VARIANCE-TYPE ESTIMATION OF LONG MEMORY

by

Liudas Giraitis*, Peter M Robinson*
London School of Economics and Political Science
and
Donatas Surgailis
Institute of Mathematics and Informatics, Vilnius, Lithuania

Contents:

The Suntory Centre
Suntory and Toyota International Centres
for Economics and Related Disciplines
London School of Economics and Political Science
Houghton Street
London WC2A 2AE
Tel.: 0171-405 7686

Discussion Paper
No. EM/98/363
October 1998

* Research supported by ESRC Grant R000235892.

Abstract

The aggregation procedure when a sample of length N is divided into blocks of length $m = o(N)$, $m \to \infty$ and observations in each block are replaced by their sample mean, is widely used in statistical inference. Taqqu, Teverovsky and Willinger (1995), Teverovsky and Taqqu (1997) introduced an aggregate variance estimator of the long memory parameter of a stationary sequence with long range dependence and studied its empirial performance. With respect to autovariance structure and marginal distribution, the aggregated series is closer to Gaussian fractional noise than the initial series. However, the variance type estimator based on aggregated data is seriously biased. A refined estimator, which employs least squares regression across varying levels of aggregation, has much smaller bias, permitting derivation of limiting distributional properties of suitably centered estimates, as well as of a minimum mean squared error choice of bandwidth m. The results vary considerably with the actual value of the memory parameter.

Keywords: Long memory; aggregation; semiparametric model.

JEL No.: C22

1. Introduction

A number of estimates are available of the memory parameter of a long range dependent stationary Gaussian process $\{X_t, t \in \mathbf{Z}\}$, where $\mathbf{Z} = \{t : t \in 0, \pm 1, \ldots\}$. Let X_t have lag-j autocovariance

$$r(j) = cov(X_0, X_j) \sim \sigma^2 j^{-\theta}, \tag{1.1}$$

as $j \to \infty$, where $0 < \sigma^2 < \infty$ and $0 < \theta < 1$. Then X_t is said to be long range dependent. The relation (1.1) holds in case of fractionally differenced autoregressive integrated moving average processes with differencing parameter $(1-\theta)/2$, or fractional noise with self-similarity parameter $1 - \theta/2$, which specify $r(j)$ for all j. In case no such finite-parameter model is specified, robust 'semiparametric' estimation can be based on (1.1).

On the basis of observations $X_1, \ldots, X_N$, Taqqu, Teverovsky and Willinger (1995), Teverovsky and Taqqu (1997) have proposed the variance type estimate

$$\widehat{\theta}_m = -\frac{\log S_m^2}{\log m}, \tag{1.2}$$

where

$$S_m^2 = [\frac{N}{m}]^{-1} \sum_{k=1}^{[N/m]} \left(X_k^{(m)} - [\frac{N}{m}]^{-1} \sum_{j=1}^{[N/m]} X_j^{(m)} \right)^2,$$

where $[\cdot]$ denotes integer part and $X_k^{(m)}$ is the aggregated series of order m,

$$X_k^{(m)} = \frac{1}{m} \sum_{t=1}^{m} X_{t+(k-1)m}, \quad k = 1, 2, \ldots. \tag{1.3}$$

It is intended that both m and N/m be large. A natural connection between the aggregation and the wavelets method for analysing long-memory signals was discussed in the recent paper by Abry, Veitch and Flandrin (1998). Teverovsky and Taqqu (1997) applied $\hat{\theta}_m$ to the analysis of ethernet data, while Taqqu, Teverovsky and Willinger (1995) compared its finite sample performance to that of other estimates of θ by means of Monte Carlo simulations.

In Section 2 we examine asymptotic properties of $\hat{\theta}_m$. We find that whereas $X_k^{(m)}$ is closer to Gaussian fractional noise than X_t, $\hat{\theta}_m$ has a bias of order $(\log m)^{-1}$ as $m \to \infty$ and $N/m \to \infty$. Thus the bias is of order no less then $(\log N)^{-1}$, so that only in very long series can $\hat{\theta}_m$ be a useful estimate. We point out in Section 2 that $\hat{\theta}_m$ can be viewed as merely a special case of a more general class of estimates, but these have similar asymptotic properties. Prompted by a related idea of Taqqu, Teverovsky and Willinger (1995), Teverovsky and Taqqu (1997), who proposed a plotting S_m^2 against m on a $\log - \log$ scale and fitting a straight line, we also consider in Section 2

$$\widehat{\theta}_{m_0,m_1} = -\frac{\sum_{j=m_0}^{m_1} \nu_j \log S_j^2}{\sum_{j=m_0}^{m_1} \nu_j^2}, \tag{1.4}$$

where

$$\nu_j := \log j - \frac{1}{m_1 - m_0} \sum_{i=m_0}^{m_1} \log i$$

for $m_0 < m_1$, such that $m_0 \to \infty$ as $N \to \infty$ but $N/m_1 \to \infty$. We find that $\widehat{\theta}_{m_0,m_1}$ is less biased than $\widehat{\theta}_m$, and moreover establish its limiting distributional behaviour and optimal (minimum mean squared error) choice of m. We compare the properties of $\widehat{\theta}_{m_0,m_1}$, with those of some rival estimates of θ. Proofs depend on a sequence of lemmas contained in Section 3. Since S_j^2 is invariant to location shift in X_t, we take $EX_t = 0$ with no loss of generality.

2. Asymptotic properties of $\widehat{\theta}_m$ and $\widehat{\theta}_{m_0,m_1}$.

To describe the limiting behaviour of $\widehat{\theta}_m$ and $\widehat{\theta}_{m_0,m_1}$, we introduce a number of definitions. Put

$$\widetilde{\sigma}_m^2(\theta) = m^\theta E S_m^2. \tag{2.1}$$

Write $\stackrel{\text{law}}{=}$ for equality of distributions, and $\xi = o_2(1)$ whenever $E\xi^2 = o(1)$. For $a > 1$, put

$$d_a = d_a(\theta) = \begin{cases} a^\theta, & \text{if } 0 < \theta < 1/2, \\ (a/\log a)^{1/2}, & \text{if } \theta = 1/2, \\ a^{1/2}, & \text{if } 1/2 < \theta < 1. \end{cases}$$

Introduce the *fractional Brownian motion* $J_1(t)$ and the *Rosenblatt process* $J_2(t)$, for $t \in \mathbf{R}$, as stochastic Ito-Wiener integrals

$$J_1(t) = k_1(\theta) \int_{\mathbf{R}} e_t(x)|x|^{(\theta-1)/2} W(dx),$$

$$J_2(t) = k_2(\theta) \int_{\mathbf{R}^2} e_t(x_1 + x_2)|x_1|^{(\theta-1)/2}|x_2|^{(\theta-1)/2} W(dx_1)W(dx_2),$$

respectively, where $e_t(x) = (e^{itx} - 1)/(ix)$, where $W(dx) = \overline{W(-dx)}$ is a complex-valued Gaussian spectral measure ("white noise"), with zero mean and covariance $EW(dx)\overline{W}(dy) = dx$ if $x = y$; $= 0$ otherwise,

$$k_1(\theta) = (D(\theta)w(\theta))^{-1/2}, \quad k_2(\theta) = D(\theta)^{-1}(2w(2\theta))^{-1/2},$$

$$w(\theta) = \frac{2}{(1-\theta)(2-\theta)}, \quad D(\theta) = 2\Gamma(\theta)\cos(\frac{\theta\pi}{2}),$$

see Dobrushin and Major (1979), Taqqu (1979). The processes $J_1(t), J_2(t)$ are well-defined (on the same probability space) for $0 < \theta < 1$ and $0 < \theta < 1/2$, respectively (see e.g. Taqqu (1979) for more on these processes). Let $J_i \equiv J_i(1)$, $i = 1, 2$ and

$$Z(\theta) \stackrel{\text{law}}{=} \begin{cases} \sigma_*^2(\theta)(\{\frac{2w(2\theta)}{w(\theta)^2}\}^{1/2} J_2 - J_1^2 + EJ_1^2), & \text{if } 0 < \theta < 1/2, \\ \mathcal{N}(0, s^2(\theta)), & \text{if } 1/2 \le \theta < 1, \end{cases}$$

$$s^2(\theta) = \begin{cases} 4\sigma^4, & \text{if } \theta = 1/2, \\ 2\sigma_*^4(\theta)\sum_{t\in Z}\rho_t^2, & \text{if } 1/2 < \theta < 1, \end{cases}$$

where $\sigma_*^2(\theta) = \sigma^2 w(\theta)$, and where $\rho_t = \rho_t(\theta)$ is the autocovariance of fractional Gaussian noise,

$$\rho_t = \mathrm{Cov}(J_1(t) - J_1(t-1), J_1(0) - J_1(-1)) = \frac{1}{2}(|t+1|^{2-\theta} - 2|t|^{2-\theta} + |t-1|^{2-\theta}).$$

Theorem 2.1. *Let (1.1) hold and $\sigma_*^2(\theta) \neq 0$. Then, as $N \to \infty$, $m \to \infty$, $N/m \to \infty$*

$$\widehat{\theta}_m = \theta - (\log m)^{-1}\Big\{\log \sigma_*^2(\theta) + o(1) + \frac{Z(\theta) + o_2(1)}{\sigma_*^2(\theta) d_{N/m}}\Big\}$$

Proof. The proof follows from Lemmas 3.1 and 3.3.

We deduce from Therorem 2.1 that

$$E\widehat{\theta}_m = \theta - \frac{\log \sigma_*^2(\theta)}{\log m}(1 + o(1))$$

and moreover that

$$E(\widehat{\theta}_m - \theta)^2 = \Big(\frac{\log \sigma_*^2(\theta)}{\log m}\Big)(1 + o(1)),$$

so that, in particular

$$(\log m)(\widehat{\theta}_m - \theta) \Rightarrow_P - \log \sigma_*^2 \neq 0.$$

Thus not only are the bias and variance of $\widehat{\theta}_m$ large (and dependent on σ^2 as well as θ) but Theorem 2.1 does not provide useful inference rules because $\widetilde{\sigma}_*^2(\theta)$ is unknown.

