View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Warwick Research Archives Portal Repository

University of Warwick institutional repository: http://go.warwick.ac.uk/wrap

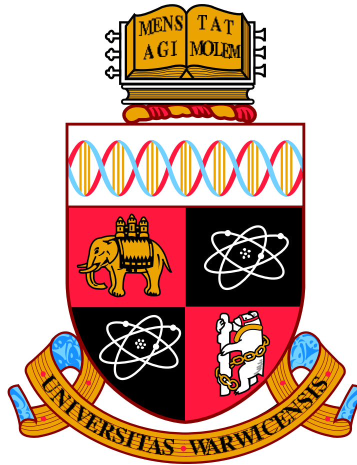
A Thesis Submitted for the Degree of PhD at the University of Warwick

http://go.warwick.ac.uk/wrap/49186

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.



Algorithms for Polycyclic-by-Finite Groups

by

Shavak Sinanan

Thesis

Submitted to The University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

July, 2011

THE UNIVERSITY OF
WARWICK

To my parents,

Avory & Ousha

Table of Contents

List of Tables **vii**

List of Algorithms **viii**

Listings **ix**

Acknowledgements **x**

Declaration **xiii**

Abstract **xiv**

Notation **xv**

1 Introduction **1**

1.1 The Class of Polycyclic-by-Finite Groups 1

1.2 Overview of Project 2

1.2.1 Objectives 2

1.2.2 Results 3

1.2.3 Notes and Assumptions 5

2 Background Material **6**

2.1 Permutation Groups 6

2.1.1 Definitions and Notation 7

2.1.2 Computation of Orbits and Stabilisers 8
2.1.3 Bases and Strong Generating Sets 11
2.2 Polycyclic Groups 17
2.2.1 Finite Soluble Groups 19
2.2.2 Infinite Polycyclic Groups 21
2.3 Representation Theory and Extensions 22
2.3.1 The Terminology of Representation Theory 23
2.3.2 Semidirect Products, Complements, Derivations and First Cohomology Groups 24
2.3.3 Extensions of Modules and The Second Cohomology Group . 26

3 Multiplication **29**

3.1 Representation of Elements 30
3.1.1 The Normal Form 30
3.1.2 The Multiplication Problem 31
3.2 The Multiplication Algorithm 32
3.2.1 Strategy 32
3.2.2 Data 33
3.2.3 The Algorithm 37
3.3 Applications 44
3.3.1 Transversal Images 45
3.3.2 Element Inversion 45
3.3.3 Orders of Elements 46
3.3.4 Transfer to the Category of Polycyclic Groups 46

4 Subgroups **50**

4.1 The Subgroup Generation Problem 51
4.2 Construction 52
4.2.1 The Extended Schreier–Sims Algorithm 52

4.2.2 Data 64
4.2.3 Membership Testing 66
4.3 Applications 67
4.3.1 The Soluble Radical 67
4.3.2 Sylow Subgroups 68

5 Conjugacy **70**
5.1 The Centre 71
5.2 The Conjugacy Problem 75
5.2.1 Centralisers 75
5.2.2 Conjugacy Testing 80

6 Conclusion **83**
6.1 Examples and Run-times 83
6.2 Implementation 101
6.2.1 List of Functions 101
6.2.2 Technical Considerations 102
6.2.3 Native Support 103
6.3 Future Work 104

A Package Documentation **106**
A.1 Introduction 106
A.2 Getting Started 106
A.3 Creation of a Group 107
A.3.1 Construction Functions 107
A.4 Basic Group Properties 107
A.4.1 Infrastructure 107
A.4.2 Numerical Invariants 108
A.4.3 Predicates 108

A.5 Elements 108
A.5.1 Definition of Elements 109
A.5.2 Arithmetic Operations on Elements 109
A.5.3 Properties of Elements 110
A.5.4 Predicates for Elements 110
A.5.5 Set Operations 111
A.6 Subgroups 111
A.6.1 Definition of Subgroups by Generators 112
A.6.2 Membership and Coercion 112
A.6.3 Standard Subgroup Constructions 112
A.6.4 Sylow Subgroups 113
A.6.5 Centralisers 113
A.7 Normal Subgroups and Subgroup Series 113
A.7.1 Normal Structure 113
A.7.2 Characteristic Subgroups 113
A.7.3 Subgroup Series 114
A.8 Conjugacy 114
A.9 Transfer Between Group Categories 114
A.9.1 Transfer to GrpPC or GrpGPC 114

B Program Listing **116**

Bibliography **238**

List of Tables

6.1 Run-times for perfect groups 95

6.2 Run-times for subgroups of AGL$(3, 5)$ 96

6.3 Run-times for subgroups of AGL$(4, 3)$ 97

6.4 Run-times for reducible matrix groups 98

6.5 Run-times for module extensions 99

6.6 Run-times for finitely presented groups that map onto A_5 100

List of Algorithms

3.1 Element Multiplication 42

4.1 Sifting 55

4.2 Standard version of the extended Schreier–Sims algorithm 59

4.3 Normal closure version of the extended Schreier–Sims algorithm . . . 63

4.4 Base-preserving version of the extended Schreier–Sims algorithm . . . 65

5.1 Finding the centre 74

5.2 Finding centralisers 79

5.3 Testing Element Conjugacy 82

Listings

B.1 grputils.m 116
B.2 permsv.m 117
B.3 recfrmtdef.m 122
B.4 permdata.m 124
B.5 pcbfconstruct.m 130
B.6 pcbfarithmetic.m 159
B.7 pcbfcattrans.m 173
B.8 pcbfschreiersims.m 180
B.9 pcbfsubgroup.m 197
B.10 pcbfderived.m 213
B.11 pcbfsylow.m 218
B.12 pcbfcentre.m 221
B.13 pcbfcentraliser.m 225
B.14 pcbfconjugacy.m 229
B.15 pcbfmain.m 233

Acknowledgements

First and foremost I would like to express my deepest gratitude to my supervisor, Prof. Derek Holt, who has, at all times, been a source of inspiration and encouragement. Derek has been an excellent supervisor, and is an outstanding mathematician. His keen insights and sharp observations have always kept me interested and motivated, and I owe every ounce of mathematical knowledge and skill that I have acquired over the last four years to him.

A key ingredient to success is a productive, challenging yet comfortable environment, and I am fortunate to have found that and so much more in, where I now call my mathematical home, the Warwick Mathematics Institute. The support provided by the Institute is all that one could hope for as a graduate student, and I wish to sincerely thank the staff for the magnificent job that they have done, and continue to do, in managing us mathematicians. Special mention must be made of the current and former Postgraduate Coordinators, Carole Fisher and Carol Wright, for whose patience and understanding over the last four years I am eternally grateful. I would also like to acknowledge the University of Warwick for the financial support that I have received via the Warwick Postgraduate Scholarship and the Chancellor's International Scholarship during my time as a doctoral candidate.

I must also express my gratitude to Prof. John Cannon of the Computational Algebra Group at the University of Sydney, for providing me with the opportunity to visit Sydney to collaborate directly with the Magma developers, and for his keen interest in my work.

Special acknowledgement must be made of those who have contributed to my earlier academic development, in particular I would like to mention Prof. Michael Vaughan-Lee of Christ Church College, Oxford, and Prof. E. J. Farrell of the University of the West Indies, St. Augustine.

As I near the end of my time at Warwick, I think about the reasons why I am where I am now, and there are just two: Avory and Ousha. To my parents: your unwavering support at all stages of my life means more to me than you shall ever know, and you will *always* be the reason why I am where I am.

A famous quote of William Butler Yeats goes: “Think where man’s glory most begins and ends, and say my glory was I had such friends.” To those most important to me:

Rajiv ‘Fliz’ Sinanan — “You’re the best man I know.”

Rishi ‘Ding’ Ramsumair — C _ c a _ _ l _ ×3. SBW. Blauuw.

Travers ‘JT’ Sinanan — “Be resourceful man. Bring me some champagne and oysters.”

Kerwin ‘The Prophet’ Cumberbatch — Salaam brother Numsi.

Avinash ‘T.C.’ Ramdass — The real man in short pants.

Shivana ‘Tea Poog’ Maharaj — For F.R.I.E.N.D.S and crêpes.

Mandana ‘Mana Joon’ Namdar — For halloumi cheese distractions!

Tasneem ‘Joggers’ Sharafally — Gorillas, Shirts and Permitted.

Andrew ‘Koopa Troopa’ Ferguson — Red shell first, cigar later.

Shivani ‘Nanny Goat’ Jacelon — You’re ugly.

Sasha ‘The Popo’ Nath — For Jay Sean and Apple Sourz bonding.

Machel Pantin — $\triangleright + \triangledown + \triangleright +$ Any punch button.

Anja Dass — Wha’s the scene horse?

Megan ‘Chinchilla’ Lee Yuen — For Chinese delights.

Clint ‘Tuffzool’ Sieunarine — My brother of the sword.

Michael Doré — Time for an I.M.O. Q6?

Deepika ‘Special K’ Khurana — For more patience, love and understanding than one has ever deserved.

Declaration

The author declares that the material contained in this thesis is entirely his own work. As is standard practice, the subject matter developed builds on existing theory, and clear citations and references are provided where necessary.

Although material contained herein may be submitted for publication at a later date, the author has not published any work which forms part of this thesis.

No part of this thesis has been submitted, for the purposes of a degree or otherwise, to any other university or educational institution.

Abstract

A set of fundamental algorithms for computing with polycyclic-by-finite groups is presented here. Polycyclic-by-finite groups arise naturally in a number of contexts; for example, as automorphism groups of large finite soluble groups, as quotients of finitely presented groups, and as extensions of modules by groups.

No existing mode of representation is suitable for these groups, since they will typically not have a convenient faithful permutation representation.

A mixed mode is used to represent elements of such a group; utilising a polycyclic presentation or a power-conjugate presentation for the elements of the normal subgroup, and a permutation representation for the elements of the quotient.

Notation

$\mathbb{Z}$	the set of integers, $\{\dots, -1, 0, 1, \dots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, \dots\}$
$\mathbb{Z}^-$	the set of negative integers, $\{\dots, -2, -1\}$
$\mathbb{Z}_n$, $\mathbb{Z}/n\mathbb{Z}$	the set of integers modulo n
$\mathbb{N}$	the set of natural numbers, $\{0, 1, \dots\}$
$\mathbb{F}_{p^n}$	the finite field of order p^n where p is prime and $n \in \mathbb{Z}^+$
$\mathrm{GL}(d, R)$	the group of invertible $d \times d$ matrices over a ring, R
$\mathrm{SL}(d, R)$	the group of $d \times d$ matrices with determinant 1 over a ring, R
$\mathrm{PGL}(d, q)$	the projective general linear group of degree d over the field, $\mathbb{F}_q$, q a prime power
$\mathrm{PSL}(d, q)$	the projective special linear group of degree d over the field, $\mathbb{F}_q$, q a prime power
$\mathrm{GU}(d, q^2)$	the general unitary group of degree d over the field, $\mathbb{F}_{q^2}$, q a prime power
$\mathrm{SU}(d, q^2)$	the special unitary group of degree d over the field, $\mathbb{F}_{q^2}$, q a prime power
$\mathrm{PGU}(d, q^2)$	the projective general unitary group of degree d over the field, $\mathbb{F}_{q^2}$, q a prime power
$\mathrm{PSU}(d, q^2)$	the projective special unitary group of degree d over the field, $\mathbb{F}_{q^2}$, q a prime power

$\mathrm{AGL}(d, q)$	the group corresponding to the action of $\mathrm{GL}(d, \mathbb{F}_q)$ on the affine points of the d-dimensional vector space over the field, $\mathbb{F}_q$, q a prime power
$[3 \ldots 8]$	the array, [3, 4, 5, 6, 7, 8]
$[11 \ldots -1 \text{ by } -3]$	the array, [11, 8, 5, 2, −1]
[]	the empty array

Chapter 1

Introduction

This chapter aims to acquaint the reader with the topic and scope of the research undertaken towards the completion of this thesis. Firstly, the class of *polycyclic-by-finite* groups is discussed briefly, after which the objectives and results of the project are outlined.

1.1 The Class of Polycyclic-by-Finite Groups

Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of groups. A group is said to be *poly-*$\mathcal{P}$ if it admits a finite subnormal series such that every factor group has property $\mathcal{P}$. A group, G, is a $\mathcal{P}$*-by-*$\mathcal{Q}$ *group* if it has a normal subgroup, N, such that N has property $\mathcal{P}$, and G/N has property, $\mathcal{Q}$.

Thus, a group is *polycyclic-by-finite* if it has a normal polycyclic subgroup of finite index. That is, if it has normal subgroup of finite index that admits a subnormal series with cyclic factors.

Segal (1983) proves that the property "polycyclic-by-finite" is equivalent to the properties:

(i) poly-(cyclic or finite),

(ii) (poly-C_∞)-by-finite,

where C_∞ is the property of being infinite cyclic.

By a well-known theorem of P. Hall, every polycyclic-by-finite group is finitely presented — and in fact, polycyclic-by-finite groups form the largest known section-closed class of finitely presented groups. It is this fact that makes polycyclic-by-finite groups natural objects of study from the algorithmic standpoint. For an excellent theoretical account of the algorithmic decision theory of polycyclic-by-finite groups, the reader is referred to (Baumslag et al., 1991).

In contrast to the paper of Baumslag et al. (1991), this thesis explores the computational properties of polycyclic-by-finite groups from a practical perspective, attempting to produce algorithms which are conducive to computer implementation.

In this context, the algorithms developed are targeted not only at infinite polycyclic-by-finite groups, but, in fact, primarily at finite insoluble groups with a large soluble normal subgroup, as such groups do not often have a convenient permutation representation. These groups, although trivially polycyclic-by-finite, may be viewed as polycyclic-by-finite with non-trivial polycyclic normal subgroup, leading to a computationally effective representation. Groups of this type arise naturally in many applications such as automorphism groups of large finite soluble groups, as quotients of finitely presented groups, and as extensions of modules by groups.

1.2 Overview of Project

The objectives and outcome of the research governing this thesis are outlined in this section.

1.2.1 Objectives

The central goal of this thesis is to define a data structure, and develop fundamental, machine implementable algorithms, for the class of polycyclic-by-finite groups. By a data structure, one means a mode by which such groups may be represented on

the computer. Fundamental algorithms for a class of groups include algorithms to perform the tasks of multiplication, element inversion and subgroup generation.

The material developed also aims to use the fundamental algorithms developed to perform more advanced structural computations in polycyclic-by-finite groups such as finding Sylow subgroups (in the finite case), computing centralisers and testing element conjugacy.

Focus was initially restricted to finite insoluble groups, as many of the interesting examples of polycyclic-by-finite groups that arise in practice are in fact finite (see Section 1.1). However, taking into account the recent developments and ongoing research in the area of computing with infinite polycyclic groups (see Eick, 2001), it was deemed prudent to allow for the possibility of infinite groups, and the parameters of the problem were broadened accordingly.

1.2.2 Results

This subsection contains a synopsis of the results emerging from this thesis.

Theoretical

The main body of the thesis begins by defining a normal form for the elements of a polycyclic-by-finite group, after which, a data structure which facilitates multiplication of elements is introduced. This material forms the content of Chapter 3, which culminates with a detailed description of the multiplication algorithm.

The question of subgroup generation is addressed in Chapter 4. In the discussion of this topic, the underlying problem of generating a subgroup is extracted and treated separately, after which, there follows a description of how the data structure is computed for a subgroup. This dissection results in the chapter exhibiting a more pedagogical style. Several applications of subgroup generation are presented in the final section of this chapter.

A selection of advanced structural computations in polycyclic-by-finite groups is discussed in Chapter 5. Specifically, algorithms to compute the centre of the group and the centraliser of an element are presented, along with a constructive method to test element conjugacy. Substantial use of linear algebra and representation theory is made here.

Practical

In keeping with the "machine implementable" ethos of this course of research, every function described in thesis has been fully implemented using the Magma Computational Algebra System[1]. This suite of functions forms a major portion of a complete package for computing with polycyclic-by-finite groups in Magma. The implementation was done in parallel with the development of the algorithms, and forms a significant part of the project.

An up-to-date version of the source code (written in the Magma language) fulfilling the implementation may be found on the compact disc accompanying this thesis. The reader should consult the documentation provided Appendix A for directions on how to load and use the package. Some example computations are given in Section 6.1 of Chapter 6. Although extensive testing has been done, the implementation is still in developmental stage and, in rare circumstances, bugs may arise, especially when attempting to compute with infinite polycyclic-by-finite groups. Source code listings are given in Appendix B.

[1]Magma is a large, well-supported software package designed for computations in algebra, number theory, algebraic geometry and algebraic combinatorics. See `http://magma.maths.usyd.edu.au/magma/` for details.

1.2.3 Notes and Assumptions

Existing Computational Representations

At points in the thesis, the reader may encounter the phrase "without performing any element arithmetic in E", where E is a polycyclic-by-finite group. This simply means that, while there exists some (possibly inefficient) representation for E on the computer, this representation is not used to perform element arithmetic. Indeed, the first and most important goal of the theory is to set up machinery so that elements of E can be manipulated by performing operations only within the normal polycyclic subgroup and the associated finite quotient, without appealing to any existing representation of E.

On a related point, while, for a given polycyclic-by-finite group, E, there may be many different decompositions of E as an extension of a polycyclic group by a finite group, the algorithms presented in this thesis are targeted at those decompositions in which the finite quotient is relatively small and therefore admits a permutation representation of manageable degree. Thus, it shall hereinafter be assumed that the polycyclic-by-finite groups in this thesis can be so decomposed.

Complexity

Time and space complexity analyses are provided for the main algorithms around which the theory is built. The time complexity of an algorithm shall be stated in terms of the number of multiplications performed using the given computational representation of the group in question; in the case of later algorithms, the time cost is assessed by counting the number of multiplications performed using the newly devised representation for the polycyclic-by-finite group.

Chapter 2

Background Material

This chapter contains a summary of the required background material, and introduces notation that is used throughout the rest of the thesis.

2.1 Permutation Groups

Permutation group algorithms are among the best developed parts of computational group theory. The base and strong generating set data structure and the Schreier–Sims algorithm introduced by Sims (1970, 1971) form the backbone of this area, and the resulting methods enable detailed structural computations to be carried out routinely in permutation groups of degree up to about 10^7.

As discussed in Chapter 1, the chosen representation for polycyclic-by-finite groups involves viewing the finite quotients primarily as permutation groups. Thus, the algorithms developed here for polycyclic-by-finite groups rely on the existing and well established methods for manipulating finite groups of permutations; the fundamental concepts of which are presented in this section. For a detailed account of the material discussed here, the reader is referred to (Holt et al., 2005, Chap. 4) and (Seress, 2003).

2.1.1 Definitions and Notation

In the context of permutation groups, the notation used is similar to that of Seress (2003) and Holt et al. (2005). The required definitions are reproduced concisely here.

The cycle notation shall be used for permutations, with the identity permutation denoted by $()$. The group of all permutations of an n-element set, Ω, is denoted by $\mathrm{Sym}(\Omega)$, or S_n if only the size of Ω is relevant. The image of $\alpha \in \Omega$ under the permutation $g \in \mathrm{Sym}(\Omega)$ is written as α^g. The alternating group on Ω is denoted by $\mathrm{Alt}(\Omega)$ (or A_n in the case of a generic n-element set). The *support* of $g \in \mathrm{Sym}(\Omega)$, denoted by $\mathrm{supp}(g)$, consists of those elements of Ω that are actually displaced by g, and the *degree* of g, $\deg(g)$, is defined to be the size of this set, i.e. $\mathrm{supp}(g) = \{\omega \in \Omega \mid \omega^g \neq \omega\}$, $\deg(g) = |\mathrm{supp}(g)|$. The set of fixed points of g is defined as $\mathrm{fix}(g) = \Omega \setminus \mathrm{supp}(g)$.

A group, G, is said to *act on* a set, Δ, if there exists a homomorphism (called an action), $\varphi\colon G \to \mathrm{Sym}(\Delta)$, and the *degree* of φ is defined to be $|\Delta|$ (if $G \leqslant \mathrm{Sym}(\Delta)$, then one speaks of the degree of G). The action, φ, is *faithful* if its kernel, $\ker\varphi$, is the singleton containing the identity. The image, $\varphi(G)$, is denoted by G^Δ. In the special case where $G \leqslant \mathrm{Sym}(\Omega)$, $\Delta \subseteq \Omega$ is fixed by G, and φ is the restriction of permutations to Δ, G^Δ is also denoted by $G|_\Delta$.

Let $G \leqslant \mathrm{Sym}(\Omega)$. The orbit of $\omega \in \Omega$ under G is the set of images, $\omega^G = \{\omega^g \mid g \in G\}$. For $\Delta \subseteq \Omega$ and $g \in G$, $\Delta^g = \{\delta^g \mid \delta \in \Delta\}$. The group, G, is said to be *transitive* on Ω if it has only one orbit, and G is *k-transitive* ($k \leqslant n$) if the action of G induced on the set of ordered k-tuples of distinct elements of Ω is transitive. The largest such k is called the *degree of transitivity* of G.

If G is transitive and $\Delta \subseteq \Omega$, then Δ is called a *block of imprimitivity* for G if for all $g \in G$ either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \varnothing$. The group, G, is called *primitive* if all blocks have 0, 1 or $|\Omega|$ elements. If Δ is a block then the set of images of Δ is

a partition of Ω, which is called a *block system*, and an action of G is induced on the block system. A block is called *minimal* if it has more than one element and its proper subsets of size at least 2 are not blocks. A block is called *maximal* if the only block properly containing it is Ω. A block system is *maximal* if it consists of minimal blocks, whereas a block system is *minimal* if it consists of maximal blocks. The action of G on a minimal block system is primitive.

The *pointwise* and *setwise stabilisers* of $\Delta \subseteq \Omega$ are denoted by $G_{(\Delta)}$ and G_Δ respectively. i.e. $G_{(\Delta)} = \{g \in G \mid (\forall\, \delta \in \Delta)(\delta^g = \delta)\}$, $G_\Delta = \{g \in G \mid \Delta^g = \Delta\}$. If Δ has only one or two elements, then the set braces and parentheses are often dropped from the notation of the pointwise stabiliser; in particular G_δ denotes the stabiliser of $\delta \in \Omega$. If $\Delta = (\delta_1, \ldots, \delta_m)$ is a *sequence* of elements of Ω, then G_Δ denotes the pointwise stabiliser of that sequence.

The group, G, is said to be *semiregular* if $G_\delta = 1$ for all $\delta \in \Omega$, and G is *regular* if it is transitive and semiregular. The group, G, is a *Frobenius group* if it is transitive, not regular, but for which $G_{\alpha,\beta} = 1$ for distinct $\alpha, \beta \in \Omega$.

If $g \in \mathrm{Sym}(\Omega)$ then a bijection $\phi\colon \Omega \to \Delta$ naturally defines a permutation $\bar{\phi}(g) \in \mathrm{Sym}(\Delta)$ by the rule $\phi(\omega)^{\bar{\phi}(g)} = \phi(\omega^g)$ for all $\omega \in \Omega$. The groups, $G \leqslant \mathrm{Sym}(\Omega)$ and $H \leqslant \mathrm{Sym}(\Delta)$, are *permutation isomorphic*, written $H \sim G$, if there is a bijection, $\phi\colon \Omega \to \Delta$, such that $\bar{\phi}(G) = \{\bar{\phi}(g) \mid g \in G\} = H$.

2.1.2 Computation of Orbits and Stabilisers

In algorithms involving permutation groups, calculations of point orbits and transversals of the associated stabilisers are performed frequently.

The Basic Orbit Algorithm

Let $G = \langle X \rangle$ be a finite group acting on a finite set, Ω, and let $\alpha \in \Omega$. In a computational context, the orbit, α^G, may be viewed as the vertex set of a breadth-

first-search tree rooted at α in the directed graph, $D(\Omega, \overrightarrow{E})$, with vertex set Ω and edge set $\overrightarrow{E} = \{\overrightarrow{(\beta, \gamma)} \mid \beta, \gamma \in \Omega \wedge (\exists\, x \in X \cup X^{-1})(\beta^x = \gamma)\}$. Assuming that the images, β^x, can be computed, and that points of Ω can be compared, a standard breadth-first-search method may be employed to find this vertex set along with a transversal of G_α in G.

The algorithm starts with $L_0 = \{\alpha\}$ and computes the sets, L_i, according to the recursive definition

$$L_i = \{\gamma \in \Omega \mid (\exists\, \beta \in L_{i-1})(\overrightarrow{(\beta, \gamma)} \in \overrightarrow{E})\} \backslash \bigcup_{j<i} L_j,$$

terminating when $L_m = \varnothing$ for some m. The finiteness of Ω guarantees that the algorithm does in fact terminate. The L_i are called the *levels* of the breadth-first-search tree.

