FINITE SAMPLE IMPROVEMENTS IN STATISTICAL INFERENCE WITH I(1) PROCESSES*

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Discussion Paper
No.EM/01/422
July 2001

* The first author's research was supported by MURST and by ESRC Grant R000235892. The second author's research was supported by a Leverhulme Trust Personal Research Professorship and ESRC Grant R000235892. We are grateful to four referees for comments which have significantly improved the paper.

Abstract

Robinson and Marinucci (1998) investigated the asymptotic behaviour of a narrow-band semiparametric procedure termed Frequency Domain Least Squares (FDLS) in the broad context of fractional cointegration analysis. Here we restrict to the standard case when the data are *I*(1) and the cointegrating errors are *I*(0), proving that modifications of the Fully-Modified Ordinary Least Squares (FM-OLS) procedure of Phillips and Hansen (1990) which use the FDLS idea have the same asymptotically desirable properties as FM-OLS, and, on the basis of a Monte Carlo study, find evidence that they have superior finite-sample properties; the new procedures are also shown to compare satisfactorily with parametric estimates.

Keywords: Fully-modified ordinary least squares; finite sample improvements; statistical inference with *I*(1) processes; Monte Carlo study; parametric estimates.

JEL No.: C22

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1. INTRODUCTION

The contention that jointly dependent macroeconomic series often have unit roots, and may be cointegrated, has considerably influenced econometric research in recent years. Consider the model ("cointegrated system") for the jointly dependent $p_1 \times 1$ and $p_2 \times 1$ vector of observables y_t and x_t, respectively,

$$y_t = \Pi x_t + u_{1t} \ , \ t \geq 1 \ , \ x_t = x_{t-1} + u_{2t} \ , \ t \geq 1 \ , \ x_0 = 0 \ , \tag{1.1}$$

where Π is an unknown $p_1 \times p_2$ matrix, and $u_t' = (u_{1t}', u_{2t}')$ is a $p \times 1$ $I(0)$ vector sequence, that is, it is covariance stationary with zero mean and has a spectral density matrix that is bounded and nonsingular at the origin, with $p = p_1 + p_2$; $z_t' = (y_t', x_t')$ are then said to be $I(1)$, and cointegrated of orders (1,0) ($CI(1,0)$); we do not assume u_{1t} to be uncorrelated with u_{2t} (or, hence, with x_t). Cointegrated systems like (1.1) have been considered in the analysis of various economic hypotheses, in the context of, for example, purchasing power parity and other models of exchange rate determination, present value models, life-cycle models of consumers' behaviour, and the quantity theory of money. In simultaneously modelling short- and long-run behaviour, deviations from cointegrating relationships can capture short-term adjustments, in the spirit of the Error-Correction Mechanism (ECM) representation of Sargan (1964) and Davidson et al. (1978). Inference on $I(1)$ cointegrated processes in the framework of a fully parametric ECM representation has been developed by, for example, Phillips (1991a) and Johansen (1988,1995), where asymptotically optimal rules are presented, based on maximum likelihood estimation (MLE), and requiring a complete specification of the system, with standard chi-squared testing procedures justified asymptotically.

Complete specification is unnecessary, however, for the achievement of such desirable first-order asymptotic properties. Indeed semiparametric inference rules based on instrumental variables (see Sargan (1959)) were considered for the analysis of $I(1)$

processes by Phillips and Hansen (1990), who introduced a Fully Modified Ordinary Least Squares (FM-OLS) technique which is optimal in the context of nonparametric autocorrelation in u_t. Also, the efficient frequency domain approach originally introduced by Hannan (1963a,b), which adapts to disturbance autocorrelation of nonparametric form and permits inclusion of only a proper subset of frequencies, has been developed by Phillips (1991b) in the presence of unit roots.

A common thread of many such semiparametric procedures is their reliance on initial estimates of Π, typically by Ordinary Least Squares (OLS). In the far broader framework of fractional cointegration analysis, Robinson and Marinucci (1998) showed that the performance of OLS can be improved on in several circumstances, including the $CI(1,0)$ case, by a semiparametric narrow-band procedure termed Frequency Domain Least Squares (FDLS), originally suggested in the context of a stationary (bivariate) sequence z_t with long memory by Robinson (1994); here, correlation between x_t and u_{1t} in (1.1) renders OLS inconsistent due to simultaneous equation bias, but if u_{1t} has less memory than x_t (for example, if it is $I(0)$), FDLS is consistent. This proposal was further developed in the stationary case by Robinson and Marinucci (1998), but especially when z_t, and possibly u_{1t}, are nonstationary. Here, OLS is typically consistent if the memory in x_t exceeds that of u_{1t}, but, depending on the orders of integration of x_t and u_{1t}, FDLS can have the same limiting distributional behaviour as OLS, perhaps with less "second order bias", or even converges faster, indicating that the medium and high-frequency "information" discarded by FDLS is at best unimportant and at worst harmful. Our present case of $I(1)$ x_t and $I(0)$ u_{1t} is one in which Robinson and Marinucci (1998) found FDLS to have less second-order bias than OLS, and corresponding finite-sample improvements in Monte Carlo simulations. However, both OLS and FDLS share the disadvantage of having a nonstandard limit distribution which is inconvenient for inference. The present paper combines the FM-OLS and FDLS ideas to provide estimates with the same desirable asymptotic properties as FM-OLS, but, according to our Monte Carlo investigation, superior finite-sample properties. Intuitively, the latter finding is due to the presence, to some higher-order, of simultaneous equation bias in FM-OLS, which is reduced by stressing a narrow band of frequencies around the origin.

The following section, after some discussion of theoretical background, reviews a modification of FM-OLS previously considered by Robinson and Marinucci (1998), introduces a "Fully Modified Frequency Domain Least Squares" procedure, and establishes its asymptotic behaviour. Section 3 investigates via a Monte Carlo experiment their finite-sample performance, comparing them with the more traditional FM-OLS and MLE procedures in terms of bias, standard deviation and distributional properties. We find the results for the new procedures to be encouraging.

