mathcal{F} \colon B_{\boldsymbol{E} \times \boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2) \to B_{\boldsymbol{Y}_{r_0}}(\widehat{\rho}) \tag{8.51}$$

(in the role of f) so that $\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K}, \mathcal{H}), \mathcal{K}, \mathcal{H}) = \mathcal{F}(\mathcal{K}, \mathcal{H})$ for any

$$(\mathcal{K}, \mathcal{H}) \in B_{\boldsymbol{E} \times \boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2),$$

and

$$\mathcal{F} \in \widetilde{C}^m(B_{\boldsymbol{E} \times \boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2), \boldsymbol{Y}), \tag{8.52}$$

satisfying (8.2) whenever $(\mathcal{K}, \mathcal{H}) \in B_{\boldsymbol{E} \times \boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2)$ and $j, \ell \in \mathbb{N}_0$ with $\ell + j \le m$. Here, the estimates (8.2) follow from the bounds (E.8). □

8.2. Properties of the map $\mathcal{H}$

Using our results in the previous section we finally obtain a map $\mathcal{H}$ mapping a neighbourhood of the origin in $\boldsymbol{E}$ to $\boldsymbol{M}_0$ so that $\boldsymbol{\mathcal{T}}(\mathcal{F}(\mathcal{K}, \mathcal{H}(\mathcal{K})), \mathcal{K}, \mathcal{H}(\mathcal{K})) = \mathcal{F}(\mathcal{K}, \mathcal{H}(\mathcal{K}))$ and $\Pi(\mathcal{F}(\mathcal{K}, \mathcal{H}(\mathcal{K}))) = \mathcal{H}(\mathcal{K})$. This requires another application of the implicit function theorem, this time for the composition of the projection Π with the map $\mathcal{F}$ in Proposition 8.1. We write $\mathcal{G} := \Pi \circ \mathcal{F}$ in the following. The projection $\Pi \colon \boldsymbol{Y}_{r_0 - 2n} \to \boldsymbol{M}_0$ is a bounded linear mapping for any $0 \le n \le m$. Using Proposition 8.1 we obtain, in particular, that $\mathcal{G} \in C_*^m(B_{\boldsymbol{E} \times \boldsymbol{M}_0}(\widehat{\rho}_1, \widehat{\rho}_2), \boldsymbol{M}_0)$. Note that $\mathcal{F}(0, \mathcal{H}) = 0$ because $\boldsymbol{\mathcal{T}}(0, 0, \mathcal{H}) = 0$ for all $\mathcal{H} \in \mathcal{V}_\rho$ (see (ii) in Lemma 8.2), and thus $\mathcal{G}(0, \mathcal{H}) = 0$ and $D_{\mathcal{H}}\mathcal{G}(0, 0) = 0$. Therefore, by standard implicit function theorem, there exists a C^m_*-map $\mathcal{H}\colon B_{\boldsymbol{E}}(\rho_1) \to B_{\boldsymbol{M}_0}(\rho_2)$ with a suitable $\rho_1 \le \hat{\rho}_1$ and $\rho_2 = \hat{\rho}_2$ such that $\mathcal{G}(\mathcal{K}, \mathcal{H}(\mathcal{K})) = \mathcal{H}(\mathcal{K})$.

APPENDIX A

Discrete Sobolev Estimates

For the convenience of the reader we recall a discrete version of the Sobolev inequality. Discrete Sobolev inequalities are classical, see, e.g., Sobolev's original work [**Sob40**]. Let $B_n = [0,n]^d \cap \mathbb{Z}^d$, and for $p > 0$ define the norm

$$\|f\|_p = \|f\|_{p,B_n} = \Big(\sum_{x\in B_n} |f(x)|^p \Big)^{1/p} \tag{A.1}$$

for any function $f\colon B_n \to \mathbb{R}$.

Proposition A.1. *For every $p \geq 1$ and $m, M \in \mathbb{N}$ there exists a constant $\mathfrak{C} = \mathfrak{C}(p, M, m)$ such that:*

(i) If $1 \leq p \leq d$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$, and $q \leq p^*$, $q < \infty$, then

$$n^{-\frac{d}{q}} \|f\|_q \leq \mathfrak{C} n^{-\frac{d}{2}} \|f\|_2 + \mathfrak{C} n^{1-\frac{d}{p}} \|\nabla f\|_p. \tag{A.2}$$

(ii) If $p > d$, then

$$\big|f(x) - f(y)\big| \leq \mathfrak{C} n^{1-\frac{d}{p}} \|\nabla f\|_p \qquad \text{for all } x, y \in B_n. \tag{A.3}$$

(iii) If $m \in \mathbb{N}$, $1 \leq p \leq \frac{d}{m}$, $\frac{1}{p_m} = \frac{1}{p} - \frac{m}{d}$, and $q \leq p_m$, $q < \infty$, then

$$n^{-\frac{d}{q}} \|f\|_q \leq \mathfrak{C} n^{-\frac{d}{2}} \sum_{k=0}^{M-1} \|(n\nabla)^k f\|_2 + \mathfrak{C} n^{-\frac{d}{p}} \|(n\nabla)^M f\|_p. \tag{A.4}$$

(iv) If $M = \lfloor \frac{d+2}{2} \rfloor$, the integer value of $\frac{d+2}{2}$, then

$$\max_{x\in B_n} |f(x)| \leq \mathfrak{C} n^{-\frac{d}{2}} \sum_{k=0}^{M} \|(n\nabla)^k f\|_2. \tag{A.5}$$

Remark A.2.

(i) In the proof of (iv) we actually get

$$\max_{x\in B_n} |f(x)| \leq (n+1)^{-\frac{d}{2}} \sum_{x\in B_n} |f(x)|^2 + \mathfrak{C} n^{-\frac{d}{2}} \sum_{k=1}^{M} \|(n\nabla)^k f\|_2. \tag{A.6}$$

(ii) As written, the higher derivatives on the RHS of (i)-(iv) require the values of f outside B_n. If one traces the dependence more carefully then one sees that $(\nabla_1^{\alpha_1} \ldots \nabla_d^{\alpha_d} f)(x)$ is only needed for x such that $x + \alpha_1 e_1 + \cdots + \alpha_d e_d \in B_n$, so that only the values of f inside B_n are needed.

The proof may be reduced to the continuous case by interpolation. Let $n = 1$, $B_1 = \{0,1\}^d$, $f : B_1 \to \mathbb{R}_+$, and let $\widetilde{f}$ be the interpolation of f which is affine in each coordinate direction, i.e., $\widetilde{f}$ is the unique function of the form

$$\widetilde{f}(x) = \prod_{i=1}^{d}(a_i x_i + b_i), \quad \widetilde{f}(x) = f(x) \quad \text{for } x \in \{0,1\}^d. \tag{A.7}$$

The Proposition A.1 will be proven with help of the following Lemma.

Lemma A.3.

(i) $\frac{1}{(p+1)^d 2^d} \sum_{x\in B_1} f^p(x) \le \int_{(0,1)^d} \widetilde{f}^p(x)\mathrm{d}x \le \frac{1}{2^d}\sum_{x\in B_1} f^p(x)$.

(ii) $\sup_{x\in(0,1)^d} |\partial_i \widetilde{f}(x)| \le \max_{x\in B_1, x_i=0} |f(x+e_i)-f(x)| \le \Big(\sum_{x\in B_1, x_i=0} |f(x+e_i) - f(x)|^p\Big)^{1/p}$ *for any* $i = 1,\dots,d$.

Proof. (i) The integrand is a product of functions of one variable. Taking into account that

$$\frac{1}{2^d}\sum_{x\in B_1} f^p(x) = \prod_{i=1}^{d}\Big(\frac{1}{2}(a_i+b_i)^p + \frac{1}{2}b_i^p\Big), \tag{A.8}$$

it suffices to prove the claim for $d = 1$. Considering thus a nonnegative function on the interval $[0,1]$ of the form $ax + b$ and assuming w.l.o.g. that $a, b \ge 0$, we get

$$\int_0^1 (ax+b)^p \mathrm{d}x = \sum_{k=0}^{p}\binom{p}{k}\frac{1}{k+1}a^k b^{p-k} \le b^p + \sum_{k=1}^{p}\binom{p}{k}\frac{1}{2}a^k b^{p-k} = \frac{1}{2}b^p + \frac{1}{2}(a+b)^p. \tag{A.9}$$

On the other hand,

$$\begin{aligned}\sum_{k=0}^{p}\binom{p}{k}\frac{1}{k+1}a^k b^{p-k} \ge \frac{1}{p+1}\sum_{k=0}^{p}\binom{p}{k}a^k b^{p-k} &= \frac{1}{p+1}(a+b)^p \\ \ge \frac{1}{p+1}\Big(\frac{1}{2}(a+b)^p + \frac{1}{2}b^p\Big).&\end{aligned} \tag{A.10}$$

(ii) For $\widetilde{f}$ of the form (A.7) we have $\partial_i \widetilde{f}(x) = a_i \prod_{j\ne i}^{d}(a_j x_j + b_j)$ while, on the other hand, we have $a_i \prod_{j\ne i}^{d}(a_j x_j + b_j) = \widetilde{f}(x+e_i) - \widetilde{f}(x) = f(x+e_i) - f(x)$ for any $x \in B_1$ such that $x_i = 0$. □

Proof of Proposition A.1. (i) and (ii) follow from Lemma A.3 and the continuous embedding Theorem.

The claim (iii) follows from (i) by iteration.

To prove (iv), assume first that d is odd and thus $M = \lfloor \frac{d+2}{2} \rfloor = \frac{d}{2} + \frac{1}{2}$. Let us apply (iii) with $p = 2$, $m = M - 1$, and

$$\frac{1}{p_m} = \frac{1}{2} - \frac{M-1}{d} = \frac{d-(d-1)}{2d} = \frac{1}{2d}. \tag{A.11}$$

Hence,

$$n^{-\frac{d}{2d}}\|\nabla f\|_{2d} \le \mathfrak{C} n^{-\frac{d}{2}-1}\sum_{k=1}^{M}\|(n\nabla)^k f\|_2. \tag{A.12}$$

Further,

$$\big|f(x) - f(y)\big| \le \mathfrak{C} n^{1-\frac{d}{2d}} \|\nabla f\|_{2d} = \mathfrak{C} n^{\frac{1}{2}} \|\nabla f\|_{2d} \tag{A.13}$$

for all $x, y \in B_n$ by (ii). Averaging over y yields

$$\Big|f(x) - (n+1)^{-d} \sum_{y \in B_n} f(y)\Big| \le \mathfrak{C} n^{\frac{1}{2}} \|\nabla f\|_{2d}. \tag{A.14}$$

On the other hand,

(A.15)

$$\Big|(n+1)^{-d} \sum_{y\in B_n} f(y)\Big| \le (n+1)^{-d} \Big(\sum_{y\in B_n} f(y)^2\Big)^{1/2} \Big(\sum_{y \in B_n} 1\Big)^{1/2} \le (n+1)^{-d/2} \|f\|_2$$

yielding

$$|f(x)| \le \mathfrak{C} n^{\frac{1}{2}} \|\nabla f\|_{2d} + (n+1)^{-d/2} \|f\|_2 \tag{A.16}$$

for all $x \in B_n$. The assertion (iv) for odd d follows.

Similarly for even d when $M = \lfloor \frac{d+2}{2} \rfloor = \frac{d}{2} + 1$ and we use $m = M - 2$ and $q = 2d > p_m = d$. $\square$

APPENDIX B

Integration by Parts and Estimates of the Boundary Terms

For the convenience on the reader we spell out the estimates of the boundary terms in detail.

a) $d = 1$

The forward and backward derivative are $\partial v(x) = v(x+1) - v(x)$ and $\partial^* v(x) = v(x-1) - v(x)$.

Proposition B.1 (**Integration by parts**). *Let $g, v, u\colon \mathbb{Z} \to \mathbb{R}$ and $m \in \mathbb{N}$. Then:*

(i)

$$\sum_{x=-m}^{m} g(x)\partial v(x) = \sum_{x=-m}^{m} \partial^* g(x)v(x) + g(m)v(m+1) - g(-m-1)v(-m).$$

(ii)

$$\sum_{x=-m}^{m} \partial u(x)\partial v(x) = \sum_{x=-m}^{m} (\partial^*\partial u)(x)v(x) + \partial u(m)v(m+1) - \partial u(-m-1)v(-m).$$

Proposition B.2 (**Evaluation of the boundary terms**). *There exist a constant $\mathfrak{c} < 3\sqrt{2}$ such that for any $v\colon \mathbb{Z} \to \mathbb{R}$ and any $m \in \mathbb{N}$, $m > 1$, one has*

$$v(-m)^2 + v(m+1)^2 \le \frac{\mathfrak{c}}{2m+1} \sum_{x=-m}^{m} v(x)^2 + \mathfrak{c}(2m+1) \sum_{x=-m}^{m} \partial v(x)^2. \tag{B.1}$$

Proof. Assume first that the number of those $x \in \{-m, \dots, m\}$ for which $v(x)^2 \ge \frac{1}{3}\big(v(-m)^2 + v(m+1)^2\big)$ is at least $\frac{2m+1}{\sqrt{2}}$. Then $\sum_{x=-m}^{m} v(x)^2 \ge \frac{1}{3\sqrt{2}}(2m+1)\big(v(-m)^2 + v(m+1)^2\big)$.

On the other hand, if the number of such x's is less then $\frac{2m+1}{\sqrt{2}}$, then there exists x such that $\partial v(x)^2 \ge \frac{\sqrt{2}}{6}\frac{v(-m)^2+v(m+1)^2}{2m+1}$, implying

$$\sum_{x=-m}^{m} \partial v(x)^2 \ge \frac{1}{3\sqrt{2}} \frac{v(-m)^2 + v(m+1)^2}{2m+1}.$$

Indeed, having assured the existence of y and z such $v(y)^2 < \frac{1}{3}\big(v(-m)^2 + v(m+1)^2\big)$ (the existence of such y is obvious for $m > 1$ implying that $\big(1 - \frac{1}{\sqrt{2}}\big)(2m+1) > 1$) and $v(z)^2 \ge \frac{1}{2}\big(v(-m)^2 + v(m+1)^2\big)$ (again, its existence follows since

$\frac{1}{2}\big(v(-m)^2+v(m+1)^2\big) \leq \max\{v(-m)^2, v(m+1)^2\}\big)$ implying that the interval $\big[\frac{1}{3}\big(v(-m)^2+v(m+1)^2\big), \frac{1}{2}\big(v(-m)^2+v(m+1)^2\big)\big]$ has to be spanned within at most $\frac{2m+1}{\sqrt{2}}$ increments $\partial v(x)^2$.

In both cases,

$$\text{(B.2)}\quad \frac{1}{2m+1}\sum_{x=-m}^{m} v(x)^2 + (2m+1)\sum_{x=-m}^{m}\partial v(x)^2 \geq \frac{1}{3\sqrt{2}}\big(v(-m)^2+v(m+1)^2\big)$$

implying the claim. ☐

The combination of Proposition B.1 and B.2 yields:

Proposition B.3. *Let $u, v\colon \mathbb{Z}\to\mathbb{R}$ and $m\in\mathbb{N}$. With the constant $\mathfrak{c}$ from Proposition B.2 and any $\eta>0$, one has*

(B.3)

$$\Big|\sum_{x=-m}^{m}\partial u(x)\partial v(x)\Big| \leq \frac{1}{2}(2m+1)^2\frac{1}{\eta}\sum_{x=-m}^{m}\big|(\partial^*\partial u)(x)\big|^2+\frac{1}{2}\frac{\eta}{(2m+1)^2}\sum_{x=-m}^{m}v(x)^2+$$
$$+\frac{2m+1}{2\eta}\Big[\partial u(-m-1)^2+\partial u(m)^2\Big]+\frac{\mathfrak{c}\,\eta}{2}\Big[\frac{1}{(2m+1)^2}\sum_{x=-m}^{m}v(x)^2+\sum_{x=-m}^{m}\partial v(x)^2\Big].$$

b) Multidimensional case

Let $X\in\mathcal{P}_k$ be a union of k-blocks. Further, let $\partial^{\pm}X=\cup_{i=1}^{d}\partial_i^{\pm}X$, where, for any $i=1,\dots,d$,

$$\text{(B.4)}\qquad \partial_i^- X := \{x\in\mathbb{Z}^d\colon x\notin X, x+e_i\in X \text{ or } x\in X, x+e_i\notin X\}$$

and

$$\text{(B.5)}\qquad \partial_i^+ X = \partial_i^- X + e_i := \{x+e_i\colon x\in\partial_i^- X\}.$$

Notice that $\partial^- X\cup\partial^+ X=\partial X$, the boundary defined in (4.31).

Lemma B.4. *Let B be a k-block and let $v: B\cup\partial B\to\mathbb{R}$. Then, for any $i=1,\dots,d$,*

$$\text{(B.6)}\qquad \sum_{x\in\partial_i^+ B} v(x)^2 \leq \mathfrak{c}\Big(\frac{1}{L^k}\sum_{x\in B}v(x)^2 + L^k\sum_{x\in B}|\nabla_i v(x)|^2\Big)$$

and

$$\text{(B.7)}\qquad \sum_{x\in\partial_i^- B} v(x)^2 \leq \mathfrak{c}\Big(\frac{1}{L^k}\sum_{x\in B}v(x)^2 + L^k\sum_{x\in B}|\nabla_i^* v(x)|^2\Big),$$

where c is the constant from Proposition B.2.

Proof. Applying Proposition B.2 to all lines in B that are parallel to e_i, we get (B.6). Similarly for (B.7), when considering the sites on these lines in the opposite order. ☐

Notice that, using $\nabla_i^* v(x) = -\nabla_i v(x-e_i)$, the last term in (B.7) can be actually replaced by $L^k\sum_{x\in B-e_i}|\nabla_i v(x)|^2$

To formulate the following immediate corollary of Lemma B.4, let, for any $X\in\mathcal{P}_k$ and $\ell\in\mathbb{N}$, the neighbourhood $U_\ell(X)$ be defined iteratively with $U_1(X)=X\cup\partial X$ and $U_{\ell+1}(X)=U_\ell(X)\cup\partial U_\ell(X)$.

Proposition B.5. *Let $X \in \mathcal{P}_k$ and $u : U_4(X) \to \mathbb{R}$. With the constant $\mathfrak{c}$ from Proposition B.2,*

(a)

$$L^k \sum_{x\in\partial X} |\nabla v(x)|^2 \le 2\mathfrak{c}\Big(\sum_{x\in X} |\nabla v(x)|^2 + L^{2k} \sum_{x\in U_1(X)} |\nabla^2 v(x)|^2\Big),$$

(b)

$$L^{3k} \sum_{x\in\partial X} |\nabla^2 v(x)|^2 \le 2\mathfrak{c}\Big(L^{2k}\sum_{x\in X} |\nabla^2 v(x)|^2 + L^{4k} \sum_{x\in U_1(X)} |\nabla^3 v(x)|^2\Big),$$

and

(c)

$$L^{5k} \sum_{x\in\partial X} |\nabla^3 v(x)|^2 \le 2\mathfrak{c}\Big(L^{4k}\sum_{x\in X} |\nabla^3 v(x)|^2 + L^{6k} \sum_{x\in U_1(X)} |\nabla^4 v(x)|^2\Big).$$

Proof. Let $B_1, \dots, B_n$ denote the k-blocks contained in X. Applying Lemma B.4 to each B_ℓ, $\ell = 1, \dots, n$, $i = 1, \dots, d$, observing that

$$\partial X \subset \bigcup_{\ell=1}^{n} \partial B_\ell, \tag{B.8}$$

and summing over i, we get

$$L^k \sum_{x\in\partial X} |\nabla v(x)|^2 \le \mathfrak{c}\Big(2\sum_{x\in X} |\nabla v(x)|^2 + L^{2k} \sum_{x\in X}\sum_{i=1}^{d}\big(|\nabla_i^2 v(x)|^2 + |\nabla_i^*\nabla_i v(x)|^2\big)\Big). \tag{B.9}$$

Using

$$\sum_{x\in X}\sum_{i=1}^{d} |\nabla_i^*\nabla_i v(x)|^2 = \sum_{x\in X-e_i}\sum_{i=1}^{d} |\nabla_i^2 v(x)|^2 \le \sum_{x\in U_1(X)} |\nabla^2 v(x)|^2, \tag{B.10}$$

we get the first claim.

The second and the third claim follow in a similar way. □

Notice that the sums over $x \in U_1(X)$ on the right hand side of the bounds in Proposition B.5 can be actually replaced by the sums over $x \in (X \cup \partial^- X) \setminus (X \cap \partial^- X)$.

Proposition B.6. *Let $u, v : X \cup \partial X \to \mathbb{R}$ and $X \in \mathcal{P}_k$. With the constant $\mathfrak{c}$ from Proposition B.2 and any $\eta > 0$, we get*

$$\Big|\sum_{x\in X} \nabla u(x)\nabla v(x)\Big| \le \frac{\eta(1+\mathfrak{c}d)}{2L^{2k}} \sum_{x\in X\cup\partial^- X} v(x)^2 + \frac{L^k}{2\eta}\sum_{x\in\partial^- X} |\nabla u(x)|^2 + \\ + \frac{\mathfrak{c}\eta}{2}\sum_{x\in X} |\nabla v(x)|^2 + \frac{L^{2k}}{2\eta}\sum_{x\in X\cup\partial^- X} |\nabla^2 u(x)|^2. \tag{B.11}$$

Proof. For any $x \in \partial_i^- X$, let $\epsilon_i(x) = +1$ if $x \in X$ and $\epsilon_i(x) = -1$ if $x \notin X$. By Proposition B.1, for each $i \in \{1, \dots, d\}$, we have

$$\sum_{x\in X} \nabla_i u(x)\nabla_i v(x) = \sum_{x\in X} \nabla_i^*\nabla_i u(x) v(x) + \sum_{x\in\partial_i^- X} \epsilon_i(x)\nabla_i u(x) v(x+e_i). \tag{B.12}$$

Summing over $i = 1, \dots, d$, we get

(B.13)

$$\Big|\sum_{x\in X} \nabla u(x)\nabla v(x)\Big| \le \sum_{i=1}^{d} \sum_{x\in X-e_i} |\nabla_i^2 u(x)v(x)| + \sum_{i=1}^{d} \sum_{x\in\partial_i^- X} |\nabla_i u(x)v(x+e_i)| \le$$

$$\le \frac{L^{2k}}{2\eta} \sum_{x\in X\cup\partial^- X} |\nabla^2 u(x)|^2 + \frac{\eta}{2L^{2k}} \sum_{i=1}^{d} \sum_{x\in X-e_i} v(x)^2 + \frac{L^k}{2\eta} \sum_{x\in\partial^- X} |\nabla u(x)|^2 + \frac{\eta}{2L^k} \sum_{i=1}^{d} \sum_{x\in\partial_i^+ X} v(x)^2.$$

Applying now Lemma B.4 on the last term, we get the claim. □

Lemma B.7. *Let* $Y \subset X$, $X, Y \in \mathcal{P}_k$, *and* $u : U_4(X) \to \mathbb{R}$. *Then*

$$\max_{x\in X} u(x)^2 \le \frac{2}{|Y|} \sum_{x\in Y} u(x)^2 + 2(\mathrm{diam} X)^2 \max_{x\in X} |\nabla u(x)|^2. \tag{B.14}$$

Proof. Cf. [**Bry09**, Lemma 6.20]. Considering the shortest path from any $x \in X$ to $y \in Y$, we have

$$|u(x)| \le |u(y)| + |x-y|_\infty \max_{z\in X} |\nabla u(z)|. \tag{B.15}$$

Using that $|x-y|_\infty \le \mathrm{diam} X$ (with the diameter taken in $|\cdot|_\infty$ metric on $\mathbb{Z}^d$), using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, and averaging both sides over Y, we get

$$u(x)^2 \le \frac{2}{|Y|} \sum_{y\in Y} u(y)^2 + 2(\mathrm{diam} X)^2 \max_{z\in X} |\nabla u(z)|^2 \tag{B.16}$$

yielding the claim. □

APPENDIX C

Gaussian Calculus

Here we recall the formulae for the derivative of a Gaussian integral with respect to the covariance matrix. The arguments are classical, but we provide proofs for the convenience of the reader. We begin with the first derivative. We will make the following general assumptions throughout this appendix.

Let V be a finite dimensional Euclidean vector space with scalar product $(\cdot,\cdot)$ and Lebesgue measure λ. Denote by $\mathrm{Sym}^{(+)}(V)$ *and* $\mathrm{Sym}^{(\geq)}(V)$ *the set of positive definite respectively of positive semi-definite symmetric operators on V. For* $\mathcal{C} \in \mathrm{Sym}^{(+)}(V)$ *denote by $\mu_{\mathcal{C}}$ the Gaussian measure with covariance $\mathcal{C}$.* Let $g\colon V \to \mathbb{R}$ be measurable and assume that there exists a $\mathcal{B} \in \mathrm{Sym}^{(\geq)}(V)$ and a constant $M \in \mathbb{R}$ such that

$$|g(x)| \leq M \mathrm{e}^{\frac{1}{2}(\mathcal{B}x,x)} \quad \text{for all } x \in V.$$

For $\mathcal{C}^{-1} > \mathcal{B}$ define

$$H(\mathcal{C}) := \int_V g(x)\, \mu_{\mathcal{C}}(\mathrm{d}x) = \frac{1}{\det(2\pi\mathcal{C})^{1/2}} \int_V g(x) \mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}\, \lambda(\mathrm{d}x). \tag{C.1}$$

We first recall that H is real-analytic in the set $\{\mathcal{C} \in \mathrm{Sym}^{(+)}(V)\colon \mathcal{C}^{-1} > \mathcal{B}\}$. In fact we will extend H to a complex analytic function as follows. Let $\widetilde{V}$ denote the complexification of V with the canonical sesquilinear-form $(\cdot,\cdot)$, let $GL(\widetilde{V})$ denote the set of all invertible $\mathbb{C}$-linear maps from $\widetilde{V}$ to itself and let

$$\mathcal{U} := \{\mathcal{C} \in GL(\widetilde{V})\colon \mathrm{Re}\,(\mathcal{C}^{-1}x,x) > (\mathcal{B}x,x)\ \forall x \in V \setminus \{0\}\}.$$

Define H on $\mathcal{U}$ by the right hand side of (C.1).

