First Order Feynman-Kac Formula

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Abstract

We study the parabolic integral kernel associated with the weighted Laplacian and the Feynman-Kac kernels. For manifold with a pole we deduce formulas and estimates for them and for their derivatives, given in terms of a Gaussian term and the semi-classical bridge. Assumptions are on the Riemannian data.

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1 Introduction

Let M be a complete connected smooth Riemannian manifold of dimension n and Δ the Laplace-Beltrami operator with the sign convention that Δ is negative definite. Let h be a smooth real valued function on M, define $\Delta^h = \Delta + 2L_{\nabla h}$. We study the solutions of the parabolic equation $\frac{\partial}{\partial t} f = (\frac{1}{2}\Delta^h - V)f$ where $t > 0$, V a real valued bounded Hölder continuous function on M, and $\lim_{t\downarrow 0} f(t,x) = f(x)$. Without loss of generality we may assume that $V \geq 0$. Denote by $P_t^{h,V} f$ its solution, which is also denoted by P_t^V if $h = 0$, by P_t^h if $V = 0$, and by $P_t f$ if both h and V vanish. Their corresponding integral kernels are denoted by the lower case functions: $p_t^{h,V}$, p_t^V, p_t^h, and p_t. Also, if Z is an additional C^1 vector field the notations $p_t^{h,Z,V}$ etc. will be used. All stochastic processes are assumed to have infinite life time.

Set $\mathbb{V}_t = \mathbb{E}\left[e^{-\int_0^t V(x_s)ds}\right]$ where (x_t) is the canonical h-Brownian motion, by an h-Brownian motion we mean a strong Markov processes with generator $\frac{1}{2}\Delta^h$. We deduce the following first order Feynman-Kac formula

$$d(P_T^{h,V} f)(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T)\left(\int_0^t \langle W_s(v), u_s dB_s\rangle - \int_0^t\int_0^r dV(W_s(v))\,ds\,dr\right)\right],$$

where u_s and W_s are respectively the stochastic parallel and stochastic damped parallel translations along its sample paths, under the assumption that Ric $-2\,\mathrm{Hess}(h) \geq K$ where K is a constant. This formula can be obtained by filtering out redundant noise from the equation in [19, section 2]. However it is fairly easy to deduce it directly, avoiding some assumptions made in [19]. From this formula, it is clear that $dP_t^{h,V} f$ is uniformly close to

$dP_t^h f$. We will in fact show that, for explicit constants $C_1(t,K)$ and $C_2(t,K)$ depending on t and K,

$$|\nabla P_t^{h,V} f|_{x_0} \le \frac{1}{\sqrt{t}}\Big(2C_1\mathbf{E}\left[(f(x_t)\mathbb{V}_t \log(f(x_t)\mathbb{V}_t))^+\right]\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2.$$

Choosing f to be the Feynman-Kac kernel leads to the following estimates:

$$|\nabla \log p_t^{h,V}|_{x_0} \le \frac{\sqrt{2C_1}}{\sqrt{t}}\Big(\sup_{y\in M} \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2.$$

Together with estimates on $\sup_{y\in M}\log\frac{p_t(y,y_0)}{p_{2t}(x_0,y_0)}$, this leads to the estimate of S.-J. Sheu [41]. Sheu's estimates and extensions can be found in [28, 44, 20], all for the case $V = 0$ and $h = 0$. Feynman-Kac formula is a popular subject see [21, 42, 45, 26, 5, 37, 34, 14]. Differentiating Feynman-Kac semigroups has been studied in connection with Hamiltonian-Jacobian equation, see [15, 12, 11].

If M is a manifold with a pole y_0, by which we mean that the exponential map $\exp_{y_0}$ is a diffeomorphism, it makes sense to compare the Feynman-Kac kernel and its derivatives with the 'Gaussian kernel'. We do not make assumptions on uniform ellipticity of the operator; all assumptions will be on Riemmannian data. Let J_{y_0} denote the Jacobian determinant of the exponential map at y_0 and $\Phi(y) = \frac{1}{2}J_{y_0}^{\frac{1}{2}}(y)\Delta J_{y_0}^{-\frac{1}{2}}(y)$. The subscript y_0 will be omitted from time to time. For $T>0$ fixed, a semi-classical bridge $\tilde{x}_s$ is a time dependent diffusion with generator $\frac{1}{2}\triangle + \nabla \log k_{T-s}(\cdot, y_0)$ where, for d the Riemannian distance function,

$$k_t(x_0,y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0).$$

On $\mathbb{R}^n$, the semi-classical bridge agrees with the conditioned Brownian motion. Moreover the process $r_t = d(\tilde{x}_t, y_0)$ is the n-dimensional Bessel bridge. In [15], K. D. Elworthy and A. Truman proved the following formula, whose consideration comes from classical mechanics and semi-classical limits,

$$p_T^V(x_0,y_0) = k_T(x_0,y_0)\mathbf{E}\left[e^{\int_0^T (\Phi - V)(\tilde{x}_s)\,ds}\right]. \tag{1.1}$$

We give a simple proof for the following formula, see [49] , for all $x_0 \in M$,

$$p_T^{h,V}(x_0,y_0) = e^{h(y_0)-h(x_0)} k_T(x_0,y_0)\mathbf{E}\beta_T^h,$$
$$\beta_T^h = \exp\left(\int_0^t \left(\Phi - V - \frac{1}{2}|\nabla h|^2 - \frac{1}{2}\Delta h\right)(\tilde{x}_s)ds\right),$$

The methods in [49] are similar to that in [15, 17, 16]. Here we use a different method, which allows us to deduce a first order formula. The function Φ is bounded on manifolds of constant negative curvature. If $-\frac{1}{2}|\nabla h|^2 - \frac{1}{2}\Delta h$ is bounded, the formula leads easily to a Gaussian upper bound for the integral kernel p_t^h. In this formula, an additional non-gradient type drift Z is also allowed for which we follow a beautiful idea of Watling [49].

The semi-classical bridge will be replaced by the semi-classical Riemannian bridge whose Markov generator is $\frac{1}{2}\triangle + \nabla \log k_{T-s}(\cdot, y_0) + \nabla S$, where

$$S(x) = \int_0^1 \langle \dot\gamma(u), Z(\gamma(u))\rangle du,$$

for $\gamma : [0,1] \to M$ the unique geodesic from x to y_0, representing the path average of the radial part of Z.

Let $\tilde u_t$ denote the solution to the stochastic differential equation (2.2) on the orthonormal frame bundle with $\tilde u_0 \in \pi^{-1}(x_0)$ and set $\tilde x_t = \pi(\tilde u_t)$. We often need the condition that $\mathrm{Ric} - 2\,\mathrm{Hess}(h)$ is bounded from below. Following the notation in [30, 31], set $\rho^h = \inf_{|v|=1}\{Ric(v,v) - 2\,\mathrm{Hess}\, h(v,v)\}$.

Theorem *Assume y_0 is a pole, $V \in C^{1,\alpha} \cap BC^1$, Φ and $f \in L_\infty$ and $\rho^h \geq K$. Then*

$$dp_T^{h,V}(\cdot, y_0) = \frac{1}{T} e^{h(y_0)-h(x_0)} k_T(x_0,y_0)\mathbf{E}\left[\beta_T^h \int_0^T \langle \tilde W_r(\cdot), \tilde u_r d\tilde B_r - (t-r)\nabla V dr\rangle\right]$$

where $d\tilde B_r = dB_r + \tilde u_r^{-1}\nabla \log(e^{-h}k_{T-r})(\tilde x_r)\, dr$ and $\tilde W$ is the solution to (2.4).