In the sequel, we use $\Rightarrow$ to denote weak convergence, $\|.\|$ to denote Euclidean norm, and C for a generic, positive constant; " $>$ " will be taken to signify positive definite when referring to matrices.

2. FULLY-MODIFIED FREQUENCY DOMAIN LEAST SQUARES

For the purpose of deriving asymptotical statistical properties we first introduce

Assumption A (1.1) holds, with $u_t = \Psi(L)\varepsilon_t$, $\Psi(L) = \sum_{j=0}^{\infty} \Psi_j L^j$, where L is the lag operator, $\det\{\Psi(1)\} \neq 0$, $\sum_{j=0}^{\infty} j\|\Psi_j\| < \infty$, and $\{\varepsilon_t\}$ is a sequence of independent and identically distributed ($i.i.d.$) $p \times 1$ vectors such that $E\varepsilon_t = 0$, $E\varepsilon_t\varepsilon_t' = \Sigma$, $\det\{\Sigma\} \neq 0$, $E\|\varepsilon_t\|^4 < \infty$.

Assumption A covers a wide class of short memory linear processes, including stationary and invertible vector autoregressive moving averages driven by Gaussian innovations.

Let $B'(r) = (B_1'(r), B_2'(r))$ be $p-$dimensional Brownian motion with covariance matrix $\Omega = \Psi(1)\Sigma\Psi(1)'$; under Assumption A, as $n \to \infty$,

$$n^{-2}\sum_{t=1}^{n}(x_t - \overline{x})(x_t - \overline{x})' \;\Rightarrow\; \int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr \;, \tag{2.1}$$

$$n^{-1}\sum_{t=1}^{n} u_{1t}(x_t - \overline{x})' \;\Rightarrow\; \int_0^1 dB_1(r)\overline{B}_2'(r) + \Lambda_{12} \;, \tag{2.2}$$

see Phillips (1988), where $\overline{B}_2(r) = B_2(r) - \int_0^1 B_2(r)dr$, $\Lambda_{12} = \sum_{j=0}^{\infty}\Gamma_{12}(j)$, $\Gamma_{12}(j) = Eu_{1j}u_{20}'$ and $\overline{x} = n^{-1}\sum_{t=1}^{n} x_t$. Unless Ω is block diagonal, $B_2(.)$ and $B_1(.)$ are not independent and the distributions of the right-hand sides of (2.1)/(2.2) are non-standard. Now define $\Omega_{12} = \sum_{j=-\infty}^{\infty}\Gamma_{12}(j)$, $\Gamma_{22}(j) = Eu_{2j}u_{20}'$, $\Omega_{22} = \sum_{j=-\infty}^{\infty}\Gamma_{22}(j)$ and $B_{1.2}(r) = B_1(r) - \Omega_{12}\Omega_{22}^{-1}B_2(r)$, the Gaussian process orthogonal to $B_2(r)$. The asymptotic distribution of the OLS estimates

$$\widehat{\Pi} = \sum_{t=1}^{n} y_t(x_t - \overline{x})'\left\{\sum_{t=1}^{n}(x_t - \overline{x})(x_t - \overline{x})'\right\}^{-1},$$

as $n \to \infty$ is readily seen to be given by

$$\begin{aligned} n(\widehat{\Pi} - \Pi) \;&\Rightarrow\; \left\{\int_0^1 dB_1(r)\overline{B}_2(r) + \Lambda_{12}\right\}\left\{\int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr\right\}^{-1} \\ &= A_1 + A_2 + A_3 \;, \end{aligned} \tag{2.3}$$

where

$$A_1 = \left\{\int_0^1 dB_{1.2}(r)\overline{B}_2(r)'\right\}\left\{\int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr\right\}^{-1} \tag{2.4a}$$

$$\equiv \int_{G>0} N(0, \Omega_{11.2}\otimes G)dP(G) \;,\; G = \left\{\int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr\right\}^{-1}$$

$$A_2 = \Omega_{12}\Omega_{22}^{-1}\left\{\int_0^1 dB_2(r)\overline{B}_2(r)'\right\}\left\{\int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr\right\}^{-1} \tag{2.4b}$$

$$A_3 = \Lambda_{12}\left\{\int_0^1 \overline{B}_2(r)\overline{B}_2(r)'dr\right\}^{-1} , \tag{2.4c}$$

for $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$; hence A_1 is a median-unbiased mixture of normal distributions, A_2 is proportional to the "unit root distribution" (note $dB_2 = d\overline{B}_2$) and A_3 is a second-order bias component. The Ω_{ab}, $a, b = 1, 2$ and Λ_{12} can be consistently estimated from

$$\widehat{u}_{1t} = y_t - \widehat{\Pi}x_t \ , \ \widehat{u}_{2t} = u_{2t} = x_t - x_{t-1} \ , \tag{2.5}$$

and using techniques from the rich statistical literature on nonparametric spectral density estimation (for a review see Robinson and Velasco (1997)). One type stressed in recent econometric literature is

$$\widehat{\Omega}_{ab} = \sum_{j=-\ell}^{\ell} k(\frac{j}{\ell})\widehat{\Gamma}_{ab}(j) \ , \ \widehat{\Lambda}_{ab} = \sum_{j=0}^{\ell} k(\frac{j}{\ell})\widehat{\Gamma}_{ab}(j) \ , \tag{2.6}$$

where for $a, b = 1, 2$,

$$\widehat{\Gamma}_{ab}(j) = \frac{1}{n}\sum_{t=1}^{n-j} \widehat{u}_{a,t+j}\widehat{u}'_{b,t} \ \ j \geq 0; \ \ = \widehat{\Gamma}_{ba}(-j)', \ j < 0 \ , \tag{2.7}$$

and we impose:

Assumption B The kernel function $k(.)$ satisfies

$$k(.) : R \to [-1, 1], \ k(0) = 1, \ k(x) = k(-x), \ \int_{-\infty}^{\infty} k^2(x)dx < \infty \ ,$$

and $k(.)$ is continuous at 0 and at all but at most finitely many other points, and the bandwidth sequence ℓ satisfies $\ell \to \infty$, $\ell = O(n^{1/2})$ as $n \to \infty$.