Lemma C.1. *(i) The map $H\colon \mathcal{U} \to \mathbb{C}$ is analytic and the derivative at $\mathcal{C}$ in direction $\dot{\mathcal{C}}$ reads as*

$$DH(\mathcal{C},\dot{\mathcal{C}}) = \int_V g(x) \frac{1}{2}\big((\mathcal{C}^{-1}\dot{\mathcal{C}}\mathcal{C}^{-1}x,x) - \mathrm{Tr}(\mathcal{C}^{-1}\dot{\mathcal{C}})\big)\, \mu_{\mathcal{C}}(\mathrm{d}x). \tag{C.2}$$

(ii) Assume in addition that g is continuous and that there exists a continuous function $w\colon V \to (0,\infty)$ such that

$$g(x+y) \leq M \mathrm{e}^{\frac{1}{2}(\mathcal{B}x,x)} w(y), \quad x,y \in V. \tag{C.3}$$

Define

$$\widetilde{H}(\mathcal{C})(y) := \int_V g(x+y)\, \mu_{\mathcal{C}}(\mathrm{d}x) \quad \text{for all } y \in V. \tag{C.4}$$

Then $\widetilde{H}$ is an analytic map from $\mathcal{U}$ to the space

$$C^0_w := \{h \in \mathcal{C}^0(V)\colon \|h\|_w < \infty\},$$

where

$$\|h\|_w := \sup_{y\in V} \frac{|h(y)|}{|w(y)|},$$

and the derivative at $\mathcal{C}$ *in direction* $\dot{\mathcal{C}} \in GL(\widetilde{V})$ *is given as*

$$D\widetilde{H}(\mathcal{C},\dot{\mathcal{C}})(y) = \int_V g(x+y) D_1 f(\mathcal{C},x,\dot{\mathcal{C}})\,\lambda(\mathrm{d}x), \quad y\in V,$$

where

$$f(\mathcal{C},x) := \frac{\mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}}{\det(2\pi\mathcal{C})^{1/2}}.$$

Proof. (i) Set

$$f(\mathcal{C},x) := \frac{\mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}}{\det(2\pi\mathcal{C})^{1/2}}. \tag{C.5}$$

Then for every $x\in V$ the map $\mathcal{C}\mapsto f(\mathcal{C},x)$ is complex differentiable in $\mathcal{U}$, and (using Jacobi's formula for the derivative of determinants) we get that

$$D_1 f(\mathcal{C},x,\dot{\mathcal{C}}) = \frac{1}{2}\big((\mathcal{C}^{-1}\dot{\mathcal{C}}\mathcal{C}^{-1}x,x) - \mathrm{Tr}(\mathcal{C}^{-1}\dot{\mathcal{C}})\big) f(\mathcal{C},x). \tag{C.6}$$

In particular for each $\varepsilon>0$ there exists $M'>0$ such that

$$\big|D_1 f(\mathcal{C},x,\dot{\mathcal{C}})\big| \le M' \mathrm{e}^{\frac{1}{2}\varepsilon|x|^2}\mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}|\dot{\mathcal{C}}|. \tag{C.7}$$

Since $\mathrm{Re}\,(\mathcal{C}^{-1}) > \mathcal{B}$ and since V is finite-dimensional we also have that $\mathrm{Re}\,(\mathcal{C}^{-1}) > \mathcal{B}+\varepsilon\mathrm{Id}$ and thus the function

$$g(x)\big|D_1 f(\mathcal{C},x,\dot{\mathcal{C}})\big|$$

is integrable. Now for any $\dot{\mathcal{C}}\neq 0$ we estimate

$$\begin{aligned} &\frac{1}{|\dot{\mathcal{C}}|}\Big|H(\mathcal{C}+\dot{\mathcal{C}}) - H(\mathcal{C}) - \int_V g(x) D_1 f(\mathcal{C},x,\dot{\mathcal{C}})\,\lambda(\mathrm{d}x)\Big| \\ &\quad\le \int_V |g(x)|\Big|\frac{f(\mathcal{C}+\dot{\mathcal{C}}) - f(\mathcal{C}) - D_1 f(\mathcal{C},x,\dot{\mathcal{C}})}{|\dot{\mathcal{C}}|}\Big|\,\lambda(\mathrm{d}x). \end{aligned} \tag{C.8}$$

For $\dot{\mathcal{C}}\to 0$ the integrand on the right hand side of (C.8) goes to zero for every $x\in V$. It remains to find an integrable majorant. We have

$$f(\mathcal{C}+\dot{\mathcal{C}},x) - f(\mathcal{C},x) = \int_0^1 D_1 f(\mathcal{C}+s\dot{\mathcal{C}},x)\,\mathrm{d}s.$$

Now for every $\mathcal{C}\in\mathcal{U}$ and every $\varepsilon>0$ there exist $\delta>0$ and $M''>0$ such that for all $\widetilde{\mathcal{C}}\in B_\delta(\mathcal{C})$ we have

$$\big|D_1 f(\widetilde{\mathcal{C}},x,\dot{\mathcal{C}})\big| \le M'' \mathrm{e}^{\frac{1}{2}\varepsilon|x|^2}\mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}|\dot{\mathcal{C}}|.$$

Hence for $|\dot{\mathcal{C}}|<\delta$ the integrand in (C.8) is bounded by the integrable function

$$|g(x)|(M'+M'')\mathrm{e}^{\frac{1}{2}\varepsilon|x|^2}\mathrm{e}^{-\frac{1}{2}(\mathcal{C}^{-1}x,x)}.$$

Thus by the dominated convergence theorem the right hand side of (C.8) goes to zero as $\dot{\mathcal{C}}\to 0$. This concludes the proof of (i).

(ii) The continuity of the map $y \mapsto \widetilde{H}(\mathcal{C})(y)$ follows directly from the dominated convergence theorem. Indeed, assume that $y_k \to \overline{y}$ in V as $k \to \infty$. Using the continuity of g we obtain

$$g(x+y_k)f(\mathcal{C},x) \to g(x+\overline{y})f(\mathcal{C},x) \quad \text{for every } x \in V \text{ as } k \to \infty.$$

Moreover, for $|y_k - \overline{y}| < \delta$ we have

$$\bigl|g(x+y_k)f(\mathcal{C},x)\bigr| \le M\mathrm{e}^{\frac{1}{2}(\mathcal{B}x,x)}\bigl(\sup_{z\in B_\delta(\overline{y})} w(z)\bigr)f(\mathcal{C},x),$$

and the right hand side is integrable. Hence

$$\widetilde{H}(\mathcal{C})(y_k) \to \widetilde{H}(\mathcal{C})(\overline{y}) \quad \text{as } k \to \infty$$

by the dominated convergence theorem. To verify complex differentiability define first the linear map

$$(L\dot{\mathcal{C}})(y) := \int_V g(x+y)D_1 f(\mathcal{C},x,\dot{\mathcal{C}})\,\lambda(\mathrm{d}x).$$

Then one sees as above that $y \mapsto (L\dot{\mathcal{C}})(y)$ is continuous. Moreover it follows from the bounds (C.3) and (C.7) that

$$\|L\dot{\mathcal{C}}\|_w \le MM'|\dot{\mathcal{C}}| \int_V \mathrm{e}^{\frac{1}{2}((\mathcal{B}+\varepsilon\mathrm{Id}-\mathcal{C}^{-1})x,x)}\,\lambda(\mathrm{d}x) < \infty.$$

Thus L is a bounded linear map from $GL(\widetilde{V})$ to $C^0_w(V)$. Finally we check differentiability. We have

$$\begin{aligned}&\Bigl|\widetilde{H}(\mathcal{C}+\dot{\mathcal{C}})(y) - \widetilde{H}(\mathcal{C})(y) - L\dot{\mathcal{C}}(y)\Bigr|\\ \le&\int_V |g(x+y)|\bigl|f(\mathcal{C}+\dot{\mathcal{C}},x) - f(\mathcal{C},x) - D_1f(\mathcal{C},x,\dot{\mathcal{C}})\bigr|\,\lambda(\mathrm{d}x)\\ \le& Mw(y)\int_V \mathrm{e}^{\frac{1}{2}(\mathcal{B}x,x)}\bigl|f(\mathcal{C}+\dot{\mathcal{C}},x) - f(\mathcal{C},x) - D_1f(\mathcal{C},x,\dot{\mathcal{C}})\bigr|\,\lambda(\mathrm{d}x).\end{aligned}$$

Dividing by $w(y)|\dot{\mathcal{C}}|$ and taking the supremum over y we get

$$\begin{aligned}&\|\widetilde{H}(\mathcal{C}+\dot{\mathcal{C}}) + \widetilde{H}(\mathcal{C}) - L\dot{\mathcal{C}}\|_w\\ &\quad\le M\int_V \mathrm{e}^{\frac{1}{2}(\mathcal{B}x,x)}\frac{\bigl|f(\mathcal{C}+\dot{\mathcal{C}},x) - f(\mathcal{C},x) - D_1f(\mathcal{C},x,\dot{\mathcal{C}})\bigr|}{|\dot{\mathcal{C}}|}\,\lambda(\mathrm{d}x).\end{aligned}$$

Now as in (i) it follows from the dominated convergence theorem that the right hand side goes to zero as $\dot{\mathcal{C}} \to 0$. Thus $\widetilde{H}$ is complex differentiable at $\mathcal{C}$ with derivative $D\widetilde{H}(\mathcal{C}) = L$. □

We will apply Lemma C.1 with $\mathcal{C} = \mathcal{C}_k^{(\boldsymbol{q})}$, the covariance matrices which arise in the finite range decomposition (see Proposition 4.1), and $\mathcal{B} = \varkappa\mathcal{B}_k = 2\overline{C}h^{-2}\mathcal{B}_k$ where $\mathcal{B}_k$ is as in Lemma 5.3. Now an important point is that the finite range decomposition in Proposition 4.1 does not yield a bound on terms like

$$\mathrm{Tr}\bigl(\mathcal{C}_k^{(\boldsymbol{q})}\bigr)^{-1} D_{\boldsymbol{q}}\dot{\mathcal{C}}_k^{(\boldsymbol{q})}\dot{\boldsymbol{q}}$$

which are independent of k and N.

In order to derive bounds on the derivatives of $\boldsymbol{q} \mapsto H(\mathcal{C}_k^{(\boldsymbol{q})})$ which are independent of k and N we now derive different expressions for the derivatives of H which do not involve $\mathcal{C}^{-1}$ but which require derivatives of g. This leads to a loss of regularity when we consider the convolution operator $g \mapsto \int g(\cdot + x)\, \mu_{\mathcal{C}}(\mathrm{d}x)$ as an operator between function spaces and we shall see later how to deal with this loss of regularity.

In the following we assume that

$$e_1, \dots, e_{\dim(V)} \text{ is an orthonormal basis of } V. \tag{C.9}$$

Lemma C.2. *Let $\mathcal{B} \in \mathrm{Sym}^{(\geq)}(V)$ and let $g \in C^2(V)$ with*

$$\sup_{x \in V} \sum_{s=0}^{2} |D^s g(x)| \mathrm{e}^{-\frac{1}{2}(\mathcal{B}x, x)} < \infty\,. \tag{C.10}$$

Furthermore, let $\mathcal{C} \in \mathrm{Sym}^{(+)}(V)$ be given with $\mathcal{C}^{-1} > \mathcal{B}$. Let $\dot{\mathcal{C}} \in \mathrm{Sym}^{(+)}(V)$ and define

$$h(t) := \int_V g(x)\, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x).$$

Then h is a C^1-function on some interval $(-a_0, a_0)$ and

$$h'(t) = \int_V \big(Ag\big)(x)\, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x), \tag{C.11}$$

where

$$Ag(x) := \frac{1}{2} \sum_{i,j=1}^{\dim(V)} \dot{\mathcal{C}}_{i,j} D^2 g(x, e_i, e_j), \qquad \textit{with } \dot{\mathcal{C}}_{i,j} := (\dot{\mathcal{C}} e_i, e_j). \tag{C.12}$$

Remark C.3. In coordinate free notation the map A in (C.12) can be written as

$$Ag(x) = \mathrm{Tr}\big(\mathrm{Hess}\,(g(x))\dot{\mathcal{C}}\big),$$

where $\mathrm{Hess}\,(g(x))$ is the linear map $V \to V$ defined by

$$\big(\mathrm{Hess}\,(g(x))a, b\big) = D^2 g(x, a, b) \quad \text{ for all } a, b \in V.$$

Sometimes it is more convenient to use an orthonormal basis of the complexification $\widetilde{V}$ of V to evaluate Ag. If we extend $\mathrm{Hess}\,(g(x))$ as a $\mathbb{C}$-linear map and $D^2 g(x, \cdot, \cdot)$ as a $\mathbb{C}$-bilinear map, then

$$\big(\mathrm{Hess}\,(g(x))a, b\big) = D^2 g(x, a, \overline{b}) \quad \text{ for all } a, b \in \widetilde{V}$$

since the sesquilinear form $(\cdot, \cdot)$ on $\widetilde{V} \times \widetilde{V}$ is anti-linear in the second argument. If we also extend $\dot{\mathcal{C}}$ as a $\mathbb{C}$-linear map and if $f_1 \dots, f_{\dim(V)}$ is an orthonormal basis of $\widetilde{V}$, then

$$\mathrm{Tr}_V\big(\mathrm{Hess}\,(g(x))\dot{\mathcal{C}}\big) = \mathrm{Tr}_{\widetilde{V}}\big(\mathrm{Hess}\,(g(x))\dot{\mathcal{C}}\big) = \sum_{i=1}^{\dim V} \big(\mathrm{Hess}\,(g(x))\dot{\mathcal{C}} f_i, f_i\big).$$

Hence

$$Ag(x) = \sum_{i=1}^{\dim(V)} D^2 g(x, \dot{\mathcal{C}} f_i, \overline{f}_i). \tag{C.13}$$

◇

Proof. One can easily check that the definition of A is independent of the choice of the orthonormal basis. The whole statement is invariant under isometries. Hence we may assume that $V = \mathbb{R}^n$ with the standard scalar product and that $e_1, \dots, e_n$ is the standard basis. Furthermore, we write $\mathcal{C}(t) := \mathcal{C} + t\dot{\mathcal{C}}$ in the following. The starting point is the formula for the Fourier transform of a Gaussian

$$\int_{\mathbb{R}^n} \mathrm{e}^{-i(\xi,x)}\, \mu_{\mathcal{C}(t)}(\mathrm{d}x) = \mathrm{e}^{-\frac{1}{2}(\mathcal{C}(t)\xi,\xi)}\,. \tag{C.14}$$

By continuity of $t \mapsto \mathcal{C}(t)$ we may assume that there is an $a_0 > 0$ and a $\delta > 0$ such that for $t \in (-a_0, a_0)$ we have $\mathcal{B} \le \mathcal{C}^{-1}(t) - \delta \mathrm{Id}$ and $\mathcal{C}(t) \ge \delta \mathrm{Id}$. From now on we consider $h(t)$ only on the interval $(-a_0, a_0)$.

Now assume first that g belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ of smooth and rapidly decreasing functions. By Plancherel's formula we have

$$h(t) = \int_{\mathbb{R}^n} g(x)\, \mu_{\mathcal{C}(t)}(\mathrm{d}x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) \mathrm{e}^{-\frac{1}{2}(\mathcal{C}(t)\xi,\xi)}\, \mathrm{d}\xi\,. \tag{C.15}$$

Since $g \in \mathcal{S}(\mathbb{R}^n)$, the right hand side is differentiable with respect to t and the identity $\widehat{\partial_j g}(\xi) = i\xi_j \hat{g}(\xi)$ yields, with another application of Plancherel's formula,

$$\begin{aligned}
\dot{h}(t) &= -\frac{1}{2}\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{g}(\xi) \sum_{j,k=1}^n \dot{\mathcal{C}}_{jk}\xi_j\xi_k e^{-\frac{1}{2}(\mathcal{C}(t)\xi,\xi)}\, \mathrm{d}\xi \\
&= \frac{1}{2}\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{j,k=1}^n \dot{\mathcal{C}}_{jk} (\widehat{\partial_j\partial_k g})(\xi) e^{-\frac{1}{2}(\mathcal{C}(t)\xi,\xi)}\, \mathrm{d}\xi \\
&= \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j,k=1}^n \dot{\mathcal{C}}_{jk} (\partial_j\partial_k g)(x)\, \mu_{\mathcal{C}(t)}(\mathrm{d}x) = \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}(\dot{\mathcal{C}} D^2 g(x))\, \mu_{\mathcal{C}(t)}(\mathrm{d}x) \\
&= \int_{\mathbb{R}^n} Ag(x)\, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x).
\end{aligned}$$

This proves assertion (C.11) and (C.12) for $g \in \mathcal{S}(\mathbb{R}^n)$. For a general g we use a cut-off and a convolution with a mollifier. To do so we first rewrite the result for $g \in \mathcal{S}(\mathbb{R}^n)$ in the integral form

$$\int_{\mathbb{R}^n} g(x)\, \mu_{\mathcal{C}(t)}(\mathrm{d}x) - \int_{\mathbb{R}^n} g(x)\, \mu_{\mathcal{C}(0)}(\mathrm{d}x) = \int_0^t \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}(\dot{\mathcal{C}}(s) D^2 g(x))\, \mu_{\mathcal{C}(s)}(\mathrm{d}x)\, \mathrm{d}s. \tag{C.16}$$

Now, for $g \in C^2_{\mathrm{c}}(\mathbb{R}^n)$ consider the Gaussian measure $h_k(x)\mathrm{d}x$ on $\mathbb{R}$ with covariance $\frac{1}{k}$ and define $g_k := h_k * g \in \mathcal{S}(\mathbb{R}^n)$. Hence (C.16) holds for g_k and we have a uniform convergence $g_k \to g$ and $D^2 g_k \to D^2 g$. Since $\mathcal{C}(s) \ge \delta \mathrm{Id}$ we can pass to the limit using the dominated convergence theorem which proves (C.16) whenever $g \in C^2_{\mathrm{c}}(\mathbb{R}^n)$. Finally, for g as in the lemma we let $\eta \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ to be a cut-off function that vanishes outside the unit ball $B(0,1)$ and equals 1 in the ball $B(0, \frac{1}{2})$. Let $g_k(x) = \varphi(\frac{x}{k})g(x)$. Then $g_k \in C^2_{\mathrm{c}}(\mathbb{R}^n)$ with $g_k \to g$ and $D^2 g_k \to D^2 g$ uniformly on compact subsets and

$$\sup |g_k(x)| + \sup |D^2 g_k(x)| \le C \sup \sum_{s=0}^2 |\nabla^s g(x)|. \tag{C.17}$$

Since $\mathcal{C}^{-1}(s) \ge \mathcal{B} + \delta \mathrm{Id}$ we may pass to the limit by the dominated convergence theorem. This shows that (C.16) holds for all $g \in C^2(\mathbb{R}^n)$ which satisfy (C.10) with $r = 1$. Finally continuity of $t \mapsto \mathcal{C}(t)$, the bound $\mathcal{B} \le \mathcal{C}^{-1}(s) - \delta \mathrm{Id}$ and the dominated convergence theorem imply that $s \mapsto \int_{\mathbb{R}^n} \mathrm{Tr}(\dot{\mathcal{C}} D^2 g(x)) \mu_{\mathcal{C}(s)}(\mathrm{d}x)$ is continuous. This finishes the proof. $\square$

Lemma C.4. *Let $\mathcal{B} \in \mathrm{Sym}^{(\geq)}$ and assume that $g \in C^{2\ell}(V), \ell \in \mathbb{N}$, satisfies*

$$\sup_{x \in V} \sum_{s=0}^{2\ell} \left| D^s g(x) \right| \mathrm{e}^{-\frac{1}{2}(\mathcal{B}x, x)} < \infty. \tag{C.18}$$

Assume that $\mathcal{C} \in \mathrm{Sym}^{(+)}$ with $\mathcal{C}^{-1} > \mathcal{B}$. Then the function H defined by (C.1) *satisfies*

$$D^\ell H(\mathcal{C}, \dot{\mathcal{C}}_1, \ldots, \dot{\mathcal{C}}_\ell) = \int_V \left(A_{\dot{\mathcal{C}}_1} \cdots A_{\dot{\mathcal{C}}_\ell} g \right)(x) \, \mu_{\mathcal{C}}(\mathrm{d}x), \tag{C.19}$$

where for $f \in C^2(V)$ the operator $A_{\dot{\mathcal{C}}_i}$ is defined by

$$(A_{\dot{\mathcal{C}}_i} f)(x) = \frac{1}{2} \sum_{i,j=1}^{\dim(V)} \dot{\mathcal{C}}_{i,j} D^2 f(x, e_i, e_j). \tag{C.20}$$

Proof. Since we already know that H is analytic in $\mathcal{U}$ it suffices to show the result for $\dot{\mathcal{C}}_1 = \cdots = \dot{\mathcal{C}}_\ell = \dot{\mathcal{C}}$. The full result follows by polarization. It thus suffices to show that the function h in Lemma C.2 satisfies

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} h(t) = \int_V \left(A^k g \right)(x) \, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x) \quad \text{for } 1 \le k \le \ell, \tag{C.21}$$

where $A = A_{\dot{\mathcal{C}}}$. We prove this by induction. The case $k = 1$ is just Lemma C.2. Thus assume that $k \le \ell - 1$ and (C.21) holds for k. Let $\widetilde{g} := A^k g$. Then $\widetilde{g}$ satisfies the assumptions of Lemma C.2. Thus by the induction assumption and Lemma C.2, we obtain

$$\begin{aligned} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} h(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_V \widetilde{g}(x) \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x) = \int_V \left(A\widetilde{g} \right)(x) \, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x) \\ &= \int_V \left(A^{k+1} g \right)(x) \, \mu_{\mathcal{C}+t\dot{\mathcal{C}}}(\mathrm{d}x). \end{aligned}$$

$\square$

We finally collect formulae for the derivatives up to the third order for a general dependence, that is, we now let $(-\delta, \delta) \ni t \mapsto \mathcal{C}(t) \in \mathrm{Sym}^{(+)}(V)$ be a C^ℓ map with $\mathcal{C}(0)^{-1} > \mathcal{B}$ and let g satisfies the assumptions of Lemma C.4. Then

$$\widetilde{h}(t) := \int_V g(x) \, \mu_{\mathcal{C}(t)}(\mathrm{d}x) \tag{C.22}$$

is a C^ℓ map on some interval $(-\delta', \delta')$ and the derivatives of $\widetilde{h}$ can be computed by the chain rule. In particular we obtain the following formulae.

$$\dot{\widetilde{h}}(t) = DH(\mathcal{C}(t), \dot{\mathcal{C}}(t)), \tag{C.23}$$

$$\ddot{\widetilde{h}}(t) = D^2 H(\mathcal{C}(t), \dot{\mathcal{C}}(t), \dot{\mathcal{C}}(t)) + DH(\mathcal{C}(t), \ddot{\mathcal{C}}(t)), \tag{C.24}$$

$$\begin{aligned} \dddot{\widetilde{h}}(t) &= D^3 H(\mathcal{C}(t), \dot{\mathcal{C}}(t), \dot{\mathcal{C}}(t), \dot{\mathcal{C}}(t)) + 3D^2 H(\mathcal{C}(t), \dot{\mathcal{C}}(t), \ddot{\mathcal{C}}(t)) \\ &\quad + DH(\mathcal{C}(t), \dddot{\mathcal{C}}(t)). \end{aligned} \tag{C.25}$$

In general $D^k\widetilde{h}(t)$ is a sum of terms of the form

$$D^\ell H(\mathcal{C}(t), A_1, \ldots, A_k) \tag{C.26}$$

with

$$A_i = D^{j_1}\mathcal{C}(t) \quad \text{and} \quad \sum_{i=1}^{\ell} j_i = k. \tag{C.27}$$

APPENDIX D

Chain Rules

Here we formulate and prove a chain rule with loss of regularity for a composition of two maps. It turns out that proving the needed claims as well as checking their assumptions in particular cases is much simpler when formulated in terms of higher order one-dimensional directional derivatives and the related Peano derivatives. We first review their properties and the mutual relations.[1]

D.1. Motivation

Before we enter into the precise statement of the setting and the results we consider a simple example how loss of regularity can easily arise even for seemingly innocuous maps and we sketch the key calculation in the proof of the main result. Consider the space $C^k(S^1)$ of 2π-periodic k times continuously differentiable functions and the map $F : C^k(S^1) \times \mathbb{R} \to C^k(S^1)$ defined by

$$F(y,p)(t) = \sin(y(t-p)).$$

It is easy to see that F is continuous and that the map $y \mapsto F(y,p)$ is smooth (in fact real-analytic) as a map from $C^k(S^1)$ to itself. For a fixed $y \in C^k(S^1) \setminus C^{k+1}(S)$ the map $p \mapsto F(y,p)$ is, however, not differentiable as a map from $\mathbb{R}$ to $C^k(S^1)$. It is only differentiable as a map from $\mathbb{R}$ to $C^{k-1}(S^1)$ and we have

$$\frac{\partial}{\partial p} F(y,p)(\cdot) = -\cos y(\cdot - p)\ y'(\cdot - p).$$

Similarly $p \mapsto F(y,p)$ is a C^l map to C^{k-l} for $l \le k$. Thus each derivative with respect to p leads to loss of one derivative in y. A similar phenomenon occurs if we use formula (C.11) to compute the derivative of the convolution maps $G(g, \mathcal{C}) := g * \mu_{\mathcal{C}}$ with respect to the covariance $\mathcal{C}$. Our renormalisation step involves a composition of several maps of this type and one might think that this leads to a multiple loss of regularity. The main result of this appendix, Theorem D.29 below, shows that this is not the case. The behaviour of the composed map is no worse than the behaviour of the individual maps.

To state the result informally consider scales of of Banach spaces $\boldsymbol{X}_m \subset \boldsymbol{X}_{m-1} \subset \ldots \subset \boldsymbol{X}_0$, $\boldsymbol{Y}_m \subset \ldots \subset \boldsymbol{Y}_0$ and $\boldsymbol{Z}_m \subset \ldots \subset \boldsymbol{Z}_0$ as well as a Banach space $\boldsymbol{P}$ and maps

$$G : \boldsymbol{X}_m \times \boldsymbol{P} \to \boldsymbol{Y}_m, \qquad F : \boldsymbol{Y}_m \times \boldsymbol{P} \to \boldsymbol{Z}_m$$

[1]The present version of this Appendix is based on notes written by David Preiss. He has not only provided a suitable framework for smoothness, in terms of classes C^m_* and $\widetilde{C}^m$ introduced below, with particularly clear proofs of chain rule with loss of regularity, but he has also shown (Theorem D.10) that functions from C^m_* have continuous, multilinear, and symmetric directional derivatives. Nevertheless, all deficiencies of the present Appendix are the author's fault.

and the composed map

$$H(x,p) := F(G(x,p),p).$$

Informally, the assumptions on F and G are that these maps are well-behaved with respect to the first argument, but each derivative with respect to the second argument leads to a loss of order one in the scale of Banach spaces, i.e., that for all $0 \le n \le m-l$

$$D_1^j D_2^l F(y,p) : \boldsymbol{Y}_{n+l}^j \times \boldsymbol{P}^l \to \boldsymbol{Z}_n \quad \text{is bounded} \tag{D.1}$$

and

$$D_1^j D_2^l G(x,p) : \boldsymbol{X}_{n+l}^j \times \boldsymbol{P}^l \to \boldsymbol{Y}_n \quad \text{is bounded.} \tag{D.2}$$

Then we want to show that

$$D_1^j D_2^l H(y,p) : \boldsymbol{X}_{n+l}^j \times \boldsymbol{P}^l \to \boldsymbol{Z}_n \quad \text{is bounded.} \tag{D.3}$$

If we assume that all natural expressions make sense this can be seen as follows. From the chain rule we deduce inductively that $D_2^l H(x,p,\dot p^l) := D_2^l H(x,p,\dot p,\dots,\dot p)$ is a weighted sum of the terms

$$D_1^k D_2^i F(G(x,p),\, p,\, D_2^{l_1} G(x,p,\dot p^{l_1}),\dots,\, D_2^{l_k} G(x,p,\dot p^{l_k}),\, \dot p^i)$$

with $k \ge 0$ and $i + \sum_{s=1}^k l_s = l$. Another application of the chain rule shows that $D_1^j D_2^l H(x,p,\dot x^j,\dot p^l)$ is a weighted sum of the terms

$$D_1^{k+\bar k} D_2^i F(G(x,p),\, p,\, D_1^{\bar j_1} G(x,p,\dot x^{\bar j_1}),\dots,\, D_1^{j_k} D_2^{l_k} G(x,p,\dot x^{j_k},\dot p^{l_k}),\, \dot p^i)$$

with $\bar j_r \ge 1$, $j_s \ge 0$, $l_s \ge 1$ and

$$\sum_{r=1}^{\bar k} \bar j_r + \sum_{s=1}^{k} j_s = j, \quad i + \sum_{s=1}^{k} l_s = l.$$

In particular we have $l_s \le l - i$ and hence

$$D_1^{j_s} D_2^{l_s} G : \boldsymbol{X}_{n+l}^{j_s} \times \boldsymbol{P}^{l_s} \to \boldsymbol{Y}_{n+l-(l-i)} = \boldsymbol{Y}_{n+i} \quad \text{is bounded.}$$

Moreover

$$D_1^{k+\bar k} D_2^i F : \boldsymbol{Y}_{n+i}^{k+\bar k} \times \boldsymbol{P}^i \to \boldsymbol{Z}_n \quad \text{is bounded.}$$

Thus $\|D_1^j D_2^l H(x,p,\dot x^j,\dot p^l)\|_{\boldsymbol{Z}_n}$ is bounded in terms of $\|\dot x\|_{\boldsymbol{X}_{n+l}}^j$ and $\|\dot p\|_{\boldsymbol{P}}^l$. By polarization we get the desired assertion (D.3). The main point in the proof of Theorem D.29 is to give a precise definition of the informal assumptions (D.1) and (D.2) and to show that under these assumptions all the operations performed above make sense.