Teverovsky and Taqqu (1997) motivated $\hat{\theta}_m$ by noting that

$$\widetilde{\sigma}_m^2(\theta) \sim \sigma_*^2(\theta), \quad \text{as } m, N/m \to \infty \tag{2.2}$$

due to

$$E(X_k^{(n)})^2 \sim \sigma_*^2(\theta) n^{-\theta} \quad \text{as } N \to \infty \quad \text{for all } k. \tag{2.3}$$

Our alternative interpretation suggests a broader class of estimates. Suppose X_t has a spectral density, $f(\lambda)$, satisfying

$$r(j) = \int_{-\pi}^{\pi} \cos(j\lambda) f(\lambda) d\lambda, \quad j = 0, \pm 1, \dots$$

Were X_t weakly dependent, in the sense that $0 < f(0) < \infty$, then S_m^2 would be an (unmodified) Bartlett (1950) nonparametric spectral estimate of $2\pi f(0)$, with truncation point m. Under (1.1) $f(0)$ is typically infinite but S_m^2 estimates

$$\sum_{j=1-m}^{m-1} (1 - \frac{|j|}{m}) r(j) \sim \sigma_*^2(\theta) m^{-\theta}, \quad \text{as} \quad m \to \infty.$$

$S_m^2/2\pi$ could be replaced in (1.2) by any one of a wide range of alternative smoothed nonparametric spectral estimates, at zero frequency (see Brillinger, 1975), for example if

$$S_m^2(K) = \sum_{1-m}^{m-1} K(\frac{|j|}{m})\hat{r}(j),$$

where $\hat{r}(j) = N^{-1}\sum_{t=1}^{N-j}(X_t - \bar{X})(X_{t+j} - \bar{X})$, $\hat{r}(-j) = \hat{r}(j)$, $j \geq 0$, then

$$ES_m^2(K) \sim \sigma^2 \int_{-1}^{1} K(x)|x|^{-\theta}dx \cdot m^{-\theta}$$

for a kernel function K. However though finite sample properties can be influenced by choice of K, rates of convergence will typically remain unchanged under standard conditions on K (including $K(0) = 1$).

Note that while $\hat{\theta}_m$ is invariant to location shift, it has the disadvantage of not being invariant to a change in the scale of X_t, the scale factor being absorbed in the $O((\log n)^{-1})$ bias, which also depends on θ. Scale-invariance, but not necessarily a reduction in bias, could be achieved by considering the estimate - $\log(S_m^2/S_1^2)/\log m$, where S_m^2/S_1^2 is one of the forms of variance-ratio statistic used in econometrics. Due to the weights ν_j summing to zero, the estimate $\hat{\theta}_{m_0,m_1}$, given in (1.4), is both scale-invariant and has bias of smaller order.

The leading term of the bias of $\hat{\theta}_{m_0,m_1}$ is studied by refining (1.1) in two different ways. Writing

$$r(j) = \sigma^2|j|^{-\theta} + r_1(j), \quad j \neq 0,$$

we assume either

Assumption (A):

$$\sum_{t\in Z} |r_1(t)| < \infty,$$

or

Assumption (B):

$$r_1(t) = c_1 t^{-\theta-\beta}(1 + o(1)) \quad (t \to \infty)$$

with $\beta \in (0, 1-\theta)$ and $|c_1| < \infty$.

Assumption A holds in case of fractional autoregressive integrated moving average and fractional noise models. Assumption B entails $\sum_{t\in\mathbf{Z}} |r_1(t)| = \infty$ and is similar to the requirement

$$f(\lambda) \sim \lambda^{\theta-1}(1 + O(\lambda^\beta)), \quad \text{as } \lambda \to 0$$

employed in study of alternative estimates of θ by Robinson (1994a, 1995).

Define

$$\bar{r}_1 = \sum_{t\in\mathbf{Z}} r_1(t),$$

and

$$I(\theta) = \lim_{m\to\infty}\Big(\int_0^m x^{-\theta}dx - \sum_{x=1}^m x^{-\theta}\Big),$$

(see Bender and Orszag (1978), p.305 for an expression of $I(\theta)$ in terms of Bernoulli polynomials). Now define

$$q_A(\theta, r, \sigma^2) = \frac{(1-\theta)}{\theta^2\sigma_*^2(\theta)}(\bar{r}_1 - 2\sigma^2 I(\theta)),$$

$$q_B(\theta, \beta, c) = \frac{2\beta c_1}{\sigma_*^2(\theta)(1-\beta)^2(1-\theta-\beta)(2-\theta-\beta)}.$$

To characterize limit distributional properties define

$$\widetilde{Z}(\theta) \stackrel{\text{law}}{=} \begin{cases} \zeta(\theta)((\frac{2w(2\theta)}{w(\theta)^2})^{1/2}J_2 - J_1^2 + EJ_1^2), & \text{if } 0<\theta<1/2,\\ \mathcal{N}(0, 0.09), & \text{if } \theta = 1/2,\\ \mathcal{N}(0, \tilde{s}^2(\theta)), & \text{if } 1/2<\theta<1,\end{cases}$$

where

$$\zeta(\theta) = \frac{\theta}{(1+\theta)^2}$$

and

$$\tilde{s}^2(\theta) \;=\; \int_0^1\int_0^1 (1+\log u)(1+\log v)(uv)^{1/2}D(u,v)dudv,$$

where

$$D(u,v) = \frac{\pi^2}{w^2(\theta)\cos^2(\frac{\pi\theta}{2})|\cos(\pi\theta)|\Gamma^2(\theta)\Gamma(6-2\theta)}$$

$$(|\sqrt{u/v}+\sqrt{v/u}|^{5-2\theta} + |\sqrt{u/v}-\sqrt{v/u}|^{5-2\theta} - 2(\sqrt{u/v})^{5-2\theta} - 2(\sqrt{v/u})^{5-2\theta}). \tag{2.4}$$

Theorem 2.2. *Let (1.1) hold, $\sigma_*^2(\theta)\neq 0$ and $m_0, m_1\to\infty, m_0 = o(m_1/\log^3 m_1)$, $m_1 = o(N)$ as $N\to\infty$. Then as $N\to\infty$, under Assumption A*

$$\widehat{\theta}_{m_0,m_1} = \theta + \Big(\zeta(\theta)(\frac{m_1}{N})^\theta + q_A(\theta, \bar{r}_1, \sigma^2)m_1^{\theta-1}\Big)(1+o(1)) - \frac{\widetilde{Z}(\theta)+o_2(1)}{d_{N/m_1}(\theta)},$$

and under Assumption B

$$\widehat{\theta}_{m_0,m_1} = \theta + \Big(\zeta(\theta)(\frac{m_1}{N})^\theta + q_B(\theta, \beta, c_1)m_1^{-\beta}\Big)(1+o(1)) - \frac{\widetilde{Z}(\theta)+o_2(1)}{d_{N/m_1}(\theta)}.$$

Proof. The proof follows from Lemmas 3.5, 3.7 and 3.9.

The implications of Theorem 2.2 vary depending on θ and m_1. We deduce the following corollary, omitting arguments from $\zeta, \widetilde{Z}, q_A$ and q_B and subscripts from $\widehat{\theta}_{m_0,m_1}$.

(i) $0 < \theta < 1/2$.
Under Assumption A:

$$(2.5)\qquad (\frac{N}{m_1})^{\theta}(\hat{\theta}-\theta) \Rightarrow \zeta - \widetilde{Z}, \quad \text{if } \frac{N^{\theta}}{m_1} \to 0,$$

$$(2.6)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow a\zeta + q_A - a\widetilde{Z}, \quad \text{if } m_1 \sim aN^{\theta},$$

$$(2.7)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow_P q_A, \quad \text{if } \frac{m_1}{N^{\theta}} \to 0;$$

Under Assumption B:

$$(2.8)\qquad (\frac{N}{m_1})^{\theta}(\hat{\theta}-\theta) \Rightarrow \zeta - \widetilde{Z}, \quad \text{if } \frac{N^{\theta}}{m_1^{\theta+\beta}} \to 0,$$

$$(2.9)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow a^{\theta+\beta}\zeta + q_B - a^{\theta+\beta}\widetilde{Z}, \quad \text{if } m_1 \sim aN^{\frac{\theta}{\theta+\beta}},$$

$$(2.10)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow_P q_B, \quad \text{if } \frac{m_1^{\theta+\beta}}{N^{\theta}} \to 0;$$

(ii) $\theta = 1/2$.
Under Assumption A:

$$(2.11)\qquad (\frac{N}{m_1}/\log\frac{N}{m_1})^{1/2}(\hat{\theta}-\theta) \Rightarrow -\widetilde{Z}, \quad \text{if } \frac{N}{m_1^{3-2\theta}}/\log\frac{N}{m_1} \to 0,$$

$$(2.12)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow q_A - a^*\widetilde{Z}, \quad \text{if } m_1^{1-\theta} \sim a^*\Big(\frac{N}{m_1}/\log\frac{N}{m_1}\Big)^{1/2},$$

$$(2.13)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow_P q_A, \quad \text{if } (\log\frac{N}{m_1})\frac{m_1^{3-2\theta}}{N} \to 0;$$

Under Assumption B:

$$(2.14)\qquad (\frac{N}{m_1}/\log\frac{N}{m_1})^{1/2}(\hat{\theta}-\theta) \Rightarrow -\widetilde{Z}, \quad \text{if } \frac{N}{m_1^{1+2\beta}}/\log\frac{N}{m_1} \to 0,$$

$$(2.15)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow q_B - a^*\widetilde{Z}, \quad \text{if } m_1^{\beta} \sim a^*\Big(\frac{N}{m_1}/\log\frac{N}{m_1}\Big)^{1/2},$$

$$(2.16)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow_P q_B, \quad \text{if } (\log\frac{N}{m_1})\frac{m_1^{1+2\beta}}{N} \to 0;$$

(iii) $1/2 < \theta < 1$.
Under Assumption A:

$$(2.17)\qquad (\frac{N}{m_1})^{1/2}(\hat{\theta}-\theta) \Rightarrow -\widetilde{Z}, \quad \text{if } \frac{N}{m_1^{3-2\theta}} \to 0,$$

$$(2.18)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow q_A - a^{3/2-\theta}\widetilde{Z}, \quad \text{if } m_1 \sim aN^{\frac{1}{3-2\theta}},$$

$$(2.19)\qquad m_1^{1-\theta}(\hat{\theta}-\theta) \Rightarrow_P q_A, \quad \text{if } \frac{m_1^{3-2\theta}}{N} \to 0;$$