Phillips and Hansen (1990) proposed a two-step estimate (FM-OLS) that eliminates A_2 and A_3 from (2.3),

$$\widetilde{\Pi}_{FM} \stackrel{def}{=} \left\{\sum_{t=1}^{n} \widehat{y}_t^{+}(x_t - \overline{x})' - n\widehat{\delta}\right\} \left\{\sum_{t=1}^{n}(x_t - \overline{x})(x_t - \overline{x})'\right\}^{-1} , \tag{2.8}$$

where

$$\widehat{y}_t^{+} = y_t - \widehat{\Omega}_{12}\widehat{\Omega}_{22}^{-1}\widehat{u}_{2t} \ , \ \widehat{\delta} = \widehat{\Lambda}_{12} - \widehat{\Omega}_{12}\widehat{\Omega}_{22}^{-1}\widehat{\Lambda}_{22} \tag{2.9}$$

and established the convergence

$$n(\widetilde{\Pi}_{FM} - \Pi) \Rightarrow A_1 \text{ as } n \to \infty \ . \tag{2.10}$$

Hence $\widetilde{\Pi}_{FM}$ belongs to the $LAMN$ (Locally Asymptotically Mixed Normal) family of distributions introduced by Jeganathan (1980,1988), and as such it shares its nice asymptotic statistical properties: distributions are centred around zero, nuisance parameters involve only scale effects and can be easily eliminated for the purpose of inference, an optimal theory of inference applies (LeCam (1986)), and hypothesis testing can be conducted within the usual asymptotic chi-squared paradigm.

We now describe the FDLS procedure. For $\lambda_j = 2\pi j/n$, $j = 1, ..., n-1$, we introduce the discrete Fourier transforms

$$w_x(\lambda_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} x_t \exp\{i\lambda_j t\} \ , \ w_y(\lambda_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} y_t \exp\{i\lambda_j t\} \ ,$$

and the periodogram and cross-periodogram matrices

$$I_{xx}(\lambda_j) = w_x(\lambda_j) w_x(\lambda_j)^*, \ I_{yx}(\lambda_j) = w_y(\lambda_j) w_x(\lambda_j)^*,$$

the asterisk denoting transposition combined with complex conjugation. Also, we define the averaged periodogram matrices

$$\widehat{F}_{xx}(1,m) = \frac{2\pi}{n} \sum_{j=1}^{m} Re\{I_{xx}(\lambda_j)\} \ , \ \widehat{F}_{yx}(1,m) = \frac{2\pi}{n} \sum_{j=1}^{m} Re\{I_{yx}(\lambda_j)\} \ . \tag{2.11}$$

where $Re\{.\}$ denotes real part. The FDLS statistic is then defined for $1 < m < n-1$ as

$$\widehat{\Pi}_m = \widehat{F}_{yx}(1,m) \left\{ \widehat{F}_{xx}(1,m) \right\}^{-1} , \tag{2.12}$$

(assuming the inverse exists), which for $m = n - 1$ yields OLS (with intercept correction) $\widehat{\Pi}$, in view of Parseval's equality. We are interested, however, in the behaviour of FDLS under the bandwidth condition

$$\frac{1}{m} + \frac{m}{n} \to 0 \text{ , as } n \to \infty \text{ ,} \tag{2.13}$$

which rules out OLS and moreover includes only a band of frequencies around zero that degenerates slowly as n increases, as is required in spectral density estimation for stationary series. Phillips (1991b) proposed a system-type estimates of Π based in effect on periodogram averages across a band of m frequencies about zero satisfying (2.13), though his proofs actually pertain to weighted autocovariance estimates. We take $m < n/2$, due to periodicity of period π, and evenness of the periodogram.

FDLS dominate OLS in several circumstances under (2.13), for example it is still consistent in cases where OLS is not, i.e., when x_t and u_{1t} in (1.1) are stationary long memory processes, labelled the "stationary cointegration" case, and it exhibits a faster rate of convergence when x_t, u_{1t} have "less than unit-root-nonstationarity", i.e. when the collective memory in x_t and u_{1t} in (1.1) is more limited than in the $CI(1,0)$ case, see Robinson and Marinucci (1998). In the more standard, unit root framework considered in this paper, Robinson and Marinucci (1998) established under Assumption A and (2.13) the convergence (see 2.3,2.4a,2.4b)

$$n(\widehat{\Pi}_m - \Pi) \Rightarrow A_1 + A_2$$

because the second-order bias component

$$\frac{m}{2n} h(0) \left\{ \widehat{F}_{xx}(1,m) \right\}^{-1} \to_p 0,$$

under (2.13), where

$$h(0) = \frac{1}{2\pi} \sum_{j=0}^{\infty} (2j+1) \left\{ \Gamma_{12}(j) - \Gamma_{12}(-j-1) \right\} \tag{2.14}$$

is finite because Assumption A implies that

$$\sum_{j=0}^{\infty} j \|\Gamma_{12}(j)\| < \infty \ . \tag{2.15}$$

Thus FDLS has a the second-order bias of smaller order than that of OLS.

