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A MODEL FOR THE BEHAVIOR OF FLUID DROPLETS BASED ON MEAN CURVATURE FLOW*

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Abstract. The authors of [W. D. Ristenpart et al., *Nature*, 461 (2009), pp. 377–380] have observed the following remarkable phenomenon during their experiments. If two oppositely charged droplets of fluid are close enough, at first they attract each other and eventually touch. Surprisingly after that the droplets are repelled from each other, if the initial strength of the charges is high enough. Otherwise they coalesce and form a big drop, as one might expect. We present a theoretical model for these observations using mean curvature flow. The local asymptotic shape of the touching fluid droplets is that of a double cone, where the angle corresponds to the strength of the initial charges. Our model yields a critical angle for the behavior of the touching droplets, and numerical estimates of this angle agree with the experiments. This shows, contrary to general belief (see [W. D. Ristenpart et al., *Nature*, 461 (2009), pp. 377–380] and [W. D. Ristenpart et al., *Phys. Rev. Lett.*, 103 (2009), 164502]), that decreasing surface energy can explain the phenomenon. To determine the critical angle within our model, we construct appropriate barriers for the mean curvature flow. In [*Comm. Partial Differential Equations*, 20 (1995), pp. 1937–1958] Angenent, Chopp, and Ilmanen manage to show the existence of one-sheeted and two-sheeted self-expanding solutions with a sufficiently steep double cone as an initial condition. Furthermore they provide arguments for nonuniqueness even among the one-sheeted solutions. We present a proof for this, yielding a slightly stronger result. Using the one-sheeted self-expanders as barriers, we can determine the critical angle for our model.

Key words. mean curvature flow, nonlinear partial differential equations, fluid dynamics

AMS subject classifications. 35R01, 35K55, 53C44, 53Z05

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1. Introduction.

In [RB] the behavior of oppositely charged droplets of fluid is investigated. Droplet motion induced by electrical charges occurs in a vast number of applications, including storm cloud formation, commercial ink-jet printing, petroleum and vegetable oil dehydration, electrospray ionization for use in mass spectrometry, electrowetting, and lab-on-a-chip manipulations (see [RB]).

The phenomenon can be described as follows. Two closely positioned oppositely charged droplets of fluid attract each other and converge. Experiments and numerical simulations (see [RB1]) both provide evidence that a short-lived bridge is formed between the droplets, which instantly causes the charges to be exchanged. The bridge between the touching droplets has the local asymptotic shape of a double cone. Furthermore the charges determine the angle of the double cone, where a lower charge corresponds to a steeper cone (i.e., having a larger acute angle with the rotation axis).

We want to study the behavior of the system after the droplets have touched. One might think that coalescence occurs and that one big drop is formed. However, experiments in [RB] have shown that above a critical field strength the droplets do not coalesce after touching but are repelled from each other.

Following an idea by P. Topping, we present a theoretical model for this phenomenon. It is assumed that after the two droplets have touched and exchanged their charges, the motion of the system is driven by minimization of energy. To model

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this mathematically we use the mean curvature flow, which is the gradient flow of the surface area (see, e.g., [EC]). Our model has two main advantages over the theoretical approaches presented in [RB] and [RB1]. First, we are able to show that minimization of surface energy can explain the observations from the experiments. Second, our results are mostly independent from any assumption on the precise local conical shape formed by the touching droplets.

We define the double cone in $\mathbb{R}^n$ for $0 < \alpha < \frac{\pi}{2}$ by rotating around the x_1-axis:

$$D_\alpha = \left\{ x \in \mathbb{R}^n : \; |(x_2, \ldots, x_n)|^2 = \tan^2 \alpha \cdot x_1^2 \right\}.$$

The results we obtain from our model can be summarized as follows.

Assume that two initially oppositely charged droplets of fluid after touching have the local shape of a smoothing of a double cone D_α in $\mathbb{R}^3$. Also assume that their motion is governed by minimization of area which we model using the mean curvature flow.

Then there is a critical angle α_{crit} with the following properties. If $\alpha < \alpha_{crit}$, the droplets are repelled from each other. If the associated smoothing lies outside of D_α and $\alpha > \alpha_{crit}$, the droplets coalesce and form one big drop.

Using appropriate barriers and a level-set flow argument, we can conclude that α_{crit} is precisely the critical angle for the existence of one-sheeted self-expanding evolutions of $D_\alpha \subset \mathbb{R}^3$, which means $\alpha_{crit} \approx 66°$.

These formulations are made precise in what follows. The critical angle of 60°–70°, observed during experiments (see [RB]), agrees with our prediction.

A family of smooth, immersed hypersurfaces $(M_t)_{t \in I}$ (with I a real interval) in $\mathbb{R}^n$ is called a solution of the mean curvature flow if

$$\frac{\partial x}{\partial t} = \mathbf{H}, \quad x \in M_t, \; t \in I. \tag{1}$$

$\mathbf{H} = -H\upsilon$ is the mean curvature vector and υ a choice of unit normal. Equation (1) is equivalent to $\frac{\partial x}{\partial t} = \triangle_{M_t} x$ ($\triangle_{M_t}$ is the Laplacian of M_t).

Based on the ideas in [AI] we present a new proof for the existence of one-sheeted self-expanders with the double cone D_α for α large enough as an initial condition. We call a solution of (1) self-expanding if

$$M_t = \sqrt{t} \cdot M_1, \quad t \in (0, \infty). \tag{2}$$

The singular initial condition D_α is here understood to be attained locally in the sense of Hausdorff distance.

THEOREM 1.1. *For $n \geq 3$ there exists a critical angle $\alpha^*_{crit}(n) \in \left(0, \frac{\pi}{2}\right)$ with the following properties.*

For any angle $\alpha > \alpha^*_{crit}$ there exist at least three distinct, smooth, rotationally symmetric evolutions of the double cone D_α which are self-expanding. Two of these evolutions are one-sheeted and one is two-sheeted.