Under Assumption B:

$$(2.20)\qquad (\frac{N}{m_1})^{1/2}(\hat{\theta}-\theta) \Rightarrow -\widetilde{Z}, \quad \text{if } \frac{N}{m_1^{1+2\beta}} \to 0,$$

$$(2.21)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow q_B - a^{\beta+1/2}\widetilde{Z}, \quad \text{if } m_1 \sim aN^{\frac{1}{1+2\beta}},$$

$$(2.22)\qquad m_1^{\beta}(\hat{\theta}-\theta) \Rightarrow_P q_B, \quad \text{if } \frac{m_1^{1+2\beta}}{N} \to 0.$$

The choice of m_1, in cases (2.6), (2.9), (2.12), (2.15), (2.18) and (2.21) minimizes the order of the mean squared error (MSE) $E(\widehat{\theta}_{m_0,m_1}-\theta)^2$. The leading term in this minimized MSE can be further minimized with respect to a and a^*. For $0<\theta<\frac{1}{2}$ and $\frac{1}{2}<\theta<1$ the optimal a are given as follows.

a) For $0<\theta<\frac{1}{2}$, under Assumption A

$$a = \frac{(1-2\theta)q_A\zeta + \{(1-2\theta)^2q_A^2\zeta^2 + 4\theta(1-\theta)q_A^2(\zeta^2+E\widetilde{Z}(\theta)^2)\}^{1/2}}{2\theta(\zeta^2+E\widetilde{Z}(\theta)^2)}.$$

b) For $0<\theta<\frac{1}{2}$, under Assumption B

$$a = \Big[\frac{(\beta-\theta)q_B\zeta + \{(\beta-\theta)^2q_B^2\zeta^2 + 4\theta\beta q_B^2(\zeta^2+E\widetilde{Z}(\theta)^2)\}^{1/2}}{2\theta(u^2+E\widetilde{Z}(\theta)^2)}\Big]^{\frac{1}{\theta+\beta}}.$$

c) For $\frac{1}{2}<\theta<1$, under Assumption A

$$a = \Big\{\frac{E\widetilde{Z}(\theta)^2}{2(1-\theta)q_A^2}\Big\}^{\frac{1}{2\theta-3}}.$$

d) For $\frac{1}{2}<\theta<1$, under Assumption B

$$a = \Big\{\frac{E\widetilde{Z}(\theta)^2}{2\beta q_B^2}\Big\}^{\frac{1}{2\beta+1}}.$$

Note that $E\widetilde{Z}(\theta)=0$. Thus, in various of the cases, the limiting distribution of the normalized $\hat{\theta}-\theta$ is centered at a non-zero mean. The modified estimate

$$\hat{\theta}^* = \hat{\theta} - (\frac{m_1}{N})^{\hat{\theta}}\zeta(\hat{\theta})$$

satisfies

$$(\frac{N}{m_1})^{\theta}(\hat{\theta}^*-\theta) \Rightarrow -\widetilde{Z}$$

to correspond to (2.5) and (2.8), while replacing $\hat{\theta}$ by $\hat{\theta}^*$ in (2.14), (2.17) and (2.20) makes no difference to the limit distribution.

For $\frac{1}{2} < \theta < 1$, $\hat{\theta}$ has a constant rate of convergence and a limiting normal distribution, although with variance of complicated form that would have to be estimated by numerical approximation. For $0 < \theta < \frac{1}{2}$ (see (2.5), (2.6), (2.7), (2.8)) the rate of convergence varies with θ and the limit distribution, depending on the Rosenblatt process, is relatively intractable. Moreover, the outcome is also determined by conditions on m_1 that vary with θ. Such properties are shared by the average periodogram (AP) estimate of Robinson (1994a), further analyzed by Lobato and Robinson (1996), and the log-autocovariance (LA) estimate proposed by Robinson (1994b) and modified and analyzed by Hall, Koul and Turlach (1997) (HKT). By contrast, the log- periodogram (LP) estimate of Geweke and Porter-Hudak (1983) and the semiparametric Gaussian (SG) estimate of Künsch (1987) have been subsequently found to be asymptotically normal for all $\theta \in (0, 1)$, with constant rate of convergence, and simple asymptotic variance that is independent of θ, under conditions on the bandwidth that do not depend on θ, The dichotomy in distributional behaviour of $\hat{\theta}_{m_0,m_1}$ and the AP and LA estimates is due to $r(j)$ being square- summable for $\theta > 1/2$ only, while the relative complexity of the asymptotic variance formula, so far as $\hat{\theta}_{m_0,m_1}$ and the LA estimate are concerned, corresponds to failure to attain even approximately the property of uncorrelated regression errors, so that regression is not a very natural technique here. The LP and SG estimates, on the other hand, result from an approximate "whitening" of the X_t and thus mimic the classical properties of regression and maximum likelihood estimates. It may also be noted that the asymptotic properties of the AP and SG estimates were established without our assumption of Gaussianity. The convergence rates of the AP and SG estimates depend on the smoothness of the spectral density at zero frequency. Instead, we impose assumptions on the asymptotic behaviour of autocovariances, which do not have a precise frequency-domain correspondence. Our assumption (A) is substantially weaker than that of HKT; under their assumption, which is similar to our assumption (B), our estimator achieves at least as fast a rate of convergence as the LA one they study. Our results could likely be extended to cover linear processes or non-Gaussian processes of the type discussed by HKT.

3. Technical Lemmas

Lemma 5.1. *Under the conditions of Theorem 1.1,*

$$Z_m(\theta) = d_{N/m} m^{\theta} (S_m^2 - ES_m^2) \tag{3.1}$$

satisfies, as $N \to \infty$,

$$Z_m(\theta) \Longrightarrow Z(\theta), \tag{3.2}$$

and moreover

$$EZ_m^2(\theta) \to EZ^2(\theta). \tag{3.3}$$

Proof. Write $Z_m(\theta)$ in terms of the aggregated Gaussian sequence $(X_k^{(m)})$ (1.3):

$$Z_m(\theta) = U_2(m) - d_{N/m}(m/N)^{\theta} \big(U_1^2(m) - EU_1^2(m)\big), \tag{3.4}$$

where

$$U_1(m) = N^{\theta/2}[\frac{N}{m}]^{-1} \sum_{k=1}^{[N/m]} X_k^{(m)} = N^{\theta/2} X_1^{(m[N/m])}, \tag{3.5}$$

$$U_2(m) = d_{N/m} m^{\theta} [\frac{N}{m}]^{-1} \sum_{k=1}^{[N/m]} \big((X_k^{(m)})^2 - E(X_k^{(m)})^2 \big). \tag{3.6}$$

Let $1/2 \le \theta < 1$. Note that $d_{N/m}(m/N)^\theta \to 0$; indeed, $d_{N/m}(m/N)^\theta = (m/N)^{\theta - 1/2}$ if $1/2 < \theta < 1$, $= \log^{-1/2}(N/m)$ if $\theta = 1/2$. Since $U_1(m)$ is Gaussian, and (2.3) implies that

$$\lim_{N\to\infty} EU_1^2(m) = \sigma_*^2(\theta), \tag{3.7}$$

it follows that

$$E\Big(d_{N/m}(m/N)^\theta (U_1^2(m) - EU_1^2(m))\Big)^2 = O(d_{N/m}^2 (m/N)^{2\theta}) = o(1).$$

Let $\theta > 1/2$. Rewrite $U_2(m)$ as

$$U_2(m) = c_N \sum_{k=1}^{[N/m]} \big((Y_k^{(m)})^2 - E(Y_k^{(m)})^2 \big), \tag{3.8}$$

where

$$c_N = (\frac{N}{m})^{1/2}(1 + O(\frac{m}{N})^{1/2}))$$

and

$$Y_k^{(m)} = m^{\theta/2} X_k^{(m)} \tag{3.9}$$

is the normalized aggregated series converging as $m \to \infty$ to fractional Gaussian noise (see Lemma 2.1 Dobrushin and Major (1979)). By easy computation, for any $k \in Z$,

$$\lim_{m\to\infty} \rho_k^{(m)} \equiv \lim_{m\to\infty} \mathrm{Cov}(Y_0^{(m)}, Y_k^{(m)}) = \sigma_*^2(\theta)\rho_k, \tag{3.10}$$

and

$$|\rho_k^{(m)}| \le C|k|^{-\theta} \quad (|k| \ge 1), \tag{3.11}$$

uniformly in $m \ge 1$, where C is a generic constant. Consequently, for $1/2 < \theta < 1$,

$$\lim_{m\to\infty} \sum_{k\in Z} (\rho_k^{(m)})^2 = \sigma_*^4(\theta) \sum_{k \in Z} \rho_k^2 \equiv s^2(\theta)/2 < \infty. \tag{3.12}$$

By evaluating cumulants of the sum on the right hand of (3.9), similarly to Giraitis and Surgailis (1985, Theorem 5), or Breuer and Major (1983), from (3.9)-(3.13) one infers $U_2(m) \Longrightarrow \mathcal{N}(0, s^2(\theta))$, and $EU_2^2(m) \to s^2(\theta)$, thus proving the theorem in the case $1/2 < \theta < 1$.

Let $\theta = 1/2$. Here, the series in (3.12) logarithmically diverges:

$$\sum_{|k| \le N/m} \big(\rho_k^{(m)}\big)^2 = 4\sigma^4 \log(N/m)(1 + o(1)). \tag{3.13}$$

By (3.10), (3.13), using the argument in Giraitis and Surgailis (1985, Th. 6), we obtain $U_2(m) \Longrightarrow \mathcal{N}(0, 4\sigma^4)$ and $EU_2^2(m) \to 4\sigma^4$, proving the case $\theta = 1/2$.

Finally, let $0 < \theta < 1/2$. Using the argument of Dobrushin and Major (1979, Th. 1′), from (3.10) and (3.11) one has the convergence

$$(U_1(m), U_2(m)) \Longrightarrow (\sigma_*(\theta) J_1, \sqrt{2}\sigma\sigma_*(2\theta) J_2),$$

together with the convergence of variances $EU_i^2(m)$, $i = 1, 2$ to the corresponding variances of the limiting random variables. This yields

$$Z_m(\theta) \Longrightarrow \sqrt{2}\sigma\sigma_*(2\theta) J_2 - \sigma_*^2(\theta)(J_1^2 - EJ_1^2) = \sqrt{2}\sigma_*^2(\theta)\Big(\big(\frac{2w(2\theta)}{w^2(\theta)}\big)^{1/2} J_2 - J_1^2 + 1\Big),$$

and (3.3), to complete the proof.

