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Gini, Deprivation and Complaints

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Abstract

Recent insights from the philosopher Larry Temkin have suggested a new basis for the measurement of income inequality, founded on the notion of individual complaints. about income distribution. Under certain specifications of the relationship between complaints and personal incomes it can be shown that a concept similar to the concept of deprivation then emerges. In turn deprivation is related to the Gini index and to poverty. The paper examines the relationships between the Gini index and Lorenz orderings on the one hand and deprivation, poverty and complaints on the other hand.

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1 Introduction

The early contributions of Gini and Lorenz have shaped the way the whole subject of the income-distribution analysis has developed (Gini 1912, Lorenz 1905). Yet, over the last three or four decades, the Gini-Lorenz insights have been largely reinterpreted using the welfare-based approaches pioneered by Atkinson (1970) and Kolm (1969) taking their cue from the early work by Dalton (1920). This explicitly welfarist approach to the analysis of income distribution has influenced the development of research methodology and practical policy tools.

However, it is clear that a welfarist approach may not be necessary or even desirable: some have difficulty with issues such as the type of social consensus that supposedly underpins a social-welfare function; others may feel that coherent statements can be made about inequality comparisons without any reference to welfare. So alternative approaches to the subject have used analogies with information theory or on an explicit axiomatisation of inequality that does not use the device of the social-welfare function.[1]

More recent work has attempted to reconsider the fundamental nature of income inequality and to examine the meaning of particular concepts in distributional analysis that lie outside the territory familiar social-welfare analysis and information theory. Typically these focus on income differences rather than on individual income levels. The purpose of this paper is to draw together results from a number of these recent contributions, to show the relationships between them and related work on deprivation and poverty, and to discuss the relationship with the original insights by Gini and Lorenz.

2 The setting

Let us begin by setting out a simplified framework of analysis for discussing the interconnected topics that form the theme of this paper. For present purposes it is convenient to work with a fixed, finite population of economic agents who are identical in every respect other than income. The analysis can be extended to other empirically relevant cases by, for example, adjusting income using an equivalence scale and reweighting family units accordingly. Also, for many of the measures, we could easily use a more general distribution-function approach to present the results.

[1] See for example Theil (1967) on information theory, Bourguignon (1979), Cowell (1980), Shorrocks (1980) on axiomatic approaches.

2.1 Population and income

Consider a given population of individuals

$$N := \{1, ..., n\}.$$

For each individual i there is an exogenously determined quantity to be known as "income," x_i.

The income distribution in the population is given by an n-vector

$$\mathbf{x} := (x_1, x_2, ..., x_n).$$

Let $\mathbb{R}$ denote the set of real numbers and let Ω_n^* be the set of ordered n-vectors:

$$\Omega_n^* := \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n,\ x_1 \leq x_2 \leq ... \leq x_n\} \tag{1}$$

We may consider $\mathbf{x}$ to be taken from a connected subset $\mathbb{D}$ of Ω_n^*: one possibility is that $\mathbb{D}$ is the set of non-negative (ordered) n-vectors. This would be appropriate if "income" is to be defined in a way that automatically rules out negative numbers; for example if "income" were in reality expenditure then it would be natural to assume $x_i \geq 0$. Some approaches choose to focus on a concept of individual welfare or utility which may, perhaps, be taken as a simple transform of individual income, in which case all that is required is a reinterpretation of $\mathbb{D}$.[2] Given the precise specification of $\mathbb{D}$ one can then, for example, represent inequality, poverty and other indices as functions from $\mathbb{D}$ to $\mathbb{R}$.

The methodology broadly consists in setting out a fairly parsimonious set of axioms that characterise the essential tools of distributional analysis. These tools can be summarised as:

- *Evaluation functions.* Perhaps the best known example of such a function is the Gini coefficient itself. This is one representative of a wide class that includes social-welfare functions, poverty measures and inequality measures.

- *Ranking criteria.* The prime example here is obviously the set of second-order ranking criteria associated with the Lorenz curve.

[2] However, this is not an innocuous assumption. Individual utility may well be a function of other people's incomes or utilities as well, in which case the relationship between simple properties of oredrergins and welfare properties may no longer hold – see Amiel and Cowell (1994).

More specifically the idea is to find, for any specific problem in distributional analysis, a set of axioms that appropriately capture the principles that have an intuitive or ethical appeal for the problem in question and then to show that this specific set of axioms is necessary and sufficient for a particular evaluation function or ranking criterion to satisfy the stated principles.

2.2 The axiomatic approach

It might seem that the axiomatic approach is somewhat arbitrary. However, one of the arguments of this paper is that there is a commonality of axioms across a number of topics that yield important insights on the connections between various principles of distributional analysis.

To begin with there are a few basic axioms that are frequently invoked to define the structure of evaluation functions and rankings. They are used so frequently that it makes sense to state a version of them here, before we have examined any of the specific distributional issues.

Let Φ be some evaluation function used for comparing income distributions, let δ, λ be scalars and $\mathbf{1} \in \mathbb{R}^n$ denote the vector $(1, 1, ..., 1)$.