For $\alpha = \alpha^*_{crit}$ at least one one-sheeted and one two-sheeted self-expanding, smooth, rotationally symmetric evolution of D_α exist.

Here "one-sheeted" and "two-sheeted" refer to the number of connected components of the solutions. The existence of one-sheeted self-expanders was first proved in [AI]. Additionally our proof provides nonuniqueness among one-sheeted solutions, which was also first stated in [AI].

Nonuniqueness here corresponds to fattening of the level-set flow. For $\alpha < \alpha^*_{crit}$ the two-sheeted evolution of D_α is unique, and therefore we have nonfattening of the level-set flow (see [AI]).

Using the one-sheeted self-expanders we can show that $\alpha_{crit} = \alpha^*_{crit}(3)$.

2. Existence of self-expanders. In this section we want to study the evolution by mean curvature of the double cone D_α, following [AI]. We are interested in solutions $(M_t)_{t\in(0,\infty)}$ that satisfy

$$M_t \text{ is rotationally symmetric and self-expanding.} \tag{3}$$

From (1) and (2) one can see that for a solution of the mean curvature flow to move self-expanding is equivalent to the self-expanding equation

$$H = -\frac{x \cdot \upsilon}{2}, \quad x \in M_1. \tag{4}$$

Under condition (3) this equation becomes an ODE.

We write $x = (x_1, x_2, \dots, x_n) = (x_1, \hat{x})$ for $x \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ $(n \geq 3)$. For a curve γ in $\mathbb{R}^2$, which is symmetric with respect to the axis of rotation, we define the corresponding surface of rotation

$$M(\gamma) = \{(x_1, \hat{x}) \in \mathbb{R}^n : \ (x_1, |\hat{x}|) = (y, u) \in \gamma\}.$$

For $-\pi \leq \alpha \leq \pi$ let σ_α be the closed ray $\{t(\cos\alpha, \sin\alpha) : \ t \geq 0\}$ in $\mathbb{R}^2$. The cone in $\mathbb{R}^n$ with angle α is defined as $C_\alpha = M(\sigma_\alpha \cup \sigma_{-\alpha})$. We get $D_\alpha = C_\alpha \cup C_{\pi-\alpha}$.

Following [AI] we get an equation for γ from (4). Parametrizing γ by arclength s we define $\theta \in [0, 2\pi)$ along γ by setting $\gamma_s = (\cos\theta, \sin\theta)$ for the tangent vector γ_s. The left-handed unit normal is given by $\overline{\upsilon} = (-\sin\theta, \cos\theta)$. Let $\mathbf{k}$ be the curvature vector of γ and $k = \mathbf{k} \cdot \overline{\upsilon}$. Equation (4) for $M(\gamma)$ becomes

$$k - \frac{n-2}{u}\cos\theta + \frac{y}{2}\sin\theta - \frac{u}{2}\cos\theta = 0, \quad (y, u) \in \gamma. \tag{5}$$

A solution of this equation creates a smooth surface $M(\gamma)$.

The following lemma is due to Angenent, Chopp, and Ilmanen (see [AI]). The first part is a consequence of results from Ecker and Huisken on graphical mean curvature flow (see [EH]).

LEMMA 2.1. (i) *(Two-sheeted case.) For $\alpha \in (0, \pi)$ there exists a unique, smooth, connected curve $\gamma(\alpha)$, solving* (5) *and asymptotic to C_α. Furthermore, unless γ is the u-axis, γ is the graph of a positive, convex (or negative, concave) even function $y = y(u)$.*

(ii) *(One-sheeted case.) If γ is a smooth, connected curve solving* (5) *which meets $\{u > 0\}$, then γ lies in $\{u > 0\}$ and there exist α, β with $0 < \alpha < \beta < \pi$ such that γ is asymptotic to $\sigma_\alpha \cup \sigma_\beta$. Moreover, if γ meets the u-axis at a right angle, then $\beta = \pi - \alpha$ and γ is the graph of a positive, even function $u = u(y)$ which is monotone for $y \neq 0$.*

We define

$$A = \left\{\alpha \in \left(0, \frac{\pi}{2}\right) : \ \exists \text{ connected } \gamma \text{ solving (5), asymptotic to } \sigma_\alpha \cup \sigma_{\pi-\alpha}\right\}$$

and $\alpha^*_{crit}(n) = \inf A$.

Let γ be a connected solution of (5) which is asymptotic to $\sigma_\alpha \cup \sigma_{\pi-\alpha}$. There must be point x on γ with $\theta \in \{0, \pi\}$. The proof of part (ii) of Lemma 2.1 shows that there are at most two points on γ with $\theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. Therefore γ must be a graph $u(y)$, and we have shown

$$A = \left\{\alpha \in \left(0, \frac{\pi}{2}\right) : \exists \text{ a graph } \gamma \text{ solving (5), asymptotic to } \sigma_\alpha \cup \sigma_{\pi-\alpha}\right\}.$$

Hence we now want to focus on smooth graphs u solving (5). By symmetry it is enough to consider solutions with a nonnegative derivative in 0. After imposing initial conditions we get the following initial value problem for u:

$$(6) \quad u_{yy} = \left(1 + (u_y)^2\right)\left(\frac{1}{2}(u - yu_y) + \frac{n-2}{u}\right), \qquad u(0) = C > 0,\ u_y(0) = K \geq 0.$$

For symmetry reasons it is enough to consider $u|_{[0,\infty)}$, which we denote again by u. For $y > 0$ we denote by $\alpha(y) = \arctan\left(\frac{u}{y}\right)$ the signed angle, which u makes with the positive y-axis. First we note the following observations regarding the solution u (for (i), (iii), and (iv), see also [AI]).