The set, $\Delta = \bigcup_{j<m} L_j$, is the required orbit; the containment $\Delta \subseteq \alpha^G$ clearly holds, and the reverse inclusion follows from the fact that, by construction, $\Delta^x = \Delta$ for all $x \in X$, hence $\Delta^g = \Delta$ for all $g \in G$. A transversal of G_α in G is found by keeping a record, at each level, the permutations of $X \cup X^{-1}$ that are used to compute the points that lie in the subsequent level.

Assuming that the tasks of finding an image, β^x, and testing element membership in subsets of Ω can both be performed in constant time, the orbit computation method presented here has time complexity $O(|\Delta||X|)$.

Schreier Vectors

Computing and storing a stabiliser transversal explicitly may require $\Theta(n^2)$ memory. This approach becomes impractical when dealing with permutation groups of large degree. The Schreier vector data structure offers an improvement on this space complexity.

Definition 2.1. Let $G = \langle X \rangle$ be a finite permutation group acting on a finite set,

Ω, and let $\alpha \in \Omega$. A *Schreier vector* (or *Schreier tree*) for α relative to X is a directed labelled tree, T, with all edges directed to the root, α, and edge labels taken from the set, $X \cup X^{-1}$. The vertices of T are the points of the orbit, α^G, and the edge labels are such that if $\overrightarrow{(\gamma, \delta)}$ is an edge with label x then $\gamma^x = \delta$.

Let G be as described in Definition 2.1, let $|\Omega| = n$ and let $X = \{x_1, \ldots, x_r\}$. Let T be a Schreier vector for α relative to X. If γ is a vertex of T then the sequence of edge labels along the unique path from γ to α is a word in the elements of $X \cup X^{-1}$ which, when viewed as a product of permutations, moves γ to α. Thus, Schreier vectors define inverses of a set of coset representatives for G_α in G.

The Schreier vector data structure is usually implemented using two arrays of length equal to the size of the orbit in question. For the group, G, and the point, α, the first array, Δ, holds the elements of α^G, with $\Delta[1] = \alpha$. The second, v, is an integer array whose entries satisfy:

(i) $v[1] = 0$.

(ii) For $j = 2, \ldots, |\alpha^G|$, let e_j be the edge of T with first component $\Delta[j]$, then

$$v[j] = \begin{cases} i & \text{if } e_j \text{ has label } x_i, \\ -i & \text{if } e_j \text{ has label } {x_i}^{-1}. \end{cases}$$

The maximum length of an orbit is n, hence, storing a Schreier vector in the manner described above requires only $O(n)$ memory. There is a time-space trade-off associated with Schreier vectors, for, reconstructing a coset representative from a Schreier vector (by retracing a sequence of edge labels as described above) requires $O(n)$ multiplications in the permutation group, in contrast to the constant time required when the transversals are stored explicitly.

2.1.3 Bases and Strong Generating Sets

The notion of a base and strong generating set of a permutation group is fundamental in the theory. The resulting data structure allows for efficient manipulation of permutation groups.

Basic Definitions

Let $\Omega = \{1, \ldots, n\}$ and $G \leqslant \mathrm{Sym}(\Omega)$. A sequence, $B = (\beta_1, \ldots, \beta_l)$, of elements belonging to Ω is called a *base* for G if the only element of G to fix B pointwise is the identity. The sequence B defines a stabiliser subgroup chain,

$$G = G^{[1]} \geqslant G^{[2]} \geqslant \cdots \geqslant G^{[l]} \geqslant G^{[l+1]} = 1, \tag{2.1}$$

where $G^{[i]} = G_{(\beta_1,\ldots,\beta_{i-1})}$ $(i > 1)$ is the pointwise stabiliser of $\{\beta_1, \ldots, \beta_{i-1}\}$. The base, B, is called *non-redundant* if $G^{[i+1]} < G^{[i]}$ for all $i = 1, \ldots, l$. The orbits, ${\beta_i}^{G^{[i]}}$, are called the *basic orbits* or *fundamental orbits* of G (relative to B).

By repeated applications of Lagrange's Theorem and the Orbit–Stabiliser Theorem, one obtains

$$|G| = \prod_{i=1}^{l} \left[G^{[i]} \colon G^{[i+1]}\right] = \prod_{i=1}^{l} |\beta_i^{G^{[i]}}|. \tag{2.2}$$

Now, for each i, clearly $|\beta_i^{G^{[i]}}| \leqslant n$. Moreover, if B is non-redundant then $|\beta_i^{G^{[i]}}| \geqslant 2$. These inequalities, combined with Equation (2.2) immediately yield

$$2^{|B|} \leqslant |G| \leqslant n^{|B|},$$

or

$$\frac{\log_2 |G|}{\log_2 n} \leqslant |B| \leqslant \log_2 |G|.$$

That is, the length of a base is logarithmic in the size of the permutation group. This inequality is used frequently in the complexity calculations of this, and subsequent

chapters.

A *strong generating set* for G relative to B is a generating set, S, for G with the property that

$$\langle S \cap G^{[i]} \rangle = G^{[i]} \tag{2.3}$$

for $i = 1, \ldots, l+1$.

The *base image* of an element $g \in G$ is the sequence $B^g = (\beta_1^g, \ldots, \beta_l^g)$.

Observation 2.2. The base image of g uniquely determines the element, g. To see this, suppose that $B^g = B^h$ for some $h \in G$. Then $B^{gh^{-1}} = B$ whence $gh^{-1} = 1$ by the definition of a base.

The Sifting Procedure

Given a base for a permutation group, one may define a normal form for group elements relative to this base.

Proposition 2.3. *Let U_i be a right transversal of $G^{[i+1]}$ in $G^{[i]}$ for $i = 1, \ldots, l$. Then, every element $g \in G$ may be expressed uniquely as*

$$g = u_l u_{l-1} \cdots u_1$$

where $u_i \in U_i$.

Proof. Induction on the length of a base may be used to prove the existence of the asserted decomposition. The case $l = 1$ is trivial, for the required decomposition is simply the element itself.

Assume that the decomposition exists for bases of length $l-1$, and let $g \in G$. The Orbit–Stabiliser Theorem provides an element, $u_1 \in U_1$, such that $\beta_1^g = \beta_1^{u_1}$. The element, gu_1^{-1}, belongs to the group $G^{[2]}$ which has base, $(\beta_2, \ldots, \beta_l)$ of length $l-1$, and right transversals, U_i, for $i = 2, \ldots, l$. By the inductive hypothesis, gu_1^{-1}

has decomposition $u_l u_{l-1} \cdots u_2$, where $u_i \in U_i$. This gives

$$g = u_l u_{l-1} \cdots u_1$$

as required.

To prove uniqueness, assume that $u_l u_{l-1} \cdots u_1$ and $u'_l u'_{l-1} \cdots u'_1$ are different decompositions of an element, $g \in G$. Let j be the smallest index such that $u_j \neq u'_j$. Then, by the Orbit–Stabiliser Theorem, $\beta_j^{u_j} \neq \beta_j^{u'_j}$, whence $u_l u_{l-1} \cdots u_1$ and $u'_l u'_{l-1} \cdots u'_1$ give rise to different base images, contradicting Observation 2.2. It follows that $u_i = u'_i$ for each i. □

By Proposition 2.3, the transversals, U_i, provide a convenient normal form for elements of g.

The decomposition in the statement of the proposition can be done algorithmically as follows. Given $g \in G$, find the coset representative, $u_1 \in U_1$, such that $\beta_1^g = \beta_1^{u_1}$ and compute $g_2 = g u_1^{-1} \in G^{[2]}$. Then find $u_2 \in U_2$ such that $\beta_2^{g_2} = \beta_2^{u_2}$ and compute $g_3 = g_2 u_2^{-1}$. Iterate l times to obtain the required factorisation. This procedure is called *sifting* or *stripping*. Sifting involves constructing exactly one transversal element for each base point, and thus requires $O(n \log_2 |G|)$ multiplications in the permutation group.

Sifting can also be used to test membership in G. Given $h \in \mathrm{Sym}(\Omega)$, one attempts to factor h as a product of coset representatives, u_i. If the factorisation is successful then $h \in G$. Two things may go awry; it is possible that for some $i \leqslant l$, the ratio $h_i = h u_1^{-1} u_2^{-1} \cdots u_{i-1}^{-1}$ computed by the sifting procedure carries β_i out of the orbit $\beta_i^{G^{[i]}}$, or $h_{l+1} = h u_1^{-1} u_2^{-1} \cdots u_l^{-1} \neq 1$. In either case, $h \notin G$. The last ratio h_i computed by the sifting procedure is called the *siftee* of h.

The Schreier–Sims Algorithm

The Schreier–Sims algorithm is used to construct a base and strong generating set for a given permutation group. The method is based on the following lemmas, taken from (Seress, 2003, Chap. 4).

Lemma 2.4 (Schreier). *Let $H \leqslant G = \langle S \rangle$ and let R be a right transversal of H in G with $1 \in R$. For $g \in G$, denote the unique element of $Hg \cap R$ by $\overline{g}$. Then the set,*

$$T = \{rs(\overline{rs})^{-1} \mid r \in R, s \in S\},$$

generates H.

Proof. By definition, the elements of T are in H, so it suffices to show that $T \cup T^{-1}$ generates H. Note that $T^{-1} = \{rs(\overline{rs})^{-1} \mid r \in R, s \in S^{-1}\}$.

Let $h \in H$ be arbitrary. Since $H \leqslant G$, h can be written in the form $h = s_1 \cdots s_k$ for some non-negative integer, k, where $s_i \in S \cup S^{-1}$ for each i. Define a sequence $h_0, h_1, \ldots, h_k$ of group elements such that

$$h_j = t_1 \cdots t_j r_{j+1} s_{j+1} \cdots s_k \tag{2.4}$$

with $t_i \in T \cup T^{-1}$ for each $i \leqslant j$, $r_{j+1} \in R$, and $h_j = h$. Let $h_0 = 1s_1 \cdots s_k$. Recursively, if h_j is already defined, then let $t_{j+1} = r_{j+1}s_{j+1}(\overline{r_{j+1}s_{j+1}})^{-1}$ and $r_{j+2} = \overline{r_{j+1}s_{j+1}}$. Clearly, $h_{j+1} = h_j = h$, and h_{j+1} has the form of Equation (2.4) as required.

Thus, $h = h_k = t_1 \cdots t_k r_{k+1}$. Since $h \in H$ and $t_1 \cdots t_k \in \langle T \rangle \leqslant H$, it follows that $r_{k+1} \in H \cap R = \{1\}$, whence $h \in \langle T \rangle$. Thus $H \leqslant \langle T \rangle$. □

The elements of the set, T, in Lemma 2.4 are called *Schreier generators* for the subgroup H.

Lemma 2.5 (Sims, 1970). *Let* $G \leqslant \mathrm{Sym}(\Omega)$ *and* $\{\beta_1, \ldots, \beta_l\} \subseteq \Omega$. *For each* j *in* $\{1, \ldots, l+1\}$, *let* $S_j \subseteq G_{(\beta_1,\ldots,\beta_{j-1})}$ *such that* $\langle S_j \rangle \geqslant \langle S_{j+1} \rangle$ *holds for* $j \leqslant l$. *If* $G = \langle S_1 \rangle$, $S_{l+1} = \varnothing$, *and*

$$\langle S_j \rangle_{\beta_j} = \langle S_{j+1} \rangle \tag{2.5}$$

holds for each j, *then* $B = (\beta_1, \ldots, \beta_l)$ *is a base for* G *and* $S = \bigcup_{j=1}^{l} S_j$ *is a strong generating set for* G *relative to* B.

Proof. Induction on l is used here. The case $l = 1$ is trivial, for $\bigcup_{j=1}^{1} S_j$ clearly fulfils the requirements of a strong generating set (relative to the base containing the single point β_1) for the group, $\langle S_1 \rangle$.

Assume that the result holds for bases of length $l-1$. Then, in particular, $S^* = \bigcup_{j=2}^{l} S_j$ is a strong generating set for $\langle S_2 \rangle$ relative to the base $B^* = (\beta_2, \ldots, \beta_l)$.

Let $G^{[i]} = G_{(\beta_1,\ldots,\beta_{i-1})}$ for $i = 2, \ldots, l+1$. To prove the lemma, it is required to verify that Equation (2.3) holds for each i. Setting $j = 1$ in Equation (2.5) yields

$$G_{\beta_1} = \langle S_1 \rangle_{\beta_1} = \langle S_2 \rangle,$$

which implies the containment,

$$G_{\beta_1} \leqslant \langle S \cap G_{\beta_1} \rangle.$$

The reverse inclusion is obvious, thus

$$\langle S \cap G^{[2]} \rangle = G^{[2]},$$

and Equation (2.3) is satisfied for $i = 2$.

For $i > 2$, the inductive hypothesis implies that $S^* \cap G_{(\beta_1,\ldots,\beta_{i-1})}$ generates $\langle S_2 \rangle_{(\beta_1,\ldots,\beta_{i-1})}$, and so $G^{[i]} \geqslant \langle S \cap G_{(\beta_1,\ldots,\beta_{i-1})} \rangle \geqslant \langle S^* \cap G_{(\beta_1,\ldots,\beta_{i-1})} \rangle = \langle S_2 \rangle_{(\beta_1,\ldots,\beta_{i-1})} =$

$(G_{\beta_1})_{(\beta_1,\ldots,\beta_{i-1})} = G^{[i]}$. Therefore,

$$\langle S \cap G^{[i]} \rangle = G^{[i]}$$

for each i, as required. □

Given a permutation group, $G = \langle T \rangle$, acting on a set, Ω, a base and strong generating set can be constructed in the following way. A data structure containing a list, $B = (\beta_1, \ldots, \beta_k)$, of already known elements of a non-redundant base is maintained, along with an approximation, S_i, for a generator set of the stabiliser, $G_{(\beta_1,\ldots,\beta_{i-1})}$, for each $i \in \{1, \ldots, k\}$. Throughout execution, the S_i satisfy the property that, for all i, $\langle S_i \rangle \geqslant \langle S_{i+1} \rangle$. The data structure is said to be *up-to-date below level* j if Equation (2.5) holds for each i in the range $j < i \leqslant k$.

In the case where the data structure is up-to-date below level j, a transversal, R_j, of $\langle S_j \rangle_{\beta_j}$ in $\langle S_j \rangle$ is computed. Then a check is made to determine whether Equation (2.5) is satisfied for $i = j$. By Lemma 2.4, this can be done by sifting the Schreier generators obtained from R_j and S_j in the group, $\langle S_{j+1} \rangle$. (In this group, membership testing is possible, since Lemma 2.5 implies that $\bigcup_{i=j+1}^{k} S_i$ is a strong generating set for $\langle S_{j+1} \rangle$.) If all Schreier generators are in $\langle S_{j+1} \rangle$ then the data structure is up-to-date below level $j - 1$; otherwise a non-trivial siftee, h, at level t for some t in the range, $j + 1 \leqslant t \leqslant k + 1$, is added to S_t, and the data structure is now up-to-date below level t. If $t = k + 1$, a new base point, β_{k+1}, is chosen from $\mathrm{supp}(h)$.

The algorithm initialises B to contain a single point, $\beta_1 \in \Omega$, that is moved by at least one generator in T and sets S_1 to T. At that moment, the data structure is up-to-date below level 1; the algorithm terminates when the data structure becomes up-to-date below level 0. Lemma 2.5 immediately implies correctness.

Bounds on the time-space requirements of the algorithm are given in Theorem 2.6. For a proof and a detailed discussion, the reader is referred to (Seress,

2003, Chap. 4).

Theorem 2.6. *Given a finite permutation group $G = \langle X \rangle$ acting on a set of cardinality n, a base and strong generating set for G can be computed by deterministic algorithms in $O((n \log_2 |G|)^3 + |X| n^3 \log_2 |G|)$ time using $O(n(\log_2 |G|)^2 + |X|n)$ memory.*

2.2 Polycyclic Groups

A group, G, is said to be *polycyclic* if it has a descending chain of subgroups

$$G = G_1 \trianglerighteq \cdots \trianglerighteq G_r \trianglerighteq G_{r+1} = 1,$$

in which G_i/G_{i+1} is cyclic for $i = 1, 2, \ldots, r$. Such a chain of subgroups is called a *polycyclic series*.

If $G_i/G_{i+1} = \langle x_i G_{i+1} \rangle$ for each i, then $G = \langle x_1, \ldots, x_r \rangle$. Thus, every polycyclic group is finitely generated. The sequence, $(x_1, \ldots, x_r)$, is called a *polycyclic generating sequence* for G.

Subgroups and quotients of polycyclic groups are themselves polycyclic, for if $H \leqslant G$ and $N \trianglelefteq G$, then

$$H = H \cap G_1 \trianglerighteq \cdots \trianglerighteq H \cap G_r \trianglerighteq H \cap G_{r+1} = 1$$

and

$$G/N = G_1 N/N \trianglerighteq \cdots \trianglerighteq G_r N/N \trianglerighteq G_{r+1} N/N = 1,$$

are polycyclic series for H and G/N respectively,

Lemma 2.8 gives a characterisation of polycyclic groups as a subclass of the soluble groups. First, a necessary and sufficient condition for an abelian group to be polycyclic:

Lemma 2.7. *An abelian group is polycyclic if and only if it is finitely generated.*

Proof. One direction is clear, for every polycyclic group is finitely generated, and so, in particular, every abelian polycyclic group is finitely generated.

Conversely, suppose that $A = \langle a_1, \ldots, a_r \rangle$ is abelian. Then

$$A = \langle a_1, \ldots, a_r \rangle \trianglerighteq \langle a_1, \ldots, a_{r-1} \rangle \cdots \trianglerighteq \langle a_1, a_2 \rangle \trianglerighteq \langle a_1 \rangle \trianglerighteq 1$$

is a series with cyclic factors, showing A to be polycyclic. □

Lemma 2.8. *The polycyclic groups are exactly the soluble groups for which every subgroup is finitely generated.*

Proof. Suppose that G is polycyclic. Then, it follows immediately from the definition of a polycyclic group that G has a subnormal series with abelian factors, and is hence soluble. If $H \leqslant G$, then H is polycyclic and hence finitely generated.

Conversely, suppose that G is a soluble group for which every subgroup is finitely generated. Let

$$G = G_1 \trianglerighteq \cdots \trianglerighteq G_r \trianglerighteq G_{r+1} = 1,$$

be a soluble series for G. By hypothesis, the abelian factors, G_i/G_{i+1}, are finitely generated, and hence polycyclic by Lemma 2.7. Thus, the soluble series above may be refined to obtain a polycyclic series for G, by inserting isomorphic copies of a polycyclic series for each factor, G_i/G_{i+1}. □

In contrast to Lemma 2.7, not every finitely generated soluble group is polycyclic. A counterexample is constructed here. Take

$$A = \prod_{i=-\infty}^{\infty} \langle a_i \rangle$$

to be the restricted direct product of infinitely many infinite cyclic groups, and let

x be the automorphism of A defined by

$$a_i^x = a_{i+1} \quad (-\infty < i < \infty).$$

Then x has infinite order and generates a group, $\langle x \rangle \cong \mathbb{Z}$, of automorphisms of A. Now form the semidirect product (see Definition 2.10),

$$G = A \rtimes \langle x \rangle.$$

Then G is soluble, and G is generated by two elements, namely a_0 and x. But G is not polycyclic as the subgroup, A, of G is not finitely generated. The group, G, is the so-called *wreath product*,

$$G = \langle a_0 \rangle \wr \langle x \rangle \cong \mathbb{Z} \wr \mathbb{Z}.$$

Every polycyclic group admits a specific type of finite presentation that allows for efficient structural computation within the group. Finite presentations for polycyclic groups are discussed in the subsections below. Segal (1983) provides an excellent treatise on the beautiful theory of polycyclic groups.

2.2.1 Finite Soluble Groups

It is an easy corollary of Lemma 2.8 that, in the finite case, the properties "polycyclic" and "soluble" are equivalent. To prove this directly, argue as follows. Observe that a soluble series for a finite group has finite abelian factors. Therefore, the Basis Theorem for finite abelian groups may then be applied to decompose the finite abelian factors into direct sums of cyclic groups, thereby yielding a refinement of the given soluble series with cyclic factors as required. Conversely, polycyclicity implies solubility.

Thus, every finite soluble group has a subnormal series with cyclic factors. Such a series gives rise to various finite presentations reflecting the polycyclic structure of the group. These presentations are useful because the Word Problem in such presentations can be solved in an algorithmic fashion.

Let G be a finite soluble group. A presentation for G of the form,

$$\begin{aligned} \langle a_1, \ldots, a_r \mid a_j^{p_j} &= w_{j,j} \quad \text{for } 1 \leqslant j \leqslant r, \\ a_j^{a_i} &= w_{i,j} \quad \text{for } 1 \leqslant i < j \leqslant r \rangle, \end{aligned}$$

where

(i) p_j is the least prime such that $a_j^{p_j} \in \langle a_{j+1}, \ldots, a_r \rangle$ for $j < r$, and $a_r^{p_r}$ is the identity, and

(ii) $w_{i,j}$ is a word in the generators $a_{i+1}, \ldots, a_r$,

shall be called a *power-conjugate presentation* for G. The generators of G corresponding to $a_1, \ldots, a_r$ in this presentation are known as a *power-conjugate generating sequence* for G. The relations of the first type are called *power relations*, while those of the second type are called *conjugate relations*.

Let $G_i = \langle a_i, \ldots, a_r \rangle$ for each $i \leqslant r$, and define G_{r+1} to be the trivial group. The presentation above is said to be *consistent* if $|G_i/G_{i+1}| = p_i$ for each i. In this case, every element of G can be written uniquely in the *normal form* $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$, where $0 \leqslant \alpha_i < p_i$ for $i = 1, \ldots, r$.

It is straightforward to show that every finite soluble group possesses a consistent power-conjugate presentation, and conversely, that every power-conjugate presentation defines a finite soluble group. Given a consistent power-conjugate presentation for a group, there exists an algorithm (the *collection algorithm*), which, when given an arbitrary word over the power-conjugate generating sequence, determines the corresponding normal word. In particular, collection can be used to

compute the normal word which is equal to the product of two given normal words, thus implementing the group multiplication.

Power-conjugate presentations are an effective way of representing finite soluble groups, and, over the past two decades, a considerable body of efficient algorithms has been developed for computing with soluble groups defined in terms of power-conjugate presentations. For a survey of the algorithms currently in use for power-conjugate presentations, the reader is referred to (Holt et al., 2005, Chap. 8), and, for a discussion of computation in soluble permutation groups, see (Seress, 2003, Chap. 7).

2.2.2 Infinite Polycyclic Groups

A generalisation of the power-conjugate presentation is used to represent infinite polycyclic groups.

Let G be a polycyclic group. A presentation for G of the form,

$$\begin{aligned} \langle a_1, \ldots, a_r \mid a_i^{m_i} &= w_{i,i} \quad \text{for } i \in I, \\ a_j^{a_i} &= w_{i,j} \quad \text{for } 1 \leqslant i < j \leqslant r, \\ a_j^{a_i^{-1}} &= w_{-i,j} \quad \text{for } 1 \leqslant i < j \leqslant r,\ i \notin I \rangle, \end{aligned}$$

where

(i) $I \subseteq \{1, \ldots, r\}$,

(ii) $m_i > 1$ for $i \in I$, and

(iii) $w_{i,j}$ is of the form $w_{i,j} = a_{|i|+1}^{\epsilon(i,j,|i|+1)} \cdots a_r^{\epsilon(i,j,r)}$, with $0 \leqslant \epsilon(i,j,k) < m_k$ if $k \in I$.

shall be called a *polycyclic presentation* for G. The generators of G corresponding to $a_1, \ldots, a_r$ in this presentation are known as a *polycyclic generating sequence* for G, and the values, m_i $(i \in I)$, are called the corresponding *polycyclic exponents*. The

relations of the first type are called *power relations*, while those of the second and third types are called *conjugate relations*.

Let $G_i = \langle a_i, \ldots, a_r \rangle$ for each $i \leqslant r$, and define G_{r+1} to be the trivial group. The presentation above is said to be *consistent* if the quotient, G_i/G_{i+1}, has order m_i whenever $i \in I$, and is infinite whenever $i \notin I$. In this case, every element of G can be written uniquely in the *normal form* $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$, where $0 \leqslant \alpha_i < m_i$ for $i \in I$.

It is straightforward to show that every polycyclic group possesses a consistent polycyclic presentation, and conversely, that every polycyclic presentation defines a polycyclic group. Given a consistent polycyclic presentation for a group, there exists a version of the collection algorithm, which, when given an arbitrary word over the polycyclic generating sequence, determines the corresponding normal word. In particular, as in the case of power-conjugate presentations, collection can be used to compute the normal word which is equal to the product of two given normal words, thus implementing the group multiplication.

