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Early Detection Techniques for Market Risk Failure

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EARLY DETECTION TECHNIQUES FOR MARKET RISK FAILURE

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Abstract

The implementation of appropriate statistical techniques for monitoring conditional VaR models, i.e, backtesting, reported by institutions is fundamental to determine their exposure to market risk. Backtesting techniques are important since the severity of the departures of the VaR model from market results determine the penalties imposed for inadequate VaR models. In this paper we make six contributions to backtesting techniques. In particular, we show that the Kupiec test can be viewed as a combination of CUSUM change point tests; we detail the lack of power of CUSUM methods in detecting violations of VaR as soon as these occur; we develop an alternative technique based on weighted U-statistic type processes that have power against wrong specifications of the risk measure and early detection; we show these new backtesting techniques are robust to the presence of estimation risk; we construct a new class of weight functions that can be used to weight our processes; and our methods are applicable both under conditional and unconditional VaR settings.

Keywords and Phrases: Backtesting; Basel Accord; Change-Point tests; Conditional Quantile; Risk management; Value at Risk.

JEL codes: G15, G18, G21, G28.

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1 Introduction

In the aftermath of a series of bank failures that occurred during the seventies a group of ten countries (G-10) decided to create a committee to set up a regulatory framework to be observed by internationally active banks operating in these member countries. This committee coined as Basel Committee on Banking Supervision (BCBS) was intended to prevent financial institutions, in particular banks, from operating without effective supervision. The subsequent documents derived from this commitment focused on the imposition of capital requirements for internationally active banks which would serve as provisions for losses from adverse market fluctuations, concentration of risks or simply bad management of institutions. The risk measure agreed upon was the Value-at-Risk (VaR). In financial terms, this is the maximum loss on a trading portfolio for a period of time given a confidence level, and in practice, determines restrictions on the minimum amount of capital held as reserves by financial institutions. In statistical terms, VaR is a (conditional) quantile of the conditional distribution of returns on the portfolio given agent's information set.

The computation of these VaR measures has become of paramount importance in risk management since financial institutions are monitored to ensure the accuracy of the quantile measures reported. This implies that banks with sufficiently highly developed risk management systems can decide on their own internal risk models as long as these satisfy requirements set by the Basel Accord (1996) for computing capital reserves. The main toolkit for measuring and testing the performance of different VaR methodologies proposed in the Basel Accord was a statistical device denoted backtesting that consisted of out-of-sample comparisons between the actual trading results with internally model-generated risk measures. The magnitude and sign of the difference between the model-generated measure and trading results indicate whether the VaR model reported by an institution is correct for forecasting the underlying market risk and if this is not so, whether the departures are due to over- or under-risk exposure of the institution. The implications of over- or under- risk exposure being diametrically different: either extra penalties on the level of capital requirements or bad management of the outstanding equity by the institution. These backtesting techniques are usually interpreted as statistical parametric tests for the coverage probability α defining the conditional quantile VaR measure. The seminal papers in this area of research are due to Kupiec (1995) and Christoffersen (1998) who proposed asymptotic standard Gaussian tests and likelihood ratio tests respectively.

In a parametric VaR setting and when the set of parameters is known these tests are optimal against alternatives given by wrong specifications of the VaR measure. These tests, however, suffer from estimation risk and do not provide any information about the timing of the rejection of VaR. An alternative monitoring technique, that to the best of our knowledge has not been exploited in the risk assessment literature, is the use of CUSUM-type tests. Pioneering papers in the use of this technique are Page (1954), Gombay, Horváth and Hušková (1996) or Csörgő and Horváth (1988a, 1988b). In a financial econometrics context Kuan and Hornik (1995), Leisch, Hornik and Kuan (2000) or Altissimo and Corradi (2003) propose CUSUM tests for monitoring changes in fluctuation tests. In particular, in a similar spirit to this paper Altissimo and Corradi propose CUSUM-type tests sensitive to early and late in the sample change-point detection. However, in contrast to the methods proposed in this paper their CUSUM statistics depend on a nuisance parameter that needs to be tabulated via simulation. In a related paper Andreou and Ghysels (2006) review this and other monitoring techniques for the volatility process when this is estimated using high-frequency data. These authors explore in detail this alternative and discuss the relative power of CUSUM-type tests when computed at different frequencies. They also discuss the implementation of historical and sequential CUSUM procedures and study the asymptotic properties of related change-point hypothesis tests.

This paper contributes to this literature in six different aspects. First, it shows that the standard Kupiec test used for VaR monitoring can be viewed as a CUSUM change-point test. In this way we show that, by simply manipulating the standard backtesting test we can obtain very useful information regarding the timing of the failure of VaR in assessing risk exposure. The second contribution consists of acknowledging the failure of this CUSUM technique in detecting violations of VaR that occur early in the testing period, and more importantly in introducing an alternative technique based on weighted U-statistics type processes that not only shows power against wrong specifications of the risk measure but is also capable of early detections. This is very important in a risk management context since it allows financial institutions to monitor closely their risk exposure and propose alternative VaR models when the current ones are not well specified to measure risk. Our third contribution is to show that these new backtesting techniques are robust to the presence of estimation risk and therefore can be implemented with no alteration to the asymptotic theory in the realistic case that the parameters of the model need to be estimated. If this is the case the standard backtesting procedure consists of an in-sample period where the parameters of the risk model are estimated, and an out-of-sample period for monitoring purposes. Our study of weight functions leads us to introduce a rich class of weight functions that have desirable properties; continuity for example, extend the current family of weight functions and are better for early detection. We are also able to show that our results hold both for unconditional and conditional Var settings. Our sixth and final contribution along these lines is to analyze the power of the weighted statistics proposed by Gombay, Horváth and Hušková(1996) in terms of local alternatives. In order to do this, we must tabulate the cumulative distribution function (*cdf*) for some members of their family of statistics. The tables corresponding to these *cdfs* are collected in Section 6 entitled Tabulated *cdf*s for Gombay, Horváth, Hušková's Weighted Statistics - this information may be of interest to researchers interested in using these statistics as well as the ones proposed here.

The rest of the paper is structured as follows. Section 2 introduces the standard backtesting monitoring techniques used to assess the validity of conditional VaR models. The section also discusses change-point tests and shows that standard CUSUM tests can complement the standard backtesting techniques by providing information about the timing of rejection of the VaR model. In this section we also introduce a novel family of test statistics based on change-point detection techniques and the relevant asymptotic theory. In Section 3 we carry out a simulation exercise where we compare the power of these tests to the power of standard CUSUM tests. We stress the early detection property exhibited by our test statistic in respect to other CUSUM-type competitors. Section 4 concludes; and proofs of theorems, propositions and corollaries stated herein are collected in the section named Mathematical Appendix. Finally Section 6 collects the Tabulated *cdf*s for Gombay, Horváth, Hušková's Weighted Statistics.

2 Risk Monitoring: Backtesting techniques

We will start this section by formally defining the Value-at-Risk at an α coverage probability. Denote the real-valued time series of portfolio returns by Y_t, and assume that at time $t-1$ the agent's information set is given by $\Im_{t-1}$, which may contain past values of Y_t and other relevant explanatory variables. Also, by assuming that the conditional distribution of Y_t given $\Im_{t-1}$ is continuous, we can define the α-th conditional VaR of Y_t given $\Im_{t-1}$ as the $\mathcal{F}_{t-1}$-measurable function $q_\alpha(\Im_{t-1})$ satisfying the equation

$$\mathbb{P}(Y_t \leq q_\alpha(\Im_{t-1}) \mid \Im_{t-1}) = \alpha, \text{ almost surely (a.s.)}, \alpha \in (0,1), \ \forall t \in \mathbb{Z}, \tag{1}$$

with $\mathcal{F}_{t-1}$ the sigma-algebra generated by the set of information available at $t-1$.

In *parametric* VaR inference one assumes the existence of a parametric family of functions $\mathcal{M} = \{m_\alpha(\cdot,\theta) : \theta \in \Theta \subset \mathbb{R}^p\}$ and proceeds to make VaR out-of-sample forecasts using the model $\mathcal{M}$. In these parametric VaR models the nuisance parameter θ belongs to a compact set Θ embedded in a finite-dimensional Euclidean space $\mathbb{R}^p$, and can be estimated by a $\sqrt{R}$-consistent estimator, with R denoting the (in-)sample size in the backtesting exercise. The most popular parametric VaR models are those derived from traditional location-scale models such as ARMA-GARCH models, but other models include quantile regression models such as those of Koenker and Xiao (2006), autoregressive quantile regression models of Engle and Manganelli (2004) and models that specify the dynamics of higher moments of the conditional distribution of Y_t. Therefore, under the null hypothesis of the correct specification of the conditional VaR by a parametric model $m_\alpha(\cdot,\theta)$, expression (1) reads as

$$\mathbb{E}[I_{t,\alpha}(\theta) \mid \Im_{t-1}] = \alpha \text{ a.s. for some } \theta \in \Theta, \tag{2}$$

with $I_{t,\alpha}(\theta) := 1(Y_t \leq m_\alpha(\Im_{t-1},\theta))$, and $1(\cdot)$ an indicator function that takes the value one if $Y_t \leq m_\alpha(\Im_{t-1},\theta)$ and zero otherwise.

Christoffersen (1998) proposed two different likelihood ratio tests for this hypothesis. For the unconditional version of (2), $\mathbb{E}[I_{t,\alpha}(\theta)] = \alpha$, this author proposes an unconditional likelihood ratio test with statistic given by

$$LR_u(P,R) = -2log\frac{L(\alpha;\{I_{t,\alpha}(\theta_0)\}_{t=R+1}^n)}{L(\hat{\pi};\{I_{t,\alpha}(\theta_0)\}_{t=R+1}^n)},$$

with $L(\hat{\pi};\{I_{t,\alpha}(\theta_0)\}_{t=R+1}^n) = (1-\hat{\pi})^{n_0}\hat{\pi}^{n_1}$, $\hat{\pi} = n_1/(n_0+n_1)$, and where n_1 denotes the number of VaR exceedances and $n_0 = P - n_1$. R is used for estimating the VaR model and P is the out-of-sample size for testing the risk model, and $n = R + P$.

Under the null hypothesis this test statistic is asymptotically distributed as a χ^2 distribution with one degree of freedom. This unconditional test is an appropriate method for gauging the risk exposure reported by the VaR models if and only if the exceedances are serially independent. To test the assumption of serially independence, Chirstoffersen proposes a conditional likelihood ratio test with power against Markov processes of order one. This author also combines both tests to define a composite hypothesis test with power against wrong specifications of the coverage probability and against the occurrence of clustering in the violations of VaR. For illustration purposes we will concentrate first on the unconditional testing framework and will show later that our results also apply to the conditional framework. Thus, in the unconditional setting Kupiec developed an alternative, asymptotically equivalent, test statistic to Christoffersen. This method is based on the absolute value of the following standardized partial sum:

$$K_n \equiv K(n, R) := \frac{1}{\sqrt{n-R}} \sum_{t=R+1}^{n} (I_{t,\alpha}(\theta) - \alpha). \tag{3}$$

Assuming serial independence of $\{I_{t,\alpha}\}_{R+1}^{n}$, the standardized version of $K(n, R)$ converges to a standard normal random variable. Mathematically,

$$(\alpha(1-\alpha))^{-1/2} K_n \xrightarrow{\mathcal{D}} N(0, 1). \tag{4}$$

Christoffersen and Kupiec tests are optimal in terms of asymptotic power if θ is known. In practice these parameters are rarely known and therefore must be estimated by some $\sqrt{R}$-consistent estimator of θ which can, in certain circumstances, yield a different asymptotic variance of the test statistic K_n. Escanciano and Olmo (2008) propose an alternative to this backtesting statistic that corrects for distortions in the asymptotic size of the test produced by model parameter estimation.