Lemma 3.2 *Under the conditions of Theorem 2.1, uniformly in* $m_0 \le m \le m_1$,

$$E\big(m^\theta(S_m^2 - ES_m^2)\big)^4 \le C(m/N)^{4\min(\theta,1/2)-\delta} \quad (\forall \delta > 0).$$

Proof. Since (3.7) holds uniformly in $m_0 \le m \le m_1$, and $U_1(m)$ is Gaussian, it follows that as $N \to \infty$ $E\big(U_1^2(m) - EU_1^2(m)\big)^4$ is bounded uniformly in $m_0 \le m \le m_1$. In view of (3.1) and (3.4) it remains to check that $EU_2^4(m)$ is similarly bounded. Using the (diagram) formula for moments of $U_2(m)$ in terms of covariances $\rho_{t-s}^{(m)}$, $t, s = 1, \ldots, [N/m]$ (see Giraitis and Surgailis (1985), or the proof of Lemma 3.3 below), one obtains

$$EU_2^4(m) \le \Big(\big(\frac{m}{N}\big) \sum_{t,s=1}^{[N/m]} (\rho_{t-s}^{(m)})^2\Big)^2 \le C,$$

where the last inequality follows from (3.11). The proof is completed.

Lemma 3.2. *Under the conditions of Theorem 2.1, uniformly in* $m_0 \le m \le m_1$,

$$(\widehat{\theta}_m - \theta)\log m = -\log \tilde{\sigma}_m^2(\theta) - (\tilde{\sigma}_m^2(\theta))^{-1} m^\theta\big(S_m^2 - ES_m^2\big) + R_m, \tag{3.14}$$

where

$$ER_m^2 = o\big((m/N)^{\min(2\theta,1)+\delta}\big) \quad (\exists \delta > 0).$$

Proof. We have

$$(\widehat{\theta}_m - \theta)\log m \;=\; -\log\big(m^\theta S_m^2\big) \;=\; -\log\Big(\tilde{\sigma}_m^2(\theta) + m^\theta\big(S_m^2 - ES_m^2\big)\Big).$$

Set

$$\begin{aligned} W_1 &= \{(m/N)^3 < m^\theta S_m^2 \le (m/N)^{\theta/4}\}, \\ W_2 &= \{0 \le m^\theta S_m^2 \le (m/N)^3\}, \\ W_3 &= \{m^\theta S_m^2 > (m/N)^{\theta/4}\}. \end{aligned}$$

Then

$$(\widehat{\theta}_m - \theta)\log m \;=\; \sum_{i=1}^{3}(-1)\log\big(m^\theta S_m^2\big)\mathbf{1}(W_i) \equiv \sum_{i=1}^{3}\xi_{m,i},$$

where $\mathbf{1}(.)$ denotes the indicator function. We show that uniformly in $m_0 \le m \le m_1$,

$$E\xi_{m,i}^2 = O\big((m/N)^{\min(2\theta,1)+\delta}\big), \quad i = 1,2, \tag{3.15}$$

while

$$\xi_{m,3} \;=\; -\log\tilde{\sigma}_m^2(\theta) - (\tilde{\sigma}_m^2(\theta))^{-1}(S_m^2 - ES_m^2) + R_m', \tag{3.16}$$

where

$$E(R_m')^2 = O\big((m/N)^{\min(2\theta,1)+\delta}\big).$$

Clearly, (3.15) and (3.16) imply (3.14).

Consider $E\xi_{m,1}^2$. By the definition of W_1,

$$E\xi_{m,1}^2 = E\log^2\big(m^\theta S_m^2\big)\mathbf{1}(W_1) \;\le\; 9\log^2(m/N)P\{W_1\}.$$

Here, $P\{W_1\} \le P\{W_3^c\}$, W_3^c being the complement of W_3, and

$$\begin{aligned} P\{W_3^c\} &= P\{m^\theta S_m^2 \le (m/N)^{\theta/4}\} = P\{m^\theta(S_m^2 - ES_m^2) \le (m/N)^{\theta/4} - \tilde{\sigma}_m^2(\theta)\} \\ &\le P\{m^\theta(S_m^2 - ES_m^2) \le -\sigma_*^2(\theta)/2\} \end{aligned}$$

since (3.7) holds uniformly in $m_0 \le m \le m_1$. Thus, for any $\epsilon > 0$,

$$P\{W_1\} \le P\{W_3^c\} \le Cm^{4\theta}E(S_m^2 - ES_m^2)^4 \le C(m/N)^{4\min(\theta,1/2)-\epsilon}, \tag{3.17}$$

from Lemma 3.2. Hence by taking $\epsilon > 0$ small enough, (3.15) follows for $i = 1$.

Next, consider $\xi_{m,2}$. By the inequality $\sum_1^n \nu_j a_j \ge \prod_1^n a_j^{\nu_j}$, which is true for any $a_j > 0, \nu_j > 0, j = 1,\dots,n$, $\sum_1^n \nu_j = 1$, we obtain with $Y_k := m^{\theta/2}(X_k^{(m)} - [N/m]^{-1}\sum_{j=1}^{[N/m]} X_j^{(m)})$

$$m^\theta S_m^2 = \frac{1}{[N/m]}\sum_{k=1}^{[N/m]} Y_k^2 \ge \prod_{k=1}^{[N/m]} Y_k^{2/[N/m]}.$$

On the other hand, $0 \le m^\theta S_m^2 = \frac{1}{[N/m]}\sum_{k=1}^{[N/m]} Y_k^2 \le (m/N)^3 < 1$ on W_2. Thus, $Y_k^2 \le (m/N)^2$ on W_2. Consequently,

$$\begin{aligned}
E\xi_{m,2}^2 &= E\mathbf{1}(W_2)\log^2(m^\theta S_m^2) \le E\mathbf{1}(W_2)\Big(\log \prod_{k=1}^{[N/m]} Y_k^{2/[N/m]}\Big) \\
&\le [N/m]^{-2}\sum_{k,k'=1}^{[N/m]} E\mathbf{1}(W_2)|\log Y_k^2|\,|\log Y_{k'}^2| \\
&\le [N/m]^{-2}2^{-1}\sum_{k,k'=1}^{[N/m]}\big(E\mathbf{1}(W_2)\log^2 Y_k^2 + E\mathbf{1}(W_2)\log^2 Y_{k'}^2\big) \\
&\le \max_{1\le k\le [N/m]} E\mathbf{1}(W_2)\log^2 Y_k^2 \\
&\le P^{3/4}\{W_2\}\max_{1\le k\le [N/m]}(E\log^8 Y_k^2)^{1/4} \\
&\le C(m/N)^{3\min(\theta,1/2)-3\epsilon/4}\max_{1\le k\le [N/m]} E^{1/4}(|Y_k|^\epsilon + |Y_k|^{-\epsilon}) \\
&\le C(m/N)^{\min(2\theta,1)+\delta},
\end{aligned}$$

because $P\{W_2\} \le P\{W_3^c\}$ is estimated by (3.17) and because $Y_k \sim \mathcal{N}(0,d)$ where $d \to \sigma_*^2(\theta)$ from (2.1) and (2.2).

To show (3.16) we use $\log(x+y) - \log x = y/x + O(y^2\max((x^{-2},(x+y)^{-2}))$, for $x > 0, x+y > 0$, with $x = \tilde\sigma_m^2(\theta), y = m^\theta(S_m^2 - ES_m^2)$, $x + y = m^\theta S_m^2 \ge (m/N)^{\theta/4}$ on W_3, to obtain

$$\begin{aligned}
\xi_{m,3} &= -\mathbf{1}(W_3)\big(\log\tilde\sigma_m^2(\theta) - (\tilde\sigma_m^2(\theta))^{-1}m^\theta(S_m^2 - ES_m^2)\big) + q_1 \\
&= -\log\tilde\sigma_m^2(\theta) - (\tilde\sigma_m^2(\theta))^{-1}m^\theta(S_m^2 - ES_m^2) + q_1 + q_2 + q_3,
\end{aligned}$$

where

$$\begin{aligned}
Eq_1^2 &\le C(m/N)^{-\theta}E\big(m^\theta(S_m^2 - ES_m^2)\big)^4 \\
&\le O((m/N)^{-\theta}(m/N)^{4\min(\theta,1/2)-\epsilon}) = o((m/N)^{\min(2\theta,1)+\epsilon})
\end{aligned}$$

by Lemma 3.2, for any $\epsilon > 0$ small enough,

$$Eq_2^2 = \log^2\tilde\sigma_m^2(\theta)P(W_3^c) = O((m/N)^{\min(2\theta,1)+\delta}),$$

using (2.3) and (3.17), and, finally,

$$\begin{aligned}
Eq_3^2 &= (\tilde\sigma_m^2(\theta))^{-2}E|m^\theta(S_m^2 - ES_m^2)|^2\mathbf{1}(W_3^c) \le CE^{1/2}\big(m^\theta(S_m^2 - ES_m^2)\big)^4P^{1/2}(W_3^c) \\
&\le C\big(m/N\big)^{2\min(\theta,1/2)-\epsilon}\big(m/N\big)^{2\min(\theta,1/2)-\epsilon} \le C\big(m/N\big)^{\min(2\theta,1)+\epsilon}
\end{aligned}$$

for any sufficiently small $\epsilon > 0$. Lemma 3.3 is proved.

Lemma 3.4. Defining

$$\mu_j = \frac{\nu_j}{\sum_{j=m_0}^{m_1} \nu_j^2}.$$

we have

$$\sum_{j=m_0}^{m_1} \mu_j = 0, \tag{3.18}$$

$$\sum_{j=m_0}^{m_1} \mu_j^2 = m_1^{-1}(1+o(1)), \tag{3.19}$$

and , for any $\gamma > -1$, and uniformly in $m_0 \le t \le m_1$,

$$\sum_{j=m_0}^{m_1} \mu_j j^\gamma = \frac{\gamma}{(1+\gamma)^2} m_1^\gamma + o(m_1^\gamma), \tag{3.20}$$

$$\sum_{j=m_0}^{t} |\mu_j| j^\gamma \le C t^{\frac{1}{2}+\gamma} m_1^{-1/2} \tag{3.21}$$

Proof. (3.18) is obvious; (3.19) is proved in Robinson (1995). (3.20) follows easily from (3.19) and the fact that

$$\sum_{j=m_0}^{m_1} \log j = m_1(\log m_1 + 1)(1+o(1)),$$

$$\sum_{j=m_0}^{m_1} j^\gamma = \frac{m_1^{\gamma+1}}{\gamma+1}(1+o(1)), \tag{3.22}$$

$$\sum_{j=m_0}^{m_1} j^\gamma \log j = \frac{m_1^{\gamma+1} \log m_1}{\gamma+1}(1+o(1)) + \frac{m_1^{\gamma+1}}{(\gamma+1)^2}(1+o(1)).$$

Finally (3.21) follows from (3.9), using relations above and the Cauchy inequality.