Computing with infinite polycyclic groups are a comparatively new topic in computational group theory and the number of available algorithms is much smaller than in the case of finite polycyclic groups. For an accessible introduction to the algorithmic theory of polycyclic groups, the reader is referred to (Sims, 1994, Chap. 9). A practical account of computing with polycyclic groups can be found in (Eick, 2001).

2.3 Representation Theory and Extensions

This section concerns group representation theory, together with the basic theory of extensions of abelian groups. A survey similar to that of Holt et al. (2005) is provided, focusing on the definitions and results needed for later chapters. James and Liebeck (2001) provide an accessible introductory account of Representation Theory, while the reader may consult (Rotman, 2002) for proofs of the more advanced results, in particular those involving representations over finite fields. For a detailed

description of the theory of extensions and cohomology, the reader is referred to Chapter 7 of (Rotman, 1995).

Computation with group representations is a significant subtopic within computational group theory. Some of the methods in this area, particularly those related to group representation theory for its own sake, involve advanced theory. Even for computations that are concerned only with the group-theoretical structure of finite groups, some of the more sophisticated algorithms require some familiarity with representation theory.

The basic reason for this is that if a finite group, G, has normal subgroups, $N < M$, for which M/N is an elementary abelian p-group for some prime, p, then the conjugation action of G on M gives rise to a representation of G/M over the field of order p, and properties of that representation translate into group-theoretical properties of G. For example, the representation is irreducible if and only if M/N is a chief factor of G.

2.3.1 The Terminology of Representation Theory

Let K be a commutative ring with unity, and let G be a finite group. The *group ring*, KG, of G over K is defined to be the ring of finite formal sums,

$$\left\{ \sum_{g \in G} r_g g \mid r_g \in K \right\},$$

with the obvious addition and multiplication inherited from that of G. In fact KG is an associative algebra with unity, and thus it is a ring with unity and a module over K. The group ring, KG, is also known as the *group algebra* of G over K.

Let M be a right (unital) KG-module. The module product of $m \in M$ and $x \in KG$ shall be denoted by $m \cdot x$, but when $x \in K$, and M is viewed primarily as a K-module, then the product may be written as xm. The commutativity of K ensures that this causes no problems. From the module axioms, and the fact

that $(m \cdot g) \cdot g^{-1} = m$ for $m \in M$, $g \in G$, one observes that multiplication by a group element $g \in G$ defines an automorphism of M as a K-module. Therefore there is an associated action $\varphi \colon G \to \mathrm{Aut}_K(M)$, and group action notation, m^g, shall sometimes be used as alternative to $m \cdot g$. Conversely, if M is a K-module, then any action $\varphi \colon G \to \mathrm{Aut}_K(M)$ can be used to make M into a KG-module.

It shall always be assumed that M is finitely generated and free as a K-module, and so, after fixing on a free basis of M, one may identify M with K^d for some d. Then, using the same free basis of M, $\mathrm{Aut}_K(M)$ may be identified with the group $\mathrm{GL}(d, K)$. So the action homomorphism, φ, is $\varphi \colon G \to \mathrm{GL}(d, K)$, which is the standard definition of a representation of G of degree d over K.

According to basic results from representation theory, two KG-modules are isomorphic if and only if the associated representations, φ_1, φ_2, are *equivalent*, which means that they have the same degree and there exists $\alpha \in \mathrm{GL}(d, K)$ with $\alpha \cdot \varphi_2(g) = \varphi_1(g) \cdot \alpha$ for all $g \in G$.

2.3.2 Semidirect Products, Complements, Derivations and First Cohomology Groups

Recall the following definitions.

Definition 2.9. An extension, G, of a group, N, is called a *split extension* if there is a subgroup, C, of G with $NC = G$ and $N \cap C = 1$. Here C is called a *complement* of N in G.

Definition 2.10. Let G and M be groups, and suppose that there is a given homomorphism, $\varphi \colon G \to \mathrm{Aut}(M)$. For $g \in G$, $m \in M$, abbreviate $m^{\varphi(g)}$ to m^g. The *semidirect product* of M by G using φ, denoted by $G \ltimes M$ or $G \ltimes_\varphi M$, is the set, $G \times M$, endowed with the multiplication, $(g_1, m_1)(g_2, m_2) = (g_1 g_2, m_1^{g_2} m_2)$, for $g_1, g_2 \in G$, $m_1, m_2 \in M$.

Proposition 2.11. *Any split extension, E, of a group, M, by a group, G, is isomorphic to the semidirect product $G \ltimes_\varphi M$, where the action, φ, of G on M is defined by the conjugation action of a complement of M in E on M.*

Proof. Observe that the semidirect product, $G \ltimes M$, is an extension of M by G, using the maps, $M \to G \ltimes M$ and $G \ltimes M \to G$, defined by $m \mapsto (1_G, m)$ and $(g, m) \mapsto g$ respectively. It is a split extension, with complement, $\{(g, 1_M) \mid g \in G\} \cong G$.

Now, if a group, E, has a normal subgroup, M, with a complement, G then any $e \in E$ can be written uniquely as $e = gm$ where $g \in G$, $m \in M$, and

$$g_1 m_1 \cdot g_2 m_2 = g_1 g_2 m_1^{g_2} m_2$$

for $g_1, g_2 \in G$, $m_1, m_2 \in M$.

Denote the conjugation action of G on M by $\varphi\colon G \to \mathrm{Aut}(M)$. Then E is isomorphic to $G \ltimes_\varphi M$ via the map $gm \mapsto (g, m)$. □

In general, different complements could give rise to different actions, φ. However, if M is abelian, then the actions coming from different complements are the same. It shall be assumed for the remainder of this subsection that M is abelian, and additive notation shall be employed where appropriate. It shall also be assumed that M is a K-module for some commutative ring K with unity. There is no loss of generality here, because any abelian group can be regarded as a $\mathbb{Z}$-module in the obvious manner. In the case where M is an elementary abelian p-group for some prime, p, K is taken to be the field $\mathbb{F}_p$.

As discussed in Subsection 2.3.1, an action, $\varphi\colon G \to \mathrm{Aut}_K(M)$, of G on the K-module, M, corresponds to endowing M with the structure of a KG-module, and so one may speak about the semidirect product $G \ltimes M = G \ltimes_\varphi M$ of the KG-module, M, with G. The multiplication rule in $G \ltimes M$, using additive notation in M, is

$$(g_1, m_1)(g_2, m_2) = (g_1 g_2, m_1^{g_2} + m_2).$$

A left transversal in $G \ltimes M$ of the subgroup $\hat{M} = \{(1_G, m) \mid m \in M\}$ isomorphic to M has the form $T_\delta = \{(g, \delta(g)) \mid g \in G\}$, for a map $\delta \colon G \to M$. The transversal, T_δ, is a complement of $\hat{M}$ in $G \ltimes M$ if and only if $(g, \delta(g))(h, \delta(h)) = (gh, \delta(gh))$ for all $g, h \in G$ or, equivalently,

$$\delta(gh) = \delta(g)^h + \delta(h) \quad \forall g, h \in G. \tag{2.6}$$

If M is a KG-module, then a map $\delta \colon G \to M$ is called a *derivation* or a *crossed homomorphism* or a 1-*cocycle* if Equation (2.6) holds. Observe that setting $h = 1_G$ in Equation (2.6) yields $\delta(1_G) = 0_M$ for any derivation, δ.

The set, $\{\delta \colon G \to M\}$, of derivations is denoted by $Z^1(G, M)$. By using the obvious pointwise addition and scalar multiplication, $Z^1(G, M)$ can be made into a K-module. The set, T_δ, is a complement of $\hat{M}$ in $G \ltimes M$ if and only if $\delta \in Z^1(G, M)$.

Notice that for a fixed $m \in M$, $\{(g, 0_M)^{(1_G, m)} = (g, m - m^g) \mid g \in G\}$ is a complement of $\hat{M}$ in $G \ltimes M$, and so $g \mapsto m - m^g$ is a derivation. Such a map is called a principal derivation or 1-coboundary. The set of all principal derivations is denoted by $B^1(G, M)$ and forms a K-submodule of $Z^1(G, M)$. The quotient K-module, $H^1(G, M) = Z^1(G, M)/B^1(G, M)$, is called the *first cohomology group* of G, M and the associated action. By construction, $H^1(G, M)$ is in one-one correspondence with the set of conjugacy classes of complements of $\hat{M}$ in $G \ltimes M$.

2.3.3 Extensions of Modules and The Second Cohomology Group

Let E be any extension of an abelian subgroup, M, (regarded as a subgroup of E) by a group, G. So there exists an epimorphism $\rho \colon E \to G$ with kernel M. For $g \in G$, choose $\hat{g} \in E$ with $\rho(\hat{g}) = g$ and, for $m \in M$, define $m^g = m^{\hat{g}}$. Since M is abelian, this definition is independent of the choice of $\hat{g}$, and it defines an action of G on M. In general, this action makes M into a $\mathbb{Z}G$-module, but if M happens to be a

module over a commutative ring K with unity, and the conjugation actions of $g \in G$ define K-automorphisms of M, then M becomes a KG-module. In particular, this is true with $K = \mathbb{F}_p$ in the case when M is an elementary abelian p-group for some prime, p.

Definition 2.12. Let G be a group and M a KG-module for some commutative ring, K. *A KG-module extension* of M by G is defined to be a group extension, E, of M by G in which the given KG-module, M, is the same as the KG-module defined by conjugation within E.

Given E as above, the set, $\{\hat{g} \mid g \in G\}$, forms a transversal of M in G. For $g, h \in G$, one has $\hat{g}\hat{h} = \widehat{gh}\tau(g,h)$, for some function, $\tau\colon G \times G \to M$, where the associative law in E implies that, for all $g, h, k \in G$,

$$\tau(g, hk) + \tau(h, k) = \tau(g, h)^k + \tau(gh, k).$$

A function $\tau\colon G \times G \to M$ satisfying this identity is called a 2-*cocycle*, and the additive group of such functions forms a K-module and is denoted by $Z^2(G, M)$.

Conversely, it is straightforward to check that, for any $\tau \in Z^2(G, M)$, the group $E = \{(g, m) \mid g \in G, m \in M\}$ with multiplication defined by

$$(g_1, m_1)(g_2, m_2) = (g_1 g_2, \tau(g_1, g_2) + m_1^{g_2} + m_2)$$

is a KG-module extension of M by G that defines the 2-cocycle τ on choosing $\hat{g} = (g, 0)$.

A general transversal of M in E has the form $\hat{g} = (g, \delta(g))$ for a function, $\delta\colon G \to M$, and it can be checked that this transversal defines the 2-cocycle, $\tau + c_\delta$, where c_δ is defined by $c_\delta(g, h) = \delta(gh) - \delta(g)^h - \delta(h)$. A 2-cocycle of the form, c_δ, for a function, $\delta\colon G \to M$, is called a 2-*coboundary*, and the additive group of such functions is a K-module and it denoted by $B^2(G, M)$.

Two KG-module extensions, E_1 and E_2 of a KG-module, M, by G are said to be *equivalent* if there is an isomorphism from E_1 to E_2 that maps the copy of M in E_1 to the copy of M in E_2, and induces the identity map on both M and on G. From the above discussion, it is not difficult to show that the extensions, E_1 and E_2 with respective 2-cocycles, τ_1 and τ_2 are equivalent if and only if $\tau_1 - \tau_2 \in B^2(G, M)$ and, in particular, an extension splits if and only if its corresponding 2-cocycle belongs to $B^2(G, M)$.

The quotient K-module, $H^2(G, M) = Z^2(G, M)/B^2(G, M)$, is called the *second cohomology group* of G, M and the associated action. It follows from the discussion above that $H^2(G, M)$ is in one-one correspondence with the equivalence classes of KG-module extensions of M by G.

Chapter 3

Multiplication

Multiplication is the most fundamental operation that one can perform within a group. In order to design a multiplication algorithm which produces consistent results, a normal form for group elements must be defined.

This chapter contains a detailed description of the multiplication algorithm developed for the class of polycyclic-by-finite groups.

Firstly, the proposed normal form for elements of polycyclic-by-finite groups is introduced. After a brief analysis of the technicalities involved in designing a feasible multiplication method, the data structure used to represent polycyclic-by-finite groups is presented, followed by the multiplication algorithm itself. The chapter concludes with a survey of useful functions that follow as straightforward applications of the multiplication method.

The definition of the normal form, and the subsequent theory developed, relies on the presupposition that it is computationally feasible to represent the finite quotient of the polycyclic-by-finite group in question faithfully by a group of permutations or matrices. Specifically, a base and strong generating set data structure for the quotient is required. Thus, it shall hereinafter be assumed that, in all cases, such a representation exists and, for the sake of clarity, the finite quotient shall be viewed as a permutation group.

3.1 Representation of Elements

The preliminary aspects of computing with polycyclic-by-finite groups are discussed in this section.

3.1.1 The Normal Form

The first stage in representing polycyclic-by-finite groups on the computer involves defining a suitable normal form for elements of such groups. The design of such a normal form can be based on the structure of the group as an extension of a polycyclic group by a finite group, and should facilitate effective manipulation of elements. In particular, it must be possible to multiply elements written in normal form efficiently.

Let E be a polycyclic-by-finite group, let $N \lhd E$ be polycyclic of finite index in E and denote the quotient E/N by G. An element $e \in E$ can be uniquely represented as an ordered pair (g, n) where $g \in G$ and $n \in N$, and conversely each such ordered pair determines an element of E. A base and strong generating set data structure for the permutation group G, combined with a power-conjugate or polycyclic presentation for N, automatically induce a normal form for elements of E written in this manner.

The ordered pair normal form is observed to be an efficacious way of storing elements of a polycyclic-by-finite group, for its components are elements of a finite permutation group and a polycyclic group respectively — groups for which there are well-developed, optimised algorithms available, thereby fully exploiting the structure of the group in question.

In the context of computing with algebraic structures, representing the polycyclic-by-finite group E as a pair (G, N) (with an associated action) not only provides one with a tractable form for elements, but also elucidates structural information about the group. For instance, a check for solubility amounts simply to testing G for the

property. The transparent nature of the representation aids in the design of potentially complex algorithms whose operation centres around structural computation.

3.1.2 The Multiplication Problem

Let E, N and G be as in Subsection 3.1.1, let $\rho\colon E \to G$ be the natural map, and fix a left transversal, L, of N in E with $1_E \in L$. For each $g \in G$, denote the unique element of $\rho^{-1}(g) \cap L$ by $\overline{g}$.

Let $e \in E$ and suppose that $\rho(e) = g$. Then $e = \overline{g}n$, where $n \in N$ is uniquely determined by the left transversal, L, and, as in Subsection 3.1.1, e can be represented by the ordered pair (g, n).

Consider multiplying two such elements, $e_1 = \overline{g}_1 n_1$ and $e_2 = \overline{g}_2 n_2$:

$$e_1 \cdot e_2 = (\overline{g}_1 n_1) \cdot (\overline{g}_2 n_2) = \overline{g}_1 \overline{g}_2 \cdot n_1^{\overline{g}_2} n_2.$$

The first component in the ordered pair representation of the product $e_1 \cdot e_2$ is $g = \rho(e_1 \cdot e_2) = g_1 g_2$. The second component (relative to the transversal, L) is thus

$$(\overline{g_1 g_2})^{-1} e_1 e_2 = (\overline{g_1 g_2})^{-1} \overline{g}_1 \overline{g}_2 \cdot n_1^{\overline{g}_2} n_2 = n_{(e_1, e_2)} \in N,$$

and the product $e_1 \cdot e_2$ is represented by the ordered pair $(g, n_{(e_1, e_2)})$.

The computation above illustrates the difficulty in formulating a feasible multiplication algorithm for elements of E written as ordered pairs. For any pair e_1, e_2 of elements of E, such an algorithm must be able to compute the corresponding $n_{(e_1, e_2)} \in N$ without performing any element arithmetic in E. This is achieved by precomputing and storing these values for a set of key elements, as described in the following section.

3.2 The Multiplication Algorithm

This section contains a detailed description of the multiplication method.

3.2.1 Strategy

The approach used to solve the multiplication problem is described here.

Keeping the notation of Subsection 3.1.2, let $S = \{x_1, \ldots, x_m\}$ be a non-redundant strong generating set relative to a base, $B = (\beta_1, \ldots, \beta_l)$, for the finite group, G. For each i, denote the i-th basic stabiliser relative to B by $G^{[i]}$ and denote the i-th basic orbit relative to B by $\Delta_i = \{\delta_{i,1}, \ldots, \delta_{i,d_i}\}$, where $\delta_{i,1} = \beta_i$. Additionally, let $S_i = S \cap G^{[i]} = \{x_{i_1}, \ldots, x_{i_{s_i}}\}$ and denote by S_i^{-1} the set, $\{x_{i_1}^{-1}, \ldots, x_{i_{s_i}}^{-1}\}$.

For each i, let U_i be a right transversal of $G^{[i+1]}$ in $G^{[i]}$. Then, as described in Subsection 2.1.3, each element of G can be represented uniquely as a product of transversal elements $u_l u_{l-1} \cdots u_1$ where $u_i \in U_i$.

Fix i and let $u \in U_i$ be the permutation taking β_i to $\delta_{i,j} \in \Delta_i$. Take $x \in S_i$, and let h, h' be the permutations in U_i which map β_i to $\beta_i^{ux} = \delta_{i,j}^{x}$, $\beta_i^{ux^{-1}} = \delta_{i,j}^{x^{-1}}$ respectively. Then

$$u \cdot x = y_1 y_2 \cdots y_k \cdot h, \tag{3.1a}$$

$$u \cdot x^{-1} = z_1 z_2 \cdots z_{k'} \cdot h', \tag{3.1b}$$

where $y_1, \ldots, y_k, z_1, \ldots, z_{k'}$ are elements of $S_{i+1} \cup S_{i+1}^{-1}$. The words, $y_1 y_2 \cdots y_k$ and $z_1 z_2 \cdots z_{k'}$, are called the *tails* of u relative to x and x^{-1} respectively.

Now regard G as a quotient of the larger group, E. Let L be a left transversal of N in E with $1_E \in L$ and, as in Subsection 3.1.2, for each $g \in G$, denote the unique

element of $g \cap L$ by $\overline{g}$. Then Equations (3.1) become

$$\overline{u} \cdot \overline{x} = \overline{y_1}\,\overline{y_2} \cdots \overline{y_k} \cdot \overline{h} \cdot n, \tag{3.2a}$$

$$\overline{u} \cdot \overline{x^{-1}} = \overline{z_1}\,\overline{z_2} \cdots \overline{z_{k'}} \cdot \overline{h'} \cdot n', \tag{3.2b}$$

respectively, for some $n, n' \in N$. The elements n and n' are called the *heads* of u relative to x and x^{-1} respectively and the Equations (3.2) are called the *shift* equations.

The shift equations suggest a scheme by which elements of E may be multiplied. Suppose that the elements, $\overline{u_i}$ (where $u_i \in U_i$ for each i), and the tails, $y_1 y_2 \cdots y_k$, $z_1 z_2 \cdots z_{k'}$, can be computed consistently. Furthermore, suppose that the conjugate of each element of the normal subgroup, N, by elements, $\overline{x}$ and $\overline{x^{-1}}$ (where $x \in S$), can be calculated without performing any element arithmetic in E. Then the shift equations can be utilised to design an iterative function which performs multiplications of the form:

$$(\overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_1} n_1) \cdot (\overline{x} n_2)$$

where $n_1, n_2 \in N$, conjugating within N whenever necessary. This method can be extended to handle element multiplications in full generality.

The tasks of conjugation in N and calculation of the elements, $\overline{u_i} \in L$, are performed via a precomputed data set, the specification of which forms the content of the next subsection.

3.2.2 Data

The multiplication algorithm relies directly on a precomputed set of data to calculate the elements of the normal polycyclic subgroup that arise when elements of the chosen transversal are shifted past strong generator preimages in the polycyclic-by-

finite group. A technical description of the contents of this data set is given in this subsection.

The Finite Quotient

The first segment of required data is computed wholly within the finite quotient G of the given polycyclic-by-finite group. For each $i \in \{1, \ldots, |B|\}$ and each $\delta_{i,j} \in \Delta_i$, the tails of u, the permutation in U_i mapping β_i to $\delta_{i,j}$, relative to each $x \in S_i$ and $x^{-1} \in S_i^{-1}$ are calculated and stored.

In the implementation, a fixed Schreier vector data structure is used to encode the transversal elements, $u_i \in U_i$, as words over $S \cup S^{-1}$. Since this data structure remains unchanged, a given permutation in G can be written as a word over $S \cup S^{-1}$ consistently by using the set of stored Schreier vectors to expressing said permutation in normal form $u_l u_{l-1} \cdots u_1$ (where $u_i \in U_i$), storing each u_i as an integer sequence representing a word over $S \cup S^{-1}$.

Tails are computed for each $i \in \{1, \ldots, |B|\}$ and each element in Δ_i, for each applicable strong generator. It follows from the inequalities,

$$|B| \leqslant \log_2 |G|,$$

$$|\Delta_i| \leqslant n$$

and

$$|S_i| \leqslant |S|,$$

that the number of tails is $O(|S| n \log_2 |G|)$. The calculation of each tail requires one application of the sifting procedure, which has time complexity $O(n \log_2 |G|)$ (see Subsection 2.1.3). Hence, the time complexity of computing all tails is $O(|S|(n \log_2 |G|)^2)$.

A tail is formed by concatenating $|B|$ integer sequences, each of length less than n, and so, each tail requires $O(n \log_2 |G|)$ storage, hence the space complexity

associated with storing all tail elements is identical to its time complexity, namely $O(|S|(n\log_2|G|)^2)$.

All algorithms presented in this chapter, and indeed in the rest of the thesis, assume the existence of a method which operates as outlined to write permutations of G as words over $S \cup S^{-1}$, and the tails of each u, relative to each x, x^{-1} are computed in a consistent manner using this method. This uniformity is critical when passing from the quotient, G, to the polycyclic-by-finite group, E.

The Polycyclic-by-Finite Group

Three distinct sectors of data are computed by performing element arithmetic within the polycyclic-by-finite group in question, using its existing representation.

Firstly, the heads of the shift equations are computed and recorded. In order to do so consistently, an explicit definition of the transversal, L, is required. The construction is as follows. For each $x \in S$, the element $\overline{x} \in L$ is fixed by choosing exactly one member of each corresponding preimage set $\rho^{-1}(x)$. The set

$$\overline{S} = \{\overline{x} \mid x \in S\}$$

is held in memory and is used to define the inverse image set

$$\overline{S^{-1}} = \{\overline{x^{-1}} \mid x \in S\}$$

by the equation:

$$\overline{x^{-1}} = \begin{cases} \overline{x} & \text{if } x^2 = 1, \\ \overline{x}^{-1} & \text{otherwise.} \end{cases}$$

Identifiying strong generators of order 2 as a special case is necessary to eliminate the possibility of a strong generator being assigned two distinct images in L which would violate uniqueness of coset representatives in a transversal.

The remaining elements of L are calculated when required using the sets $\overline{S}$ and $\overline{S^{-1}}$, as prescribed by the Schreier vector data structure held in memory. Given a permutation $g \in G$, the transversal element $\overline{g} \in L$ is computed by first expressing g as a word over $S \cup S^{-1}$ (via the Schreier vector data structure), and then multiplying through this word, substituting recorded values from the sets $\overline{S}$ and $\overline{S^{-1}}$.

As discussed above, the Schreier vector data structure furnishes one with a method by which a permutation in G can be consistently represented as word over $S \cup S^{-1}$. This ensures that the method proposed here for computing transversal elements, in turn, produces consistent results.

For each u, x pair as above, n and n' of Equations 3.2 are then calculated in the obvious manner:

$$n = \overline{h}^{-1} \cdot \overline{y_k}^{-1}\,\overline{y_{k-1}}^{-1} \cdots \overline{y_1}^{-1} \cdot \overline{u} \cdot \overline{x}, \tag{3.3a}$$

$$n' = \overline{h'}^{-1} \cdot \overline{z_{k'}}^{-1}\,\overline{z_{k'-1}}^{-1} \cdots \overline{z_1}^{-1} \cdot \overline{u} \cdot \overline{x^{-1}}. \tag{3.3b}$$

The products above are evaluated using the existing representation for E.

Each tail is of length $O(n \log_2 |G|)$. Thus, the calculation of the heads, n and n', above require $O(n \log_2 |G|)$ multiplications using the existing representation of E; for all of the $O(|S| n \log_2 |G|)$ tails, the total time cost is therefore $O(|S|(n \log_2 |G|)^2)$.

This step of data collection stores one element of the polycyclic group, N, for each of the $O(|S| n \log_2 |G|)$ tails. Assuming that an element of N requires $O(r)$ storage, the memory cost here is $O(|S| r n \log_2 |G|)$.