These tests are designed to evaluate the specification of the conditional VaR measure after P out-of-sample periods. Neither method, however, is devised to exhibit power against the timing of the rejection of the null hypothesis. A potential solution to this is the use of CUSUM-type tests. The historical version of this test would replace the in-sample stage, where the parameters of the risk model are estimated. The interest in terms of risk management, on the other hand, lies in the out-of-sample version of the CUSUM test where the change-point statistic is designed to detect deviations in the risk forecasts given by VaR. This version of the test can be devised as a sequential test where the deviations are detected on-line (as they occur) or as a historical off-line test implemented for the out-of-sample period P. The latter version corresponds to the standard backtesting exercise used for regulatory purposes nowadays. We elaborate more on this at the end of the following subsection.

2.1 Change-point detection techniques

In this section we study more refined and powerful versions of the backtesting tests introduced above for the correct specification of the VaR model. We start with the standard CUSUM test as benchmark and show that this test can be built as a simple combination of the Kupiec test for two different sample periods.

Now we are not only interested in detecting failure of the risk model but also in the timing of the failure. In order to do this we will use different change-point detection techniques. For sake of exposition we will assume in this and next subsection that the vector of parameters θ is known, and therefore there is no need to use an in-sample period, hence $R = 0$ and $n = P$.

Let us consider the following process:

$$X_t = \begin{cases} 1(Y_t \leq m_\alpha(\Im_{t-1}, \theta)), & 1 \leq t \leq k^\star, \\ 1(Y_t \leq m^\star_\alpha(\Im_{t-1}, \theta)), & k^\star < t \leq P, \end{cases} \tag{5}$$

with $m^\star_\alpha(\Im_{t-1}, \theta) \neq m_\alpha(\Im_{t-1}, \theta)$.

The process $m^\star_\alpha(\Im_{t-1}, \theta)$ describes the actual conditional VaR process with α coverage probability, after the structural break. Thus, in contrast to backtesting tests the expected value of this random variable is $\mathbb{E}[X_t] = \alpha$ for all $1 \leq t \leq P$. Here, we are not interested in detecting deviations in the target coverage probability but in the parametric quantile process $m_\alpha(\Im_{t-1}, \theta)$ and the location of the change, $k^\star$. Hence the relevant hypothesis test is

$$H_O : k^\star \geq P$$

versus the alternative of wrong specification of the risk model given by

$$H_A : 1 \leq k^\star < P.$$

In this context (3) can be expressed as

$$K_P = \frac{1}{\sqrt{P}} \left(\sum_{t=1}^{P} I_{t,\alpha}(\theta) - P\alpha \right). \tag{6}$$

Moreover, we can construct a process $\widetilde{K}_P$ indexed by a parameter τ, with $0 \leq \tau \leq 1$, as follows:

$$\widetilde{K}_P(\tau) = \frac{1}{\sqrt{P}} \left(\sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\theta) - \lceil \tau P \rceil \alpha \right), \tag{7}$$

with $\lceil \cdot \rceil$ denoting the integer part of τP.

Under H_O, Donsker's (1951) theorem restated on D$[0,1]$ now applies and we conclude with the following result regarding the process $\widetilde{K}_P$:

$$\widetilde{K}_P(\tau) \xrightarrow{\mathcal{D}} \sqrt{\alpha(1-\alpha)} W(\tau), \quad 0 \leq \tau \leq 1. \tag{8}$$

Here $\{W(\tau);\, 0 \leq \tau \leq 1\}$ represents a standard Wiener process with zero mean and covariance function given by $\mathbb{E}[W(\tau_1)W(\tau_2)] = \min(\tau_1, \tau_2)$.

We now consider the class of CUSUM change-point tests and show that the Kupiec test can be regarded as a member of this class. To make any statements regarding the asymptotic properties of the CUSUM change-point tests, we need to construct a piecewise constant CUSUM process. With this in mind, define $X_P(\tau)$, a family of partial sums, as

$$X_P(\tau) = \frac{1}{\sqrt{P}} \left(\sum_{t=1}^{\lceil \tau P \rceil} X_t - \lceil \tau P \rceil \alpha \right). \tag{9}$$

The CUSUM test for uniformity of the process $\{X_t\}_{t=1}^{P}$ is based on deviations of the partial sum $X_P(\tau)$ from the total sum $X_P(1) = \frac{1}{\sqrt{P}}(\sum_{t=1}^{P} X_t - P\alpha)$, which leads to the following test statistic:

$$M_P^{(CS)}(\tau) := X_P(\tau) - \tau X_P(1). \tag{10}$$

By construction, this process satisfies, as $P \to \infty$, the following

$$\sup_{0 \leq \tau \leq 1} M_P^{(CS)}(\tau) \xrightarrow{\mathcal{D}} \sqrt{\alpha(1-\alpha)} \sup_{0 \leq \tau \leq 1} (W(\tau) - \tau W(1)) \equiv \sqrt{\alpha(1-\alpha)} \sup_{0 \leq \tau \leq 1} B(\tau), \tag{11}$$

where $B(\tau)$ denotes a Brownian bridge.

Moreover, under H_O, $X_t = I_{t,\alpha}(\theta)$ for $1 \leq t \leq P$, and $M_P^{(CS)}(\cdot)$ is denoted $M_P^{(CS,O)}(\cdot)$. This process is expressed as

$$M_P^{(CS,O)}(\tau) := \frac{1}{\sqrt{P}} \sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\theta) - \frac{\tau}{\sqrt{P}} \sum_{t=1}^{P} I_{t,\alpha}(\theta), \tag{12}$$

since the α terms cancel, and therefore, in terms of the Kupiec test for different sample periods as

$$M_P^{(CS,O)}(\tau) = \widetilde{K}_P(\tau) - \tau \widetilde{K}_P(1). \tag{13}$$

We now have our piece-wise continuous partial sum process with jump points at $\tau = \frac{k}{P}$, that satisfies, as $P \to \infty$,

$$\sup_{0 \leq \tau \leq 1} M_P^{(CS,O)}(\tau) \xrightarrow{\mathcal{D}} \sqrt{\alpha(1-\alpha)} \sup_{0 \leq \tau \leq 1} B(\tau). \tag{14}$$

This result is immediate by applying (8).

Thus far τ was restricted to the interval [0, 1] which led to historical out-of-sample (sample size P) and in-sample (R replaces P in the formulas above) CUSUM tests. Alternatively to the out-of-sample procedure (off-line monitoring), one can devise a sequential CUSUM test defined by $1 < \tau \leq \infty$ that extends the in-sample method, defined by $0 \leq \tau \leq 1$, and allows to detect deviations of the risk model as soon as these occur (on-line monitoring). In this method, however, and as discussed in Andreou and Ghysels, the limiting distribution of the relevant CUSUM statistic is determined by the boundary crossing probability of $B(\tau)$ on $[1, \tau]$. This boundary can be any strictly positive function $\xi b(\tau)$, where ξ is a scaling factor satisfying $\mathbb{P}\{|B(\tau)| < \xi b(\tau) \text{ for all } \tau > 1\} = \xi b(\tau)$, hence the problem of choosing appropriate boundary functions $b(\tau)$ in this testing environment. Andreou and Ghysels discuss, in a conditional volatility assessment context, a battery of boundary function candidates.

Also note that it is the historical out-of-sample version of the CUSUM test, and not the sequential one, which is in the spirit of the standard backtesting methodology proposed in Basel Accord (1996) and implemented by regulators. In the remainder of the article we will concentrate on historical out-of-sample CUSUM-type tests.

2.2 An Alternative Change-Point Detection Test

Section 2.1 introduced the CUSUM test and then showed how the Kupiec test could be considered as a special case of this test. As the CUSUM figures prominently in this research, it is interesting to make a few observations regarding the CUSUM test; one regarding the similarity of this test with that of the Kolmogorov-Smirnov (K-S) statistic; and the other regarding the consistency of the power function of the K-S statistic to deviations from the hypothesized distribution that may occur in the tails. When the summand in (10) is set to the indicator of some random event, i.e. $X_t = I(Y_t < y)$, where Y_t, $t = 1, \ldots, P$ are *IID*, this results in the right-continuous empirical distribution function. This function is an important ingredient from which the K-S statistic is fashioned. Indeed, the K-S statistic calculates, uniformly, the distance between the empirical distribution function and the distribution function specified under the null hypothesis, and rejects said null hypothesis when this distance is too great. Mason and Schuenemeyer (M-S) (1983) have shown that, both in finite and large sample theory, the K-S statistic exhibits poor sensitivity to deviations that may occur in the tails: M-S establish that the K-S statistic is inconsistent against such deviations. A similar fate holds true here for our above mentioned CUSUM test: the CUSUM test is also insensitive to deviations of this nature.

In an attempt to rectify this apparent insensitivity of the K-S test to deviations from the hypothesized distribution that may occur in the tails, M-S apply weights to their statistics and find that these weighted statistics perform much better than the K-S statistic - they are consistent against such deviations. Moreover, they state that, while they are unable to find uniformly good weight functions, there do exist weight functions, dependent upon P, that make weighted versions of the K-S statistic consistent with respect to deviations that may occur in the tails. Hence, just as in the K-S setting where it was shown to be useful to employ weights, it is also desirable in our setting to introduce weights that may remedy this situation somewhat on the tails, i.e., in particular for early detection of deviations of VaR models.

More interestingly, Orasch and Pouliot (2004) study the empirical power of statistics constructed from the CUSUM, as well as the CUSUM statistic itself, that test for a change in the location parameter that occurs early on in the sample. They find that the CUSUM test is completely insensitive to such deviations. More specifically, the family of partial sum processes defined in (10) is more powerful for detecting changes in the distribution that occur near $P/2$ than noticing changes near the endpoints, 1 and P of the sample. These observations would indicate the value of constructing test statistics that are more sensitive to tail alternatives or, in the case of this research, early detection, yet remain sensitive to departures that may occur later on as well. Altissimo and Corradi and Andreou and Ghysels, in a different context, also present alternatives to the standard CUSUM designed to have power in the tails.

There remains, however, the heretofore answered question of what form the optimally selected weights should take and how to weight the partial sum processes detailed in (10) and (12). The choice of weight functions remains an active area of research, work by Csörgő and Horváth (1997, 1988a, 1988b) provide a detailed account of the use of weight functions and site some of the many interesting properties of weighted statistics. For our purposes, however, we focus on Theorem 2.1. in Szyszkowicz (1991), which is referred to here as Theorem S. Before this we need to introduce some definitions and basic notation.