From this theorem we immediately see that, for an explicit constant $C(K, h, |\nabla \log J|_\infty)$, depending only on K, h, and $|\nabla \log J|_\infty$, the following estimate holds,

$$\begin{aligned}&|\nabla p_T^{h,V}(\cdot, y_0)|_{x_0}\\ &\leq C e^{h(y_0)-h(x_0)}\,(2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0,y_0)}{2t}} J_{y_0}^{-\frac{1}{2}}(x_0)|\beta_T^h|_\infty \left(\frac{d(x_0,y_0)}{T} + 1 + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + T|dV|_\infty\right).\end{aligned}$$

Letting $Z_T = \frac{\beta_T^h}{\mathbf{E}(\beta_T^h)}$, we also have:

$$\left|\nabla \log p_T^{h,V}(\cdot, y_0)\right|_{x_0} \leq C(K, |\nabla \log J|_\infty)\, |Z_T|_{L_2(\Omega)} \left(\frac{d(x_0,y_0)}{T} + 1 + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + T|dV|_\infty\right).$$

There is also a version of the above formula and estimates for Hölder continuous potential V, which however involves a $\ln T|V|_\infty$ term. Note that precise Gaussian estimates for heat kernels and their derivatives using semi-classical bridge were obtained by S. Aida [1] for asymptotically flat Riemannian manifolds with a pole where the derivaties of $\log J$ up to order 4, the Riemmanina curvature and the derivative of the Ricci curvature are assumed to be bounded. Heat kernel formula for Schrödinger type operator acting on sections of vector bundles can be found in M. Ndumu [38] and M. Braverman [8]. The study of the probability measure induced by the semi-classical bridge has been followed up in [32].

2 Preliminaries and First Order Feynman-Kac Formula

In this section we introduce the notation and the preliminary results. Denote by BC^r the space of bounded C^r functions on M with bounded derivatives, C_K^r its subspace of functions with compact supports and $C^{1,\alpha}$ the space of functions whose first derivatives are locally Hölder continuous.

The h-Brownian motion we use will be given by the canonical construction below. Let $\{B_t^i\}$ be a family of independent one-dimensional Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, set $B_t = (B_t^1, \dots, B_t^n)$. Let $\{H_i\}$ be canonical horizontal vector fields on the orthonormal frame bundle OM of M, associated to an orthonormal basis of $\mathbb{R}^n$. The tilde sign over a vector field on M indicates its horizontal lift to OM. If the h-Brownian motion is complete, which holds if Ric $-2\,$Hess(h) is bounded from below, then the following stochastic differential equation (SDE) is complete,

$$du_t = \sum_{i=1}^n H_i(u_t) \circ dB_t^i + \widetilde{\nabla h}(u_t)dt. \tag{2.1}$$

Furthermore if π is the projection from OM to M, $x_t := \pi(u_t)$ is a h-Brownian motion on M starting at $x_0 := \pi(u_0)$. If h vanishes it is sufficient to assume that $\underline{\mathrm{Ric}}_x \geq -\alpha(r(x))$ where α grows at most quadratically and Ric_x is the Ricci curvature at $x \in M$ and $\underline{\mathrm{Ric}} = \inf_{v \in T_xM:|v|=1}\{\mathrm{Ric}_x(v,v)\}$. The corresponding equation,

$$d\tilde{u}_t = \sum_{i=1}^n H_i(\tilde{u}_t) \circ dB_t^i + \widetilde{\nabla h}(\tilde{u}_t)dt + \nabla \widetilde{\log k_{T-t}}(\tilde{u}_t)dt, \quad t < T, \tag{2.2}$$

gives rise to the semi-classical bridge in the same way, $\tilde{x}_t = \pi(\tilde{u}_t)$.

Let $\mathrm{Ric}_x^\sharp : T_xM \to T_xM$ be the linear map defined by $\langle \mathrm{Ric}_x^\sharp u, v\rangle = \mathrm{Ric}_x(u,v)$. Denote by (W_t) and $(\tilde{W}_t)$ respectively the solutions to the following two equations, the first, along (x_t), being

$$\frac{D}{dt}W_t = -\frac{1}{2}\mathrm{Ric}_{x_t}^\#(W_t) + \mathrm{Hess}(h)(W_t), \quad W_0 = \mathrm{id}_{T_{x_0}M} \tag{2.3}$$

and the second, along $(\tilde{x}_t)$, being

$$\frac{D}{dt}\tilde{W}_t = -\frac{1}{2}\mathrm{Ric}_{\tilde{x}_t}^\#(\tilde{W}_t) + \mathrm{Hess}(h)(\tilde{W}_t), \quad \tilde{W}_0 = \mathrm{id}_{T_{x_0}M}. \tag{2.4}$$

Here $\frac{D}{dt}W_t = u_t \frac{d}{dt} u_t^{-1} W_t$ and $\frac{D}{dt}\tilde{W}_t = \tilde{u}_t \frac{d}{dt}\tilde{u}_t^{-1}\tilde{W}_t$, and so the first equation, for example, is interpreted as follows: $u_t^{-1}W_t$ solves the equation

$$\frac{d}{dt}w_t = -\frac{1}{2}(u_t^{-1}\mathrm{Ric}_{x_t}^\sharp u_t)w_t + {u_t}^{-1}\,\mathrm{Hess}(h)(u_t w_t), \quad w_0 = \mathrm{id}_{\mathbb{R}^n}.$$

Throughout this section we assume that the h-Brownian motions do not explode.

If h is a smooth real valued function on M then $\Delta^h = (d + d^*)^2$ where d^* is the L^2 adjoint of the exterior differential operator d with respect to the measure $e^{2h}d\mathrm{vol}$ and the initial domain of Δ^h consists of smooth and compactly supported differential forms. Then $d+d^*$ and all its powers are essentially self-adjoint, [30, ch.2], and we denote by the same notation their closures. Suppose that V is bounded, then the operator $f \to Vf$ is Δ^h bounded and by the Kato-Rellich theorem, $\frac{1}{2}\Delta^h - V$ is self-adjoint on the domain of Δ^h and essentially self-adjoint on C_K^∞. By functional calculus $e^{-t(\frac{1}{2}\Delta^h - V)}$ is a strongly continuous contraction semi-group on $L_2 \cap \mathcal{B}_b$, where the L^2 space is defined by the weighted measure $e^{2h}dx$. Furthermore if $f \in L^2(M; e^{2h}dx)$ then $e^{-t(\frac{1}{2}\Delta^h - V)}f$ belongs to the domain of Δ^h. By direct computation $\mathbf{E}\left(f(x_t)\mathbb{V}_t\right)$, where $\mathbb{V}_t = e^{-\int_0^t V(x_r)dr}$, is also a strongly continuous contraction semi-group on $L_2 \cap L_\infty$ with generator $\frac{1}{2}\Delta^h - V$ and core C_K^∞. Consequently

$$e^{-t(\frac{1}{2}\Delta^h - V)}f = \mathbf{E}\left(f(x_t)\mathbb{V}_t\right), \qquad f \in L_2 \cap L_\infty.$$

Furthermore they solve the variation of constant formula:

$$g_t = P_t^h f + \int_0^t P_{t-s}^h(V g_s)\, ds,$$

and consequently they are $C^{1,2}$ functions and solve the parabolic equation. The solution measure has a density $p_t^{h,V}$ with respect to the volume measure. We need the following formulation for the Feynman-Kac formula.

Lemma 2.1 *If $P_t^{h,V} f$ is a $C^{1,2}$ solution to the parabolic equation, then*

$$\mathbb{V}_s P_{t-s}^{h,V} f(x_s) = P_t^{h,V} f(x_0) + \int_0^s \mathbb{V}_r d P_{t-r}^{h,V} f(u_r dB_r), \quad 0 \le s \le t. \tag{2.5}$$

Proof By the assumption on the h-Brownian motion, u_s exists for all time. We may therefore apply Itô's formula to $\mathbb{V}_s P_{t-s}^{h,V} f(x_s)$, using $d\mathbb{V}_s = -V(x_s)\mathbb{V}_s ds$ and $dx_s = u_s \circ dB_s + \nabla h(x_s) ds$ to obtain (2.5). $\square$

For $k \in 0, 1, \cdots$ denote by H^k the completion of H_0^k, where

$$H_0^k = \left\{ f \in C^\infty : |f|_{H_k}^2 = \sum_{j=0}^k |\nabla^{(j)} f|_{L^2}^2 < \infty \right\},$$

under the norm $|\cdot|_{H^k}$. Denote by $\overline{C_K^\infty}^{H_k}$ the closure of C_K^∞ under $|\cdot|_{H^k}$. Denote also by $d^\star$ the dual of d in $L^2(M; dx)$, taking $h=0$, the latter having initial domain C_K^∞. Then the Laplace-Beltrami operator on functions is the closure of $-d^\star d$ and for any complete Riemannian manifold Dom$(d) = \overline{C_K^\infty}^{H^1} = H^1$. For higher order derivatives the corresponding statements hold for manifolds with bounded geometry, see [4]. For $k=2$, $\overline{C_K^\infty}^{H^2} = H^2$ if the injectivity radius of M is positive and if the Ricci curvature is bounded below, see E. Hebey [27]. We avoid these assumptions.