Lemma 3.5.

$$\widehat{\theta}_{m_0,m_1} = \theta - b_{m_0,m_1}(\theta) - (\widetilde{\sigma}_m^2(\theta))^{-1} \Sigma_{m_0,m_1}(\theta) + R_{m_0,m_1},$$

where

$$\Sigma_{m_0,m_1}(\theta) = \sum_{j=m_0}^{m_1} \mu_j j^\theta (S_j^2 - ES_j^2)$$

and

$$ER_{m_0,m_1}^2 = O((m_1/N)^{\min(2\theta,1)+\delta}) \quad (\exists \delta > 0).$$

Proof. As $R_{m_0,m_1} = \sum_{j=m_0}^{m_1} \mu_j R_j$ (see (3.14)), so by Lemma 3.3 and Lemma 3.4 (3.21),

$$ER^2_{m_0,m_1} \le C\Big(\sum_{j=m_0}^{m_1} |\mu_j| E^{1/2} R_j^2\Big)^2$$
$$\le C\Big(\sum_{m=m_0}^{m_1} |\mu_j| (j/N)^{\min(\theta,1/2)+\delta/2}\Big)^2 \le C(m_1/N)^{\min(2\theta,1)+\delta},$$

proving the lemma.

Lemma 3.6 *As $m \to \infty$,*

$$\tilde\sigma^2_m(\theta) = \sigma^2_*(\theta) + \big(Q(m) - (\frac{m}{N})^\theta \sigma^2_*(\theta)\big)(1+o(1)) \tag{3.22}$$

and

$$\log(\tilde\sigma^2_m(\theta)) = \log \sigma^2_*(\theta) + \big(\frac{Q(m)}{\sigma^2_*(\theta)} - (\frac{m}{N})^\theta\big)(1+o(1)), \tag{3.23}$$

where

$$Q(m) = \begin{cases} (\bar r_1 - 2\sigma^2 I(\theta)) m^{\theta-1}, & \textit{under Assumption (A)},\\ \frac{2c_1}{(1-\theta-\beta)(2-\theta-\beta)} m^{-\beta}, & \textit{under Assumption (B)}. \end{cases}$$

Proof. From (2.3)

$$\tilde\sigma^2_m(\theta) = m^\theta E\{(X_1^{(m)})^2 - (X_1^{([N/m]m)})^2\}$$
$$= m^\theta E(X_1^{(m)})^2 - (\frac{m}{N})^\theta \sigma^2_*(\theta)(1+o(1)).$$

We have

$$E(X_1^{(m)})^2 = \frac{1}{m^2}\sum_{t,s=1}^{m} r(t-s) = \frac{r(0)}{m} + \frac{2}{m^2}\sum_{j=1}^{m-1}(m-j)r(j).$$

Using the relations

$$\sum_{t=1}^{m-1} t^{-\theta} = (1-\theta)^{-1} m^{1-\theta} - I(\theta) + o(1),$$
$$\sum_{t=1}^{m-1} t^{1-\theta} = (2-\theta)^{-1} m^{2-\theta} + O(m^{1-\theta}),$$

for $0 < \theta < 1$, the proof of (3.22) is completed under Assumption B, whereas under Assumption A it remains only to deduce from the Toeplitz lemma that $\sum_1^m (1-j/m) r_1(j) = \bar r_1 + o(1)$. Then (3.23) follows because $\log\{\tilde\sigma^2_m(\theta)/\sigma^2_*(\theta)\} = (\tilde\sigma^2_m(\theta) - \sigma^2_*(\theta))(1+o(1))$.

Lemma 3.7. *Under the conditions of Theorem 2.1*

$$b_{m_0,m_1}(\theta) = -\Big(\zeta(\theta)(\frac{m_1}{N})^\theta + Q(m_1)\frac{h(\theta,\beta)}{\sigma_*^2(\theta)}\Big)(1+o(1)),$$

where

$$h(\theta,\beta) = \begin{cases} (1-\theta)/\theta^2, & \text{if Assumption (A) holds,} \\ \beta/(1-\beta)^2, & \text{if Assumption (B) holds.} \end{cases}$$

Proof. By Lemma 3.6 and (3.18),

$$\begin{aligned} b_{m_0,m_1}(\theta) &= \sum_{j=m_0}^{m_1} \mu_j \log \widetilde{\sigma}_j^2(\theta) \\ &= \Big\{\sigma_*^{-2}(\theta) \sum_{j=m_0}^{m_1} \mu_j Q(j) - \sum_{j=m_0}^{m_1} \mu_j (j/N)^\theta\Big\}\big(1+o(1)\big). \end{aligned} \tag{3.24}$$

By Lemma 3.5 (3.20),

$$\sum_{j=m_0}^{m_1} \mu_j (j/N)^\theta = \zeta(\theta)(m_1/N)^\theta\big(1+o(1)\big), \tag{3.25}$$

and, similarly, with Lemma 3.6 in mind,

$$\sum_{j=m_0}^{m_1} \mu_j Q(j) = -h(\theta,\beta)Q(m_1)\big(1+o(1)\big). \tag{3.26}$$

(3.24)-(3.26) imply the lemma.

Lemma 3.8. *Let* $1/2 < \theta < 1$. *For almost every* $(u,v) \in [0,1]^2$, *as* $N, m \to \infty,\ m = o(N)$,

$$D_m(u,v) \equiv \sigma_*^{-4}(\theta) EU_2([um])U_2([vm]) \to D(u,v), \tag{3.27}$$

and moreover, for any $\epsilon > 0$ *there is a constant* $C = C_\epsilon < \infty$ *such that*

$$\big|EU_2([um])U_2([vm])\big| \le C, \quad (u,v) \in (\epsilon, 1]^2. \tag{3.28}$$

Proof. Write $N_{u,m} = [N/[um]]$. Then

$$U_2([um]) = N_{u,m}^{-1/2} \sum_{t=1}^{N_{u,m}} \big((Y_t^{([um])})^2 - E(Y_t^{([um])})^2\big),$$

where $Y_t^{(m)}$ is given by (3.9). We have

$$\sigma_*^4(\theta) D_m(u,v) = \big(N_{u,m} N_{v,m}\big)^{-1/2} \sum_{t=1}^{N_{u,m}} \sum_{s=1}^{N_{v,m}} \big(\rho_{[mu],[mv]}(t,s)\big)^2,$$

where

$$\begin{aligned} \sigma_*^4(\theta)\rho_{[mu],[mv]}(t,s) &:= EY_t^{[mu]} Y_s^{[mv]} \\ &= ([mu][mv])^{\theta/2-1} \sum_{i=1}^{[mu]} \sum_{l=1}^{[mv]} r(i-l+(t-1)[mu]-(s-1)[mv]). \end{aligned}$$

Observe that, as $m, |tu-sv| \to \infty$ and uniformly in $(u,v) \in (\epsilon,1]^2$,

$$\rho_{[mu],[mv]}(t,s) \sim \frac{\sigma^2 (uv)^{\theta/2}}{|tu-sv|^\theta}. \tag{3.29}$$

Indeed,

$$\sigma^{-2}|tu-sv|^\theta \rho_{[mu],[mv]}(t,s) = \Big(\frac{[mu]}{m}\frac{[mv]}{m}\Big)^{\theta/2-1} \sum_{i=1}^{[mu]} \sum_{l=1}^{[mv]} m^{-2} B(i/m, l/m)\big(1+o(1)\big),$$

where

$$B(x,y) \equiv B(x,y;t,s,u,v,m) := \frac{|tu-sv|^\theta}{|x-y+(tu-sv)+u-v|^\theta} \to 1$$

as $|tu-sv| \to \infty$, and uniformly in $x,y,u,v \in [0,1]$. Hence, (3.29) clearly follows. Similarly, for any $t,s \in \mathbf{Z}$ *fixed*,

$$\rho_{[mu],[mv]}(t,s) \sim \gamma_{u,v}(t,s), \quad \text{as } m \to \infty, \tag{3.30}$$

uniformly in $(u,v) \in (\epsilon,1]^2$, where

$$\begin{aligned} \gamma_{u,v}(t,s) &= \sigma^2 (uv)^{\theta/2-1} \int_0^u \int_0^v |x-y+(t-1)u-(s-1)v|^{-\theta} dx dy \\ &= \sigma_*^2(\theta)(uv)^{\theta/2-1} \mathrm{Cov}\big(J_1(tu)-J_1((t-1)u), J_1(sv)-J_1((s-1)v)\big). \end{aligned} \tag{3.31}$$

From (3.30)-(3.31) and the uniform boundedness of $\widetilde{\rho}_{[mu],[mv]}(t,s)$ it follows, uniformly in $(u,v) \in (\epsilon,1]^2$, that

$$D_m(u,v) = \Gamma_m(u,v)\big(1+o(1)\big),$$

where

$$\Gamma_m(u,v) = \big(N_{u,m} N_{v,m}\big)^{-1/2} \sum_{t=1}^{N_{u,m}} \sum_{s=1}^{N_{v,m}} \gamma_{u,v}^2(t,s). \tag{3.32}$$

By (3.31),

$$\gamma_{u,v}(t,s) = \int_{\mathbf{R}} e^{i(tu-sv)x} h_{u,v}(x)dx,$$

where

$$h_{u,v}(x) = k_1^2(\theta)(uv)^{\theta/2-1}(1-e^{-iux})(1-e^{ivx})|x|^{\theta-3}, \quad x \in \mathbf{R}.$$