The second portion of data computed within the polycyclic-by-finite group relates to conjugation of elements of the normal subgroup N by elements of E. During its operation, the multiplication algorithm frequently calls for conjugations of the form n^w, where $n \in N$ and w is a word over $\overline{S} \cup \overline{S^{-1}}$. Thus, it is necessary that such elements be computable without appealing to the original representation of the

group E. To enable such functionality, the conjugates

$$a^{\overline{x}}, a^{\overline{x^{-1}}} \in N$$

are computed and stored for each polycyclic generator, a, of the stored polycyclic presentation for N, and each $x \in S$. This information is sufficient to devise a straightforward iterative procedure which calculates conjugates, n^w, without performing element arithmetic in E. During this step, $O(r|S|)$ multiplications are performed using the existing representation of E, where r is the size of the stored polycyclic generating sequence for N. Assuming $O(r)$ storage for each of the $O(r|S|)$ elements of N, the total memory requirement is $O(r^2|S|)$.

Finally, sets, $\hat{S}$ and $\hat{S}^{-1}$, representing the elements of $\overline{S}$ and $\overline{S^{-1}}$ respectively in normal form are stored. In this case, these are:

$$\hat{S} = \{(y, 1_N) : y \in S\}, \tag{3.4a}$$

$$\hat{S}^{-1} = \{(y, 1_N) : y \in S^{-1}\}. \tag{3.4b}$$

Computing an element of $\hat{S}$ or $\hat{S}^{-1}$ requires one application of the sifting procedure, which has time complexity $O(n \log_2|G|)$, so the time complexity associated with this step of data collection is $O(|S| n \log_2|G|)$.

The first component of the normal form for an element of E consists of $|B|$ integer sequences, each of length less than n, and thus requires $O(n \log_2|G|)$ storage. The second component of the normal form is an element of N, for which it is assumed that $O(r)$ storage is necessary. Thus, $\hat{S}$ and $\hat{S}^{-1}$ require $O(|S|(r + n \log_2|G|))$ storage.

3.2.3 The Algorithm

Equipped with the data set of Subsection 3.2.2, it is possible to design an algorithm which, when given two elements of E written in the normal form defined in Sub-

section 3.1.1, computes the normal form of their product without performing any extemporaneous element arithmetic in E.

Let e_1 and e_2 be two elements of E with normal form representations, (g_1, n_1) and (g_2, n_2), respectively. Keeping the notation of Subsection 3.2.1, write e_1 as

$$(\overline{u_l}\,\overline{u_{l-1}}\cdots\overline{u_1})\cdot n_1$$

where $u_i \in U_i$ for each i, and $n_1 \in N$. Let $e_2 = \overline{g}_2 n_2$ where $n_2 \in N$, $\overline{g}_2 \in L$.

The multiplication algorithm initially expresses $\overline{g}_2$ as a word over $\overline{S} \cup \overline{S^{-1}}$, say $w = \overline{q_1}\,\overline{q_2}\cdots\overline{q_\sigma}$ where $q_j \in S \cup S^{-1}$ for each j. The desired product may then be written as

$$e_1 \cdot e_2 = (\overline{u_l}\,\overline{u_{l-1}}\cdots\overline{u_1})\cdot n_1 \cdot (\overline{q_1}\,\overline{q_2}\cdots\overline{q_\sigma})\cdot n_2$$

The algorithm proceeds to rearrange the terms of the product by conjugating the element n_1 by the word w. As discussed in Subsection 3.2.2, this conjugation is performed using a precomputed data set, with no element arithmetic carried out in E. The product then becomes

$$e_1 \cdot e_2 = \underbrace{(\overline{u_l}\,\overline{u_{l-1}}\cdots\overline{u_1})}\cdot\overline{q_1}\,\overline{q_2}\cdots\overline{q_\sigma}\cdot n_1^{\overline{q_1}\,\overline{q_2}\cdots\overline{q_\sigma}} n_2 \tag{3.5}$$

$$= \overline{u_l}\,\overline{u_{l-1}}\cdots\overline{u_2}\,\underbrace{\overline{u_1}\cdot\overline{q_1}}\,\overline{q_2}\cdots\overline{q_\sigma}\cdot n_1^w n_2. \tag{3.6}$$

Note 3.1. Initially, the leftmost sequence of transversal elements, $\overline{u_i}$, in the expression for the product (under-bracketed in Equation (3.5)) has l terms.

The under-bracketed segment of Equation (3.6) can be recognised as the left-hand side of a shift equation, say

$$\overline{u_1}\cdot\overline{q_1} = \overline{y_1}\,\overline{y_2}\cdots\overline{y_\kappa}\cdot\overline{h_1}\cdot n_3 \tag{3.7}$$

for some $n \in N$, $h_1 \in U_1$ and where $y_j \in S_2 \cup S_2^{-1}$ for each j. Via direct substitution, the Equation (3.7) can be used to "shift" the transversal element, $\overline{q_1}$, past $\overline{u_1}$ without performing any element arithmetic in the group E. The algorithm executes this shifting procedure as illustrated below.

$$\begin{aligned} e_1 \cdot e_2 &= \overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_2}\; \underbrace{\overline{u_1} \cdot \overline{q_1}}_{\text{replace by } \overline{y_1}\,\overline{y_2} \cdots \overline{y_\kappa} \cdot \overline{h_1} \cdot n_3}\; \overline{q_2} \cdots \overline{q_\sigma} \cdot n_1^w n_2 \\ &= \overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_2} \cdot (\overline{y_1}\,\overline{y_2} \cdots \overline{y_\kappa} \cdot \overline{h_1} \cdot n_3) \cdot \overline{q_2}\,\overline{q_3} \cdots \overline{q_\sigma} \cdot n_1^w n_2 \\ &= \overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_2} \cdot \overline{y_1}\,\overline{y_2} \cdots \overline{y_\kappa} \cdot \underline{\overline{h_1} \cdot \overline{q_2}}\; \overline{q_3} \cdots \overline{q_\sigma} \cdot n_3^{\overline{q_1}^{-1} w} n_1^w n_2 \end{aligned} \tag{3.8}$$

The final rearrangement in Equation (3.8) is the result of conjugating n_3 by

$$\overline{q_2}\,\overline{q_3} \cdots \overline{q_\sigma} = \overline{q_1}^{-1} w.$$

The under-bracketed segment of the product in Equation (3.8) can again be recognised as the left-hand side of a shift equation. The algorithm repeats the procedure above to shift $\overline{q_2}$ past $\overline{h_1}$.

Observation 3.2. Each time a shift is made, the word over $\overline{S} \cup \overline{S^{-1}}$ immediately to the left of the sequence of normal subgroup elements in the expression for the product is reduced by exactly one symbol.

In light of Observation 3.2, the algorithm is able to continue iterating through the word, w, shifting at every step, arriving at the following equation:

$$\begin{aligned} e_1 \cdot e_2 &= \overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_2} \cdot \overline{y_1}\,\overline{y_2} \cdots \overline{y_\zeta} \cdot \overline{u_1'} \cdot n_{\sigma+2} n_{\sigma+1}^{\overline{q_\sigma}} \cdots n_3^{\overline{q_1}^{-1} w} n_1^w n_2 \\ &= \underline{\overline{u_l}\,\overline{u_{l-1}} \cdots \overline{u_2}} \cdot \overline{y_1}\,\overline{y_2} \cdots \overline{y_\zeta} \cdot \overline{u_1'} \cdot n', \end{aligned} \tag{3.9}$$

for some $u_1' \in U_1$, $n_3, \ldots, n_{\sigma+2} \in N$; where $y_j \in S_2 \cup S_2^{-1}$ for each j and

$$n' = n_{\sigma+2} n_{\sigma+1}^{\overline{q_\sigma}} \cdots n_3^{\overline{q_1}^{-1} w} n_1^w n_2.$$

At this stage, the algorithm has decreased the under-bracketed sequence of transversal elements, $\overline{u_i}$, in Equation (3.5) by one term, as indicated in Equation (3.9), thus reducing the problem to a smaller case. The algorithm restarts its inner loop to process the word, $\overline{y_1}\,\overline{y_2}\cdots\overline{y_\zeta}$, by, as before, recognising $\overline{u_2}\cdot\overline{y_1}$ as the left-hand side of a shift equation.

Observation 3.3. Each time a word over $\overline{S}\cup\overline{S^{-1}}$ is processed fully as described above, the leftmost sequence of transversal elements, $\overline{u_i}$, in the expression for the product is decreased by exactly one term.

The word-processing procedure is repeated for each $\overline{u_i}$ yielding an expression of the form

$$\overline{u'_l}\,\overline{u'_{l-1}}\cdots\overline{u'_1}\cdot n''$$

where $u'_i \in U_i$ for each i, for the product.

Remark 3.4. Whenever a shift is made at level i (with respect to the base and strong generating set hierarchy), the elements of $\overline{S}\cup\overline{S^{-1}}$ that are placed to the left of the new element, $\overline{h_i}$ (where $h_i \in U_i$), belong to $\overline{S}_{i+1}\cup\overline{S^{-1}_{i+1}}$. In particular, since $S_{l+1}=\varnothing$, no non-trivial elements of $\overline{S}\cup\overline{S^{-1}}$ are placed to the left of any possible $\overline{h_l}$ where $h_l \in U_l$.

The multiplication method is presented in Algorithm 3.1. The function takes as input two elements, (g_1,n_1), (g_2,n_2), written in normal form, of a polycyclic-by-finite group, E, and operates as described above to compute the normal form for the product $(g_1,n_1)\cdot(g_2,n_2)$.

The notation employed thus far shall be used in the description of the algorithm, viz. the normal polycyclic subgroup of E and associated quotient, via which the normal form is defined, shall be denoted by N and G respectively. It is assumed during execution that the data set of Subsection 3.2.2 is held in memory, with the tails of the shift equations stored as arrays of strong generators. The base

and strong generating set relative to which the data is computed are denoted by $S = \{x_1, \ldots, x_m\}$ and $B = (\beta_1, \ldots, \beta_l)$ respectively, and the left transversal of N in E by L.

The algorithm also assumes the existence of a function LENGTH which, when supplied with an array (or sequence), returns the number of non-null entries of that array (or sequence).

The central step in measuring the run-time of MULTIPLY is counting the number of times that Lines 16–21 are executed. To calculate this, one must establish a bound for LENGTH(*rightword*), for each of the l iterations of the outer loop. Such a bound is derived in the proof of Theorem 3.5. The most expensive computations within Lines 16–21 occur in Line 18 and in Line 20. Line 18 consists of an application of a straightforward iterative procedure to compute the conjugate of an element of N by an element of $\overline{S} \cup \overline{S^{-1}}$ using the precomputed data set (see Subsection 3.2.2). Line 20 is a standard Schreier vector computation in G, whose time complexity is discussed in Subsection 2.1.2. The operations of Line 18 and Line 20 dominate the memory retrievals and variable reassignment in Lines 16–21, and so, in Theorem 3.5, the time complexity of MULTIPLY is stated in terms of the number of multiplications that must be performed within the finite quotient, and conjugations of elements of the normal polycyclic subgroup by preimages of strong generators (and inverses).

Theorem 3.5. *The multiplication algorithm terminates with the correct value for the desired product, and requires $O(n(n\log_2|G|)^{\log_2|G|})$ multiplications using the permutation representation of the finite quotient, G, and $O((n\log_2|G|)^{\log_2|G|})$ conjugations of elements of the normal polycyclic subgroup by strong generator preimages, where n is the degree of the permutation representation of G.*

Proof. Termination is guaranteed by Observations 3.2 and 3.3 combined with Remark 3.4, while Equations (3.5)–(3.9) imply correctness.

The rate of growth of *rightword* is investigated as follows. Subsection 3.2.2 shows

Algorithm 3.1 Element Multiplication

1: **function** MULTIPLY((g_1, n_1), (g_2, n_2))
2:     **if** $g_2 = 1_G$ **then**
3:         **return** $(g_1, n_1 n_2)$
4:     **end if**
5:     Write g_1 in normal form $u_l u_{l-1} \cdots u_1$ where $u_i \in U_i$ for each i
6:     Write g_2 as a word, $w = q_1\, q_2 \cdots q_t$, where $q_i \in S \cup S^{-1}$ for each i
7:     *rightword* $\leftarrow [q_1,\, q_2,\, \ldots,\, q_t]$
8:     *leftword* $\leftarrow$ []
9:     $n'_1 \leftarrow n_1$
10:     **for** $i \leftarrow 2$ **to** l **do**
11:         $n'_i \leftarrow 1_N$
12:     **end for**
13:     **for** $i \leftarrow 1$ **to** l **do**
14:         $u'_i \leftarrow u_i$
15:         **for** $j \leftarrow 1$ **to** LENGTH(*rightword*) **do**
16:             Retrieve from memory the tail array, *tailword*, and tail element, n_σ, of the shift equation corresponding to the pair: u'_i, *rightword*$[j]$
17:             *leftword* $\leftarrow$ *leftword* cat *tailword*
18:             Find the conjugate, n_γ, of n'_i by the image of *rightword*$[j]$ in L
19:             $n'_i \leftarrow n_\sigma \cdot n_\gamma$
20:             Find $h \in U_i$ which maps $\beta_i^{u'_i}$ to the image of $\beta_i^{u'_i}$ under *rightword*$[j]$
21:             $u'_i \leftarrow h$
22:         **end for**
23:         *rightword* $\leftarrow$ *leftword*
24:         *leftword* $\leftarrow$ []
25:     **end for**
26:     $g_* \leftarrow 1_G$, $n_* \leftarrow 1_N$, $i \leftarrow l$
27:     **while** $i > 0$ **do**
28:         $n_* \leftarrow n_*^{\overline{u'_i}} \cdot n'_i$ ▷ The conjugation $n_*^{\overline{u'_i}}$ is performed using the data set
29:         $g_* \leftarrow g_* \cdot u'_i$
30:         $i \leftarrow i - 1$
31:     **end while**
32:     $n_* \leftarrow n_* \cdot n_2$
33:     **return** (g_*, n_*)
34: **end function**

that, in the first iteration of the outer loop of MULTIPLY, *rightword* has length at most $n \log_2|G|$. As illustrated in Subsection 3.2.3, each shift made in the inner loop of the algorithm appends a tail of length at most $n \log_2|G|$ to the variable, *leftword*. This operation is performed in Line 17. Since a shift is made for each element of *rightword*, it follows that, at the end of the first iteration of the outer loop, *leftword* has length at most

$$\textsc{Length}(\mathit{rightword}) \cdot n \log_2|G| \leqslant (n \log_2|G|)^2.$$

Hence, in the second iteration of the outer loop, *rightword* has length at most $(n \log_2|G|)^2$. Applying an similar argument, one may deduce that, in the third iteration of the outer loop *rightword* has length at most $(n \log_2|G|)^3$, and, in general, *rightword* has length at most $(n \log_2|G|)^i$ in the i-th iteration of the outer loop.

Thus, in a run of MULTIPLY, Lines 16–21 are executed no more than

$$\sum_{i=1}^{l} (n \log_2|G|)^i$$

times, where l is the length of the base of the finite quotient, G.

By virtue of the inequality $l \leqslant \log_2|G|$, the sum, $\sum_{i=1}^{l} (n \log_2|G|)^i$ is $O((n \log_2|G|)^{\log_2|G|})$.

It follows immediately that MULTIPLY performs $O((n \log_2|G|)^{\log_2|G|})$ conjugations of polycyclic group elements by strong generator preimages.

As discussed in Subsection 2.1.2, the Schreier vector computation of Line 20 requires $O(n)$ multiplications in the finite quotient, G. Thus a run of MULTIPLY requires a total of $O(n(n \log_2|G|)^{\log_2|G|})$ multiplications in G. □

In light of the exponential growth (illustrated in the proof Theorem 3.5) of the words through which MULTIPLY must iterate, small base representations for the quotient group are highly desirable for effective performance of the multiplication

algorithm. As discussed in Subsection 1.2.3 of Chapter 1, in constructing the representation for the given polycyclic-by-finite group, one attempts to "factor out" the largest possible N, minimising $|G|$ and consequently restricting the size of any base B for G via the inequality $|B| \leqslant \log_2 |G|$ (see Subsection 2.1.3). Finding a quotient which admits a base of moderate size is a critical safeguard against the potential run-time cost associated with the accumulation of long words.

In practice, sparing cases where an injudicious choice is made for N (and hence G), the arrays that appear are of manageable span, yielding positive empirical results.

Multiplication of elements written in normal form shall hereinafter be indicated with the use of standard infix notation, omitting the '$\cdot$' operator when context permits.

3.3 Applications

With element arithmetic in place, one may design straightforward functions to perform standard operations on an element of E written in normal form, such as finding its image in the transversal, L, computing its inverse and calculating its order. Methods to perform these tasks are discussed in Subsections 3.3.1–3.3.3.

In many applications involving subgroups of polycyclic-by-finite groups, one often encounters subgroups that are polycyclic. In such situations, a polycyclic presentation for the subgroup in question is desirable, for this would allow one to utilise the host of well-developed algorithms available for polycyclic groups. Subsection 3.3.4 describes a method which finds a consistent polycyclic (or power-conjugate) presentation for a polycyclic-by-finite group that is in fact polycyclic.

3.3.1 Transversal Images

Given an element, $g \in G$, the normal form of the element, $\overline{g} \in L$, is computed according to the construction of L given in Subsection 3.2.2, using the sets $\hat{S}$ and $\hat{S}^{-1}$ defined in Equations (3.4).

To find $\overline{g}$, one first expresses g as a word, $y_1 \cdots y_k$, over $S \cup S^{-1}$, by the procedure described in Subsection 3.2.2. Then the element, $\hat{y}_i \in \hat{S} \cup \hat{S}^{-1}$, corresponding to y_i is retrieved from memory. The normal form for $\overline{g}$ is then given by the product:

$$\hat{y}_1 \cdots \hat{y}_k.$$

Writing the elements of L in their normal form is useful from a technical perspective, as it provides one with a method of inverting the natural map, ρ.

As in Subsection 3.2.2 and the proof of Theorem 3.5, the integer, k, is bounded above by $n \log_2 |G|$, so, finding a transversal image requires $O(n \log_2 |G|)$ memory retrievals and $O(n \log_2 |G|)$ applications of MULTIPLY.

3.3.2 Element Inversion

Let (g, n_1) be an element of E, written in normal form. To compute $(g, n_1)^{-1}$, first use the procedure outlined in Subsection 3.3.1 to write $\overline{g^{-1}}$ in normal form, say (g^{-1}, n_2). Then

$$(g, n_1) \cdot (g^{-1}, n_2) = (1_G, n_3),$$

and hence

$$\begin{aligned}(g, n_1)^{-1} &= (g^{-1}, n_2) \cdot (1_G, n_3)^{-1} \\ &= (g^{-1}, n_2) \cdot (1_G, n_3^{-1}) \\ &= (g^{-1}, n_2 n_3^{-1}).\end{aligned}$$

The inversion procedure performs exactly one more multiplication after writing

$\overline{g^{-1}}$ in normal form. Thus, the time complexity associated with inverting an element is identical to that of the method described in Subsection 3.3.1, namely $O(n \log_2 |G|)$ multiplications.

3.3.3 Orders of Elements

Let (g, n) be an element of E, written in normal form. To calculate $o(g, n)$, first compute the power

$$(g, n)^{o(g)} = (1_G, n').$$

Then

$$o(g, n) = o(g)o(n').$$

Using the method of repeated squaring to compute powers, one easily observes that finding the order of an element requires $O(\log_2 |G|)$ applications of Multiply.

3.3.4 Transfer to the Category of Polycyclic Groups

Suppose that the polycyclic-by-finite group, E, is in fact polycyclic. Then G is polycyclic and one may use the already available methods for permutation groups to compute a power-conjugate presentation for G. A consistent polycyclic or power-conjugate presentation for E may be found by "glueing together" the presentations for G and N as described below.

For the sake of clarity, it shall be assumed in the following discussion that N (and hence E) is finite, and a power-conjugate presentation for E shall be constructed; a similar procedure may be employed in the infinite case.

Let N and G have consistent power-conjugate presentations,

$$\begin{aligned} \langle a_1, \ldots, a_r \mid a_j^{p_j} &= w_{j,j} \quad \text{for } 1 \leqslant j \leqslant r, \\ a_j^{a_i} &= w_{i,j} \quad \text{for } 1 \leqslant i < j \leqslant r \rangle \end{aligned}$$

and

$$\langle b_1, \ldots, b_s \mid b_j^{q_j} = v_{j,j} \quad \text{for } 1 \leqslant j \leqslant s,$$
$$b_j^{b_i} = v_{i,j} \quad \text{for } 1 \leqslant i < j \leqslant s \rangle$$

respectively, where

(i) p_j, q_j are the least primes such that $a_j^{p_j} \in \langle a_{j+1}, \ldots, a_r \rangle$ for $j < r$ and $b_j^{q_j} \in \langle b_{j+1}, \ldots, b_s \rangle$ for $j < s$, and $a_r^{p_r}$, $b_s^{q_s}$ are the identity elements in N, G respectively, and

(ii) $w_{i,j}$, $v_{i,j}$ are words in the generator sets $\{a_{i+1}, \ldots, a_r\}$, $\{b_{i+1}, \ldots, b_s\}$ respectively.

A power-conjugate presentation for E can be constructed on the set of generators $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$ as follows. Notation is abused here in regarding the b_i both as elements of E and G, however, the meaning shall be clear from the context as explained below.

For each b_i in the power-conjugate presentation for G, the method of Subsection 3.3.1 is used to find the normal form of $\overline{b_i}$, say (b_i, n), after which, the multiplication algorithm is used to compute the power, $(b_i, n)^{q_i} = (b_i^{q_i}, n') = (v_{i,i}, n')$. The power-conjugate presentation for N may be used to express n' as a word, $a_1^{\alpha_1} \cdots a_r^{\alpha_r}$, and thus one may (with abuse of notation) write the power relation:

$$b_i^{q_i} = v_{i,i} a_1^{\alpha_1} \cdots a_r^{\alpha_r}.$$

Power relations with left-hand-side $a_i^{p_i}$ remain unchanged in the new presentation.

Conjugate relations are derived in a similar manner; the normal form of each element in $\{\overline{b_1}, \ldots, \overline{b_s}, \overline{a_1}, \ldots, \overline{a_r}\}$ is found (the normal form of $\overline{a_i}$ is simply $(1_G, a_i)$), and the multiplication algorithm is used to perform the required conjugations according to the hierarchy induced by G and N. The resulting elements are expressed

as words over $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$ as above.

Proposition 3.6. *The presentation so obtained is a consistent power-conjugate presentation for E.*

Proof. The presentation constructed clearly satisfies the conditions outlined in the definition of a power-conjugate presentation given in Subsection 2.2.1. That it is consistent follows immediately from the fact that the presentations for N and G above are consistent. It remains now to show that the presentation constructed indeed defines a group isomorphic to E.

Let $E^\star$ be the group defined by the presentation constructed over the generating set $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$.

Define the map, $\psi\colon \{b_1, \ldots, b_s, a_1, \ldots, a_r\} \to \{\overline{b_1}, \ldots, \overline{b_s}, \overline{a_1}, \ldots, \overline{a_r}\}$, by:

$$\psi(b_i) = \overline{b_i}, \tag{3.10}$$

$$\psi(a_j) = \overline{a_j}, \tag{3.11}$$

for $i = 1, \ldots, s$, $j = 1, \ldots, r$. By construction, every relation in the presentation for $E^\star$ over $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$ also holds in E, with the symbol, b_i, replaced by $\psi(b_i)$ and the symbol, a_j, replaced by $\psi(a_j)$. Since $\{\overline{b_1}, \ldots, \overline{b_s}, \overline{a_1}, \ldots, \overline{a_r}\}$ generates E, ψ may be extended in the obvious manner to a homomorphism from $E^\star$ onto E. That is, E is a homomorphic image of $E^\star$. It shall now be shown that ψ is in fact an isomorphism.

Suppose that

$$\psi(b_1^{\beta_1} \cdots b_s^{\beta_s} a_1^{\alpha_1} \cdots a_r^{\alpha_r}) = 1,$$

for some $\beta_1, \ldots, \beta_s$, $\alpha_1, \ldots, \alpha_r$, with $0 \leqslant \beta_i < q_i$ for each i, and $0 \leqslant \alpha_j < p_j$ for each j. Then,

$$\overline{b_1}^{\beta_1} \cdots \overline{b_s}^{\beta_s} \cdot \overline{a_1}^{\alpha_1} \cdots \overline{a_r}^{\alpha_r} = 1. \tag{3.12}$$

Equation (3.12) implies that $\overline{b_1}^{\beta_1} \cdots \overline{b_s}^{\beta_s} N$ is the identity in the quotient, $G = E/N$.