D.2. Derivatives and their relations

Directional derivatives.

Definition D.1. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be normed linear spaces, $\mathcal{U} \subset \boldsymbol{X}$ open and $G : \mathcal{U} \to \boldsymbol{Y}$ be a function. *Directional derivatives* of G at $x \in \mathcal{U}$ in directions $\dot x_1,\dots,\dot x_j \in \boldsymbol{X}$ are defined by

$$D^j G(x,\dot x_1,\dots,\dot x_j) = \frac{\mathrm{d}}{\mathrm{d}t_j} \dots \frac{\mathrm{d}}{\mathrm{d}t_1} G(x + \sum t_k \dot x_k)\Big|_{t_1=\dots=t_j=0}. \tag{D.4}$$

We will use the shorthand $D^jG(x,\dot{x}^j) = D^jG(x,\underbrace{\dot{x},\ldots,\dot{x}}_{j})$, and later, similarly,

$$D^jG(x,\dot{x}_1^{j_1},\ldots,\dot{x}_k^{j_k}) = D^jG(x,\underbrace{\dot{x}_1,\ldots,\dot{x}_1}_{j_1},\ldots,\underbrace{\dot{x}_k,\ldots,\dot{x}_k}_{j_k})$$

with $j = \sum_{s=1}^k j_s$.

Definition D.2. We use $C_*^m(\mathcal{U},\boldsymbol{Y})$ to denote the set of continuous functions $G:\mathcal{U}\to\boldsymbol{Y}$ such that for each $j\le m$ and $\dot{x}\in X$, the derivative $D^jG(x,\dot{x}^j)$ exists and the map $(x,\dot{x})\in\mathcal{U}\times\boldsymbol{X}\to D^jG(x,\dot{x}^j)\in\boldsymbol{Y}$ is continuous.

Remark D.3. The star $*$ is added just to indicate that this is not the standard class C^m of m-differentiable functions. Also, this definition is formally much weaker than that by Hamilton [**Ham82**] who takes G to be m-times differentiable if $D^mf : \mathcal{U}\times\underbrace{\boldsymbol{X}\times\cdots\times\boldsymbol{X}}_{m}\to\boldsymbol{Y}$ exists and is continuous (jointly as a function on the product space). However, Theorem D.10 below shows that it actually yields the same space. Note that for $X=\mathbb{R}$ it follows directly from the definition of $C_*^m(\mathcal{U},\boldsymbol{Y})$ that $C_*^m(\mathcal{U},\boldsymbol{Y}) = C^m(\mathcal{U},\boldsymbol{Y})$. We will see in Proposition D.17 that this identity holds whenever X is finite dimensional. $\diamond$

In proofs, especially when proving chain rules, it is often useful to rely on the notion of Peano derivatives.

Definition D.4. The Peano derivatives $G^{(n)}(x,\dot{x})$ of a function G at x in direction $\dot{x}$ are defined inductively by

$$G^{(n)}(x,\dot{x}) = n!\lim_{t\to 0}\frac{G(x+t\dot{x}) - \sum_{j=0}^{n-1}\frac{G^{(j)}(x,\dot{x})}{j!}t^j}{t^n} \tag{D.5}$$

whenever the derivative exists. Equivalently,

$$\Big\|G(x+t\dot{x}) - \sum_{j=0}^{n}\frac{G^{(j)}(x,\dot{x})}{j!}t^j\Big\|_Y = o(t^n) \text{ as } t\to 0. \tag{D.6}$$

Lemma D.5. *We notice the following obvious properties of these derivatives.*

(a) $G^{(0)}(x,\dot{x})$ *exists iff* G *is continuous at* x *in direction* $\dot{x}$*; then* $G^{(0)}(x,\dot{x}) = G(x)$.

(b) $G^{(n)}(x,t\dot{x}) = t^nG^{(n)}(x,\dot{x})$.

We show that $C_*^n(\mathcal{U},\boldsymbol{Y})$ can be equivalently defined using the Peano derivatives.

Lemma D.6. *Suppose* G *is* m*-times Peano differentiable at every point of the line segment* $[x,x+\dot{x}]$ *in the direction of* $\dot{x}$*. Then for any* $0\le j\le n\le m$,

$$\Big\|G^{(j)}(x+\dot{x},\dot{x}) - \sum_{i=0}^{n-j}\frac{G^{(j+i)}(x,\dot{x})}{i!}\Big\|_{\boldsymbol{Y}} \le \sup_{0\le\tau\le 1}\Big\|\frac{G^{(n)}(x+\tau\dot{x},\dot{x}) - G^{(n)}(x,\dot{x})}{(n-j)!}\Big\|_{\boldsymbol{Y}}.$$

Proof. The case $j=n$ is obvious. When $j<n$, $\boldsymbol{X}=\boldsymbol{Y}=\mathbb{R}$ and $\dot{x}=1$, the inequality follows immediately from the mean value statement of [**Oli54**, Theorem 2(ii)]. To prove the general case, find $y^*\in\boldsymbol{Y}^*$ realizing the norm on the left and use the special case for the map $t\in\mathbb{R}\to y^*G(x+t\dot{x})\in\mathbb{R}$. $\square$

Proposition D.7. $G\in C_*^m(\mathcal{U},\boldsymbol{Y})$ *iff* $G^{(n)}(x,\dot{x})$, $n\le m$ *exist and are continuous on* $\mathcal{U}\times\boldsymbol{X}$*. Moreover, for such* G, $D^nG(x,\dot{x}^n) = G^{(n)}(x,\dot{x})$ *on* $\mathcal{U}\times\boldsymbol{X}$ *for* $n\le m$.

Proof. If $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$ and the segment $[x, x+\dot{x}] \subset \mathcal{U}$, then the function $(-\epsilon, 1+\epsilon) \ni t \mapsto G(x+t\dot{x}) \in \boldsymbol{Y}$ is m-times continuously differentiable, and, in view of [**Die60**, 8.14.3 and 8.14, Problem 5],

$$\Big\| G(x+t\dot{x}) - \sum_{j=0}^{n} \frac{D^j G(x,\dot{x}^j)}{j!} t^k \Big\|_{\boldsymbol{Y}} = o(t^n) \text{ as } t \to 0, \tag{D.7}$$

for each $n \le m$, yielding $G^{(j)}(x,\dot{x}) = D^j G(x,\dot{x}^j)$, $j = 0, 1, \dots, m$.

For the opposite implication, suppose $G^{(m)}$ exists and is continuous on $\mathcal{U} \times \boldsymbol{X}$. Given any $(x,\dot{x}) \in \mathcal{U} \times \boldsymbol{X}$, for small enough $|t|$ we may use Lemma D.6 with $n = m$ and $t\dot{x}$ instead of $\dot{x}$ to infer that for each $0 \le j < n = j+1 \le m$,

$$\Big\| G^{(j)}(x+t\dot{x},\dot{x}) - \sum_{i=0}^{1} G^{(j+i)}(x,\dot{x}) t^j \Big\|_{\boldsymbol{Y}} = o(t) \text{ as } t \to 0,$$

which says that $\frac{d}{dt} G^{(j)}(x+t\dot{x},\dot{x})\big|_{t=0} = G^{(j+1)}(x,\dot{x})$. Hence $D^n G(x,\dot{x}^n)$ exists and equals to $G^{(n)}(x,\dot{x})$ for every $(x,\dot{x}) \in \mathcal{U} \times \boldsymbol{X}$ and $0 \le n \le m$. Since $G^{(n)}$ are continuous, $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$. □

We also show that in the presence of continuity it suffices to require the existence of the Peano derivatives in a rather weak sense.

Lemma D.8. *Suppose $G \colon \mathcal{U} \to \boldsymbol{Y}$ and $g_j : \mathcal{U} \times \boldsymbol{X} \to \boldsymbol{Y}$, $0 \le j \le m$, are continuous functions such that for a weak* dense set of $y^* \in \boldsymbol{Y}^*$, $y^* \circ G$ is m-times Peano differentiable on $\mathcal{U}$ with its jth Peano derivative being $y^* \circ g_j$. Then $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$ and $D^k G(x,\dot{x}^j) = G^{(j)}(x,\dot{x}) = g_j(x,\dot{x})$.*

Proof. For the y^* for which the assumption holds, Proposition D.7 shows that $y^* \circ G \in C^m_*(\mathcal{U}, \mathbb{R})$ and $D^j(y^* \circ G)(x,\dot{x}^j) = y^* \circ g_j(x,\dot{x})$. Hence, whenever the segment $[x, x+t\dot{x}]$ is contained in $\mathcal{U}$,

$$y^*\Big(G(x+t\dot{x}) - \sum_{j=0}^{m} \frac{g_j(x,\dot{x})}{j!} t^j \Big) = \frac{1}{m!} \int_0^t (t-s)^m y^*\big(g_m(x+s\dot{x},\dot{x}) - g_m(x,\dot{x}) \big)\, \mathrm{d}s.$$

The function $s \in [0,t] \to (t-s)^m (g_m(x+s\dot{x},\dot{x}) - g_m(x,\dot{x}))$ is continuous, hence its Riemann integral, say I, exists as an element of the completion of $\boldsymbol{Y}$. But since by the above $y^*(I) = y^*\big(G(x+t\dot{x}) - \sum_{j=0}^{m} \frac{g_j(x,\dot{x})}{j!} t^j \big)$ for a weak* dense set of $y^* \in \boldsymbol{Y}^*$,

$$G(x+t\dot{x}) - \sum_{j=0}^{m} \frac{g_j(x,\dot{x})}{j!} t^j = \frac{1}{m!} \int_0^t (t-s)^m \big(g_m(x+s\dot{x},\dot{x}) - g_m(x,\dot{x}) \big)\, \mathrm{d}s.$$

Since g_m is continuous, G is m times Peano differentiable at every $x \in \mathcal{U}$ as a mapping of $\mathcal{U}$ to $\boldsymbol{Y}$, with continuous $G^{(j)}(x,\dot{x}) = g_j(x,\dot{x})$. So the statement follows from Proposition D.7. □

The previous Lemma will be used in the situation when $G : \mathcal{U} \to \boldsymbol{Y}$ and $\boldsymbol{Y} \hookrightarrow \boldsymbol{V}$ (meaning $\boldsymbol{Y}$ is a linear subspace of $\boldsymbol{V}$ and $\|\cdot\|_{\boldsymbol{V}} \le \|\cdot\|_{\boldsymbol{Y}}$) to require differentiability for the map $G : \mathcal{U} \to \boldsymbol{V}$ only.

Corollary D.9. *Suppose $\boldsymbol{Y} \hookrightarrow \boldsymbol{V}$ and $G \colon \mathcal{U} \to \boldsymbol{Y}$ is m times Peano differentiable when considered as a map to $\boldsymbol{V}$ and such that each function $G^{(j)}(x,\dot{x})$, $0 \le j \le m$, has values in $\boldsymbol{Y}$ and is continuous as a map of $\mathcal{U} \times \boldsymbol{X}$ to $\boldsymbol{Y}$. Then $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$ and $D^j G(x,\dot{x}^j) = G^{(j)}(x,\dot{x})$.*

PROOF. Since $\boldsymbol{V}^*$ is weak* dense in $\boldsymbol{Y}^*$, Lemma D.8 is applicable with

$$g_j(x,\dot{x}) = G^{(j)}(x,\dot{x}).$$

□

Multilinearity and symmetry of derivatives.

THEOREM D.10. *$\boldsymbol{X}$, $\boldsymbol{Y}$ be normed linear spaces with $\mathcal{U} \subset \boldsymbol{X}$ open, and let $G \in C_*^m(\mathcal{U}, \boldsymbol{Y})$. Then, for every $1 \le j \le m$, the directional derivative $D^j G(x, \dot{x}_1, \dots, \dot{x}_j)$ exists for all $x \in \mathcal{U}$ and $\dot{x}_1, \dots, \dot{x}_j \in \boldsymbol{X}$.*

Moreover, it is a continuous, symmetric, j-linear map in the variables $\dot{x}_1, \dots, \dot{x}_j$ and $D^j G \in C_*^{m-j}(\mathcal{U} \times \boldsymbol{X}^j, \boldsymbol{Y})$.

The main idea is to get information on the map $s \mapsto G^{(j)}(x + sv, \dot{x}, \dots, \dot{x})$ by writing

$$G(x + s(v + t\dot{x})) = G(x + sv + st\dot{x})$$

and using Peano differentiability of G at x on the left hand side and Peano differentiability at $x + sv$ on the right hand side. A key tool is the following polynomial interpolation lemma. Theorem D.10 will then be a consequence of Proposition D.12 below.

LEMMA D.11. *For any $j = 0, \dots, m$, let $\Phi_j : (-s_0, s_0) \to \boldsymbol{X}$ be bounded and $\Psi_j : \mathbb{R} \to \boldsymbol{X}$. Suppose that*

$$\sum_{j=0}^{m} s^j (\Psi_j(t) - \Phi_j(s) t^j) = o(s^m) \text{ as } s \to 0 \tag{D.8}$$

for every $t \in \mathbb{R}$. Then for each $j = 0, \dots, m$:

(a) *The function Ψ_j is a polynomial of degree at most j and*

(b) *there exists a polynomial $p_j : \mathbb{R} \to \boldsymbol{X}$ of degree at most $m - j$ such that $\Phi_j(s) = p_j(s) + o(s^{m-j})$ as $s \to 0$.*

(c) *Moreover, if $\widehat{\Phi}_j, \widehat{\Psi}_j$ also satisfy* (D.8) *then*[2]

$$\|\widehat{\Phi}_j - \Phi_j\|_{\text{poly}} \le C \limsup_{s \to 0} \sup_{t \in (0,1)} \Big\| \sum_{j=0}^{m} s^j (\widehat{\Psi}_j - \Psi_j(t)) \Big\|.$$

PROOF. Fix different $t_0, \dots, t_m \in (0,1)$ and let q_j be the corresponding Lagrange basis polynomials, $q_j(t_k) = \delta_{k,j}$. Then for every $t \in \mathbb{R}$,

$$\begin{aligned} \sum_{j=0}^{m} s^j \Big(\Psi_j(t) - \sum_{k=0}^{m} \Psi_j(t_k) q_k(t) \Big) &= \\ = \sum_{j=0}^{m} s^j (\Psi_j(t) - \Phi_j(s) t^j) - \sum_{k=0}^{m} q_k(t) \sum_{j=0}^{m} s^j (\Psi_j(t_k) - \Phi_j(s) t_k^j) &= o(s^m), \end{aligned} \tag{D.9}$$

implying that $\Psi_j(t) - \sum_{k=0}^{m} \Psi_j(t_k) q_k(t) = 0$ for each $j = 0, 1, \dots, m$ and thus each $\Psi_j(t)$ is a polynomial of degree at most m. Only now we use that Φ_j are bounded, yielding from (D.8) that $\sum_{k=0}^{j} s^k (\Psi_k(t) - \Phi_k(s) t^k) = o(s^j)$ for every $j = 0, \dots, m$, and the above argument with j instead of m shows that Ψ_j has degree at most j.

[2]For $p(s) = \sum_{\ell=0}^{n} p_\ell s^\ell$ we define $\|p\|_{\text{poly}} = \max_{\ell=0,\dots,n} |p_\ell|$.

For (b), let $0 \le \ell \le m$ and find a_k so that $\sum_{i=0}^m a_k t_k^j = \delta_{j,\ell}$. By the degree estimate on Ψ_j, $\sum_{k=0}^m a_k \Psi_j(t_k) = 0$ for $j < \ell$. Hence
(D.10)

$$\Phi_\ell(s) - \sum_{j=0}^{m-\ell} s^j \sum_{k=0}^m a_k \Psi_{j+\ell}(t_k) = -s^{-\ell} \sum_{k=0}^m a_k \sum_{j=0}^m s^j (\Phi_j(s) t_k^j - \Psi_j(t_k)) = o(s^{m-\ell}).$$

For (c), we just notice that, in view of (D.10), the coefficients of $p_k(s)$ are linear combinations (with fixed coefficients) of the values $\Psi_{j+k}(t_k)$ with $t_k \in (0,1)$.

□

Proposition D.12. *Let $G \in C_*^m(\mathcal{U}, \boldsymbol{Y})$. Then for every $1 \le j \le m$, the directional derivative $D^j G(x, \dot{x}_1, \dots, \dot{x}_j)$ exists for all $x \in \mathcal{U}$ and $\dot{x}_1, \dots, \dot{x}_j \in \boldsymbol{X}$, it is symmetric and j-linear in the variables $\dot{x}_1, \dots, \dot{x}_j$, and $D^j G \in C_*^{m-j}(\mathcal{U} \times \boldsymbol{X}^j, \boldsymbol{Y})$.*

Proof. We show that $f(x,\dot{x}) := G^{(1)}(x,\dot{x})$ belongs to $C_*^{m-1}(\mathcal{U} \times \boldsymbol{X}, \boldsymbol{Y})$ and is linear in $\dot{x}$. Used recursively, this shows that for each $1 \le j \le m$, $(x, \dot{x}_1, \dots, \dot{x}_j) \to D^j G(x, \dot{x}_1, \dots, \dot{x}_j)$ is j-linear in $\dot{x}_1, \dots, \dot{x}_j$ and belongs to $C_*^{m-j}(\mathcal{U} \times \boldsymbol{X}^j, \boldsymbol{Y})$. Recall that by Proposition D.7, G is m-times Peano differentiable and $G^{(j)}(x,\dot{x}) = D^j G(x, \dot{x}^j)$ for $j \le m$, $x \in \mathcal{U}$, and $\dot{x} \in \boldsymbol{X}$.

Fix $x, \dot{x}, v \in \boldsymbol{X}$ and denote $\Phi_j(s) = G^{(j)}(x+sv, \dot{x})/j!$ and $\Psi_j(t) = G^{(j)}(x, v + t\dot{x})/j!$. By definition, for each $t \in \mathbb{R}$, $G(x + s(v+t\dot{x})) = \sum_{j=0}^m \Psi_j(t) s^j + o(s^m)$. Also, by Lemma D.6,

$$\text{(D.11)} \quad \|G((x+sv)+st\dot{x}) - \sum_{j=0}^m \Phi_j(s)(st)^j\| \le$$
$$\le (st)^m \sup_{0 \le \tau \le 1} \|G^{(m)}(x+sv+\tau st\dot{x}, u) - G^{(m)}(x+sv)\| = o(s^m).$$

Hence $\sum_{j=0}^m s^j(\Psi_j(t) - \Phi_j(s)t^j) = o(s^m)$ and we see from Lemma D.11(a) that $G^{(1)}(x, v+t\dot{x}) = a + bt$ for some a, b. For $t = 0$ we get $a = G^{(1)}(x,v)$ and by continuity, $b = \lim_{t\to\infty} G^{(1)}(x, v/t + \dot{x}) = G^{(1)}(x,\dot{x})$. Hence $G^{(1)}(x, v+\dot{x}) = G^{(1)}(x,v) + G^{(1)}(x,\dot{x})$, and we infer that $f(x,\dot{x}) = G^{(1)}(x,\dot{x})$ is linear in the second variable.

By Lemma D.11(b), for each fixed $x, \dot{x}$ the function $g_{\dot{x}}(x) = f(x,\dot{x})$ has the Peano derivative $g_{\dot{x}}^{(j)}(x,v)$, $j = 1, \dots, m-1$. Moreover, continuity of Peano derivatives $G^{(n)}$ and Lemma D.11(c) imply that $(x,\dot{x},v) \to g_{\dot{x}}^{(j)}(x,v)$ is continuous on $\mathcal{U} \times \boldsymbol{X}^2$. Since $f(x,\dot{x})$ is linear in $\dot{x}$,

$$\text{(D.12)} \qquad f((x,\dot{x}) + t(u,\dot{u})) - f((x,\dot{x})) = g_{\dot{x}}(x+tu) - g_{\dot{x}}(x) + t g_{\dot{u}}(x+tu),$$

showing that f is $m-1$ times continuously Peano differentiable. Hence f belongs to $C_*^{m-1}(\mathcal{U} \times \boldsymbol{X}, \boldsymbol{Y})$ by Proposition D.7.

Symmetry of the directional derivatives follows from the following lemma. □

Lemma D.13. *Let $G\colon \mathcal{U} \to \boldsymbol{Y}$ and fix (not necessarily distinct) $\dot{x}_1, \dots, \dot{x}_k \in \boldsymbol{X}$. Suppose that the directional derivative $x \in \mathcal{U} \to D^j G(x, \dot{x}_1^{j_1}, \dots, \dot{x}_k^{j_k})$ exists and is continuous whenever $j := j_1 + \dots + j_k \le m$. Then for any $t_1, \dots, t_k \in \mathbb{R}$,*

$$\text{(D.13)} \qquad G^{(j)}(x, \sum_{s=1}^k t_s \dot{x}_s) = j! \sum_{j_1+\dots+j_k=j} D^j G(x, \dot{x}_1^{j_1}, \dots, \dot{x}_k^{j_k}) \frac{t_1^{j_1} \dots t_k^{j_k}}{j_1! \cdots j_k!}.$$

In particular, $D^kG(x,(\sum_{s=1}^k t_s\dot x_s)^k) = G^{(k)}(x,\sum_{s=1}^k t_s\dot x_s)$ *exists and*

$$D^kG(x,\dot x_1,\dots,\dot x_k) = D^kG(x,\dot x_{\pi(1)},\dots,\dot x_{\pi(k)}) \tag{D.14}$$

for every permutation π *of* $\{1,\dots,k\}$.

Proof. Expanding recursively and estimating errors by Lemma D.6, we get

$$G(x+t\sum t_s\dot x_s) = \sum_{j:=j_1+\cdots+j_k\le m} D^jG(x,\dot x_1^{j_1},\dots,\dot x_k^{j_k})\frac{t_1^{j_1}\dots t_k^{j_k}}{j_1!\cdots j_k!}t^j + o(t^m), \tag{D.15}$$

which shows (D.13). Since the right hand side of (D.13) is continuous in x, Proposition D.7 used separately on each line in the direction $\sum_{s=1}^k t_s\dot x_s$ implies that the iterated derivative $D^kG(x,(\sum_{s=1}^k t_s\dot x_s)^k)$ exists and equals $G^{(k)}(x,\sum_{s=1}^k t_s\dot x_s)$.

Using the equality (D.13) with $\sum_{s=1}^k t_s\dot x_s$ replaced by $\sum_{s=1}^k t_{\pi(s)}\dot x_{\pi(s)}$ gives the same left hand side. Since the right side is a polynomial, the coefficients in front of $t_1\cdots t_k$ are equal, giving the last statement. □

Remark D.14. Notice that the order of directions in the recursive expansion can be chosen. As a result, the assumption can be narrowed, say in the case of two directions $\{\dot x_1,\dot x_2\}$, to the assumption that the directional derivative $x\in\mathcal U\to D^jG(x,\dot x_1^{j_1},\dot x_2^{j_2},\dot x_1^{j_3})$ exists and is continuous whenever $j := j_1+j_2+j_3\le m$ and $j_3\in\{0,1\}$. ⋄

The following Corollary is a useful criterion for proving that a given function on a product space belongs to C_*^m. It involves partial derivatives which are defined and denoted in the standard way. In particular, $D_1^jD_2^\ell G((x,p),\dot p^\ell,\dot x^j) = D^{j+\ell}G((x,p),(0,\dot p)^\ell,(\dot x,0)^j)$.

Corollary D.15. *Suppose* $G:\mathcal O\subset \boldsymbol X\times\boldsymbol P\to\boldsymbol Y$, $m\in\mathbb N$, *and for each* $j+\ell\le m$, *the derivative* $(x,p,\dot x,\dot p)\to D_1^jD_2^\ell G((x,p),\dot p^\ell,\dot x^j)$ *exists and is continuous on* $\mathcal O\times\boldsymbol X\times\boldsymbol P$. *Then* $G\in C_*^m(\mathcal O,\boldsymbol Y)$.

Proof. Lemma D.13 shows that for each $j\le m$ the Peano derivative

$$\begin{aligned} G^{(j)}((x,p),(\dot x,\dot p)) &= D^jG((x,p),((\dot x,0)+(0,\dot p))^j) = \\ &= \sum_{k=0}^j\binom{j}{k}D^jG((x,p),(0,\dot p)^k,(\dot x,0)^{j-k}) = \\ &= \sum_{k=0}^j\binom{j}{k}D_1^{j-k}D_2^kG((x,p),\dot p^k,\dot x^{j-k}) \end{aligned}$$

exists and is continuous. Hence $G\in C_*^m(\mathcal O,Y)$ by Proposition D.7. □

Remark D.16. Notice that in view of Remark D.14, there is also a flexibility in the demanded order of partial derivatives in the condition in the Corollary. ⋄

Relation to usual derivatives.

Proposition D.17. *Using* $C^m(\mathcal U,\boldsymbol Y)$ *to denote the usual spaces of Fréchet differentiable functions (with operator norms on multilinear forms from* $L_m(\boldsymbol X,\boldsymbol Y)$*) and* $m\ge 0$, *we have*

$$C^m(\mathcal U,\boldsymbol Y) = \{G\in C_*^m(\mathcal U,\boldsymbol Y) : D^mG\in C(\mathcal U,L_m(\boldsymbol X,\boldsymbol Y))\}\supset C_*^{m+1}(\mathcal U,\boldsymbol Y).$$

If $\boldsymbol X$ *is finite dimensional then* $C^m(\mathcal U,\boldsymbol Y) = C_*^m(\mathcal U,\boldsymbol Y)$.