Below, we prove the convergence (3.29), or

$$\Gamma_m(u,v) \to \Gamma(u,v)$$

for $0 < u, v \le 1$ such that the ratio u/v is *irrational* number. As the Lebesgue measure of such pairs $(u,v) \in [0,1]^2$ equals 1, this proves (3.27). Put $\widetilde{h}_{u,v}(x) = (1-e^{-iux})(1-e^{ivx})|x|^{\theta-3}$, $\widetilde{\gamma}_{u,v}(t,s) = \int_{\mathbf{R}} e^{i(tu-sv)x}\widetilde{h}_{u,v}(x)dx$. Then

$$\widetilde{\gamma}_{u,v}(t,s) = \widetilde{g}_{u,v}(tu-sv),$$

where

$$\widetilde{g}_{u,v}(z) = \int_{\mathbf{R}} e^{izx}\widetilde{h}_{u,v}(x)dx, \quad z \in \mathbf{R}$$

is the Fourier transform. Write $\widetilde{\Gamma}_m(u,v)$ for the right hand side of (3.32) with $\gamma_{u,v}(t,s)$ replaced by $\widetilde{\gamma}_{u,v}(t,s)$. Then

$$\widetilde{\Gamma}_m(u,v) = \widetilde{\Gamma}_{m,K}(u,v) + \Lambda_{m,K},$$

where

$$\widetilde{\Gamma}_{m,K}(u,v) = \big(N_{u,m}N_{v,m}\big)^{-1/2} \sum_{t=1}^{N_{u,m}} \sum_{s=1}^{N_{v,m}} \widetilde{\gamma}_{u,v}^2(t,s)\mathbf{1}(|tu-sv| \le K).$$

According to (3.29) and (3.30),

$$\begin{aligned}
|\Lambda_{m,K}| &\le C\big(N_{u,m}N_{v,m}\big)^{-1/2} \sum_{t=1}^{N_{u,m}} \sum_{s=1}^{N_{v,m}} |tu-sv|^{-2\theta}\mathbf{1}(|tu-sv| > K) \\
&\le C(m/N)\int_0^{N/m}\int_0^{N/m} |t-s|^{-2\theta}\mathbf{1}(|t-s|>K)dsdt \le CK^{1-2\theta} = o(1)
\end{aligned}$$

uniformly in $N, m, N/m \to \infty$.

Consider $\widetilde{\Gamma}_{m,K}(u,v)$. Note that $|tu-sv| \le K$ $(t,s \in \mathbf{Z})$ is equivalent to $t = [sv/u] + l$, $|l| \le [K/u]$, $l \in \mathbf{Z}$. Hence

$$\widetilde{\Gamma}_{m,K}(u,v) = \sum_{l:|l|\le[K/u]} d_m(l),$$

where

$$d_m(l) \equiv d_m(l; u, v)$$
$$= \big(N_{u,m} N_{v,m}\big)^{-1/2} \sum_{s=1}^{N_{v,m}} \widetilde{g}^2_{u,v}([sv/u]u + lu - sv) \mathbf{1}(1 - l \le [sv/u] \le N_{u,m} - l).$$

Write $d_m(l) = \widetilde{d}_m(l) + \chi_m(l)$, where

$$\widetilde{d}_m(l) = \big(N_{u,m} N_{v,m}\big)^{-1/2} \sum_{s=1}^{N_{v/m}} \widetilde{g}^2_{u,v}(([sv/u] - sv/u + l)u).$$

As $|l|$ is bounded ($|l| \le [K/u]$) and $|\widetilde{g}_{u,v}(z)| \le \int_{\mathbf{R}} |\widetilde{h}_{u,v}(x)| dx \equiv C(u,v) < \infty$, so $|\chi_m(l)| = O(C^2(u,v)(m/N)) = o(1)$. To show the limit of $\widetilde{d}_m(l)$, we use the fact that the *fractional part* $\{sv/u\} := sv/u - [sv/u]$, $s \in \mathbf{Z}$ of the irrational number sv/u is asymptotically uniformly distributed in the interval $[0, 1]$ (see e.g. Drobot (1964, Theorem on Uniform Distribution)). Namely, for any interval $I \subset [0, 1]$,

$$\lambda_N(I) := (1/N) \sum_{s=1}^{N} \mathbf{1}(\{sv/u\} \in I) \to \lambda(I), \qquad (N \to \infty), \tag{3.33}$$

where $\lambda(I)$ is the Lebesgue measure. From (3.33) and the continuity of $\widetilde{g}_{u,v}(z)$ it follows that for any $l \in \mathbf{Z}$ and any pair $0 < u, v \le 1$ such that v/u is irrational,

$$\widetilde{d}_m(l) = \big(N_{v,m}/N_{u,m}\big)^{1/2} \int_0^1 \widetilde{g}^2_{u,v}((l - \nu)u) \lambda_{N_{v,m}}(d\nu)$$
$$\to (u/v)^{1/2} \int_0^1 \widetilde{g}^2_{u,v}((l - \nu)u) d\nu.$$

By the argument above,

$$\widetilde{\Gamma}_{m,K}(u,v) \to (u/v)^{1/2} \sum_{|l| \le [K/u]} \int_0^1 \widetilde{g}^2_{u,v}((l - \nu)u) d\nu. \tag{3.34}$$

It remains to show that the right hand side of (3.34) (times $(uv)^{\theta-2} \sigma_*^4 k_1^4(\theta)$) approaches $D(u, v)$ as $K \to \infty$.

Write

$$\widetilde{g}_{u,v}((l - \nu)u) = u^{-1} \int_\Pi e^{ily} H(y, \nu) dy,$$

where $\Pi = (-\pi, \pi]$ and where

$$H(y, \nu) \equiv H(y, \nu; u, v) = \sum_{k \in \mathbf{Z}} \widetilde{h}_{u,v}((y + 2k\pi)/u) e^{-i(y+2k\pi)\nu} \tag{3.35}$$

is a continuous function on $\Pi\backslash\{0\}$ satisfying $|H(y,\nu)| \le C|y|^{\theta-1}$, $y \in \Pi$, where the constant $C = C(u,v) < \infty$ does not depend on ν. The last bound implies $H(\cdot,\nu) \in L^2(\Pi)$ uniformly in ν; it follows from (3.35) and the bound $|\widetilde{h}_{u,v}(y)| \le C|y|^{\theta-1}$ if $|y| \le 1; \le C|y|^{\theta-3}$ if $|y| > 1$. Consequently, by Parseval's theorem,

$$\lim_{K\to\infty} \sum_{|l|\le[K/u]} \widetilde{g}^2_{u,v}((l-\nu)u) = 2\pi u^{-2} \int_\Pi |H(y,\nu)|^2 dy,$$

uniformly in $\nu \in [0,1]$, and therefore

$$\lim_{K\to\infty} (u/v)^{1/2} \sum_{|l|\le[K/u]} \int_0^1 \widetilde{g}^2_{u,v}((l-\nu)u)d\nu = 2\pi u^{-3/2} v^{-1/2} \int_\Pi \int_0^1 |H(y,\nu)|^2 dy d\nu.$$

The last integral is

$$\sum_{k,j\in\mathbf{Z}} \int_\Pi \Big\{ \int_0^1 e^{i2(j-k)\pi\nu} d\nu \Big\} \widetilde{h}_{u,v}((y+2k\pi)/u)\overline{\widetilde{h}_{u,v}((y+2j\pi)/u)}dy$$
$$= \sum_{k\in\mathbf{Z}} \int_\Pi |\widetilde{h}_{u,v}((y+2k\pi)/u)|^2 dy = u \int_{\mathbf{R}} |\widetilde{h}_{u,v}(y)|^2 dy$$

to prove the convergence (3.27) for

$$D(u,v) = 2\pi k_1^4(\theta)(uv)^{\theta-5/2} \int_{\mathbf{R}} |(1-e^{-iux})(1-e^{ivx})|^2 |x|^{2\theta-6} dx$$

where the formula (2.4) is deduced by repeated integration by parts.

It remains to show the bound (3.28) which follows from the Cauchy-Schwartz inequality and

$$EU_2^2([um]) \le C \tag{3.36}$$

uniformly in $u \in (\epsilon,1]$. Here, (3.36) follows from

$$D_m(u,u) \equiv N_{u,m}^{-1} \sum_{t,s=1}^{N_{u,m}} \big(\rho_{[um],[um]}(t,s)\big)^2 \le C, \quad u \in (\epsilon,1], \tag{3.37}$$

where $\rho_{[um],[um]}(t,s) = \rho_{t-s}^{([um])}$ satisfies $|\rho_{[um],[um]}(t,s)| \le C|t-s|_+^{-\theta}$ uniformly in $u \in (\epsilon,1]$; see (3.11). Hence, (3.37) easily follows (recall $\theta > 1/2$). Lemma 3.8 is proved.

Lemma 3.9. *Under the conditions of Theorem 2.2 (without assuming (A) or (B)),*

$$d_{N/m_1}\Sigma_{m_0,m_1}(\theta) = \sigma_*^2(\theta)\widetilde{Z}(\theta) + o_2(1),$$

where

$$\Sigma_{m_0,m_1}(\theta) = \sum_{j=m_0}^{m_1} \mu_j d_{N/m_1}^{-1} Z_j(\theta).$$

Proof. Let $0 < \theta \le 1/2$. Write

$$\Sigma_{m_0,m_1}(\theta) = \sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1} Z_i(\theta) = \Big(\sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1}\Big) Z_{m_1}(\theta) + \widetilde{R}_{m_0,m_1},$$

where

$$\widetilde{R}_{m_0,m_1} = \sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1}(Z_i(\theta) - Z_{m_1}(\theta)).$$

We show that

$$E\widetilde{R}_{m_0,m_1}^2 = o\big(d_{N/m_1}^{-2}\big). \tag{3.38}$$

Then the statement of Lemma 3.9 follows from Lemma 3.1 and

$$\sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1} = \zeta(\theta) d_{N/m_1}^{-1}\big(1 + o(1)\big), \tag{3.39}$$

$$\sum_{i=m_0}^{m_1} |\mu_i| d_{N/i}^{-1} \le C d_{N/m_1}^{-1}. \tag{3.40}$$

Let us first check (3.39)-(3.40). For $\theta < 1/2$, they follow from Lemma 3.4, (3.20) and (3.21). If $\theta = 1/2$,

$$\begin{aligned}\sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1} &= (\frac{\log(N/m_1)}{N})^{1/2} \sum_{i=m_0}^{m_1} \mu_i i^{1/2} + O((\frac{\log m_1}{N})^{1/2} \sum_{i=m_0}^{m_1} |\mu_i| i^{1/2})\\ &= \frac{2}{9} d_{N/m_1}^{-1}(1 + o(1))\end{aligned}$$

and (3.40) follow in the same way.