Recall that $\rho^h(x) = \inf_{v \in ST_xM}\{\text{Ric}(v,v) - 2\,\text{Hess}(h)(v,v)\}$.

Lemma 2.2 *Fix $T > 0$. Assume that V is bounded with $V \in C^{1,\alpha}$. If $f \in L^2 \cap BC^1$ then for all $0 \le t < T$ and $v \in T_{x_0}M$ we have*

$$\begin{aligned}\mathbb{V}_t \cdot \left(dP^{h,V}_{T-t}f\right)(W_t(v)) = \; & dP^{h,V}_T f(v) + \int_0^t \mathbb{V}_s \cdot \left(\nabla dP^{h,V}_{T-s}f\right)(u_s dB_s, W_s(v)) \\ & + \int_0^t \mathbb{V}_s \cdot dV(W_s(v)) \cdot P^{h,V}_{T-s}f(x_s)\, ds.\end{aligned} \tag{2.6}$$

If furthermore $|dV|$ is bounded and ρ^h is bounded from below, then for all $t \in [0,T)$,

$$\mathbf{E}\left[\mathbb{V}_t(dP^{h,V}_{T-t}f)(W_t(v))\right] = d(P^{h,V}_T f)(v) + \mathbf{E}\left[\int_0^t \mathbb{V}_s dV(W_s(v)) P^{h,V}_{T-s}f(x_s)\, ds\right]. \tag{2.7}$$

Proof Since $V \in C^{1,\alpha}$, the solution $P^{h,V}_t f$ is three times differentiable in space and we may differentiate both sides of the parabolic equation. Since $P^{h,V}_t f \in \text{Dom}(d)$, it follows that $d\Delta(P^{h,V}_t f) = \Delta^{1,h} d(P^{h,V}_t f)$, see [30, Chapter 2] or [22] for $h = 0$ case. Consider the function $(t, \alpha, (x, v)) \in [0,T] \times \mathbb{R}_+ \times TM \mapsto (\alpha, dP^{h,V}_{T-t}f(v))$ and apply Itô's formula to it and to the process $(t, \mathbb{V}_t, W_t(v))$ to obtain

$$\begin{aligned} & \mathbb{V}_t dP^{h,V}_{T-t}f(W_t(v)) - dP^{h,V}_T f(v) \\ = & \int_0^t \mathbb{V}_s \nabla (dP^{h,V}_{T-s}f)(u_s dB_s, W_s(v)) + \int_0^t \nabla \mathbb{V}_s (dP^{h,V}_{T-s}f)(\nabla h(x_s), W_s(v)) ds \\ & + \int_0^t \mathbb{V}_s \left(\frac{\partial}{\partial s} dP^{h,V}_{T-s}f\right)(W_s(v))\, ds + \frac{1}{2}\int_0^t \mathbb{V}_s \operatorname{tr} \nabla^2 (dP^{h,V}_{T-s}f)(W_s(v))\, ds \\ & + \int_0^t \mathbb{V}_s dP^{h,V}_{T-s}f\left(\frac{D}{ds}W_s(v)\right) ds - \int_0^t V(x_s) \mathbb{V}_s dP^{h,V}_{T-s}f(W_s(v))\, ds. \end{aligned} \tag{2.8}$$

Using Bochner's formula, $\Delta^{1,h} = \text{trace}\, \nabla^2 + 2L_{\nabla h} - \text{Ric}^\sharp$ for the Laplace-Beltrami operator Δ^1 on differential 1-forms, the definition of the Lie derivative and equation (2.3), we thus have

$$\begin{aligned} & \mathbb{V}_t dP^{h,V}_{T-t}f(W_t(v)) - dP^{h,V}_T f(v) \\ = & \int_0^t \mathbb{V}_s \nabla (dP^{h,V}_{T-s}f)(u_s dB_s, W_s(v)) - \int_0^t V(x_s) \mathbb{V}_s dP^{h,V}_{T-s}f(W_s(v))\, ds \\ & + \int_0^t \mathbb{V}_s \left(\frac{\partial}{\partial s} dP^{h,V}_{T-s}f\right)(W_s(v))\, ds + \frac{1}{2}\int_0^t \mathbb{V}_s \Delta^{1,h}(dP^{h,V}_{T-s}f)(W_s(v))\, ds. \end{aligned}$$

We can commute the time and space derivatives and also commute Δ^h with d to obtain

$$\frac{\partial}{\partial s} d(P^{h,V}_{T-s}f) + \frac{1}{2}\Delta^{1,h}(dP^{h,V}_{T-s}f) = V d(P^{h,V}_{T-s}f) + (dV) P^{h,V}_s f.$$

By substituting into equation (2.8) we then obtain, after some cancellation, the required formula. □

Lemma 2.3 *Assume that ρ^h is bounded from below and V is a bounded Hölder continuous function. Then for all $f \in L_\infty$ and $v \in T_{x_0}M$,*

$$\begin{aligned}(dP_t^{h,V} f)(v) =&\frac{1}{t}\mathbf{E}\left[f(x_t)\int_0^t \langle W_s(v), u_s dB_s\rangle\right]\\ &+\mathbf{E}\left[f(x_t)\left(\int_0^t e^{-\int_{t-s}^t V(x_u)du}\frac{V(x_{t-s})}{t-s}\int_0^{t-s}\langle W_r(v), u_r dB_r\rangle\right)ds\right].\end{aligned}$$

Proof Let $f \in BC^1 \cap L^2$ and we first assume that V belongs also to BC^1. We differentiate both sides of the variation of constants formula $P_t^{h,V} f = P_t^h f + \int_0^t P_{t-s}^h (V P_s^{h,V} f)\,ds$ to obtain for $v \in T_{x_0}M$,

$$\begin{aligned}dP_t^{h,V} f(v) =&dP_t^h f(v) + \int_0^t dP_{t-s}^h (V P_s^{h,V} f)(v)\,ds\\ =&dP_t^h f(v) + \int_0^t \frac{1}{t-s}\mathbf{E}\left[(V P_s^{h,V} f)(x_{t-s})\int_0^{t-s}\langle W_r(v), u_r dB_r\rangle\right]ds.\end{aligned}$$

Since the first two terms in the equation make sense, so does the last term, which by the standard Feynman-Kac formula, Lemma 2.1, and the Markov property, has the following expression:

$$\begin{aligned}&\mathbf{E}\left[(V P_s^{h,V} f)(x_{t-s})\int_0^{t-s}\langle W_r(\cdot), u_r dB_r\rangle\right]\\ &=\mathbf{E}\left[V(x_{t-s})f(x_t)e^{-\int_{t-s}^t V(x_u)du}\int_0^{t-s}\langle W_r(\cdot), u_r dB_r\rangle\right].\end{aligned}$$

The formula follows from the corresponding formula for $P_t^h f$which is well known and can be easily seen by multiply both sides of (2.5) by the martingale $\int_0^t \langle u_s dB_s, W_s(v)\rangle$:

$$\begin{aligned}\mathbf{E}\left(\int_0^t \langle u_s dB_w, W_s(v)\rangle P_{T-t}^h f(x_t)\right) &= \mathbf{E}\left(\int_0^t dP_{T-r}^h f(u_r dB_r)\int_0^t \langle u_s dB_s, W_s(v)\rangle\right)\\ &=\mathbf{E}\int_0^t dP_{T-r}^h f(W_r(v))dr = t\,dP_T^h f(v),\end{aligned}$$

The last identity follows from the second formula in Lemma 2.2. Then using the normalised geodesic $\sigma : [0,1] \to M$ connecting two points x and y in M so that $f(x) - f(y) = \int_0^1 \frac{d}{ds}[f(\sigma(s))]ds$ and by the formula above, and the fact that V is bounded, $|W_t|$ is bounded to see that $|dP_t^{h,V} f(x) - dP_t^{h,V} f(y)| \le C|f|_\infty d(x,y)$. In particular if $Q_t^{h,V}(x,\cdot)$ denotes the probability measure associated with $DP_t^{h,V} f$, then $|Q_t^{h,V}(x,M) - Q_t^{h,V}(y,M)| \le Cd(x,y)$. Thus the total variation norm of $|Q_t^{h,V}(x,\cdot) - Q_t^{h,V}(y,\cdot)| \le Cd(x,y)$ which means that the required formula holds for a bounded measurable function f. For a bounded Hölder continuous V it is sufficient to approximate with regular functions. □