PROOF. We first show the inclusion

$$\{G \in C^m_*(\mathcal{U}, \boldsymbol{Y}) : D^m G \in C(\mathcal{U}, L_m(\boldsymbol{X}, \boldsymbol{Y})\} \supset C^{m+1}_*. \tag{D.16}$$

Let $G \in C^{m+1}_*(\mathcal{U}, \boldsymbol{Y})$. Given $x \in \mathcal{U}$ find $\delta > 0$ with

$$\|D^{m+1}G(x + \dot{x}, \dot{x}_1, \dots, \dot{x}_{m+1})\| \le 1 \quad \text{whenever} \quad \max\{\|\dot{x}\|, \|\dot{x}_i\|\} \le \delta. \tag{D.17}$$

Hence for $\|\dot{x}\| < \varepsilon\delta^{m+1}$ and $\max_i \|\dot{x}_i\| \le 1$,

$$\begin{aligned} &\|D^m G(x + \dot{x}, \dot{x}_1, \dots, \dot{x}_m) - D^m G(x, \dot{x}_1, \dots, \dot{x}_m)\| = \\ &= \delta^{-m}\|D^m G(x + \dot{x}, \delta\dot{x}_1, \dots, \delta\dot{x}_m) - D^m G(x, \delta\dot{x}_1, \dots, \delta\dot{x}_m)\| \le \\ &\le \delta^{-m-1} \sup_{0<t<1} \|D^{m+1}G(x + t\dot{x}, \delta\dot{x}_1, \dots, \delta\dot{x}_m, \delta\dot{x}/\|\dot{x}\|)\| \, \|\dot{x}\| < \varepsilon, \end{aligned} \tag{D.18}$$

yielding the inclusion.

Now we show by induction that

$$C^m(\mathcal{U}, \boldsymbol{Y}) \supset \{G \in C^m_*(\mathcal{U}, \boldsymbol{Y}) : D^m G \in C(\mathcal{U}, L_m(\boldsymbol{X}, \boldsymbol{Y})\} \tag{D.19}$$

since the other inclusion is obvious. For $m = 1$ the inclusion follows from the linearity of the derivative $DG(x, \cdot)$, Proposition D.7 and Lemma D.6 applied with $n = 1$ and $j = 0$. Now assume that (D.19) holds for $m - 1$ and let $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$ with $D^m G \in C(\mathcal{U}, L_m(\boldsymbol{X}, \boldsymbol{Y})$. By (D.16) applied with $m-1$ instead of m we have $D^{m-1}G \in C(\mathcal{U}, L_{m-1}(\boldsymbol{X}, \boldsymbol{Y}))$ and thus by induction assumption $G \in C^{m-1}(\mathcal{U}, \boldsymbol{Y})$.

Define the maps $F : \mathcal{U} \to L_{m-1}(\boldsymbol{X}, \boldsymbol{Y})$ and $K : \mathcal{U} \to L(\boldsymbol{X}, L_{m-1}(\boldsymbol{X}, \boldsymbol{Y}))$ by

$$F(x)(\dot{x}_1, \dots, \dot{x}_{m-1}) := D^{m-1}G(x, \dot{x}_1, \dots \dot{x}_{m-1}), \tag{D.20}$$

$$K(x)(\dot{x}_m)(\dot{x}_1, \dots, \dot{x}_{m-1}) := D^m G(x, \dot{x}_1, \dots \dot{x}_m). \tag{D.21}$$

Our aim is to show that F is Fréchet differentiable at $x \in \mathcal{U}$ and its Fréchet derivative agrees with K. Then $F \in C^1(\mathcal{U}, L_{m-1}(\boldsymbol{X}, \boldsymbol{Y}))$ and thus $G \in C^m(\mathcal{U}, \boldsymbol{Y})$.

For a fixed $\dot{x}_1, \dots, \dot{x}_{m-1} \in \boldsymbol{X}$, let $\Phi(t) := F(x+t\dot{x}_m)(\dot{x}_1, \dots, \dot{x}_{m-1})$ and assume that $[x, x + \dot{x}_m] \subset \mathcal{U}$. Since $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$, the function Φ is in $C^1((-\varepsilon, 1 + \varepsilon), \boldsymbol{Y})$ and by Lemma D.6,

$$\begin{aligned} \|\Phi(1) - \Phi(0) - \Phi'(0)\|_{\boldsymbol{Y}} &\le \sup_{\tau \in (0,1)} \|\Phi'(\tau) - \Phi'(0)\|_{\boldsymbol{Y}} \le \\ &\le \sup_{\tau \in (0,1)} \|D^m G(x + \tau\dot{x}_m) - D^m G(x)\|_{L_m(\boldsymbol{X}, \boldsymbol{Y})} \|\dot{x}_1\| \dots \|\dot{x}_m\|. \end{aligned} \tag{D.22}$$

Now $\Phi'(0) = K(x)(\dot{x}_m)(\dot{x}_1, \dots, \dot{x}_{m-1})$ and taking the supremum over all

$$\dot{x}_1, \dots, \dot{x}_{m-1}$$

with $\|\dot{x}_i\| \le 1$ we get

$$\begin{aligned} &\|F(x + \dot{x}_m) - F(x) - K(x)(\dot{x}_m)\|_{L_{m-1}(\boldsymbol{X}, \boldsymbol{Y})} \le \\ &\le \sup_{\tau \in (0,1)} \|D^m G(x + \tau\dot{x}_m) - D^m G(x)\|_{L_m(\boldsymbol{X}, \boldsymbol{Y})} \|\dot{x}_m\|. \end{aligned} \tag{D.23}$$

It follows from the continuity of $D^m G$ (as a map with values in $L_m(\boldsymbol{X}, \boldsymbol{Y})$) that F is Fréchet differentiable with derivative K.

Finally assume that $\boldsymbol{X}$ is finite dimensional and let $G \in C^m_*(\mathcal{U}, \boldsymbol{Y})$. By multilinearity of $D^m G(x, \cdot)$ and polarization we see that

$$\|D^m G(x) - D^m G(x')\|_{L_m(\boldsymbol{X}, \boldsymbol{Y})} \le C(m) \sup_{v \in X : \|v\|=1} \|D^m G(x, v^m) - D^m G(x', v^m)\|_{\boldsymbol{Y}}.$$

Since $(x, v) \to D^m G(x, v^m)$ is continuous and $\{v \in \boldsymbol{X} : \|v\| = 1\}$ is compact it follows that $D^m G \in C(\mathcal{U}, L_m(\boldsymbol{X}, \boldsymbol{Y}))$. This finishes the proof of the proposition. □

D.3. Chain rule with a loss of regularity

Here we consider the chain rule showing that $F \circ G \in C^m_*(\mathcal{U}, \boldsymbol{Z})$ in the situation when $G : \mathcal{U} \to \mathcal{Y}$, $F : \mathcal{Y} \to \boldsymbol{Z}$, where $\mathcal{U}$ and $\mathcal{Y}$ are open subsets of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively, and $G \in C^m_*(\mathcal{U}, \boldsymbol{V})$ for some $\boldsymbol{Y} \hookrightarrow \boldsymbol{V}$ (meaning, as above, that $\boldsymbol{Y}$ is a linear subspace of $\boldsymbol{V}$ and $\|\cdot\|_{\boldsymbol{V}} \leq \|\cdot\|_{\boldsymbol{Y}}$). This generalizes the chain rule of [**Ham82**, Theorem 3.6.4] where $\boldsymbol{V} = \boldsymbol{Y}$ and F is assumed to belong to $C^m_*(\mathcal{Y}, \boldsymbol{Z})$. In our situation, although $F \circ G$ obviously makes sense, expressions such as $DF(G(x), DG(x, \dot{x}))$ may not, since derivatives of G belong to $\boldsymbol{V}$ and so not to the domain of the derivative of F. So for the chain rule to hold, a natural assumptions are that $\boldsymbol{Y}$ is dense in $\boldsymbol{V}$ and $D^j F$ has a continuous extension from $\mathcal{Y} \times \boldsymbol{Y}^j$ to $\mathcal{Y} \times \boldsymbol{V}^j$. (The density of $\boldsymbol{Y}$ in $\boldsymbol{V}$ is not really needed, but is convenient since it guarantees that the extension is unique and j-multilinear in the last variables.)

Definition D.18. We use $C^m_{\boldsymbol{V}}(\mathcal{Y}, \boldsymbol{Z})$ to denote the space of maps $F : \mathcal{Y} \subset \boldsymbol{Y} \to \boldsymbol{Z}$ such that for any $j \leq m$, the derivative $D^j F$ exists and can be extended to a continuous map $D^j_{\boldsymbol{V}} F$ of $\mathcal{Y} \times \boldsymbol{V}^j$ to $\boldsymbol{Z}$ (with a slight abuse of notation we usually skip the subscript $\boldsymbol{V}$ from $D^j_{\boldsymbol{V}}$).

Remark D.19.

(a) For $j = 0$ this requires only that $F \colon \mathcal{Y} \to \boldsymbol{Z}$ be continuous.
(b) Proposition D.7 and the polarization formula show that it suffices to extend the maps $(y, \dot{y}) \in \mathcal{Y} \times \boldsymbol{Y} \to D^j F(y, \dot{y}^j)$ to continuous maps defined on $\mathcal{Y} \times \boldsymbol{V}$.
(c) By Proposition D.7, $C^m_{\boldsymbol{V}}(\mathcal{Y}, \boldsymbol{Z}) \subset C^m_*(\mathcal{Y}, \boldsymbol{Z})$ with equality when $\boldsymbol{V} = \boldsymbol{Y}$. ⋄

Lemma D.20. *Let* $F \in C^m_{\boldsymbol{V}}(\mathcal{Y}, \boldsymbol{Z})$ *and* $j \leq m$. *Then* $D^j_V F \in C^{m-j}_{\boldsymbol{V}^{j+1}}(\mathcal{Y} \times \boldsymbol{V}^j, \boldsymbol{Z})$.

Proof. By the polarization formula it suffices to show that $(y, v) \to \Phi(y, v) := D^j_{\boldsymbol{V}} F(y, v^j)$ belongs to $C^{m-j}_{\boldsymbol{V}^2}(\mathcal{Y} \times \boldsymbol{V}, \boldsymbol{Z})$. Considering first Φ as a map of $\mathcal{Y} \times \boldsymbol{Y}$ to $\boldsymbol{Z}$ and using multilinearity of the derivative, we have

$$D^k_1 D^\ell_2 \Phi((y, v), \dot{v}^\ell, \dot{y}^k) = j \cdots (j - \ell + 1) D^{j+k} F(y, v^{j-\ell}, \dot{v}^\ell, \dot{y}^k) \tag{D.24}$$

for $\ell \leq j$ and $k \leq m - j$. Since these derivatives are zero for $\ell > j$, we have $\Phi \in C^{m-j}_*(\mathcal{Y} \times \boldsymbol{Y}, \boldsymbol{Z})$ by Corollary D.15 and Theorem D.10. Moreover, expressing $D^s \Phi$, $0 \leq s \leq m - j$, with the help of partial derivatives, we see that these derivatives have continuous extensions to maps $(\mathcal{Y} \times \boldsymbol{V}) \times (\boldsymbol{V} \times \boldsymbol{V})^s \to \boldsymbol{Z}$ implying the statement. □

Theorem D.21. *Suppose* $\mathcal{U} \subset \boldsymbol{X}$ *and* $\mathcal{Y} \subset \boldsymbol{Y}$ *are open,* $\boldsymbol{Y} \hookrightarrow \boldsymbol{V}$, $G : \mathcal{U} \to \boldsymbol{Y}$, $G(\mathcal{U}) \subset \mathcal{Y}$, $G \in C^m_*(\mathcal{U}, \boldsymbol{V})$, *and* $F : \mathcal{Y} \to \boldsymbol{Z}$, $F \in C^m_{\boldsymbol{V}}(\mathcal{Y}, \boldsymbol{Z})$. *Then* $F \circ G \in C^m_*(\mathcal{U}, \boldsymbol{Z})$ *and* $D^j(F \circ G)(x, \dot{x}^j)$ *is a linear combination of terms*

$$D^k_V F(G(x), D^{j_1} G(x, \dot{x}^{j_1}), \ldots, D^{j_k} G(x, \dot{x}^{j_k})) \tag{D.25}$$

where $j_s \geq 1$ *and* $\sum_{s=1}^k j_s = j$.

Proof. We will show existence and continuity of Peano derivatives of $F \circ G$. Let $x \in \mathcal{U}$, $\dot{x} \in \boldsymbol{X}$. For any t, working just on the segment

$$I_t := [G(x), G(x + t\dot{x})] \subset \boldsymbol{Y}$$

we have an estimate

$$\text{(D.26)}\qquad \Big\|F(G(x+t\dot x)) - \sum_{s=0}^{j} \frac{D^s F(G(x),(G(x+t\dot x)-G(x))^s)}{s!}\Big\| \le \sup_{y\in I_t}\Big\|\frac{D^j F(y,(G(x+t\dot x)-G(x))^j) - D^j F(x,(G(x+t\dot x)-G(x))^j)}{j!}\Big\|$$

for any $j \le m$. Here all derivatives of F are applied to elements of $\boldsymbol{Y}$, so the extension has not been used yet. Since $(G(x+t\dot x)-G(x))/t$ converge, in the norm $\|\cdot\|_{\boldsymbol{V}}$, to $G'(x,\dot x)$, $G'(x,\dot x) \in \boldsymbol{V}$ and, using continuity of the extended $D^j F$,

$$D^j F(y_t, ((G(x+t\dot x)-G(x))/t)^j) \to D^j F(x, G'(x,\dot x)^j) \text{ as } t \to 0$$

whenever $y_t \in I_t$. Hence the right side of (D.26) is $o(t^j)$. Since $x, \dot x$ are fixed, expanding $D^s F(G(x),(G(x+t\dot x)-G(x))^s)$ is standard: $D^s F(y,\dot y_1,\dots,\dot y_s)$ has been extended to a continuous s-linear form on $\boldsymbol{V}^s$, into which one plugs a C^j function $\mathbb{R} \to \boldsymbol{Y} \subset \boldsymbol{V}$, namely $t \to G(x+t\dot x)-G(x)$.

It follows that $F\circ G$ is m-times Peano differentiable with derivatives given by the terms from the expansion of $D^s F(G(x),(G(x+t\dot x)-G(x))^s)$, giving (D.25). These formulas show that $(F\circ G)^{(s)}$ is continuous as a map $\mathcal{U}\times\boldsymbol{X}\to\boldsymbol{Z}$. Consequently, $F\circ G \in C^m_*(\mathcal{U},\boldsymbol{Z})$ by Proposition D.7. □

D.4. Chain rule with parameter and a graded loss of regularity

In the chain rule of this section, the main point is that the inner and/or outer function depend on an additional parameter, the regularity of partial derivatives depends on the order of the derivative with respect to the parameter, and the resulting composition has the same regularity properties as the functions we are composing. In principle, this chain rule is very different from the one in Theorem D.29, although we will reduce its proof to is.

Proposition D.22. *Suppose $\boldsymbol{P},\boldsymbol{Q},\boldsymbol{Y},\boldsymbol{V}$ are normed linear spaces, $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{Y}$ are open subsets of $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{Y}$, respectively, $\boldsymbol{Y} = \boldsymbol{Y}_m \hookrightarrow \boldsymbol{Y}_{m-1} \hookrightarrow \dots \hookrightarrow \boldsymbol{Y}_0$, $\Phi\colon \mathcal{P}\to\boldsymbol{Y}$ and $F:\mathcal{Y}\times\mathcal{Q}\to\boldsymbol{V}$ are such that $\Phi(\mathcal{P})\subset\mathcal{Y}$ and for each $0\le\ell\le m$,*

(i) *$\Phi \in C^{m-\ell}_*(\mathcal{P},\boldsymbol{Y}_\ell)$;*
(ii) *for each $j\le m-\ell$, $D_1^j D_2^\ell F$ exists on $\mathcal{Y}\times\mathcal{Q}\times\boldsymbol{Q}^\ell\times\boldsymbol{Y}^j$ and has a continuous extension to $\mathcal{Y}\times\mathcal{Q}\times\boldsymbol{Q}^\ell\times\boldsymbol{Y}_\ell^j$.*

Then the map $\Psi(p,q) := F(\Phi(p),q)$ belongs to $C^m_*(\mathcal{P}\times\mathcal{Q},\boldsymbol{V})$ and for each $j+\ell\le m$ the derivative $D_1^j D_2^\ell\Psi((p,q),\dot q^\ell,\dot p^j)$ is a combination of terms

$$\text{(D.27)}\qquad D_1^k D_2^\ell F((\Phi(p),q),\dot q^\ell, D^{j_1}\Phi(p,\dot p^{j_1}),\dots,D^{j_k}\Phi(p,\dot p^{j_k}))$$

where $j_s\ge 1$, $\sum_{s=1}^k j_s = j$ and $D_1^i D_2^\ell F$ denotes the extension from (ii).

Proof. Clearly, $D_2^\ell\Psi((p,q),\dot q^\ell) = D_2^\ell F((\Phi(p),q),\dot q^\ell)$ exists for each $0\le\ell\le m$, and with fixed q and $\dot q$ it is a composition $f_{q,\dot q}\circ\Phi$, where $f_{q,\dot q}(y) = D_2^\ell F((y,q),\dot q^\ell)$. By (i), $\Phi\in C^{m-\ell}_*(\mathcal{P},\boldsymbol{Y}_\ell)$, and by (ii), $f_{q,\dot q}\in C^{m-\ell}_{\boldsymbol{Y}_\ell}(\mathcal{Y},\boldsymbol{V})$. Hence by Theorem D.21, the function $p\to D_2^\ell\Psi((p,q),\dot q^\ell)$ belongs to $C^{m-\ell}_*(\mathcal{P},\boldsymbol{V})$ and its jth derivative is a combination of the terms specified in (D.27).

It remains to observe that $(p,q)\to\big((\Phi(p),q),\dot q, D^{j_1}\Phi(p,\dot p^{j_1}),\dots,D^{j_k}\Phi(p,\dot p^{j_k})\big)$ maps, by the condition $j_s\le j\le m-\ell$ and (i), $\mathcal{P}\times\mathcal{Q}$ continuously to $(\mathcal{Y}\times\mathcal{Q})\times\boldsymbol{Q}\times\boldsymbol{Y}_\ell^k$ and this space is mapped by $\big((y,q),\dot q,\dot y_1,\dots,\dot y_k\big)\to D_1^i D_2^\ell F\big((y,q),\dot q^\ell,\dot y_1,\dots,\dot y_k\big)$

continuously to $\boldsymbol{V}$ by (ii). Hence each of the functions in (D.27) maps $\mathcal{P} \times \mathcal{Q}$ continuously to $\boldsymbol{V}$, implying that $\Psi \in C^m_*(\mathcal{P} \times \mathcal{Q}, \boldsymbol{V})$. □

Corollary D.23. *If, under the assumptions of Proposition D.22 we are also given a function* $\Upsilon \in C^m_*(\mathcal{P}, \boldsymbol{Q})$ *with* $\Upsilon(\mathcal{P}) \subset \mathcal{Q}$*, the map* $\Theta(p) := F(\Phi(p), \Upsilon(p))$ *belongs to* $C^m_*(\mathcal{P}, \boldsymbol{V})$ *and for each* $n \leq m$*, the derivative* $D^n\Theta(p, \dot{p}^n)$ *is a combination of terms*

$$D^i_1 D^k_2 F\big((\Phi(p), \Upsilon(p)), D^{j_1}\Upsilon(p, \dot{p}^{j_1}), \dots, D^{j_i}\Upsilon(p, \dot{p}^{j_i}), D^{\ell_1}\Phi(p, \dot{p}^{\ell_1}), \dots, D^{\ell_k}\Phi(p, \dot{p}^{\ell_k})\big)$$

where $j_s, \ell_s \geq 1$ *and* $\sum_{s=1}^{i} j_s + \sum_{s=1}^{k} \ell_s = n$.

Proof. Observe that $\Theta = \Psi \circ \kappa$ where Ψ comes from Proposition D.22 and $\kappa : \mathcal{P} \to \boldsymbol{P} \times \boldsymbol{Q}$ is $\kappa(p) = (p, \Upsilon(p))$. Since $\kappa \in C^m_*(\mathcal{P}, \boldsymbol{P} \times \boldsymbol{Q})$, $\kappa(\mathcal{P}) \subset \mathcal{P} \times \mathcal{Q}$ and $\Psi \in C^m_*(\mathcal{P} \times \mathcal{Q}, V)$, the statement follows from Theorem D.21. □

The following main chain rule is a 'symmetric' version of the above, which is capable of being iterated. It will be stated in the following situation. Let $\boldsymbol{P}$, $\boldsymbol{X} = \boldsymbol{X}_m \hookrightarrow \dots \hookrightarrow \boldsymbol{X}_0$, $\boldsymbol{Y} = \boldsymbol{Y}_m \hookrightarrow \dots \hookrightarrow \boldsymbol{Y}_0$ and $\boldsymbol{Z} = \boldsymbol{Z}_m \hookrightarrow \dots \hookrightarrow \boldsymbol{Z}_0$ be normed linear spaces, $\mathcal{U} \subset \boldsymbol{X}$, $\mathcal{V} \subset \boldsymbol{P}$, and $\mathcal{Y} \subset \boldsymbol{Y}$ are open. We will use $\widetilde{\boldsymbol{X}}_n$ to denote the closure of $\boldsymbol{X}$ in $\boldsymbol{X}_n$, and similarly for $\widetilde{\boldsymbol{Y}}_n$ and $\widetilde{\boldsymbol{Z}}_n$. Also, we use $\mathbb{X}$ (and similarly $\mathbb{Y}$ and $\mathbb{Z}$) for the sequence $(\boldsymbol{X}_m, \dots, \boldsymbol{X}_0)$.

The class of functions we will consider may be informally described as those $G\colon \mathcal{U} \times \mathcal{V} \to \boldsymbol{Y}$ for which $D^j_1 D^\ell_2 G$ is a continuous map $\mathcal{U} \times \mathcal{V} \times \boldsymbol{P}^\ell \times \widetilde{\boldsymbol{X}}^j_n \to \boldsymbol{Y}_{n+\ell}$, i.e., ℓ derivatives in the parameter $p \in \mathcal{V}$ lead to a loss of regularity of order ℓ in the scale of Banach spaces. Since this description has several interpretations, we give a rather detailed one as a formal definition.

Definition D.24. For any $0 \leq k \leq m$, we define $\widetilde{C}^k(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ as the set of all maps $G : \mathcal{U} \times \mathcal{V} \to \boldsymbol{Y}$ such that

(a) $G \in C^k_*(\mathcal{U} \times \mathcal{V}, \boldsymbol{Y}_0)$.
(b) For each $j + \ell \leq k$, the function

$$(x, p, \dot{x}_1, \dots, \dot{x}_j, \dot{p}_1, \dots, \dot{p}_\ell) \to D^j_1 D^\ell_2 G((x, p), \dot{p}_1, \dots, \dot{p}_\ell, \dot{x}_1, \dots, \dot{x}_j),$$

which is by (a) defined as a map $\mathcal{U} \times \mathcal{V} \times \boldsymbol{X}^j \times \boldsymbol{P}^\ell \to \boldsymbol{Y}_0$ has a (necessarily unique) extension to a continuous mapping $\mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}^j_\ell \times \boldsymbol{P}^\ell \to \boldsymbol{Y}_0$. This extension is also denoted $D^j_1 D^\ell_2 G$.
(c) For each $0 \leq j \leq k - \ell$ and each $0 \leq n \leq m - \ell$ the restriction of $D^j_1 D^\ell_2 G$ (which has been already extended by (b)) to $\mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}^j_{n+\ell} \times \boldsymbol{P}^\ell$ has values in $\boldsymbol{Y}_n$ and is continuous as a mapping between these spaces.

Notice that, clearly, $\widetilde{C}^i(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})) \subset \widetilde{C}^k(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ for $k \leq i$. For proving that $G \in \widetilde{C}^k(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ the following simplification of this definition is rather useful.

Lemma D.25. *Assume that* $0 \leq k \leq m$*. Then* $G\colon \mathcal{U} \times \mathcal{V} \to \boldsymbol{Y}$ *belongs to* $\widetilde{C}^k(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ *iff*

(i) *as a map of* $\mathcal{U} \times \mathcal{V}$ *to* $\boldsymbol{Y}_0$*,* G *has derivatives* $D^j_1 D^\ell_2 G((x, p), \dot{p}^\ell, \dot{x}^j)$ *for all* $j + \ell \leq k$*,* $(x, p) \in \mathcal{U} \times \mathcal{V}$*,* $\dot{p} \in \boldsymbol{P}$ *and* $\dot{x} \in \boldsymbol{X}$*;*

(ii) *for* $0 \le j \le k-\ell$ *and all* $0 \le n \le m-\ell$ *there is continuous map* $\Psi_{j,\ell,n} \colon \mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}_{n+\ell} \times \boldsymbol{P} \to \boldsymbol{Y}_n$ *such that* $D_1^j D_2^\ell G((x,p),\dot p^\ell,\dot x^j) = \Psi_{j,\ell,n}(x,p,\dot x,\dot p)$ *for every* $(x,p) \in \mathcal{U} \times \mathcal{V}$, $\dot p \in \boldsymbol{P}$, *and* $\dot x \in \boldsymbol{X}$.