(i) Every critical point of u is a strict local minimum.
- To see this compute $u_{yy} = \frac{u}{2} + \frac{n-2}{u} > 0$ from (6) whenever $u_y = 0$.

(ii) $u_y > 0$ for all $y > 0$ and every critical point of u_y is a strict local maximum.
- The first part is a direct consequence of (i) and the fact that $u_{yy}(0) = \frac{C}{2} + \frac{n-2}{C} > 0$. For the second part compute $u_y^{(3)} = -(1 + (u_y)^2)(n-2)\frac{u_y}{u^2} < 0$ from (6) using the first part whenever $u_{yy} = 0$.

(iii) Every critical point of $\alpha(y)$ is a strict local minimum. Therefore u is asymptotic to a ray σ_α, $\alpha = \lim_{y\to\infty} \alpha(y)$.
- Let $0 = \alpha_y = \frac{1}{u^2+y^2}(yu_y - u)$, $y > 0$. Using (6) we get $0 = yu_y - u = \frac{2(n-2)}{u} - \frac{2u_{yy}}{1+(u_y)^2}$ and therefore $u_{yy} > 0$. But then $\alpha_{yy} = \frac{yu_{yy}}{u^2+y^2} > 0$.

(iv) $0 < \alpha = \lim_{y\to\infty} \alpha(y) < \frac{\pi}{2}$.
- Any set that is close to C_0 or C_π in some region must disappear rapidly in this region under mean curvature flow by the clearing out lemma (see Appendix A). But solutions of (6) correspond to self-expanding evolutions of double cones.

(v) $u_y \to \tan\alpha$ as $y \to \infty$ and hence $|u_y|$ is bounded.
- Since $u_{yy}(0) > 0$, (ii) shows that u_y is either strictly monotone increasing or has one strict local maximum. Therefore (iii) and (iv) yield the claim.

(vi) u_{yy} has at most one zero and $u_{yy} \to 0$ as $y \to \infty$.
- In (v) it was shown that u_{yy} can have at most one zero. By (v) and (iii) the right-hand side of (6) and therefore u_{yy} goes to 0 as $y \to \infty$.

To prove Theorem 1.1 we show first that for any fixed $K \geq 0$ the asymptotic angle $\alpha = \lim_{y\to\infty} \alpha(y)$ of u goes to $\frac{\pi}{2}$ when C goes to 0 or ∞ in (6). Furthermore we show that the asymptotic angle of any sequence u_k of solutions of (6), for which $(u_k)_y$ blows up somewhere on $[0,\infty)$, must go to $\frac{\pi}{2}$. Together with continuous dependence of the asymptotic angle on $C > 0$ and $K \geq 0$ we then show that this is enough to prove Theorem 1.1.

As a first step we show that the first derivative of the solutions u blows up when C goes to 0 or ∞ in (6). In view of (v) this is useful for investigating the behavior of the corresponding asymptotic angles.

LEMMA 2.2. *The solutions u of* (6) *satisfy* $\sup_{[0,\infty)} u_y \to \infty$ *if* $C \to 0$ *or* $C \to \infty$.

Proof. Let $y_0 > 0$ and $D = \sup_{(0,y_0)} u_y$. We can write (6) as

$$\frac{u_{yy}}{1+(u_y)^2} + \frac{yu_y}{2} = \frac{u}{2} + \frac{n-2}{u}. \tag{7}$$

Integrating from 0 to y_0 noting that $u_y \le D$ and $y \le y_0$, we can bound the left-hand side of (7) by

$$\int_0^{y_0} \frac{u_{yy}}{1+(u_y)^2} + \frac{yu_y}{2} dy \le \arctan\left(u_y\left(y_0\right)\right) + \frac{y_0 D}{2} \le \frac{\pi}{2} + \frac{y_0 D}{2}. \tag{8}$$

By (ii) the integral of the right-hand side of (7) is bounded below as follows:

$$\int_0^{y_0} \frac{u}{2} + \frac{n-2}{u} dy \ge \int_0^{y_0} \frac{u}{2} dy \ge \frac{Cy_0}{2}. \tag{9}$$

Inequalities (8) and (9) yield $\frac{Cy_0}{2} \le \frac{\pi}{2} + \frac{y_0 D}{2}$, and therefore $D \to \infty$ as $C \to \infty$. Since $u_y \le D$ we have $u \le C + Dy$ on $(0, y_0)$, and therefore

$$\int_0^{y_0} \frac{u}{2} + \frac{n-2}{u} dy \ge \int_0^{y_0} \frac{n-2}{C+Dy} dy = \frac{n-2}{D} \log\left(1 + \frac{Dy_0}{C}\right). \tag{10}$$

Combining inequalities (8) and (10) and solving for C yields

$$C \ge \frac{Dy_0}{\exp\left(\frac{D}{n-2}\left(\frac{\pi}{2} + \frac{Dy_0}{2}\right)\right) - 1}.$$

This shows that $C \nrightarrow 0$ if D remains bounded. Therefore $D \to \infty$ as $C \to 0$. ∎

From the last lemma we know that u_y blows up somewhere on $(0, \infty)$ as $C \to 0$ or $C \to \infty$. The next lemma shows that the asymptotic angle α of solutions of (6) has to go to $\frac{\pi}{2}$ if the first derivative blows up somewhere on $[0, \infty)$.

LEMMA 2.3. *For any sequence of solutions u_k of* (6) *with* $\sup_{[0,\infty)} (u_k)_y \to \infty$ *as* $k \to \infty$ *we have* $\alpha_k \to \frac{\pi}{2}$ *as* $k \to \infty$.

Proof. We want to show that for any $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ for all $k \ge k_0$ holds for the asymptotic angles of the u_k.

Let $\varepsilon > 0$. Since $\sup_{[0,\infty)} (u_k)_y \to \infty$ as $k \to \infty$, (v) implies that we can find $\tilde{k}_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ holds for all $k \ge \tilde{k}_0$ for which the solutions satisfy $(u_k)_{yy} > 0$ on $[0, \infty)$.