Definition 1.

1.) Let Q be the class of positive functions on $(0,1)$ which are non-decreasing in a neighborhood of zero and non-increasing in a neighborhood of one, where a function q defined on (0,1) is called positive if

$$\inf_{\delta \leq \tau \leq 1-\delta} q(\tau) > 0 \quad for\ all\ \delta \in (0, 1/2). \tag{15}$$

2.) Let $c > 0$ be a constant value. Then for $q \, \epsilon \, Q$,

$$\Psi(q, c) \;=\; \int_0^1 \frac{1}{\tau(1-\tau)} \exp(-\frac{c}{\tau(1-\tau)q^2(\tau)}) d\tau. \tag{16}$$

Let kernel $h(x,y)$ satisfy the following property: $h(x,y) = -h(y,x)$, *i.e.*, the kernel is antisymmetric. We have under H_0 that $\mathbb{E}h(X_1, X_2) = 0$. Let $\tilde{h}(t) = \mathbb{E}h(X_1, t)$ and assume that

$$\mathbb{E}h^2(X_1, X_2) \;<\; \infty \tag{17}$$

$$0 < \sigma^2 = \mathbb{E}\tilde{h}^2(X_1), \tag{18}$$

set

$$Z_k := \sum_{i=1}^{k} \sum_{j=k+1}^{P} h(X_i, X_j), \qquad 1 \leq k < P. \tag{19}$$

Theorem S. *Assume that Y_t for $t = 1, \ldots, P$ are IID, $h(x, y) = -h(x, y)$, (17) and (18) are satisfied. Then a sequence of Brownian bridges $\{B_P(\tau), 0 \leq \tau \leq 1\}$ can be defined such that, as $P \to \infty$,*

(i) $$\sup_{0<\tau<1} \frac{\frac{P^{-3/2}}{\sigma} Z_{\lceil \tau P \rceil} - B_P(\tau)}{q(\tau)} = \begin{cases} o_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for all } \; c > 0 \\ O_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for some } c > 0. \end{cases}$$

(ii) *Let $\{B(\tau);\ 0 \leq \tau \leq 1\}$ be a Brownian bridge. Then*

$$\sup_{0<\tau<1} \frac{1}{\sigma} \frac{|Z_{\lceil \tau P \rceil}|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{|B(\tau)|}{q(\tau)}$$

if and only if $\Psi(q, c) < \infty$ for some $c > 0$.

To connect (19) to our family of partial sum processes $M_P^{(CS,O)}(\cdot)$ we will assume $\tau = \frac{k}{P}$, set $h(x, y) = x - y$ and note that this kernel is antisymmetric. Replace $h(X_i, X_j)$ in (19) with $X_i - X_j$ which, after some algebra, reduces to the following;

$$Z_k = \sum_{i=1}^{k} X_i - k \sum_{j=1}^{P} X_j. \tag{20}$$

Set $X_i = I_{i,\alpha}(\theta)$ in (20) and normalize by $P^{3/2}$; after which we arrive at the following representation:

$$\frac{Z_k}{P^{3/2}} = \frac{\sum_{i=1}^{k} I_{i,\alpha}(\theta) - \frac{k}{P} \sum_{j=1}^{P} I_{j,\alpha}(\theta)}{P^{1/2}} \tag{21}$$

which corresponds to $M_P^{(CS,O)}(\tau)$, with $\tau = \frac{k}{P}$ and appropriate subscript t, as detailed in (12).

Using Theorem S, we are now able to make the following statements regarding a weighted version of $M_P^{(CS,O)}(\tau)$ the nature of which are detailed in Proposition 1.

Proposition 1. *Let H_O hold and $q \in Q$. Then we can define a sequence of Brownian bridges $\{B_P(\tau);\ 0 \leq \tau \leq 1\}$ such that, as $P \to \infty$, the following hold:*

i) $$\sup_{0<\tau<1} \frac{\frac{1}{(\alpha(1-\alpha))^{1/2}} M_P^{(CS,O)}(\tau) - B_P(\tau)}{q(\tau)} = \begin{cases} o_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for all } c > 0 \\ O_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for some } c > 0, \end{cases}$$

and

ii)

$$\sup_{0<\tau<1} \frac{\left| \frac{1}{(\alpha(1-\alpha))^{1/2}} M_P^{(CS,O)}(\tau) \right|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{\left| B(\tau) \right|}{q(\tau)}$$

if only if $\Psi(q,c) < \infty$ for some c.

Two important observations are made here and are detailed in Remark 1 and Corollary 1 given below. Corollary 1 ensures that Proposition 1 applies to the conditional VaR setting developed in Section 2.

Remark 1. *Let $\left\{ B_P(\tau) := \frac{W(P\tau) - \tau W(P)}{\sqrt{P}}; 0 \leq \tau \leq 1 \right\}$ be a version of a Brownian Bridge. Then, for $P = 1, 2, \ldots$, we have*

$$\{B_P(\tau);\ 0 \leq \tau \leq 1\} \stackrel{\mathcal{D}}{=} \{B(\tau);\ 0 \leq \tau \leq 1\}.$$

Corollary 1. *Let $\{m_\alpha(\Im_{t-1}, \theta)\}_{t=1}^P$ be a stochastic process satisfying, under H_O, that $I\!P\{Y_t \leq m_\alpha(\Im_{t-1}, \theta) | \Im_{t-1}\} = \alpha$. Then the sequence $\{I_{t,\alpha}(\theta)\}_{t=1}^P$ of indicator functions are IID.*

Proof. The proof of this result is immediate by observing that under H_O, $E[I_{t,\alpha}(\theta) - \alpha | \Im_{t-1}] = 0$. Now, by the law of iterated expectations it is immediate to obtain that $COV(I_{t,\alpha}(\theta) - \alpha, I_{t-j,\alpha}(\theta) - \alpha) = 0$ for all $j \neq 0$, and with it the mutual independence of the sequence of indicator functions. This last statement is consequence of the statistical properties of the bernoulli distribution function followed by the sequence of indicators.

Remark 2. *Corollary 1 shows that all propositions made in Section 2.2, 2.3 and 2.4 under H_O hold in the conditional VaR context introduced in* (2).

2.3 Population Parameter Unknown

The weighted partial sum process developed in (12) depends on an unknown vector of population parameters θ, that in practice is usually unknown. A natural solution is to replace θ by any consistent estimator. As we are interested in functionals of CUSUM test statistics, it would be of interest to know if such substitutions affect their limiting distribution. Such substitutions, as can be the case, increase the randomness of such functionals of these processes, and then cause the thus altered process to have a limiting distribution different from that of the functional of the original partial sum process. In what follows we show the convergence of the estimated CUSUM process to the same limiting distribution as the original CUSUM statistic, and with it the absence of the so-called estimation risk. For our purposes, we focus on the estimated version of the test statistic $M_P^{(CS,O)}(\tau)$ defined in (12) as

$$\widehat{M}_P^{(CS,O)}(\tau) := \frac{\sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\widehat{\theta}_{T+t}) - \tau \sum_{t=1}^{P} I_{t,\alpha}(\widehat{\theta}_{T+t})}{P^{1/2}}, \quad 0 \leq \tau \leq 1, \tag{22}$$

with $\{\widehat{\theta}_{T+t}\}_{t=1}^{P}$ any sequence of consistent estimators of the vector of parameters θ encompassing the three different schemes used in the backtesting literature, namely, the recursive, fixed and rolling forecasting schemes. They differ in how the parameter θ is estimated. In the recursive scheme the estimator $\widehat{\theta}_{T+t}$ is computed with all the sample available up to $T+t-1$ for $t = 1, \ldots, P$, and T denoting the in-sample size. The latter is also true for the fixed forecasting scheme, where the estimator is not updated when new observations become available, and therefore leaves $\{\widehat{\theta}_{T+t}\}_{t=1}^{P} \equiv \widehat{\theta}_T$. Finally, for the rolling estimator the subscript T denotes the number of observations used in the estimation process, in this case the sequence of estimators $\{\widehat{\theta}_{T+t}\}_{t=1}^{P}$ is constructed from the sample $t-1, \ldots, t+T-1$.

Now, following West (1996) and McCraken (2000) we assume $T, P \to \infty$, as $T+P \to \infty$, and satisfying $\lim_{T+P\to\infty} P/T = \pi$, with $0 \leq \pi < \infty$. This assumption allows us to state the consistency of the sequence of relevant estimators for each forecasting scheme and also, in this context, the application of functional central limit theorem results.

Proposition 2. *$I_{t,\alpha}(\theta)$ for $t = 1, \ldots, P$ be IID random variables, $q \in Q$ satisfy integral conditions $\Psi(q, c) < \infty$ for some $c > 0$, and let $\{\widehat{\theta}_{T+t}\}_{t=1}^{P}$ be the sequence of recursive, fixed or rolling* *estimators. Then, as $T, P \to \infty$,*

$$\sup_{0<\tau<1} \frac{\left|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)\right|}{q(\tau)} = o_P(1).$$

Proof. For the fixed forecasting scheme only T needs go to infinity and so the proposition follows by the continuous mapping theorem for all points θ of continuity of $I_{t,\alpha}(\theta)$. In the case of the recursive and rolling forecasting schemes both T and P need to go to infinity, as $T + P \to \infty$, consult the Mathematical Appendix for details.

Using Proposition 2, we are able to make the following statements concerning (22). The nature of these statements include one concerning approximation in probability and one regarding the asymptotic distribution of the supremum over τ of these processes. These are all detailed in Proposition 3, and are similar in nature to those statements made regarding the partial sum process (9) and detailed in Proposition 1.

Proposition 3. *Let H_O hold and $q \varepsilon Q$. Then we can define a sequence of Brownian Bridges $\{B_P(\tau);\ 0 \le \tau \le 1\}$ such that, as $T, P \to \infty$, the following hold:*

i) $$\sup_{0<\tau<1} \frac{\frac{1}{(\alpha(1-\alpha))^{1/2}}\widehat{M}_P^{(CS,O)}(\tau) - B_P(\tau)}{q(\tau)} = \begin{cases} o_P(1), & \textit{if and only if } \Psi(q,c) < \infty \quad \textit{for all } c > 0 \\ O_P(1), & \textit{if and only if } \Psi(q,c) < \infty \quad \textit{for some } c > 0, \end{cases}$$

and

ii)

$$\sup_{0<\tau<1} \frac{\left|\frac{1}{(\alpha(1-\alpha))^{1/2}}\widehat{M}_P^{(CS,O)}(\tau)\right|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{\left|B(\tau)\right|}{q(\tau)}$$

only if $\Psi(q,c) < \infty$ for some c.

Under the alternative, H_A, there remains one additional parameter to estimate, $k^\star$. A number of estimators of this parameter have been proposed in the literature but we provide only one, as it is intuitive and some of its large sample properties have been detailed in the literature.

$$\widehat{k^\star} := \min\left\{k : \frac{|M_P^{(CS,O)}(\frac{k}{P})|}{q(\frac{k}{P})} = \max_{1\le i<P} \frac{|M_P^{(CS,O)}(\frac{i}{P})|}{q(\frac{i}{P})}\right\}. \tag{23}$$

The asymptotic properties of this estimator have been studied by Antoch and Hušková(1995). They also show that the Bootstrap approximation to this distribution is asymptotically valid. For more on this, we refer those interested to their paper. This estimator of location of change in the VaR model should greatly assist risk managers in understanding the reasons for changes in their VaR model as they are now able to locate the timing of change and isolate major causes that contributed to said change.

