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SOME VANISHING SUMS INVOLVING BINOMIAL COEFFICIENTS IN THE DENOMINATOR

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Abstract. We obtain expressions for sums of the form $\sum_{j=0}^{m}(-1)^j \frac{j^d\binom{m}{j}}{\binom{n+j}{j}}$ and deduce, for an even integer $d \geq 0$ and $m = n > d/2$, that this sum is 0 or $\frac{1}{2}$ according as to whether $d > 0$ or not. Further, we prove for even $d > 0$ that $\sum_{l=1}^{d} c_{l-1} \frac{(-1)^l\binom{n}{l} l!}{(l+1)\binom{2n}{l+1}} = 0$ where $c_r = \frac{1}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^{d-1}$. Similarly, we show when $d > 0$ is even that $\sum_{r=0}^{d} a_r \frac{r!\binom{n}{r+1}}{\binom{2n}{r+1}} = 0$ where $a_r = \frac{(-1)^{d+r}}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^d$.

Introduction

Identities involving binomial coefficients usually arise in situations where counting is carried out in two different ways. For instance, some identities obtained by William Horrace [1] using probability theory turn out to be special cases of the Chu-Vandermonde identities. Here, we obtain some generalizations of the identities observed by Horrace and give different types of proofs; these, in turn, give rise to some other new identities. In particular, we evaluate sums of the form $\sum_{j=0}^{m}(-1)^j j^d \frac{\binom{m}{j}}{\binom{n+j}{j}}$ and deduce that they vanish when d is even and $m = n > d/2$. It is well-known [2] that sums involving binomial coefficients can usually be expressed in terms of the hypergeometric functions but it is more interesting if such a function can be evaluated explicitly at a given argument. Identities such as the ones we prove could perhaps be of some interest due to the explicit evaluation possible. The papers [3], [4] are among many which deal with identities for sums where the binomial coefficients occur in the denominator and we use similar methods here.

1. Horrace's identities - other proofs and generalizations

We start with the identities in Horrace's paper which he deduced using probability theory.

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Lemma 1.1. *For $m \geq 1, n \geq 0$; we have*
$\sum_{j=0}^{m}(-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{n}{n+m}$*; and*
$\sum_{j=1}^{m}(-1)^{j-1} j \frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{mn}{(n+m)(n+m-1)}$.

The lemma can be easily deduced by induction or using the method of [3].

Remark 1.2. *We give another expression for the left hand sides of these identities. Recall the forward difference operator Δ defined on a function f by $(\Delta f)(x) = f(x+1) - f(x)$. As usual, one defines $\Delta^{k+1} f = \Delta(\Delta^k f)$ etc. It is easily seen by induction on m that*

$$(\Delta^m f)(x) = \sum_{r=0}^{m}(-1)^r \binom{m}{r} f(x+m-r).$$

Now, the left hand side of the first identity of Lemma 1.1 is

$$\sum_{j=0}^{m}(-1)^j \frac{\binom{m}{j}}{\binom{n+j}{j}}$$

which is $(\Delta^m g)(0)$ where

$$g(x) = \frac{n!}{(m+1-x)(m+2-x)\cdots(m+n-x)}.$$

Now, one can express $g(x)$ as a partial fraction $\sum_{i=1}^{n} \frac{a_i}{m+i-x}$. Also, each a_j can be found by multiplying both sides by the product $(m+1-x)(m+2-x)\cdots(m+n-x)$ and evaluating at $x = m+j$; we have $a_j \prod_{i \neq j}(i-j) = n!$ for each $j \leq n$. Now, we compute $(\Delta^m g)(x) = \sum_{i=1}^{n}(\Delta^m g_i)(x)$ where $g_i(x) = \frac{a_i}{m+i-x}$. Computing, we see that

$$(\Delta^m g)(0) = n! \sum_{i=1}^{n} \sum_{r=0}^{m} \prod_{j \leq n; j \neq i} \frac{1}{j-i} \frac{(-1)^r \binom{m}{r}}{r+i}$$

which easily simplifies to

$$(\Delta^m g)(0) = n \sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1}\binom{n-1}{i-1}\binom{m}{r}}{r+i}.$$

It is worth noting that although the left hand sides of these identities can be thought of as the action by the $(m+n)$-th difference operator, it does not give anything new and merely reproduces the left hand sides again. Now, by Lemma 1.1, we get $(\Delta^m g)(0) = \frac{n}{m+n}$ and we have the following corollary.

Corollary 1.3.

$$\sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1}\binom{n-1}{i-1}\binom{m}{r}}{r+i} = \frac{1}{m+n}.$$

Doing the same process with the second identity in Lemma 1.1, we have :

$$\sum_{i=1}^{n} \sum_{r=0}^{m} \frac{(-1)^{r+i-1} i \binom{n-1}{i-1}\binom{m}{r}}{r+i} = \frac{mn}{(m+n)(m+n-1)}.$$

As a matter of fact, the identity of Corollary 1.3 can be proved in a much more general form by another manner as follows.