Consider (3.38). We have:

$$\begin{aligned}E\widetilde{R}_{m_0,m_1}^2 &\le \sum_{j,j'=m_0}^{m_1} |\mu_j| d_{N/j}^{-1} |\mu_{j'}| d_{N/j'}^{-1} E^{1/2}|Z_{m_1}(\theta) - Z_j(\theta)|^2\, E^{1/2}|Z_{m_1}(\theta) - Z_{j'}(\theta)|^2\\ &\le 2 \max_{m_0 \le j \le [\epsilon m_1]} E|Z_{m_1}(\theta) - Z_j(\theta)|^2 \Big(\sum_{j=m_0}^{[\epsilon m_1]} |\mu_j| d_{N/j}^{-1}\Big)^2\\ &+ 2 \max_{[\epsilon m_1] \le j \le m_1} E|Z_{m_1}(\theta) - Z_j(\theta)|^2 \Big(\sum_{j=[\epsilon m_1]}^{m_1} |\mu_j| d_{N/j}^{-1}\Big)^2\\ &=: p'_{m_0,m_1} + p''_{m_0,m_1}.\end{aligned}$$

We show below that, in the case $0 < \theta \le 1/2$, for each $\epsilon > 0$,

$$\max_{[\epsilon m_1] \le j \le m_1} E(Z_{m_1}(\theta) - Z_j(\theta))^2 \to 0. \tag{3.41}$$

Together with (3.40), this implies

$$p''_{m_0,m_1} = o\Big(\sum_{j=[\epsilon m_1]}^{m_1} |\mu_j| d^{-1}_{N/j} \Big)^2 = o(d^{-2}_{N/m_1}). \tag{3.42}$$

Consequently, (3.38) follows from (3.42) and the inequality

$$p'_{m_0,m_1} \le \delta(\epsilon) d^{-2}_{N/m_1} \tag{3.43}$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. In turn, (3.43) follows from $\max_{j\ge 1} EZ_j^2(\theta) < \infty$ (see (3.3)) and

$$\sum_{j=m_0}^{[\epsilon m_1]} |\mu_j| d^{-1}_{N/j} \le C\epsilon d^{-1}_{N/m_1}$$

which follows from (3.21).

Let us turn to the proof of (3.41). To that end, according to (3.4)-(3.6), it suffices to show that, for any $\epsilon > 0$,

$$\max_{[\epsilon m_1] \le j \le m_1} E\big(U_2(m_1) - U_2(j)\big)^2 \to 0, \quad \text{if} \quad 0 < \theta \le 1/2, \tag{3.44}$$

$$\max_{[\epsilon m_1] \le j \le m_1} E\big(U_1^2(m_1) - U_1^2(j)\big)^2 \to 0, \quad \text{if} \quad 0 < \theta < 1/2. \tag{3.45}$$

As $E\big(U_2(m_1) - U_2(j)\big)^2 = EU_2^2(m_1) - 2EU_2(m_1)U_2(j) + EU_2^2(j)$, so (3.44) follows from

$$q_{m,j} := EU_2(m)U_2(j) \to q \tag{3.46}$$

uniformly in $[\epsilon m_1] \le j, m \le m_1$, where the limit q does not depend on m, j. We have

$$\begin{aligned} q_{m,j} &= \frac{d_{N/m}}{[N/m]} \frac{d_{N/j}}{[N/j]} \sum_{t=1}^{[N/m]} \sum_{s=1}^{[N/j]} EH_2(Y_t^{(m)}) H_2(Y_s^{(j)}) \\ &= 2 \frac{d_{N/m}}{[N/m]} \frac{d_{N/j}}{[N/j]} \sum_{t=1}^{[N/m]} \sum_{s=1}^{[N/j]} \rho^2_{m,j}(t,s), \end{aligned}$$

where $H_2(Y_t^{(j)}) = (Y_t^{(j)})^2 - E(Y_t^{(j)})^2$, and where

$$\rho_{m,j}(t,s) = EY_t^{(m)} Y_s^{(j)} = m^{1-\theta/2} j^{1-\theta/2} \sum_{i=1}^{m} \sum_{l=1}^{j} r(i - l + (t-1)m - (s-1)j). \tag{3.47}$$

Let us split the right hand side of (3.47) into two parts $q'_{m,j}$ and $q''_{m,j}$ according to whether $|(t-1)m-(s-1)j| > Km_1$ or $|(t-1)m-(s-1)j| \le Km_1$ holds, where $K > 2$ a given number. By assumption (1.1), if $|(t-1)m-(s-1)j| > Km_1$, then $r(i-l+(t-1)m-(s-1)j) = \sigma^2|i-l+(t-1)m-(s-1)j|^{-\theta}(1+o_K(1)) = \sigma^2|(t-1)m-(s-1)j|^{\theta}(1+o_K(1))$, where $o_K(1) \to 0$ $(K \to \infty)$ uniformly in t, s, m, j and uniformly in $1 \le i, l \le m_1$. Hence, if $0 < \theta < 1/2$, for each $K < \infty$ one obtains

$$\begin{aligned} q'_{m,j} &= 2\sigma^4 \frac{d_{N/m}}{[N/m]} \frac{d_{N/j}}{[N/j]} m^\theta j^\theta \sum_{t=1}^{[N/m]} \sum_{s=1}^{[N/j]} |(t-1)m-(s-1)j|^{-2\theta} \\ &\qquad \mathbf{1}(|(t-1)m-(s-1)j| \ge Km_1)\big(1+o_K(1)\big) \\ &= 2\sigma^4 \int_0^1 \int_0^1 |t-s|^{-2\theta} dt ds + o_K(1), \end{aligned}$$

uniformly in $\epsilon m_1 \le m, j \le m_1$. Similarly, if $\theta = 1/2$, then

$$\begin{aligned} q'_{m,j} &= 2\sigma^4 \frac{1}{(\log(N/m)\log(N/j))^{1/2} N} \int_0^N \int_0^N |t-s|^{-1} 1(|t-s| \ge Km_1) dt ds \big(1+o_K(1)\big) \\ &= 4\sigma^4 \frac{\log(N)}{(\log(N/m)\log(N/j))^{1/2}} \big(1+o_K(1)\big) = 4\sigma^4 + o_K(1), \end{aligned}$$

uniformly in $\epsilon m_1 \le m, j \le m_1$. To end the proof of (4.46), it remains to estimate

$$q''_{m,j} = 2 \frac{d_{N/m}}{[N/m]} \frac{d_{N/j}}{[N/j]} \sum_{t=1}^{[N/m]} \sum_{s=1}^{[N/j]} \rho^2_{m,j}(t,s) \mathbf{1}(|(t-1)m-(s-1)j| < Km_1). \tag{3.48}$$

From $|(t-1)m-(s-1)j| < Km_1$ and $\epsilon m_1 \le m, j \le m_1$ one has $|(t-1)-(s-1)j/m| \le K$ and therefore, for fixed s, the sum over t in (3.48) has a bounded number of terms. Using the boundedness of $|\rho_{m,j}(t,s)| = |EY_t^{(m)} Y_s^{(j)}| \le (E(Y_t^{(m)})^2 E(Y_s^{(j)})^2)^{1/2} = \widetilde{\sigma}_m(\theta)\widetilde{\sigma}_j(\theta)$, (see Lemma 3.6), one obtains, in the case $0 < \theta < 1/2$,

$$q''_{m,j} \le C(m/N)^{1-\theta}(j/N)^{1-\theta} \sum_{s=1}^{N/j} 1 \le C m^{1-\theta} j^{-\theta} N^{2\theta-1} \le C(m_1/N)^{1-2\theta} = o(1);$$

the case $\theta = 1/2$ follows analogously. This proves (3.46) and hence (3.44). Similarly, (3.45) follows from

$$EU_1^2(m)U_1^2(j) \to p \tag{3.49}$$

uniformly in $\epsilon m_1 \le j, m \le m_1$, where limit p does not depend on m, j. Relation (3.49) with $p = 3\sigma_*^4(\theta)$ follows easily from Gaussianity of $U_1(m), U_1(j)$ and (3.5). This proves the lemma for $0 < \theta \le 1/2$.

Let now $1/2 < \theta < 1$. Write

$$\Sigma_{m_0,m_1}(\theta) = \sum_{i=m_0}^{m_1} \mu_i d_{N/i}^{-1} Z_i(\theta) = \sum_{i=m_0}^{[\epsilon m_1]-1} \ldots + \sum_{i=[\epsilon m_1]}^{m_1} \ldots \equiv \Sigma' + \Sigma''.$$

Here, similarly as in the proof of (3.43), one can show that

$$E(\Sigma')^2 \le \delta(\epsilon) d_{N/m_1}^{-2},$$

where $\delta(\epsilon) \to 0\,(\epsilon \to 0$. It remains to show that, for each $\epsilon > 0$,

(3.50) $$d_{N/m_1}\Sigma'' \Longrightarrow \mathcal{N}(0, \sigma_*^4(\theta)\tilde{s}_\epsilon^2(\theta)),$$

and that there exists the limit

(3.51) $$\lim_{\epsilon \to 0} \tilde{s}_\epsilon^2(\theta) = \tilde{s}^2(\theta).$$

According to (3.4),

$$\Sigma'' = \sum_{j=[\epsilon m_1]}^{m_1} \mu_j d_{N/j}^{-1} U_2(j) - \sum_{j=[\epsilon m_1]}^{m_1} \big(\mu_j d_{N/j}^{-1}\big) d_{N/j}(j/N)^\theta (U_1^2(j) - EU_1^2(j)) \equiv \widetilde{\Sigma}'' - \mathcal{X},$$

where $E\big(d_{N/j}(j/N)^\theta (U_1^2(j) - EU_1^2(j))\big)^2 = O((m_1/N)^{2\theta-1}) = o(1)$ uniformly in $[\epsilon m_1] \le i \le m_1$; see (5.11′). Hence $E\mathcal{X}^2 = o\big(\sum_{j=[\epsilon m_1]}^{m_1} |\mu_j| d_{N/j}^{-1}\big)^2 = o(d_{N/m_1}^{-2})$; see (3.21), and it remains to show that (3.50)-(3.51) hold with Σ'' replaced by $\widetilde{\Sigma}''$.