2.4 Study of Power function of the different CUSUM tests

To obtain the asymptotic results detailed in Section 2.3, $q(\cdot)$ was required to satisfy the integral equation (16) but, there has been neither discussion on the form these weight functions may take nor from among those weight functions that satisfy (16) - for some or all c - which ones are optimal. This section explores this by comparing the asymptotic statistical power of standard and weighted versions of the CUSUM test. Section 2.4 also studies the power functions for different families of weight functions specially tailored for change-point detection tests. In particular, the family of weight functions introduced by Gombay, Horváth and Hušková(GHH), the extension of this family introduced by Orash and Pouliot (OP) and a novel further extension that we explore in this paper. The rationale of OP and our new family of functions is to propose alternative weighted CUSUM tests exhibiting more statistical power against rejections of H_O early on in the out-of-sample backtesting period. This is detailed as follows.

Remember that the process X_t is defined as

$$X_t = \begin{cases} 1(Y_t \leq m_\alpha(\Im_{t-1}, \theta)), & 1 \leq t \leq k^\star, \\ 1(Y_t \leq m^\star_\alpha(\Im_{t-1}, \theta)), & k^\star < t \leq P, \end{cases} \tag{24}$$

with $m^\star_\alpha(\Im_{t-1}, \theta) \neq m_\alpha(\Im_{t-1}, \theta)$, and both belonging to M. Also note from Section 2.1 that $M_P^{(CS)}(\tau)$ converges in distribution to $\sqrt{\alpha(1-\alpha)} \sup_{0<\tau<1} B(\tau)$.

Now, under the alternative hypothesis H_A and after some algebra the $M_P^{(CS,O)}(\tau)$ CUSUM test can be expressed as

$$M_P^{(CS,O)}(\tau) \;=\; M_P^{(CS)}(\tau) \;+\; \tau(1-\tau^\star)(\widetilde{\alpha}-\alpha)\sqrt{P} \;-\; \frac{\tau}{\sqrt{P}}\sum_{t=\tau^\star P+1}^{P}(I_{t,\alpha}(\theta)-\alpha)$$
$$+\; \frac{\tau}{\sqrt{P}}\sum_{t=\tau^\star P+1}^{P}(I^\star_{t,\alpha}(\theta)-\widetilde{\alpha}) \;+\; \frac{1}{\sqrt{P}}\sum_{t=\tau^\star P+1}^{\lceil \tau P\rceil}\left(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta)\right), \tag{25}$$

with $I^\star_{t,\alpha}(\theta) = 1(Y_t \le m^\star_\alpha(\Im_{t-1},\theta))$.

Without loss of generality, we have assumed that $\tau > \frac{k^\star}{P}$ to obtain equation (25). We make the two additional assumptions:

$$\delta(P) \to 0 \qquad \text{as} \;\; P \to \infty \tag{26}$$
$$\delta(P)P \to 0 \qquad \text{as} \;\; P \to \infty. \tag{27}$$

Using (25), we have the following result:

$$\sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} \;=\; \sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \Bigg| \frac{M_P^{(CS)}(\tau)}{q(\tau)} + \frac{\tau(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\tau)}\sqrt{P}$$
$$-\; \frac{\tau}{\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I_{t,\alpha}(\theta)-\alpha)}{q(\tau)} \;+\; \frac{\tau}{\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I^\star_{t,\alpha}(\theta)-\widetilde{\alpha})}{q(\tau)}$$
$$+\; \frac{1}{\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{\lceil\tau P\rceil}\left(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta)\right)}{q(\tau)}\Bigg|. \tag{28}$$

But

$$\sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} \;=\; \frac{|M_P^{(CS,O)}(\frac{k^\star}{P}))|}{q(\frac{k^\star}{P})}$$

which leads to the following simplification of (28);

$$\frac{|M_P^{(CS,O)}(\frac{k^\star}{P})|}{q(\frac{k^\star}{P})} \;=\; \Bigg| \frac{M_P^{(CS)}(\frac{k^\star}{P})}{q(\frac{k^\star}{P})} \;+\; \frac{k^\star}{P}\frac{(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\frac{k^\star}{P})}\sqrt{P}$$
$$-\; \frac{k^\star}{P\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I_{t,\alpha}(\theta)-\alpha)}{q(\frac{k^\star}{P})} \;+\; \frac{k^\star}{P\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I^\star_{t,\alpha}(\theta)-\widetilde{\alpha})}{q(\frac{k^\star}{P})}$$
$$+\; \frac{1}{\sqrt{P}}\frac{\sum_{t=k^\star+1}^{k^\star}\left(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta)\right)}{q(\frac{k^\star}{P})}\Bigg| \tag{29}$$

The last term in (29) is zero which leaves only the first four terms, i.e.,

$$\frac{|M_P^{(CS,O)}(\frac{k^\star}{P})|}{q(\frac{k^\star}{P})} = \left| \frac{M_P^{(CS)}(\frac{k^\star}{P})}{q(\frac{k^\star}{P})} + \frac{k^\star}{P}\frac{(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\frac{k^\star}{P})}\sqrt{P} - \frac{k^\star}{P\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I_{t,\alpha}(\theta)-\alpha)}{q(\frac{k^\star}{P})} + \frac{k^\star}{P\sqrt{P}}\frac{\sum_{t=\tau^\star P+1}^{P}(I_{t,\alpha}^\star(\theta)-\widetilde{\alpha})}{q(\frac{k^\star}{P})} \right|. \quad (30)$$

Expression (30) provides the rate at which the mean of $\frac{|M_P^{(CS,O)}(\frac{k^\star}{P})|}{q(\frac{k^\star}{P})}$ is increasing; it is precisely $\frac{k^\star}{P}\frac{(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\frac{k^\star}{P})}\sqrt{P}$. Hence this must be subtracted to ensure that the statistic has zero mean. Moreover, we can remove the absolute value sign in (30) by noting that if the sum is negative, then multiply by -1. This does not, however, alter the limiting distribution which is $N(0,\cdot)$; if, on the other hand it is positive, then it can be removed without consequence. Hence, the absolute value sign does not affect the limiting distribution and can be removed.

Proposition 4. *Under H_A, (5), (26), (27) and (30), then, as $P \to \infty$,*

$$\begin{bmatrix} \frac{1}{\sqrt{P}}\sum_{t=1}^{k^\star}(I_{t,\alpha}(\theta)-\alpha) \\ \frac{1}{\sqrt{P}}\sum_{t=1}^{k^\star}(I_{t,\alpha}(\theta)-\alpha) + \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I_{t,\alpha}^\star(\theta)-\alpha) \\ \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha}) \\ \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I_{t,\alpha}^\star(\theta)-\alpha) \end{bmatrix} \xrightarrow{\mathcal{D}} N(0,\Sigma), \quad (31)$$

where

$$\Sigma = \begin{bmatrix} \tau^\star\alpha(1-\alpha) & \tau^\star\alpha(1-\alpha) & 0 & 0 \\ & \alpha(1-\alpha) & (1-\tau^\star)COV(I_{k^\star+1,\alpha}(\theta), I_{k^\star+1,\alpha}^\star(\theta)) & (1-\tau^\star)\alpha(1-\alpha) \\ & & (1-\tau^\star)\widetilde{\alpha}(1-\widetilde{\alpha}) & (1-\tau^\star)COV(I_{k^\star+1,\alpha}(\theta), I_{k^\star+1,\alpha}^\star(\theta)) \\ & & & (1-\tau^\star)\alpha(1-\alpha) \end{bmatrix} \quad (32)$$

Proof. This is a direct consequence of the multivariate version of the Lindberg-Levy CLT.

The matrix Σ depends on α $\widetilde{\alpha}$, $\tau^\star$ and $COV(I_{k^\star+1,\alpha}(\theta), I^\star_{k^\star+1,\alpha}(\theta))$. Note that

$$\mathbb{E}I_{k^\star+1,\alpha}I^\star_{k^\star+1,\alpha} = \mathbb{P}\left\{Y_{k^\star+1} \leq \min\{m_\alpha(\theta), m^\star_\alpha(\theta)\} \mid \Im_{k^\star}\right\}, \tag{33}$$

and hence

$$COV(I_{k^\star+1,\alpha}(\theta), I^\star_{k^\star+1,\alpha}(\theta)) = \begin{cases} \alpha(1-\widetilde{\alpha}), & \text{if } m_\alpha(\theta) > m^\star_\alpha(\theta), \\ \widetilde{\alpha}(1-\alpha), & \text{otherwise.} \end{cases} \tag{34}$$

Now, using Proposition 4, the following statement can be made regarding the statistic detailed in (12). Define $h^{'} = [1, -\tau^\star, -\tau^\star, \tau^\star]$.

Theorem 1. *Assume H_A, (26), (27) all hold, then as, $P \to \infty$,*

$$\frac{q(\tau^\star)}{\sqrt{h^{'}\Sigma h}}\left[\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} - \frac{\tau^\star(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\tau^\star)}\sqrt{P}\right] \xrightarrow{\mathcal{D}} N(0,1), \tag{35}$$

with $h^{'}\Sigma h = \tau^\star(1-\tau^\star)\left\{\alpha(1-\alpha) + \tau^\star\left[\widetilde{\alpha}(1-\widetilde{\alpha}) - \alpha(1-\alpha)\right]\right\}$.

Proof. This follows as a result of Proposition 4, Theorem 1 and equation (30).

The next result follows as a consequence of the above theorem.

Corollary 2. *Theorem 1 establishes that*

$$\sup_{0<\tau<1}\frac{|M^{(CS,O)}(\tau)|}{P^{1/2}q(\tau)} \xrightarrow{P} \frac{\tau^\star(1-\tau^\star)(\widetilde{\alpha}-\alpha)}{q(\tau^\star)}$$

as $P \to \infty$ and, as a result, the consistency of each CUSUM-type test.

With these results in place we can study the power function of each CUSUM-type test. In order to do this we derive first the expression for the size of the different change-point tests characterized by the family of weight functions $q(\tau)$:

$$\lim_{P\to\infty} P_{H_0}\left\{\frac{1}{\sqrt{\alpha(1-\alpha)}}\sup_{0<\tau<1}\frac{\left|M_P^{(CS,0)}(\tau)\right|}{q(\tau)} > C^q_{1-\beta}\right\} = \beta, \tag{36}$$

with $C^q_{1-\beta}$ the relevant critical value of the asymptotic distribution of $\sup\limits_{0<\tau<1}\frac{|B(\tau)|}{q(\tau)}$.