Proposition 2.4 *Suppose that* $\mathrm{Ric} - 2\,\mathrm{Hess}(h) \geq K$ *and that* $V \in C^{1,\alpha} \cap BC^1$*. Then for all* $0 \leq t < T$, $v \in T_{x_0}M$ *and* $f \in L_\infty$*, (2.7) holds.*

Proof We prove the formula using equation (2.6). By the boundedness of V, dV and f it is clear that $\int_0^t \mathbb{V}_s dV(W_s(v)) P_{T-s}^{h,V} f(x_s)\, ds$ is bounded. Since $|W_t(v)|$ is bounded, we use Lemma 2.3 to conclude that $|d(P_{T-t}^{h,V} f| \in L_2$, and so does $\mathbb{V}_t d(P_{T-t}^{h,V} f)(W_t(v))$ which means that the stochastic integral appearing in equation (2.6) is L^2-bounded and therefore a true martingale with vanishing expectation. □

Theorem 2.5 *Assume that* $V \in C^{1,\alpha} \cap BC^1$ *and* $f \in L_\infty$*. Suppose that* $\mathrm{Ric} - 2\,\mathrm{Hess}(h) \geq K$*. Then for all* $0 < t \leq T$ *and* $v \in T_{x_0}M$ *we have*

$$\begin{aligned} dP_T^{h,V} f(v) = &\ \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \langle W_s(v), u_s dB_s\rangle\right] \\ &- \frac{1}{t}\mathbf{E}\left[\mathbb{V}_T f(x_T) \int_0^t \int_0^r dV(W_s(v))\, ds\, dr\right]. \end{aligned}$$

Proof By Lemma 2.1 we have

$$\mathbb{V}_t P_{T-t}^{h,V} f(x_t) = P_T^{h,V} f(x_0) + \int_0^t \mathbb{V}_r d(P_{T-r}^{h,V} f)(u_r dB_r). \tag{2.9}$$

Next we multiply the above equation by the L^2 martingale $\int_0^t \langle u_s dB_s, W_s(v)\rangle$ and use Itô's isometry to obtain

$$\mathbf{E}\left[\mathbb{V}_t P_{T-t}^{h,V} f(x_t) \int_0^t \langle u_s dB_s, W_s(v)\rangle\right] = \mathbf{E}\left[\int_0^t \mathbb{V}_r d(P_{T-r}^{h,V} f)(W_r(v))\, dr\right]$$

using the fact that the last term in equation (2.9) is an L^2 martingale. We now apply equation (2.7) to deduce that for any $0 < t \leq T$,

$$\begin{aligned} dP_T^{h,V} f(v) = &\ \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t P_{T-t}^{h,V} f(x_t) \int_0^t \langle u_r dB_r, W_r(v)\rangle\right] \\ &- \frac{1}{t}\mathbf{E}\left[\int_0^t \int_0^r \mathbb{V}_s dV(W_s(v)) P_{T-s}^{h,V} f(x_s)\, ds\, dr\right] \end{aligned} \tag{2.10}$$

and the rest follows from the Markov property. □

This extends the formula in [7] for the logarithmic derivative of the heat kernel (on a compact manifold) proved using Malliavin calculus, the latter was extended to non-compact manifolds in [30, 19] by an elementary stochastic calculus. For manifolds whose second fundamental form of an isometric embedding satisfies suitable conditions, the formula can be deduced from that in [19] using the techniques of filtering our redundant noise. Here we do not make any assumptions on the second fundamental form. See also [18] and [33] for reaction-diffusion equation on $\mathbb{R}^n$ and [46].

For estimations we need the following lemma, which is [43, Lemma 6.45].

Lemma 2.6 *Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\phi \geq 0$ is a measurable function on Ω. If Ψ is a measurable function on Ω such that $\phi\Psi$ is integrable then*

$$\mathbf{E}\left[\phi\Psi\right] \leq \mathbf{E}\left[\phi \log \phi\right] + \log \mathbf{E}\left[\exp(\Psi)\right].$$

Proposition 2.7 *Suppose that* $\mathrm{Ric} - 2\,\mathrm{Hess}(h) \geq K$ *and* $V \in C^{1,\alpha} \cap BC^1$. *Then for a non-negative bounded measurable function f we have*

$$|\nabla P_t^{h,V} f|_{x_0} \leq \frac{1}{\sqrt{t}}\Big(2C_1(t,K)\mathbf{E}\left[(f(x_t)\mathbb{V}_t \log(f(x_t)\mathbb{V}_t))^+\right]\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K)$$

for all $t > 0$ where

$$C_1(t,K) := \frac{1-e^{-Kt}}{Kt}, \quad C_2(t,K) := \frac{2}{Kt}\left(1 + \left(\frac{e^{-Kt/2}-1}{Kt/2}\right)\right).$$

Proof For $\gamma \in \mathbb{R}$ set

$$\phi := f(x_t)\mathbb{V}_t, \quad \psi_t := \gamma \int_0^t \langle W_s(v_0), u_s dB_s - (t-s)\nabla V(x_s)\, ds\rangle$$

to see, by Theorem 2.5, Fubini's theorem and Lemma 2.6, that for $v_0 \in T_{x_0}M$,

$$\gamma t d(P_t^{h,V} f)(v_0) \leq \mathbf{E}\left[\phi \log \phi\right] + \log \mathbf{E}\left[\exp\left[\gamma \int_0^t \langle W_s(v_0), u_s dB_s - (t-s)\nabla V(x_s)\, ds\rangle\right]\right].$$

Since $\frac{D}{ds}W_s = -\frac{1}{2}\mathrm{Ric}^\sharp(W_s) + \nabla^2 h(W_s,\cdot)^\sharp$ we have $|W_s| \leq e^{-\frac{Ks}{2}}$ so,

$$\begin{aligned}
&\log \mathbb{E}\left[\exp\left[\gamma \int_0^t \langle W_s(v_), u_s dB_s - (t-s)\nabla V(x_s) ds\rangle\right]\right]\\
\leq\ &\log \mathbb{E}\left[\exp\left(|\gamma| \int_0^t \langle W_s(v_0), u_s dB_s\rangle + |\ \gamma| \left|\int_0^t (t-s)\langle -\nabla V(\xi_s), W_s(v_0)\rangle ds\right|\right)\right]\\
\leq\ &\gamma^2 \int_0^t e^{-Ks}|v_0| ds + |\gamma||\nabla V|_\infty \int_0^t (t-s)e^{-\frac{Ks}{2}}|v_0| ds\\
\leq\ &\gamma^2 t C_1(t,K)|v_0| + |\gamma||\nabla V|_\infty t^2 C_2(t,K)|v_0|.
\end{aligned}$$

Thus we obtain

$$\gamma t d(P_t^{h,V} f)(v_0) \leq \mathbf{E}\left[(\phi \log \phi)^+\right] + \gamma^2 t C_1(t,K)|v_0| + |\gamma||\nabla V|_\infty t^2 C_2(t,K)$$

which after minimizing over γ yields

$$t|\nabla P_t^{h,V} f|_{x_0} \leq \left(2tC_1(t,K)\mathbf{E}\left[(\phi \log \phi)^+\right]\right)^{\frac{1}{2}} + t^2|\nabla V|_\infty C_2(t,K),$$

as claimed. $\square$

If $P_t^{h,V} f(x_0) > 0$ we define the non-negative quantity

$$\mathcal{H}_t(f, x_0) := \mathbb{E}\left[\frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)} \log\left(\frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)}\right)\right],$$

and replacing the non-negative function f in Proposition 2.7 by $\frac{f}{P_t^{h,V} f(x_0)}$, we deduce the following corollary.