Axiom 1 (Continuity) *Φ is a continuous function $\mathbb{D} \to \mathbb{R}$.*

Axiom 2 (Linear homogeneity) *For all $\mathbf{x} \in \mathbb{D}$, $\lambda > 0$*

$$\Phi(\lambda \mathbf{x}) = \lambda \Phi(\mathbf{x})$$

Axiom 3 (Translation invariance) *For all $\mathbf{x}, \mathbf{x} + \delta \mathbf{1} \in \mathbb{D}$*

$$\Phi(\mathbf{x} + \delta \mathbf{1}) = \Phi(\mathbf{x})$$

Axioms 1 to 3 can readily be adapted to the characterisation of ranking criteria rather than evaluation functions. These axioms, or modifications of them, endow the evaluation function Φ with a structure that turns out to be very useful for building a variety of tools for distributional analysis. Consider the use of these in characterising a familiar inequality index. The absolute Gini coefficient

$$I_{\mathrm{AG}}(\mathbf{x}) := \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} [x_j - x_i] \tag{2}$$

has contours as illustrated in Figure 1. The triangular area is the simplex with the centroid $\mathbf{1} = (1, 1, 1)$ at which there is perfect equality. Let point $\mathbf{x}^*$ be some arbitrary income distribution; by Axiom 1, along any path from $\mathbf{x}^*$

to **1** inequality continuously (but not necessarily monotonically) approaches the value of perfect equality, conventionally normalised at 0; a point $\mathbf{x}^{\lambda}$ lying along the ray through $\mathbf{x}^*$ such that $\mathbf{x}^{\lambda} = \lambda \mathbf{x}^*$ will represent a distribution with inequality $\lambda I_{\text{AG}}(\mathbf{x}^*)$ (Axiom 2); a point lying along a line through $\mathbf{x}^*$ parallel to the ray through **1** represents a distribution with the same inequality as $\mathbf{x}^*$ (Axiom 3).

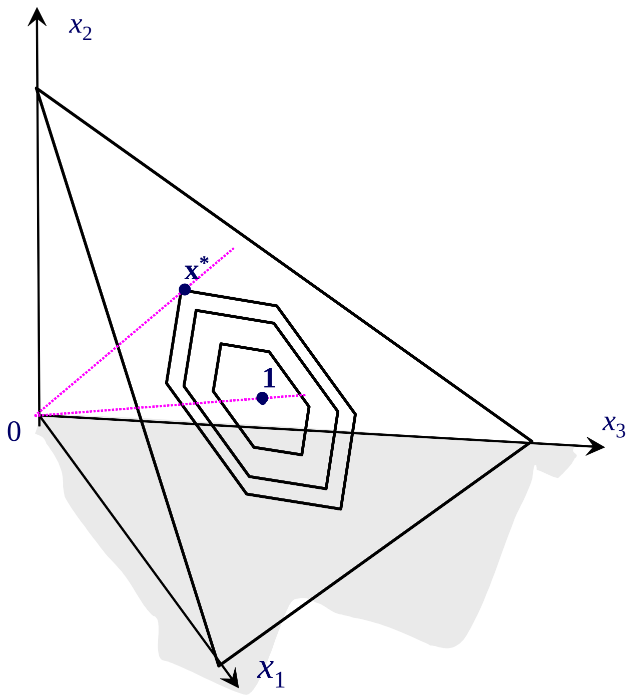

Figure 1: Contour map for the Gini, $n = 3$

What about ethical criteria such as the transfer principle? There are two approaches that have been adopted in the literature. The first is to build such a requirement in as an explicit principle for Φ. The second, and perhaps more satisfactory, is to allow it emerge from the axiomatic structure: here the idea is to allow the structural axioms and other essential properties to characterise a general class of functions to which Φ belongs and then to consider the members of this class that may satisfy the particular ethical principle.

3 Deprivation

Income differences lie at the heart of the Gini approach to inequality. They are also central to the concept of "relative deprivation" that has its origins in sociology (Runciman 1966). In the economics context the concept of deprivation can be seen as emerging naturally from the relationship between inequality measures and social-welfare functions. There are also intuitive and formal connections between deprivation and poverty analysis; in fact deprivation is structurally similar to the problem poverty measurement which we will briefly consider first.

3.1 Poverty

Sen's approach to poverty (Sen 1976, 1979) makes the connection between poverty and deprivation explicit. His approach also assists the present discussion by making clear two key components of the problem: the identification of the poor and the aggregation of information about the poor. The aggregation of information can be into a single poverty index (an example of an evaluation function) or into a poverty ranking.

The question of income can be considered to lie outside the scope of the present discussion although in any application the distinction between, say, total family income and consumption expenditure may be crucial for the identification of issue. Here income is just the quantity x used in section 2.

3.1.1 Identification and the reference point

The definition of a poverty line is a particularly convenient device because it automatically defines a reference point. Given a specific poverty line $z \in \mathbb{R}_+$ we can introduce the concept of the *poverty gap* for any person i

$$g_i(\mathbf{x}, z) = \begin{cases} z - x_i & \text{if } x_i \leq z \\ 0 & \text{otherwise.} \end{cases} . \tag{3}$$

The poverty gap concept plays a central rôle in Sen's approach and is also at the heart of the Foster et al. (1984) analysis that developed the family of poverty indices (evaluation functions) given by

$$\frac{1}{n} \sum_{i=1}^{n} \left[\frac{g_i(\mathbf{x}, z)}{z} \right]^{\alpha} \tag{4}$$

where α is a sensitivity index.

Let us define the cumulative poverty gap as

$$G_i(\mathbf{x}, z) := \frac{1}{n} \sum_{j=1}^{i} g_j(\mathbf{x}, z), \; i = 1, 2, ..., n \tag{5}$$

This then yields a key concept used for the purposes of ranking – the TIP curve (Jenkins and Lambert 1997) or poverty profile (Shorrocks 1998). This curve is formed by joining the points $\left(\frac{i}{n}, G_i(\mathbf{x}, z)\right)$ and must be increasing and concave.

3.1.2 Axiomatic approach

The seminal paper by Sen (1976) was remarkable for its introduction of the Gini coefficient into the analysis of poverty. It is clear that this emerges from the specific assumptions that Sen introduced about the nature of poverty including an explicit introduction of an assumption that the weight to be placed on the gap $g_i(\mathbf{x}, z)$ in the aggregation process is to be proportional to i itself – i.e. proportional to i's position in the income distribution.