Proposition 5. *Assume a set of local alternative hypotheses defined by a VaR model with coverage probability $\widetilde{\alpha}$, from $\tau^\star$, that satisfies $\widetilde{\alpha} - \alpha = \frac{a}{P^\gamma}$, with $a, \gamma > 0$ constant values. Then, the power function (pf) at a β significance level is defined by*

$$pf_\beta = 1 - \Phi\left(q(\tau^\star) C^q_{1-\beta} \sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}} - \frac{\tau^\star(1-\tau^\star)}{\sqrt{h'\Sigma h}} a P^{1/2-\gamma} \right), \tag{37}$$

with $\Phi(\cdot)$ the cdf of a standard normal distribution.

The power of the change point test statistic $\sup\limits_{0<\tau<1} \left| M_P^{(CS,O)}(\tau) \right|$ is a function of the distance between the processes $m_\alpha(\Im_{t-1}, \theta)$ and $m^\star_\alpha(\Im_{t-1}, \theta)$, the timing of the break $\tau^\star$ and also the weight function $q(\tau)$. It is interesting to observe that pf_β is independent of the covariance between the null and alternative conditional quantile processes, since the covariance terms in Σ cancel out when computing the asymptotic variance $h'\Sigma h$ of the CUSUM-type statistics.

In order to compare the power of each test in rejecting the null hypothesis of no break point we can use, for $\gamma > 1/2$, a Taylor expansion of the different power functions about the constant value $q(\tau^\star) C^q_{1-\beta} \sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}}$, see for example Lehmann (18) for more on this technique. Then

$$pf_\beta = 1 - \Phi\left(q(\tau^\star) C^q_{1-\beta} \sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}} \right) + \phi\left(q(\tau^\star) C^q_{1-\beta} \sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}} \right) \frac{\tau^\star(1-\tau^\star)}{\sqrt{h'\Sigma h}} a P^{1/2-\gamma} + o\left(P^{1/2-\gamma} \right), \tag{38}$$

with $\phi(\cdot)$ the density function of a standard normal distribution.

Corollary 3. *As $P \to \infty$, the weighted CUSUM-type test exhibits more power than the standard CUSUM ($q(\tau) \equiv 1$ for all τ) when*

$$q(\tau^\star) < \frac{C_{1-\beta}}{C^q_{1-\beta}}. \tag{39}$$

Following similar arguments, it is simple to see that the same condition holds for ranking CUSUM tests in terms of the power function for $\gamma = 1/2$. Also, this condition can be extended to compare CUSUM tests determined by different weight functions $q_1(\tau)$ and $q_2(\tau)$. Thus, the first method exhibits more statistical power than the second weighted CUSUM if the following holds

$$\frac{q_1(\tau^\star)}{q_2(\tau^\star)} < \frac{C^{q_2}_{1-\beta}}{C^{q_1}_{1-\beta}}, \tag{40}$$

with $C^{q_1}_{1-\beta}$ and $C^{q_2}_{1-\beta}$ the relevant critical values.

These conditions can be further refined if one uses a specific family of weight functions $q(\cdot)$ satisfying certain smoothing conditions. Thus, in a different context GHH proposed $q(\tau,\nu) = \{(\tau(1-\tau))^{\nu};\ 0 \leq \nu \leq 1/2\}$ to develop weighted CUSUM-type tests for testing for at most one change in the variance. In what follows we will concentrate on versions of this family in order to obtain results on local and global optimal power for the hypothesis test of change-point detection. Thus, following (40) one can construct a condition to discriminate between weight functions within the $GHH-$family indexed by ν. A weight function determined by ν_1 is preferred to ν_2 in terms of power for certain fixed $\tau^\star$ if the following condition holds

$$\tau^\star(1-\tau^\star) - \left(\frac{C^{q(\cdot,\nu_2)}_{1-\beta}}{C^{q(\cdot,\nu_1)}_{1-\beta}}\right)^{\frac{1}{\nu_1-\nu_2}} < 0. \tag{41}$$

The differentiability on τ of this family of functions also allows us to extend this local result to the entire domain of τ. A sufficient condition for this is

$$4^{\nu_1-\nu_2}\left(\frac{C^{q(\cdot,\nu_2)}_{1-\beta}}{C^{q(\cdot,\nu_1)}_{1-\beta}}\right) > 1, \tag{42}$$

if $\nu_1 > \nu_2$.

This result is immediately obtained by taking first derivatives of the difference in (41) and observing that it attains a global maximum at $\tau = 1/2$. In general condition (41) does not hold and one needs to check condition (41) for different values of τ and choose the optimal function locally, depending on the detection region of interest. For this one needs the relevant critical values. The complete asymptotic distributions of the CUSUM-type statistics defined by this family of weight functions are tabulated for $\nu = 0$ in Orasch and Pouliot (2004); for $\nu = 1/16$, 3/16, 5/16 and 7/16 in Section 6 of this paper. Also, the asymptotic distribution of the weighted CUSUM test for $\nu = 1/2$ can be found in closed form in Gombay, Horváth and Hušková(1996). Thus, for example, the relevant critical values at 10%, 5% and 1% are [1.232, 1.366, 1.640] for the standard CUSUM ($\nu = 0$) and [2.943, 3.660, 5.293] for the weighted CUSUM for $\nu = 1/2$.

It remains unanswered whether one can design a weight function that is optimal against alternatives produced by change-points at a certain known $\tau^\star$. The following panels shed light on this by plotting the first derivative function of the power function in (38) adapted to $q(\tau,\nu)$;

$$-\phi\left(q(\tau,\nu)C_{1-\beta}^{q(\cdot,\nu)}\sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}}\right)\left[\frac{\partial q(\tau,\nu)}{\partial\tau}C_{1-\beta}^{q(\cdot,\nu)}\sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}}-\frac{1}{2}q(\tau,\nu)C_{1-\beta}^{q(\cdot,\nu)}\sqrt{\alpha(1-\alpha)}\frac{\partial(h'\Sigma h)/\partial\tau}{(h'\Sigma h)^{3/2}}\right]. \tag{43}$$

There are two conclusions that one can extract from these plots. First, as ν's proximity to 1/2 increases, the test becomes more sensitive to alternatives in the tails, and second, this sensitivity is directional. In other words, equation (43) depends on $h'\Sigma h$ and therefore on the value of α and $\widetilde{\alpha}$; thus, the unique root of the above equation corresponds to either early detection for over-conservative[1] models ($\alpha = 0.05$ and $\widetilde{\alpha} = 0.01$, in the example reported on the left panel) or late detection for under-conservative[2] models ($\alpha = 0.01$ and $\widetilde{\alpha} = 0.05$, on the right panel).

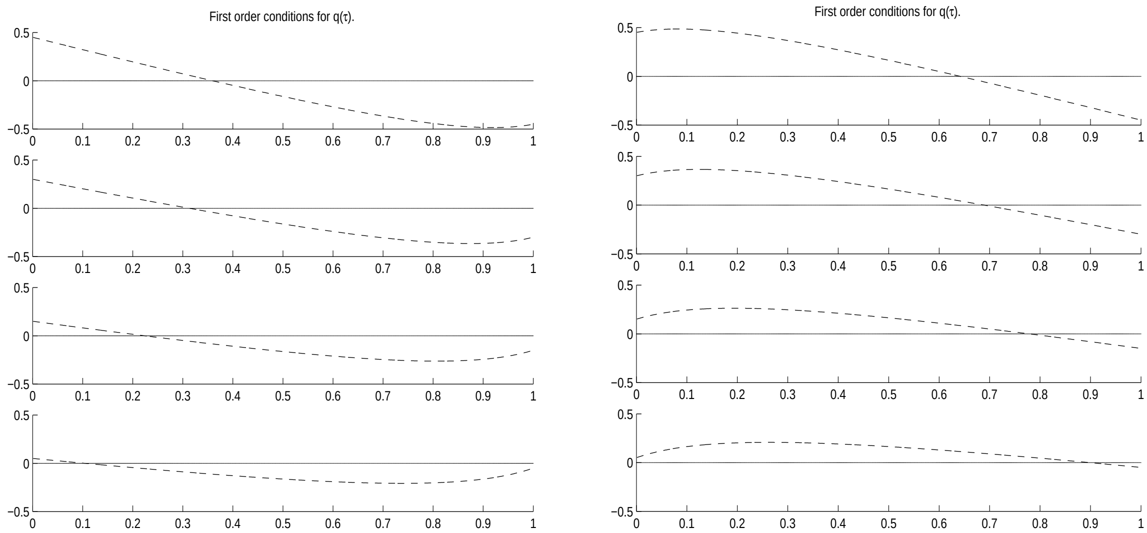

Figure 1. First order conditions of the power function in (43) for $q(\tau,\nu) = (\tau(1-\tau))^{\nu}$, with $\nu = 0.05$, 0.20, 0.35, 0.45 (Up-down panels). Left panel corresponds to $\alpha = 0.05$ and $\widetilde{\alpha} = 0.01$ and right panel to $\alpha = 0.01$ and $\widetilde{\alpha} = 0.05$

In an extensive simulation study and in a different context, Orasch and Pouliot show that the optimal weight function for early detection determined by $\nu = 1/2$ can be further improved by giving extra weight to both tails. We extend the function proposed by these authors in the same

[1]overly conservative in the sense that the model was specified to be wrong 5% of the time but is in fact wrong only 1% of the time.

[2]under conservative in the sense that the model was specified to be wrong only 1% of the time but is in fact wrong 5% of the time.

spirit of the GHH-family of functions, thus, our new family of functions is

$$q^{step}(\tau,\nu) := \begin{cases} (\tau(1-\tau))^{\nu} & \text{if } \tau\ \varepsilon\ (a,b) \\ \left(\tau(1-\tau)\log\log\frac{1}{\tau(1-\tau)}\right)^{\nu} & \text{if } \tau\ \varepsilon\ \ [0,a]\ \sqcup\ [b,1), \end{cases} \tag{44}$$

where $a = 0.071033$ and $b = 0.92896$ are derived by imposing continuity on the q^{step} function. This family of weight functions also satisfies the integrability condition for some c and, therefore, the results in Propositions 1 and 3 for the associated CUSUM tests defined by q^{step} hold.

Intuitively, one might expect $q^{step}(\tau,\nu)$ to out-perform the weight function of GHH for similar ν. To see this, $q^{step}(\tau,\nu) = (\tau(1-\tau))^{\nu}$ for $a < \tau < b$; hence for changes to the model that occur for τ in this range both weight functions should have similar power. This is not the case for $0 < \tau \leq a$ or $b \leq \tau < 1$, since $q^{step}(\tau,\nu)$ places more weight than $(\tau(1-\tau))^{\nu}$; hence $q^{step}(\tau,\nu)$ should be more powerful for τ in this range. This is an intuitive explanation and it may not hold mathematically. Indeed, the extent to which one function is in fact better than another can be quantified in the case of local alternatives and that is the next task. Thus, this family of functions has more statistical power than $q(\tau,\nu)$ in the tails defined by a and b if

$$\frac{(\tau(1-\tau))^{\nu_1}}{\left(\tau(1-\tau)\log\log\frac{1}{\tau(1-\tau)}\right)^{\nu_2}} - \frac{C_{1-\beta}^{q^{step}(\cdot,\nu_2)}}{C_{1-\beta}^{q(\cdot,\nu_1)}} < 0, \quad \text{for } 0<\tau\leq a, \text{ and } b\leq\tau<1, \tag{45}$$

with $C_{1-\beta}^{q(\cdot,\nu_1)}$ and $C_{1-\beta}^{q^{step}(\cdot,\nu_2)}$ the relevant critical values, and for $a < \tau < b$ if (41) holds with the appropriate critical values.