Corollary 2.8 *Suppose that* Ric $-2\,$Hess$(h) \geq K$ *and* $V \in C^{1,\alpha} \cap BC^1$*. Then for any bounded measurable function* $f \geq 0$ *we have*

$$|\nabla \log P_t^{h,V} f|_{x_0} \leq \frac{1}{\sqrt{t}}\Big(2C_1(t,K)\mathcal{H}_t(f,x_0)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K), \quad t > 0.$$

and

$$|\nabla \log p_t^{h,V}|_{x_0} \leq \frac{1}{\sqrt{t}}\sqrt{2C_1(t,K)}\Big(\sup_{y\in M} \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V)\Big)^{\frac{1}{2}} + t|\nabla V|_\infty C_2(t,K).$$

Indeed, choosing $f(\cdot) = p_t^{h,V}(\cdot, y_0)$, the Feynman-Kac kernel associated to $P_t^{h,V}$, we have

$$\begin{aligned}
\mathcal{H}_t(f, x_0) &\leq \sup_{y\in M} \log \frac{f(y)\mathbf{E}\left[e^{-\int_0^t V(x_s)ds}|x_t = y\right]}{P_t^{h,V} f(x_0)} \mathbb{E}\left[\frac{f(x_t)\mathbb{V}_t}{P_t^{h,V} f(x_0)}\right] \\
&\leq \sup_{y\in M} \log \frac{p_t^{h,V}(y,y_0)e^{-t\inf V}}{p_{2t}^{h,V}(x_0,y_0)} \\
&= \sup_{y\in M} \log \frac{p_t^h(y,y_0)\mathbb{E}\left[e^{-\int_0^t V(b_s^{t,y,y_0})ds}\right] e^{-t\inf V}}{p_{2t}^h(x_0,y_0)\mathbb{E}\left[e^{-\int_0^{2t} V(b_s^{t,x_0,y_0})ds}\right]} \\
&\leq \sup_{y\in M} \log \frac{p_t^h(y,y_0)}{p_{2t}^h(x_0,y_0)} + 2t(\sup V - \inf V)
\end{aligned}$$

where $b_s^{t,x,y}$ is the process obtained by conditioning the h-Brownian motion by $x_t = y$ and $x_0 = x$. Given suitable heat kernel upper and lower bounds, or more generally a Harnack inequality for p_t^h, we would have an estimate on the logarithmic derivative of $p_t^{h,V}$. These two types of assumptions are naturally related. For the first see [10, 29, 9, 25, 13, 39], [29, Corollary 3.1], and [24, Thms 7.4, 7.5, 7.9]. If the Sobolev inequality $|f|^2_{2n/(n-2)} \leq C\int_M |\nabla f|^2\,dx$ holds for $f \in C_K^\infty$ then the heat kernel satisfies the on-diagonal estimate $p_t(x,x) \leq Ct^{-\frac{n}{2}}$ (leading to off-diagonal estimates). In fact, N. Th. Varopoulos proved that the Sobolev inequality is also necessary for the on-diagonal upper bound, [23, Lemma 5.1, Thm 6.1]. See also [24, 40] for a clear account on the relation between various functional inequalities and the on diagonal Gaussian upper bounds of the type vol$(B(x,\sqrt{t}))^{-1}e^{-\frac{d(x,y)^2}{t}}$. For the case $h = 0$ and Ric $\geq -K$, $K \geq 0$, a global

Harnack inequality exists, [29, Thm. 2.2], for positive solutions of the heat equation. For example, [6, Corollary 2],

$$\frac{f_t(x)}{f_{t+s}(y)} \le \left(\frac{t+s}{t}\right)^{\frac{n}{2}} \exp\left[\frac{(d(x,y)+\sqrt{nK}s)^2}{4s} + \frac{\sqrt{nK}}{2}\min\{(\sqrt{2}-1)d(x,y), \frac{\sqrt{nK}}{2}s\}\right].$$

From this can be deduced the following theorem,

Theorem 2.9 *Suppose that the Ricci curvature is bounded from below and $V \in C^{1,\alpha} \cap BC^1$. Then for all $T > 0$ there exists a positive constant $C(T)$ such that*

$$|\nabla \log p_t^V(\cdot, y_0)(x_0)| \le C(T)\left(\frac{1}{t} + \frac{d^2(x_0,y_0)}{t^2} + |V|_\infty\right)^{\frac{1}{2}} + C(T)t|\nabla V|_\infty$$

for all $t \in (0,T]$ and $x_0, y_0 \in M$.

When $V = 0$, this recovers the estimates in J. Sheu [41], see also [28, 35, 44]. For brevity we do not write down the estimate involving h it is however worth noticing that gradient formula for $P_t^h f$ would lead to a parabolic Harnack inequality for p_t^h, see [3] for such an estimate.

3 On Manifolds with a Pole

Let $n \ge 2$. Assume that y_0 is a pole for M. That is, we assume that the exponential map $\exp_{y_0}$ is a diffeomorphism between $T_{y_0}M$ and M. If the Jacobian $J : M \to \mathbb{R}$ defined by

$$J(y) \equiv J_{y_0}(y) = |\det D_{\exp_{y_0}^{-1}(y)} \exp_{y_0}|$$

is non-singular, then we denote by J^{-1} the reciprocal of J and for $t > 0$ define

$$\Phi(y) = \frac{1}{2}J^{\frac{1}{2}}(y)\Delta J^{-\frac{1}{2}}(y), \qquad k_t(y) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{r^2(y)}{2t}} J^{-\frac{1}{2}}(y)$$

for $y \in M$, where r denotes the distance to the pole y_0. The function $J^{\frac{1}{2}}$ is called Ruse's invariant (see for example A. G. Walker [48]). The objective is to obtain probabilistic representation for the kernel p^V and its derivatives, involving the distance function, J and Φ. On the standard n-dimensional hyperbolic space, the heat kernel $p_t(x,y)$ depends on x and y through $r = d(x,y)$ and is given by iterative formulas:

$$p_t(x,y) = C(m)\frac{1}{\sqrt{t}}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m e^{-m^2 t - \frac{r^2}{4t}}, \qquad n = 2m+1,$$

$$p_t(x,y) = C(m)t^{-\frac{3}{2}} e^{-\frac{(2m+1)^2}{4}t}\left(\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m \int_\rho^\infty \frac{se^{-\frac{s^2}{4t}}}{\cosh s - \cosh r}\, ds, \quad n = 2m+2.$$

If its sectional curvature is $-R^{-2}$, then [17],

$$\begin{aligned} J &= \left(\frac{R}{r}\sinh\frac{r}{R}\right)^{n-1} \\ \Phi &= -\frac{(n-1)^2}{8R^2} + \frac{(n-1)(n-3)}{8}\left(r^{-2} - \left(R^2\sinh^2\left(\frac{r}{R}\right)\right)^{-1}\right). \end{aligned}$$

In particular Φ is bounded above.

If M is a model space, i.e. M is a manifold with a pole p such that for every linear isometry $\phi : T_pM \to T_pM$ there exists an isometry $\Phi : M \to M$ such that $\phi(p) = p$ and $d\Phi_p = \phi$. In the geodesic polar coordinates, $(r,\theta) \in (0,\infty) \times S^{n-1}$, the pull back metric in $(0,\infty) \times S^{n-1}$ can be written as $dr^2 + f(r)^2 d\theta^2$. The function f is C^∞ and satisfies $f(0) = 0$, $f'(0) = 1$, $f(r) > 0$ for $r > 0$ and $f''(r) = -R(r)f(r)$ where $R(r)$ is the sectional curvature in a plane containing the radial direction ∂_r at a point x with $d(x,p) = r$. Then $\log J_p(x) = (n-1)\log\frac{f(r)}{r}$. If $R(r) = R$, a constant, then $f(r) = \frac{\sinh(\sqrt{R}r)}{\sqrt{R}}$ and $\log f$ has bounded derivatives of all order. In general,

$$|\nabla \log J| = (n-1)\left|(\log f)'(r) - \frac{1}{r}\right|, \qquad \Delta r = (n-1)(\log f)'(r),$$

$$\Delta(\log J) = (n-1)^2(\log f)'(r)\left((\log f)'(r) - \frac{1}{r}\right) + (n-1)\left((\log f)''(r) + \frac{1}{r^2}\right),$$

$$\Phi = \frac{1}{4}(n-1)\left[\frac{n-2}{r^2} - \frac{n-1}{r}(\log f)'(r) - (\log f)''(r)\right].$$

For general manifolds, volume comparison theorem has seen studied in [50]. But we are not aware of any comparison theorems for $\nabla \log J$ and Φ.

3.1 Girsanov Transform and Zeroth Order Formula

Let T be a positive number and $x_0, y_0 \in M$. The semi-classical bridge process, between x_0 and y_0 in time T, is the diffusion process starting at x_0 with generator $\frac{1}{2}\Delta + \nabla \log k_{T-s}$ where $k_t(x_0,y_0) := (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x_0,y_0)}{2t}} J^{-\frac{1}{2}}(x_0)$. If $\tilde u_s$ satisfies

$$d\tilde u_s = \sum H_i(\tilde u_s)\circ dB_s^i + \nabla \widetilde{\log k}_{T-s}(\tilde u_s)\, ds$$

with $\tilde u_0 = u_0$ and $\tilde A_s$ the horizontal lift of A_s then $\tilde x_s = \pi(\tilde u_s)$ is a semi-classical bridge.