To prove the asymptotic normality (3.50), it is enough to show that the corresponding cumulants of order $k \ge 3$ vanish, i.e. that

(3.52) $$\mathrm{Cum}_k(\widetilde{\Sigma}'') = o(d_{N/m_1}^{-k}), \quad k \ge 3,$$

and, moreover, the convergence of variances:

(3.53) $$(N/m_1)\mathrm{Var}(\widetilde{\Sigma}'') \to \sigma_*^4(\theta)\tilde{s}_\epsilon^2(\theta).$$

We prove now (3.51) using the argument in Giraitis and Surgailis (1985), Theorem 5. By the multilinearity property of the cumulants (see Brillinger (1975), Theorem 2.3.1),

(3.54) $$\mathrm{Cum}_k(\widetilde{\Sigma}'') = \sum_{j_1 \ldots, j_k = [\epsilon m_1]}^{m_1} \mathrm{Cum}(U_2(j_1), \ldots, U_2(j_k)) \prod_{p=1}^{k} \mu_{j_p} d_{N/j_p}^{-1}$$

where on the right hand side is the joint cumulant of random variables $U_2(j_1), \ldots, U_2(j_k)$. We claim that

(3.55) $$\xi_k \equiv \mathrm{Cum}(U_2(j_1), \ldots, U_2(j_k)) = o(1)$$

uniformly in $[\epsilon m_1] \le j_1, \dots, j_k \le m_1$. Hence (3.52) follows by (3.54) and (3.21). To prove (3.55), write

$$U_2(j) = [N/j]^{-1/2} \sum_{t=1}^{[N/j]} H_2(Y_t^{(j)}).$$

By the well-known (diagram) formula,

$$\xi_k = \sum_{\mathcal{P}_{k-1}} 2^k \prod_{r=1}^{k} [N/j_r]^{-1/2} \sum_{t_1=1}^{[N/j_{p(1)}]} \cdots \sum_{t_k=1}^{[N/j_{p(k)}]} \rho_{j_{p(1)}, j_{p(2)}}(t_1, t_2)$$
$$\cdots \rho_{j_{p(k-1)}, j_{p(k)}}(t_{k-1}, t_k) \rho_{j_{p(k)}, j_{p(1)}}(t_k, t_1) \equiv \sum_{\mathcal{P}_{k-1}} \xi_k^{(p)_k},$$

where the sum $\sum_{\mathcal{P}_{k-1}}$ is taken over all permutations $(p)_k \equiv (p(1), \dots, p(k))$ of $(1, 2, \dots, k)$ such that $p(1) = 1$. Consider an arbitrary term $\xi_k^{(p)_k}$, e.g. $(p)_k = (1, 2, \dots, k)$ for simplicity. Given $K > 0$, write

$$T_k \equiv \sum_{t_1=1}^{[N/j_1]} \cdots \sum_{t_k=1}^{[N/j_k]} \rho_{j_1, j_2}(t_1, t_2) \cdots \rho_{j_k, j_1}(t_k, t_1) = T'_{k,K} + T''_{k,K},$$

where the sum $T'_{k,K}$ is taken over $t_1, \dots, t_k$ which satisfy $|j_p t_p - j_{p+1} t_{p+1}| < K m_1$ for all $p = 1, \dots, k$, with the convention $j_{k+1} = j_1$, $t_{k+1} = t_1$. As $[\epsilon m_1] \le j_p \le m_1$, $p = 1, \dots, k$, so $|T'_{k,K}| \le C(K)(N/m_1)$. To estimate $T''_{k,K}$ we need the inequalities: for any $p = 1, \dots, k$,

$$\beta_p \equiv \sum_{t=1}^{[N/j_p]} \sum_{s=1}^{[N/j_{p+1}]} \rho^2_{j_p, j_{p+1}}(t, s) \le CN/m_1 \tag{3.56}$$

because the $\rho_{m,j}(t, s)$ are uniformly bounded, and

$$\beta_{p,K} \equiv \sum_{t=1}^{[N/j_p]} \sum_{s=1}^{[N/j_{p+1}]} \rho^2_{j_p, j_{p+1}}(t, s) \mathbf{1}(|j_p t - j_{p+1} s| > K m_1) \le \delta(K) N/m_1, \tag{5.57}$$

where $\delta(K) \to 0$ $(K \to \infty)$. Relations (3.56)-(3.57) follow easily from the bound: $|\rho_{m,j}(t, s)| \le C(mj)^\theta |tm - sj|^{-\theta}$, $|tm - sj| > K$; see (3.39).

Using Cauchy - Schwartz inequality (see eq. (2.14) of Giraitis and Surgailis (1985)), by (3.56)-(3.57) one obtains

$$|T''_{k,K}| \le \sum_{p=1}^{k} \beta_{p,K}^{1/2} \prod_{q=1,\dots,k: q \ne p} \beta_p^{1/2} \le C \delta^{1/2}(K)(N/m_1)^{k/2}.$$

Consequently, for $(p)_k = (1, 2, \dots, k)$,

$$|\xi_k^{(p)_k}| \le C \Big\{ \prod_{p=1}^{k} (j_p/N)^{1/2} \Big\} \big(C(K)(N/m_1) + \delta^{1/2}(K)(N/m_1)^{k/2} \big)$$
$$\le C(K)(m_1/N)^{(k-2)/2} + C\delta^{1/2}(K) = o(1),$$

provided $k \geq 3$. The above estimate clearly applies to abitrary permutations $(p)_k \in \mathcal{P}_{k-1}$, thus proving (3.55).

Finally, let us prove (3.53). We have

$$\tilde{s}^2_{\epsilon,N}(\theta) := \frac{1}{\sigma^4_*(\theta)}\mathrm{Var}(d_{N/m_1}\widetilde{\Sigma}'') = \frac{d^2_{N/m_1}}{\sigma^4_*(\theta)} \sum_{i,j=[\epsilon m_1]}^{m_1} \mu_i\mu_j(ij/N^2)^{1/2}EU_2(i)U_2(j)$$
$$= \int_\epsilon^1\int_\epsilon^1 m_1\mu_{[um_1]}m_1\mu_{[vm_1]}([um_1][vm_1]/m_1^2)^{1/2}D_{m_1}(u,v)dudv,$$

where $D_{m_1}(u,v) = \sigma_*^{-4}(\theta)EU_2([um_1])U_2([vm_1])$. Observe (see the proof of Lemma 3.4) that, uniformly in $u \in (\epsilon, 1]$,

$$m_1\mu_{[um_1]} \to 1 + \log u \qquad (m_1 \to \infty).$$

According to Lemma 3.8, for almost every $(u,v) \in (0,1]^2$,

$$D_{m_1}(u,v) \to D(u,v),$$

and, furthermore, $|D_{m_1}(u,v)|$ is uniformly bounded on $(u,v) \in (\epsilon,1]^2$. Therefore, one can pass to the limit under the signs of integral in (3.59), yielding

$$\tilde{s}^2_{\epsilon,N}(\theta) \to \tilde{s}^2_\epsilon(\theta) \ = \int_\epsilon^1\int_\epsilon^1 (1+\log u)(1+\log v)(uv)^{1/2}D(u,v)dudv.$$

The limit (3.51) is obtained by putting $\epsilon = 0$ in the integral above, as the limiting integral on $(0,1]^2$ is well-defined and finite.

References

Abbry, P., Veitch, D. and Flandrin, P. (1998). Long-range dependence: revisiting aggregation with wavelets. *J. Time Ser. Anal.* **19**, 253-266.

Bartlett, M.S. (1950). Periodogram analysis and continuous spectra. *Biometrika* **37**, 1-16.

Bender, C. M. and Orszag, S. A. (1978). *Mathematical methods for scientists and engineers.* McGraw-Hill.

Breuer, P. and Major, P. (1983). Central limit theorems for non-linear functions of Gaussian fields. *J. Multivar. Anal.* **13**, 425-441.

Brillinger, D. (1975) *Time Series. Data Analysis and Theory.* Holden Day.

Dobrushin, R.L. and Major, P. (1979). Non-central limit theorems for non-linear functions of Gaussian fields. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **50**, 27-52.

Drobot, S. *Real Numbers.* (1964) Prentice Hall, Inc. Englewood Cliffs, New Jersey.

Geweke, J. and Porter-Hudak, S. (1983). The estimation and application of long memory time series models. *J. Time Ser. Anal.*, **4**, 221-238.

Giraitis, L. and Surgailis, D. (1985). CLT and other limit theorems for functionals of Gaussian processes. *Probab. Th. Rel. Fields* **70**, 191-212.

Giraitis, L. and Surgailis, D. (1986). Multivariate Appell polynomials and the central limit theorem. In: Eberlein, E. and Taqqu, M.S. (eds.), *Dependence in Probability and Statistics.* Birkhauser, New York.

Hall, P., Koul H. L. and Turlach B. A. (1997). Note on convergence rates of semiparametric estimators of dependence index. *Ann. Stat.* **25**, 1725-1739.

Künsch, H. (1987). Statistical properties of self-similar processes. *Proceedings of the 1st World Congress of the Bernoulli Society*, **4**, 67-74, VNU Science Press.

Lobato, I. and Robinson, P.M. (1996). Averaged periodogram estimation of long memory. *J. of Econometrics* **73**, 303-324

Robinson, P.M. (1994a). Rates of convergence and optimal spectral bandwidth for long range dependence *Probab. Th. Rel. Fields* **99**, 443-473.

Robinson, P.M. (1994b). Time series with strong dependence. In: *Advances in Econometrics* Vol. 1 (C.A. Sims, ed.), Cambridge University Press, 47-95.

Robinson, P.M. (1995). Log-periodogram regression of time series with long range dependence. *Ann. Stat.* **23**, 1048-1072.

Taqqu, M. (1979). Convergence of integrated process of arbitrary Hermite rank. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **50**, 53-83.

Taqqu, M., Teverovsky, V. and Willinger, W. (1995). Estimators for long-range dependence: an empirical study. *Fractals*, **3**, 785-798.

Teverovsky, V. and Taqqu, M. (1997). Testing for long-range dependence in the presence of shifting mean or a slowly declining trend, using a variance-type estimator. *J. Time Series Analysis*, **18**, 279-304.