Lemma 1.4.

$$\sum_{i_1,\cdots,i_k} \frac{(-1)^{i_1+\cdots+i_k}\binom{n_1}{i_1}\cdots\binom{n_k}{i_k}}{i_1+i_2+\cdots+i_k+1} = \frac{1}{n_1+n_2+\cdots+n_k+1}.$$

Proof. Writing $(1-t)^{n_1+\cdots+n_k} = (1-t)^{n_1}\cdots(1-t)^{n_k}$ and integrating both sides from 0 to 1 after expanding the right side binomially, we have the identity asserted. □

2. A vanishing theorem

A natural generalization of Lemma 1.1 would be to consider the sums of the form $\sum_{j=1}^{m}(-1)^{j-1}j^d\frac{\binom{m}{j}}{\binom{n+j}{j}}$ for various $d>1$. We have the following result which first shows how the roles of m and n are interchanged and then implies a vanishing result when $m=n$. In between, we also adopt a method used in [3] for evaluating sums where binomial coefficients appear in the denominator.

Theorem 2.1. *Let θ be a polynomial and let $m+n>deg(\theta)$. Then, the sum*

$$P_{m,n}(\theta) := \sum_{j=0}^{m}(-1)^j\frac{\theta(j)\binom{m}{j}}{\binom{n+j}{j}}$$

satisfies

$$\binom{m+n}{n}P_{m,n}(\theta) = \sum_{j=0}^{m}(-1)^j\theta(j)\binom{m+n}{m-j} = \sum_{i=0}^{n}(-1)^{i-1}\theta(-i)\binom{m+n}{n-i} + \theta(0).$$

Further, if θ is an even function and if $m=n$, then $P_{m,n}(\theta)=\theta(0)/2$.
In particular, for $n>2k\geq 0$, $\sum_{j=0}^{n}(-1)^j\frac{j^{2k}\binom{n}{j}}{\binom{n+j}{j}} = 0$ if $k>0$ and $=\frac{1}{2}$ if $k=0$.

Proof. Now $P_{m,n}(\theta) = \sum_{j=0}^{m}(-1)^j\frac{\theta(j)\binom{m}{j}}{\binom{n+j}{j}} = (\Delta^m\Phi)(0)$ where

$$\Phi(x) = \frac{\theta(m-x)n!}{(m+1-x)(m+2-x)\cdots(m+n-x)}.$$

Now, we divide $\theta(x)$ by the polynomial $\prod_{i=1}^{n}(x+i)$ and write

$$\theta(x) = u(x)\prod_{i=1}^{n}(x+i)+v(x)$$

and $\deg(v)<n$.
Note that if u is not the zero polynomial, we have $\deg(u)<m$ by hypothesis. In particular, $(\Delta^m u)$ is the zero polynomial.
Now, we expand in partial fractions as in Remark 1.2 :

$$\frac{v(m-x)n!}{(m+1-x)(m+2-x)\cdots(m+n-x)} = \sum_{r=1}^{n}\frac{c_r}{m+r-x}.$$

The coefficients c_r are obtained easily as before; we get

$$c_i = \frac{v(-i)n!}{(-1)^{i-1}(i-1)!(n-i)!}.$$

Note that $v(-i) = \theta(-i)$ for all $i = 1, \cdots, n$. Thus,

$$P_{m,n}(\theta) = (\Delta^m \Phi)(0) = (\Delta^m w)(0)$$

where $w(x) = \frac{v(m-x)n!}{(m+1-x)(m+2-x)\cdots(m+n-x)} = \sum_{r=1}^{n} \frac{c_r}{m+r-x}$.

For $i = 1, \cdots, n$ we evaluate $(\Delta^m \frac{1}{m+i-x})(0) = \sum_{r=0}^{m}(-1)^r \frac{\binom{m}{r}}{r+i}$ as in [3] as follows.

$$\sum_{r=0}^{m}(-1)^r \frac{\binom{m}{r}}{r+i} = \sum_{r=0}^{m}(-1)^r \binom{m}{r} \int_0^1 (1-t)^{r+i-1} dt$$
$$= \int_0^1 t^{i-1}(1-t)^m dt = \beta(i, m+1) = \frac{(i-1)!m!}{(m+i)!}.$$

Therefore,

$$P_{m,n}(\theta) = \sum_{i=1}^{n} c_i \frac{(i-1)!m!}{(m+i)!} = \sum_{i=1}^{n} \frac{v(-i)n!}{(-1)^{i-1}(i-1)!(n-i)!} \frac{(i-1)!m!}{(m+i)!}$$
$$= \frac{1}{\binom{m+n}{n}} \sum_{i=1}^{n}(-1)^{i-1} v(-i)\binom{n+m}{n-i} = \frac{1}{\binom{m+n}{n}} \sum_{i=1}^{n}(-1)^{i-1}\theta(-i)\binom{n+m}{n-i}$$

because $v(-i) = \theta(-i)$ for all $i = 1, \cdots, n$. which is Adding and subtracting the term corresponding to $i = 0$, we get the expression asserted in the theorem, viz.,

$$P_{m,n}(\theta) = \frac{1}{\binom{m+n}{n}} \sum_{i=0}^{n}(-1)^{i-1}\theta(-i)\binom{m+n}{n-i} + \theta(0).$$