Proof. If $G \in \widetilde{C}^k(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$, (i) and (ii) are obvious. For the opposite implication, assuming (i) and (ii) we see that for each $j+\ell \le k$, $(x,p,\dot x,\dot p) \to D_1^j D_2^\ell G((x,p),\dot p^\ell,\dot x^j)$ is a continuous map $\mathcal{U} \times \mathcal{V} \times \boldsymbol{X} \times \boldsymbol{P} \to \boldsymbol{Y}_0$. Hence $G \in C_*^k(\mathcal{U} \times \mathcal{V}, \boldsymbol{Y}_0)$ by Corollary D.15, yielding D.24(a). Lemma D.13 and the polarization formula establish the function

$$(x,p,\dot x_1,\dots,\dot x_j,\dot p_1,\dots,\dot p_\ell) \to D_1^j D_2^\ell G((x,p),\dot p_1,\dots,\dot p_\ell,\dot x_1,\dots,\dot x_j)$$

as a combination of terms

$$(x,p,\dot x_1,\dots,\dot x_j,\dot p_1,\dots,\dot p_\ell) \to D_1^j D_2^\ell G((x,p),(\sum_{k\in I}\sigma_k \dot p_k)^\ell,(\sum_{k\in J}\tau_k \dot x_k)^j)$$

where $I \subset \{1,\dots,\ell\}$, $J \subset \{1,\dots,j\}$, and $\sigma_k,\tau_k = \pm 1$. This shows that for each $0 \le n \le m-\ell$, the derivative $D_1^j D_2^\ell G$ can be extended to a continuous map $\widetilde{\Psi}_{j,\ell,n}$, from $\mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}_{n+\ell}^j \times \boldsymbol{P}^\ell$ to $\boldsymbol{Y}_n$. With $n=0$ this shows D.24(b). For $0 \le n \le m-\ell$ we see from $\boldsymbol{X} = \boldsymbol{X}_m \hookrightarrow \boldsymbol{X}_{n+\ell} \hookrightarrow \boldsymbol{X}_\ell$ that both $\widetilde{\Psi}_{j,\ell,n}$ and the restriction of $\widetilde{\Psi}_{j,\ell,0}$ to $\mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}_{n+\ell}^j \times P^\ell$ are continuous as maps of $U := \big(\mathcal{U} \times \mathcal{V} \times \widetilde{\boldsymbol{X}}_{n+\ell}^j \times \boldsymbol{P}^\ell, \|\cdot\|_{\boldsymbol{X}_\ell}\big)$ to $\boldsymbol{Y}_0$. Since $\boldsymbol{X}$ is dense in $(\widetilde{\boldsymbol{X}}_{n+\ell}, \|\cdot\|_{X_{n+\ell}})$, and so also in $(\widetilde{\boldsymbol{X}}_{n+\ell}, \|\cdot\|_{X_\ell})$, the maps $\widetilde{\Psi}_{j,\ell,n}$ and $\widetilde{\Psi}_{j,\ell,0}$ coincide on a dense subset of U, hence on all of U, proving D.24(c). □

Remark D.26. Clearly, the claim remains true if one replaces

$$D_1^j D_2^\ell G((x,p),\dot p^\ell,\dot x^j)$$

with the derivatives taken in the opposite order (see Remark D.16). In the present and the following appendices, in the notation $\widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ we indicate, somehow pedantically but usefully for clarity in proofs, the sequences $\mathbb{X}$, $\mathbb{Y}$ of Banach spaces. When using this notion in particular applications, the sequences $\mathbb{X}$ and $\mathbb{Y}$ will be clear from the context and we will skip them from the notation writing just $\widetilde{C}^m(\mathcal{U} \times \mathcal{V})$. ◇

For working with functions from $\widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ it is useful to know that they have properties stronger than those given in the definition.

Lemma D.27. *Let* $G \in \widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$ *and* $0 \le j,n \le m-\ell$. *Then*

(1) *for fixed* $x \in \mathcal{U}$, *the map* $p \to G(x,p)$ *belongs to* $C_*^\ell(\mathcal{V}, \widetilde{\boldsymbol{Y}}_{m-\ell})$;

(2) *for fixed* $p \in \mathcal{V}$ *and* $\dot p_1,\dots,\dot p_\ell \in \boldsymbol{P}$, *the (extended) map*

$$(x,\dot x_1,\dots,\dot x_j) \to D_1^j D_2^\ell G((x,p),\dot p_1,\dots,\dot p_\ell,\dot x_1,\dots,\dot x_j)$$

belongs to $C^{m-\ell-j}_{\boldsymbol{X}_{n+\ell}^{j+1}}(\mathcal{U} \times \widetilde{\boldsymbol{X}}_{n+\ell}^j, \widetilde{\boldsymbol{Y}}_n)$.

Proof. (1) By Corollary D.9 and D.24(c) with $n = m-\ell$, the map $p \to G(x,p)$ belongs to $C_*^\ell(\mathcal{P}, \boldsymbol{Y}_{m-\ell})$. Hence the derivative $D_2^\ell G$ is an iterated limit of elements of $\boldsymbol{Y}$ taken in the norm of $\boldsymbol{Y}_{m-\ell}$, and so it belongs to $\widetilde{\boldsymbol{Y}}_{m-\ell}$.

(2) By Lemma D.20 it suffices to show that the function

$$x \to D_2^\ell G((x,p),\dot p_1,\dots,\dot p_\ell)$$

belongs to $C^{m-\ell}_{\boldsymbol{X}_{n+\ell}}(\mathcal{U}, \widetilde{\boldsymbol{Y}}_n)$. But this follows by the same argument as in the proof of (1). □

REMARK D.28. Since (2) puts the values of the (extended) derivatives into the corresponding closures of $\boldsymbol{Y}$, G belongs to $C^m(\mathcal{U}\times\mathcal{V}, \mathbb{X}, \mathbb{Y})$ iff and only if it belongs to this space when $\boldsymbol{X}_n$ and $\boldsymbol{Y}_n$ are replaced by $\widetilde{\boldsymbol{X}_n}$ and $\widetilde{\boldsymbol{Y}}_n$, respectively. So, at least in proofs, we may always assume that $\boldsymbol{X}$ is dense in $\boldsymbol{X}_n$ and $\boldsymbol{Y}$ in $\boldsymbol{Y}_n$. ⋄

THEOREM D.29. *Let* $G \in \widetilde{C}^m(\mathcal{U}\times\mathcal{V}, \mathbb{X}, \mathbb{Y})$, $G(\mathcal{U}\times\mathcal{V}) \subset \mathcal{Y}$, $F \in \widetilde{C}^m(\mathcal{Y}\times\mathcal{V}, \mathbb{Y}, \mathbb{Z})$ *and define* $F \diamond G\colon \mathcal{U}\times\mathcal{V} \to \boldsymbol{Z}$ *by* $F \diamond G(x,p) := F(G(x,p),p)$. *Then* $F \diamond G \in \widetilde{C}^m(\mathcal{U}\times\mathcal{V}, \mathbb{X}, \mathbb{Z})$.

PROOF. By Remark D.28, we may assume $\widetilde{\boldsymbol{X}_n} = \boldsymbol{X}_n$, and similarly for $\boldsymbol{Y}_n$ and $\boldsymbol{Z}_n$. Set $H := F \diamond G$. For fixed $x \in \mathcal{U}$, the function $p \to H(x,p)$ is of the form of a composition $F(\Phi(p), \Upsilon(p))$ where the outer function $F\colon \mathcal{Y}\times\mathcal{V} \to \boldsymbol{Z}$ and the inner functions $\Phi(p) = G(x,p)$ and $\Upsilon(p) = p$ satisfy the assumptions of Corollary D.23 with $\boldsymbol{Q} = \boldsymbol{P}$, $\mathcal{Q} = \mathcal{P}$ and $\boldsymbol{V} = \boldsymbol{Z}_0$. Hence $p \to H(x,p)$ belongs to $C^m_*(\mathcal{P}, \boldsymbol{Z}_0)$ and for each $\ell \le m$, the derivative $D^\ell_2 H((x,p),\dot{p}^\ell)$ is a combination of terms

$$D^k_1 D^i_2 F\big((G(x,p),p), \dot{p}^i, D^{m_1}_2 G((x,p),\dot{p}^{m_1}), \dots, D^{m_k}_2 G((x,p),\dot{p}^{m_k})\big) \tag{D.28}$$

where $m_s \ge 1$ and $i + \sum_{s=1}^k m_s = \ell$.

We now fix $p, \dot{p}$ and differentiate the function in (D.28) with respect to x. We set

$$K(x) := \big(G(x,p), D^{m_1}_2 G((x,p),\dot{p}^{m_1}), \dots, D^{m_k}_2 G((x,p),\dot{p}^{m_k})\big)$$

and

$$L(y, \dot{y}_1, \dots, \dot{y}_k) = D^k_1 D^i_2 F\big((y,p), \dot{p}^i, \dot{y}_1, \dots, \dot{y}_k\big).$$

Then the expression in (D.28) is given by the composition $(L \circ K)(x)$. Since $m_s \le l - i \le m - i$ we have $m - m_s \ge i$ and it follows from Lemma D.27 (2) (applied to the s-th component of K with $n = m - m_s$) that

$$K \in C^{m-l}_*(\mathcal{U}; \mathcal{Y} \times \boldsymbol{Y}^k_i).$$

Application of Lemma D.27 (2) to F yields that

$$L \in C^{m-i-k}_{\boldsymbol{Y}^{k+1}_i}(\mathcal{Y} \times \boldsymbol{Y}^k_i, \boldsymbol{Z}_0) \subset C^{m-\ell}_{\boldsymbol{Y}^{k+1}_i}(\mathcal{Y} \times \boldsymbol{Y}^k_i, \boldsymbol{Z}_0).$$

where the inclusion follows from the relation $l \ge i + k$. Hence, Theorem D.21 shows $L \circ K \in C^{m-l}_*(\mathcal{U}, \boldsymbol{Z}_0)$ and for each $j \le m - \ell$ the derivative of $D^j(L \circ K)$ (and hence the derivative $D^j_1 D^\ell_2 H$) exists and is given by a sum of terms of the form

$$D^k_1 D^i_2 F\Big((G(x,p),p), \dot{p}^i, D^{j_1}_1 D^{\ell_1}_2 G((x,p),\dot{p}^{\ell_1},\dot{x}^{j_1}), \dots, D^{j_k}_1 D^{\ell_k}_2 G((x,p),\dot{p}^{\ell_k},\dot{x}^{j_k})\Big) \tag{D.29}$$

where $j_s + \ell_s \ge 1$, $i + \sum_{s=1}^k \ell_s = \ell$ and $\sum_{s=1}^k j_s = j$.

Finally, we rely on Lemma D.27 once more. For any $s = 1, \dots, k$, the map $(x, p, \dot{x}, \dot{p}) \to D^{j_s}_1 D^{\ell_s}_2 G((x,p),\dot{p}^{\ell_s},\dot{x}^{j_s})$ is a continuous map from $\mathcal{U}\times\mathcal{V}\times\boldsymbol{X}_{n_s+\ell_s}\times\boldsymbol{P}$ to $\boldsymbol{Y}_{n_s}$ whenever $n_s \le m - \ell_s$. Choosing $n_s = n + \ell - \ell_s$ for any fixed $n \le m - \ell$, we get a map $\mathcal{U}\times\mathcal{V}\times\boldsymbol{X}_{n+\ell}\times\boldsymbol{P} \to \boldsymbol{Y}_{n+\ell-\ell_s}$. Using that $\ell_s \le \ell - i$, the derivatives have been extended so that the function of $(x, p, \dot{x}, \dot{p})$ defined in (D.29) is a composition of continuous maps

$$\mathcal{U}\times\mathcal{V}\times\boldsymbol{X}_{n+\ell}\times\boldsymbol{P} \to \mathcal{Y}\times\boldsymbol{P}\times\boldsymbol{P}^i\times\boldsymbol{Y}^k_{n+i}$$

and

$$\mathcal{Y} \times \boldsymbol{P} \times \boldsymbol{P}^i \times \boldsymbol{Y}^k_{n+i} \to \boldsymbol{Z}_n.$$

Hence $(x, p, \dot{x}, \dot{p}) \to D_1^j D_2^\ell H((x,p), \dot{p}^\ell, \dot{x}^j)$ is continuous as a map of $\mathcal{U} \times \mathcal{V} \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P}$ to $\boldsymbol{Z}_n$ and we conclude from Lemma D.25 that $H \in \widetilde{C}^m(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Z})$. □

REMARK D.30. Let $p_0 \in \mathcal{V}$ and assume that $G(\mathcal{U} \times B_\delta(p_0)) \subset \mathcal{Y}$,

$$\|D_1^j D_2^\ell G((x,p), \dot{p}^\ell, \dot{x}^j)\|_{\boldsymbol{Y}_n} \le C_1 \|\dot{x}\|^j_{\boldsymbol{X}_{n+\ell}} \|\dot{p}\|^\ell \tag{D.30}$$

for any $(x, p, \dot{x}, \dot{p}) \in \mathcal{U} \times B_\delta(p_0) \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P}$ and any $0 \le j + \ell \le m,\ 0 \le n \le m - l$ and

$$\|D_1^j D_2^\ell F((y,p), \dot{p}^\ell, \dot{y}^j)\|_{\boldsymbol{Z}_n} \le C_2 \|\dot{y}\|^j_{\boldsymbol{Y}_{n+\ell}} \|\dot{p}\|^\ell \tag{D.31}$$

for any $(y, p, \dot{y}, \dot{p}) \in \mathcal{Y} \times\ B_\delta(p_0) \times \boldsymbol{Y}_{n+\ell} \times \boldsymbol{P}$ and any $0 \le j + \ell \le m,\ 0 \le n \le m - l$. Then

$$\|D_1^j D_2^\ell H((x,p), \dot{p}^\ell, \dot{x}^j)\|_{\boldsymbol{Z}_n} \le C_3 \|\dot{x}\|^j_{\boldsymbol{X}_{n+\ell}} \|\dot{p}\|^\ell \tag{D.32}$$

for any $(x, p, \dot{x}, \dot{p}) \in \mathcal{U} \times B_\delta(p_0) \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P}$ and any $0 \le j + \ell \le m,\ 0 \le n \le m - l$, where C_3 depends only on C_1, C_2 and m. In fact, since $D_1^j D_2^\ell H((x,p), \dot{p}^\ell, \dot{x}^j)$ is a weighted sum of the terms in (D.29) it is easy to see that there exists a constant $C(m)$ such that $C_3 \le C(m)\, C_1(1 + C_2^m)$. ◇

If we the introduce the norm

(D.33)

$$\|G\|_{\widetilde{C}^m(\mathcal{U}\times\mathcal{V},\mathbb{X},\mathbb{Y})} := \inf \Big\{ M : \|D_1^j D_2^\ell G((x,p), \dot{p}^\ell, \dot{x}^j)\|_{\boldsymbol{Y}_n} \le M \|\dot{x}\|^j_{\boldsymbol{X}_{n+\ell}} \|\dot{p}\|^\ell,$$
$$\forall (x, p, \dot{x}, \dot{p}) \in \mathcal{U} \times \mathcal{V} \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P} \text{ and any } 0 \le j + \ell \le m,\ 0 \le n \le m - l \}$$

then the remark implies that $\|H\|$ can be controlled in terms of $\|F\|$ and $\|G\|$.

D.5. A special case of a function G that is linear in its first argument

Here we discuss conditions assuring that $G \in \widetilde{C}^m$ in a special case of linear dependence on the first variable:

LEMMA D.31. *Let $G : \boldsymbol{X} \times \mathcal{V} \to \boldsymbol{Y}$ and assume that:*

(i) For any $p \in \mathcal{V}$, the map $x \mapsto G(x,p)$ is linear.
(ii) For any $0 \le \ell \le m$ and any $x \in \boldsymbol{X}$, the map $p \mapsto G(x,p)$ is in $C_*^\ell(\mathcal{V}, \boldsymbol{Y}_{m-\ell})$.
(iii) For any $p_0 \in \mathcal{V}$ there exists $\delta, C > 0$ such that

$$\|D_2^\ell G((x,p), \dot{p}^\ell)\|_{\boldsymbol{Y}_n} \le C \|x\|_{\boldsymbol{X}_{n+\ell}} \|\dot{p}\|^\ell$$

for any $0 \le \ell \le m$, $0 \le n \le m - \ell$, and $(x, p, \dot{p}) \in \boldsymbol{X} \times B_\delta(p_0) \times \boldsymbol{P}$.

Then $G \in \widetilde{C}^m(\boldsymbol{X} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$. Moreover

$$\|G\|_{\widetilde{C}^m(B_R\times\mathcal{V},\mathbb{X},\mathbb{Y})} \le C(m)(1 + R)M', \tag{D.34}$$

where

$$M' := \inf \big\{ M : \|D_2^\ell G((x,p), \dot{p}^\ell)\|_{\boldsymbol{Y}_n} \le M \|\dot{x}\|_{\boldsymbol{X}_{n+\ell}} \|\dot{p}\|^\ell,$$
$$\textit{for any } (x, p, \dot{x}, \dot{p}) \in \boldsymbol{X} \times \mathcal{V} \times \boldsymbol{P} \textit{ and any } 0 \le n + \ell \le m \} \tag{D.35}$$

PROOF. We will verify the conditions of Lemma D.25.

The conditions (i) and (ii) above imply the condition Lemma D.25(i). Indeed, taking into account the linearity of G in the first variable, the derivative $D_1G((x,p),\dot{x})$ exists and equals $G(\dot{x},p)$ (with any norm $\|\cdot\|_{\boldsymbol{Y}_n}$, $0 \le n \le m$ (in particular, also $n = m-\ell$) on the target space $\boldsymbol{Y}$). Thus $D_2^\ell D_1 G((x,p),\dot{x},\dot{p}^\ell) = D_2^\ell G((\dot{x},p),\dot{p}^\ell)$ and $D_2^\ell D_1^j G((\dot{x},p),\dot{x}^j,\dot{p}^\ell) = 0$ for $j \ge 2$.

Further, we show that the derivatives $(x,p,\dot{p}) \to D_2^\ell G((x,p),\dot{p}^\ell)$ can be extended to continuous maps $\Phi_{\ell,n} : \widetilde{\boldsymbol{X}}_{n+\ell} \times \mathcal{V} \times \boldsymbol{P} \to \boldsymbol{Y}_n$. Indeed, consider fixed $p \in \mathcal{V}, \dot{p} \in \boldsymbol{P}$, $x \in \widetilde{\boldsymbol{X}}_{n+\ell}$, and a sequence $x_k \in \boldsymbol{X}_m$ converging to x in the norm of $\boldsymbol{X}_{n+\ell}$, $\|x_k - x\|_{\boldsymbol{X}_{n+\ell}} \to 0$. The derivative $D_2^\ell G((x_k,p),\dot{p}^\ell)$ belongs to $\boldsymbol{Y}_{m-\ell} \hookrightarrow \boldsymbol{Y}_n$ for each x_k, and in view of the bound (iii) we get

$$\|D_2^\ell G((x_k,p),\dot{p}^\ell) - D_2^\ell G((x_{k'},p),\dot{p}^\ell)\|_{\boldsymbol{Y}_n} \le C\|x_k - x_{k'}\|_{\boldsymbol{X}_{n+\ell}}\|\dot{p}\|^\ell, \tag{D.36}$$

yielding the existence of the limit $\Phi_{\ell,n}(x,p,\dot{p}) := \lim_{k\to\infty} D_2^\ell G((x_k,p),\dot{p}^\ell) \in \boldsymbol{Y}_n$. This also gives the continuity of the map $x \to \Phi_{\ell,n}(x,p,\dot{p})$. Combined with the continuity $(p,\dot{p}) \to D_2^\ell G((x,p),\dot{p}^\ell)$ from the condition (ii), we get the continuity of $\Phi_{\ell,n}$ as stated above.

To conclude, we introduce the continuous $\Psi_{0,\ell,n} : \boldsymbol{X} \times \mathcal{V} \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P} \to \boldsymbol{Y}_n$ defined by $\Psi_{0,\ell,n}(x,p,\dot{x},\dot{p}) = \Phi_{\ell,n}(p,x,\dot{p})$ and $\Psi_{1,\ell,n} : \boldsymbol{X} \times \mathcal{V} \times \boldsymbol{X}_{n+\ell} \times \boldsymbol{P} \to \boldsymbol{Y}_n$ defined by $\Psi_{1,\ell,n}(x,p,\dot{x},\dot{p}) = \Phi_{\ell,n}(p,\dot{x},\dot{p})$. For $j \ge 2$ we take $\Psi_{j,\ell,n}(x,p,\dot{x},\dot{p}) = 0$.

The assumptions of Lemma D.25 are thus satisfied, allowing us to conclude that $G \in \widetilde{C}^m(\boldsymbol{X} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$. □

D.6. A special case of function G not depending on the parameter p

In applications of the chain rule it is convenient to also consider the case of maps that do not explicitly depend on the parameter p. We get

LEMMA D.32. *Suppose that* $G : \mathcal{U} \times \mathcal{V} \to \boldsymbol{Y}$ *and* $\tilde{G} : \mathcal{U} \to \boldsymbol{Y}$ *satisfy*

$$G(x,p) = \tilde{G}(x) \quad \forall (x,p) \in \mathcal{U} \times \mathcal{V}. \tag{D.37}$$

Assume that

(1) $\tilde{G} \in C_*^m(\mathcal{U}, \boldsymbol{Y}_m)$ *and*
(2) *for* $1 \le \ell \le m$ *the map* $(x,\dot{x}) \mapsto D^\ell \tilde{G}(x,\dot{x}^\ell)$ *can be extended to a continuous map from* $\mathcal{U} \times \boldsymbol{X}_0$ *to* $\boldsymbol{Y}_0$ *and for* $1 \le n \le m-1$ *the restriction of this map to* $\mathcal{U} \times \boldsymbol{X}_n$ *is continuous as a map with values in* $\boldsymbol{Y}_n$.

Then $G \in \tilde{C}^m(\mathcal{U} \times \mathcal{V}, \mathbb{X}, \mathbb{Y})$. *Moreover*

$$\|G\|_{\tilde{C}^m(\mathcal{U}\times\mathcal{V},\mathbb{X},\mathbb{Y})} \le M' \tag{D.38}$$

with

(D.39)
$$M' = \inf\left\{M : \|D^j G(x,\dot{x}^\ell)\|_{\boldsymbol{Y}_n} \le M\|\dot{x}\|^l_{X_n} \ \forall (x,\dot{x}) \in \mathcal{U} \times X_n \ \forall\, 0 \le n \le m\right\}.$$

PROOF. First note that $D_2^\ell G = 0$ for $\ell \ne 0$. Let $\phi_{l,0} : \mathcal{U} \times X_0 \to Y_0$ denote the extension of $D^l G$ to $\mathcal{U} \times X_0$ and let $\phi_{l,n}$ denote the restriction of $\phi_{l,0}$ to $\mathcal{U} \times X_n$. Set

$$\psi_{j,0,n}(x,p,\dot{x},\dot{p}) := \phi_{l,n}(x,\dot{x}), \qquad \psi_{j,l,n}(x,p,\dot{x},\dot{p}) = 0 \quad \text{if } l \ne 0. \tag{D.40}$$

Then the assertion follows from Lemma D.25 □

D.7. A map in $C^1 \setminus C^1_*$ and failure of the inverse functions theorem in C^1_*

Proposition D.33. *Let H be an infinite dimensional separable Hilbert space. Then there exists $G \in C^1_*(H,H) \cap C^\infty(H \setminus \{0\}, H)$ such that G is not Fréchet differentiable at zero. Moreover the exists a function $F \in C^1_*(H,H)$ which satisfies $DF(0,\dot{x}) = \dot{x}$ but which is not invertible in any neighbourhood of 0.*

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of H. We will construct G as a convergent sum

$$G(x) = \sum_{k\in\mathbb{N}} G_k(x) e_k \tag{D.41}$$

such that

- $G_k \in C^\infty(H)$,
- the support $\operatorname{supp} G_k$ of G_k is concentrated near $2^{-k}e_k$,
- $\operatorname{supp} G_k \cap \operatorname{supp} G_l = \varnothing$ for $k \neq l$,
- the gradients ∇G_k are uniformly bounded and converge weakly, but not strongly, to 0 as $k \to \infty$.

Specifically G_k can be defined as follows. Let P_k denote the orthogonal projection of H onto the subspace

$$X_k := \{x \in H : (x, e_j) = 0 \quad \forall j \leq k-1\}. \tag{D.42}$$

Let

$$\varphi \in C^\infty_c\big((-\tfrac{1}{16}, \tfrac{1}{16})\big), \quad 0 \leq \varphi \leq 1, \quad \varphi(0) = 1, \tag{D.43}$$

$$G_k(x) = 2^{-k}\varphi(\|2^k P_k x - e_k\|^2) \prod_{j\leq k-1} \varphi\left(2^{\frac{j+k}{2}}(x, e_j)\right). \tag{D.44}$$

For $k = 0$ the product $\prod_{j\leq k-1}$ is replaced by 1. Clearly $G_k \in C^\infty(H)$. Moreover

$$\begin{aligned} \operatorname{supp} G_k \subset K_k := \Big\{x : |(x,e_j)| \leq \tfrac{1}{4}2^{-\frac{k+j}{2}} &\text{ if } j \leq k-1, \\ |(x,e_k) - 2^{-k}| \leq \tfrac{1}{4} &\text{ and } |P_{k+1}x| \leq \tfrac{1}{4}2^{-k}\Big\}. \end{aligned} \tag{D.45}$$

We claim that

$$K_k \cap K_l = \varnothing \quad \text{if } k \neq l. \tag{D.46}$$

To show this we may assume that $k < l$. If $x \in K_k \cap K_l$ then the definition of K_k implies that $(x, e_k) \geq \frac{3}{4}2^{-k}$ while the definition of K_l yields $|(x,e_k)| \leq \frac{1}{4}2^{-\frac{k+l}{2}}$. Since both inequalities cannot hold simulateneously we get $K_k \cap K_l = \varnothing$. Note also that

$$x \in K_k \quad \Longrightarrow \quad |x|^2 \leq \frac{1}{8}2^{-k} + \frac{25}{16}2^{-2k} + \frac{1}{8}2^{-2k} \leq 2^{-k+1} \tag{D.47}$$

In particular if $x_0 \neq 0$ then the ball $B_{|x_0|/2}(x_0)$ intersects only finitely many of the sets K_k. Hence the sum $G = \sum_k G_k e_k$ is a finite sum in $B_{|x_0|/2}(x_0)$ and thus defines a C^∞ map on that set. Thus

$$G \in C^\infty(H \setminus \{0\}, H). \tag{D.48}$$

Moreover $G_k(0) = 0$ and thus $G(0) = 0$.

We now show that

(D.49) the directional derivative $D^1G(0,\dot{x})$ exists and equals 0; and that

(D.50) the map $(x, \dot{x}) \mapsto D^1 G(x, \dot{x})$ is a continuous map from $H \times H$ to H.

To prove (D.49) we note that $G_k(x) = 0$ if $|(x, e_k)| \le \frac{1}{2}$ and $|G_k(x)| \le 1$ for all $x \in H$. Thus

$$|G_k(x)| \le 2|(x, e_k)|. \tag{D.51}$$

Since each function G_k is in $C^\infty(H)$ it suffices to show that for each $\dot{x} \in H$

$$\lim_{m\to\infty} \limsup_{t\to 0} \frac{1}{t} \Big| \sum_{k\ge m} G_k(t\dot{x}) e_k \Big| = 0. \tag{D.52}$$

Now by (D.51) and orthogonality

$$\Big| \sum_{k\ge m} G_k(t\dot{x}) e_k \Big|^2 = \sum_{k\ge m} |G_k(t\dot{x})|^2 \le 4t^2 \sum_{k\ge m} |(\dot{x}, e_k)|^2 = 4t^2 |P_m \dot{x}|^2. \tag{D.53}$$

Thus

$$\limsup_{t\to 0} \frac{1}{t} \Big| \sum_{k\ge m} G_k(t\dot{x}) e_k \Big| \le 2|P_m \dot{x}| \tag{D.54}$$

and the assertion (D.52) follows.