By (vi) it is therefore enough to find $\overline{k}_0 \in \mathbb{N}$ such that $\left|\alpha_k - \frac{\pi}{2}\right| < \varepsilon$ for all $k \ge \overline{k}_0$ for which the solutions u_k have precisely one inflection point. Hence it is enough to prove the claim for a sequence of solutions of (6) with precisely one inflection point, $\hat{y}_k$.

We have from (6) for $y > 0$

$$\left(\frac{u_k}{y}\right)_y = \frac{(u_k)_y}{y} - \frac{u_k}{y^2} = \frac{2}{y^2}\left(\frac{n-2}{u_k} - \frac{(u_k)_{yy}}{1 + \left((u_k)_y\right)^2}\right).$$

This means $\left(\frac{u_k}{y}\right)_y > 0$ on $[\hat{y}_k, \infty)$ for any $k \in \mathbb{N}$. By (iii) we have $\frac{u_k}{y} \to \tan \alpha_k$ as $y \to \infty$, so to prove the claim it is enough to show $\frac{u_k}{y} \to \infty$ at some point in $[\hat{y}_k, \infty)$ when $k \to \infty$.

Suppose first that $C_k \geq C_0 > 0$ for $k \in \mathbb{N}$ holds for the initial conditions. At $\hat{y}_k$ we can write (6) as

$$\frac{u_k(\hat{y}_k)}{\hat{y}_k} = (u_k)_y(\hat{y}_k) - \frac{2(n-2)}{u_k(\hat{y}_k)\hat{y}_k}.$$

So by (ii)

$$\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq (u_k)_y(\hat{y}_k) - \frac{2(n-2)}{C_k\hat{y}_k} \tag{11}$$

and $\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq \frac{C_k}{\hat{y}_k}$, which yields

$$\frac{1}{C_k\hat{y}_k} \leq \frac{u(\hat{y}_k)}{C_k^2\hat{y}_k}. \tag{12}$$

Combining inequalities (11) and (12) yields

$$\left(1 + \frac{1}{C_k^2}\right)\frac{u_k(\hat{y}_k)}{\hat{y}_k} \geq (u_k)_y(\hat{y}_k).$$

By assumption we have $(u_k)_y(\hat{y}_k) \to \infty$ as $k \to \infty$, so $\frac{u_k(\hat{y}_k)}{\hat{y}_k} \to \infty$ as $k \to \infty$.

To show the claim for the case that the sequence C_k is not bounded below, without loss of generality it is enough to show the claim for a sequence of solutions u_k with $C_k \to 0$ as $k \to \infty$.

Assume not; then there is a subsequence $(u_{k_i})_{i\in\mathbb{N}}$ with $\frac{u_{k_i}}{y} \leq D$ on $[\hat{y}_{k_i}, \infty)$ for all $i \in \mathbb{N}$ for a constant $D > 0$.

Equation (6) for $y > 0$ can be written as

$$\frac{(u_{k_i})_{yy}}{y\left(1 + \left((u_{k_i})_y\right)^2\right)} + \frac{(u_{k_i})_y}{2} = \frac{(u_{k_i})}{2y} + \frac{n-2}{yu_{k_i}}. \tag{13}$$

Integrating (13) from $\hat{y}_{k_i}$ to $2\hat{y}_{k_i}$ and estimating each side using the bound on $\frac{u_{k_i}}{y}$ yields

$$\int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})_{yy}}{y\left(1 + \left((u_{k_i})_y\right)^2\right)} + \frac{(u_{k_i})_y}{2}dy \leq \int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})_y}{2}dy \leq \frac{\hat{y}_{k_i}D}{2} \tag{14}$$

and

$$\int_{\hat{y}_{k_i}}^{2\hat{y}_{k_i}} \frac{(u_{k_i})}{2y} + \frac{n-2}{yu_{k_i}}dy \geq \frac{n-2}{2u_{k_i}(2y_{k_i})} \geq \frac{n-2}{4\hat{y}_{k_i}D}. \tag{15}$$

Combining inequalities (14) and (15) yields $\hat{y}_{k_i}^2 \geq \frac{n-2}{2D}$ for $i \in \mathbb{N}$. To get a contradiction we show that $\hat{y}_{k_i} \to 0$ as $i \to \infty$.

Assume not; so there is a subsequence of $(k_i)_{i\in\mathbb{N}}$ which we call, without loss of generality, $(k_i)_{i\in\mathbb{N}}$ and an $\tilde{\varepsilon} > 0$ such that $\hat{y}_{k_i} \geq \tilde{\varepsilon}$ for all $j \in \mathbb{N}$.

Evaluating (6) at $\hat{y}_{k_i}$ yields

$$\frac{u_{k_i}(\hat{y}_{k_i})}{\hat{y}_{k_i}} = (u_{k_i})_y(\hat{y}_{k_i}) - \frac{2(n-2)}{u_{k_i}(\hat{y}_{k_i})\hat{y}_{k_i}} \leq D.$$

By assumption we have $(u_{k_i})_y(\hat{y}_{k_i}) \to \infty$ as $i \to \infty$, and therefore $u_{k_i}(\hat{y}_{k_i})\,\hat{y}_{k_i} \to 0$ as $i \to \infty$. Since $\hat{y}_{k_i} \geq \tilde{\varepsilon}$ and by (ii) we therefore have $u_{k_i} \to 0$ uniformly on $[0,\tilde{\varepsilon}] \subseteq [0,\hat{y}_{k_i}]$ as $i \to \infty$.