Since $|\nabla r| = 1$ away from y_0, Itô's formula implies for $0 \le t < T$ that

$$r(\tilde x_t) - r(x_0) = \beta_t + \int_0^t \frac{1}{2}\Delta r(\tilde x_s) ds - \int_0^t \frac{r(\tilde x_s)}{T-s} ds - \frac{1}{2}\int_0^t dr\left(\nabla \log J(\tilde x_s)\right) ds$$

where β_t is a standard one-dimensional Brownian motion. It is clear from this and the formula

$$\Delta r = \frac{n-1}{r} + dr(\nabla \log J) \tag{3.1}$$

that $r(\tilde{x}_t)$ is distributed as a Bessel bridge, starting at $r(x_0)$ and ending at 0 at time T, from which it follows that $\lim_{t\uparrow T}\tilde{x}_t = y_0$, almost surely.

We follow a beautiful idea of Watling [49] and use a modification of the construction by Elworthy-Truman to define the semi-classical Riemannian bridge for a given vector field Z. For $\gamma : [0,1] \to M$ the unique geodesic from x to y_0,

$$S(x) = \int_0^1 \langle \dot{\gamma}(u), Z(\gamma(u))\rangle du,$$

and naturally $S(y_0) = 0$.

Lemma 3.1 *[Watling] Let x_t be a $\frac{1}{2}\Delta + Z + \nabla S + \nabla(\log(k_{T-s}))$ diffusion (the semi-classical Riemannian bridge). Then $d(y_0, \tilde{x}_t)$ is the n-dimensional Bessel bridge process.*

In the formula for $r_t := d(\tilde{x}_t, y_0)$, the additional drift $Z + \nabla S$ appears in the form of $\langle \nabla d, Z+\nabla S\rangle$ and this vanishes. Indeed, $S(\gamma_x(t)) = \int_t^1 \langle \dot{\gamma}(u), Z(\gamma(u))\rangle du$ so $\frac{d}{dt}|_{t=0} S(\gamma_x(t)) = -\langle \dot{\gamma}_x(0), Z(x)\rangle$ on one hand, and $\frac{d}{dt}|_{t=0} S(\gamma_x(t)) = \langle \nabla S, \dot{\gamma}_x(0)\rangle$ on the other hand.

The following basic lemma will be used repeatedly, in which we denote by $\mathbb{P}$ the probability distribution of the Brownian motion with drift $\nabla h + Z$ and by $\mathbb{Q}$ the probability distribution of a semi-classical Riemannian bridge. Note that $\Phi = \frac{1}{2} J^{\frac{1}{2}} \Delta(J^{-\frac{1}{2}}) = \frac{1}{2}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J)$. Define

$$\Phi^h = -\frac{1}{2}|\nabla h|^2 - \frac{1}{2}\Delta h + \Phi, \quad \Psi = \frac{1}{2}|\nabla S|^2 + \frac{1}{2}\Delta S + \langle Z, \nabla S + \nabla h\rangle.$$

Lemma 3.2 *Suppose that $h \in C^2(M;\mathbb{R})$ and Z is a C^1 vector field. Suppose that the $\frac{1}{2}\Delta^h + Z$ diffusion x_t is complete. Fix $t \in [0,T)$. Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathcal{F}_t$ with Radon-Nikodym derivative*

$$M_t := \frac{k_T(x_0) e^{(h-S)(x_0)}}{k_{T-t}(\tilde{x}_t) e^{(h-S)(\tilde{x}_t)}} \exp\left[\int_0^t (\Phi^h + \Psi)(\tilde{x}_s)\, ds\right]. \tag{3.2}$$

Proof For any $u_0 \in OM$ with $\pi(u_0) = x_0$, let $\tilde{u}_s$ be the solution of the following equation with initial value u_0:

$$du_s = \sum H_i(u_s) \circ dB_s^i + (\widetilde{Z} + \widetilde{\nabla h})(u_s)\, ds. \tag{3.3}$$

Then (u_t) exists for all time and $x_t = \pi(u_t)$. Let $\tilde{u}_t$ be the solution to

$$d\tilde{u}_s = \sum H_i(\tilde{u}_s) \circ dB_s^i + (\nabla \widetilde{\log k_{T-s}} + \widetilde{Z} + \widetilde{\nabla S})(\tilde{u}_s)\, ds, \quad \tilde{u}_0 = u_0 \tag{3.4}$$

Then $\tilde{x}_s = \pi(\tilde{u}_s)$ is a semi-classical Riemannian bridge. It has generator $\frac{1}{2}\Delta + Z + \nabla S + \nabla(\log(k_{T-s}))$. One crucial observation is that

$$\frac{1}{2}\Delta^h + Z + \nabla \log(k_{T-s} e^S e^{-h}) = \frac{1}{2}\Delta + \nabla S + Z + \nabla \log k_{T-s}(\cdot, y_0),$$

and it is possible to treat $\tilde{x}_t$ as the (x_t) process by adding the additional drift $\nabla \log(k_{T-s}e^{S-h})$. Since both processes are well defined before time T, they are equivalent on $\mathcal{F}_t$ for any $t < T$.

Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$ and let

$$\tilde{B}_t^i := B_t^i + \int_0^t d\log(k_{T-s}e^{S-h})(\tilde{u}_s e_i)\, ds.$$

By the Girsanov-Cameron-Martin theorem we obtain,

$$M_t = \exp\left[-\sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i - \frac{1}{2}\int_0^t |\nabla \log(k_{T-s}e^{S-h})|^2_{(\tilde{x}_s)} ds\right].$$

Now Itô's formula implies

$$\begin{aligned}\log(e^{S-h}k_{T-t})(\tilde{x}_t) = {} & \log(e^{S-h}k_T)(x_0) + \sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i \\ & + \int_0^t |\nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s)|^2\, ds + \int_0^t \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right) \log(k_{T-s}e^{S-h})(\tilde{x}_s)\, ds\end{aligned}$$

so the stochastic integral appearing in the formula for M_t can be eliminated. Then, by the relation (3.1) we see that

$$\begin{aligned}\frac{\partial}{\partial s}\log k_{T-s} &= \frac{n}{2(T-s)} - \frac{r^2}{2(T-s)^2}, \\ |\nabla \log(k_{T-s}e^{-h})|^2 &= \frac{r^2}{(T-s)^2} + \frac{1}{4}|\nabla \log J|^2 + \frac{r dr(\nabla \log J)}{T-s} - 2\langle \nabla h, \nabla \log(k_{T-s})\rangle + |\nabla h|^2, \\ \Delta \log(k_{T-s}e^{-h}) &= -\frac{n}{T-s} - \frac{r dr(\nabla \log J)}{T-s} - \frac{1}{2}\Delta(\log J) - \Delta h.\end{aligned}$$

For $S = 0$, we have,

$$\begin{aligned}&\frac{1}{2}|\nabla \log(k_{T-s}e^{-h})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right)\log(k_{T-s}e^{-h}) \\ &= \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) + \frac{1}{2}|\nabla h|^2 - \frac{1}{2}\Delta h - |\nabla h|^2 \\ &= \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2,\end{aligned}$$

from which we deduce the formula with non-vanishing S ,

$$\begin{aligned}&\frac{1}{2}|\nabla \log(k_{T-s}e^{-h+S})|^2 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta^h + Z\right)\log(k_{T-s}e^{-h+S}) \\ =&\frac{1}{2}|\nabla S|^2 + \langle \nabla S, \nabla \log(k_{T-s}e^{-h})\rangle + \frac{1}{2}\Delta^h S + \langle Z, \nabla S\rangle + \langle Z, \nabla \log(k_{T-s}e^{-h})\rangle \\ &+ \frac{1}{8}|\nabla \log J|^2 - \frac{1}{4}\Delta(\log J) - \frac{1}{2}\Delta h - \frac{1}{2}|\nabla h|^2.\end{aligned}$$