However, an alternative approach to the axiomatisation of poverty has been provided by Ebert and Moyes (2002). The approach effectively examines the structure of rankings on $n + 1$ incomes – the n incomes of the agents $(x_1, x_2, ..., x_n)$ plus the poverty line z. They use Axioms 1 and 3 but replace Axiom 2 with scale invariance so that

$$P(\lambda\mathbf{x}, \lambda z) = P(\mathbf{x}, z) \tag{6}$$

where P is the ordinal function representing the poverty ranking. In addition the following axioms are required where, for convenience we define p as the number of of people who are poor:

$$p(\mathbf{x}, z) := \#\{i : x_i \leq z\}$$

Axiom 4 (Focus) *For $\mathbf{x} \in \mathbb{D}$ and $x_i > z$ P is constant in x_i.*

Axiom 5 (Monotonicity) *For $\mathbf{x} \in \mathbb{D}$ and $x_i \leq z$ P is strictly decreasing in x_i.*

Axiom 6 (Independence) *Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{D}$ be such that $P(\mathbf{x}, z) = P(\mathbf{y}, z)$, and $x_j = y_j$ for all $j \leq p(\mathbf{x}, z)$. Then, for any $i \leq p(\mathbf{x}, z)$, and any x° such that $x_{i-1} \leq x^\circ \leq x_{i+1}$*

$$P(x_1, x_2, ..., x_{i-1}, x^\circ, x_{i+1}..., x_n, z) = P(y_1, y_2, ..., y_{i-1}, x^\circ, y_{i+1}..., y_n, z)$$

Then, Ebert and Moyes (2002) show the following:

Theorem 1 *Given Axioms 1, 3, 4-6, and the scale-invariance property (6) the function P representing the poverty ranking must satisfy*

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\left[\frac{g_i\left(\mathbf{x},z\right)}{z}\right]^{\alpha},z\right) \tag{7}$$

or

$$\varphi\left(\frac{1}{n}\sum_{i=1}^{n}g_i\left(\mathbf{x},z\right)^{\alpha},z\right) \tag{8}$$

where $\alpha > 0$ and φ is continuous and increasing in its first argument.

Clearly (7) is just a transformation of the Foster et al. (1984) index (4), while (8) is an "absolute" counterpart of the "relative" index (4).[3]

3.2 Individual deprivation

The elements of a theory of individual deprivation are essentially the definition of income, the reference group, and an evaluation method. Again the definition of income can be set aside, for the same reasons as in section 3.1. The specification of the reference group can be based on intuition, on theories from the social sciences, or on an explicit axiomatisation.

3.2.1 The Yitzhaki approach

The key insight for our purposes was provided by Yitzhaki (1979). With hindsight this can be seen as a natural extension of one aspect of the Sen approach to the structure of poverty. Yitzhaki originally specified his individual deprivation measure using a fairly general formulation. If x is an individual's income and F is the distribution function for the economy in question then, assuming that individuals are alike in all respects other than income, the deprivation felt by someone with income x is

$$d\left(x\right) := \int_{x}^{\infty}\left[1 - F(y)\right]\, dy \tag{9}$$

Expression (9) is equivalent to

$$d\left(x\right) = \int_{x}^{\infty}\left[y - x\right]\, dF(y) \tag{10}$$

[3]The terms "absolute" and "relative" index are in quotes because both (7) and (8) satisfy both scale and translation invariance in transformations of $(\mathbf{x}, z)$.

In terms of the present notation, for a finite population (10) can be expressed as follows. Given the income distribution represented by the vector $\mathbf{x}$, the deprivation experienced by individual i is

$$d_i\left(\mathbf{x}\right) = \frac{1}{n}\sum_{j=i+1}^{n}\left[x_j - x_i\right] \tag{11}$$

Furthermore, define the conditional mean

$$\mu_i(\mathbf{x}) := \frac{1}{n-i}\sum_{j=i+1}^{n} x_j; \tag{12}$$

where we note in passing that the conventional mean is given by $\mu_0(\mathbf{x})$. Then (11) can be written equivalently as

$$d_i\left(\mathbf{x}\right) = \frac{n-i}{n}\left[\mu_i(\mathbf{x}) - x_i\right] \tag{13}$$

The individual deprivation index $d_i\left(\mathbf{x}\right)$ is evidently the counterpart to the "gap" concept (3) used in poverty analysis.

3.2.2 An axiomatic approach

However, the deprivation problem differs from the poverty in one important respect. The poverty line can be taken as exogenous information defining the poverty problem, but there is no counterpart to that in the deprivation problem. The poverty gap $g_i\left(\mathbf{x}, z\right)$ from this definition and scarcely needs axiomatisation – although in the Ebert and Moyes (2002) formulation it follows immediately from Axiom 3. In the case of deprivation one either has to assume (13) arbitrarily or find an appropriate method of axiomatising it.

Nevertheless a suitable axiomatisation of $g_i\left(\cdot\right)$ can be found using some of the same structure as for the characterisation of poverty. Clearly the crucial component of the problem is the definition of a reference group.

Ebert and Moyes (2000) provide an axiomatisation of individual deprivation whereby the index is to be defined for all logically possible reference groups for a given N. As an alternative Bossert and D'Ambrosio (2004) axiomatise the Yitzhaki index using an approach that differs from Ebert and Moyes (2000) in the way the reference group of an individual is to be represented. Although it is otherwise similar to Ebert and Moyes (2000), some of the other axioms in have to be modified or replaced as a result of this alternative way of characterising the reference group.