Since $(u_{k_i})_{yy} > 0$ on $[0,\hat{y}_{k_i})$ we can estimate

$$u_{k_i}(\hat{y}_{k_i}) \geq \int_{\frac{\hat{y}_{k_i}}{2}}^{\hat{y}_{k_i}} (u_{k_i})_y\,dy \geq \frac{\hat{y}_{k_i}}{2}(\hat{u}_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right) \geq \frac{\tilde{\varepsilon}}{2}(\hat{u}_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right).$$

Hence $(u_{k_i})_y\left(\frac{\hat{y}_{k_i}}{2}\right) \to 0$ as $i \to \infty$. We get $(\hat{u}_{k_i})_y \to 0$ as $i \to \infty$ uniformly on $\left[0,\frac{\tilde{\varepsilon}}{2}\right] \subseteq \left[0,\frac{\hat{y}_{k_i}}{2}\right]$.

Since $u_{k_i} \to 0$ and $(u_{k_i})_y \to 0$ uniformly on $\left[0,\frac{\tilde{\varepsilon}}{2}\right]$ as $i \to \infty$, (6) implies that $(u_{k_i})_{yy} \to \infty$ uniformly on $\left[0,\frac{\tilde{\varepsilon}}{2}\right]$ as $i \to \infty$. But

$$(u_{k_i})_y\left(\frac{\tilde{\varepsilon}}{2}\right) = (u_{k_i})_y(0) + \int_0^{\frac{\tilde{\varepsilon}}{2}} (u_{k_i})_{yy}\,dy \geq \frac{\tilde{\varepsilon}}{2}\min_{\left[0,\frac{\tilde{\varepsilon}}{2}\right]}(u_{k_i})_{yy}. \qquad \square$$

The next lemma is a stability result for (6).

Lemma 2.4. *The asymptotic angle of solutions of* (6), $\alpha = \alpha(C,K) : (0,\infty) \times [0,\infty) \to \left(0,\frac{\pi}{2}\right)$, *is a continuous function of the initial conditions* $C > 0$ *and* $K \geq 0$.

Proof. Assume not. Then there exist $\tilde{\varepsilon} > 0, (C_0,K_0) \in (0,\infty) \times [0,\infty)$, and a bounded sequence $(C_k,K_k)_{k\in\mathbb{N}}$ in $(0,\infty)\times[0,\infty)$ with $(C_k,K_k) \to (C_0,K_0)$ as $k \to \infty$ and $|\tan\alpha_k - \tan\alpha_0| \geq \tilde{\varepsilon}$ holds for all $k \in \mathbb{N}$ for the associated solutions u_k and u_0 of (6).

We establish appropriate bounds on $\left|(u_k)_y\right|$ and $\left|(u_k)_{yy}\right|$ to get a contradiction. In view of (vi) we assume that $(u_0)_{yy}$ has precisely one zero $\hat{y}_0$. The case $(u_0)_{yy} > 0$ on $[0,\infty)$ can be handled in the same way.

Since we have continuous dependence of u_k and $(u_k)_y$ on the initial conditions on compact intervals, we may assume that all (C_k,K_k) are close enough to (C_0,K_0), so that each $(u_k)_{yy}$ has precisely one zero $\hat{y}_k$ close to $\hat{y}_0$. So there is a compact interval J containing all $\hat{y}_k$ and $\hat{y}_0$.

By continuous dependence we may assume that $(u_k)_y$ is bounded independently of $k \in \mathbb{N}$ on J. Since all $\hat{y}_k$ lie in J we get $0 < \sup_{y\geq 0}(u_k)_y \leq D$ for all $k \in \mathbb{N}$, $D > 0$ a constant. In the following we will adjust the constant $D > 0$ as necessary without explicitly mentioning it.

The bound on $\left|(u_k)_y\right|$ and (ii) (we may assume that $(C_k)_{k\in\mathbb{N}}$ is bounded below) imply that the right-hand side of (6) is bounded on J independently of $k \in \mathbb{N}$. Therefore $(u_k)_{yy}$ is bounded on J, and since the zeros of $(u_k)_{yy}$ all lie in J, we get $\sup_{y\geq 0}(u_k)_{yy} \leq D$ for all $k \in \mathbb{N}$.

Differentiating (6) with respect to $y > 0$, setting $u_y^{(3)} = 0$, and employing (ii) yields

$$(u_k)_{yy} = \frac{-2(n-2)\,(u_k)_y}{y\,(u_k)^2} + \frac{4\,(u_k)_y\left((u_k)_{yy}\right)^2}{y\left(1+\left((u_k)_y\right)^2\right)^2} \geq \frac{-2(n-2)\,(u_k)_y}{y\,(u_k)^2}.$$

Since $\left|(u_k)_y\right|$ is bounded independently of $k \in \mathbb{N}$ and u_k is bounded below independently of $k \in \mathbb{N}$ and the zeros of $(u_k)_{yy}$ lie close to $\hat{y}_0 > 0$ and $(u_k)_{yy}(0) > 0$, this means $(u_k)_{yy}$ cannot have arbitrarily small local extrema. Therefore $(u_k)_{yy}$ must be bounded below. Together with our upper bound on $(u_k)_{yy}$ we get $\sup_{y\geq 0}\left|(u_k)_{yy}\right| \leq D$ for all $k \in \mathbb{N}$.