We also tabulate the asymptotic *cdf* of $q^{step}(\cdot)$ for $\nu = 1/16$, 3/16, 5/16 and 7/16. These are not reported in the paper for sake of space, but can be obtained from the authors upon request. Also, for $\nu = 1/2$ this can be found in Orash and Pouliot (22). In particular the critical values at 10%, 5% and 1% of the associated CUSUM test for $\nu = 1/2$ are [2.940, 3.180, 3.680].

Figure 2 illustrates these findings. On the left panel we find evidence of better power of both $q(\tau,1/2)$ and $q^{step}(\tau,1/2)$ weighted CUSUM tests compared to standard CUSUM; on the right panel we plot condition (45), and observe the outperformance of $q^{step}(\tau,1/2)$ over $q(\tau,1/2)$. The examples corresponding to other values of ν can be obtained from the authors upon request.

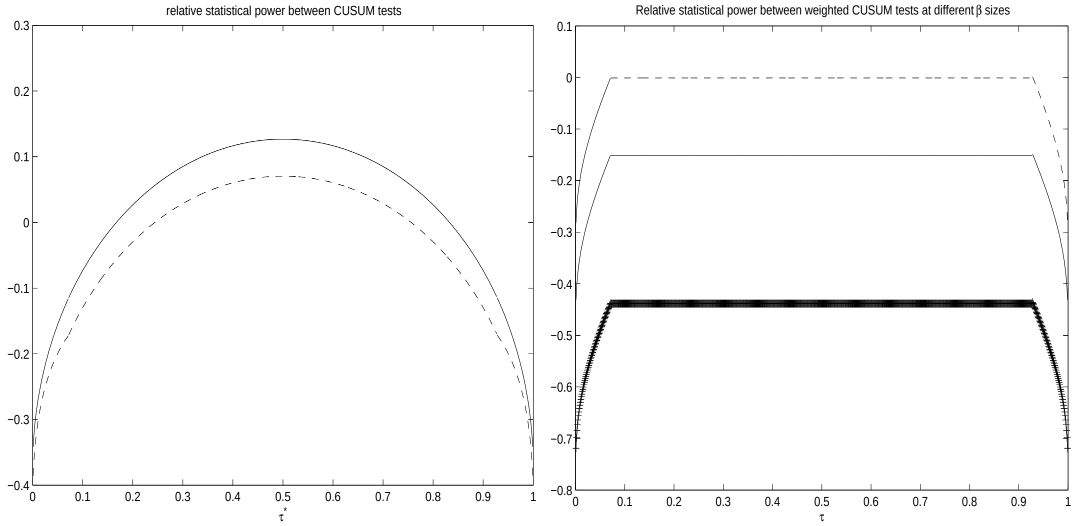

Figure 2. Relative power functions. Left panel plots left term of expression (41) for $q(\tau, 1/2)$ and $q^{step}(\tau, 1/2)$ against standard CUSUM at $\beta = 0.05$ size. Solid line for $q(\tau, 1/2)$ and dashed line for $q^{step}(\tau, 1/2)$. Right panel plots left term of expression (45) for $q(\tau, 1/2)$ against $q^{step}(\tau, 1/2)$. Dashed line for $\beta = 0.10$, solid line for 0.05 and + line for 0.01.

The new weight function boosts the statistical power of the associated CUSUM test compared to the standard CUSUM and GHH-CUSUM tests. The following section introduces a simulation study for different conditional VaR models widely employed in the literature, that sheds some light about the importance of these weight functions for realistic conditional VaR models used in risk management.

3 Monte-Carlo Experiments

The aim of this section is to see the significance of our theoretical findings in finite samples. In order to do this we carry out a study of the finite-sample size and power of the different CUSUM-type tests developed in the previous section. We will concentrate on three features of these change point tests: first, we will see that the three tests are well designed in the sense of yielding empirical sizes close to the nominal size. The second aim will be the study of potential finite-sample distorting effects in the empirical size of the test derived from estimating the relevant parameters of the VaR model. Finally, we will gauge the power of the tests against different departures of the null hypothesis.

The null hypothesis is given by assuming no structural break in a conditional VaR modelled by a location-scale process. In particular we consider a pure GARCH(1,1) model with Gaussian innovations. The different alternative hypotheses consist of out-of-sample structural breaks in the risk model produced by switches of the data generating process to location-scale models of different form. We consider an IGARCH(1,1) process with parameters close to those of the null hypothesis, an ARCH(1) process, and a GARCH(1,1) model with the same structure as the null model but with innovations driven by a Student-t distribution with five degrees of freedom. All these models are enclosed within the following structure:

$$Y_t = \mu(\Im_{t-1}, \beta_0) + \sigma(\Im_{t-1}, \beta_0)\varepsilon_t, \tag{46}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are specifications for the conditional mean and standard deviation of Y_t given $\Im_{t-1}$, respectively, and ε_t are the standardized innovations which are usually assumed to be *IID,* and independent of $\Im_{t-1}$. Note that under such assumptions the α-th conditional VaR is given by

$$m_\alpha(\Im_{t-1}, \theta_0) = \mu(\Im_{t-1}, \beta_0) + \sigma(\Im_{t-1}, \beta_0)F_\varepsilon^{-1}(\alpha), \tag{47}$$

where $F_\varepsilon^{-1}(\alpha)$ denotes a univariate quantile function of ε_t and the nuisance parameter vector is $\theta_0 = (\beta_0, F_\varepsilon^{-1}(\alpha))$. Thus, the α-th conditional VaR under H_0 in our case is given by

$$m_\alpha(\Im_{t-1}, \theta_0) = \sqrt{0.05 + 0.1Y_{t-1}^2 + 0.85\sigma^2(\Im_{t-1}, \beta_0)}\Phi_\varepsilon^{-1}(\alpha), \tag{48}$$

with $\Phi^{-1}(\cdot)$ the inverse of the standard normal *cdf*.

The true conditional VaR under the alternatives are

$$m_\alpha(\Im_{t-1}, \theta_0) = \sqrt{0.05 + 0.15Y_{t-1}^2 + 0.85\sigma^2(\Im_{t-1}, \beta_0)}\Phi_\varepsilon^{-1}(\alpha), \tag{49}$$

for the IGARCH process;

$$m_\alpha(\Im_{t-1}, \theta_0) = \sqrt{0.05 + 0.85Y_{t-1}^2}\Phi_\varepsilon^{-1}(\alpha), \tag{50}$$

for the ARCH(1) process, and finally

$$m_\alpha(\Im_{t-1}, \theta_0) = \sqrt{0.05 + 0.1Y_{t-1}^2 + 0.85\sigma^2(\Im_{t-1}, \beta_0)}t_{5,\varepsilon}^{-1}(\alpha), \tag{51}$$

with $t_{5,\varepsilon}(\cdot)$ denoting the *cdf* of a Student-t distribution with five degrees of freedom.

Tables 1 and 2 report the empirical size for the standard CUSUM, for the GHH-CUSUM ($CUSUM_{GHH}$) and for the new alternative CUSUM ($CUSUM_{q^{step}}$) tests, the last two tests computed for $\nu = 1/2$. The hypothesis test is computed for VaR measures computed at $\alpha = 1\%$ and $\alpha = 5\%$ coverage probability, and derived from model (48). The results are as follows:

$\alpha = 0.01$	CUSUM			$CUSUM_{GHH}$			$CUSUM_{q^{step}}$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
n=100	0.049	0.035	0.018	0.076	0.059	0.040	0.089	0.081	0.056
n=300	0.040	0.019	0.006	0.084	0.060	0.034	0.095	0.077	0.052
n=500	0.035	0.027	0.007	0.085	0.057	0.036	0.088	0.070	0.041
$\alpha = 0.05$	CUSUM			$CUSUM_{GHH}$			$CUSUM_{q^{step}}$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
n=100	0.046	0.027	0.004	0.094	0.076	0.007	0.106	0.096	0.024
n=300	0.038	0.016	0.006	0.077	0.068	0.008	0.087	0.074	0.019
n=500	0.042	0.019	0.001	0.082	0.067	0.010	0.090	0.065	0.017

Table 1. Empirical size of CUSUM-type tests for (48). θ parameters are assumed to be known.

Whereas the weighted CUSUM tests approximate rather well the different nominal sizes for both $\alpha = 1\%, 5\%$ the standard CUSUM test exhibits quite poor results. For these sample sizes the empirical size underestimates considerably the nominal size in the tails. In the following table 2 we study the impact of estimation effects on the size of the test. For this we compute empirical sizes of the relevant out-of-sample tests using P observations, and assume a previous in-sample period of $R = 500$ observations to estimate, by quasi-maximum likelihood, the parameters of the GARCH(1,1) model (fixed forecasting scheme). In contrast to standard backtesting tests, see Escanciano and Olmo (2008), the results of this table lends support to the hypothesis of no estimation risk for CUSUM-type tests. Similar analyses have been carried out for null hypotheses given by heavy-tailed error distributions. These are available from the authors upon request.

$\alpha = 0.01$	CUSUM			$CUSUM_{GHH}$			$CUSUM_{q^{step}}$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
P=100	0.057	0.037	0.018	0.087	0.072	0.048	0.112	0.094	0.063
P=300	0.072	0.048	0.029	0.127	0.095	0.060	0.140	0.113	0.078
P=500	0.072	0.043	0.017	0.109	0.087	0.049	0.131	0.099	0.070
$\alpha = 0.05$	CUSUM			$CUSUM_{GHH}$			$CUSUM_{q^{step}}$		
	10%	5%	1%	10%	5%	1%	10%	5%	1%
P=100	0.047	0.023	0.007	0.071	0.063	0.009	0.081	0.074	0.030
P=300	0.067	0.039	0.010	0.109	0.089	0.018	0.121	0.098	0.031
P=500	0.061	0.029	0.013	0.109	0.089	0.018	0.121	0.098	0.031

Table 2. Empirical size of CUSUM-type tests for (48). θ parameters are estimated under H_O.

The following set of tables illustrates the power of the different CUSUM-type tests at $\beta = 5\%$ against departures of the null hypothesis given by the alternative models discussed above. We analyze single structural breaks occurring at four fractions of the out-of-sample period: $\tau = 0.05$, 0.1, 0.3 and 0.5. The power is computed assuming, as in the latter two tables, a previous in-sample period of $R = 500$ observations where the parameters of the null model are estimated using the fixed forecasting scheme.

The results for the IGARCH and ARCH(1) alternatives are reported in Tables 3 and 4. Note that whereas for $\tau = 0.05$, 0.1 and 0.3 the weighted CUSUM methods outperform the standard CUSUM, for $\tau = 0.5$ the latter improves considerably over the other two methods. The power of the standard CUSUM for $\tau = 0.1$ is very small supporting the fact widely documented in the literature of the lack of power of these tests for early detection.