To prove (D.50) it suffices to prove continuity at $(0, \dot{x})$ since we already know that $G \in C^\infty(H \setminus \{0\}, H)$. Thus we need to show

$$\lim_{(x,v)\to(0,\dot{x})} D^1 G(x, v) = 0. \tag{D.55}$$

Since $D^1 G$ is linear in the second argument and since finite linear combinations $\sum_{l=0}^M a_l e_l$ are dense in H it suffices to establish the following two properties

$$\|D^1 G(x, v)\| \le C\|v\| \quad \forall (x, v) \in H \times H, \tag{D.56}$$

$$\lim_{x\to 0} D^1 G(x, e_m) = 0 \quad \forall m \in \mathbb{N}. \tag{D.57}$$

To prove the bound on $D^1 G$ note that (for $x \ne 0$)

$$\begin{aligned} \nabla G_k(x) = {} & 2\varphi'(\|2^k P_k x - e_k\|^2)(2^k P_k x - e_k) \prod_{j\le k-1} \varphi\big(2^{\frac{j+k}{2}}(x, e_j)\big) \\ & + \varphi(\|2^k P_k x - e_k\|^2) \sum_{l\le k-1} \varphi'\big(2^{\frac{l+k}{2}}(x, e_l)\big) 2^{\frac{l-k}{2}} e_l \prod_{j\le k-1, j\ne l} \varphi\big(2^{\frac{j+k}{2}}(x, e_j)\big). \end{aligned} \tag{D.58}$$

Since the vectors $e_1, \ldots, e_{k-1}, 2^k P_k x - e_k$ are orthogonal this yields, with $C' = \sup|\varphi'|^2$,

$$|\nabla G_k(x)|^2 \le 4C' \frac{1}{4} + C' \sum_{l\le k-1} 2^{l-k} \le 2C'. \tag{D.59}$$

Since the G_k have disjoint support and since $D^1 G(0, v) = 0$ it follows that

$$\|D^1 G(x, v)\| \le \sqrt{2} \sup|\varphi'| \, \|v\| \quad \forall (x, v) \in H \times H \tag{D.60}$$

and thus (D.56).

To prove (D.57) note that $G_k(x) = 0$ if $\|x\| \le \frac{3}{4} 2^{-k}$. Thus for $\|x\| \le \frac{3}{4} 2^{-m}$ we have

$$|D^1 G(x, e_m)| \begin{cases} \le 2^{\frac{m-k}{2}} & \text{if } x \in \operatorname{supp} G_k \text{ for some } k, \\ = 0 & \text{else.} \end{cases} \tag{D.61}$$

Now if $x \in \operatorname{supp} G_k$ and $x \to 0$ then $k \to \infty$. This implies (D.57).

Thus we have shown that

$$G \in C^1_*(H,H) \quad \text{with} \quad D^1G(0,\dot{x}) = 0 \quad \forall \dot{x} \in H. \tag{D.62}$$

We finally show that G is not Fréchet differentiable at 0. If G was Fréchet differentiable at 0 the Fréchet derivative $DG(0)$ would satisfy $DG(0) = 0$. Thus Fréchet differentiability would give

$$\lim_{x \to 0} \frac{\|G(x)\|}{\|x\|} = 0. \tag{D.63}$$

On the other hand we have

$$G(2^{-k}e_k) = G_k(2^{-k}e_k)e_k = 2^{-k}e_k. \tag{D.64}$$

Taking $k \to \infty$ we get a contradiction to (D.63).

To get a counterexample to the inverse function theorem in $C^1_*(H,H)$ set

$$F(x) := x - G(x). \tag{D.65}$$

Then $F \in C^1_*(H,H)$ and by (D.62)

$$D^1F(0,\dot{x}) = \dot{x} \quad \forall \dot{x} \in H. \tag{D.66}$$

Now (D.64) imlies that

$$F(2^{-k}e_k) = 0 = F(0) \tag{D.67}$$

and hence there exists no neighbourhood of 0 in which F is invertible. □

APPENDIX E

Implicit Function Theorem with Loss of Regularity

Here we state and prove a version of the implicit function theorem which incorporates a loss of regularity and is tailored for the use in Chapters 4.5 and 8.

We consider a function of three variables (rather than a function of two variables as in the standard version of the implicit function theorem). The implicit function we are looking for expresses the first variable as a function of the second and the third variable. The reason for this set-up is that the second and the third variable play very different roles. Differentiation with the respect to the third variable (which in our application is the renormalised coefficient in the difference operator) leads to a loss of regularity, while differentiation with respect to the second variable does not. This bad behaviour with respect to the third variable is partially compensated by the fact that we know that $F(0,0,p)=0$ for all values of the third variable in a neighbourhood of 0 (and not just for $p=0$) and that we have uniform control of $D_1F(0,0,p)$.

THEOREM E.1. *Let $m \geq 2$. Let $\boldsymbol{X} = \boldsymbol{X}_m \hookrightarrow \ldots \hookrightarrow \boldsymbol{X}_0$, $\boldsymbol{E}$, and $\boldsymbol{P}$ be normed spaces, with $\mathbb{X} = (\boldsymbol{X}_m, \ldots, \boldsymbol{X}_0)$, $\mathbb{E} = (\boldsymbol{E}, \ldots, \boldsymbol{E})$, and $\mathbb{X} \times \mathbb{E} = (\boldsymbol{X}_m \times \boldsymbol{E}, \ldots, \boldsymbol{X}_0 \times \boldsymbol{E})$. Further, let $\mathcal{U} \subset \boldsymbol{X}$, $\mathcal{V} \subset \boldsymbol{E}$, and $\mathcal{W} \subset \boldsymbol{P}$ be open and assume that $F \in \tilde{C}^m((\mathcal{U} \times \mathcal{V}) \times \mathcal{W}; \mathbb{X} \times \mathbb{E}, \mathbb{X})$, i.e., $F \in C^m_*(\mathcal{U} \times \mathcal{V} \times \mathcal{W}, \boldsymbol{X}_0)$, for any $j' + j'' + \ell \leq m$ the derivative*

$$D_1^{j'} D_2^{j''} D_3^{\ell} F \text{ can be extended to a continuous map}$$

$$\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \boldsymbol{X}_\ell^{j'} \times \boldsymbol{E}^{j''} \times \boldsymbol{P}^\ell \to \boldsymbol{X}_0 \tag{E.1}$$

and

$$\text{the restriction of } D_1^{j'} D_2^{j''} D_3^{\ell} F \text{ defines a continuous map}$$

$$\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \boldsymbol{X}_{n+\ell}^{j'} \times \boldsymbol{E}^{j''} \times \boldsymbol{P}^\ell \to \boldsymbol{X}_n \text{ if } 0 \leq n \leq m-\ell. \tag{E.2}$$

Assume, moreover, that $(0,0,0) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ and

$$F(0,0,p) = 0 \text{ for all } p \in \mathcal{W}, \tag{E.3}$$

and, there exists $\gamma \in (0,1)$ such that

$$\|D_1 F(0,0,p)\|_{L(\boldsymbol{X}_n, \boldsymbol{X}_n)} \leq \gamma \text{ for any } n \leq m \text{ and } p \in \mathcal{W}. \tag{E.4}$$

Then there exist open subsets $\widetilde{\mathcal{U}} \subset \mathcal{U}$, $\widetilde{\mathcal{V}} \subset \mathcal{V}$, and $\widetilde{\mathcal{W}} \subset \mathcal{W}$ with $0 \in \widetilde{\mathcal{U}}$, $0 \in \widetilde{\mathcal{V}}$, $0 \in \widetilde{\mathcal{W}}$, and a unique function $f : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \to \widetilde{\mathcal{U}}$ such that

$$F(f(\varpi,p),\varpi,p) = f(\varpi,p) \text{ for any } (\varpi,p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}. \tag{E.5}$$

Moreover $f \in \tilde{C}^m(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X})$, i.e.,

$$f \in C^n_*(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_{m-n}) \quad \text{for all } 0 \leq n \leq m \tag{E.6}$$

and

$$D_1^{j''} D_2^l f : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times E^{j''} \times P^l \to \boldsymbol{X}_{m-l} \quad \textit{is continuous} \tag{E.7}$$

for $j'' + l \leq m$.

Finally if $F(x, \varpi, p) = x$ *and* $(x, \varpi, p) \in \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$ *then* $x = f(\varpi, p)$. *The derivatives of* f *are given by the usual formulae, see* (E.28) *for the first derivative and the inductive definitions* (E.33) *and* (E.34) *for the higher derivatives.*

If

$$\|D_1^{j'} D_2^{j''} D_3^{\ell} F(x, \varpi, p, \dot{x}^{j'}, \dot{\varpi}^{j''}, \dot{p}^l)\|_{X_n} \leq C_1 \|\dot{x}\|_{X_{n+l}}^{j'} \|\dot{\varpi}\|_E^{j''} \|\dot{p}\|_P^{\ell}.$$

for all $(x, \varpi, p) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *and all* $0 \leq n \leq m - \ell$, *then there exists a constant* $C_2 = C_2(C_1, \gamma, m)$ *such that*

$$\|D_1^j D_2^{\ell} f(\varpi, p, \dot{\varpi}^j, \dot{p}^{\ell})\|_{X_{m-l}} \leq C_2 \|\dot{\varpi}\|^j \|\dot{p}\|^{\ell} \tag{E.8}$$

for all $(\varpi, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$.

The examples in Proposition D.33 shows that the inverse function theorem (and hence the implicit function theorem) in general does not hold in C^1_*, even when there is no loss of regularity. This is why we assume $m \geq 2$ in Theorem E.1.

Remark E.2. The usual implicit function theorem also holds in the C^m_* spaces instead of the C^m spaces as long as $m \geq 2$. More specifically, let $\mathcal{U} \subset \boldsymbol{X}$, $\mathcal{V} \subset \boldsymbol{E}$ and assume that $F \in C^m_*(\mathcal{U} \times \mathcal{V}, \boldsymbol{X})$ with $F(0,0) = 0$ and $\|D_1 F(0,0)\| \leq \gamma < 1$. Then there exist $\widetilde{\mathcal{U}} \subset \mathcal{U}$ and $\widetilde{\mathcal{V}} \subset \mathcal{V}$ and $f \in C^m_*(\widetilde{\mathcal{V}}, \boldsymbol{X})$ with $f(\widetilde{\mathcal{V}}) \subset \widetilde{\mathcal{U}}$ such that $F(f(\varpi), \varpi) = f(\varpi)$ for all $\varpi \in \widetilde{\mathcal{V}}$. This follows directly from Theorem E.1. Indeed, it suffices to consider the situation where $\boldsymbol{X}_m = \ldots = \boldsymbol{X}_0 = \boldsymbol{X}$ and to extend F trivially to a function on $\mathcal{U} \times \mathcal{V} \times \boldsymbol{P}$ which is independent of the third argument. Then F satisfies all the hypothesis of Theorem E.1 and the conclusion of the theorem gives the desired assertion. ◇

Remark E.3. Let $\hat{\mathcal{U}} = \mathcal{U} \times \mathcal{V}$, $\hat{\boldsymbol{X}}_{\ell} = \boldsymbol{X}_{\ell} \times \boldsymbol{E}$. Then, strictly speaking, the definition of $\widetilde{C}^m((\mathcal{U} \times \mathcal{V}) \times \mathcal{W}, \mathbb{X} \times \mathbb{E}, \mathbb{X})$ requires that

$$D^j_{(x,\varpi)} D^{\ell}_p F \quad \text{can be extended to a continuous map}$$

$$\hat{\mathcal{U}} \times \mathcal{W} \times \hat{\boldsymbol{X}}_{n+l}^{j'} \times \boldsymbol{P}^{\ell} \to \boldsymbol{X}_n \quad \text{if } 0 \leq n \leq \ell - m \text{ and } j + \ell \leq m. \tag{E.9}$$

In view of Corollary D.15 this is equivalent to (E.2). ◇

Proof.

Step 1. Prelimary estimates.

We claim that there exist subsets $\widetilde{\mathcal{U}} \subset \mathcal{U}, \widetilde{\mathcal{V}} \subset \mathcal{V}, \widetilde{\mathcal{W}} \subset \mathcal{W}$ that are balls around 0 and a constant M such that the following estimates hold:

$$\|D_1^{j'} D_2^{j''} D_3^{\ell} F((x, \varpi, p), \dot{x}^{j'}, \dot{\varpi}^{j''}, \dot{p})\|_{\boldsymbol{X}_{n+\ell}} \leq M \|\dot{x}\|_{\boldsymbol{X}_n}^{j'} \|\dot{\varpi}\|_{\boldsymbol{E}}^{j''} \|\dot{p}\|_{\boldsymbol{P}}^{\ell} \tag{E.10}$$

for all $(x, \varpi, p) \in \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$, all $\dot{x} \in \boldsymbol{X}, \dot{\varpi} \in \boldsymbol{E}, \dot{p} \in \boldsymbol{P}$, and all $j' + j'' + \ell = 2$, $0 \leq n + \ell \leq m$,

$$\|D_2 F((x, \varpi, p), \dot{\varpi})\|_{\boldsymbol{X}_m} \leq M \|\dot{\varpi}\|_{\boldsymbol{E}} \quad \text{for all } (x, \varpi, p) \in \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \tag{E.11}$$

$$\|F(0, \varpi, p)\|_{\boldsymbol{X}_m} \leq M \|\varpi\|_{\boldsymbol{E}} \quad \text{for all } (\varpi, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \text{ and} \tag{E.12}$$

(E.13) $\|D_1F(x,\varpi,p)\|_{L(\boldsymbol{X}_n,\boldsymbol{X}_n)} \le \frac{1+\gamma}{2}$ for all $(x,\varpi,p) \in \widetilde{\mathcal{U}}\times\widetilde{\mathcal{V}}\times\widetilde{\mathcal{W}}, 0 \le n \le m.$

Indeed, using the joint continuity in (E.2) at $(x,\varpi,p)=0$ and $(\dot{x},\dot{\varpi},\dot{p})=0$ we see that for $\varepsilon = 1$ there exists a $\delta \in (0,1]$ such that

$$\|D_1^{j'}D_2^{j''}D_3^{\ell}F((x,\varpi,p),\dot{x}^{j'},\dot{\varpi}^{j''},\dot{p})\|_{\boldsymbol{X}_{n+\ell}} < 1$$

if $\max(\|\dot{x}\|_{\boldsymbol{X}_n},\|\dot{\varpi}\|_{\boldsymbol{E}},\|\dot{p}\|_{\boldsymbol{P}}) < \delta$ and $\max(\|x\|_{\boldsymbol{X}},\|\varpi\|_{\boldsymbol{E}},\|p\|_{\boldsymbol{P}}) < \delta$. By the multilinearity of $D_1^{j'}D_2^{j''}D_3^{\ell}$ this implies (E.10) if $M \ge \delta^{-2}$. Similarly we see that (E.11) holds. Now (E.12) follows from (E.11), the assumption $F(0,0,p)=0$ and Lemma D.6. Finally (E.13) follows from the assumption $\|D_1F(0,0,p)\|_{L(\boldsymbol{X}_n,\boldsymbol{X}_n)} \le \gamma$ and (E.10) (applied with $\ell = 0$) provided that the radius of $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ is chosen sufficiently small.

Step 2. Existence, uniqueness and continuity of f.

First, observe that, according to (E.2), the derivative D_1F defines a continuous map $D_1F : \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times \boldsymbol{X}_m \to \boldsymbol{X}_m$. Taking into account the inequality (E.13) and, possibly, shrinking the diameters of balls $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$, and $\widetilde{\mathcal{W}}$, we have

(E.14) $$\|F(x_1,\varpi,p) - F(x_2,\varpi,p)\|_{\boldsymbol{X}_m} \le \tfrac{1+\gamma}{2}\|x_1 - x_2\|_{\boldsymbol{X}_m}$$

for any $x_1, x_2 \in \widetilde{\mathcal{U}}$ and any $\varpi \in \widetilde{\mathcal{V}}$ and $p \in \widetilde{\mathcal{W}}$. Employing now the Banach fixed point theorem [**Die60**, (10.1.1)] (and possibly shrinking $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$ further) we get the existence of a unique map $f : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \to \widetilde{\mathcal{U}}$ such that $F(f(\varpi,p),\varpi,p) = f(\varpi,p)$ for any $(\varpi,p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$; moreover, $f \in C^0(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_m)$.

Step 3. Differentiability of f, i.e., $f \in C^1_*(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_{m-1})$.

Using the characterisation in terms of Peano derivatives, Proposition D.7, we need to find a continuous function $f^{(1)} : (\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}) \times (\boldsymbol{E} \times \boldsymbol{P}) \to \boldsymbol{X}_{m-1}$ so that, for any $\varpi \times p \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$ and $\dot{\varpi} \times \dot{p} \in \boldsymbol{E} \times \boldsymbol{P}$, we have

(E.15) $$\lim_{t\to 0}\Big\|\frac{\xi(t)}{t} - f^{(1)}\Big\|_{\boldsymbol{X}_{m-1}} = 0$$

with

(E.16) $$\xi(t) := f(\varpi + t\dot{\varpi}, p + t\dot{p}) - f(\varpi,p).$$

Introducing

(E.17) $$G(x,\varpi,p) := F(x,\varpi,p) - x,$$

the function f is defined by

(E.18) $$G(f(\varpi,p),\varpi,p) = 0 \text{ for all } (\varpi,p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}.$$

Differentiating now formally the equation

(E.19) $$G(f((\varpi,p) + t(\dot{\varpi},\dot{p})), \varpi + t\varpi, p + t\dot{p})) = 0$$

with respect to t and setting

(E.20) $$R_1^{(1)} := D_2G((x,\varpi,p),\dot{\varpi}) + D_3G((x,\varpi,p),\dot{p})$$

we expect that

(E.21) $$f^{(1)}((\varpi,p),(\dot{\varpi},\dot{p})) = -D_1G(x,\varpi,p)^{-1}R_1^{(1)}$$

with $x = f(\varpi,p)$.

The mapping $D_1G(x,\varpi,p) : \boldsymbol{X}_n \to \boldsymbol{X}_n$ is bounded and invertible for any $n \le m$ since, according to (E.13),

$$\|D_1G(x,\varpi,p) - \mathbb{1}\|_{L(\boldsymbol{X}_n,\boldsymbol{X}_n)} \le \frac{1+\gamma}{2} < 1 \tag{E.22}$$

and thus

$$\|D_1G(x,\varpi,p)^{-1}\|_{L(\boldsymbol{X}_n,\boldsymbol{X}_n)} \le \frac{2}{1-\gamma} \tag{E.23}$$

for any $(x,\varpi,p) \in \widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$. Hence, the function $f^{(1)}$ introduced by (E.21) is well defined.

To verify the claim (E.15), we recall that ξ is continuous (with values in X_m) and use the first assertion in Lemma D.27 with $l = 1$ and Lemma D.6 to estimate

(E.24)
$$\begin{aligned}
&\|\underbrace{G(x+\xi(t),\varpi+t\dot\varpi,p+t\dot p)}_{=0} - G(x+\xi(t),\varpi+t\dot\varpi,p) - D_3G(x+\xi(t),\varpi+t\dot\varpi,p,t\dot p)\|_{X_{m-1}}\\
&\le t \sup_{\tau\in[0,1]} \|D_3G(x+\xi(t),\varpi+t\dot\varpi,p+\tau t\dot p,\dot p) - D_3G(x+\xi(t),\varpi+t\dot\varpi,p,\dot p)\|_{X_{m-1}}\\
&= o(t).
\end{aligned}$$

Similarly, using the second assertion in Lemma D.27 and Lemma D.6 we get

(E.25)
$$\begin{aligned}
&\|G(x+\xi(t),\varpi+t\dot\varpi,p) - \underbrace{G(x,\varpi,p)}_{=0} - D_1G(x,\varpi,p,\xi(t)) - D_2G(x,\varpi,p,t\dot\varpi)\|_{X_{m-1}}\\
&= o(t) + o(\|\xi(t)\|_{X_{m-1}}).
\end{aligned}$$

Combining these two estimate we deduce that

$$\|D_1G(x,\varpi,p)\xi(t) + tR_1^{(1)}\|_{X_{m-1}} \le o(t) + o(\|\xi(t)\|_{X_{m-1}}) \tag{E.26}$$

. and using (E.23) and the definition of $f^{(1)}$ it follows that

$$\|\xi(t) - tf^{(1)}\|_{X_{m-1}} = o(t) + o(\|\xi(t)\|_{X_{m-1}}). \tag{E.27}$$

This implies first that $\|\xi(t)\|_{X_{m-1}} \le Ct$ for small $|t|$ and then division by t yields the desired assertion (E.15).

We finally show that

(E.28)
$$f^{(1)}((\varpi,p),(\dot\varpi,\dot p)) = -D_1G(x,\varpi,p)^{-1}(D_2G((x,\varpi,p),\dot\varpi) + D_3G((x,\varpi,p),\dot p))$$

defines a continuos map from $\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times \times E \times \boldsymbol{P}$ to X_{m-1}. Together with (E.15) this show that $f \in C^1_*(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}; \boldsymbol{X}_{m-1})$. Clearly the map

$$(\varpi,p),(\dot\varpi,\dot p)) \mapsto D_2G((x,\varpi,p),\dot\varpi) + D_3G((x,\varpi,p),\dot p) \tag{E.29}$$

has the desired continuity properties.

It thus suffices to verify the following continuity property of D_1G^{-1} for any n with $0 \le n \le m$:

(E.30) Whenever $(x_j,\varpi_j,p_j,y_j) \to (x,\varpi,p,y)$ in $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times \boldsymbol{X}_n$ then $D_1G(x_j,\varpi_j,p_j)^{-1}y_j \to D_1G(x,\varpi,p)^{-1}y$ in $\boldsymbol{X}_n$.

This would be obvious if were able to assume that $(x,\varpi,p) \to D_1G(x,\varpi,p)$ is continuous as a map with values in $L(\boldsymbol{X}_n,\boldsymbol{X}_n)$. However, we only have continuity of $(x, \varpi, p, \dot{x}) \to D_1G((x, \varpi, p), \dot{x})$ as a map from $\widetilde{\mathcal{U}} \times \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times \boldsymbol{X}_n$ to $\boldsymbol{X}_n$. To show that (E.30) holds under this weaker assumption let $z := D_1G(x, \varpi, p)^{-1}y$ and $z_j := D_1G(x_j, \varpi_j, p_j)^{-1}y_j$. Then

$$D_1G((x_j, \varpi_j, p_j), z_j - z) = (y_j - y) - (D_1G((x_j, \varpi_j, p_j), z) - y) \to 0 \quad \text{in } \boldsymbol{X}_n. \tag{E.31}$$

Since $\|D_1G(x_j, \varpi_j, p_j)^{-1}\|_{L(\boldsymbol{X}_n, \boldsymbol{X}_n)} \leq 2/(1-\gamma)$ it follows that $z_j \to z$ in $\boldsymbol{X}_n$.

Step 4. Higher Peano derivatives and proof of (E.6).

Let $2 \leq k \leq m$. Employing Proposition D.7 again, we will prove that $f \in C^k_*(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_{m-k})$ by showing that $f : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \to \boldsymbol{X}_{m-k}$ has continuous Peano derivatives up to order k. As before $(\varpi, p) \in \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}$ and for sufficiently small t let $\xi(t) := f(s + t\dot{\varpi}, p + t\dot{p}) - f(\varpi, p)$. We will show by induction that $\xi(t)$ is Peano differentiable at 0 and that the Peano derivatives up to order k can be computed by expanding the identity

$$0 = G(x + \xi(t), \varpi + t\dot{\varpi}, p + t\dot{p})), \quad \text{where } x = f(\varpi, p), \tag{E.32}$$

to order k in t.

Define $f^{(1)}$ by (E.21). For $k \geq 2$ define inductively $R_k = R_k(t) = R_k(t, \varpi, p, \dot{\varpi}, \dot{p})$ and $f^{(k)} = f^{(k)}(\varpi, p, \dot{\varpi}, \dot{p})$ as follows,

$$\begin{aligned} R_k(t) :=& \\ &\sum_{\substack{j'+j''+\ell \leq k \\ j''+\ell \geq 1}} \tfrac{1}{j'!\,j''!\,\ell!} D_1^{j'} D_2^{j''} D_3^{\ell} G\left((x, \varpi, p), \left(\sum_{q=1}^{k-\ell-j''} \frac{f^{(q)}}{q!} t^q\right)^{j'}, \dot{\varpi}^{j''}, \dot{p}^{\ell}) t^{j''+\ell}\right) + \\ &\qquad + \sum_{2 \leq j' \leq k} \tfrac{1}{j'!} D_1^{j'} G\left((x, \varpi, p), \left(\sum_{q=1}^{k-1} \frac{f^{(q)}}{q!} t^q\right)^{j'}\right). \end{aligned} \tag{E.33}$$

Note that R_k is a polynomial in t. We use $R_k^{(j)}$ to denote its j-th order derivative at $t = 0$, i.e., $R_k^{(j)}/j!$ is the coefficient of t^j in the polynomial R_k. Also, notice that in the right hand side of the equation above, only terms $f^{(q)}$ of the order $q \leq k-1$ occur. Note also that $R_k(t)$ contains all the terms of order t^j with $j \leq k$ of the joint Taylor expansion of G and $\xi(t)$ except for the term $D_1G(x, \varpi, p, \xi(t))$. Thus looking on the coefficients of t^k it is natural to define

$$f^{(k)} := -D_1G(x, \varpi, p)^{-1} R_k^{(k)}, \tag{E.34}$$

i.e., $f^{(k)}$ is the unique solution of the linear equation $D_1G(x, \varpi, p, \dot{x}) + R_k^{(k)} = 0$ (we will see below that $R_k^{(k)} \in \boldsymbol{X}_{m-k}$ and that this equation has indeed a unique solution in $\boldsymbol{X}_{m-k}$).

For $k \leq m$, we will prove by induction that

$$f^{(k)} \in \boldsymbol{X}_{m-k} \tag{E.35}$$

and that $f^{(k)}$ is the sought Peano derivative since

$$\left\| \xi(t) - \sum_{q=1}^{k} \frac{f^{(q)}}{q!} t^q \right\|_{\boldsymbol{X}_{m-k}} = o(t^k). \tag{E.36}$$

For $k = 1$ the definitions of $R_1^{(1)}$ and $f^{(1)}$ agree with those given in Step 3. The claims (E.35) and (E.36) for $k = 1$ were also established in Step 3.

Assume now that (E.35) and (E.36) hold for $k-1$ and that $k \leq m$. Then it is easy to see that for all t we have $R_k(t) \in \boldsymbol{X}_{m-k}$ and in particular $R_k^{(k)} \in \boldsymbol{X}_{m-k}$. Indeed, if $\ell + j'' \geq 1$ then $\sum_{q=1}^{k-\ell-j''} \frac{f^{(q)}}{q!} t^q \in \boldsymbol{X}_{m-k+\ell}$ and, since

$$\text{(E.37)} \qquad D_1^{j'} D_2^{j''} D_3^{\ell} G \quad \text{maps} \quad \mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \boldsymbol{X}_{m-k+\ell}^{j'} \times \boldsymbol{E}^{j''} \times \boldsymbol{P}^{\ell} \text{ to } \boldsymbol{X}_{m-k},$$

the first sum in the definition of $R_k(t)$ is in $\boldsymbol{X}_{m-k}$. If $\ell = j'' = 0$, then $\sum_{q=1}^{k-1} \frac{f^{(q)}}{q!} t^q \in \boldsymbol{X}_{m-k+1}$ which is mapped by $D_1^{j'} G(x, \varpi, p)$ into $\boldsymbol{X}_{m-k+1}$ implying that the second sum in the definition of $R_k(t)$ is contained in $\boldsymbol{X}_{m-k+1} \subset \boldsymbol{X}_{m-k}$. We have seen in Step 3 that the map $\dot{x} \mapsto D_1 G((x, \varpi, p), \dot{x})$ is bounded and invertible as a map from $\boldsymbol{X}_n$ to $\boldsymbol{X}_n$ for all $0 \leq n \leq m$. Hence, the definition (E.34) implies that $f^{(k)}$ is well defined and lies in $\boldsymbol{X}_{m-k}$.