Since all zeros of $(u_k)_{yy}$ lie in a compact interval, (v) implies that all $(u_k)_y$ are bounded below by a positive constant outside a compact interval. Hence we can estimate for $y > 0$ large enough, using (6) and the bounds on $\left|(u_k)_{yy}\right|$,

$$\left(\frac{u_k}{y}\right)_y = \frac{1}{y}\left((u_k)_y - \frac{u_k}{y}\right) = \frac{2}{y^2}\left(\frac{n-2}{u_k} - \frac{(u_k)_{yy}}{1+(u_y)^2}\right) \leq \frac{D}{y^2}.$$

So by (iii) for any $\delta > 0$ there exists $y_\delta > 0$ with

$$\left|\frac{u_k}{y} - \tan\alpha_k\right| < \delta, \quad y \geq y_\delta,\ k \in \mathbb{N}.$$

Using our assumption on $\alpha_{C_0}, \alpha_{C_k}$ we can choose y_δ large enough such that

$$|\tan\alpha_{C_k} - \tan\alpha_{C_0}| \geq \frac{\tilde{\varepsilon}}{2},$$
$$\left|\frac{u_k}{y} - \tan\alpha_{C_k}\right| < \delta,$$
$$\left|\frac{u_0}{y} - \tan\alpha_{C_0}\right| < \delta$$

for $y \geq y_\delta$ and $k \in \mathbb{N}$. The triangle inequality yields for small enough $\delta > 0$

$$\left|\frac{u_k}{y} - \frac{u_0}{y}\right| \geq \frac{\tilde{\varepsilon}}{4}, \quad y \geq y_\delta,\ k \in \mathbb{N}.$$

We can write the difference of the ODEs for u_0 and u_k at $y > 0$ as $\xi_1 = \xi_2 + \xi_3 + \xi_4$, where

$$\xi_1 = \frac{1}{y}\left(u_0 - u_k\right),$$
$$\xi_2 = (u_0)_y - (u_k)_y,$$
$$\xi_3 = \frac{2}{y}\left(\frac{(u_0)_{yy}}{1+(u_0)_y^2} - \frac{(u_k)_{yy}}{1+(u_k)_y^2}\right),$$
$$\xi_4 = \frac{2}{y}\left(\frac{n-2}{u_k} - \frac{n-2}{u_0}\right).$$

Using the bounds on $\left|(u_k)_y\right|$ and $\left|(u_k)_{yy}\right|$ we can find $\tilde{y} > y_\delta$ such that $|\xi_3| < \frac{\tilde{\varepsilon}}{12}$ and $|\xi_4| < \frac{\tilde{\varepsilon}}{12}$ for any $k \in \mathbb{N}$. By continuous dependence we can find $\tilde{k} \in \mathbb{N}$ large enough such that $|\xi_2| < \frac{\tilde{\varepsilon}}{12}$ at $\tilde{y}$. Therefore we get $|\xi_2 + \xi_3 + \xi_4| < \frac{\tilde{\varepsilon}}{4}$—a contradiction of our previous estimate for ξ_1. ∎

We can now finish the proof of Theorem 1.1.

Proof. The last three lemmas imply that the continuous function $\alpha = \alpha(C,K) : (0,\infty)\times[0,\infty) \to \left(0, \frac{\pi}{2}\right)$ attains its minimum at some $(C_{crit}, K_{crit}) \in (0,\infty)\times[0,\infty)$. The solution u of (6) with initial conditions C_{crit}, K_{crit} must have asymptotic angle α^*_{crit}. Continuity and Lemma 2.2 imply that for $\alpha > \alpha^*_{crit}$ there are at least two one-sheeted, rotationally symmetric, self-expanding evolutions of D_α and that there is at least one such evolution of $D_{\alpha^*_{crit}}$.

$\alpha^*_{crit} < \frac{\pi}{2}$ is a direct consequence of (iv) and $\alpha^*_{crit} > 0$ is proved in [AI]. This finishes the proof of Theorem 1.1. ∎

The level-set flow of any D_α is a rotationally symmetric set with a smooth boundary (see [AI]). This implies that for $\alpha < \alpha^*_{crit}$ there can be no one-sheeted self-expanding evolutions of D_α. As stated in Lemma 2.1 the existence of a two-sheeted, rotationally symmetric, self-expanding evolution of D_α (for any cone angle $\alpha \in \left(0, \frac{\pi}{2}\right)$) is asserted using results of Ecker and Huisken (see [EH]) which ensure existence of self-expanding evolutions of the two Lipschitz graphs C_α and $C_{\pi-\alpha}$.

We can only prove a suitable comparison principle for solutions of (6) for $C \geq \sqrt{2(n-2)}$. But we believe that there are no more than two one-sheeted, rotationally symmetric, self-expanding evolutions of D_α for $\alpha > \alpha^*_{crit}$ (both of these evolutions must then correspond to symmetric solutions of (6), i.e., $K = 0$). Additionally there might be nonrotationally symmetric evolutions of D_α (see [AI]).

Remark 2.5. Solutions of (4) are stationary for the functional

$$\mathbf{K}[M] = \int_M \exp\left(\frac{|x|^2}{4}\right) d\mathcal{H}^{n-1}(x).$$

The authors of [AI] sketch a proof for the existence of one-sheeted self-expanders, asymptotic to D_α which are minimizers of $\mathbf{K}$. Furthermore they indicate how one might prove a version of Theorem 1.1 using this approach.

3. Touching fluid droplets. As mentioned before we want to apply the previous results to study the behavior of touching fluid droplets. These are assumed to have locally conical, rotationally symmetric shape, i.e., the shape of a smoothing of D_α. The following definition makes this formulation precise. From now on we set the dimension to $n = 3$.

DEFINITION 3.1. *We call $M_\alpha = M(u_\alpha)$ a smoothing of the double cone D_α with angle $0 < \alpha < \frac{\pi}{2}$, $\gamma = \tan\alpha$, if for $a < 0 < b$*

$$u_\alpha(y) = \begin{cases} \gamma |y| & \text{if } y \leq a \text{ or } b \leq y, \\ s(y) & \text{if } a \leq y \leq b \end{cases}$$

such that $s > 0$ and $u_\alpha \in C^{2,\beta}(\mathbb{R})$, $\beta > 0$. M_α is said to lie outside of the double cone D_α if $s(y) \geq \gamma|y|$ for $a < y < b$. u is the generating function of M_α.