$\alpha = 0.01$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0	0.064	0.145	0.086	0.07	0.080	0.091	0.095	0.156	0.090	0.111	0.113
P=300	0	0.026	0.074	0.085	0.084	0.076	0.099	0.105	0.091	0.176	0.144	0.134
P=500	0.001	0.020	0.089	0.094	0.123	0.102	0.134	0.120	0.146	0.094	0.184	0.157
$\alpha = 0.05$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0.028	0.004	0.055	0.089	0.522	0.096	0.085	0.077	0.522	0.102	0.115	0.115
P=300	0.06	0.018	0.173	0.257	0.10	0.114	0.142	0.139	0.10	0.130	0.233	0.216
P=500	0.001	0.016	0.321	0.522	0.171	0.124	0.242	0.216	0.164	0.160	0.342	0.376

Table 3. Empirical power of CUSUM-type tests against model (49). θ parameters are estimated under H_O.

$\alpha = 0.01$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0.000	0.017	0.043	0.057	0.07	0.071	0.092	0.094	0.156	0.087	0.111	0.120
P=300	0.009	0.024	0.044	0.065	0.084	0.092	0.094	0.105	0.091	0.100	0.119	0.131
P=500	0.001	0.018	0.057	0.083	0.123	0.094	0.094	0.112	0.146	0.115	0.057	0.142
$\alpha = 0.05$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0.022	0.007	0.033	0.081	0.068	0.069	0.073	0.076	0.074	0.084	0.102	0.113
P=300	0.009	0.013	0.090	0.171	0.062	0.118	0.096	0.124	0.073	0.131	0.160	0.191
P=500	0.019	0.015	0.185	0.303	0.073	0.103	0.153	0.158	0.099	0.148	0.215	0.243

Table 4. Empirical power of CUSUM-type tests against model (50). θ parameters are estimated under H_O.

For the heavy-tailed GARCH(1,1) model the results in Table 5 also show the consistent outperformance of the q^{step}-CUSUM over the standard and GHH-CUSUM test statistics; this holds across fractions of the testing period. It is also striking the very low power of the standard CUSUM test across all fractions. It also can be observed from the tables that the GHH-CUSUM test loses ability to detect changes in the model for larger fractions. This is in contrast to the q^{step}-CUSUM which shows improved ability to detect changes in the model for larger fractions. This simulation confirms the intuitive motivation for combining weight functions which leads to better performance of q^{step}-CUSUM relative to both GHH-CUSUM and standard CUSUM.

$\alpha = 0.01$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0.022	0.082	0.048	0.047	0.061	0.110	0.083	0.080	0.088	0.136	0.103	0.102
P=300	0.012	0.064	0.048	0.025	0.091	0.082	0.086	0.073	0.08	0.116	0.099	0.084
P=500	0.021	0.058	0.030	0.028	0.091	0.080	0.082	0.094	0.098	0.104	0.098	0.101
$\alpha = 0.05$	CUSUM				$CUSUM_{GHH}$				$CUSUM_{q^{step}}$			
τ	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5	0.05	0.1	0.3	0.5
P=100	0.075	0.026	0.020	0.037	0.09	0.100	0.073	0.075	0.121	0.110	0.092	0.105
P=300	0.074	0.026	0.043	0.040	0.108	0.060	0.085	0.078	0.138	0.066	0.103	0.098
P=500	0.058	0.022	0.041	0.059	0.088	0.064	0.073	0.103	0.109	0.082	0.093	0.123

Table 5. Empirical power of CUSUM-type tests against model (51). θ parameters are estimated under H_O.

4 Conclusion

Backtesting techniques are of paramount importance for risk managers and regulators concerned with assessing the risk exposure of a financial institution to market risk. We have shown in this paper that by combining the standard backtesting Kupiec test statistic computed over different periods one can develop alternative backtesting procedures that allow not only to detect deviations of the risk model from the actual risk exposure but also to determine the timing of these departures. This is possible by using certain combinations of the Kupiec test that result in CUSUM-type tests.

Within this class of nonparametric statistics we have been shown via power calculation as well as through simulations that weighted versions of these U-statistic type processes significantly improve the sensitivity of change-point detection tests to changes in parameters that underly the VaR model; in particular we have shown this for a large class of location-scale models in the simulation section. We also show that refinements of the weighted CUSUM family of tests introduced in Gombay, Horváth and Hušková (1996) by Orasch and Pouliot (2004) perform, in most cases, better in this risk monitoring framework for early detection of a change in the VaR model. Moreover, this statistic still remains sensitive to changes in the middle of the out-of-sample period as well. It was noticed that there were some finite sample effects that led to larger empirical size than the nominal size. This, however, can be mitigated through the use of critical values that correspond to simulated finite sample critical values.

The use of the family of weight functions $q(\tau, \nu)$, and in particular $q^{step}(\tau, \nu)$, which displayed power for all fractions of τ, should facilitate risk managers whose task is to ensure that their model accurately reflects the risk exposure of the returns process. With these techniques in place they should be able to detect breaks in the relevant conditional VaR process at the earliest possible time; thereby enabling financial institutions to manage their capital requirements as well as avoid penalties from failing to report accurately their VaR. In addition, an estimator of the time of change, which has been overlooked in backtesting, has been provided as well which should provide additional diagnostic techniques for risk managers.

5 Mathematical Appendix

Proposition 2

Proof.

$$
\begin{aligned}
\sup_{0<\tau<1} \frac{\left|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)\right|}{q(\tau)} &\leq \sup_{0<\tau<\frac{1}{P+1}} \frac{\left|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)\right|}{q(\tau)} \\
&\quad + \sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}} \frac{\left|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)\right|}{q(\tau)} + \sup_{\frac{P}{P+1}<\tau<1} \frac{\left|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)\right|}{q(\tau)} \\
&= J_1(P) + J_2(P) + J_3(P) \qquad (52) \\
&= o_P(1), \quad T, P \to \infty. \qquad (53)
\end{aligned}
$$

The result claimed in (53) can be established by verifying that each term in (52) is $o_P(1)$, as $T, P \to \infty$. To do this, first note that $I_{t,\alpha}(\widehat{\theta}_{T+t})$ converges in probability to $I_{t,\alpha}(\theta)$, as $T \to \infty$, at all θ's that are points of continuity, and for $\{\widehat{\theta}_{T+t}\}_{t=1}^{P}$, a sequence of consistent estimators of θ. $J_1(P)$ and $J_3(P)$ satisfy (53) which can be established along the lines of GHH page 155, or, if more details are required, one can consult Pouliot (2001, Chapter 4) and relevant sections therein.

To establish $J_2(P) = o_P(1)$, as $T, P \to \infty$, requires a slightly different argument, one that we provide the details for here. For every $\epsilon > 0$, consider the following:

$$
\begin{aligned}
\mathbb{P}\left\{\frac{|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)|}{q(\tau)} > \epsilon\right\} &\leq \mathbb{P}\left\{P^{-1/2}\frac{|\sum_{t=1}^{\lceil\tau P\rceil} I_{t,\alpha}(\widehat{\theta}_{T+t}) - \sum_{t=1}^{\lceil\tau P\rceil} I_{t,\alpha}(\theta)|}{q(\tau)} > \frac{\epsilon}{2}\right\} \qquad (54) \\
&\quad + \mathbb{P}\left\{P^{-1/2}\frac{\tau|\sum_{t=1}^{P} I_{t,\alpha}(\widehat{\theta}_{T+t}) - \sum_{t=1}^{P} I_{t,\alpha}(\theta)|}{q(\tau)} > \frac{\epsilon}{2}\right\} \\
&\leq \frac{8}{\epsilon}\mathbb{E}\sum_{t=1}^{P}|I_{t,\alpha}(\widehat{\theta}_{T+t}) - I_{t,\alpha}(\theta)| \sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}} \frac{\tau}{P^{1/2}q(\tau)} \qquad (55)
\end{aligned}
$$

The inequality (54) follows from basic results in probability theory, while (55) follows from

Markov's inequality. (55) implies immediately the following inequality:

$$\lim_{P\to\infty}\lim_{T\to\infty}\mathbb{P}\left\{\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{|\widehat{M}_P^{(CS,O)}(\tau)\ -\ M_P^{(CS),O}(\tau)|}{q(\tau)}\ >\ \epsilon\right\}\ \leq \tag{56}$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\lim_{T\to\infty}\left[\mathbb{E}\left[\sum_{t=1}^{P}|I_{t,\alpha}(\widehat{\theta}_{T+t})-I_{t,\alpha}(\theta)|\right]\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau}{P^{1/2}q(\tau)}\right]=$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\lim_{T\to\infty}\left[\mathbb{E}\left[\sum_{t=1}^{\infty}I(0<t\leq P)\,|I_{t,\alpha}(\widehat{\theta}_{T+t})-I_{t,\alpha}(\theta)|\right]\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau}{P^{1/2}q(\tau)}\right]=0. \tag{57}$$

The result detailed in equality (57) follows from the consistency of the sequence of estimators of $\{\widehat{\theta}_{T+t}\}_{t=1}^{P}$ and from a Proposition from Cohn (1980) [Proposition 3.1.5, page 89].

Proposition 3

Proof.

This follows from statement ii) of Theorem S and Proposition 2 (Proposition 2 requires the integral condition to hold only for some $c>0$). This proposition in conjunction with Theorem S, as $T,P\to\infty$, provide the following result;

$$\begin{aligned}\left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|\widehat{M}_P^{(CS,O)}(\tau)|}{q(\tau)}\leq x\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|B(\tau)|}{q(\tau)}\leq x\right\}\right| &\leq \left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|\widehat{M}_P^{(CS,O)}(\tau)|}{q(\tau)}\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)}\leq x\right\}\right|\\ &+\left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)}\leq x\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|B(\tau)|}{q(\tau)}\leq x\right\}\right|\\ &=\ 0,\end{aligned}$$

for all $x\in\mathbb{R}$. The last line establishes statement ii) of Proposition 3.