To prove (E.36), we first define

$$\text{(E.38)} \quad \widetilde{R}_k(t) := \sum_{\substack{j'+j''+\ell \leq k \\ j''+\ell \geq 1}} \tfrac{1}{j'!\,j''!\,\ell!} D_1^{j'} D_2^{j''} D_3^{\ell}((x, \varpi, p), \xi(t)^{j'}, \dot{\varpi}^{j''}, \dot{p}^{\ell}) t^{j''+\ell} + \\ + \sum_{2 \leq j' \leq k} \tfrac{1}{j'!} D_1^{j'} G((x, \varpi, p), \xi(t)^{j'}).$$

Similar to the estimate for the first derivative, it follows from Lemma D.27, Lemma D.6 and Proposition D.7 (c.f. also Lemma D.13) that

(E.39)

$$\Big\| \underbrace{G(x + \xi(t), \varpi + t\dot{\varpi}, p + t\dot{p})}_{=0} - \underbrace{G(x, \varpi, p)}_{=0} - D_1 G((x, \varpi, p), \xi(t)) - \widetilde{R}_k(t) \Big\|_{\boldsymbol{X}_{m-k}} \leq$$

$$\leq \sup_{\tau \in [0,1]} \Big\| \sum_{j''+\ell = k} \tfrac{1}{j''!\,\ell!} \Big(D_2^{j''} D_3^{\ell} G((x + \tau\xi(t), \varpi + \tau t\dot{\varpi}, p + \tau t\dot{p}), \dot{\varpi}^{j''}, \dot{p}^{\ell}) - \\ - D_2^{j''} D_3^{\ell} G((x, \varpi, p), \dot{\varpi}^{j''}, \dot{p}^{\ell}) \Big) \Big\|_{\boldsymbol{X}_{m-k}} t^k$$

$$+ \sup_{\tau \in [0,1]} \Big\| \sum_{\substack{j'+j''+\ell = k \\ j' \geq 1}} \tfrac{1}{j'!\,j''!\,\ell!} \Big(D_1^{j'} D_2^{j''} D_3^{\ell} G((x, +\tau\xi(t), \varpi + \tau t\dot{\varpi}, p + \tau t\dot{p}), (\tfrac{\xi(t)}{t})^{j'}, \dot{\varpi}^{j''}, \dot{p}^{\ell}) - \\ - D_1^{j'} D_2^{j''} D_3^{\ell} G((x, \varpi, p), (\tfrac{\xi(t)}{t})^{j'}, \dot{\varpi}^{j''}, \dot{p}^{\ell}) \Big) \Big\|_{\boldsymbol{X}_{m-k}} t^k$$

The first term on the right hand side is $o(t^k)$ since $D_2^{j''} D_3^{\ell} G$ is continuous in all of its arguments and since $\xi(t) \to 0$ in $\boldsymbol{X}_m$. For the second term we use that $\ell \leq k-1$ since $j' \geq 1$ and that, as proven in the Step 3, the function $\xi(t)/t$ converges to $f^{(1)}$ in $\boldsymbol{X}_{m-1}$. As a result, observing that $D_1^{j'} D_2^{j''} D_3^{\ell}$ is a continuous map from $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \times \boldsymbol{X}_{m-1}^{j'} \times E^{j''} \times \boldsymbol{P}^{\ell}$ to $\boldsymbol{X}_{m-1-\ell} \hookrightarrow \boldsymbol{X}_{m-k}$, the second term is also $o(t^k)$. In summary,

$$\text{(E.40)} \qquad \| D_1 G((x, \varpi, p), \xi(t)) + \widetilde{R}_k(t) \|_{\boldsymbol{X}_{m-k}} = o(t^k).$$

Combining the induction assumption,

$$\text{(E.41)} \qquad \Big\| \xi(t) - \sum_{q=1}^{k-j''-\ell} \frac{f^{(q)}}{q!} t^q \Big\|_{\boldsymbol{X}_{m-k+\ell+j''}} = o(t^{k-j''-\ell})$$

valid for any $j'' + \ell \geq 1$ with the estimate $\|\sum_{q=1}^{k-j''-\ell} \frac{f^{(q)}}{q!} t^q \|_{\boldsymbol{X}_{m-k+\ell+j''}} \leq 3Ct$ which follows from (E.41) and the bound $\|\xi(t)\|_{X_{m-1}} \leq Ct$ proven in Step 3, we can evaluate every term occurring in $R_k - \widetilde{R}_k$. Namely, we bound

(E.42)
$$\left\| D_1^{j_1'+j_2'} D_2^{j''} D_3^{\ell} G((x,\varpi,p), \Big(\sum_{q=1}^{k-\ell-j''} \tfrac{f^{(q)}}{q!} t^q - \xi(t)\Big)^{j_1'}, \xi(t)^{j_2'}, \dot{\varpi}^{j''}, \dot{p}^{\ell}) t^{j''+\ell} \right\|_{\boldsymbol{X}_{m-k}} = o(t^k).$$

Here we took into account that the difference $R_k - \widetilde{R}_k$ contains only terms with $j_1' \geq 1$ implying that $o((t^{k-j''-\ell})^{j_1'}) t^{j''+\ell} t^{j_2'} = o(t^k)$ since $(k-j''-\ell)j_1' + j'' + \ell + j_2' \geq k + (j_1'-1)(k-j''-\ell) + j_2' \geq k$. Similarly for the remaining terms,

$$\left\| D_1^{j_1'+j_2'} G((x,\varpi,p), \Big(\sum_{q=1}^{k-1} \tfrac{f^{(q)}}{q!} t^q - \xi(t)\Big)^{j_1'}, \xi(t)^{j_2'}) \right\|_{\boldsymbol{X}_{m-k}} = o(t^k) \tag{E.43}$$

since $j_1' \geq 1$ and $j_1' + j_2' \geq 2$ and thus $o((t^{k-1})^{j_1'}) t^{j_2'} = o(t^k) t^{(k-1)(j_1'-1)+j_2'-1} = o(t^k)$.

As a result, we can conclude that

$$\|R_k(t) - \widetilde{R}_k(t)\|_{\boldsymbol{X}_{m-k}} = o(t^k) \tag{E.44}$$

and thus

$$\|D_1 G((x,\varpi,p), \xi(t)) + R_k(t)\|_{\boldsymbol{X}_{m-k}} = o(t^k). \tag{E.45}$$

Moreover one can easily check that for any $q \leq k$

$$\|R_q(t) - R_k(t)\|_{\boldsymbol{X}_{m-k}} = o(t^q) \tag{E.46}$$

and thus the derivatives of order q at 0 satisfy $R_q^{(q)} = R_k^{(q)}$. Now the definition of $f^{(q)}$ for $q \leq k$ implies that

$$D_1 G((x,\varpi,p), f^{(q)}) = -R_q^{(q)} = -R_k^{(q)}. \tag{E.47}$$

Thus

$$\|D_1 G((x,\varpi,p), \sum_{q=1}^{k} \frac{f^{(q)}}{q!} t^q) + R_k(t)\|_{\boldsymbol{X}_{m-k}} = o(t^k) \tag{E.48}$$

since R_k is a polynomial with values in $\boldsymbol{X}_{m-k}$. Comparison with (E.45) yields

$$\|D_1 G((x,\varpi,p), \xi(t) - \sum_{q=1}^{k} \frac{f^{(q)}}{q!} t^q)\|_{\boldsymbol{X}_{m-k}} = o(t^k) \tag{E.49}$$

and this implies the claim (E.36) since $\dot{x} \mapsto G((x,\varpi,p),\dot{x})$ is a bounded and invertible map from $\boldsymbol{X}_{m-k}$ to itself.

We have thus shown that for any $n \leq m$ the map $f : \mathcal{V} \times \mathcal{W} \to \boldsymbol{X}_{m-n}$ has Peano derivatives for any $k \leq n$ given by

$$f^{(k)}((\varpi,p),(\dot{\varpi},\dot{p})) = f^{(k)}, \tag{E.50}$$

where $f^{(k)}$ is inductively defined by (E.33) and (E.34) with $x = f(\varpi,p)$. It follows by induction that the maps

$$(\varpi, p, (\dot{\varpi},\dot{p})) \mapsto R_k^{(k)}, \tag{E.51}$$

$$(\varpi, p, (\dot{\varpi},\dot{p})) \mapsto f^{(k)} \tag{E.52}$$

are continuous as maps from $\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times E \times \boldsymbol{P}$ to $\boldsymbol{X}_{m-n}$ (here we use again (E.30)).

Thus $f^{(n)}$ exists and is continuous on $(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_{m-n})$. By Proposition D.7, the existence and continuity of Peano derivatives $f^{(n)}$ thus finally implies that $f \in C^n_*(\widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}}, \boldsymbol{X}_{m-n})$ for all $n \leq m$.

Step 5. Improved estimates for $D_1^j D_2^\ell f$ and proof of (E.7).
For $j = 0$ there is nothing to show since $D_2^l f(\varpi, p, \dot{p}^\ell) = f^{(l)}(\varpi, p, 0, \dot{p})$ and thus (E.7) follows from (E.6). For $j \geq 1$ set

$$n := j + \ell$$

and note that

$$\frac{1}{n!} f^{(n)}(\varpi, p, \dot{\varpi}, s\dot{p}) = \sum_{l=0}^{n} s^l \frac{1}{j!} \frac{1}{\ell!} D_1^j D_2^l f(\varpi, p, \dot{\varpi}^j, \dot{p}^\ell) \tag{E.53}$$

Thus, up to a constant factor, $D_1^j D_2^\ell f$ is given by the coefficient of s^l in the polynomial $s \mapsto f^{(n)}(\varpi, p, \dot{\varpi}, s\dot{p})$. Using this observation we will now prove (E.7) by induction over n.

For $n = 1$ the assertion follows directly from (E.28).

Assume the assertion has been shown for $j + l \leq n - 1$ (where $n \leq m$). We will show the assertion for $j + l = n$. In view of (E.34) it suffices to show the following: If $R^{(n)}_{n,l}(\varpi, p, \dot{\varpi}, \dot{p})$ is the coefficient of s^l in the polynomial

$$h(s) := R^{(n)}_n(\varpi, p, \dot{\varpi}, s\dot{p})$$

then

$$R^{(n)}_{n,l} : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times E \times P \to X_{m-l} \quad \text{is continuous.}$$

To see this note that $h(s)$ is a weighted sum of terms of the form

$$D_1^{j'} D_2^{j''} D_3^{\ell'} F(x, \varpi, p, f^{(q_1)}, \ldots, f^{(q_{j'})}, \dot{\varpi}^{j''}, \dot{p}^{\ell'})\ s^{\ell'}$$

with $f^{(q_i)} = f^{(q_i)}(\varpi, p, \dot{\varpi}, s\dot{p})$ and terms of the form

$$D_1^{j'} F(x, \varpi, p, f^{(q_1)}, \ldots, f^{(q_{j'})}).$$

Using (E.53) we see that $R^{(n)}_{n,l}$ is a weighted sum of terms

$$T_1 := D_1^{j'} D_2^{j''} D_3^{\ell'} F(x, \varpi, p, D_1^{a_1} D_2^{\ell_1} f, \ldots, D_1^{a_{j'}} D_2^{\ell_{j'}} f, \dot{\varpi}^{j''}, \dot{p}^{\ell'}) \quad \text{with } \ell_i \leq \ell - \ell'$$

and of terms

$$T_2 := D_1^{j'} F(x, \varpi, p, D_1^{a_1} D_2^{\ell_1} f, \ldots, D_1^{a_{j'}} D_2^{\ell_{j'}} f) \quad \text{with } q_i \leq \ell$$

where

$$D_1^{a_i} D_2^{\ell_i} f = D_1^{a_i} D_2^{\ell_i} f(\varpi, p, \dot{\varpi}^{a_i}, \dot{p}^{l_i}).$$

Now by induction assumption

$$D_1^{a_i} D_2^{\ell_i} f : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times E^{a_i} \times P^{l_i} \to X_{m-(\ell-\ell')}$$

is continuous if $\ell_i \leq \ell - \ell'$. Thus $T_1 : \widetilde{\mathcal{V}} \times \widetilde{\mathcal{W}} \times E \times P \to X_{m-\ell}$ is continuous. Similarly one shows continuity of T_2.

Step 6. Proof of (E.8).
This is proved by induction over $n = j + l$ very similar to Step 5.

□

APPENDIX F

Geometry of Course Graining

We will use two combinatorial lemmas (Lemma 6.15 and 6.16 from [**Bry09**]) proven by Brydges that are for completeness summarised below.

Lemma F.1. *Let $X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k$. Then*

$$|X|_k \geq (1+2\alpha(d))|\overline{X}|_{k+1} \quad \text{with} \quad \alpha(d) = \tfrac{1}{(1+2^d)(1+6^d)}. \tag{F.1}$$

For any $X \in \mathcal{P}_k$ we have

$$|X|_k \geq (1+\alpha(d))|\overline{X}|_{k+1} - (1+\alpha(d))2^{d+1}|\mathcal{C}(X)| \quad \text{with} \quad \alpha(d) = \tfrac{1}{(1+2^d)(1+6^d)}. \tag{F.2}$$

Lemma F.2. *There exist $\delta = \delta(d, L) < 1$ such that*

$$\sum_{\substack{X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ \overline{X} = U}} \delta^{|X|_k} \leq 1 \tag{F.3}$$

for any $k \in \mathbb{N}$ and any $U \in \mathcal{P}_{k+1}^{\mathrm{c}}$.

Proof. For any X contributing to the sum we have $|X|_k \geq (1+2\alpha(d))|\overline{X}|_{k+1}$ and thus

$$\sum_{\substack{X \in \mathcal{P}_k^{\mathrm{c}} \setminus \mathcal{S}_k \\ \overline{X} = U}} \delta^{|X|_k} \leq 2^{L^d |U|_{k+1}} \delta^{(1+2\alpha(d))|U|_{k+1}} \leq 1 \tag{F.4}$$

once $\delta \leq 2^{-\frac{L^d}{1+2\alpha(d)}}$. □

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List of Symbols

$\mathcal{A}^{(\boldsymbol{q})}$	$= \sum_{i,j=1}^{d} \big(\delta_{i,j} + q_{i,j}\big) \nabla_i^* \nabla_j$, page 19
$\boldsymbol{A}_k^{(\boldsymbol{q})}$	$(\boldsymbol{A}_k^{(\boldsymbol{q})} \dot{H}, 0) = D\boldsymbol{T}_k(0,0,\boldsymbol{q})(\dot{H},0)$, linearisation of $\boldsymbol{T}_k(\cdot,\cdot,\boldsymbol{q})$ at $(0,0)$, $(\boldsymbol{A}_k^{(\boldsymbol{q})}\dot{H})(B',\varphi) = \sum_{B\in\mathcal{B}_k(B')} \big[\dot{H}(B,\varphi) + \sum_{x\in B}\sum_{i,j=1}^{d} \dot{d}_{i,j} \nabla_i \nabla_j^* \mathcal{C}_{k+1}^{(\boldsymbol{q})}(0)\big]$, page 35
α	a parameter in the norm $\\|\cdot\\|_{\boldsymbol{Y}_r}$, page 38
$\boldsymbol{\alpha}$	$= (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}, i = 1, \dots, d$, a multiindex, page 10
$\alpha(d)$	$= \frac{1}{(1+2^d)(1+6^d)}$ from the bound $\|X\|_k \geq (1+\alpha(d))\|\overline{X}\|_{k+1} - (1+\alpha(d))2^{d+1}\|\mathcal{C}(X)\|$ for any $X \in \mathcal{P}_k$, page 71
$\alpha(d)$	$= \frac{1}{(1+2^d)(1+6^d)}$ from the bound $\|X\|_k \geq (1+\alpha(d))\|\overline{X}\|_{k+1} - (1+\alpha(d))2^{d+1}\|\mathcal{C}(X)\|$ for any $X \in \mathcal{P}_k$, page 141
$\|\boldsymbol{\alpha}\|$	$= \sum_{i=1}^{d} \alpha_i$ (for a multiindex $\boldsymbol{\alpha}$), page 10
$B_\delta(0)$	$= \{u \in \mathbb{R}^d \mid \|u\| < \delta\}$, page 10
B_n	$= [0,n]^d \cap \mathbb{Z}^d$, page 99
B_{r_0}	$\leq r_0^{r_0}$, bound on the number of partitions (for Bruno di Faà formula), page 57
B_x	the k-block containing x, page 27
B^*	= the cube of the side $(2^{d+1}-1)L^k$ centered at B, the small set neighbourhood of B, page 25
$\boldsymbol{B}_k^{(\boldsymbol{q})}$	$(\boldsymbol{B}_k^{(\boldsymbol{q})} \dot{K}, 0) = D\boldsymbol{T}_k(0,0,\boldsymbol{q})(0,\dot{K})$, linearisation of $\boldsymbol{T}_k(\cdot,\cdot,\boldsymbol{q})$ at $(0,0)$, $(\boldsymbol{B}_k^{(\boldsymbol{q})}\dot{K})(B',\varphi) = -\sum_{B\in\mathcal{B}_k(B')} \Pi_2 \sum_{\substack{X\in\mathcal{S}_k \\ X\supset B}} \frac{1}{\|X\|_k} \Big(\int_{\boldsymbol{\mathcal{X}}} \dot{K}(X,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\Big)$, page 35
$\mathcal{B}_k = \mathcal{B}_k(\Lambda_N)$	the set of all k-blocks in Λ_N, page 25
$\mathcal{B}_k(X)$	the set of k-blocks in X, page 25
$\boldsymbol{\mathcal{B}}_{\boldsymbol{\mathcal{X}}_N}$	σ-algebra on $\boldsymbol{\mathcal{X}}_N$ induced by the Borel σ-algebra with respect to the product topology, page 7
β	inverse temperature, page 8
$c_{\boldsymbol{\alpha},a}$	coefficients in bounds of derivatives of finite range covariance function, page 23
$\mathcal{C}^{(\boldsymbol{q})}$	the inverse of the operator $\mathcal{A}^{(\boldsymbol{q})}$, page 19
$\mathcal{C}_k^{(\boldsymbol{q})}$	finite range covariance operator, page 23
$\boldsymbol{C}_k^{(\boldsymbol{q})}$	$(0, \boldsymbol{C}_k^{(\boldsymbol{q})}\dot{K}) = D\boldsymbol{T}_k(0,0,\boldsymbol{q})(0,\dot{K})$, linearisation of $\boldsymbol{T}_k(\cdot,\cdot\boldsymbol{q})$ at $(0,0)$, $\boldsymbol{C}_k^{(\boldsymbol{q})}(\dot{K})(U,\varphi) = \sum_{B:\overline{B^*}=U} \big(1-\Pi_2\big) \sum_{\substack{Y\in\mathcal{S}_k \\ Y\supset B}} \frac{1}{\|Y\|} \Big(\int_{\boldsymbol{\mathcal{X}}} \dot{K}(Y,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)\Big) + {} + \sum_{\substack{X\in\mathcal{P}_k^c\setminus\mathcal{S}_k \\ \overline{X}=U}} \int_{\boldsymbol{\mathcal{X}}} \dot{K}(X,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)$, page 35

$\widetilde{C}^m(\mathcal{U}\times\mathcal{V})$	the class of functions $G\colon \mathcal{U}\times\mathcal{V}\to \boldsymbol{M}'$ for which the derivative $D_1^{j'}D_2^{j''}D_3^{\ell}G$ is a continuous map $\mathcal{U}\times\mathcal{V}\times\boldsymbol{M}_0^{j''}\times\widetilde{\boldsymbol{M}}_r^{j'}\times(\mathbb{R}^{d\times d}_{\mathrm{sym}})^{\ell}\to \boldsymbol{M}'_{r-2\ell}$, page 34
$\mathcal{C}_k^{(\boldsymbol{q})}$	finite range covariance function, page 23
$\widehat{\mathcal{C}}_k^{(\boldsymbol{q})}(p)$	discrete Fourier transform of the kernel $\mathcal{C}_k^{(\boldsymbol{q})}$, page 24
$\mathcal{C}(X)$	the set of all connected components of X, page 25
$\mathfrak{c}$	$< 3\sqrt{2}$, the constant from the bound $v(-m)^2+v(m+1)^2\le \frac{\mathfrak{c}}{2m+1}\sum_{x=-m}^{m}v(x)^2+\mathfrak{c}(2m+1)\sum_{x=-m}^{m}\partial v(x)^2$, page 103
$\mathfrak{C}$	$=\mathfrak{C}(p,M,m)$, the constant from the discrete Sobolev estimates, e.g., $\max_{x\in B_n}\lvert f(x)\rvert\le \mathfrak{C}n^{-\frac{d}{2}}\sum_{k=0}^{M}\lVert(n\nabla)^k f\rVert_2$, page 99
$\partial^{\boldsymbol{\alpha}}$	$=\prod_{i=1}^{d}\partial_i^{\alpha_i}$, page 10
e_i	unit coordinate vectors in $\mathbb{R}^d$, page 7
E	the map $E\colon(\boldsymbol{M}_0,\lVert\cdot\rVert_{k,0})\to(\boldsymbol{M}_{\vert\vert\vert},\vert\vert\vert\cdot\vert\vert\vert_{k,\cdot})$ defined by $E(H)(B,\varphi)=\exp\{-H(B,\varphi)\}$, page 54
$\boldsymbol{E}$	the Banach space with the norm $\lVert\cdot\rVert_\zeta$, page 10
$\mathbb{E}_k$	expectation with respect to $\mu_k=\mu_{\mathcal{C}_k^{(\boldsymbol{q})}}$, page 24
$\mathcal{E}_N(\varphi)$	$=\frac{1}{2}\sum_{x\in\mathbb{T}_N}\sum_{i=1}^{d}\big(\nabla_i\varphi(x)\big)^2$, page 7
$\mathcal{E}_{\boldsymbol{q}}(\varphi)$	$=\frac{1}{2}(\mathcal{A}^{(\boldsymbol{q})}\varphi,\varphi)=\frac{1}{2}\sum_{x\in\mathbb{T}_N}\sum_{i,j=1}^{d}\big(\delta_{i,j}+q_{i,j}\big)\nabla_i\varphi(x)\nabla_j\varphi(x)$, page 20
$f_p(x)$	$=L^{-dN/2}e^{i\langle p,x\rangle}$, Fourier basis functions, page 23
$F(X)(\varphi)$	$=F(X,\varphi)$ for $F\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$, page 27
$F^X(\varphi)$	$=\prod_{B\in\mathcal{B}_k(X)}F(B,\varphi)$, page 26
$F(X,\varphi)$	$=F^X(\varphi)$ for $F\in M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$, page 26
$\mathcal{F}_1$	ideal Hamiltonian map, page 38
$\mathcal{F}_{2N}$	irrelevant term of the solution map, page 38
$g_{k,x}(\varphi)$	$=\frac{1}{h^2}\sum_{s=2}^{4}L^{(2s-2)k}\sup_{y\in B_x^*}\lvert\nabla^s\varphi(y)\rvert^2$, page 28
$g_{k:k+1,x}(\varphi)$	$=\frac{1}{h^2}\sum_{s=2}^{4}L^{(2s-2)(k+1)}\sup_{y\in B_x^*}\lvert\nabla^s\varphi(y)\rvert^2$, page 28
$G_{k,x}(\varphi)$	$=\frac{1}{h^2}\big(\lvert\nabla^2\varphi(x)\rvert^2+L^{2k}\lvert\nabla^2\varphi(x)\rvert^2+L^{4k}\lvert\nabla^3\varphi(x)\rvert^2\big)$, page 27
$\gamma_{N,\beta}^u(\mathrm{d}\varphi)$	$=\frac{1}{Z_{N,\beta}(u)}\exp\big(-\beta H_N^u(\varphi)\big)\lambda_N(\mathrm{d}\varphi)$, random gradient field with Hamiltonian H_N^u (with tilt u), page 8
$\Gamma_{k,\mathsf{A}}(X)$	$=\begin{cases}\mathsf{A}^{\lvert X\rvert} & \text{if } X\in\mathcal{P}_k^{\mathrm{c}}\setminus\mathcal{S}_k\\ 1 & \text{if } X\in\mathcal{S}_k.\end{cases}$, page 28
h	a parameter in the norms $\vert\vert\vert\cdot\vert\vert\vert_{k,X}$ or $\lVert\cdot\rVert_{k,X,r}$ (via the weight functions $G_{k,x}$ and $g_{k,x}$), page 28
$H(B,\varphi)$	ideal Hamiltonian of the form $H(B,\varphi)=\lambda\lvert B\rvert+\ell(\varphi)+Q(\varphi)$, page 26
$H_{k+1}(B',\varphi)$	$=\sum_{B\in\mathcal{B}_k(B')}\Pi_2\Big((\boldsymbol{R}_{k+1}H_k)(B,\varphi)-\sum_{\substack{X\in\mathcal{S}^{(k)}\\ X\supset B}}\frac{1}{\lvert X\rvert_k}(\boldsymbol{R}_{k+1}K_k)(X,\varphi)\Big)$, page 32
$\overline{H}_k$	$=\boldsymbol{A}_k^{-1}\big(H_{k+1}-\boldsymbol{B}_kK_k\big)$, page 38
$\widetilde{H}_k(B,\varphi)$	$=\Pi_2\Big((\boldsymbol{R}_{k+1}H_k)(B,\varphi)-\sum_{\substack{X\in\mathcal{S}^{(k)}\\ X\supset B}}\frac{1}{\lvert X\rvert_k}(\boldsymbol{R}_{k+1}K_k)(X,\varphi)\Big)$, page 32
$H_N(\varphi)$	$=\mathcal{E}_N(\varphi)+\sum_{x\in\mathbb{T}_N}\sum_{i=1}^{d}V(\nabla_i\varphi(x))$, Hamiltonian on $\mathbb{T}_N$ (with no tilt), page 7