In the Bossert and D'Ambrosio (2004) approach the reference group for individual i is the "better-than" set

$$B_i(\mathbf{x}) = \{j \in N : x_j > x_i\}$$

Axiom 7 (Focus) *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{D}$ *such that* $B_i(\mathbf{x}) = B_i(\mathbf{y})$ *and* $x_j = y_j$ *for all* $j \in B_i(\mathbf{x})$ *and* $x_i = y_i$ *then*

$$d_i(\mathbf{x}) = d_i(\mathbf{y})$$

Axiom 8 (Normalisation) *For all* $x \in \mathbb{D}$, *and* $j \neq i$ *such that* $x_j = 1$ *and* $x_i = 0$, $i \neq j$

$$d_i(\mathbf{x}) = \frac{1}{n}$$

Axiom 9 (Additive decomposition) *For all* $x \in \mathbb{D}$, *let* B^1, B^2 *be any two mutually exclusive and exhaustive subsets of* $B_i(\mathbf{x})$ *and define vectors* x^1 *and* x^2 *such that*

$$x_j^t = \begin{cases} x_i & \textit{if } i \in B^t \\ x_j & \textit{otherwise} \end{cases}, \ t = 1, 2.$$

Then

$$d_i(\mathbf{x}) = d_i\left(\mathbf{x}^1\right) + d_i\left(\mathbf{x}^2\right).$$

Then Bossert and D'Ambrosio (2004) show:

Theorem 2 *Axioms 7 to 9, along with Axioms 2, 3 for the case* $\Phi = d_i$, *give*

$$d_i(\mathbf{x}) = \frac{1}{n} \sum_{j \in B_i(\mathbf{x})} [x_j - x_i]$$

The above expression is clearly the Yitzhaki index of individual deprivation (11) again.

3.3 Aggregate deprivation

Now consider the required elements for an approach to a concept of aggregate deprivation: clearly we need the definition of individual deprivation and an aggregation method.

3.3.1 A standard approach

Perhaps the most obvious way to derive a measure of aggregate deprivation from the individual deprivation measures is just to add them up. Using the Yitzhaki notation, suppose the deprivation experienced by a person with income x is measured by (9) or (10). Writing $\mu(F)$ for the mean of the distribution given by the distribution function F, expressions (9) or 10) can also be written as

$$\mu(F) - x + xq - C(F; q) \tag{14}$$

where

$$q = F(x)$$

and C is the income-cumulation function

$$C(F; q) := \int_0^q x(t)dt.$$

Integrating (14) over the distribution F we get

$$= -\int_0^1 C(F; q)dq + \int_0^\infty x \int_0^x dF(y)dF(x);$$

then the aggregated value of deprivation for the distribution F is

$$\mu(F) - 2\int_0^1 C(F; q)dq \tag{15}$$

which is the absolute Gini. In terms of the notation of section 2 we would have the simplified form of aggregate deprivation given by

$$\frac{1}{n}\sum_{i=1}^{n} d_i(\mathbf{x}) \tag{16}$$

which, on rearrangement, gives (2).

The form (15) shows the close relationship between this interpretation of deprivation and generalised-Lorenz rankings (Hey and Lambert 1980). Finally, note that (16) is an absolute index (because of translation invariance); we can obviously convert it into a relative index by dividing by the mean, in which case one gets the conventional Gini coefficient, which exhibits scale invariance and will decrease under uniform additions to all incomes.

3.3.2 Extensions

However, it may be worth considering alternative forms of aggregation of individual deprivation. In a manner similar to the Foster et al. (1984) aggregation of poverty gaps, the deprivation index suggested by Chakravarty and Chakraborty (1984) and developed further in Chakravarty and Mukherjee (1999b) aggregates individual deprivation as follows:

$$\left[\frac{1}{n}\sum_{i=1}^{n} w_i d_i(\mathbf{x})^{\varepsilon}\right]^{\frac{1}{\varepsilon}} \tag{17}$$

where $d_i(\mathbf{x})$ is given by (13), w_i is the weight $[1 - i/n]^{1-\varepsilon}$ and $\varepsilon \geq 1$ is a sensitivity parameter. A similar relative concept for aggregate deprivation has been suggested by Chakravarty and Mukherjee (1999a):[4]

$$1 - \left[\frac{1}{n}\sum_{i=1}^{n}\left[1 - \frac{d_i(\mathbf{x})}{\mu_0(\mathbf{x})}\right]^{\varepsilon}\right]^{\frac{1}{\varepsilon}} \tag{18}$$

where $\mu_0(\mathbf{x})$ is the mean for the whole distribution.[5]

3.3.3 Relationship with Gini

The rôle of the Gini coefficient in characterising deprivation has become familiar in the literature. The fact that individuals' rank is incorporated into

[4] D'Ambrosio and Frick (2004) take the concept a stage further. They examine the relationship between (a) relative deprivation/satisfaction, i.e. the gaps between the individual's income and the incomes of all individuals richer/poorer than him and (b) self-reported level of satisfaction with income and life.

[5] The approach is similar to the paper by Duclos and Grégoire (2002) who use the so-called S-Gini coefficient in the specification of a class of poverty indices that combine normative concerns for absolute and relative deprivation. Their indices are distinguished by a parameter that captures the ethical sensitivity of poverty measurement to "exclusion" or "relative-deprivation" aversion. The connection with the Chakravarty approaches can be seen if one writes the S-Gini as

$$\frac{1}{n}\sum_{i=1}^{n}\frac{d_i(\mathbf{x})}{\mu_0(\mathbf{x})} w_i(v)$$

where the weights are given by

$$w_i(v) := v[v-1]\left[1 - \frac{i}{n}\right]^{v-2}$$

its definition can be seen as a natural interpretation of social disadvantage.[6] However, a further lesson from the deprivation literature is the fundamental importance of differences – a concept that underlies all the approaches[7] and is also central to the Gini coefficient and the various types of generalised Gini coefficients. Furthermore the comparison between the poverty and relative deprivation approaches clearly draws attention to the concept of reference group and reference income, a point that is essential in the argument of Section 4.