6 Tabulated CDFs for Gombay, Horváth and Hušková's Weighted Statistics

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\{\sup \lvert \frac{W(t)-tW(1)}{(t(1-t))^{7/16}} \rvert \le x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.146	0.01	1.734	0.34	2.112	0.67
1.206	0.02	1.744	0.35	2.123	0.68
1.251	0.03	1.754	0.36	2.136	0.69
1.284	0.04	1.764	0.37	2.149	0.70
1.315	0.05	1.776	0.38	2.164	0.71
1.345	0.06	1.787	0.39	2.180	0.72
1.370	0.07	1.797	0.40	2.194	0.73
1.390	0.08	1.809	0.41	2.209	0.74
1.408	0.09	1.822	0.42	2.227	0.75
1.426	0.10	1.833	0.43	2.244	0.76
1.443	0.11	1.844	0.44	2.261	0.77
1.458	0.12	1.855	0.45	2.278	0.78
1.473	0.13	1.865	0.46	2.297	0.79
1.489	0.14	1.876	0.47	2.315	0.80
1.504	0.15	1.887	0.48	2.333	0.81
1.517	0.16	1.897	0.49	2.353	0.82
1.531	0.17	1.909	0.50	2.375	0.83
1.545	0.18	1.921	0.51	2.397	0.84
1.558	0.19	1.931	0.52	2.421	0.85
1.572	0.20	1.941	0.53	2.446	0.86
1.586	0.21	1.953	0.54	2.472	0.87
1.599	0.22	1.964	0.55	2.502	0.88
1.611	0.23	1.976	0.56	2.532	0.89
1.623	0.24	1.988	0.57	2.563	0.90
1.633	0.25	2.001	0.58	2.595	0.91
1.644	0.26	2.014	0.59	2.631	0.92
1.655	0.27	2.026	0.60	2.675	0.93
1.666	0.28	2.037	0.61	2.727	0.94
1.676	0.29	2.048	0.62	2.784	0.95
1.688	0.30	2.060	0.63	2.856	0.96
1.700	0.31	2.073	0.64	2.931	0.97
1.711	0.32	2.086	0.65	3.080	0.98
1.722	0.33	2.101	0.66	3.282	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\{\sup \lvert \frac{W(t)-tW(1)}{(t(1-t))^{5/16}} \rvert \le x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.818	0.01	1.27	0.34	1.593	0.67
0.857	0.02	1.28	0.35	1.604	0.68
0.887	0.03	1.29	0.36	1.616	0.69
0.913	0.04	1.30	0.37	1.629	0.70
0.936	0.05	1.31	0.38	1.642	0.71
0.957	0.06	1.32	0.39	1.655	0.72
0.975	0.07	1.33	0.40	1.669	0.73
0.993	0.08	1.34	0.41	1.684	0.74
1.008	0.09	1.34	0.42	1.697	0.75
1.022	0.10	1.35	0.43	1.711	0.76
1.039	0.11	1.36	0.44	1.726	0.77
1.051	0.12	1.37	0.45	1.743	0.78
1.062	0.13	1.38	0.46	1.758	0.79
1.074	0.14	1.39	0.47	1.772	0.80
1.086	0.15	1.40	0.48	1.787	0.81
1.097	0.16	1.41	0.49	1.803	0.82
1.109	0.17	1.42	0.50	1.819	0.83
1.121	0.18	1.43	0.51	1.840	0.84
1.131	0.19	1.44	0.52	1.862	0.85
1.142	0.20	1.45	0.53	1.886	0.86
1.152	0.21	1.46	0.54	1.908	0.87
1.162	0.22	1.47	0.55	1.932	0.88
1.171	0.23	1.48	0.56	1.959	0.89
1.180	0.24	1.49	0.57	1.987	0.90
1.189	0.25	1.50	0.58	2.021	0.91
1.197	0.26	1.51	0.59	2.065	0.92
1.206	0.27	1.52	0.60	2.109	0.93
1.215	0.28	1.53	0.61	2.148	0.94
1.225	0.29	1.54	0.62	2.201	0.95
1.234	0.30	1.55	0.63	2.268	0.96
1.244	0.31	1.56	0.64	2.345	0.97
1.253	0.32	1.57	0.65	2.449	0.98
1.262	0.33	1.58	0.66	2.624	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\{\sup \lvert \frac{W(t)-tW(1)}{(t(1-t))^{3/16}} \rvert \le x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.620	0.01	0.991	0.34	1.273	0.67
0.654	0.02	0.998	0.35	1.283	0.68
0.681	0.03	1.005	0.36	1.292	0.69
0.698	0.04	1.014	0.37	1.302	0.70
0.714	0.05	1.022	0.38	1.311	0.71
0.730	0.06	1.029	0.39	1.321	0.72
0.745	0.07	1.036	0.40	1.333	0.73
0.759	0.08	1.044	0.41	1.345	0.74
0.771	0.09	1.053	0.42	1.357	0.75
0.783	0.10	1.062	0.43	1.370	0.76
0.794	0.11	1.070	0.44	1.384	0.77
0.806	0.12	1.078	0.45	1.397	0.78
0.816	0.13	1.086	0.46	1.411	0.79
0.826	0.14	1.095	0.47	1.424	0.80
0.836	0.15	1.103	0.48	1.438	0.81
0.845	0.16	1.111	0.49	1.454	0.82
0.855	0.17	1.120	0.50	1.469	0.83
0.864	0.18	1.130	0.51	1.484	0.84
0.873	0.19	1.140	0.52	1.500	0.85
0.881	0.20	1.148	0.53	1.519	0.86
0.889	0.21	1.155	0.54	1.540	0.87
0.897	0.22	1.163	0.55	1.563	0.88
0.906	0.23	1.172	0.56	1.589	0.89
0.915	0.24	1.180	0.57	1.621	0.90
0.924	0.25	1.188	0.58	1.649	0.91
0.932	0.26	1.197	0.59	1.679	0.92
0.941	0.27	1.207	0.60	1.713	0.93
0.949	0.28	1.217	0.61	1.755	0.94
0.957	0.29	1.226	0.62	1.798	0.95
0.964	0.30	1.236	0.63	1.852	0.96
0.971	0.31	1.245	0.64	1.920	0.97
0.978	0.32	1.253	0.65	2.005	0.98
0.985	0.33	1.263	0.66	2.166	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\{\sup \lvert \frac{W(t)-tW(1)}{(t(1-t))^{1/16}} \rvert \leq x\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.482	0.01	0.795	0.34	1.032	0.67
0.514	0.02	0.802	0.35	1.040	0.68
0.534	0.03	0.808	0.36	1.049	0.69
0.551	0.04	0.814	0.37	1.058	0.70
0.566	0.05	0.820	0.38	1.067	0.71
0.578	0.06	0.826	0.39	1.076	0.72
0.589	0.07	0.833	0.40	1.086	0.73
0.600	0.08	0.839	0.41	1.095	0.74
0.610	0.09	0.845	0.42	1.105	0.75
0.619	0.10	0.852	0.43	1.116	0.76
0.629	0.11	0.858	0.44	1.128	0.77
0.638	0.12	0.865	0.45	1.140	0.78
0.646	0.13	0.873	0.46	1.152	0.79
0.655	0.14	0.880	0.47	1.165	0.80
0.663	0.15	0.887	0.48	1.177	0.81
0.671	0.16	0.893	0.49	1.190	0.82
0.678	0.17	0.900	0.50	1.203	0.83
0.686	0.18	0.908	0.51	1.218	0.84
0.693	0.19	0.916	0.52	1.232	0.85
0.700	0.20	0.924	0.53	1.248	0.86
0.707	0.21	0.932	0.54	1.265	0.87
0.714	0.22	0.940	0.55	1.284	0.88
0.722	0.23	0.948	0.56	1.306	0.89
0.729	0.24	0.955	0.57	1.330	0.90
0.735	0.25	0.962	0.58	1.355	0.91
0.742	0.26	0.969	0.59	1.381	0.92
0.749	0.27	0.977	0.60	1.410	0.93
0.756	0.28	0.984	0.61	1.443	0.94
0.763	0.29	0.992	0.62	1.483	0.95
0.769	0.30	0.999	0.63	1.531	0.96
0.775	0.31	1.008	0.64	1.591	0.97
0.781	0.32	1.016	0.65	1.653	0.98
0.788	0.33	1.024	0.66	1.795	0.99

References

[1] Altissimo, F. Corradi. V. (2003). *Strong rules for detecting the number of breaks in a time series*, Journal of Econometrics Vol. 117, 207 - 244.

[2] Andreou, E. and Ghysels, E. (2006). *Monitoring disruptions in financial markets*, Journal of Econometrics, Vol. 135, pp. 77-124.

[3] Antoch, J. and Hušková, M. (1995). *Change-Point Problem and Bootstrap*, Nonparametric Statistics, Vol. 5 pp. 123-144.

[4] Basel Committee on Banking Supervision (1996). *Supervisory Framework for the Use of Backtesting in Conjunction with the Internal Models Approach to Market Risk Capital Requirements*, Bank for International Settlements.

[5] Christoffersen, P. F. (1998). *Evaluating Interval Forecasts*, International Economic Review, Vol. 39, No. 4. November.

[6] Cohn, Donald (1980). Measure Theory, Birkhauser, Stuttgart fourth reprint.

[7] Csörgő, M. and Horváth, L. (1997). Limit Theorems in Change-Point Analysis, John Wiley, Chichester.

[8] Csörgő, M. and Horváth, L. (1988a). *Nonparametric Methods for Changepoint Problems*, Handbook of Statistics, Vol. 7, 403-425. Elsevier Science Publishers B.V. (North-Holland).

[9] Csörgő, M. and Horváth, L. (1988b). *Invariance Principles for Changepoint Problems*, Journal of Multivariate Analysis, Vol. 27, 151-168.

[10] Csörgő, M. and Horváth, L. (1987). *Nonparametric tests for the changepoint problem*, Journal of Statistical Planning and Inference, Vol. 17, 1-19.

[11] Donsker, M. (1951). *An Invariance Principle for Certain Probability Limit Theorems. Four papers on probability*, Mem. Amer. Math. Statist. 2, 277-283.

[12] Engle T., Manganelli, S. (2004). *CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles*, Journal of Business and Economic Statistics, Vol 22.

[13] Escanciano, J.C., Olmo, J. (2008). *Backtesting VaR with Estimation Risk*, Journal of Business and Economic Statistics, forthcoming.

[14] Gombay E. and Horváth L. and Hušková, M. (1996). *Estimators and Tests for Change in Variances*, Statistics & Decisions 14, 145 - 159.

[15] Koenker, and Xiao, (2006). *Quantile Autoregression*, JASA, Vol. 101, pp. 980-990.

[16] Kuan, C. M. and Hornik, K. (1995). *The generalized fluctuation test: unifying view*, Econometric Reviews 14, pp. 135-161.

[17] Kupiec, P. (1995). *Techniques for Verifying the Accuracy of Risk Measurement Models*, The Journal of Derivatives, Vol. 3 No. 2, Winter.

[18] Lehmann, E.L. (1999). *Elements of Large-Sample Theory,* Springer-Verlag, New York.

[19] Leisch, F., Hornik, K., Kuan,C.M. (2000). *Monitoring structural changes with the generzlized fluctuation test*, Econometric Theory 16. pp. 835 - 854.

[20] Mason, D. M. and Schuenemeyer, J. H. (1983). *A Modified Kolmogorov-Smirnov Test Sensitive to Tail Alternatives.* The Annals of Statistics, Vol. 11, No. 3.

[21] McCracken, M.W. (2000). *Robust Out-of-sample Inference*, Journal of Econometrics 99, 5, 195-223.

[22] Orasch, M. and Pouliot, W. (2004). *Tabulating Weighted Sup-norm Functionals Used in Change-point Analysis*, Journal of Statistical Computation and Simulation, Vol. 74, April, pp. 249 - 276.

[23] Page, E. S. (1954). *A Test for a change in a Parameter occurring at an unknown Point*, Biometrika 42, 523-526.

[24] Pouliot, W. (2001). *Non-Sequential, Non-Parametric Change-Point Analysis*, Carleton University Dissertation.

[25] Szyszkowicz, B. (1991). *Weighted Stochastic Processes Under Contiguous Alternatives*, C.R. Math. Rep. Acad. Sci. Canada, Vol XIII, No. 5, 211-216.

[26] West, K.D. (1996). *Asymptotic Inference about Predictive Ability*, Econometrica 64, 5, 1067-1084.