As in Robinson and Marinucci (1998), we consider a modification of FM-OLS (2.8), based on first step FDLS, rather than OLS, residuals, and denoted

$$\widetilde{\Pi}^*_{FM} = \left\{ \sum_{t=1}^{n} \widetilde{y}^+_t (x_t - \overline{x})' - n\widetilde{\delta} \right\} \left\{ \sum_{t=1}^{n} (x_t - \overline{x})(x_t - \overline{x})' \right\}^{-1} , \tag{2.16}$$

$$\widetilde{y}^+_t = y_t - \widetilde{\Omega}_{12}\widetilde{\Omega}_{22}^{-1} u_{2t} \ , \ \widetilde{\delta} = \widetilde{\Lambda}_{12} - \widetilde{\Omega}_{12}\widetilde{\Omega}_{22}^{-1}\widetilde{\Lambda}_{22} \ ,$$

$$\widetilde{u}_{1t} = y_t - \widehat{\Pi}_m x_t \ , \ \widetilde{u}_{2t} = u_{2t} = x_t - x_{t-1} \ , \tag{2.17}$$

for $\widetilde{\Omega}_{ab}$ and $\widetilde{\Lambda}_{ab}$, $a, b = 1, 2$, defined as in (2.6), (2.7) with $\widehat{\Gamma}_{ab}(j)$ replaced by $\widetilde{\Gamma}_{ab}(j)$ which employs the FDLS residuals (2.17) in place of $\widehat{u}_{at}$ and $\widehat{u}_{bt}$.

A further alternative is to use the FDLS idea more directly, namely a narrow-band version of FM-OLS which reproduces its capacity to achieve asymptotic mixed normality, Fully Modified Frequency Domain Least Squares (FM-FDLS):

$$\widetilde{\Pi}_{FD} = \widehat{F}_{\widetilde{y}^+x}(1, m) \left\{ \widehat{F}_{xx}(1, m) \right\}^{-1} , \tag{2.18}$$

where $\widehat{F}_{\widetilde{y}^+x}$ is defined analogously to $\widehat{F}_{yx}$ in (2.11), with y_t replaced by $\widetilde{y}^+_t$.

The estimates $\widetilde{\Pi}^*_{FM}$ and $\widetilde{\Pi}_{FD}$ share the same asymptotic distribution as $\widetilde{\Pi}_{FM}$, and are thus asymptotically optimal in the same sense. The proof of the following Theorem, which is given in the Appendix, relies on some general results on the asymptotic behaviour of the averaged (cross-) periodogram for nonstationary processes, see Robinson and Marinucci (1998, Proposition 4.2 and Lemma 5.4).

Theorem 1 Under Assumption A, B and (2.13), as $n \to \infty$

$$n(\widetilde{\Pi}^*_{FM} - \Pi) \ , \ n(\widetilde{\Pi}_{FD} - \Pi) \Rightarrow A_1 \ ,$$

where A_1 is given in (2.4a).

In view of Theorem 1 and (2.10), $\widetilde{\Pi}_{FM}$, $\widetilde{\Pi}^*_{FM}$, and $\widetilde{\Pi}_{FD}$ are asymptotically equivalent. However, because the latter two are all, in various related ways, less affected by second order bias, it seems natural to conjecture that these estimates may improve on $\widetilde{\Pi}_{FM}$ in finite samples. We now provide some Monte Carlo evidence to support this claim.

3. MONTE CARLO EVIDENCE

We start from the model of Gonzalo (1994) in Monte Carlo comparison of estimates of Johansen (1988,1995) with simple (time-domain) estimates such as OLS:

$$y_t = \Pi x_t + u_{1t} \quad , \; x_t = \gamma y_t + w_t \quad , \quad w_t = w_{t-1} + e_{2t} \quad , \quad x_0 = 0 \; ,$$

which is easily seen to be equivalent to (1.1) with $p_1 = p_2 = 1$ and

$$u_{2t} = \frac{1}{1-\gamma\Pi}(\gamma(u_{1t} - u_{1,t-1}) + e_{2t}) \quad .$$

We adopt two alternative specifications for u_{1t}, namely

$$\begin{aligned} Model\ A \quad &: \quad u_{1t} = \rho u_{1,t-1} + e_{1t} \quad , \\ Model\ B \quad &: \quad u_{1t} = \rho_1 u_{1,t-1} + \rho_2 u_{1,t-2} + e_{1t} \quad , \end{aligned}$$

where $(e_{1t}, e_{2t})' \equiv i.i.d.\ N(0, \Sigma)$. Note that the consequent univariate specification of u_{2t} is ARMA(1,1) under Model A and ARMA(2,2) under Model B, when $\gamma \neq 0$, and white noise when $\gamma = 0$. We set

$$\Pi = 2 \; , \;\; \Sigma = \begin{bmatrix} 1 & .5 \\ .5 & 1 \end{bmatrix} \; , \quad \rho_2 = -.9 \; ,$$

and allow ρ, ρ_1 and γ to vary, taking $\rho = .8, .4, .0, -.4, -.8$, $\rho_1 = .947, 34, -.34, -.947$, and $\gamma = 1, 0$. The spectral density of u_{1t} has a peak at zero frequency for $\rho > 0$ and at π for $\rho < 0$ in Model A, and at frequency $\arccos(-\rho_1(1+\rho_2)/4\rho_2)$ in Model B, that is at $\pi/3, 4\pi/9, 5\pi/9$ and $2\pi/3$, respectively, for the four ρ_1. One expects the finite-sample performance of frequency domain procedures to be largely influenced by the locations and magnitudes of such peaks and by the exogeneity parameter γ; x_t is weakly exogenous if and only if $\gamma = 0$.

The implementation of efficient estimation procedures on Models A and B requires the ECM representations

$$\begin{aligned} Model\ A &: \quad \Delta z_t = \Psi_A z_{t-1} + \varepsilon_{At} \quad , &(3.1)\\ Model\ B &: \quad \Delta z_t = \Psi_B z_{t-1} + \Gamma \Delta z_{t-1} + \varepsilon_{Bt} \quad , &(3.2)\end{aligned}$$

where Δ is the difference operator, $\varepsilon_{At}, \varepsilon_{Bt}$ are *i.i.d.* vectors and

$$\Psi_A = (\rho - 1)\begin{bmatrix} 1 \\ \gamma \end{bmatrix}\begin{bmatrix} 1 \\ -\Pi \end{bmatrix}', \ \Psi_B = (\rho_1 + \rho_2 - 1)\begin{bmatrix} 1 \\ \gamma \end{bmatrix}\begin{bmatrix} 1 \\ -\Pi \end{bmatrix}', \Gamma = \rho_2 \begin{bmatrix} 1 \\ \gamma \end{bmatrix}\begin{bmatrix} 1 \\ -\Pi \end{bmatrix}'. \qquad (3.3)$$