By construction, the element $\overline{b_i}N$ corresponds to the element b_i in the consistent power-conjugate presentation for G. By uniqueness of words written in normal form,

$$\beta_1 = \beta_2 = \cdots = \beta_s = 0.$$

This reduces Equation (3.12) to

$$\overline{a_1}^{\alpha_1} \cdots \overline{a_r}^{\alpha_r} = 1.$$

Again, by construction, $\overline{a_i}$ corresponds to element a_i in the consistent power-conjugate presentation for N. So by uniqueness,

$$\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0.$$

Thus, ψ is injective, and $E \cong E^\star$ as required. □

Data facilitating computation of the isomorphism for transfer of elements between the two representations consists of a list containing the ordered pair normal form of each element of the set, $\{\overline{b_1}, \ldots, \overline{b_s}, \overline{a_1}, \ldots, \overline{a_r}\}$, together with a list of words over $\{b_1, \ldots, b_s, a_1, \ldots, a_r\}$ defining the images of the elements, $\overline{y}$, where $y \in S \cup S^{-1}$.

Chapter 4

Subgroups

Virtually every structural computation performed within a group involves the manipulation of subgroups. Thus, in order to obtain useful results pertaining to a given polycyclic-by-finite group, one requires an efficient method by which subgroups may be computed.

This chapter contains a detailed description of the subgroup generation method developed for the class of polycyclic-by-finite groups.

Firstly, the complications involved in computing and representing subgroups of polycyclic-by-finite groups are discussed. The strategy used to overcome these difficulties is presented, after which, the procedure by which subgroups are constructed is described in detail. The chapter then proceeds to address the important question of membership testing in subgroups. Finally, a selection of straightforward applications of the subgroup generation method is given.

The theory developed in this chapter, and indeed the rest of the thesis, requires that the polycyclic-by-finite group from which subgroups are created is represented as an extension of a normal polycyclic subgroup by a finite group as described in Chapter 3, and that the multiplication algorithm is applicable. Thus, in what follows, it shall be assumed that all parent polycyclic-by-finite groups are so represented.

4.1 The Subgroup Generation Problem

The first step in defining the parameters of the subgroup generation problem involves choosing a mode of representation for subgroups. There is a logical option in the case of polycyclic-by-finite groups, for this class of groups is closed under formation of subgroups, and hence it is possible, at least in theory, to represent generic subgroups using the scheme developed in Chapter 3.

With this choice for the representation, generating a subgroup entails computing a data set which contains sufficient information to apply the multiplication technique of Chapter 3 wholly within the context of that subgroup. The problem may be formulated more concretely as follows:

Let E be a polycyclic-by-finite group, let $N \lhd E$ be polycyclic of finite index in E and denote the quotient, E/N, by G. Given a set of generators, written in normal form relative to G and N, of a subgroup, $H \leqslant E$, it is required to compute the following information relating to H:

(i) The normal polycyclic subgroup, $N_H = H \cap N$, of H.

(ii) A base, B_H, and strong generating set, T, relative to B_H, for the quotient, $G_H = H/N_H \cong HN/N$.

(iii) Elements of E representing images of T in a left transversal, L_H, of N_H in H.

(iv) Conjugates of each polycyclic generator of N_H by the images of T in L_H.

(v) Data corresponding to Equations (3.1) and (3.2) in the context of H.

Items (iv) and (v) are calculated in a manner similar to that of the parent group; the details of the respective calculations are given in Subsection 4.2.2.

The difficulty lies in computing (i) and (iii). The approach taken is to extend the well-known Schreier–Sims method described in Subsection 2.1.3 of Chapter 2, to efficiently compute the segments of data described in (i), (ii) and (iii) simultaneously.

The newly developed version of the Schreier–Sims method, which forms the core of the subgroup generation procedure, is discussed in the next subsection.

In light of the application of the Second Isomorphism Theorem in (ii), the quotient, G_H may be regarded as a subgroup of the quotient, G, for which there is a known faithful permutation representation, thus facilitating computation of the required base and strong generating set. For the rest of this chapter, G_H will be identified with an isomorphic copy in HN/N.

4.2 Construction

The technical details of subgroup construction are discussed in this section.

4.2.1 The Extended Schreier–Sims Algorithm

The extended version of the Schreier–Sims algorithm is described in detail here. Let E, N and G be as in Section 4.1. Given a set of elements of E that generate a subgroup, $H \leqslant E$, the extended Schreier–Sims algorithm aims to compute the intersection $H \cap N$, together with a base for the quotient, HN/N, and a set of elements of E (or more precisely HN) whose images in HN/N define a strong generating set relative to the computed base.

The Permutation Action of the Polycyclic-by-Finite Group

Manipulating the elements of E relative to the (not faithful) permutation action of E induced by that of the quotient, G, is central to the operation of the extended Schreier–Sims algorithm.

Denote by Ω the set of points on which G acts. Given an element, $(g, n) \in E$, and a point, $\omega \in \Omega$, one may unambiguously speak of the action of (g, n) on ω, with obvious meaning, viz.

$$\omega^{(g,n)} = \omega^g.$$

The fundamental algorithms for orbit computation can be applied without modification in the case of this permutation action. Specifically, it is possible, using the methods described in Subsection 2.1.2 of Chapter 2, to compute the orbit of a point, $\omega \in \Omega$, under a set, $X \subseteq E$, of elements along with a Schreier vector encoding a transversal of the stabiliser, $\langle X \rangle_\omega$, in $\langle X \rangle$. It is important to note here that the normal subgroup component of each element encoded by such a Schreier vector is not truncated, i.e. the transversal elements are expressed as products over X, not over the image of X in G.

The definitions given in Subsection 2.1.3 of Chapter 2 for a base and a strong generating set may be reformulated to fit the context of this class of groups.

A sequence, $B_H = (\gamma_1, \dots, \gamma_k)$, of elements belonging to Ω is called a *base* for the subgroup, H, of E if every element of H that fixes B_H pointwise belongs to $H \cap N$. The sequence, B_H, defines a stabiliser subgroup chain

$$H = H^{[1]} \geqslant H^{[2]} \geqslant \cdots \geqslant H^{[k]} \geqslant H^{[k+1]} = H \cap N \qquad (4.1)$$

where $H^{[i]} = H_{(\gamma_1,\dots,\gamma_{i-1})}$ $(i > 1)$ is the pointwise stabiliser of $\{\gamma_1, \dots, \gamma_{i-1}\}$. The base, B_H, is called *non-redundant* if $H^{[i+1]} < H^{[i]}$ for all $i = 1, \dots, k$. The orbits, ${\gamma_i}^{H^{[i]}}$, are called the *basic orbits* or *fundamental orbits* of H (relative to B_H).

A *strong generating set* for H relative to B_H is a generating set, S_H, for H with the property that

$$\langle S_H \cap H^{[i]} \rangle = H^{[i]} \qquad (4.2)$$

for $i = 1, \dots, k + 1$.

In the language of the definitions made above, the objective of the extended Schreier–Sims method may be stated more simply: given a set of generators of a subgroup, $H \leqslant E$, the method attempts to compute a base and a strong generating set for H.

Before turning to the central problem of how a base and a strong generating set

for a subgroup of a polycyclic-by-finite group are constructed, it shall be assumed for the moment that these are given, and a version of the sifting procedure for polycyclic-by-finite groups shall be introduced.

Let S_H be a strong generating set for the subgroup H, with associated base, $B_H = (\gamma_1, \ldots, \gamma_k)$. It is assumed that the basic orbits, $\Theta_i = {\gamma_i}^{H^{[i]}}$, have been calculated and that Schreier vectors encoding transversals, R_i, of $H^{[i+1]}$ in $H^{[i]}$ exist for each i, with transversal elements at level i represented as products over $S_H \cap H^{[i]}$.

The sifting algorithm for polycyclic-by-finite groups operates as follows. Given $(g, n) \in E$, the algorithm attempts to find the coset representative $(y_1, n_1') \in R_1$ such that $\gamma_1^{(g,n)} = \gamma_1^{(y_1,n_1')}$, and computes $(g_2, n_2) = (g, n)(y_1, n_1')^{-1} \in H^{[2]}$ if a coset representative is found. If no such coset representative exists, then (g, n) takes γ_1 out of the orbit, Θ_1, and so the algorithm breaks and returns the siftee, (g, n), along with an integer indicating the level at which the break occurred, in this case 1. Otherwise, the algorithm continues and attempts find $(y_2, n_2') \in R_2$ such that $\gamma_2^{(g_2,n_2)} = \gamma_2^{(y_2,n_2')}$ and then computes $(g_3, n_3) = (g_2, n_2)(y_2, n_2')^{-1}$ if possible. The algorithm attempts to perform k iterations of this type, immediately breaking if, at any level, the base point is taken out of orbit, in which case the siftee and an integer indicating the level is returned. If the algorithm is able to perform k iterations successfully, then it returns the siftee and the integer, $k + 1$.

Remark 4.1. Note that, if $(g, n) \in H$, then the sifting algorithm will perform k iterations, and the siftee returned will have trivial first component, but possibly non-trivial second component. The element, $(g, n) \in E$, belongs to H if and only if the siftee returned after k iterations has a trivial first component and second component belonging to $H \cap N$. Thus, the sifting algorithm has reduced membership testing in H to membership testing in $H \cap N$, which can be accomplished using the already available methods for polycyclic groups.

The sifting procedure is presented in Algorithm 4.1. The function takes as input

an element, (g, n), of a polycyclic-by-finite group, E, a base, B_H, for a subgroup, $H \leqslant E$, the sequence, $\Theta = (\Theta_1, \ldots, \Theta_k)$, of basic orbits and a sequence, $R = (R_1, \ldots, R_k)$, where R_i is a transversal of $H^{[i+1]}$ in $H^{[i]}$. The method operates as described above to sift (g, n) in the hierarchy defined by B_H.

As in Subsection 2.1.3, sifting involves constructing exactly one transversal element for each base point, and thus requires $O(d \log_2 |G|)$ multiplications in E, where d is the degree of the permutation representation for G.

Algorithm 4.1 Sifting

1: **function** SIFT((g, n), B_H, Θ, R)
2: $(g_s, n_s) \leftarrow (g, n)$
3: **for** $i \leftarrow 1$ **to** k **do** $\triangleright$ (g_s, n_s) fixes base points $\gamma_1, \ldots, \gamma_{i-1}$
4: $\omega \leftarrow \gamma_i^{(g_s, n_s)}$
5: **if** $\omega \notin \Theta_i$ **then**
6: **return** (g_s, n_s), i
7: **end if**
8: Find the coset representative, $(y_i, n'_i) \in R_i$, such that $\gamma_i^{(y_i, n'_i)} = \omega$
9: $(g_s, n_s) \leftarrow (g_s, n_s) \cdot (y_i, n'_i)^{-1}$
10: **end for**
11: **return** (g_s, n_s), $k + 1$
12: **end function**

Constructing the Base and Strong Generating Set

Three modified versions of the Schreier–Sims algorithm are presented, each of which, when given a list of generators of a subgroup of a polycyclic-by-finite group, computes a base and strong generating set for that subgroup. The first operates in a manner similar to that described in Subsection 2.1.3 of Chapter 2. The second is a refinement of the first, with changes made in the method by which the normal subgroup is calculated. The third version focuses on the situation where it is required to use the base of the parent group in the representation of the subgroup.

The operation of the extended Schreier–Sims algorithm is based on the following reformulation of Lemma 2.5.

Lemma 4.2. *Let E be a polycyclic-by-finite group, $H \leqslant E$ and let $N \lhd E$ be polycyclic of finite index in E. Assume that there is a known permutation representation for the quotient $G = E/N$, and that the elements of E are expressed in normal form relative to G and N. Denote the set on which G acts by Ω and let $\{\gamma_1, \ldots, \gamma_k\} \subseteq \Omega$. For each i in $\{1, \ldots, k+1\}$, let $S_{H,j} \subseteq H_{(\gamma_1,\ldots,\gamma_{j-1})}$ such that $\langle S_{H,j}\rangle \geqslant \langle S_{H,j+1}\rangle$ holds for $j \leqslant k$. If $H = \langle S_{H,1}\rangle$, $S_{H,k+1} \subseteq H \cap N$, and*

$$\langle S_{H,j}\rangle_{\gamma_j} = \langle S_{H,j+1}\rangle \tag{4.3}$$

holds for each j, then $B_H = (\gamma_1, \ldots, \gamma_k)$ is a base for H and $S_H = \bigcup_{j=1}^{k+1} S_{H,j}$ is a strong generating set for H relative to B_H.

Proof. Induction on k is used here. The case $k = 1$ is trivial, for $\bigcup_{j=1}^{1} S_{H,j}$ clearly fulfils the requirements of a strong generating set (relative to the base containing the single point γ_1) for the group, $\langle S_{H,1}\rangle$.

Assume that the result holds for bases of length $k-1$. Then, in particular, $S_H^* = \bigcup_{j=2}^{k} S_{H,j}$ is a strong generating set for $\langle S_{H,2}\rangle$ relative to the base $B_H^* = (\gamma_2, \ldots, \gamma_k)$.

Let $H^{[i]} = H_{(\gamma_1,\ldots,\gamma_{i-1})}$ for $i = 2, \ldots, k+1$. To prove the lemma, it is required to verify that Equation (4.2) holds for each i. Setting $j = 1$ in Equation (4.3) yields

$$H_{\gamma_1} = \langle S_{H,1}\rangle_{\gamma_1} = \langle S_{H,2}\rangle,$$

which implies the containment,

$$H_{\gamma_1} \leqslant \langle S_H \cap H_{\gamma_1}\rangle.$$

The reverse inclusion is obvious, thus

$$\langle S_H \cap H^{[2]}\rangle = H^{[2]},$$

and Equation (4.2) is satisfied for $i = 2$.

For $i > 2$, the inductive hypothesis implies that $S_H^* \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}$ generates $\langle S_{H,2}\rangle_{(\gamma_1,\ldots,\gamma_{i-1})}$, and so $H^{[i]} \geqslant \langle S_H \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}\rangle \geqslant \langle S_H^* \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}\rangle = \langle S_{H,2}\rangle_{(\gamma_1,\ldots,\gamma_{i-1})} = (H_{\gamma_1})_{(\gamma_1,\ldots,\gamma_{i-1})} = H^{[i]}$. Therefore,

$$\langle S_H \cap H^{[i]}\rangle = H^{[i]}$$

for each i, as required. □

Given a set, X, of generators for a subgroup, $H \leqslant E$, the extended Schreier–Sims algorithm constructs a base and strong generating set in the following way. A data structure containing a list, $B_H = (\gamma_1, \ldots, \gamma_k)$, of already known elements of a non-redundant base is maintained, along with an approximation, $S_{H,i}$, for a generator set of the stabiliser, $H_{(\gamma_1,\ldots,\gamma_{i-1})}$, for each $i \in \{1, \ldots, k+1\}$. Throughout execution, the $S_{H,i}$ satisfy the property that, for all i, $\langle S_{H,i}\rangle \geqslant \langle S_{H,i+1}\rangle$, and $S_{H,k+1} \subseteq H \cap N$. The data structure is said to be *up-to-date below level* j if Equation (4.3) holds for each i in the range $j < i \leqslant k$.

In the case where the data structure is up-to-date below level j, a transversal, R_j, of $\langle S_{H,j}\rangle_{\gamma_j}$ in $\langle S_{H,j}\rangle$ is computed. Then a check is made to determine whether Equation (4.3) is satisfied for $i = j$. By Lemma 2.4, this can be done by sifting the Schreier generators obtained from R_j and $S_{H,j}$ in the group, $\langle S_{H,j+1}\rangle$. By Remark 4.1, membership testing is possible in the group, $\langle S_{H,j+1}\rangle$, since Lemma 4.2 implies that $\bigcup_{i=j+1}^{k+1} S_{H,i}$ is a strong generating set for $\langle S_{H,j+1}\rangle$ relative to the base, $(\gamma_{j+1}, \ldots, \gamma_k)$. If all Schreier generators are in $\langle S_{H,j+1}\rangle$ then the data structure is up-to-date below level $j-1$. Otherwise there exists a non-trivial siftee, (g_s, n_s), at level t say. If $t = k+1$ and $g_s \neq 1_G$, then a new base point, γ_{k+1}, is appended to B_H from $\mathrm{supp}((g_s, n_s))$ and the set $S_{H,k+2}$ is initialised with the contents of $S_{H,k+1}$. The siftee, (g_s, n_s), is added to the sets $S_{H,j+1}, \ldots, S_{H,t}$, and the data structure is now up-to-date below level $\min(t, |B_H|)$.

The algorithm initialises B_H to contain a sequence of points $\gamma_1, \ldots, \gamma_k$ in Ω such that each point is moved by at least one generator in X with non-trivial first component, and sets $S_{H,i}$ to $X \cap \langle X \rangle_{(\gamma_1,\ldots,\gamma_{i-1})}$ for $i = 1, \ldots, k+1$. At that moment, the data structure is up-to-date below level k; the algorithm terminates when the data structure becomes up-to-date below level 0.

The first version of the extended Schreier–Sims method is presented in Algorithm 4.2. The function takes as input a set, X, of elements of a polycyclic-by-finite group, E, and operates as described above to compute a base and strong generating set for $H = \langle X \rangle$. The notation of this subsection is used in the description of this algorithm (and subsequent versions); in particular, is it assumed that the elements of E are written in normal form relative to a normal polycyclic subgroup, N, with associated quotient, G, acting on a set, Ω. The algorithm also assumes the existence of a procedure Append which, when supplied with an array (or sequence) and an element, appends the element to the end of the array (or sequence).

The normal subgroup, $H \cap N$, of H can be computed easily from the strong generating set S_H returned by SchreierSims as it is generated by the elements in S_H with trivial first component.

The most expensive computations in SchreierSims are the multiplications in E using its representation as an extension of N — from a time complexity perspective, these computations dominate those performed in G and in N (see Theorem 3.5 for a complexity analysis of the multiplication operation in E). An estimate of the number of multiplications in E performed by SchreierSims is given in Theorem 4.3.

Theorem 4.3. *The extended Schreier–Sims algorithm is correct and requires* $O(d^2(\log_2|G|)^3 + |X|(d\log_2|G|)^2)$ *multiplications in the polycyclic-by-finite group, where* X *is the generating set provided,* G *is the finite quotient used in the representation of the polycyclic-by-finite group, and* d *is the degree of the permutation representation of* G*.*

Algorithm 4.2 Standard version of the extended Schreier–Sims algorithm

1: **function** SchreierSims(X)
2: $B_H \leftarrow ()$, $k \leftarrow 0$
3: **for** $(g, n) \in X$ **do**
4: **if** $g \neq 1_G$ **and** $(g, n) \in H_{(\gamma_1,\ldots,\gamma_k)}$ **then** ▷ (g, n) fixes all points in B_H
5: Find $\gamma_{k+1} \in \Omega$ with $\gamma_{k+1}^{(g,n)} \neq \gamma_{k+1}$
6: Append(B_H, γ_{k+1})
7: $k \leftarrow k + 1$
8: **end if**
9: **end for**
10: **for** $i \leftarrow 1$ **to** k **do**
11: $S_{H,i} \leftarrow X \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}$, $H^{[i]} \leftarrow \langle S_{H,i} \rangle$, $\Theta_i \leftarrow \gamma_i^{H^{[i]}}$,
12: Compute a stabiliser transversal, R_i, in $H^{[i]}$ corresponding to the orbit, Θ_i
13: **end for**
14: $\Theta \leftarrow (\Theta_1, \ldots, \Theta_k)$, $R \leftarrow (R_1, \ldots, R_k)$
15: $S_{H,k+1} \leftarrow \{(g, n) \in X \mid g = 1_G\}$
16: $N_H \leftarrow \langle S_{H,k+1} \rangle$ ▷ This subgroup is generated in the polycyclic group N
17: $i \leftarrow k$
18: **while** $i \geqslant 1$ **do**
19: **for** $\theta \in \Theta_i$ **do**
20: Find the coset representative, $(g, n) \in R_i$, such that $\gamma_i^{(g,n)} = \theta$
21: **for** $(x, n_x) \in S_{H,i}$ **do**
22: Find the coset representative, $(g', n') \in R_i$, such that $\gamma_i^{(g',n')} = \theta^{(x,n_x)}$
23: $(g, n) \leftarrow (g, n) \cdot (x, n_x) \cdot (g', n')^{-1}$
24: **if** $(g, n) = (1_G, 1_N)$ **then**
25: **continue** (x, n_x)
26: **end if**
27: *uptodate* $\leftarrow$ **true**
28: (g, n), $j \leftarrow$ Sift((g, n), B_H, Θ, R)

```
29:         if g ≠ 1_G then
30:             uptodate ← false
31:             if j > k then
32:                 Find γ_{k+1} ∈ Ω with γ_{k+1}^{(g,n)} ≠ γ_{k+1}
33:                 Append(B_H, γ_{k+1})                    ▻ Extend base
34:                 k ← k + 1
35:                 S_{H,k+1} ← S_{H,k}                     ▻ Maintain inclusion
36:             end if
37:         else if n ∉ N_H then
38:             uptodate ← false
39:             S_{H,k} ← S_{H,k} ∪ {(g, n)}
40:             N_H ← ⟨N_H, n⟩
41:             j ← k
42:         end if
43:         if not uptodate then
44:             for t ← i + 1 to j do
45:                 S_{H,t} ← S_{H,t} ∪ {(g, n)}, H^[t] ← ⟨S_{H,t}⟩, Θ_t ← γ_t^{H^[t]},
46:                 Compute a stabiliser transversal, R_t, in H^[t] corresponding
    to the orbit, Θ_t
47:             end for
48:             Θ ← (Θ_1, . . . , Θ_k), R ← (R_1, . . . , R_k)
49:             i ← j + 1
50:             break θ
51:         end if
52:       end for
53:     end for
54:     i ← i − 1
55:   end while
56:   return B_H, ⋃_{i=1}^{k+1} S_{H,i}
57: end function
```

Proof. Lemma 4.2 implies correctness.

The number of multiplications in E may be estimated by counting the number of times that Sift is executed (Line 28). Since each Schreier generator is sifted exactly once, the number of times Sift is executed is equal to the total number of Schreier generators.

Keeping the notation of SchreierSims, after initialisation in Lines 3–13, each R_i has size at most d, each $S_{H,i}$ has size at most $|X|$, and, k is bounded by $\log_2|G|$. Thus, the number of Schreier generators before any sift operations are performed is $O(|X|d\log_2|G|)$.

Observe that although the sets R_i and $S_{H,i}$ change during the operation of the algorithm, they are always only augmented and therefore any elements of R_i and $S_{H,i}$ must be combined to a Schreier generator only once. The number of base points is at most $\log_2|G|$. For a fixed base point γ_i, the set, $S_{H,i}$, changes at most $\log_2|G|$ times during the algorithm, since the group, $\langle S_{H,i}\rangle$, must increase each time an element is added to $S_{H,i}$. Combining this with the bound $|R_i| \leqslant d$, it follows that the number of Schreier generators constructed (Line 23) throughout the operation of the algorithm is $O(d(\log_2|G|)^2)$. The total number of Schreier generators is thus $O(d(\log_2|G|)^2 + |X|d\log_2|G|)$.

Each Schreier generator is sifted once and each sift requires $O(d\log_2|G|)$ multiplications in the polycyclic-by-finite group. Therefore, the total number of multiplications that must be performed in the polycyclic-by-finite group is $O(d^2(\log_2|G|)^3 + |X|(d\log_2|G|)^2)$. □

Consider Lines 19–23 of SchreierSims. If the generator, (x, n_x), has trivial first component, i.e. $x = 1_G$, then $\theta^{(x,n_x)} = \theta$ whence $(g', n') = (g, n)$ and the multiplication of elements in Line 23 is reduced to the conjugation,

$$(g, n) \cdot (1_G, n_x) \cdot (g, n)^{-1},$$

of a generator in N by an element of H. The algorithm proceeds to sift this conjugate down to level $k + 1$, adding it to the generator set of $H \cap N$.

In light of this, one may modify SchreierSims so that strong generators with trivial first component are not added at every level. Instead, generators of this type are kept in a separate set and a normal closure computation is performed after the structure becomes up-to-date below level 0. Eliminating strong generators with trivial first component from the data structure has the obvious advantage of speeding up both the orbit computations and the membership checks at each level.

A version of the extended Schreier–Sims algorithm incorporating these changes

is presented in Algorithm 4.3. The algorithm assumes the existence of a function GENERATORS, which, when supplied with a polycyclic group, returns a set of generators for that group.

The discussion above justifies the following remark, which implies the correctness of NORMALCLOSURESCHREIERSIMS.

Remark 4.4. The subgroup, $H \cap N$, is the normal closure in $\langle X \rangle$ of the siftees appearing NORMALCLOSURESCHREIERSIMS with trivial first component.

The normal closure computation in Line 48 of NORMALCLOSURESCHREIERSIMS is performed using the standard method of repetitive adding of conjugates to a generating set. The available methods for polycyclic groups are used for membership testing of conjugates, and conjugation is of course performed using the multiplication method of Chapter 3.