Since $\nabla S + Z$ vanishes on radial directions, see the proof for Lemma 3.1, the first line on the right hand side of the equation is

$$\begin{aligned}&\frac{1}{2}|\nabla S|^2 - \langle \nabla S + Z, \nabla h\rangle + \frac{1}{2}\Delta S + \langle \nabla h, \nabla S\rangle + \langle Z, \nabla S\rangle\\ &= \frac{1}{2}|\nabla S|^2 + \frac{1}{2}\Delta S + \langle Z, \nabla S + \nabla h\rangle.\end{aligned}$$

Finally we see that

$$\begin{aligned}\log(e^{S-h}k_{T-t})(\tilde{x}_t) = \;& \log(e^{S-h}k_T)(x_0) + \sum_{i=1}^m \int_0^t \langle \nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s), \tilde{u}_s e_i\rangle dB_s^i\\ &+ \frac{1}{2}\int_0^t |\nabla \log(k_{T-s}e^{S-h})(\tilde{x}_s)|^2\, ds + \int_0^t (\Phi^h + \Psi)(\tilde{x}_s)\, ds.\end{aligned}$$

and the required identity follows. □

In particular, if $f \in \mathcal{B}_b$ then Lemma 3.2 implies that for $0 \le t < T$,

$$\int_M f(y) p_t^{h,Z,V}(x_0, y)\, dy = \mathbf{E}[\mathbb{V}_t f(x_t)] = k_T(x_0, y_0) e^{(S-h)(x_0)} \mathbb{E}\left[\frac{f(\tilde{x}_t) e^{(h-S)(\tilde{x}_t)}}{k_{T-t}(\tilde{x}_t)} \beta_t^{h,Z}\right], \tag{3.5}$$

where

$$\beta_t^{h,Z} = \exp\left(\int_0^t (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right). \tag{3.6}$$

Elworthy and Truman's proof of the following theorem, for the case $h = 0$, was inspired by semiclassical mechanics. They used a semiclassical bridge which arrives at y_0 at time $T + \lambda$ and took the limit as $\lambda \downarrow 0$. We give a slightly modified proof, generalising their result for Δ^h, the method of which will later be used to derive a derivative formula. The following generalises a formula by Elworthy-Truman, [17]. Watling [49] treated Brownian motion with a general drift Z.

Theorem 3.3 *[15, 49] Let $V \in C^{0,\alpha} \cap L_\infty$ and suppose that the Brownian motion with drift $\nabla h + Z$ does not explode. Suppose that $\Phi^h + \Psi - V$ is bounded above, or more generally the following convergence $\lim_{t\to T} \left|\beta_t^h - \beta_T^h\right| = 0$. Then*

$$p_T^{h,Z,V}(x_0, y_0) = e^{(h-S)(y_0) - (h-S)(x_0)} k_T(x_0, y_0) \mathbf{E}\left[\exp\left(\int_0^T (\Phi^h + \Psi - V)(\tilde{x}_s) ds\right)\right]. \tag{3.7}$$

Proof Let ϕ be a smooth function with compact support with $\phi(y_0) = 1$. Denote by E_t the standard Gaussian kernel in the tangent space $T_{y_0}M$. Then, by a change of variables, we see that

$$\begin{aligned}&\lim_{t\uparrow T}\lim_{r\uparrow t} p_t^{h,Z,V}(\phi k_{T-r})(x_0) = \lim_{t\uparrow T}\lim_{r\uparrow t}\int_M p_t^{h,Z,V}(x_0, y)\phi(y) k_{T-r}(y)\, dy\\ &= \lim_{t\uparrow T}\lim_{r\uparrow t}\int_{T_{y_0}M} (p_t^{h,Z,V}(x_0, \cdot)\cdot \phi \cdot J^{1/2})(\exp_{y_0} v) E_{T-r}(v)\, dv\\ &= (p_T^{h,Z,V}(x_0, \cdot)\cdot\phi\cdot J^{1/2})(\exp_{y_0} 0_{y_0}) = \; p_T^{h,Z,V}(x_0, y_0)\end{aligned}$$

where 0_{y_0} denotes the origin of the tangent space $T_{y_0}M$, using the fact that $p_t^{h,Z,V}(x_0,\cdot)\cdot\phi\cdot J^{1/2}$ has compact support with $J(\exp_{y_0} 0_{y_0}) = J(y_0) = 1$. Thus, by taking $f = \phi\cdot k_{T-r}$ in equation (3.5) for $r < t$, we observe, since $\tilde{x}_t$ converges to y_0 a.s., that

$$\begin{aligned} p_T^{h,Z,V}(x_0,y_0) =& e^{(S-h)(x_0)} k_T(x_0,y_0) \lim_{t\uparrow T}\lim_{r\uparrow t} \mathbf{E}\left[\phi(\tilde{x}_t)\frac{k_{T-r}(\tilde{x}_t)e^{(S-h)(\tilde{x}_t)}}{k_{T-t}(\tilde{x}_t)}\exp\left(\int_0^t (\Phi^h+\Psi-V)(\tilde{x}_s)ds\right)\right] \\ =\ & e^{(h-S)(y_0)-(h-S)(x_0)} k_T(x_0,y_0)\mathbf{E}\left[\exp\left(\int_0^T (\Phi^h+\Psi-V)(\tilde{x}_s)ds\right)\right], \end{aligned}$$

Since ϕ has compact support the limit $r\to t$ is trivial to justify. The second follows from the assumption. The proof is complete. $\square$

At this point we compare formula (3.7), valid for all time $t \le T$, with S. R. S. Varadhan's asymptotic relation [47] and the asymptotic expansion of S. Minakshisundaram and A. Pleijel [36] for small time. The first states that $\lim_{t\downarrow 0} 2t\log p_t \to -d^2$, uniformly on compact sets. This was proved in [47] for operators of the form $\sum_{i,j} a_{i,j}\partial^2\partial x_i\partial x_j$ in $\mathbb{R}^n$. The latter states that there are smooth functions H_i defined on $M\times M\setminus \mathrm{Cut}(M)$ such that

$$p_t(x,y) \sim (2\pi t)^{-\frac{n}{2}} e^{-\frac{d^2(x,y)}{2t}} \sum_{i=0}^{\infty} H_i(x,y)t^i$$

with $H_0(x,y) = J_x^{-\frac{1}{2}}(y)$, as $t\to 0$. Both converge uniformly on compact subsets of $M\times M\setminus \mathrm{Cut}(M)$, where $\mathrm{Cut}(M)$ denotes the cut locus of M. See also [2].

3.2 First Order Formula

We return the h-Brownian motion whose generator is $\frac{1}{2}\Delta+\nabla h$. Set $\beta_t^h = \exp\left(\int_0^t(\Phi^h-V)(\tilde{x}_s)ds\right)$.

Theorem 3.4 *Assume y_0 is a pole, $V\in C^{1,\alpha}\cap BC^1$, $\Phi^h - V$ is bounded above, $f\in L_\infty$ and $\underline{\mathrm{Ric}} - 2\,\mathrm{Hess}\, h \ge K$. Then*

$$dp_T^{h,V}(\cdot,y_0) = \frac{1}{T}e^{h(y_0)-h(x_0)}k_T(x_0,y_0)\mathbf{E}\left[\beta_T^h\left(\int_0^T \langle \tilde{W}_r(\cdot),\tilde{u}_r d\tilde{B}_r\rangle - \int_0^T (t-r)dV(\tilde{W}_r(v))dr\right)\right]$$

where $\tilde{W}$ is the solution to (2.4) and for $r\in[0,T)$

$$d\tilde{B}_r = dB_r + \tilde{u}_r^{-1}\nabla(\log(k_{T-r}e^{-h}))(\tilde{x}_r)\,dr.$$

Proof For all $0<t\le T$ and $v\in T_{x_0}M$ we have, by Theorem 2.5 and Fubini's theorem, that

$$\begin{aligned} dP_t^{h,V}f(v) =\ & \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t f(x_t)\int_0^t \langle W_s(v),u_s dB_s\rangle - (t-s)dV(W_s(v))\,ds\right] \\ =\ & \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t f(x_t)\int_0^t \langle W_s(v),u_s dB_s - (t-s)\nabla V ds\rangle\right]. \end{aligned}$$