These representation is the basis for implementing the estimate of Johansen (1988). His original procedure estimated a basis for the cointegrating space, rather than Π, but normalized estimates of Π, and their limiting distribution, can be readily obtained, see e.g. Johansen (1995, pp.179-184). For all series we estimated, by his procedure, both the equations

$$\begin{aligned} \Delta z_t &= \Upsilon \begin{bmatrix} 1 & \text{-}\Pi \end{bmatrix} z_{t-1} + \varepsilon_{At}, &(3.4)\\ \Delta z_t &= \Upsilon \begin{bmatrix} 1 & \text{-}\Pi \end{bmatrix} z_{t-1} + \Xi \Delta z_{t-1} + \varepsilon_{Bt}, &(3.5)\end{aligned}$$

where Υ is an unconstrained 2×1 vector and Ξ is an unconstrained 2×2 matrix. By comparison with (3.1)-(3.3), it is apparent that (3.4) is just-identified with respect to Model A when $\gamma \neq 0$ but over-parameterized when $\gamma = 0$, but is mis-specified with respect to Model B, whereas (3.5) is over-parameterized with respect to both Models A and B, the more so in case of Model A. We shall refer to Johansen's estimate as MLE but of course this description would only be accurate under just-identification.

The results, based on 5000 replications of series of lengths $n = 64$ and 128, are summarized in Tables I through VIII; here $\widetilde{\Pi}_{FM}$, $\widetilde{\Pi}^*_{FM}$, and $\widetilde{\Pi}_{FD}$ are defined in (2.8), (2.16), and (2.18), respectively, and $\widetilde{\Pi}_{M0}$ and $\widetilde{\Pi}_{M1}$ are the MLEs based on (3.4), (3.5), respectively. We set the bandwidth parameter ℓ equal to the closest integer to $\sqrt{n}$, hence obtaining $\ell = 8$ for $n = 64$ and $\ell = 11$ for $n = 128$. We fixed $m = 5$ for $n = 64$ and $m = 6$ for 128. (In Robinson and Marinucci (1998) the effect of varying m was considered, in a different type of simulation experiment involving FDLS). Hence we have a total of $(2 \times 5 \times 2) + (2 \times 4 \times 2) = 36$ groups of simulations for each estimate.

Tables I and II (Model A) and V and VI (Model B) illustrate our findings for Monte Carlo bias (B) and standard deviation (SD). A general feature of the results (which largely holds also in other tables) is a substantial overall improvement in performance of all estimates as n increases, and considerably better results for $\gamma = 0$ than for $\gamma = 1$, with little difference between the various FM estimates when $\gamma = 0$, but noticeable superiority in our narrow-band proposals over FM-OLS when $\gamma = 1$. Another factor is the location of the spectral peak of u_{1t} (and thence of u_{2t} also). The "traditional" estimates $\widetilde{\Pi}_{FM}$, $\widetilde{\Pi}_{M0}$ and $\widetilde{\Pi}_{M1}$ are based upon the whole frequency band $[0, \pi]$, whereas $\widetilde{\Pi}_{FD}$ and, to a lesser extent, $\widetilde{\Pi}^*_{FM}$, focus on a degenerating band around the origin. One therefore expects our new proposals to perform better the further the spectral peak of u_{1t} is shifted away from the origin.

This is found indeed to be the case for both Model A and B. Using mean squared error $MSE = B^2 + SD^2$ (not explicitly reported) as the basis for comparison, we find for Model A that $\widetilde{\Pi}_{M0}$ is best in 13 cases out of 20, $\widetilde{\Pi}_{FD}$ is best 4 times, and $\widetilde{\Pi}^*_{FM}$ thrice; remarkably, the "traditional" $\widetilde{\Pi}_{FM}$ estimate is dominated by at least one of the new procedures in all cases, often significantly, and this same finding also emerges when we examine B and SD separately. It is also noteworthy that $\widetilde{\Pi}_{FD}$ improves over $\widetilde{\Pi}_{M1}$ in 11 cases, so that the loss here due to over-parameterization is greater than that due to our semiparametric aspect. A close inspection of (3.3) and (3.4) reveals that in the weakly exogenous case $\gamma = 0$ the long-run behaviour derives from the equation for y alone; irrespective of the super-consistency result, we could hence anticipate that limited information procedures such as the various FM estimators would be relatively efficient for Model A. We refer the reader to Johansen (1995) for an explanation of weak exogeneity in the VAR context. For Model B, the results are rather similar, but the MLE does better relatively speaking, bearing in mind the mis-specification in (3.6) and the over-parameterization in (3.7); $\widetilde{\Pi}_{M1}$ is best in all 16 cases, and $\widetilde{\Pi}_{FD}$ is superior to $\widetilde{\Pi}_{M0}$ in all but two. Note that although Johansen's procedure performs best in the vast majority of cases, it can produce very high B and SD in Model A for $\rho = 0.8$. This phenomenon was previously noted by Gonzalo (1994, p. 217-219), and is due to the normalization which we adopt in order to estimate Π; in particular, $\widetilde{\Pi}_{M0}$ and $\widetilde{\Pi}_{M1}$ are ratios of random variables and need not have finite moments. Although in this sense our Monte Carlo study is somewhat unfair to MLE, for applied research the estimates of Π in the triangular representation (1.1)-(1.3) are likely to be the most useful for testing economic hypotheses, at least in a bivariate context, see for instance the examples discussed by Hamilton (1994, p.651). When more than two variables are included, however, identification might hang on long-run causal relationships or restrictions on the cointegrating vector, which are not coherent to the triangular form we use here but to some observationally equivalent form.