From a theoretical perspective, the time complexity of this version of the extended Schreier–Sims method is no different from its predecessor. However, in practice, methods used to compute normal closures often terminate rapidly, and so this version may offer a slight advantage in efficiency.

Both SCHREIERSIMS and NORMALCLOSURESCHREIERSIMS attempt to compute a new base for the subgroup H. Situations may arise where one wishes to use the base of the parent group in the representation of the subgroup. A version of the extended Schreier–Sims algorithm with this minor simplification is presented in Algorithm 4.4. The function takes as input a set, X, of elements of a polycyclic-by-finite group, E, along with the base, $B = (\gamma_1, \ldots, \gamma_k)$, of E.

As indicated in the discussion immediately following Theorem 3.5, smaller bases are preferable in relation to the performance of MULTIPLY. Taking this into consideration, it may be prudent in most instances to opt for a possible reduction in base size, as in Algorithms 4.2 and 4.3, unless the application specifically requires that the original base be retained.

Algorithm 4.3 Normal closure version of the extended Schreier–Sims algorithm

1: **function** NormalClosureSchreierSims(X)
2: $X_1 \leftarrow \{(g,n) \in X \mid g \neq 1_G\}$, $X_2 \leftarrow \{(g,n) \in X \mid g = 1_G\}$ ▷ $X = X_1 \cup X_2$
3: $B_H \leftarrow ()$, $k \leftarrow 0$
4: **for** $(g,n) \in X_1$ **do**
5: **if** $(g,n) \in H_{(\gamma_1,\ldots,\gamma_k)}$ **then** ▷ (g,n) fixes all points in B_H
6: Find $\gamma_{k+1} \in \Omega$ with $\gamma_{k+1}^{(g,n)} \neq \gamma_{k+1}$
7: Append(B_H, γ_{k+1})
8: $k \leftarrow k+1$
9: **end if**
10: **end for**
11: **for** $i \leftarrow 1$ **to** k **do**
12: $S_{H,i} \leftarrow X_1 \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}$, $H^{[i]} \leftarrow \langle S_{H,i} \rangle$, $\Theta_i \leftarrow \gamma_i^{H^{[i]}}$,
13: Compute a stabiliser transversal, R_i, in $H^{[i]}$ corresponding to the orbit, Θ_i
14: **end for**
15: $\Theta \leftarrow (\Theta_1, \ldots, \Theta_k)$, $R \leftarrow (R_1, \ldots, R_k)$
16: $i \leftarrow k$
17: **while** $i \geqslant 1$ **do**
18: **for** $\theta \in \Theta_i$ **do**
19: Find the coset representative, $(g,n) \in R_i$, such that $\gamma_i^{(g,n)} = \theta$
20: **for** $(x,n_x) \in S_{H,i}$ **do**
21: Find the coset representative, $(g',n') \in R_i$, such that $\gamma_i^{(g',n')} = \theta^{(x,n_x)}$
22: $(g,n) \leftarrow (g,n) \cdot (x,n_x) \cdot (g',n')^{-1}$
23: **if** $(g,n) = (1_G, 1_N)$ **then**
24: **continue** (x,n_x)
25: **end if**
26: (g,n), $j \leftarrow$ Sift((g,n), B_H, Θ, R)
27: **if** $g \neq 1_G$ **then**
28: **if** $j > k$ **then**
29: Find $\gamma_{k+1} \in \Omega$ with $\gamma_{k+1}^{(g,n)} \neq \gamma_{k+1}$
30: Append(B_H, γ_{k+1}) ▷ Extend base
31: $k \leftarrow k+1$
32: $S_{H,k} \leftarrow \varnothing$
33: **end if**
34: **for** $t \leftarrow i+1$ **to** j **do**
35: $S_{H,t} \leftarrow S_{H,t} \cup \{(g,n)\}$, $H^{[t]} \leftarrow \langle S_{H,t} \rangle$, $\Theta_t \leftarrow \gamma_t^{H^{[t]}}$,
36: Compute a stabiliser transversal, R_t, in $H^{[t]}$ corresponding to the orbit, Θ_t
37: **end for**

```
38:                 Θ ← (Θ_1, ..., Θ_k), R ← (R_1, ..., R_k)
39:                 i ← j + 1
40:                 break θ
41:             else if n ≠ 1_N then
42:                 X_2 ← X_2 ∪ {(g, n)})
43:             end if
44:         end for
45:     end for
46:     i ← i − 1
47:   end while
48:   N_H ← ⟨X_2⟩^⟨X_1⟩
49:   X_2 ← Generators(N)
50:   return B_H, (⋃_{i=1}^k S_{H,i}) ∪ X_2
51: end function
```

4.2.2 Data

The data set described in Subsection 3.2.2 of Chapter 3 can be computed readily from the output of the extended Schreier–Sims method. Keeping the notation of Subsection 4.2.1, let B_H be a base for H and let S_H be a strong generating set for H relative to B_H. The subgroup, $N_H = H \cap N$, of H is generated by the elements of S_H with trivial first component, and this computation may be performed using the standard methods available for polycyclic groups, possibly involving the definition of a new polycyclic presentation for N_H.

Let $\overline{T} = \{(x_1, n_1), \ldots, (x_m, n_m)\}$ be the set containing all elements of S_H with non-trivial first component. It is assumed here that $x_i \neq x_j$ for $i \neq j$. The set, $T = \{x_1, \ldots, x_m\}$, is a strong generating set for $G_H \cong H/N_H$, relative to the base, B_H. Thus, using the permutation representation of G, one may compute $G_H = \langle T \rangle \leqslant G$ along with a set of basic orbits and stabilisers (and Schreier vectors encoding basic stabiliser transversals) relative to B_H. The tails of Equations (3.1) in G_H are defined relative to this set of stabiliser transversals, and are calculated within G_H as described in Subsection 3.2.2.

The crucial point to note at this stage of the subgroup construction is that the transversal, L_H, of N_H in H (analogous to the transversal, L, of N in E) is

Algorithm 4.4 Base-preserving version of the extended Schreier–Sims algorithm

1: **function** KeepBaseSchreierSims(X, B)
2: $X_1 \leftarrow \{(g,n) \in X \mid g \neq 1_G\}$, $X_2 \leftarrow \{(g,n) \in X \mid g = 1_G\}$ ▷ $X = X_1 \cup X_2$
3: $B_H \leftarrow B$
4: **for** $i \leftarrow 1$ **to** k **do**
5: $S_{H,i} \leftarrow X_1 \cap H_{(\gamma_1,\ldots,\gamma_{i-1})}$, $H^{[i]} \leftarrow \langle S_{H,i} \rangle$, $\Theta_i \leftarrow \gamma_i^{H^{[i]}}$,
6: Compute a stabiliser transversal, R_i, in $H^{[i]}$ corresponding to the orbit, Θ_i
7: **end for**
8: $\Theta \leftarrow (\Theta_1, \ldots, \Theta_k)$, $R \leftarrow (R_1, \ldots, R_k)$
9: $i \leftarrow k$
10: **while** $i \geqslant 1$ **do**
11: **for** $\theta \in \Theta_i$ **do**
12: Find the coset representative, $(g,n) \in R_i$, such that $\gamma_i^{(g,n)} = \theta$
13: **for** $(x, n_x) \in S_{H,i}$ **do**
14: Find the coset representative, $(g', n') \in R_i$, such that $\gamma_i^{(g',n')} = \theta^{(x,n_x)}$
15: $(g,n) \leftarrow (g,n) \cdot (x,n_x) \cdot (g',n')^{-1}$
16: **if** $(g,n) = (1_G, 1_N)$ **then**
17: **continue** (x, n_x)
18: **end if**
19: (g,n), $j \leftarrow$ Sift((g,n), B_H, Θ, R)
20: **if** $j \leqslant k$ **then**
21: **for** $t \leftarrow i+1$ **to** j **do**
22: $S_{H,t} \leftarrow S_{H,t} \cup \{(g,n)\}$, $H^{[t]} \leftarrow \langle S_{H,t} \rangle$, $\Theta_t \leftarrow \gamma_t^{H^{[t]}}$,
23: Compute a stabiliser transversal, R_t, in $H^{[t]}$ corresponding to the orbit, Θ_t
24: **end for**
25: $\Theta \leftarrow (\Theta_1, \ldots, \Theta_k)$, $R \leftarrow (R_1, \ldots, R_k)$
26: $i \leftarrow j + 1$
27: **break** θ
28: **else if** $n \neq 1_N$ **then**
29: $X_2 \leftarrow X_2 \cup \{(g,n)\})$
30: **end if**
31: **end for**
32: **end for**
33: $i \leftarrow i - 1$
34: **end while**
35: $N_H \leftarrow X_2^{\langle X_1 \rangle}$
36: $X_2 \leftarrow$ Generators(N)
37: **return** B_H, $(\bigcup_{i=1}^{k} S_{H,i}) \cup X_2$
38: **end function**

predefined by the strong generating set returned by the extended Schreier–Sims method. That is, the image, $\overline{x_i}$, of $x_i \in T$ in L_H is (x_i, n_i). As in Subsection 3.2.2, $\overline{T^{-1}} = \{\overline{x^{-1}} \mid x \in T\}$ is defined by the equation:

$$\overline{x^{-1}} = \begin{cases} \overline{x} & \text{if } x^2 = 1, \\ \overline{x}^{-1} & \text{otherwise.} \end{cases}$$

The sets, $\overline{T}$ and $\overline{T^{-1}}$, correspond to the sets, $\hat{S}$ and $\hat{S}^{-1}$, respectively, defined in Subsection 3.2.2 of Chapter 3.

Equipped with this definition for L_H, the tail elements of the shift equations are calculated as described in Subsection 3.2.2 — all multiplications performed in E using the multiplication algorithm of Chapter 3.

Finally, the conjugates,

$$a^{\overline{x^{-1}}}, a^{\overline{x^{-1}}} \in N_H,$$

are computed and stored for each polycyclic generator, a, of N_H, and each $x \in T$, again with all multiplications performed in E using the multiplication algorithm of Chapter 3.

This concludes the construction, and the multiplication algorithm may now be applied to elements of H written in normal form relative to G_H and N_H.

4.2.3 Membership Testing

Remark 4.1 forms the basis of the membership testing algorithm. Keeping the notation of Subsection 4.2.1, to test membership of $(g, n) \in E$ in the subgroup, H, (g, n) is sifted relative to the base and strong generating set computed by the extended Schreier–Sims algorithm. If an element with non-trivial first component is returned, then $(g, n) \notin H$. Otherwise, a siftee of the form $(1_G, n')$ is returned, and $(g, n) \in H$ if and only if $n' \in H \cap N = N_H$. The membership test in N_H is performed

using the available methods for polycyclic groups, see (Holt et al., 2005, Sec. 8.3).

The dominant computation in a test for membership in a subgroup is the application of the sifting method, which requires $O(d \log_2 |G|)$ multiplications in E, where d is the degree of the permutation representation for G (see Subsection 4.2.1).

4.3 Applications

Following immediately from the subgroup generation algorithm is a function which computes the normal closure of a subgroup generated by a finite set of elements. This function may then be used to design algorithms which find derived subgroups and commutator subgroups by computing the normal closure of a suitable set of commutators. Using these algorithms, one may write simple iterative procedures which construct derived and lower central series. For a review of how these standard functions are implemented, see (Holt et al., 2005, Sec. 3.3).

In the subsections below, two straightforward yet useful applications of subgroup generation are given. The methods presented heavily exploit the polycyclic-by-finite structure of the group in question. Standard notation is kept in this section, viz. E is a polycyclic-by-finite group represented as an extension of a polycyclic group, N, by the finite quotient, G. The natural map, $E \to G$, is denoted by ρ.

4.3.1 The Soluble Radical

For a group, E, the *soluble radical*, denoted by $\mathbf{O}_\infty(E)$, is defined as the largest soluble normal subgroup of E. This subsection contains the outline of a method which computes the soluble radical of the polycyclic-by-finite group, E.

It follows from the definition of the soluble radical that $N \leqslant \mathbf{O}_\infty(E)$. Thus, $\mathbf{O}_\infty(E)$ is the preimage under ρ of $\mathbf{O}_\infty(G)$.

The permutation representation of G may be utilised to compute $\mathbf{O}_\infty(G)$; for a detailed discussion of the soluble radical algorithm for permutation groups, see

(Seress, 2003, Chap. 6). Given a generating set, X, of $\mathbf{O}_\infty(G)$, the method described Subsection 3.3.1 can be used to find a set, $\overline{X} \subseteq E$, which, under ρ, maps to X. The soluble radical of E is then $\langle \overline{X}, N \rangle$.

4.3.2 Sylow Subgroups

In this subsection, it is assumed that E is finite.

Construction

Let p be a prime dividing $|E|$ and let H_p be a Sylow p-subgroup of G. Observe that a Sylow p-subgroup of the preimage, $\overline{H_p} = \rho^{-1}(H_p)$, is a Sylow p-subgroup of E.

Thus, one may compute a Sylow p-subgroup of E as follows. First, find a Sylow p-subgroup, H_p, of G. This is done using the available methods for permutation groups (see Holt et al., 2005, Chap. 4). Then, using a method similar to that of Subsection 4.3.1, generate $\overline{H_p} = \rho^{-1}(H_p)$ as a subgroup of E. This group is finite and soluble, and so, one may produce a consistent power-conjugate presentation defining it using the method of Subsection 3.3.4 of Chapter 3. The existing methods for computing Sylow subgroups in finite soluble groups represented by power-conjugate presentations are now applicable (see Seress, 2003, Chap. 7). Finally, the isomorphism from the subgroup defined by the power-conjugate presentation can be used to map a generating set for a Sylow p-subgroup back into E.

Conjugacy

It is also possible to design an algorithm which, when given two Sylow p-subgroups of E, returns an element which conjugates one Sylow subgroup to the other. The first step is to find, using the available functions for permutation groups (see Holt et al., 2005, Chap. 4), an element $g \in G$ which conjugates $\rho(P_1)$ to $\rho(P_2)$ in G. Using the method of Subsection 3.3.1 a preimage, $\overline{g} \in E$, for g may be found. It follows

then that $P_1^{\bar{g}}$ is a Sylow p-subgroup of the group, $\rho^{-1}(\rho(P_2)) = P_2N$, which contains P_2 as a Sylow p-subgroup.

The group, P_2N, is finite and soluble, and so may be represented by a consistent power-conjugate presentation. One may then find, using the available methods for finite soluble groups (see Seress, 2003, Chap. 7), a conjugating element, $y \in P_2N$, such that $P_1^{\bar{g}y} = P_2$, as required.

Chapter 5

Conjugacy

The material presented in this chapter utilises the machinery developed thus far to perform some advanced structural calculations within polycyclic-by-finite groups. Specifically, the question of element conjugacy is addressed.

The chapter begins with an algorithm to compute the centre of a given polycyclic-by-finite group. Following this, a method to construct centralisers of elements belonging to a finite polycyclic-by-finite group is described. The centraliser algorithm is developed using the theory of extensions and cohomology; a synopsis of the required material is found in Section 2.3 of Chapter 2. The problem of determining whether two elements are conjugate is intimately connected to the determination of centralisers, and may be solved algorithmically with a slight modification of the centraliser method. This algorithm is presented as the final instalment of the chapter.

The underlying idea of the algorithms presented here is to transform the problem into one of linear algebra, and then use the available methods for matrix groups to perform the computation.

5.1 The Centre

The approach to computing the centre of a given polycyclic-by-finite group, E, is as follows:

1. Find a normal abelian subgroup, A, of E that contains $\mathbf{Z}(E)$.
2. For each member, x, of a generating set, X, for E, compute the matrix, M_x, that specifies the conjugation action of that generator on elements of A (written as vectors).
3. Construct a matrix whose null space corresponds to the image of $Z(E)$ in A.
4. Map a set of vectors generating $Z(E)$ in A back into E.

The following simple lemma furnishes one with a feasible choice for the normal abelian subgroup.

Lemma 5.1. *Let G be a group with soluble radical, S. Then $\mathbf{Z}(G) \trianglelefteqslant \mathbf{Z}(S) \trianglelefteqslant G$.*

Proof. The subgroup, $\mathbf{Z}(G)S$, is soluble, so $\mathbf{Z}(G) \leqslant S$ and hence, $\mathbf{Z}(G) \trianglelefteqslant \mathbf{Z}(S)$. Normality of $\mathbf{Z}(S)$ in G follows from $\mathbf{Z}(S)$ being characteristic in S. □

Let S be the soluble radical of E. Then by Lemma 5.1, $\mathbf{Z}(E)$ is contained in the normal abelian subgroup, $\mathbf{Z}(S)$.

The soluble radical, S, can be computed as a subgroup of E using the method described in Subsection 4.3.1 of Chapter 4, after which it is rewritten as a polycyclic group by the procedure given in Subsection 3.3.4 of Chapter 3. The available functions for polycyclic groups may then be utilised to compute the centre, $\mathbf{Z}(S)$, of S as an abelian group; see (Holt et al., 2005, Chap. 8) for the finite case and (Eick, 2001) for the infinite case.

Write $\mathbf{Z}(S)$ as an abelian group, A, with invariant factor decomposition,

$$\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_t} \times \mathbb{Z}^k,$$

where $m_i \mid m_{i+1}$ for each i. Writing the elements of A as row vectors, let $\vec{a}_1, \ldots, \vec{a}_{t+k}$ be the standard basis for A relative to its invariant factor decomposition. Let $x \in X$. Regarding matrices as acting on the right, the i-th row of the matrix, M_x, over $\mathbb{Z}$, defining conjugation in A by x is given by

$$\vec{a}_i x$$

expressed as a row vector, where E acts on A by conjugation.

The element, $\vec{a}_i x$, is found by first mapping $\vec{a}_i$ into the group E, and then using the multiplication algorithm of Chapter 3 to perform the conjugation. The result is mapped back into A and written as a row vector.

An element, $\vec{z} \in A$, lies in $\mathbf{Z}(E)$ if and only if, for each $x \in X$,

$$\vec{z} M_x = \vec{z} \tag{5.1}$$

A matrix whose null space corresponds to the set of all such z is constructed as follows. Let D be the $t \times (t+k)$ matrix over $\mathbb{Z}$,

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & m_t & 0 & \cdots & 0 \end{pmatrix}.$$

Let $X = \{x_1, \ldots, x_q\}$ and let $T_x = M_x - I$ for each $x \in X$, where I is the identity $(t+k) \times (t+k)$ matrix. Form the following matrix, Q, over $\mathbb{Z}$

$$\left(\begin{array}{c|c|c|c} T_{x_1} & T_{x_2} & \cdots & T_{x_q} \\ \hline D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{array}\right),$$

where blank spaces indicate zero entries.

Claim 5.2. If a vector, $\vec{z} \in A$, lies in $\mathbf{Z}(E)$, then there exists a vector $(\vec{z} \mid \vec{v}) \in \mathbb{Z}^{t(q+1)+k}$ in the null space of Q. Conversely, the first $t+k$ entries of any vector in the null space of Q define a vector belonging to $\mathbf{Z}(E)$.

Proof. Suppose that $\vec{z} \in A$ lies in $\mathbf{Z}(E)$. Then for each $x_i \in X$, $\vec{z}T_{x_i} = \vec{0}$, equality being in the abelian group, $A \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_t} \times \mathbb{Z}^k$. Viewing $\vec{z}$ as a vector in $\mathbb{Z}^{t+k}$, $\vec{z}T_{x_i} = \vec{0}$ is equivalent to $\vec{z}T_{x_i} = -\vec{v}_i D$ for some $\vec{v}_i \in \mathbb{Z}^t$. Hence, the vector $(\vec{z} \mid \vec{v}_1 \mid \cdots \mid \vec{v}_q)$ belongs to $\ker(Q)$.

Conversely let $\vec{v} \in \ker(Q)$ and partition $\vec{v}$ into segments, the first of length $t+k$, and the rest each of length t: $(\vec{z} \mid \vec{v}_1 \mid \cdots \mid \vec{v}_q)$. Then $\vec{z}T_{x_i} = -\vec{v}_i D$ for each i, which, as above, is equivalent to $\vec{z}T_{x_i} = \vec{0}$ in A, whence $\vec{z}$ lies in $Z(E)$. $\square$

Thus, the image of $Z(E)$ in A, can be recovered from a generating set of $\ker(Q)$. Mapping this image back into E completes the computation.

Pseudocode for computing the centre in this manner is presented in Algorithm 5.1. The function takes as input a polycyclic-by-finite group, E, generated by a set, $X = \{x_1, \ldots, x_q\}$.

The runtime of CENTRE is dominated by the calculation of the null space in Line 9. The fastest known algorithms to perform such computations use p-adic expansions, and were introduced by Dixon (1982). They require $O(c^4(\log_2 r)^2)$ time

Algorithm 5.1 Finding the centre

1: **function** CENTRE(E)

2: Find the soluble radical, S, of E

3: Write $Z(S)$ as an abelian group: $Z(S) \overset{\phi}{\cong} \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_t} \times \mathbb{Z}^k$ with standard basis $\{\vec{a}_1, \ldots, \vec{a}_{t+k}\}$ relative to this decomposition

4: **for** $x \in X$ **do**

5: $T_x \leftarrow \begin{pmatrix} \vec{a}_1 x \\ \vdots \\ \vec{a}_{t+k} x \end{pmatrix} - I_{t+k}$

6: **end for**

7: $D \leftarrow \begin{pmatrix} m_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & m_t & 0 & \cdots & 0 \end{pmatrix}$ ▷ D has dimension $t \times (t+k)$

8: $Q \leftarrow \left(\begin{array}{c|c|c|c} T_{x_1} & T_{x_2} & \cdots & T_{x_q} \\ \hline D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{array}\right)$

9: Compute generating set, Y, for $\ker(Q)$

10: $Y_A \leftarrow \varnothing$

11: **for** $\vec{y} \in Y$ **do**

12: Form vector, $\vec{y}_A$ defined by the first $t+k$ entries of $\vec{y}$

13: $Y_A \leftarrow Y_A \cup \{\vec{y}_A\}$

14: **end for**

15: **return** $\langle \phi^{-1}(Y_A) \rangle$

16: **end function**

for an $r \times c$ matrix. For recent developments in this area, see (Haramoto and Matsumoto, 2009). It is worth noting here that, if $k = 0$ and $m_i = m$ for each i and some $m \in \mathbb{N}$, then the computation can be performed entirely in $\mathbb{Z}_m$, offering a significant improvement in efficiency. For a practical account of computations of this type, the reader is referred to (Holt et al., 2005, Chap. 7).

5.2 The Conjugacy Problem

The Conjugacy Problem is one of the fundamental decision problems in group theory, and it is known that the problem is undecidable for many classes of groups (see Chandler and Magnus, 1982). The class of polycyclic-by-finite groups, however, is a subclass of the class of finitely presented groups that possess solvable conjugacy problem. For a systematic account of the decision problems that are solvable in the class of polycyclic-by-finite groups, the reader is referred to (Baumslag et al., 1991).

The treatment of the conjugacy problem in this section focuses on the finite case, and extends the ideas of Holt et al. (2005, Sec. 8.8). It may be possible to generalise the methods presented here to cope with infinite groups using the work of Eick (2001). However, at present, the algorithms are primarily targeted at finite insoluble groups with large soluble normal subgroup.

5.2.1 Centralisers

Let E be a finite group with soluble normal subgroup, and assume that E is represented as described in Chapter 3. In this subsection, a method of computing

$$C_E(e) = \{f \in E \mid fe = ef\}$$

for an element, $e \in E$, is introduced.

The centraliser, $C_E(e)$, is the stabiliser of e under the conjugation action of

E on its elements. Therefore, it is possible, in theory, to design a deterministic algorithm to compute $C_E(e)$ by adapting the orbit-stabiliser method described in Subsection 2.1.2 to this case. However, this approach would inevitably involve computing and storing explicitly the entire orbit, e^E. This can be time and space consuming. A more effective strategy is presented in the sequel.

Affine Actions

Before describing the algorithm itself, a digression must be made to introduce the concept of an *affine action* of a group on a vector space, which is required for the application at hand.

Definition 5.3. Let $V \cong K^d$ be a finite-dimensional vector space over a field K, let $\varphi\colon G \to \mathrm{End}_K(V)$ be a linear action of a group, G, on V, and let $\delta : G \to V$ be a derivation (see Equation (2.6)) with respect to the action, φ. Then the map, $\alpha\colon G \to \mathrm{Sym}(V)$, defined by $v^{\alpha(g)} = v^{\varphi(g)} + \delta(g)$ is called an *affine action* of G on V.