4 Complaints and income distribution

The philosopher Larry Temkin introduced an alternative way of perceiving the income distribution in terms of inequality (Temkin 1986, 1993). Once again the rôle of income differences is central to the argument and it is interesting to see how this alternative approach relates to the Gini-Lorenz approach and to the analysis of deprivation considered in section 3.

4.1 The nature of complaints

4.1.1 Individual complaints

The fundamental concept required for the Temkin approach is that of an individual agent's complaint. Like the concept of deprivation examined in Section 3.2 the Temkin concept of complaint can be naturally expressed in terms of income differences. But what differences?[8]

[6] For other developments of the basic deprivation concept and its welfare interpretation see Berrebi and Silber (1985, 1989), Chakravarty (1998), Chakravarty and Mukherjee (1998), Stark and Yitzhaki (1988), Yitzhaki (1979, 1980, 1982).

[7] Podder (1996).provides an alternative approach to aggregate deprivation that does not appear to use the basic structural axioms in that he examines utility comparisons not income differences. However, we can see this as the basic idea applied to a utility transformation of income. The main idea is preserved if one just redefines $\mathbb{D}$ in terms of the space of utilities. Likewise in Chakravarty and Moyes (2003) deprivation is formulated in terms of utility rather than just in terms of income and they use this to examine the incidence of taxation on the amount of deprivation felt in the society.

[8] Note that the complaint is not the same as the (dis)utility of deprivation, as in Podder (1996) or Chakravarty and Moyes (2003). Rather, the complaint exists as an independent entity:

"To say that the best-off have nothing to complain about is in no way to impugn their moral sensibilities. They may be just as concerned about the inequality in their world as anyone else. Nor is it to deny that, insofar as one is concerned about inequality, one might have a complaint about them being as well off as they are. It is only to recognize that, since they are at least as well off as every other member of their world, they have nothing

The answer to this again depends on the concept of the reference group. Temkin identifies three, each associated with a specific reference income level

- Best-off person
- All those better off
- The average

We will examine the way each of these relates to the standard approaches to inequality measurement and to the notions of deprivation discussed earlier.

4.1.2 Aggregate complaint

Temkin (1993) suggested two approaches, simple summation – as we did for the basic deprivation index – or a weighted aggregation. We leave this question open until we have considered the individual interpretations of "complaint."

4.2 Best-off Person (BOP)

4.2.1 Axiomatic structure

Here individual i's complaint given the income distribution $\mathbf{x}$ is specified by the difference between i's income and that of the richest person:

$$k_i(\mathbf{x}) := x_n - x_i. \tag{19}$$

Of course there may be more than one richest person; so it is useful to define $r(\mathbf{x})$ as the lowest value of $j \in N$ such that $x_j = x_n$.

We can make the Temkin idea of complaint-based inequality specific by characterisng the shape of a family of inequality measures, using the approach of Cowell and Ebert (2004). Suppose the BOP-complaint version of the Temkin inequality index is an evaluation function T. Then Cowell and Ebert (2004) use the basic structural axioms given in section 2.2 (with Φ replaced by T) plus these:

Axiom 10 (Monotonicity) *For* $\mathbf{x} \in \mathbb{D}$ *and* $i < r(\mathbf{x})$ T *is strictly decreasing in* x_i.

to complain about. Similarly, to say that the worst-off have a complaint is not to claim that they will in fact complain (they may not). It is only to recognize that it is a bad thing (unjust or unfair) for them to be worse off than the other members of their world through no fault of their own" – (Temkin 1986, p.102).

Axiom 11 (Independence) *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{D}$ *be such that* $T(\mathbf{x}) = T(\mathbf{y})$, $r(\mathbf{x}) = r(\mathbf{y}) = r > 2$ *and* $x_r = y_r$. *Then, for any* $i < r$, $x_i = y_i \Rightarrow$

$$\forall \alpha \in [x_{i-1}, x_{i+1}] \cap [y_{i-1}, y_{i+1}] \text{ and } \mathbf{x}_{-i}(\alpha), \mathbf{y}_{-i}(\alpha) \in \mathbb{D} : \; T(\mathbf{x}_{-i}(\alpha)) = T(\mathbf{y}_{-i}(\alpha)) .$$

Axiom 12 (Normalisation) $T(0, ..., 0, 1) = 1$

Note the similarity of Axioms 10 and 11 to Axioms 5 and 6 in the analysis of poverty. So a result similar to Theorem 1 emerges. Let Ω_n be the subset of Ω_n^* such that $x_{n-1} < x_n$ – there is a single richest person. Then, using the topmost income x_n as a reference point Cowell and Ebert (2004) show the following for the two cases of the space of incomes:

Theorem 3 *T satisfies Axioms 1 to 3 and 10 to 12 if and only if there are* $w_j > 0$, $j = 1, ..., n-1$, $\sum_{j=1}^{n-1} w_j = 1$ *and* $\varepsilon \in \mathbb{R}$ *such that, for all* $\mathbf{x} \in \mathbb{D}$*:*

Case 1 *($\mathbb{D} = \Omega_n$)*

$$T_\varepsilon(\mathbf{x}) = \left[\sum_{j=1}^{n-1} w_j k_j(\mathbf{x})^\varepsilon \right]^{\frac{1}{\varepsilon}} \text{ for } \varepsilon \neq 0 \tag{20}$$

$$= \prod_{j=1}^{n-1} k_j(\mathbf{x})^{w_j} \text{ for } \varepsilon = 0 \tag{21}$$

Case 2 *($\mathbb{D} = \Omega_n^*$) Condition (20) holds and* $\varepsilon > 0$.