Clearly any smoothing of a double cone stays rotationally symmetric under mean curvature flow. One can compute (see [SI]) that (1) for a mean curvature flow evolution M_t of M_α is equivalent to

$$\frac{\partial}{\partial t}u = \frac{\frac{\partial^2}{\partial^2 y}u}{1 + \left(\frac{\partial}{\partial y}u\right)^2} - \frac{1}{u}, \tag{16}$$

where $u = u(\cdot, t)$, $u(\cdot, 0) = u_\alpha$ generates M_t.

For any smoothing M_α we have short-time existence of a solution M_t of (1) on a maximal time interval $[0, T)$, $T > 0$. Furthermore the solution must be smooth for $t > 0$ and every finite time singularity must be due to pinching, i.e., $\inf_{\mathbb{R}} u(\cdot, t) \to 0$ as $t \to T$. This holds even without any growth assumption on the initial generating function (see [SI] and [LS]).

In fact a sphere comparison argument (see [EC]) shows that

$$\lim_{t\to T}\min_{\mathbb{R}} u(\cdot,t)\to 0 \text{ as } t\to T$$

must hold for finite time singularities.

This agrees with intuition about repulsion (pinching in finite time) and coalescence (long-time existence) of fluid droplets.

Using techniques by Ecker and Huisken (see [EH]) one can derive global height estimates for rotationally symmetric solutions of the mean curvature flow which yield, using the results from [BB], the following comparison principle (see [BO]).

LEMMA 3.2. *Let M_{α_1}, M_{α_2} be two smoothings of the double cone with $u_{\alpha_1}\le u_{\alpha_2}$ for the associated generating functions. Denoting the generating functions of the two evolutions with u^1 and u^2, we then have $u^1\le u^2$ as long as the solutions exist.*

COROLLARY 3.3. *Any smoothing of the double cone M_α, $0<\alpha<\frac{\pi}{2}$, has a unique evolution by mean curvature.*

Using these results, we can now define what coalescence and repulsion mean within our model.

DEFINITION 3.4. *An angle $0<\alpha<\frac{\pi}{2}$ is called a repulsion angle if there exists a smoothing of the double cone M_α which is outside of the double cone and such that the mean curvature flow evolution of M_α pinches in finite time.*

LEMMA 3.5. *An angle $0<\alpha<\frac{\pi}{2}$ is not a repulsion angle if and only if there is a smoothing of the cone M_α, for which the evolution under mean curvature flow exists for all $t>0$.*

Proof. By definition any angle $0<\alpha<\frac{\pi}{2}$ that is not a repulsion angle must have a smoothing M_α for which the evolution under mean curvature flow M_t exists for all $t>0$.

So suppose for $0<\alpha<\frac{\pi}{2}$ there exists a smoothing M_α, such that the evolution M_t exists for all $t>0$. Let $\hat{M}_\alpha$ be another smoothing that is outside of the double cone. We denote its evolution by $\hat{M}_t,\ t\in[0,T)$.

We know that the mean curvature flow is invariant under parabolic rescaling

$$x\mapsto\lambda x,\ t\mapsto\lambda^2 t$$

for any scaling parameter $\lambda>0$, $x\in M_t$, and $t\in I$. Let M_t^λ be the rescaling of M_t. Note here that any double cone D_α is invariant under the scaling $x\mapsto\lambda x$. Since $\hat{M}_\alpha$ is outside of the double cone we can therefore choose $\lambda>0$ sufficiently small in order to get initially

$$u_\alpha^\lambda\le\hat{u}_\alpha$$

for the corresponding generating functions. By Lemma 3.2 $\hat{M}_t$ must exist for all $t>0$, and therefore α is not a repulsion angle. ∎

In view of the last lemma we make the following definition.

DEFINITION 3.6. *An angle $0<\alpha<\frac{\pi}{2}$ is called a coalescence angle if it is not a repulsion angle in the sense of Definition* 3.4.

Given a field strength (respectively, a cone angle) at which coalescence occurs any lower field strength (respectively, greater cone angle) should lead to coalescence as well. The same should hold for repulsion angles with higher field strength. The next two lemmas show that this is true within our model.

LEMMA 3.7. *Let α_0 be a coalescence angle. Then for any angle $\alpha\ge\alpha_0$ and for any smoothing M_α that is outside of the double cone the mean curvature flow with* *initial data M_α exists for all $t > 0$. Therefore any angle $\alpha > \alpha_0$ is a coalescence angle.*

Proof. As in the proof of Lemma 3.5 we can scale appropriately and then use the comparison principle from Lemma 3.2. ◻

LEMMA 3.8. *Let α_0 be a repulsion angle. Then for $\alpha \leq \alpha_0$ any smoothing of the double cone M_α with angle α must pinch in finite time. This means any angle $\alpha < \alpha_0$ must be a repulsion angle.*

Proof. Again, as for Lemma 3.5 we can scale appropriately and then use Lemma 3.2. ◻

The observations from the experiments in [RB] suggest the existence of a critical angle for the behavior of the system. As the next lemma shows, this is also true for our model.

LEMMA 3.9. *There is a critical angle $0 \leq \alpha_{crit} \leq \frac{\pi}{2}$ such that any angle $\alpha < \alpha_{crit}$ is a repulsion angle and any angle $\alpha > \alpha_{crit}$ is a coalescence angle.*

Proof. Let

$$\overline{\alpha} = \sup\left\{0 < \alpha < \frac{\pi}{2} : \alpha \text{ is a repulsion angle}\right\},$$
$$\underline{\alpha} = \inf\left\{0 < \alpha < \frac{\pi}{2} : \alpha \text{ is a coalescence angle}\right\}.$$

According to Lemma 3.8 any cone angle $\alpha < \overline{\alpha}$ is not a coalescence angle. This shows that $\overline{\alpha} \leq \underline{\alpha}$. Clearly any angle is either a repulsion or a coalescence angle, and hence we have $\overline{\alpha} = \underline{\alpha} = \alpha_{crit}$. ◻

Using the constructed one-sheeted self-expanders we can now determine α_{crit}.