Keeping the notation of Definition 5.3, it is straightforward to check that α is a homomorphism:

$$\begin{aligned} v^{\alpha(gh)} &= v^{\varphi(gh)} + \delta(gh) \\ &= v^{\varphi(gh)} + \delta(g)^{\varphi(h)} + \delta(h) \\ &= (v^{\varphi(g)} + \delta(g))^{\varphi(h)} + \delta(h) \\ &= v^{\alpha(g)\alpha(h)}. \end{aligned}$$

Thus, an affine action is an action of a group, G, on a vector space, V, but not a linear action unless $\delta = 0$.

If V is identified with K^d and $\mathrm{End}_K(V)$ with $\mathrm{GL}(d, K)$, then the affine action,

α, corresponds to the homomorphism, $\hat{\alpha}\colon G \to \mathrm{GL}(d+1, K)$, given by

$$\hat{\alpha}(g) = \left(\begin{array}{c|c} \varphi(g) & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline \delta(g) & 1 \end{array} \right),$$

where, for $v \in K^n$, $(\ v^{\alpha(g)} \mid 1) = (\ v \mid 1) \cdot \hat{\alpha}(g)$.

The Centraliser Algorithm

Assume that the group, E, is represented as described in Chapter 3 with respect to the normal subgroup, N, and associated quotient, G. The normal subgroup, N, is soluble and finite, and hence possesses a normal elementary abelian series:

$$N = N_1 \trianglerighteq \cdots \trianglerighteq N_t \trianglerighteq N_{t+1} = 1.$$

The basic idea of the method is an induction downwards along this normal series. Suppose that N_t is a p-group of rank, d, say, where p is prime, and assume, by induction, that the centraliser in the factor group E/N_t has already been computed. That is, a set of generators in E whose image in E/N_t generates $\mathbf{C}_{E/N_t}(eN_t)$, is known. Define C to be the subgroup of E such that $C/N_t = \mathbf{C}_{E/N_t}(eN_t)$. The following proposition is elementary.

Proposition 5.4. *The mapping $\delta\colon C \to N_t$ defined by $c \mapsto [e, c]$ is a derivation.*

Proof. For $e \in E$ and $c_1, c_2 \in C$:

$$[e, c_1c_2] = e^{-1}(c_1c_2)^{-1}ec_1c_2 = e^{-1}c_2^{-1}e[e, c_1]c_2 = [e, c_1]^{c_2}[e, c_2].$$

The commutativity of N_t is used in the last equality. □

The group N_t may be identified with the additive group $\mathbb{F}_p^d$, and additive notation may be used for its elements. The conjugation action of C on N_t may then be written as a homomorphism, $\varphi\colon C \to \mathrm{GL}(d, \mathbb{F}_p)$, and one may consider δ as a derivation of the form $\delta\colon C \to \mathbb{F}_p^d$. These can be combined to give an affine action, $\alpha\colon C \to \mathrm{GL}(d+1, \mathbb{F}_p^d)$, defined by:

$$c \mapsto \left(\begin{array}{c|c} & 0 \\ \varphi(c) & \vdots \\ & 0 \\ \hline \delta(c) & 1 \end{array}\right).$$

This affine action yields the following characterisation of $\mathbf{C}_E(e)$.

Proposition 5.5. *The centraliser, $C_E(e)$, is the stabiliser in C of the vector,*

$$(0, \ldots, 0, 1) \in \mathbb{F}_p^{d+1},$$

where C acts on $\mathbb{F}_p^{d+1}$ via α.

Proof. The stabiliser in C of $(0, \ldots, 0, 1)$ under the action of α is the kernel of the derivation, δ. By the definition of δ, $\ker(\delta) = \mathbf{C}_C(e)$. As $\mathbf{C}_E(e) \leqslant C$, it follows that $\mathbf{C}_C(e) = \mathbf{C}_E(e)$. □

Hence one obtains an effective method to compute $\mathbf{C}_C(e)$. The algorithm commences by computing a set of elements in E whose images in the quotient group, E/N, generate the centraliser, $\mathbf{C}_{E/N}(e)$. This centraliser is computed using the known permutation representation for E/N by the already available methods for permutation groups described in (Seress, 2003) and (Holt et al., 2005, Chap. 4). The algorithm then proceeds to iterate down the normal elementary abelian series, solving the problem for successively larger quotients.

This method is similar to the method involving a single orbit-stabiliser computation, but the induction method computes relatively small orbits of vectors instead of one relatively large orbit of elements in a polycyclic-by-finite group. Thus, the induction method is usually more efficient than the single orbit-stabiliser application.

An algorithm for computing the centralisers in this manner is presented in Algorithm 5.2. The function takes as input a polycyclic-by-finite group, E, and an element, $e \in E$, and operates as described above to compute $\mathbf{C}_E(e)$. The notation of this subsection is kept in the description of the algorithm. In particular, it is assumed that the group, E, is represented as described in Chapter 3 with respect to the normal subgroup, N, and associated quotient, G.

Algorithm 5.2 Finding centralisers

1: **function** CENTRALISER(E, e)
2: Compute a set, X, of generators in E for the image of the centraliser, $\mathbf{C}_{E/N}(eN)$ in E
3: Compute a normal elementary abelian series, $N = N_1 \trianglerighteq \cdots \trianglerighteq N_t \trianglerighteq N_{t+1} = 1$, for N
4: **for** $i \leftarrow 1$ **to** t **do**
5: Express the elementary abelian section N_i/N_{i+1} as vector space, V, with standard basis $\{\vec{v}_1, \ldots, \vec{v}_d\}$
6: **for** $x \in X$ **do**
7: Express $\delta(x) = e^{-1}x^{-1}exN_{i+1}$ as a vector, $\vec{u}_x \in V$
8: $M_x \leftarrow \left(\begin{array}{c|c} \vec{v}_1 x & 0 \\ \vdots & \vdots \\ \vec{v}_{t+k} x & 0 \\ \hline \vec{u}_x & 1 \end{array}\right)$ ▷ x acts on N_i/N_{i+1} by conjugation
9: **end for**
10: Compute the stabiliser, $B_{\vec{w}}$, of $\vec{w} = (0, \ldots, 0, 1)$ in $B = \langle M_x \mid x \in X \rangle$
11: Find the image, X', in E of a generating set for $B_{\vec{w}}$
12: $X \leftarrow X'$
13: **end for**
14: **return** $\langle X \rangle$
15: **end function**

5.2.2 Conjugacy Testing

The problem of checking whether two elements $e, f \in E$ are conjugate can be solved with a variation of the centraliser algorithm.

Keeping the notation of Subsection 5.2.1, assume, by induction, that the centraliser in the factor group E/N_t has already been computed. That is, a set of generators in E whose image in E/N_t generates $C/N_t = \mathbf{C}_{E/N_t}(eN_t)$, is known. Assume also, again by induction, that e and f are equal in the factor group E/N_t. The following proposition forms the basis of the conjugacy testing algorithm.

Proposition 5.6. *If the elements $e, f \in E$ are conjugate, then the element induced by $e^{-1}f$ must lie in the orbit of the vector,*

$$(0, \ldots, 0, 1) \in \mathbb{F}_p^{d+1},$$

under C, where C acts on $\mathbb{F}_p^{d+1}$ via α.

Proof. Suppose that e and f are conjugate in E. Then there exists $y \in E$ such that $e^y = f$, or equivalently

$$[e, y] = e^{-1}f.$$

By assumption, $e^{-1}f \in N_t$ and so, from the equality above, $[e, y] \in N_t$, or equivalently $N_t y \in C/N_t$.

Therefore, $y \in C$, and

$$\delta(y) = [e, y] = e^{-1}f,$$

and the element induced by $e^{-1}f$ lies in the orbit of the vector, $(0, \ldots, 0, 1) \in \mathbb{F}_p^{d+1}$, under C. □

Each time Line 10 of Centraliser is executed, a stabiliser computation is performed. In the test for conjugacy, the underlying orbit is also computed. By Proposition 5.6, if the element induced by $e^{-1}f$ is not in this orbit, then e and f

are not conjugate. Otherwise, there exists an element, $y \in E$, such that $e^{-1}y^{-1}ey$ is equal to $e^{-1}f$ in the considered factor, i.e. e^y is equal to f in the considered factor. This conjugating element is found using a transversal of the stabiliser in image of the affine map. The element, f, is is modified to $f^{y^{-1}}$, so that e and f are now equal in the considered factor (as in Proposition 5.6), and the algorithm proceeds to the next induction step.

The algorithm commences by attempting to find an element conjugating e to f in the quotient group, E/N (see (Seress, 2003) and (Holt et al., 2005, Chap. 4) for details).

An algorithm for checking element conjugacy in this manner is presented in Algorithm 5.2. The function takes as input a polycyclic-by-finite group, E, and elements, $e, f \in E$, and operates as described above to determine whether e and f are conjugate in E, returning an element, $y \in E$, such that $e^y = f$ in the affirmative case. The notation of this subsection is kept in the description of the algorithm. In particular, it is assumed that the group, E, is represented as described in Chapter 3 with respect to the normal subgroup, N, and associated quotient, G.

Algorithm 5.3 Testing Element Conjugacy

1: **function** IsConjugate(E, e, f)
2: **if** e and f are not conjugate in E/N **then**
3: **return false**, -
4: **end if**
5: Find $y \in E$ such that e^y is equal to f in E/N
6: $f \leftarrow f^{y^{-1}}$
7: Compute a set, X, of generators in E for the image of the centraliser, $\mathbf{C}_{E/N}(eN)$ in E
8: Compute a normal elementary abelian series, $N = N_1 \trianglerighteq \cdots \trianglerighteq N_t \trianglerighteq N_{t+1} = 1$, for N
9: **for** $i \leftarrow 1$ **to** t **do**
10: Express the elementary abelian section N_i/N_{i+1} as vector space, V, with standard basis $\{\vec{v}_1, \ldots, \vec{v}_d\}$
11: **for** $x \in X$ **do**
12: Express $\delta(x) = e^{-1}x^{-1}exN_{i+1}$ as a vector, $\vec{u}_x \in V$
13: $M_x \leftarrow \left(\begin{array}{c|c} \vec{v}_1 x & 0 \\ \vdots & \vdots \\ \vec{v}_{t+k} x & 0 \\ \hline \vec{u}_x & 1 \end{array}\right)$ ▷ x acts on N_i/N_{i+1} by conjugation
14: **end for**
15: Compute the orbit, $O_{\vec{w}}$, and stabiliser, $B_{\vec{w}}$, of $\vec{w} = (0, \ldots, 0, 1)$ in $B = \langle M_x \mid x \in X \rangle$
16: Express $e^{-1}f$ as a vector, $\vec{u}_{\text{test}} \in V$
17: **if** $\vec{u}_{\text{test}} \notin O_{\vec{w}}$ **then**
18: **return false**, -
19: **end if**
20: Find $y_{\text{local}} \in E$ such that $e^{y_{\text{local}}}$ is equal to f in E/N_{i+1}
21: $f \leftarrow f^{y_{\text{local}}^{-1}}$ ▷ f is now equal to e in G/N_{i+1}
22: $y \leftarrow y_{\text{local}} \cdot y$
23: Find the image, X', in E of a generating set for $B_{\vec{w}}$
24: $X \leftarrow X'$
25: **end for**
26: **return true**, y
27: **end function**

Chapter 6

Conclusion

This chapter begins by illustrating the algorithms developed in the thesis by performing some example computations on polycyclic-by-finite groups using the package developed. The following section discusses aspects of the implementation, including native support for polycyclic-by-finite groups in the Magma Computational Algebra System based on the work of this thesis. The final section outlines the potential for continued research in this direction.

6.1 Examples and Run-times

Presented here is a selection of examples of computation with polycyclic-by-finite groups using the code developed. See Appendix A for details on its usage. Tables of run-times for various computations in several different example groups are given at the end of this section.

The first three examples use groups belonging to the database of perfect groups provided by Magma.

Example 6.1. Element arithmetic and conjugacy testing in a perfect group.

```
> D := PerfectGroupDatabase();

> E := PermutationGroup(D, 1920, 7);

> Order(E);

1920

> N := SolubleRadical(E);

> Q, rho := RadicalQuotient(E);

> G := PCBFMasterConstruct(E, N, Q, rho); // construct group

> PCBFGrpOrder(G); // order of group

1920

> g := PCBFRandom(G); // random element

> h := PCBFRandom(G);

> f := PCBFMult(G, g, h); // g * h

> finv := PCBFInv(G, f); // f^-1

> e := PCBFMult(G, f, finv);

> PCBFIsIdentity(G, e); // e should be the identity

true

> c := PCBFConjugate(G, g, h); // h^-1 * g * h

> isconj, hprime := PCBFIsConjugate(G, g, c); // check conjugacy - should be

true

> isconj;

true
```

Example 6.2. Computing the centre of a perfect group.

```
> D := PerfectGroupDatabase();

> E := PermutationGroup(D, 322560, 1);

> N := SolubleRadical(E);

> Q, rho := RadicalQuotient(E);

> G := PCBFMasterConstruct(E, N, Q, rho);

> PCBFGrpOrder(G);

322560

> Z := PCBFCentre(G); // compute centre of G

> PCBFGrpOrder(Z) eq Order(Centre(E)); // orders should match

true
```

Example 6.3. Computing a centraliser in a perfect group.

```
> D := PerfectGroupDatabase();

> E := PermutationGroup(D, 120);

> Order(E);

375000

> N := SolubleRadical(E);

> Q, rho := RadicalQuotient(E);

> G := PCBFMasterConstruct(E, N, Q, rho);

> PCBFGrpOrder(G);

375000

> g := PCBFRandom(G);

> C := PCBFCentraliser(G, g); // centraliser of g in G

> PCBFGrpOrder(C);

30
```

Example 6.4. Computing the derived group of a subgroup of AGL$(4, 3)$.

```
> n := 4; q := 3;

> G := AGL(n, q);

> N := MinimalNormalSubgroups(G)[1];

> M := MaximalSubgroups(Stabiliser(G, 1));

> gps := [ sub< G | N, m`subgroup > :  m in M ];

> #gps;

8

> E := gps[1];

> Order(E);

233280

> S := SolubleRadical(E);

> Q, rho := RadicalQuotient(E);

> Order(Q);

1440

> H := PCBFMasterConstruct(E, S, Q, rho);

> L := PCBFDerivedGroup(H);

> PCBFGrpOrder(L) eq Order(DerivedGroup(E)); // should be true

true
```

Example 6.5. In this example, the `PCBFEleOrder` function is used to compute the order of an element in an automorphism group of a finite group.

```
> S := SmallGroup(729, 102);
> A := AutomorphismGroup(S);
> P := PermutationGroup(A);
> Order(P);
221079456
> Degree(P);
702
> IsSoluble(P);
false
> N := SolubleRadical(P);
> Q, rho := RadicalQuotient(P);
> G := PCBFMasterConstruct(P, N, Q, rho);
> PCBFGrpOrder(G);
221079456
> g := PCBFRandom(G);
> PCBFEleOrder(G, g); // order of g
24
> H := PCBFsub(G, [g]); // subgroup generated by g
> PCBFGrpOrder(H); // should be 24
24
```

Example 6.6. Computing the centre of a reducible matrix group.

```
> D1 := ClassicalMaximals("L", 5, 2 :  classes := 1);

> D2 := ClassicalMaximals("L", 3, 2 :  classes := 1);

> X := [DirectProduct(d1, d2) :  d1 in D1, d2 in D2];

> IsIrreducible(X[1]);

false

> E := Image(PermutationRepresentation(X[1]));

> Order(E);

7741440

> S := SolubleRadical(E);

> Q, rho := RadicalQuotient(E);

> Order(Q);

20160

> G := PCBFMasterConstruct(E, S, Q, rho);

> Z := PCBFCentre(G);

> PCBFGrpOrder(Z) eq Order(Centre(E)); // should be true

true
```

Example 6.7. Let G be the projective special linear group acting on a vector space of dimension 5 over the field $\mathbb{F}_2$. The code that follows constructs an extension of an irreducible module of G, by G, as a record of type `PCBF`. The order of a random element of this extension is computed. See B.5 for details of constructions involving the type `ModCoho`.

```
> G := PSL(5, 2);

> I := IrreducibleModules(G, GF(2));

> C := CohomologyModule(G, I[2]); // module of dimension 5

> H2 := CohomologyGroup(C, 2);

> E, rho := Extension(C, H2.1);

> EG := PCBFModCohoConstruct(C, E, rho);

> y := PCBFRandom(EG);

> PCBFEleOrder(EG, y);

8
```

Example 6.8. In the code below, an extension of the natural permutation module over $\mathbb{Z}$ of the alternating group, A_5, is constructed as a record of type `PCBF`. The derived subgroup of this extension is then computed. This is an example of an infinite polycyclic-by-finite group represented using the methods described in this thesis. See B.5 for details of constructions involving the type `ModCoho`.

```
> G := Alt(5);
> M := PermutationModule(G, Integers());
> C := CohomologyModule(G, M);
> _ := CohomologyGroup(C, 2);
> E, rho := Extension(C, [1]);
> EG := PCBFModCohoConstruct(C, E, rho);
> PCBFNormalSubgrp(EG);
GrpGPC of infinite order on 5 PC-generators
PC-Relations:
> X := PCBFDerivedGroup(EG);
> PCBFQuotGrp(X);
Permutation group acting on a set of cardinality 5
Order = 60 = 2^2 * 3 * 5
(1, 3, 4)
(2, 4)(3, 5)
(1, 4, 3)
(2, 3, 5)
(2, 5, 3)
(2, 5, 4)
```

Example 6.9. Consider the Heineken group defined by the finite presentation,

$$\langle a, b, c \mid [a, [a, b]] = c, [b, [b, c]] = a, [c, [c, a]] = b \rangle.$$

This group has a quotient with the structure $[2^{24}] \cdot A_5$. The snippet of code that follows constructs this quotient as a record of type `PCBF` and computes Sylow 2- and 3- subgroups. See B.5 for details in constructions involving quotients of finitely presented groups.

```
> E := Group< a, b, c | (a, (a, b)) = c, (b, (b, c)) = a, (c, (c, a)) = b>; //
Heineken group
> Q := Alt(5);
> rho := Homomorphisms(E, Q)[1];
> K := Kernel(rho);
> K := Rewrite(E, K : Simplify := false);
> L, psi := pQuotient(K, 2, 5 :  Print := 1);
Lower exponent-2 central series for K
Group:  K to lower exponent-2 central class 1 has order 2^5
Group:  K to lower exponent-2 central class 2 has order 2^10
Group:  K to lower exponent-2 central class 3 has order 2^16
Group:  K to lower exponent-2 central class 4 has order 2^20
Group:  K to lower exponent-2 central class 5 has order 2^24
> G := PCBFFPQuotConstruct(E, Q, rho, L, psi);
> Factorisation(PCBFGrpOrder(G));
[<2, 26>, <3, 1>, <5, 1>]
> P2 := PCBFSylow(G, 2); // Compute Sylow 2-subgroup
> Factorisation(PCBFGrpOrder(P2)); // should be 2^26
[<2, 26>]
> P3 := PCBFSylow(G, 3);
> Factorisation(PCBFGrpOrder(P3)); // should be 3^1
[<3, 1>]
```

Tables 6.1–6.6 display a set of run-times in Magma for a variety of computations using the newly implemented suite of PCBF-group functions, over a collection of test groups. In particular, for each PCBF-group, run-times are measured for the following, each over 100 runs, choosing elements at random where applicable:

(i) Multiplying a pair of elements

(ii) Finding the inverse of an element

(iii) In the finite case, finding a Sylow 2-subgroup and a Sylow p-subgroup, where p is the largest prime dividing the order of the group

(iv) Computing the centre of the group

Tables 6.1–6.5 all follow the same format; the first and second columns of each table give the order (or in some cases, the structure) of the polycyclic normal subgroup, N, and the finite quotient, Q, respectively. The largest prime, p, dividing the order of group is recorded in the third column. The remaining columns give run-times as indicated, in seconds, and rounded to three decimal places. A "-" indicates that the table entry in question is not applicable for the particular example group, for instance, the run-time for a Sylow subgroup computation in an infinite group. In each table, the groups are ordered by increasing size of Q.

For each group listed in Tables 6.1–6.5, the first line of data records run-times using the PCBF representation of the group. For the sake of comparison, the second line of data gives run-times for similar computations using the existing representation of the group in Magma. In the case where there exists no native Magma method to perform the computation in question using the group's existing representation, the time taken is indicated by "∞". This occurs most often with finitely presented groups, for which no feasible permutation representation is available. There exist no intrinsic functions to compute the centre, or to find Sylow subgroups of a finitely

presented group in Magma. As a result, comparison run-times are unobtainable for such examples.

Table 6.1 gives run-times for a set of finite perfect groups. These groups are taken from the Magma Perfect Group Database, and their constructions follow the form of Examples 6.1–6.3.

Tables 6.2 and 6.3 investigate subgroups of $\mathrm{AGL}(3, 5)$ and $\mathrm{AGL}(4, 3)$ respectively. The standard Magma method is used to construct the groups, $\mathrm{AGL}(n, q)$, and subgroups are generated as in Example 6.4.

Table 6.4 displays run-times for a collection of reducible matrix groups. These groups are constructed by forming direct products of the maximal subgroups of the classical groups as in Example 6.6. For a survey on the subgroup structure of the finite classical groups, the reader is referred to (Kleidman and Liebeck, 1990).

Table 6.5 explores the performance of the package for extensions of modules as described in Section 2.3. In each of the finite groups of Table 6.5, the extensions are created by first performing a search for all irreducible modules of the given matrix group, after which the Magma cohomology functions are utilised. Example 6.7 illustrates such a construction. The infinite examples of Table 6.5 are constructed as in Example 6.8, where a module of a suitable matrix group is constructed explicitly. All of the extensions of Table 6.5 are non-split, except for the fifth, for which only a split extension is possible.

Table 6.6 consists of a collection of test groups, each of which map onto A_5. For each test group, an appropriate p-quotient of the kernel of the epimorphism is computed, and then an extension of that quotient by A_5 is constructed using the methods described in this thesis. See Example 6.9 for a sample construction.

The format of Table 6.6 is as follows. The first column gives the presentation of the large group, along with the map onto A_5. The second column gives data relating to the p-quotient, namely, its order and lower exponent-p class. The third column records the largest prime, q, dividing the order of the extension. The remaining

columns give run-times as indicated, in seconds, and rounded to three decimal places. No comparison run-times are available for these test groups, as there is no existing standard method in Magma to construct extensions of this type.

It is important to note here, that the current implementations of the algorithms in this thesis are subject to the overhead of the Magma interpreter, which adds a significant time cost. As discussed in Subsection 6.2.3, it is expected that, after native implementation and optimisation, run-times will improve by a factor of roughly 10^3. Tables of run-time data follow.