$H_N^u(\varphi)$	$= \mathcal{E}_N(\varphi) + \frac{1}{2}L^{Nd}\lvert u\rvert^2 + \sum_{x\in\mathbb{T}_N}\sum_{i=1}^d V(\nabla_i\varphi(x)-u_i)$, Hamiltonian on $\mathbb{T}_N$ with tilt u, page 8
$\mathscr{H}$	the initial Hamiltonian map in Theorem 4.9, page 39
$\mathcal{H}(x,\varphi)$	initial ideal Hamiltonian, page 37
η	a parameter in the norm $\lVert\cdot\rVert_{\boldsymbol{Y}_r}$, page 38
$\eta=(\eta_i)$	$\in\mathbb{R}^d$, page 67
$\eta_{i,j}$	coefficients of a quadratic test function $\dot\varphi(x) = \frac{1}{2}\sum_{i,j=1}^d \eta_{i,j}(x-\overline{x})_i(x-\overline{x})_j$, page 67
$\mathsf{\eta}(d)$	$= \mathsf{\eta}(2\lfloor\frac{d+2}{2}\rfloor+8,d)$, page 48
$\mathsf{\eta}(n,d)$	$= \max(\frac{1}{4}(d+n-1)^2, d+n+6)+10$, the decay exponent in finite range decomposition, page 23
θ	a contractivity constant for operator $\boldsymbol{C}_k^{(\boldsymbol{q})}$, page 36
$\chi(X,U)$	$= \begin{cases}\frac{\lvert\{B\in\mathcal{B}_k(X):\,\overline{B^*}=U\}\rvert}{\lvert X\rvert} \text{ if } X\in\mathcal{S}_k(\Lambda_N),\\ \mathbb{1}_{U=\overline{X}} \text{ if } X\in\mathcal{P}_k(\Lambda_N)\setminus\mathcal{S}_k(\Lambda_N),\end{cases}$ for any connected $U\in\mathcal{P}_{k+1}$, page 31
$I_k(B,\varphi)$	$= \exp\{-H_k(B,\varphi)\}$, page 30
$\tilde I_k(B,\varphi)$	$= \exp\{-\widetilde H_k(B,\varphi)\}$, page 30
$\tilde J_k(B,\varphi)$	$= 1-\tilde I(B,\varphi)$, page 30
$K_{k+1}(U,\varphi)$	$= \sum_{X\in\mathcal{P}_k(U)}\chi(X,U)\exp\{-\sum_{B\in\mathcal{B}_k(U\setminus X)}\widetilde H_k(B,\varphi)\}\int_{\boldsymbol{\mathcal{X}}}\widetilde K_k(X,\varphi,\xi)\mu_{k+1}(\mathrm{d}\xi)$, page 32
$\overline{K}_{k+1}$	$= S_k(H_k,K_k,\boldsymbol{q}) = \boldsymbol{C}_kK_k + S_k(H_k,K_k,\boldsymbol{q}) - D_2S_k((0,0,\boldsymbol{q}),K_k)$, page 38
$\mathcal{K}_{\kappa,p,u}(z)$	$= \prod_{i=1}^d\big[p+(1-p)\exp\{\frac{1}{2}(1-\kappa)(z_i-u_i)^2\}\big]-1$, Mayer function for the potential from [**BK07**], page 11
$\mathcal{K}^{(\boldsymbol{q})}(X,\varphi)$	$= \exp\Big\{\frac{1}{2}\sum_{x\in X}\sum_{i,j=1}^d q_{i,j}\nabla_i\varphi(x)\nabla_j\varphi(x)\Big\}\mathcal{K}(X,\varphi)$, page 20
$\mathcal{K}_u(X,\varphi)$	$= \prod_{x\in X}\mathcal{K}_u(\nabla\varphi(x))$ with a function $\mathcal{K}_u:\mathbb{R}^d\to\mathbb{R}$, page 9
$\mathcal{K}_{V,\beta,u}(z)$	$= \exp\{-\beta\sum_{i=1}^d U\big(\frac{z_i}{\sqrt\beta},u_i\big)\}-1$, the Mayer function for perturbation V, page 8
$\mathcal{K}_{V,\beta,u}(X,\varphi)$	$= \prod_{x\in X}\mathcal{K}_{V,\beta,u}(\nabla\varphi(x))$, page 9
$\widetilde K_k(X,\varphi,\xi)$	$= \sum_{Y\in\mathcal{P}_k(X)}(I_k(\varphi+\xi)-\tilde I_k(\varphi))^{X\setminus Y}(\varphi,\xi)K_k(Y,\varphi+\xi)$, page 30
κ	parameter in $\mathcal{K}_{\kappa,p,u}$, page 11
$\mathsf{\kappa}(d)$	$= \frac{1}{2}\big(d+\mathsf{\eta}(2\lfloor\frac{d+2}{2}\rfloor+8,d)\big)$, page 43
L	linear size of a renormalization block, page 7
$\ell(\varphi)$	$= \sum_{x\in B}\big[\sum_{i=1}^d a_i\nabla_i\varphi(x) + \sum_{i,j=1}^d \boldsymbol{c}_{i,j}\nabla_i\nabla_j\varphi(x)\big]$, linear term of ideal Hamiltonian, page 26
λ_N	$(L^{Nd}-1)$-dimensional Hausdorf measure on $\boldsymbol{\mathcal{X}}_N$, page 7
Λ_N	$= \{x\in\mathbb{Z}^d\colon \lvert x\rvert_\infty\le\frac{1}{2}(L^N-1)\}$ (identified with torus $\mathbb{T}_N$) , page 7
$M(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$	the set of all L^k-periodic maps $F:\mathcal{B}_k\times\boldsymbol{\mathcal{X}}\to\mathbb{R}$ such that $F(B,\cdot)\in M(\boldsymbol{\mathcal{X}},\nu_{k+1})$ for all $B\in\mathcal{B}_k$, page 26
$M^*(\mathcal{B}_k,\boldsymbol{\mathcal{X}})$	the set of all L^k-periodic maps $F:\mathcal{B}_k\times\boldsymbol{\mathcal{X}}\to\mathbb{R}$ such that $F(B,\cdot)\in M(\boldsymbol{\mathcal{X}},\nu_{k+1})$ for all $B\in\mathcal{B}_k$ living on $(B^*)^*$, page 26
$\widehat{\boldsymbol{M}}_r$	$\{K\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}}), \lVert K\rVert_{k,r}^{(\mathsf{A},\mathsf{B})}<\infty\}$, page 54
$\widehat{\boldsymbol{M}}_{:,r}$	$\{K\in M(\mathcal{P}_k,\boldsymbol{\mathcal{X}}), \lVert K\rVert_{k:k+1,r}^{(\mathsf{A},\mathsf{B})}<\infty\}$, page 54
$M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})$	the set of all L^k-periodic maps $F:\mathcal{P}_k\times\boldsymbol{\mathcal{X}}\to\mathbb{R}$ such that $F(X,\cdot)\in M(\boldsymbol{\mathcal{X}},\nu_{k+1})$ for all $X\in\mathcal{P}_k$, page 25

$M(\mathcal{S}_k, \boldsymbol{\mathcal{X}})$	the set of all L^k-periodic maps $F : \mathcal{S}_k \times \boldsymbol{\mathcal{X}} \to \mathbb{R}$ such that $F(X, \cdot) \in M(\boldsymbol{\mathcal{X}}, \nu_{k+1})$ for all $X \in \mathcal{S}_k$, page 26
$M(\boldsymbol{\mathcal{X}}_N)$	set of all functions on $\boldsymbol{\mathcal{X}}_N$ measurable with respect to λ_N, page 24
$M_0(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$	the set of all ideal Hamiltonians: quadratic functions of the form $H(B,\varphi) = \lambda\lvert B\rvert + \ell(\varphi) + Q(\varphi)$, page 26
$\mathcal{M}_1(\boldsymbol{\mathcal{X}}_N)$	$= \mathcal{M}_1(\boldsymbol{\mathcal{X}}_N, \boldsymbol{\mathcal{B}}_{\boldsymbol{\mathcal{X}}_N})$, the set of probability measures on $\boldsymbol{\mathcal{X}}_N$, page 7
$\mu^{(\boldsymbol{q})}(\mathrm{d}\varphi)$	$= \frac{1}{Z_N^{(\boldsymbol{q})}} \exp\bigl\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\bigr\}\lambda_N(\mathrm{d}\varphi)$, page 19
$\mu_k^{(\boldsymbol{q})}(\mathrm{d}\varphi)$	Gaussian measure with covariance $\mathcal{C}_k^{(\boldsymbol{q})}$, page 20
$\boldsymbol{M}_0$	$=\boldsymbol{M}_{k,0} = (M(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k,0})$, page 34
$\boldsymbol{M}_r$	$=\boldsymbol{M}_{k,r} = (M_r(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k,r}^{(\mathsf{A})})$, page 34
N	the power yielding the size (of the torus) L^N, page 7
$\nu(\mathrm{d}\varphi)$	$= \nu_{\beta=1}(\mathrm{d}\varphi)$, page 9
$\nu_\beta(\mathrm{d}\varphi)$	$= \frac{1}{Z_{N,\beta}^{(0)}} \exp\bigl(-\beta\mathcal{E}_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi)$, Gaussian measure on $\boldsymbol{\mathcal{X}}_N$, page 8
$\nu_k^{(\boldsymbol{q})}$	the measure on $\boldsymbol{\mathcal{X}}_N$ with covariance $\mathcal{C}_k^{(\boldsymbol{q})} + \cdots + \mathcal{C}_{N+1}^{(\boldsymbol{q})}$, page 24
$\nabla_i\varphi(x)$	$= \varphi(x+e_i) - \varphi(x)$, discrete derivative, page 7
$\nabla_i^*\varphi(x)$	$= \varphi(x-e_i) - \varphi(x)$, dual of discrete derivative ∇_i, page 7
$\lvert\nabla^s\varphi(x)\rvert^2$	$= \sum_{\lvert\boldsymbol{\alpha}\rvert=s} \lvert\nabla^{\boldsymbol{\alpha}}\varphi(x)\rvert^2$, page 27
p	$= (p_1, \ldots, p_d) \in \widehat{\mathbb{T}}_N$, dual variables, page 23
p	parameter in $\mathcal{K}_{\kappa,p,u}$ (replacing β), page 11
p_t	$= p_t(\kappa)$, corresponding phase transition value, page 12
P_1	$P_1(\widetilde{I}, \widetilde{J}, \widetilde{P})(U,\varphi) = \sum_{\substack{X_1, X_2 \in \mathcal{P}(U) \\ X_1 \cap X_2 = \varnothing}} \chi(X_1 \cup X_2, U)\widetilde{I}^{U\setminus(X_1\cup X_2)}(\varphi)\widetilde{J}^{X_1}(\varphi)\widetilde{P}(X_2,\varphi)$ mapping $(M(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), \lvert\lvert\lvert\cdot\rvert\rvert\rvert_k) \times (M(\mathcal{B}_k, \boldsymbol{\mathcal{X}}), \lvert\lvert\lvert\cdot\rvert\rvert\rvert_k) \times (M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k:k+1,r}^{(A/2)})$ into $(M((\mathcal{P}_{k+1})^{\mathrm{c}}, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k+1,r}^{(\mathsf{A})})$, page 54
P_2	$P_2(I,K) = (I-1) \circ K$ mapping , page 54
P_3	$(P_3K)(X,\varphi) = \prod_{Y \in \mathcal{C}(X)} K(Y,\varphi)$, page 54
π_i	the co-ordinate projection $\pi_i(x) = x_i$ for $x \in \mathbb{Z}^d$, page 67
Π_2	the projection from $M^*(\mathcal{B}, \boldsymbol{\mathcal{X}})$ to $M_0(\mathcal{B}, \boldsymbol{\mathcal{X}})$: $\Pi_2 F(B,\dot\varphi) = F(B,0) + \ell(\dot\varphi) + Q(\dot\varphi,\dot\varphi)$: ℓ agrees with $DF(B,0)$ on all quadratic functions $\dot\varphi$ on $(B^*)^*$ and Q agrees with $\frac{1}{2}D^2F(B,0)$ on all affine functions $\dot\varphi$ on $(B^*)^*$, page 29
$\mathcal{P}_k = \mathcal{P}_k(\Lambda_N)$	the set of all k-polymers in Λ_N, page 25
$\mathcal{P}_k(X)$	the set of all polymers Y consisting of subsets of blocks from $\mathcal{B}_k(X)$, page 25
$\mathcal{P}_k^{\mathrm{c}}$	the set of all connected k-polymers, page 25
$\boldsymbol{q}$	a symmetric $d \times d$-matrix, page 19
$\boldsymbol{q}(\mathcal{K}_u)$	the value of $\boldsymbol{q}$ yielding $H_N = 0$, page 21
$\lVert\boldsymbol{q}\rVert$	operator norm of $\boldsymbol{q}$ viewed as operator on $\mathbb{R}^d$ equipped with ℓ_2 metric, page 23
$Q(\varphi,\varphi)$	$= \frac{1}{2}\sum_{x\in B}\sum_{i,j=1}^d \boldsymbol{d}_{i,j}(\nabla_i\varphi)(x)(\nabla_j\varphi)(x)$, quadratic term of ideal Hamiltonian, page 26
r_0	a bound on the order of derivatives used in the norm $\lVert\cdot\rVert_\zeta$, page 10
R_1	$R_1(P,\boldsymbol{q})(X,\varphi) = (\boldsymbol{R}^{(\boldsymbol{q})}P)(X,\varphi) = \int_{\boldsymbol{\mathcal{X}}} P(X,\varphi+\xi)\mu_{k+1}^{(\boldsymbol{q})}(\mathrm{d}\xi)$ mapping $(M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k,r}^{(\mathsf{A})}) \times (\mathbb{R}_{\mathrm{sym}}^{d\times d}, \lVert\cdot\rVert)$ into $(M(\mathcal{P}_k^{\mathrm{c}}, \boldsymbol{\mathcal{X}}), \lVert\cdot\rVert_{k:k+1,r}^{(\mathsf{A})})$, page 54

R_2	$R_2(H,K,\boldsymbol{q})(B,\varphi)=\Pi_2\Big((\boldsymbol{R}^{(\boldsymbol{q})}H)(B,\varphi)-\sum_{X\in\mathcal{S} \atop X\supset B}\frac{1}{\lvert X\rvert}(\boldsymbol{R}^{(\boldsymbol{q})}K)(X,\varphi)\Big)$ mapping $(M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}}),\lVert\cdot\rVert_{k,0})\times(M(\mathcal{P}_k^{\rm c},\boldsymbol{\mathcal{X}}),\lVert\cdot\rVert_{k,r}^{(\mathsf{A})})\times(\mathbb{R}^{d\times d}_{\rm sym},\lVert\cdot\rVert)$ into $(M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}}),\lVert\cdot\rVert_{k,0})$, page 54
$\boldsymbol{R}_k$	renormalisation maps $(\boldsymbol{R}_kF)(\varphi)=\int_{\boldsymbol{\mathcal{X}}_N}F(\varphi+\xi)\mu_k^{(\boldsymbol{q})}(\mathrm{d}\xi)$, page 24
$\rho(x,y)$	$=\inf\{\lvert x-y+k\rvert_\infty : k\in(L^N\mathbb{Z})^d\}$, page 7
$\mathbb{R}^{d\times d}_{\rm sym}$	the set of symmetric $d\times d$-matrices, page 19
S_k	the map $S\colon M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_k^{\rm c},\boldsymbol{\mathcal{X}})\times\mathbb{R}^{d\times d}_{\rm sym}\to M((\mathcal{P}_{k+1})^{\rm c},\boldsymbol{\mathcal{X}})$ given by $S(H_k,K_k,\boldsymbol{q})=K_{k+1}$, page 33
S	the map S is composed as $S(H,K,\boldsymbol{q})=P_1\big(E(R_2(H,K,\boldsymbol{q})),1-E(R_2(H,K,\boldsymbol{q})),R_1(P_2(E(H),K),\boldsymbol{q})\big)$, page 54
$\sigma_\beta(u)$	$=-\lim_{N\to\infty}\frac{1}{\beta L^{dN}}\log Z_{N,\beta}(u)$, free energy (surface tension) with tilt u, page 8
$\varsigma(u)$	$=-\lim_{N\to\infty}\frac{1}{L^{dN}}\log\mathcal{Z}_N(u)$, the perturbative component of the surface tension, page 9
$\varsigma_N(u)$	$=-\frac{1}{L^{dN}}\log\mathcal{Z}_N(u)$, the finite volume perturbative component of the surface tension, page 19
$\mathcal{S}_k=\mathcal{S}_k(\Lambda_N)$	$=\{X\in\mathcal{P}_k^{\rm c}\colon\lvert X\rvert_k\le 2^d\}$, the set of small polymers, page 25
T_2	Taylor expansion up to the second order, $T_2F(B,\dot\varphi)=F(B,0)+DF(B,0)(\dot\varphi)+\frac12D^2F(B,0)(\dot\varphi,\dot\varphi)$, page 29
$\boldsymbol{T}_k$	map from $M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})\times\mathbb{R}^{d\times d}_{\rm sym}$ to $M_0(\mathcal{B}_{k+1},\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_{k+1},\boldsymbol{\mathcal{X}})$, $\boldsymbol{T}_k((H_k,K_k))=(H_{k+1},K_{k+1})$, page 32
$\boldsymbol{\mathcal{T}}$	the map from $\boldsymbol{Y}\times\boldsymbol{E}\times M_0$ to $\boldsymbol{Y}$, page 38
$\mathbb{T}_N$	$=\big(\mathbb{Z}/L^N\mathbb{Z}\big)^d$, torus, page 7
$\widehat{\mathbb{T}}_N$	$=\big\{p=(p_1,\dots,p_d):p_i\in\{-\frac{(L^N-1)\pi}{L^N},-\frac{(L^N-3)\pi}{L^N}\dots,0,\dots,\frac{(L^N-1)\pi}{L^N}\}\big\}$, dual torus, page 23
τ_a	a translation by a vector $a\in\mathbb{Z}^d$, page 25
$u=(u_1\dots,u_d)\in\mathbb{R}^d$	a tilt, page 7
$U(s,t)$	$=V(s-t)-V(-t)-V'(-t)s$, page 8
$\mathcal{U}_\rho$	$=\{(H,K)\in M_0(\mathcal{B}_k,\boldsymbol{\mathcal{X}})\times M(\mathcal{P}_k,\boldsymbol{\mathcal{X}})\colon\lVert H\rVert_{k,0}<\rho,\lVert K\rVert_{k,r_0}^{(\mathsf{A})}<\rho\}$, page 34
$V\colon\mathbb{R}\to\mathbb{R}$	potential perturbation, page 7
$\mathcal{V}$	$=\{\boldsymbol{q}\in\mathbb{R}^{d\times d}_{\rm sym}\colon\lVert\boldsymbol{q}\rVert<1/2\}$, page 34
$\boldsymbol{\mathcal{V}}_N$	$=\{\varphi:\mathbb{Z}^d\to\mathbb{R};\ \varphi(x+k)=\varphi(x)\ \forall k\in(L^N\mathbb{Z})^d\}$, set of fields taken as $\ell_2(\mathbb{R}^{L^{Nd}})$, page 7
$w_k^X(\varphi)$	$=\exp\Big\{\sum_{x\in X}\omega\big(2^dg_{k,x}(\varphi)+G_{k,x}(\varphi)\big)+L^k\sum_{x\in\partial X}G_{k,x}(\varphi)\Big\}$, the weak weight function, page 28
$w_{k:k+1}^X(\varphi)$	$=\exp\Big\{\sum_{x\in X}\big((2^d\omega-1)g_{k:k+1,x}(\varphi)+\omega G_{k,x}(\varphi)\big)+3L^j\sum_{x\in\partial X}G_{k,x}(\varphi)\Big\}$, the weak weight function, page 28
$W_k^X(\varphi)$	$=\exp\{\sum_{x\in X}G_{k,x}(\varphi)\}$ the strong weight function, page 27
ω	a parameter in the weightfunction $w_k^X(\varphi)$, page 28
∂X	$=\{y\notin X\vert\exists z\in X\text{ such that }\lvert y-z\rvert=1\}\cup\{y\in X\vert\exists z\notin X\text{ such that }\lvert y-z\rvert=1\}$, the boundary of X, page 27

$\overline{X}$	the closure of X: the smallest polymer $Y \in \mathcal{P}_{k+1}$ of the next generation such that $X \subset Y$, page 25
X^*	$= \cup\{B^*: B \in \mathcal{B}_k(X)\}$, the small set neighbourhood of X, page 25
$\lvert X\rvert_k$	$= \lvert\mathcal{B}_k(X)\rvert$, page 25
$\boldsymbol{\mathcal{X}}_N$	$= \{\varphi \in \boldsymbol{\mathcal{V}}_N : \sum_{x\in\mathbb{T}_N} \varphi(x) = 0\}$, page 7
ξ_k	a random field distributed according to $\mu_k = \mu_{\mathcal{C}_k^{(\boldsymbol{q})}}$, page 24
$\boldsymbol{y}$	an element $\boldsymbol{y} = (H_0, H_1, K_1, \ldots, H_{N-1}, K_{N-1}, K_N)$ of $\boldsymbol{Y}$, page 38
$\overline{\boldsymbol{y}}$	$= \boldsymbol{\mathcal{T}}(\boldsymbol{y}, K, \mathcal{H})$ the $2(N+1)$-tuple defined by $\overline{H}_k$ and $\overline{K}_k$, page 38
$Z_N^{(0)}$	$= Z_{N,\beta=1}^{(0)}$, page 9
$Z_{N,\beta}^{(0)}$	$= \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta\mathcal{E}_N(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi)$, page 8
$Z_{N,\beta}(u)$	$= \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl(-\beta H_N^u(\varphi)\bigr)\lambda_N(\mathrm{d}\varphi)$, partition function on $\mathbb{T}_N$ with tilt u, page 8
$Z_N^{(\boldsymbol{q})}$	$= \int_{\boldsymbol{\mathcal{X}}_N} \exp\bigl\{-\mathcal{E}_{\boldsymbol{q}}(\varphi)\bigr\}\lambda_N(\mathrm{d}\varphi)$, page 20
ζ	a parameter in the exponential weight of a norm (e.g. $\lVert\cdot\rVert_\zeta$) , page 10
$\mathcal{Z}_N(u)$	$= \int_{\boldsymbol{\mathcal{X}}_N} \sum_X \mathcal{K}_u(X,\varphi)\nu(\mathrm{d}\varphi)$, page 9
$\circ$	$(F_1 \circ F_2)(X,\varphi) = \sum_{Y\subset X} F_1(Y,\varphi)F_2(X\setminus Y,\varphi)$, the circle product of $F_1, F_2 \in M(\mathcal{P}_k, \boldsymbol{\mathcal{X}})$, page 26
$(\cdot,\cdot)$	the scalar product $(\varphi,\psi) = \sum_{x\in\mathbb{T}_N} \varphi(x)\psi(x)$, page 7
$\lvert x\rvert_\infty$	$\max_{i=1,\ldots,d}\lvert x_i\rvert$, page 7
$\lvert x\rvert$	$= \sqrt{\sum x_i^2}$, the Euclidean norm, page 7
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_k$	$\lvert\lvert\lvert F\rvert\rvert\rvert_k = \lvert\lvert\lvert F(B)\rvert\rvert\rvert_{k,B}$ for $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$, page 27
$\lvert\lvert\lvert\cdot\rvert\rvert\rvert_{k,X}$	the weighted strong norm, $\lvert\lvert\lvert F(X)\rvert\rvert\rvert_{k,X} = \sup_\varphi \lvert F(X,\varphi)\rvert^{k,X,r_0} W_k^{-X}(\varphi)$, page 27
$\lVert\cdot\rVert_{k,0}$	$\lVert H\rVert_{k,0} = L^{dk}\lvert\lambda\rvert + L^{\frac{dk}{2}} h\sum_{i=1}^d \lvert a_i\rvert + L^{\frac{(d-2)k}{2}} h \sum_{i,j=1}^d \lvert\boldsymbol{c}_{i,j}\rvert + \frac{h^2}{2}\sum_{i,j=1}^d \lvert\boldsymbol{d}_{i,j}\rvert$, page 29
$\lVert\cdot\rVert_{k,r}^{(\mathsf{A})}$	$\lVert F\rVert_{k,r} = \sup_{X\in\mathcal{P}_k^c}\lVert F(X)\rVert_{k,X,r}\Gamma_{k,A}(X),\ r = 1,\ldots,r_0$,, page 28
$\lVert\cdot\rVert_{k:k+1,r}^{(\mathsf{A})}$	$\lVert F\rVert_{k:k+1,r} = \sup_{X\in\mathcal{P}_k^c}\lVert F(X)\rVert_{k:k+1,X,r}\Gamma_{k,\mathsf{A}}(X),\ r = 1,\ldots,r_0$, page 28
$\lVert\cdot\rVert_{k,r}^{(\mathrm{b})}$	$\lVert F\rVert_{k,r}^{(\mathrm{b})} = \lVert F(B)\rVert_{k,B,r}$ for $F \in M(\mathcal{B}_k, \boldsymbol{\mathcal{X}})$, page 28
$\lvert\cdot\rvert^{j,X}$	$\lvert S\rvert^{j,X} = \sup_{\lvert\dot\varphi\rvert_{j,X}\le 1}\bigl\lvert S_k(\dot\varphi,\ldots,\dot\varphi)\bigr\rvert,\ j = k, k+1$, for s-linear function S_k on $\boldsymbol{\mathcal{X}} \times\cdots\times \boldsymbol{\mathcal{X}}$, page 27
$\lvert\cdot\rvert^{j,X,r}$	$\lvert F\rvert^{j,X,r} = \sum_{s=0}^r \frac{1}{s!}\lvert D^sF(\varphi)\rvert^{j,X},\ j = k, k+1$, for $F \in C^r(\boldsymbol{\mathcal{X}})$, page 27
$\lVert\cdot\rVert_{k,X,r}$	$\lVert F(X)\rVert_{k,X,r} = \sup_\varphi \lvert F(X,\varphi)\rvert^{k,X,r} w_k^{-X}(\varphi),\ r = 1,\ldots,r_0$, page 28
$\lVert\cdot\rVert_{k:k+1,X,r}$	$\lVert F(X)\rVert_{k:k+1,X,r} = \sup_\varphi \lvert F(X,\varphi)\rvert^{k,X,r} w_{k:k+1}^{-X}(\varphi),\ r = 1,\ldots,r_0$, page 28
$\lvert\cdot\rvert_{k,X}$	a norm on $\boldsymbol{\mathcal{X}}$: $\lvert\varphi\rvert_{k,X} = \max_{1\le s\le 3}\sup_{x\in X^*}\frac{1}{h}L^{k\left(\frac{d-2}{2}+s\right)}\bigl\lvert\nabla^s\varphi(x)\bigr\rvert$, page 27
$\lvert\cdot\rvert_{k+1,X}$	a norm on $\boldsymbol{\mathcal{X}}$: $\lvert\varphi\rvert_{k+1,X} = \max_{1\le s\le 3}\sup_{x\in X^*}\frac{1}{h}L^{(k+1)\left(\frac{d-2}{2}+s\right)}\bigl\lvert\nabla^s\varphi(x)\bigr\rvert$, page 27
$\lVert\boldsymbol{L}\rVert$	$= \sup\{\lVert\boldsymbol{L}(f)\rVert: \lVert f\rVert \le 1\}$, norm of a linear operator $\boldsymbol{L}$ between Banach spaces, page 35

$\|\cdot\|_{\boldsymbol{Y}_r}$	the norm on $\boldsymbol{Y}_r$, $\|\boldsymbol{y}\|_{\boldsymbol{Y}_r} = \max_{k\in\{0,\dots,N-1\}} \frac{1}{\eta^k}\|H_k\|_{k,0} \vee \max_{k\in\{1,\dots,N\}} \frac{\alpha}{\eta^k}\|K_{k,r}\|_k$, page 38
$\|\cdot\|_\zeta$	$\|\mathcal{K}\|_\zeta = \sup_{z\in\mathbb{R}^d} \sum_{\|\boldsymbol{\alpha}\|\le r_0} \zeta^{\|\boldsymbol{\alpha}\|} \left\|\partial_z^{\boldsymbol{\alpha}} \mathcal{K}(z)\right\| \mathrm{e}^{-\zeta^{-2}\|z\|^2}$, norm in the Banach space $\boldsymbol{E}$, page 10
$\varphi\big\|_{X^*}$	the restriction of φ to X^*, page 25