Therefore, it follows that, for all $0 < t < T$,

$$dP_t^{h,V}(\phi k_{T-t})(v) = \frac{1}{t}\mathbf{E}\left[\mathbb{V}_t(\phi k_{T-t})(x_t)\int_0^t \langle W_r(v), u_r dB_r - (t-r)\nabla V dr\rangle\right],$$

where ϕ is a smooth function with compact support and $\phi(y_0) = 1$, as in the proof of Theorem 3.3. By Lemma 3.2 this yields

$$dP_t^{h,V}(\phi k_{T-t})(v) = \frac{1}{t}e^{-h(x_0)}k_T(x_0,y_0)\mathbf{E}\left[\phi(\tilde{x}_t)\beta_t^h e^{h(\tilde{x}_t)}\int_0^t \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r - (t-r)\nabla V dr\rangle\right].$$

We now take limits. For the left-hand side of the previous equation we see that

$$\begin{aligned}\lim_{t\uparrow T} dP_t^{h,V}(\phi k_{T-t})(v) &= \lim_{t\uparrow T} d\left(\int_M P_t^{h,V}(\cdot,y)\phi(y)k_{T-t}(y)\,dy\right)(v)\\ &= \lim_{t\uparrow T}\int_M dP_t^{h,V}(\cdot,y)(v)\phi(y)k_{T-t}(y)\,dy = dP_T^{h,V}(\cdot,y)(v)\end{aligned}$$

where the final equality follows from the compactness of the support of ϕ and the fact that $J(y_0) = 1$. For the right-hand side, we use $\lim_{t\to T}\tilde{x}_t = y_0$, apply the dominated convergence theorem, using upper and lower bounds on Φ^h and $\mathrm{Ric} - 2\,\mathrm{Hess}\,h$, respectively, obtaining the result as claimed. □

Immediate applications of Theorems 3.3 and 3.4 are:

Corollary 3.5 *Under the assumptions of Theorem 3.4 we have,*

$$d\log p_T^{h,V}(\cdot,y_0)(v) = \frac{1}{T}\mathbf{E}\left[Z^h\int_0^T \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r - (t-r)\nabla V dr\rangle\right],$$

where $Z_T^h = \frac{\beta_T^h}{\mathbf{E}(\beta_T^h)} = \frac{\exp\left(\int_0^t(\Phi^h - V)(\tilde{x}_s)ds\right)}{\mathbf{E}\exp\left(\int_0^t(\Phi^h - V)(\tilde{x}_s)ds\right)}$.

For V bounded Hölder continuous, the above argument and Lemma 2.3 lead to:

Corollary 3.6 *Assume that y_0 is a pole for M, $\underline{\mathrm{Ric}} - 2\,\mathrm{Hess}(h)$ is bounded below and Φ^h is bounded above with V Hölder continuous and bounded. Then*

$$\begin{aligned}d\log p_T^{h,V}(\cdot,y_0)_{x_0} =& \frac{1}{T}\mathbf{E}\left[Z_T^h\int_0^T \langle \tilde{W}_s(\cdot), \tilde{u}_s d\tilde{B}_s\rangle\right]\\ &+\mathbf{E}\left[Z_T^h\int_0^T V(\tilde{x}_{T-s})\,e^{-\int_{T-s}^T V(\tilde{x}_u)du}\frac{1}{T-s}\int_0^{T-s}\langle \tilde{W}_r(\cdot), \tilde{u}_r d\tilde{B}_r\rangle\right].\end{aligned}$$

In [10, Thm.6], an estimate of the following form

$$\nabla p_t(x_0,y_0) \le C\delta^{-\alpha(n)}(x_0)t^{-(n+\frac{1}{2})}e^{-\frac{\alpha(n)d^2(x_0,y_0)}{t}}$$

is given for Riemannian manifold of bounded curvature. We have the following corresponding estimates.

Corollary 3.7 *Assume y_0 is a pole, $\Phi^h - V$ is bounded above, $\underline{\mathrm{Ric}} - 2\,\mathrm{Hess}(h) \geq K$, and ∇h, and $\nabla \log J$ are bounded. Suppose that $V \in C^{1,\alpha} \cap BC^1$. Then for an explicit constant C depending only on n, K and $|\nabla \log J|_\infty$, the following estimate holds,*

$$\frac{|\nabla p_T^{h,V}(\cdot, y_0)|_{x_0}}{k_T(x_0, y_0)} \leq C e^{h(y_0)-h(x_0)} |\beta_T^h|_\infty \left(\frac{d(x_0, y_0)}{T} + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + T|dV|_\infty \right).$$

Proof Since $|W_t(v)| \leq e^{-\frac{1}{2}Kt}|v|$,

$$\begin{aligned}
&\left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r d\tilde{B}_r \rangle - \int_0^T (t-r) dV(\tilde{W}_r(v)) dr \rangle \right| \\
&= \left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r dB_r \rangle + \int_0^T \left\langle \tilde{W}_r(v), \nabla \log(e^{-h} k_{T-r})(\tilde{x}_r) \right\rangle dr - \int_0^T (t-r) dV(\tilde{W}_r(v)) dr \rangle \right| \\
&\leq \left| \int_0^T \langle \tilde{W}_r(v), \tilde{u}_r dB_r \rangle \right| + |v| \int_0^T e^{-Kr/2} (|\nabla \log k_{T-r} - \nabla h)(\tilde{x}_r)| dr + |v|\; |dV|_\infty \int_0^T e^{-Kr/2}(t-r) dr.
\end{aligned}$$

We apply Theorem 3.4 to see that

$$\begin{aligned}
\left| T e^{h(x_0)-h(y_0)} \frac{dp_T^{h,V}(\cdot, y_0)(v)}{k_T(x_0)} \right| \leq & |v|\; |\beta_T^h|_\infty \left(\int_0^T e^{-Kr} dr \right)^{\frac{1}{2}} + |v|\; |\beta_T^h|_\infty |dV|_\infty \int_0^T e^{-Kr/2}(t-r) dr \\
& + |v|\; |\beta_T^h|_\infty \mathbf{E} \left[\int_0^T e^{-Kr/2} (|\nabla h|_\infty + |\nabla \log k_{T-r}(\tilde{x}_r)|) dr \right].
\end{aligned}$$

The first two terms on the right hand side are nicely bounded by $|v|\; |\beta_T^h|_\infty (\sqrt{C_1(K,T)} + |dV|_\infty C_2(K,T))|v|$ where C_1 and C_2 are the obvious integrals of order T and T^2 respectively. Since $\nabla \log k_t(\cdot, y_0) = -\frac{\nabla r^2}{2t} - \frac{1}{2}\nabla \log J$, the last term can be estimated using the Euclidean bridge. Then

$$\begin{aligned}
&\mathbf{E} \left[\int_0^T e^{-Kr/2} |\nabla \log k_{T-r}(\tilde{x}_r)| dr \right] \\
&\leq \frac{1}{2} \int_0^T e^{-Kr/2} |\nabla \log J(\tilde{x}_r)|_{L_1} dr + \mathbf{E} \left[\int_0^T e^{-Kr/2} \frac{d(\tilde{x}_r, y_0)}{(T-r)} dr \right] \\
&\leq \frac{1}{2} \int_0^t e^{-Kr/2} |\nabla \log J(\tilde{x}_r)|_{L_1} dr + \int_0^T e^{-Kr} \frac{\sqrt{\mathbf{E} d^2(\tilde{x}_r, y_0)}}{(T-r)} dr.
\end{aligned}$$

Since $r_t = d(\tilde{x}_t y_0)$ is the n-dimensional Bessel bridge,

$$\mathbf{E} d^2(\tilde{x}_r, y_0) = \left(\frac{T-r}{T} \right)^2 d^2(x_0, y_0) + \frac{r}{T}(T-r),$$

and consequently,

$$\int_0^t e^{-Kr/2} \frac{\sqrt{\mathbf{E} d^2((\tilde{x}_r), y_0)}}{(T-r)} dr \leq \int_0^t e^{-Kr/2} \left(\frac{1}{T} d(x_0, y_0) + \sqrt{\frac{r}{T}} \frac{1}{\sqrt{T-r}} \right) dr,$$

which is bounded by $C(d(x_0, y_0) + \sqrt{T})$. This completes the proof. □

Together with Theorem 3.3 we easily have an estimate for the gradient of the logarithmic Feynman-Kac kernel, which follows from a similar estimate as above:

$$\left|\nabla \log p_T^{h,V}(\cdot, y_0)\right|_{x_0} \leq C|Z_T|_{L_2} C\left(\frac{d(x_0, y_0)}{T} + 1 + |\nabla h|_\infty + \frac{1}{\sqrt{T}} + |dV|_\infty T\right),$$

where C is a constant depending on $|\nabla \log J|_\infty$ and on K.

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