In Tables III and IV (Model A) and VII and VIII (Model B) we report empirical sizes based on χ^2 tests with nominal size 5%,

$$\chi^2_{\widetilde{\Pi}-\Pi} = \frac{(\widetilde{\Pi} - \Pi)^2}{\widehat{s}^2_{\widetilde{\Pi}} / \widehat{F}_{xx}(1, n-1)} \ , \ \widehat{s}^2_{\widetilde{\Pi}} = \frac{1}{n} \sum_{t=1}^{n} (y_t - \widetilde{\Pi} x_t)^2 \ , \ \widetilde{\Pi} = \widetilde{\Pi}_{FM}, \ \widetilde{\Pi}^*_{FM}, \ \widetilde{\Pi}_{FD} \ ;$$

for $\widetilde{\Pi}_{M0}$ and $\widetilde{\Pi}_{M1}$, we took the denominator in the χ^2-statistics to be the (2,2)-

th element in the inverse of expression (13.12) on p.184 of Johansen (1995), which estimates their asymptotic variance. In terms of proximity to the nominal value, we find that $\widetilde{\Pi}_{FD}$ is best in 13 cases out of 20 in Model A, $\widetilde{\Pi}^*_{FM}$ is best 4 times, and only in 3 cases is $\widetilde{\Pi}_{M0}$ actually superior to the narrow-band procedures. Likewise, for Model B, $\widetilde{\Pi}_{FD}$ is best in 7 cases out of 16, $\widetilde{\Pi}_{M1}$ is best in 5 cases, $\widetilde{\Pi}_{M0}$ in 3 and $\widetilde{\Pi}_{FM}$ in the remaining one.

In general, our "narrow-band" procedures, and in particular $\widetilde{\Pi}_{FD}$, emerge as promising competitors to traditional semiparametric estimates, at the same time providing a robust and efficient alternative to parametric methods even when the correct model is known, up to a few number of lags in the specification of the ECM, though it is unsurprising that the Johansen procedures tend to perform best overall in our experiment. One is likely to improve the finite-sample performance of each of the four semiparametric procedures by iteration (for example, obtaining new cointegrating residuals by FM-FDLS and then starting each procedure anew from the estimation of Ω and δ).

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APPENDIX

Proof of Theorem 1 In view of (2.10) it suffices to prove that

$$\|\widetilde{\Pi}^{*}_{FM} - \widetilde{\Pi}_{FM}\| = o_p(n^{-1}) , \tag{A.1}$$

$$n(\widetilde{\Pi}_{FD} - \Pi) \Rightarrow A_1 . \tag{A.2}$$

We note that, as $n \to \infty$

$$\widetilde{\Omega}_{22} \to_p \Omega_{22} , \ \widetilde{\Omega}_{12} \to_p \Omega_{12} , \ \widetilde{\delta} \to_p \delta , \tag{A.3}$$

by n- consistency of $\widehat{\Pi}_m$ and standard manipulations. Thus the left-hand side of (A.1) is bounded in norm by

$$\|(\widehat{\Omega}_{12}\widehat{\Omega}_{22}^{-1} - \widetilde{\Omega}_{12}\widetilde{\Omega}_{22}^{-1}) \sum_{t=1}^{n} u_{2t}(x_t - \overline{x})' - n(\widetilde{\delta} - \widehat{\delta})\| \| \left\{ \sum_{t=1}^{n} (x_t - \overline{x})(x_t - \overline{x})' \right\}^{-1} \|,$$

which is $o_p(n)O_p(n^{-2}) = o_p(n^{-1})$, as desired, in view of (2.1), (2.2) and (A.4). For (A.2), consider

$$y_t^{+} = y_t - \Omega_{12}\Omega_{22}^{-1}u_{2t} = \Pi x_t + u_{1t} - \Omega_{12}\Omega_{22}^{-1}u_{2t} ,$$

and introduce

$$\widehat{\Pi}^{+} = \left\{ \sum_{t=1}^{n} y_t^{+} (x_t - \overline{x})' \right\} \left\{ \sum_{t=1}^{n} (x_t - \overline{x})(x_t - \overline{x})' \right\}^{-1} ;$$

from Phillips and Hansen (1990)

$$n(\widehat{\Pi}^{+} - \Pi) \Rightarrow \left\{ \int_0^1 dB_{1.2}(r) \overline{B}_2(r)' + \Lambda_{12} - \Omega_{12}\Omega_{22}^{-1}\Lambda_{22} \right\} \left\{ \int_0^1 \overline{B}_2(r)\overline{B}_2(r)' dr \right\}^{-1} .$$

In the same way we can define $\widetilde{\Pi}_{FD}^{+} = \widehat{F}_{y^+x}(1,m) \left\{ \widehat{F}_{xx}(1,m) \right\}^{-1}$. Writing $u_{1.2,t} = u_{1t} - \Omega_{12}\Omega_{22}^{-1} u_{2t}$, $\widehat{a}_{1.2} = \frac{1}{n} \sum_{t=1}^{n} u_{1.2,t}(x_t - \overline{x})'$, $\widetilde{a}_{1.2} = \widehat{F}_{u_{1.2},x}(1,m)$, and applying the argument of Robinson and Marinucci (1998, Theorem 5.3), we find that

$$\begin{aligned} n(\widetilde{\Pi}_{FD}^{+} - \Pi) &= \left[\widehat{a}_{1.2} - E\widehat{a}_{1.2} + \{ (\widetilde{a}_{1.2} - \widehat{a}_{1.2}) - E(\widetilde{a}_{1.2} - \widehat{a}_{1.2}) \} + E\widetilde{a}_{1.2} \right] \widehat{F}_{xx}(1,m) \\ &\Rightarrow A_1 \text{ , as } n \to \infty \text{ ,} \end{aligned}$$

because $\{ (\widetilde{a}_{1.2} - \widehat{a}_{1.2}) - E(\widetilde{a}_{1.2} - \widehat{a}_{1.2}) \}$ and $E\widetilde{a}_{1.2}$ are $o_p(1)$ by Proposition 4.2 and Lemma 5.4, respectively, of that paper.