Adding this expression and the expression $\frac{1}{\binom{m+n}{n}} \sum_{j=0}^{m}(-1)^j \theta(j)\binom{m+n}{m-j}$, it is evident that when $m = n$ and $\theta(i) = \theta(-i)$ for all i, the sum is $\theta(0)$. Taking $\theta(x) = x^{2k}$, the last statement follows. The proof is complete.

□

Remark 2.2. *It is important to note that although $P_{m,n}(\theta)$ can be re-expressed as a multiple of $\sum_{j=0}^{m}(-1)^j\theta(j)\binom{m+n}{m-j}$, and hence, can be viewed as the effect of the $(m+n)$-th order difference operator on a certain function, this does not give any new information but merely reproduces the expression. Thus, it is indeed worthwhile to view $P_{m,n}(\theta)$ rather as the effect of the m-th order difference operator on a certain function.*

We proved the vanishing of $P_{m,n}(\theta)$ when $m = n$ and $\theta(j) = j^{2k}$, but did not evaluate it for general m, n. As we will see, a natural method to evaluate it is to evaluate and use the following sums:

Proposition 2.3. *For $m, n \geq 1, d \geq 0$ we have*

$$T_d := \sum_{j=0}^{m}(-1)^j (j+1)(j+2)\cdots(j+d)\frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{d!\binom{n}{d+1}}{\binom{m+n}{d+1}}.$$

We also have

$$S_d := \sum_{j=0}^{m}(-1)^j j(j-1)\cdots(j-d+1)\frac{\binom{m}{j}}{\binom{n+j}{j}} = \frac{(-1)^d n\binom{m}{d}d!}{(d+1)\binom{m+n}{d+1}}.$$

As usual, the convention is that the empty product (when $d = 0$ here) is understood to be equal to 1.

Proof. As we did in the proof of Theorem 2.1, we express the denominator $\binom{n+j}{j}$ in terms of the beta function and evaluate the sums. We omit details. $\square$

Corollary 2.4.

$$\sum_{j=0}^{m}(-1)^j j^d \frac{\binom{m}{j}}{\binom{n+j}{j}} = \sum_{l=1}^{d} c_{l-1} \frac{(-1)^l n\binom{m}{l} l!}{(l+1)\binom{m+n}{l+1}}$$

where $c_r = \frac{1}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^{d-1}$ *for all* $0 \leq r < d-1$.
In particular, if $d > 0$ *is even and* $< 2n$*, then*

$$\sum_{l=1}^{d} c_{l-1} \frac{(-1)^l \binom{n}{l} l!}{(l+1)\binom{2n}{l+1}} = 0$$

with c_l*'s as above.*
Similarly, we have

$$\sum_{j=0}^{m}(-1)^j j^d \frac{\binom{m}{j}}{\binom{n+j}{j}} = \sum_{r=1}^{d} a_r \frac{r!\binom{n}{r+1}}{\binom{m+n}{r+1}}$$

where $a_r = \frac{(-1)^{d+r}}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^d$ *for all* $0 \leq r < d$.
In particular, if $d > 0$ *is even and* $< 2n$*, then*

$$\sum_{r=1}^{d} a_r \frac{r!\binom{n}{r+1}}{\binom{2n}{r+1}} = 0$$

with a_r*'s as above.*

Proof. Now $\sum_{j=0}^{m}(-1)^j j^d \frac{\binom{m}{j}}{\binom{n+j}{j}} = \sum_{l=1}^{d} c_{l-1}S_l$ where S_l is as above and where c_l' s are defined by $j^d = \prod_{k=0}^{d-1} c_k j(j-1)\cdots(j-k)$.
If we write

$$x^d = \prod_{k=0}^{d-1} c_k x(x-1)\cdots(x-k)$$

then it is easy to determine c_k's recursively and we find that for $0 \leq r < d-1$, we have

$$r!c_r = \sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^{d-1}.$$

Thus, Proposition 2.3 implies the first assertion.
Similarly, if we express $x^d = \sum_{r=0}^{d} a_r(x+1)(x+2)\cdots(x+r)$, then we have $\sum_{j=0}^{m}(-1)^j j^d \frac{\binom{m}{j}}{\binom{n+j}{j}} = \sum_{r=1}^{d} a_r T_r$. We may compute the a_r's recursively and find that for $0 \leq r < d$, we get

$$(-1)^{d+r} r! a_r = \sum_{s=0}^{r}(-1)^s\binom{r}{s}(r-s+1)^d.$$

$\square$

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