4.2.2 Inequality measures

The parameters $w_1, ..., w_{n-1}$ and ε characterise the whole family of BOP-complaint inequality indices. All of them satisfy the transfer principle (Dalton 1920) if the the richest person is included in the income transfer but not all will satisfy the transfer principle for an arbitrary pair of persons. However, Cowell and Ebert (2004) also show:

Theorem 4 *T_ε satisfies the transfer principle for any arbitrary pair of persons if and only if*

- $w_{j+1} \leq w_j$ *and* $\varepsilon > 1$ *or*
- $w_{j+1} < w_j$ *and* $\varepsilon = 1$

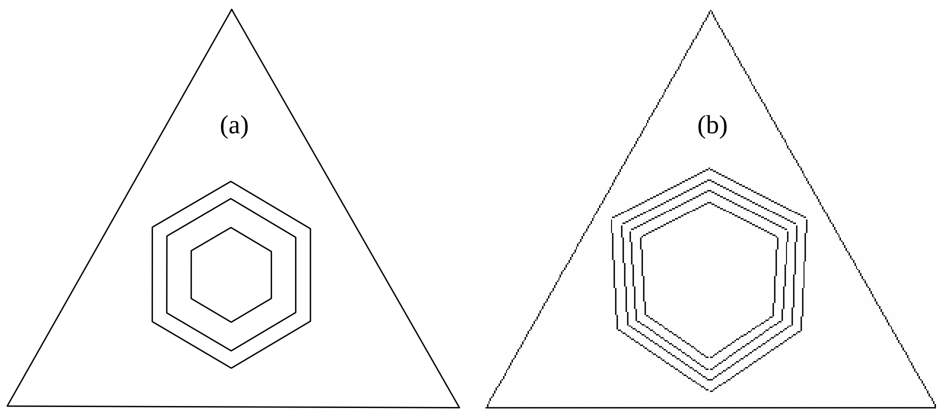

Figure 2: Contours for (a) Gini and (b) Temkin, $\varepsilon = 1$, $w_1 = 0.75$, $w_2 = 0.25$.

To illustrate the relationship of this class to the familiar Gini coefficient, examine Figure 2 that depicts iso-inequality contour maps for the case $n = 3$ with a fixed overall income level. Part (a) shows the contours for (absolute or relative) Gini, as shown in Figure 1; part (b) shows the contour map for one particular Temkin index that is clearly equivalent to those of the extended-Gini as discussed in section 3.3.

4.2.3 Ranking

Apart from the behaviour of a typical complaint-inequality index it is natural to consider how the concept of complaint may be used in providing a ranking criterion. We need the counterpart to the Lorenz insight that is provided, in the poverty context, by the TIP curve or poverty profile given in (5). So, analogously, for any $\mathbf{x} \in \mathbb{D}$ we can define the cumulation of complaints recursively as

$$K_i(\mathbf{x}) := \sum_{j=1}^{i} k_j(\mathbf{x}), \; i = 1, 2, ..., n \tag{22}$$

This concept can be used to draw the "cumulative complaint contour" (CCC) of a distribution $\mathbf{x}$, formed by joining the points $\left(\frac{i}{n}, K_i(\mathbf{x})\right)$ and has essentially the same properties as the TIP curve. If CCC($\mathbf{x}$) lies on or above CCC($\mathbf{y}$) then distribution $\mathbf{x}$ exhibits more BOP-complaint inequality than distribution $\mathbf{y}$.[9]

[9] Chakravarty et al. (2003) introduce the idea of target shortfall orderings. Here one associates with each income x° a subgroup containing all persons whose incomes are not

To see what this means consider the subclass of BOP-complaint inequality indices that satisfy the conventional transfer principle (see theorem 4 above). Let $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_1$ where

$$\mathcal{T}_0 := \left\{ T_\varepsilon : \varepsilon > 1, \sum_{j=1}^{n-1} w_j = 1, w_j \geq w_{j+1} > 0 \right\}$$

$$\mathcal{T}_1 := \left\{ T_1 : \sum_{j=1}^{n-1} w_j = 1, w_j > w_{j+1} > 0 \right\}$$

There is a close relationship between this class $\mathcal{T}$ and an inequality-ranking principle $\succcurlyeq_T$ defined in terms of the complaint cumulations:

Definition 1 *For any $\mathbf{x}, \mathbf{y} \in \mathbb{D}$ distribution $\mathbf{x}$ exhibits more complaint-inequality than $\mathbf{y}$ ($\mathbf{x} \succcurlyeq_T \mathbf{y}$) if and only if*

$$K_i(\mathbf{x}) \geq K_i(\mathbf{y}) \text{ for } i = 1, 2, ..., n$$

where K_i is given by (22).

Theorem 5 *For any $\mathbf{x}, \mathbf{y} \in \mathbb{D}$: $\mathbf{x} \succcurlyeq_T \mathbf{y} \Longleftrightarrow T_\varepsilon(\mathbf{x}) \geq T_\varepsilon(\mathbf{y})$, for all $T_\varepsilon \in \mathcal{T}$.*

The proof – reproduced in the appendix – relies on the fact that one can transform the CCC problem into one that is effectively an income-cumulation problem: the ranking $\succcurlyeq_T$ is closely related to the standard generalised-Lorenz ranking criterion $\succcurlyeq_{\text{GL}}$ (Shorrocks 1983).

4.3 All those better off (ATBO)

The analysis of Section 4.2 Theorem 3 can be adapted to the second type of complaint where each individual uses as his reference point the average income of all those who are better off. Unlike BOP the reference point is different for each person. Using the conditional mean (12) one obtains the ATBO-complaint as

$$k_i(\mathbf{x}) := \mu_i(\mathbf{x}) - x_i. \tag{23}$$

This will generate a set of ATBO-complaint indices of the form (20, 21) but with individual complaints k_i given by (23) rather than (19). This is clearly the ATBO counterpart of the family and is essentially the same as the Chakravarty version of deprivation given by (17).[10]

higher than x° and a person's target shortfall in a subgroup is the gap between the subgroup highest income and his own income. They establish an absolute target shortfall ordering, which, under constancy of population size and total income, implies the Lorenz and CCC orderings.