Now write $\widetilde{y}_t^{+} = y_t^{+} + (\Omega_{12}\Omega_{22}^{-1} - \widetilde{\Omega}_{12}\widetilde{\Omega}_{22}^{-1}) u_{2t}$, to obtain

$$\| \widetilde{\Pi}_{FD} - \widetilde{\Pi}_{FD}^{+} \| \leq \| \Omega_{12}\Omega_{22}^{-1} - \widetilde{\Omega}_{12}\widetilde{\Omega}_{22}^{-1} \| \| \left\{ \widehat{F}_{u_2 x}(1,m) \right\} \left\{ \widehat{F}_{xx}(1,m) \right\}^{-1} \|,$$

which is $o_p(1) O_p(n^{-1})$, in view of (A.4) and Theorem 5.3 of Robinson and Marinucci (1998); (A.2) follows immediately.

□

TABLE I

Model A: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 64,\ \ell = 8,\ \gamma = 1$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	-.22 (.25)	-.14 (.15)	-.13 (.13)	-.15 (.14)	-.26 (.20)
$\widetilde{\Pi}_{M0}$	-.03 (6.3)	.02 (.15)	.01 (.08)	.00 (.05)	.00 (.04)
$\widetilde{\Pi}_{M1}$	-.14 (11.3)	.01 (.50)	.01 (.42)	.00 (.05)	.00 (.04)
$\widetilde{\Pi}^{*}_{FM}$	-.21 (.24)	-.06 (.12)	-.03 (.08)	-.01 (.06)	-.03 (.07)
$\widetilde{\Pi}_{FD}$	-.26 (.22)	-.08 (.11)	-.04 (.07)	-.02 (.05)	-.01 (.04)

Model A: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 64,\ \ell = 8,\ \gamma = 0$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	.00 (.30)	.00 (.12)	.00 (.08)	.00 (.06)	.00 (.06)
$\widetilde{\Pi}_{M0}$	.24 (13.0)	.00 (.13)	.00 (.07)	.00 (.05)	.00 (.04)
$\widetilde{\Pi}_{M1}$	.69 (40.5)	.00 (.17)	.00 (.08)	.00 (.05)	.00 (.04)
$\widetilde{\Pi}^{*}_{FM}$	.01 (.30)	.00 (.12)	.00 (.08)	.00 (.05)	.00 (.04)
$\widetilde{\Pi}_{FD}$	.01 (.28)	.00 (.12)	.00 (.07)	.00 (.05)	.00 (.04)

TABLE II

Model A: Mean (standard deviation) of $\widetilde{\Pi} - \Pi$, $n = 128$, $\ell = 11$, $\gamma = 1$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	-.12 (.16)	-.07 (.09)	-.06 (.08)	-.07 (.08)	-.15 (.15)
$\widetilde{\Pi}_{M0}$	.05 (1.96)	.00 (.06)	.00 (.03)	.00 (.02)	.00 (.02)
$\widetilde{\Pi}_{M1}$	.16 (7.3)	.00 (.06)	.00 (.03)	.00 (.02)	.00 (.02)
$\widetilde{\Pi}^{*}_{FM}$	-.11 (.15)	-.02 (.06)	-.01 (.04)	-.00 (.03)	.01 (.03)
$\widetilde{\Pi}_{FD}$	-.14 (.14)	-.03 (.06)	-.01 (.03)	-.01 (.03)	.00 (.02)

Model A: Mean (standard deviation) of $\widetilde{\Pi} - \Pi$, $n = 128$, $\ell = 11$, $\gamma = 0$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	.00 (.17)	.00 (.06)	.00 (.04)	.00 (.03)	.00 (.02)
$\widetilde{\Pi}_{M0}$	.02 (1.12)	.00 (.06)	.00 (.03)	.00 (.02)	.00 (.02)
$\widetilde{\Pi}_{M1}$	.00 (.62)	.00 (.06)	.00 (.03)	.00 (.02)	.00 (.02)
$\widetilde{\Pi}^{*}_{FM}$	.00 (.16)	.00 (.06)	.00 (.04)	.00 (.03)	.00 (.02)
$\widetilde{\Pi}_{FD}$	.00 (.16)	.00 (.06)	.00 (.03)	.00 (.02)	.00 (.02)

TABLE III

Model A: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 64$, $\ell = 8$, $\gamma = 1$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	32.60%	24.34%	25.46%	27.46%	36.82%
$\widetilde{\Pi}_{M0}$	37.86%	17.60%	13.36%	11.44%	15.60%
$\widetilde{\Pi}_{M1}$	41.32%	19.72%	15.00%	12.46%	11.84%
$\widetilde{\Pi}^*_{FM}$	31.48%	15.42%	12.86%	11.48%	30.06%
$\widetilde{\Pi}_{FD}$	36.90%	18.78%	13.02%	9.84%	10.58%

Model A: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 64$, $\ell = 8$, $\gamma = 0$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	16.56%	11.56%	9.72%	8.94%	7.78%
$\widetilde{\Pi}_{M0}$	32.82%	13.16%	8.82%	7.66%	7.18%
$\widetilde{\Pi}_{M1}$	37.10%	16.54%	11.78%	10.72%	10.00%
$\widetilde{\Pi}^*_{FM}$	14.64%	8.58%	6.96%	5.74%	2.76%
$\widetilde{\Pi}_{FD}$	12.90%	7.78%	6.40%	4.98%	2.22%

TABLE IV

Model A: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 128$, $\ell = 11$, $\gamma = 1$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	25.72%	19.96%	17.68%	20.34%	31.88%
$\widetilde{\Pi}_{M0}$	23.24%	10.74%	8.12%	8.16%	10.40%
$\widetilde{\Pi}_{M1}$	24.92%	11.82%	9.28%	8.36%	8.18%
$\widetilde{\Pi}^*_{FM}$	24.30%	10.26%	8.06%	7.60%	22.20%
$\widetilde{\Pi}_{FD}$	30.50%	11.78%	7.98%	6.52%	7.34%

Model A: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 128$, $\ell = 11$, $\gamma = 0$