3.1. Determining α_{crit}.

LEMMA 3.10. *Any double cone smoothing with angle $\alpha > \alpha^*_{crit}(3)$ that lies outside of the double cone has an evolution which exists for all $t > 0$.*

Proof. Let M_α be a smoothing of D_α with $\alpha > \alpha^*_{crit}(3)$ and generating function u_α. Let s be the generating function of the self-expander asymptotic to $D_{\alpha^*_{crit}(3)}$ which exists by Theorem 1.1. As in the proof of Lemma 3.5 we can do a parabolic rescaling and therefore assume $s \leq u_\alpha$. Then the evolution of M_α must exist for all $t > 0$ by the comparison principle, Lemma 3.2, since every finite time singularity must be due to pinching. ◻

LEMMA 3.11. *Any double cone smoothing with angle $\alpha < \alpha^*_{crit}(3)$ must pinch in finite time.*

Proof. We follow a level-set flow argument from [AI].

Assume that there is a double cone smoothing M_α, $\alpha < \alpha^*_{crit}(3)$, such that its evolution by mean curvature M_t exists for all $t > 0$.

Let Γ_t, $t \geq 0$, be the level-set flow of D_α. Γ_t can be characterized as follows. $\mathbb{R}^3 \backslash \Gamma_t$ is the union of all level-set flows Δ_t such that Δ_0 is compact and lies in $\mathbb{R}^3 \backslash D_\alpha$.

First, we show that $0 \in \Gamma_t$. So let Δ_t be a level-set flow with $\Delta_0 \subset \mathbb{R}^3 \backslash D_\alpha$ compact. Δ_0 must either lie in the convex hull of one of C_α or $C_{\pi-\alpha}$ or outside of D_α. Using the maximum principle for one level-set flow and one smooth flow, we see that in the first case Δ_0 is pushed away from 0 by the graphical self-expanders of Ecker and Huisken (part (i) of Lemma 2.1) and therefore $0 \notin \Delta_t$. In the second case we can parabolically rescale M_α as in the proof of Lemma 3.5 and therefore assume $\Delta_0 \cap M_\alpha = \emptyset$. Then we can apply the maximum principle to see that $0 \notin \Delta_t$. Hence by the above characterization we must have $0 \in \Gamma_t$.

Γ_1 is rotationally symmetric, so $\Gamma_1 = M(X)$ for some closed set $X \subset \mathbb{R}^2$. In fact the boundary $\partial\Gamma_1$ is smooth and ∂X consists precisely of curves of the type in Lemma 2.1 (see [AI]).

Since $0 \in \Gamma_1$ there must be a curve of type (ii) in Lemma 2.1 in ∂X, which is asymptotic to $\sigma_\alpha \cup \sigma_{\pi-\alpha}$. This yields a smooth, rotationally symmetric, one-sheeted self-expanding evolution of D_α—a contradiction. □

In view of the definition of α_{crit} the last two lemmas yield the following.

COROLLARY 3.12. $\alpha_{crit} = \alpha^*_{crit}(3)$.

3.2. Conclusions. Using a model based on mean curvature flow, we obtain a critical cone angle α_{crit} for the behavior of oppositely charged droplets of fluid. More precisely this means any smoothing of the double cone (see Definition 3.1) with angle less than α_{crit} must pinch in finite time (repulsion). Assuming a smoothing has an angle greater than α_{crit} and lies outside of the double cone, its evolution must exist for all $t > 0$ (coalescence).

We can show that $\alpha_{crit} = \alpha^*_{crit}(3)$, where $\alpha^*_{crit}(3)$ is the critical angle for the existence of smooth, rotationally symmetric, self-expanding, one-sheeted evolutions of double cones and $\alpha^*_{crit}(3) \approx 66°$. This coincides with observations from experiments (see [RB]) which predict a critical angle of 60°–70°.

Finally we want to compare our model with the one in [RB1]. For that approach it is assumed that the bridge between the touching droplets minimizes area under a volume constraint. This corresponds to constant mean curvature surfaces of revolution (Delaunay surfaces) which are fitted to linear double cones, similarly as in Definition 3.1 (unlike in Definition 3.1 the associated generating function is only continuous). The associated capillary pressure p is assumed to determine the behavior.

The critical shape has $p = 0$ which yields a rescaled catenoid. Suitable smoothings of the associated surfaces with $p > 0$ provide double cone smoothings in the sense of Definition 3.1 which pinch in finite time. This is in agreement with [RB1]. However, Lemma 3.11 shows that down to a certain $p_0 < 0$ the (smoothings of the) associated surfaces with $p \leq 0$ still pinch in finite time under mean curvature flow. Therefore our predicted critical angle is greater than the predicted critical angle from [RB1], which is approximately 59°.

Appendix A. The clearing out lemma. This version of the clearing out lemma is due to Brakke. It says that if the area ratio of a piece of a solution of mean curvature flow in a ball is initially small, then it must leave the ball with one-quarter of the radius after a time proportional to the initial area ratio times the radius squared. Proofs can be found in [BR] or [EC].

THEOREM. *Let $(M_t)_{t>0}$ be complete, properly embedded hypersurfaces in $\mathbb{R}^n$, $n \geq 2$, evolving by mean curvature flow. There exist constants $c(n) > 0$ and $\varepsilon_0(n) > 0$ such that if M_0 satisfies*

$$\mathcal{H}^{n-1}\left(M_0 \cap B_\rho\left(x_0\right)\right) \leq \varepsilon_0 \rho^{n-1}$$

for some $x_0 \in \mathbb{R}^n$ and $\rho > 0$, then

$$\mathcal{H}^{n-1}\left(M_t \cap B_{\frac{\rho}{4}}\left(x_0\right)\right) = 0$$

for $t = c\varepsilon_0\rho^2$.

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