[10]There is a superficial difference in that the summation in (20) runs from 1 to $n-1$

4.4 Average income (AVE)

For completeness let us also consider the possibility of AVE-complaint inequality indices. Here the reference point is $\mu_0(\mathbf{x})$, the mean for the whole distribution. By analogy with (19) and (23) one now has

$$k_i(\mathbf{x}) := \mu_0(\mathbf{x}) - x_i. \tag{24}$$

as the individual "complaint" concept. But, as Devooght (2003) has pointed out, where incomes are greater than the mean, it is unclear what meaning is to be given to "complaint." Nevertheless, in this case the counterpart of (20, 21) is

$$\left.\begin{array}{ll} \left[\sum_{j=1}^{n} w_j \left|k_j(\mathbf{x})\right|^{\varepsilon}\right]^{\frac{1}{\varepsilon}} & \text{for } \varepsilon \neq 0, \\ \prod_{j=1}^{n} \left|k_j(\mathbf{x})\right|^{w_j} & \text{for } \varepsilon = 0. \end{array}\right\} \tag{25}$$

The family (25) is related to the Ebert (1988) class of inequality measures.

5 Conclusion

The focus of Gini's original contribution on income differences is fundamental. This focus is now widely recognised not only in the analysis of deprivation and of poverty but also in recent approaches to inequality that have incorporated the concept of complaint about income distribution. Likewise Lorenz's contribution, so closely associated in the literature with Gini's work, is also fundamental to recent contributions: a generalisation of the Lorenz ranking works for both poverty orderings and complaint orderings.

Recent advances in the analysis of relative deprivation, poverty and complaint inequality show that these separate problems share a common structure. As we have seen many of the same axioms are conventionally used in the approach to characterising measures for each of the three problems. However, they are not just an artefact of the methodology adopted by those who have recently worked on the formalisation of these concepts. It is clear from the original contributions in each of these areas that individual deprivation d_i, the individual poverty gap g_i and individual complaint k_i are all examples of fundamental income differences that lie at the heart of the thinking about these issues: indeed in many respects the indices that incorporate the income-difference concepts can be obtained from another with little more

whereas in (17) it is from 1 to n. However, given that complaint or deprivation is zero at the top and that (17) restricts the value of ε, this distinction is irrelevant.

than a change in notation. It is legitimate to see this modern body of work as part of the intellectual legacy of Gini and Lorenz.

References

Amiel, Y. and F. A. Cowell (1994). Monotonicity, dominance and the Pareto principle. *Economics Letters 45*, 447–450.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory 2*, 244–263.

Berrebi, Z. M. and J. Silber (1985). Income inequality indices and deprivation: a generalization. *Quarterly Journal of Economics 100*, 807–810.

Berrebi, Z. M. and J. Silber (1989). Deprivation, the Gini index of inequality and the flatness of an income distribution. *Mathematical Social Sciences 18*(12), 229–237.

Bossert, W. and C. D'Ambrosio (2004). Reference groups and individual deprivation. Working Paper 2004-10, Département de sciences économiques, Université de Montréal, C.P. 6128, succursale Centre-Ville, Montréal (Québec) H3C 3J7, Canada.

Bourguignon, F. (1979). Decomposable income inequality measures. *Econometrica 47*, 901–920.

Chakravarty, S. R. (1998). Relative deprivation and satisfaction orderings. *Keio Economic Studies 34*, 17–31.

Chakravarty, S. R. and A. B. Chakraborty (1984). On indices of relative deprivation. *Economics Letters 14*, 283–287.

Chakravarty, S. R., C. D'Ambrosio, and P. Muliere (2003, March). Target shortfall orderings and indices. DIW Discussion Paper 340, German Institute for Economic Research, Königin-Luise-Str. 5, 14195 Berlin, Germany.

Chakravarty, S. R. and P. Moyes (2003). Individual welfare, social deprivation and income taxation. *Economic Theory 21*, 843–869.

Chakravarty, S. R. and D. Mukherjee (1998). Lorenz transformation, utilitarian deprivation rule and equal sacrifice principle. *The Manchester School 66*, 521–531.

Chakravarty, S. R. and D. Mukherjee (1999a). Measures of deprivation and their meaning in terms of social satisfaction. *Theory and Decision 47*, 89–100.

Chakravarty, S. R. and D. Mukherjee (1999b). Ranking income distributions by deprivation orderings. *Social Indicators Research 46*, 125–135.

Cowell, F. A. (1980). On the structure of additive inequality measures. *Review of Economic Studies 47*, 521–531.

Cowell, F. A. and U. Ebert (2004). Complaints and inequality. *Social Choice and Welfare 23*, 71–89.

Dalton, H. (1920). Measurement of the inequality of incomes. *The Economic Journal 30*, 348–361.

D'Ambrosio, C. and J. R. Frick (2004, October). Subjective well-being and relative deprivation: An empirical link. IZA Discussion Paper 1351, Forschungsinstitut zur Zukunft der Arbeit.

Devooght, K. (2003). Measuring inequality by counting 'complaints': theory and empirics. *Economics and Philosophy 19*, 241 – 263.

Duclos, J.-Y. and P. Grégoire (2002). Absolute and relative deprivation and the measurement of poverty. *Review of Income and Wealth 48*, 471–492.

Ebert, U. (1988). A family of aggregative compromise inequality measure. *International Economic Review 29*(5), 363–376.

Ebert, U. and P. Moyes (2000). An axiomatic characterization of Yitzhaki's index of individual deprivation. *Economics Letters 68*(3), 263–270.