	$\rho = .8$	$\rho = .4$	$\rho = .0$	$\rho = -.4$	$\rho = -.8$
$\widetilde{\Pi}_{FM}$	13.56%	9.98%	8.92%	7.84%	6.20%
$\widetilde{\Pi}_{M0}$	18.50%	9.02%	6.92%	5.96%	5.90%
$\widetilde{\Pi}_{M1}$	20.02%	10.72%	8.18%	7.38%	7.26%
$\widetilde{\Pi}^*_{FM}$	10.94%	7.62%	5.98%	5.26%	2.78%
$\widetilde{\Pi}_{FD}$	10.28%	6.92%	5.24%	4.90%	2.42%

TABLE V

Model B: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 64,\ \ell = 8,\ \gamma = 1$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	-.33 (.24)	-.46 (.30)	-.30 (.22)	-.50 (.37)
$\widetilde{\Pi}_{M0}$	.02 (.47)	.00 (.07)	.00 (.04)	.00 (.03)
$\widetilde{\Pi}_{M1}$	.01 (.08)	.00 (.05)	.00 (.03)	.00 (.02)
$\widetilde{\Pi}^*_{FM}$	-.07 (.09)	.03 (.08)	-.08 (.08)	.10 (.09)
$\widetilde{\Pi}_{FD}$	-.05 (.08)	-.02 (.05)	-.01 (.04)	.00 (.03)

Model B: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 64,\ \ell = 8,\ \gamma = 0$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	.00 (.11)	.00 (.11)	.00 (.06)	.00 (.08)
$\widetilde{\Pi}_{M0}$	.02 (1.3)	.00 (.06)	.00 (.04)	.00 (.03)
$\widetilde{\Pi}_{M1}$	.00 (.07)	.00 (.04)	.00 (.03)	.00 (.02)
$\widetilde{\Pi}^*_{FM}$	.00 (.09)	.00 (.05)	.00 (.04)	.00 (.03)
$\widetilde{\Pi}_{FD}$	.00 (.08)	.00 (.05)	.00 (.03)	.00 (.03)

TABLE VI

Model B: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 128,\ \ell = 11,\ \gamma = 1$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	-.36 (.23)	-.17 (.17)	-.21 (.18)	-.35 (.29)
$\widetilde{\Pi}_{M0}$	.00 (.05)	.00 (.03)	.00 (.02)	.00 (.01)
$\widetilde{\Pi}_{M1}$	.00 (.04)	.00 (.02)	.00 (.01)	.00 (.01)
$\widetilde{\Pi}^*_{FM}$	.02 (.05)	.03 (.04)	-.08 (.07)	.07 (.06)
$\widetilde{\Pi}_{FD}$	-.01 (.04)	-.00 (.02)	.00 (.02)	.00 (.01)

Model B: Mean (standard deviation) of $\widetilde{\Pi} - \Pi,\ n = 128,\ \ell = 11,\ \gamma = 0$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	.00 (.06)	.00 (.03)	.00 (.03)	.00 (.03)
$\widetilde{\Pi}_{M0}$	.00 (.05)	.00 (.03)	.00 (.02)	.00 (.01)
$\widetilde{\Pi}_{M1}$	.00 (.04)	.00 (.02)	.00 (.01)	.00 (.01)
$\widetilde{\Pi}^*_{FM}$	.00 (.04)	.00 (.02)	.00 (.02)	.00 (.01)
$\widetilde{\Pi}_{FD}$	.00 (.04)	.00 (.02)	.00 (.02)	.00 (.01)

TABLE VII

Model B: Type I error at 5% of $\chi^2_{\tilde{\Pi}-\Pi}$, $n = 64$, $\ell = 8$, $\gamma = 1$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	39.4%	61.92%	35.30%	63.64%
$\widetilde{\Pi}_{M0}$	2.82%	0.62%	.12%	.30%
$\widetilde{\Pi}_{M1}$	21.04%	16.96%	14.96%	23.42%
$\widetilde{\Pi}^*_{FM}$	22.48%	7.71%	52.74%	74.98%
$\widetilde{\Pi}_{FD}$	13.92%	5.02%	5.18%	10.52%

Model B: Type I error at 5% of $\chi^2_{\tilde{\Pi}-\Pi}$, $n = 64$, $\ell = 8$, $\gamma = 0$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	3.08%	13.10%	2.22%	13.78%
$\widetilde{\Pi}_{M0}$	.46%	.00%	.02%	.00%
$\widetilde{\Pi}_{M1}$	7.48%	7.48%	7.56%	7.10%
$\widetilde{\Pi}^*_{FM}$	1.02%	5.24%	.34%	1.18%
$\widetilde{\Pi}_{FD}$	6.80%	4.95%	.08%	.44%

TABLE VIII

Model B: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 128$, $\ell = 11$, $\gamma = 1$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	61.34%	27.56%	31.50%	59.70%
$\widetilde{\Pi}_{M0}$	.08%	.02%	.00%	.00%
$\widetilde{\Pi}_{M1}$	14.88%	10.12%	9.88%	8.72%
$\widetilde{\Pi}^*_{FM}$	22.78%	33.32%	76.82%	83.24%
$\widetilde{\Pi}_{FD}$	10.66%	3.22%	3.22%	5.84%

Model B: Type I error at 5% of $\chi^2_{\widetilde{\Pi}-\Pi}$, $n = 128$, $\ell = 11$, $\gamma = 0$

	$\rho_1 = .947$	$\rho_1 = .34$	$\rho_1 = -.34$	$\rho_1 = -.947$
$\widetilde{\Pi}_{FM}$	9.08%	2.82%	2.48%	7.96%
$\widetilde{\Pi}_{M0}$	.00%	.00%	.00%	.00%
$\widetilde{\Pi}_{M1}$	7.00%	6.30%	6.18%	1.18%
$\widetilde{\Pi}^*_{FM}$	2.60%	.66%	.36%	.76%
$\widetilde{\Pi}_{FD}$	1.90%	.36%	.20%	.26%