Ebert, U. and P. Moyes (2002). A simple axiomatization of the Foster-Greer-Thorbecke poverty orderings. *Journal of Public Economic Theory 4*, 455–473.

Foster, J. E., J. Greer, and E. Thorbecke (1984). A class of decomposable poverty measures. *Econometrica 52*, 761–776.

Gini, C. (1912). Variabilità e mutabilità. *Studi Economico-Giuridici dell'Università di Cagliari 3*, 1–158.

Hey, J. D. and P. J. Lambert (1980). Relative deprivation and the Gini coefficient: comment. *Quarterly Journal of Economics 95*, 567–573. and reply by Yitzhaki.

Jenkins, S. P. and P. J. Lambert (1997). Three 'I's of poverty curves, with an analysis of UK poverty trends. *Oxford Economic Papers 49*, 317–327.

Kolm, S.-C. (1969). The optimal production of social justice. In J. Margolis and H. Guitton (Eds.), *Public Economics*, pp. 145–200. London: Macmillan.

Lorenz, M. O. (1905). Methods for measuring concentration of wealth. *Journal of the American Statistical Association 9*, 209–219.

Podder, N. (1996). Relative deprivation, envy and economic inequality. *Kyklos 49*, 353–376.

Runciman, W. G. (1966). *Relative Deprivation and Social Justice*. London: Routledge.

Sen, A. K. (1976). Poverty: An ordinal approach to measurement. *Econometrica 44*, 219–231.

Sen, A. K. (1979). Issues in the measurement of poverty. *Scandinavian Journal of Economics 91*, 285–307.

Shorrocks, A. F. (1980). The class of additively decomposable inequality measures. *Econometrica 48*, 613–625.

Shorrocks, A. F. (1983). Ranking income distributions. *Economica 50*, 3–17.

Shorrocks, A. F. (1998). Deprivation profiles and deprivation indices. In S. P. Jenkins, A. Kapteyn, and B. M. S. Van Praag (Eds.), *The Distribution of Welfare and Household Production: International Perspectives*, Chapter 11, pp. 250–267. Cambridge University Press.

Stark, O. and S. Yitzhaki (1988). The migration response to relative deprivation. *Journal of Population Economics 1*.

Temkin, L. S. (1986). Inequality. *Philosophy and Public Affairs 15*, 99–121.

Temkin, L. S. (1993). *Inequality*. Oxford: Oxford University Press.

Theil, H. (1967). *Economics and Information Theory*. Amsterdam: North Holland.

Yitzhaki, S. (1979). Relative deprivation and the Gini coefficient. *Quarterly Journal of Economics 93*, 321–324.

Yitzhaki, S. (1980). Relative deprivation and the Gini coefficient: reply. *Quarterly Journal of Economics 95*, 575–6.

Yitzhaki, S. (1982). Relative deprivation and economic welfare. *European Economic Review 17*(1), 99–114.

A Proof of ranking result

First introduce the following lemma:

Lemma 1 *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{D}$*:* $\mathbf{x} \succcurlyeq_T \mathbf{y} \Longleftrightarrow [\mathbf{y}{-}y_n\mathbf{1}] \succcurlyeq_{\mathrm{GL}} [\mathbf{x}{-}x_n\mathbf{1}]$.

Proof. By definition 1 we have

$$\mathbf{x} \succcurlyeq_T \mathbf{y} \Longleftrightarrow \sum_{j=1}^{i} [x_n - x_j] \geq \sum_{j=1}^{i} [y_n - y_j], i = 1, 2, ..., n-1.$$

This is equivalent to

$$\frac{1}{n}\sum_{j=1}^{i} [y_j - y_n] \geq \frac{1}{n}\sum_{j=1}^{i} [x_j - x_n], i = 1, 2, ..., n \tag{26}$$

which means that $[\mathbf{y}{-}y_n\mathbf{1}] \succcurlyeq_{\mathrm{GL}} [\mathbf{x}{-}x_n\mathbf{1}]$. ■

This then enables us to establish Theorem 5.

Proof. Consider $-T_\varepsilon(\cdot)$ as a function of $\mathbf{x}{-}x_n\mathbf{1}$: it is clearly symmetric, nondecreasing and concave in $\mathbf{x}{-}x_n\mathbf{1}$. So, using Lemma 1 and Theorem 2 of Shorrocks (1983), we find that $\mathbf{x} \succcurlyeq_T \mathbf{y}$ implies

$$-T_\varepsilon(\mathbf{y}) \geq -T_\varepsilon(\mathbf{x}).$$

Now consider a subfamily of indices with typical member $T^{\alpha,i} \in \mathcal{T}$ where

$$T^{\alpha,i}(\mathbf{x}) := \sum_{j=1}^{n-1} w_j [x_n - x_j] \tag{27}$$

$$w_j = \left\{ \begin{array}{ll} \frac{1}{i}\left[1 - \frac{2\alpha j}{[n-1][i+1]}\right] & \text{for } j = 1, ..., i \\ \\ \frac{2\alpha[n-j]}{[n-1][n-i][n-i-1]} & \text{for } j = i+1, ..., n-1. \end{array} \right\} \tag{28}$$

Each $T^{\alpha,i}$ is an instance of the case $\varepsilon = 1$ in (20). By assumption

$$T^{\alpha,i}(\mathbf{x}) \geq T^{\alpha,i}(\mathbf{y}), i = 1, ..., n-1. \tag{29}$$

However from (27) and (28) we have

$$\lim_{\alpha \to 0} T^{\alpha,i}(\mathbf{x}) = K_i(\mathbf{x})$$

and so, letting $\alpha \to 0$, (29) implies

$$K_i(\mathbf{x}) \geq K_i(\mathbf{y}), i = 1, ..., n-1. \tag{30}$$

Hence $\mathbf{x} \succcurlyeq_T \mathbf{y}$. ■