Table 6.1: Run-times for perfect groups

Group Information			Run-times (seconds)				
N	Q	p	$x \cdot y$	x^{-1}	Syl_2	Syl_p	$\mathbf{Z}$
32	A_5	5	0.060	0.180	40.040	39.000	0.840
			0.000	0.000	0.000	0.010	0.000
3888	A_5	5	0.060	0.180	52.640	48.350	0.610
			0.000	0.000	0.000	0.000	0.000
6250	A_5	5	0.050	0.140	34.930	36.190	3.480
			0.000	0.000	0.000	0.010	0.000
7776	A_5	5	0.080	0.180	70.600	39.230	1.710
			0.000	0.000	0.000	0.000	0.000
15842	A_5	89	0.120	0.360	55.360	0.190	0.200
			0.000	0.000	0.000	0.010	0.000
128	$\mathrm{PSL}(2,7)$	7	0.120	0.330	99.160	94.840	1.470
			0.000	0.000	0.000	0.000	0.000
2187	A_6	5	0.210	0.470	156.660	80.630	9.370
			0.000	0.000	0.010	0.000	0.000
128	A_7	7	0.610	0.840	319.970	547.010	13.660
			0.000	0.000	0.000	0.000	0.000

Table 6.2: Run-times for subgroups of $\mathrm{AGL}(3, 5)$

Group Information			Run-times (seconds)				
N	Q	p	$x \cdot y$	x^{-1}	Syl_2	Syl_p	**Z**
46500	1	31	0.000	0.010	0.210	0.210	0.690
			0.000	0.000	0.000	0.000	0.000
48000	1	5	0.010	0.000	0.570	0.510	1.100
			0.000	0.000	0.000	0.000	0.000
500	120	5	0.100	0.310	44.660	50.510	0.290
			0.000	0.000	0.000	0.000	0.000
50000	120	5	0.100	0.240	100.150	59.450	1.340
			0.000	0.000	0.000	0.000	0.000
50000	120	5	0.160	0.380	95.140	85.510	1.020
			0.000	0.000	0.000	0.010	0.000
250	372000	31	2.810	3.090	4599.140	7793.520	0.190
			0.000	0.000	0.000	0.010	0.000

Table 6.3: Run-times for subgroups of AGL(4, 3)

Group Information			Run-times (seconds)				
N	Q	p	$x \cdot y$	x^{-1}	Syl_2	Syl_p	**Z**
186624	1	3	0.000	0.010	1.250	1.200	1.820
			0.000	0.000	0.010	0.000	0.000
373248	1	3	0.000	0.010	1.800	1.680	2.350
			0.000	0.000	0.010	0.010	0.000
15116544	1	3	0.000	0.000	6.620	6.520	7.420
			0.000	0.000	0.010	0.010	0.000
162	1440	5	0.280	0.500	896.200	101.900	0.340
			0.000	0.000	0.000	0.000	0.000
648	1440	5	0.490	0.820	1013.610	142.860	0.500
			0.000	0.000	0.010	0.000	0.000
8748	5616	13	0.420	0.950	534.680	577.500	0.960
			0.000	0.000	0.010	0.010	0.000
8748	5616	13	0.760	1.250	546.410	905.560	0.940
			0.000	0.000	0.000	0.000	0.000

Table 6.4: Run-times for reducible matrix groups

Group Information			Run-times (seconds)				
N	Q	p	$x \cdot y$	x^{-1}	Syl_2	Syl_p	$\mathbf{Z}$
13824	1	3	0.000	0.010	0.940	0.780	1.110
			0.000	0.000	0.000	0.000	0.000
192	$\mathrm{PSL}(2,7)$	7	0.140	0.360	96.260	125.460	5.100
			0.000	0.000	0.000	0.000	0.000
9216	$\mathrm{PSL}(2,7)$	7	0.170	0.400	97.770	91.610	1.230
			0.000	0.000	0.000	0.000	0.000
221184	$\mathrm{PSL}(2,7)$	7	0.210	0.480	143.750	67.260	2.300
			0.000	0.000	0.010	0.000	0.000
221184	$\mathrm{PSL}(2,7)$	7	0.240	0.500	102.510	124.070	1.800
			0.000	0.000	0.000	0.000	0.000
384	20160	7	0.950	1.120	429.910	432.050	12.250
			0.000	0.000	0.000	0.000	0.000

Table 6.5: Run-times for module extensions

Group Information			Run-times (seconds)				
N	Q	p	$x \cdot y$	x^{-1}	Syl_2	Syl_p	**Z**
$\mathbb{Z}^5$	A_5	-	0.070	0.180	-	-	2.900
			0.000	0.000	-	-	∞
$\mathbb{Z}^7$	$\mathrm{PSL}(2,7)$	-	0.150	0.350	-	-	10.210
			0.000	0.000	-	-	∞
$\mathbb{Z}^{10}$	A_6	-	0.290	0.410	-	-	13.750
			0.000	0.000	-	-	∞
$(\mathbb{Z}/11\mathbb{Z})^3$	$\mathrm{PSL}(2,11)$	11	0.230	0.460	100.460	267.490	5.650
			0.000	0.000	∞	∞	∞
$\mathbb{Z}^5$	960	-	0.800	1.450	-	-	169.930
			0.000	0.000	-	-	∞
$\mathbb{Z}^7$	1344	-	0.570	1.020	-	-	107.550
			0.000	0.000	-	-	∞
$(\mathbb{Z}/2\mathbb{Z})^4$	$\mathrm{PSL}(4,2)$	7	0.570	0.950	548.480	415.450	14.560
			0.000	0.000	∞	∞	∞
$(\mathbb{Z}/2\mathbb{Z})^6$	$\mathrm{PSU}(4,2^2)$	5	0.610	1.100	2061.020	269.010	22.200
			0.000	0.000	∞	∞	∞

Table 6.6: Run-times for finitely presented groups that map onto A_5

Group Information				Run-times (seconds)				
Presentation and Map	p-quotient of kernel		q	$x \cdot y$	x^{-1}	Syl_2	Syl_q	$\mathbf{Z}$
	Order	p-class						
$\langle a, b, c \mid a^2 = c, b^3 = (ab)^5, cc^b = c^{ba} \rangle$ $a \mapsto (1\ 2)(3\ 4),\ b \mapsto (2\ 3\ 5),\ c \mapsto ()$	2^5	1	5	0.060	0.210	35.370	33.610	5.030
$\langle a, b, c \mid [a, [a, b]] = c, [b, [b, c]] = a, [c, [c, a]] = a \rangle$ $a \mapsto (1\ 2\ 3\ 4\ 5),\ b \mapsto (1\ 4\ 5\ 2\ 3),\ c \mapsto (1\ 4\ 2\ 3\ 5)$	2^{24}	5	5	0.150	0.270	98.840	47.360	12.910
$\langle a, b, c \mid a^2, b^3, (ab)^5, abcb^{-1}ac^{-1}b^{-1}c^{-1}bc^{-1} \rangle$ $a \mapsto (1\ 2)(3\ 4),\ b \mapsto (2\ 3\ 5),\ c \mapsto ()$	13^{55}	3	13	0.190	0.290	135.200	28.940	461.660
$\langle a, b, c \mid a^2 = (ab)^5, b^3, [a, b] = c^3, [b, c], [b, [b, c]] \rangle$ $a \mapsto (1\ 2)(3\ 4),\ b \mapsto (2\ 3\ 5),\ c \mapsto ()$	3^{210}	2	5	0.410	0.550	2772.530	2721.460	60793.650

The message to be taken from the data of Tables 6.1–6.6 is clear. Run-times increase significantly as the size of the finite quotient increases. This agrees with the complexity analysis of Theorem 3.5 and reinforces the content of the discussion immediately following that result. It should be noted that, the computation of Sylow-2 subgroups proved, in most cases, to be the most expensive computation, and that the run-time for computing the centre is dependent on the size of the normal polycyclic subgroup, as illustrated by the final example of Table 6.6.

One may conclude from the run-time data, that, with appropriate optimisations, the PCBF-group data structure is a suitable option for polycyclic-by-finite groups for which there is no available permutation or matrix representation, but whose finite quotient is small and easily represented as a permutation group.

6.2 Implementation

This section contains a description of the implementation of the polycyclic-by-finite group package using the Magma Computational Algebra System, highlighting a few technical details and efficiency considerations.

6.2.1 List of Functions

The author has implemented methods to perform the following operations in the class of polycyclic-by-finite groups:

- Multiply elements
- Compute the natural map (and its inverse) from a group to the finite quotient used in the construction
- Invert elements
- Compute the order of an element

- Transfer to category `GrpPC` or `GrpGPC`
- Find the subgroup generated by a set of elements
- Test element membership in a given group
- Coerce an element into a group
- Find the normal closure of the subgroup generated by a set of elements
- Compute the commutator subgroup of two groups having a common supergroup
- Compute Sylow subgroups (in the finite case)
- Compute the centre of a group
- Compute the derived series of a group
- Compute the lower central series of a group
- Compute the centraliser of an element
- Test whether two group elements are conjugate, and, if so, return a conjugating element

6.2.2 Technical Considerations

A major technical difficulty that arose during the course of the implementation was the inability to define Magma homomorphisms to and from polycyclic-by-finite groups represented using the newly defined structure.

The record (`Rec`) type in Magma was used to store the information defining a polycyclic-by-finite group, and since elements of type `Rec` are not recognised by Magma as algebraic structures (there is no binary operation), it was not possible to construct native homomorphisms to or from polycyclic-by-finite groups. This

problem was circumvented by maintaining data sets encoding homomorphisms, and computing images and preimages via helper methods.

The lack of a mechanism by which homomorphisms may be created proved especially problematic in the centraliser and element conjugacy functions, where, in the implementation, a finitely presented group is constructed and used to compute the maps of Line 11 in Algorithm 5.2 and Line 23 in Algorithm 5.3. At present, there is no feasible alternative to this costly solution. As a result, the implementations of the centraliser and element conjugacy functions, in their present state, should be regarded as experimental.

6.2.3 Native Support

In view of the author's progress in the area, Prof. John Cannon, Head of the Magma Computational Algebra Group, expressed interest in incorporating the methods developed by in to standard releases of the Magma Computational Algebra System, to enable built-in support of polycyclic-by-finite groups. This necessitated direct collaboration between the author and members of the Magma Computational Algebra Group.

To this end, in April, 2010, the author was invited by Prof. Cannon to visit the University of Sydney for a 5 week period. During this time, the author worked closely with Dr. William Unger to implement a set of functions for computing with polycyclic-by-finite groups.

The first stage of the implementation involved defining internal Magma data types for polycyclic-by-finite groups (New type: `GrpPCBF`) and for elements of polycyclic-by-finite groups (New type: `GrpPCBFElt`). These data types are defined in a similar fashion as for the current standard Magma group types, Permutation groups; `GrpPerm`, Matrix groups; `GrpMat`, Polycyclic groups; `GrpPC` (or `GrpGPC`), etc. The construction functions for these types mimic the construction functions of the

author's program code.

With these new types available, the author's package code was completely rewritten to correspond with the new type definitions.

Naturally, many of the methods implemented for computing with polycyclic-by-finite groups (listed in Section 6.2.1), depend heavily on element multiplication. As a result a significant portion of time was devoted to improving the performance of the multiplication algorithm.

After discussion with the author on the operation of the algorithm, a few optimisation adjustments were suggested by Dr. Unger. These yielded a noticeable improvement in efficiency. After this, to avoid the inherent overhead that is associated with the Magma interpreter, the entire multiplication algorithm was rewritten in the programming language C (the language in which the Magma Computational Algebra System is written) and then added to the standard Magma library. The implementation in C offered a significant performance gain — the speed of the function was improved by a factor of roughly 10^3.

The results achieved were encouraging. The multiplication algorithm (and its dependants) performed well under most of the tests administered.

6.3 Future Work

There are several subareas of this project that are deserving of continued research. In addition to improvement in the performance of the multiplication algorithm, the following functionality is desirable:

- Find normalisers of subgroups
- Test whether two given subgroups are conjugate
- Find all conjugacy classes of a group
- Compute the character table of a group

Finding normalisers and testing subgroup conjugacy have immediate applications to computing within automorphism groups of p-groups. This forms part of ongoing research of Prof. Derek Holt and David Howden at the Warwick Mathematics Institute.

With regard to computing character tables of groups, the author plans to seek the expertise of Dr. William Unger in adapting his recently developed methods to the category of polycyclic-by-finite groups.

Another avenue which can be explored is the possibility of centraliser computation conjugacy testing in infinite polycyclic-by-finite groups. This may perhaps involve extending the work of Eick (2001).

The author is confident that, with continued efforts at optimisation, the suite of algorithms will soon prove to be even more useful from a practical point of view, in the contexts described here, and otherwise.

Appendix A

Package Documentation

A.1 Introduction

The algorithms under development for polycyclic-by-finite groups employ current methods available for polycyclic presentations, together with the well developed base-and-strong-generating-set techniques used for computation with permutation groups. This document describes the data structure and the functions that have been implemented using the Magma Computational Algebra System. The current version at the time of writing is: Magma V2.17-5.

A.2 Getting Started

To load all necessary function and record definitions, simply start Magma from the directory in which the source code files are held and execute the command:

```
load "pcbfmain.m";
```

Alternatively, to load all function and record definitions, and execute a series of tests on the source code, execute the command:

```
load "pcbfmain-test.m";
```

A.3 Creation of a Group

To create a record of type `PCBF` representing a polycyclic-by-finite group the user is required to supply:

(i) The polycyclic-by-finite group, E.

(ii) A normal polycyclic subgroup, N, of E such that the associated quotient is finite.

(iii) The quotient, E/N, (of type `GrpPerm` or `GrpMat`) and the natural map, $E \to E/N$.

A.3.1 Construction Functions

`PCBFMasterConstruct(E, N, Q, rho)`

Constructs a record of type `PCBF` to represent the polycyclic-by-finite group, E, with normal polycyclic subgroup, N, and quotient group, Q.

In the method descriptions that follow, G, H and K denote groups of this type.

A.4 Basic Group Properties

The functions described here provide access to basic information stored for a PCBF-group, G.

A.4.1 Infrastructure

`PCBFGenerators(G)`

An enumerated sequence containing a complete set of generators for G.

`PCBFNGens(G)`

`PCBFNumberOfGenerators(G)`

The number of defining generators for G.

`PCBFNormalSubgrp(G)`

A group of type `GrpPC` or `GrpGPC` (depending on whether G is finite or infinite respectively) isomorphic to the polycyclic normal subgroup used in the construction of the PCBF representation of G.

`PCBFQuotGrp(G)`

The quotient group used in the construction on the PCBF representation of G.

A.4.2 Numerical Invariants

`PCBFGrpOrder(G)`

The order of G, returned as an ordinary integer.

A.4.3 Predicates

`PCBFIsFinite(G)`

Returns `true` if G is finite, `false` otherwise.

`PCBFIsTrivial(G)`

Returns `true` if G has order 1, `false` otherwise.

`PCBFIsSoluble(G)`

`PCBFIsSolvable(G)`

Returns `true` if G is soluble, `false` otherwise.

A.5 Elements

As discussed above, an element of a PCBF-group is stored as an ordered pair with components belonging to the quotient group and normal polycyclic subgroup respectively.

A.5.1 Definition of Elements

`PCBFCreateElement(G, x, y)`

Given elements, x, of the quotient group, and, y, of the normal subgroup used in the construction of the PCBF representation of G, construct the element of G defined by the ordered pair, (x, y).

`PCBFrho(G, g)`

Given an element, g, of G, return the image of g under the natural map onto the quotient group used in the construction of G.

`PCBFrhoinv(G, u)`

Given an element, u, of the quotient group used to construct G, return an inverse image of u in G.

`PCBFphiinv(G, n)`

Given an element, n, of the normal subgroup group used to construct G, return the image of n in G under the inclusion map.

`PCBFIdentity(G)`

`PCBFId(G)`

Construct the identity element of G.

A.5.2 Arithmetic Operations on Elements

New elements can be computed from existing ones using the following implementations of standard functions.

`PCBFMult(G, g, h)`

`PCBFMultiply(G, g, h)`

`PCBFStar(G, g, h)`

The product of the elements, g and h, of G.

`PCBFInverse(G, g)`

`PCBFInv(G, g)`

The inverse of the element, g, of G.

`PCBFPower(G, g, n)`

The n-th power of the element, g, of G, where n is an integer.

`PCBFDivide(G, g, h)`

`PCBFSlash(G, g, h)`

The quotient of the element, g, of G by the element, h, of G, i.e. the element, $g/h = gh^{-1}$.

`PCBFConjugate(G, g, h)`

The conjugate of the element, g, of G, by the element h, of G, i.e. the element $g^h = h^{-1}gh$.

`PCBFCommutator(G, g, h)`

The commutator of the elements, g and h, of G, i.e. the element $g^{-1}h^{-1}gh$.

`PCBFSeqCommutator(G, [`$\mathtt{g}_1, \ldots, \mathtt{g}_n$`])`

Given the sequence, $g_1, \ldots, g_n$, of elements of G, return the commutator $(g_1, \ldots, g_n)$. Commutators are left-normed, so they are evaluated from left to right.

A.5.3 Properties of Elements

`PCBFEleOrder(G, g)`

The order of the element, g, of G.

A.5.4 Predicates for Elements

`PCBFeq(G, g, h)`

Given g and h belonging to G return `true` if g and h are the same element,

`false` otherwise.

`PCBFne(G, g, h)`

Given g and h belonging to G return `true` if g and h are distinct elements, `false` otherwise.

`PCBFIsIdentity(G, g)`

`PCBFIsId(G, g)`

Returns `true` if g is the identity element in G and `false` otherwise.

A.5.5 Set Operations

`PCBFRandom(G)`

`PCBFRandomElement(G)`

Returns a pseudo-random element of G.

`PCBFRandomSequence(G, l)`

`PCBFRandomElementSequence(G, l)`

Given a non-negative integer, l, construct a sequence of length l of pseudo-random elements of G.

`PCBFRepresentative(G)`

`PCBFRep(G)`

A representative element of G. For a PCBF-group this always returns the identity element.

A.6 Subgroups

Subgroups of PCBF-groups are treated as independent PCBF-groups in their own right, with the subgroup relationship maintained in internal data structures.

A.6.1 Definition of Subgroups by Generators

`PCBFsub(G, L)`

Construct the subgroup of G generated by the elements of the sequence, L.

`PCBFncl(G, L)`

Construct the normal closure of the subgroup of G generated by the elements of the sequence, L.

A.6.2 Membership and Coercion

`PCBFin(G, g, H)`

Given an element, g, of G return `true` if g is an element of H, `false` otherwise.

`PCBFnotin(G, g, H)`

Given an element, g, of G return `true` if g is not an element of H, `false` otherwise.

`PCBFCoerce(G, g, H)`

Given an element, g, of G and another PCBF-group, H, attempt to write g as an element of H. If the rewrite is successful then the function returns `true` and g written as an element of H, otherwise the function returns `false` and an empty tuple.

A.6.3 Standard Subgroup Constructions

`PCBFSubgroupConjugate(G, H, g)`

Construct the conjugate, $g^{-1}Hg$, of the group, H, under the action of the element, g, of G. The group, H, is assumed to be a subgroup of G.

`PCBFCommutatorSubgroup(H, K)`

Construct the commutator subgroup of groups, H and K, where H and K are subgroups of a common PCBF-group.

`PCBFSubgroupNormalClosure(G, H)`

The normal closure of the subgroup, H, in the group, G.

A.6.4 Sylow Subgroups

`PCBFSylowSubgroup(G, p)`

`PCBFSylow(G, p)`

A Sylow p-subgroup for the finite PCBF-group, G.

`PCBFSylowConjugatingElement(G, H, K)`

Given Sylow p-subgroups, H and K, of a finite PCBF-group, G, return the element of G conjugating H into K.

A.6.5 Centralisers

`PCBFCentraliser(G, g)`

`PCBFCentralizer(G, g)`

The centraliser of the element, g, of the finite PCBF-group, G.

A.7 Normal Subgroups and Subgroup Series

A.7.1 Normal Structure

`PCBFSolubleRadical(G)`

`PCBFSolvableRadical(G)`

The soluble radical of G.

A.7.2 Characteristic Subgroups

`PCBFCentre(G)`

`PCBFCenter(G)`

The centre of G.

`PCBFCommutatorSubgroup(G)`

`PCBFDerivedSubgroup(G)`

`PCBFDerivedGroup(G)`

The derived subgroup of G.

A.7.3 Subgroup Series

`PCBFDerivedSeries(G)`

The derived series of G. The series is returned as a sequence of subgroups.

`PCBFLowerCentralSeries(G)`

The lower central series of G. The series is returned as a sequence of subgroups.

A.8 Conjugacy

`PCBFIsConjugate(G, g, h)`

Given elements, g and h, belonging to a finite PCBF-group, G, return the value true if there exists $c \in G$ such that $g^c = h$. If so, the function returns such a conjugating element as second value.

A.9 Transfer Between Group Categories

A.9.1 Transfer to GrpPC or GrpGPC

`PCBFPolycyclicGroup(G)`

Write G as a GrpPC or a GrpGPC if G is polycyclic. Attempts are made where possible to return a GrpPC. If the category transfer is successful, an ordered pair encoding the isomorphism is returned as a second value.

`PCBFPolycycliciso(G, P, isodata, g)`

Given a polycyclic group, P, isomorphic to G, return the image of the element,

g, of G in P as defined by `isodata`.

`PCBFPolycyclicisoinv(G, isodata, h)`

Given an element, h, of a polycyclic group isomorphic to G, return the image of the element, h, in G as defined by `isodata`.

Appendix B

Program Listing

Listing B.1: grputils.m

```
/*

File:                   grputils.m


Last modified:          Fri, 07 Jan 2011 17:21:11 +0000


Author(s):              Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


Company:                University of Warwick <http://www.warwick.ac.
    uk>


Description:

Miscellaneous group utilities.


Notes:

MAGMA V2.16-13.


Copyright 2006-2010, University of Warwick. All rights reserved.

*/



WordMultiply := function(posimgs, negimgs, word)
```

```
  /*

  Arguments:

    posimgs: Sequence of group elements, Type(s): SetIndx,
    SeqEnum

    negimgs: Sequence of group elements, Type(s): SetIndx,
    SeqEnum

    word: Sequence of integers representing a word in group
    elements indexed by posimgs and negimgs, Type(s): SeqEnum


  Parameters:


  Return Type(s):

    GrpElt


  Description:

    Computes the product of the sequence of group elements
    represented by word.

  */

  ans := Identity(Parent(posimgs[1]));
  for i in [1 .. #word] do
    ans *:= word[i] gt 0 select posimgs[word[i]] else negimgs[-
       word[i]];
  end for;
  return ans;
end function;
```

Listing B.2: permsv.m

```
/*

File:                permsv.m


Last modified:       Fri, 07 Jan 2011 17:21:11 +0000


Author(s):           Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>
```

```
12 Company:              University of Warwick <http://www.warwick.ac.
       uk>

   Description:

17 This file contains some auxiliary methods for working with
       orbits and Schreier
   vectors.

   Notes:
22
   MAGMA V2.16-13.

   Copyright 2006-2010, University of Warwick. All rights reserved.
27
   */

   WordInverse := func< x | [ -y : y in Reverse(x) ] >; // inverse
       of a word
32

   OrbitSV := function(alpha, X)

     /*
37
     Arguments:

       alpha: Element of the set upon which the elements of X act,
       Type(s): Elt
42
       X: Sequence of elements of a group acting on the set to
       which alpha belongs, Type(s): SeqEnum

47   Parameters:

     Return Type(s):

52     SeqEnum, SeqEnum

     Description:
```

```
57      Computes the orbit of the point alpha under the action of
           X   . The function returns the orbit and a Schreier
           vector
        corresponding to the orbit with indexing relative to the
        set , X.

62    */

    r := #X;
          v := [0];
    orb := [alpha];
67  orbsize := 1;
    c := 1;
    while c le orbsize do
      pt := orb[c];
      for i in [1 .. r] do
72      ppt := pt^X[i];
        if ppt notin orb then
          Append(~orb, ppt);
          orbsize +:= 1;
          v[orbsize] := i;
77      end if;
      end for;
      c +:= 1;
    end while;
    return v, orb;
82 end function;


 BasicOrbitsSV := function(G, B, S)

87    /*

    Arguments:

      G: Permutation group, Type(s): GrpPerm
92
      B: Base of G, Type(s): SeqEnum

      S: Strong generating set of G relative to B, Type(s):
      SetIndx
97

    Parameters:


102   Return Type(s):
```

```
    SeqEnum, SeqEnum

107   Description:

    Computes the basic orbits and Schreier vectors (relative to
    B and S) of G.

112   */

  l := #B;
  if l eq 0 then return [], []; end if;
  m := #S;
117   X := Isetseq(S);
  sv := [];
  bo := [];
  bs := [Stabiliser(G, B[1 .. i - 1]) : i in [1 .. l]];
  sv[1], bo[1] := OrbitSV(B[1], X);
122   for i in [2 .. l] do
    indexmap := [k : k in [1 .. m] | X[k] in bs[i]];
    sv[i], bo[i] := OrbitSV(B[i], X[indexmap]);
    for j in [2 .. #sv[i]] do
      sv[i][j] := indexmap[sv[i][j]];
127     end for;
  end for;
  return sv, bo;
end function;

132
SVWordInv := function(G, S, v, orb, pt)

  /*

137   Arguments:

    G: Permutation group, Type(s): GrpPerm

    S: Strong generating set for G, Type(s): SetIndx
142
    v: Schreier vector corresponding to orbit orb (with
    indexing relative to S), Type(s): SeqEnum

    orb: Sequence containing exactly the orbit of orb[1] under
147   S, Type(s): SeqEnum

    pt: Element of the set upon which G acts, Type(s): Elt
```

```
152     Parameters:


        Return Type(s):

157       SeqEnum


        Description:

162       Returns inverse of what SVWord should return, as an integer
          sequence i.e. a word in strong generators taking base point
          to pt as defined by the Schreier vector, v.

        */
167
        pos := Position(orb, pt);
        r := v[pos];
        w := [];
        while r ne 0 do
172       Append(~w, -r);
          pt := r gt 0 select pt^(S[r]^-1) else pt^(S[-r]);
          pos := Position(orb, pt);
          r := v[pos];
        end while;
177     return WordInverse(w);
      end function;


      SVPermutationInv := function(G, S, v, orb, pt)
182
        /*

        Arguments:

187       G: Permutation group, Type(s): GrpPerm

          S: Strong generating set for G, Type(s): SetIndx

          v: Schreier vector corresponding to orbit orb (with
192       indexing relative to S), Type(s): SeqEnum

          orb: Sequence containing exactly the orbit of orb[1] under
          S, Type(s): SeqEnum

197       pt: Element of the set upon which G acts, Type(s): Elt
```

```
  Parameters:

202
  Return Type(s):

    GrpPermElt

207
  Description:

    Returns inverse of what SVPermutation should return. i.e. a
    permutation (or matrix) at level of basic orbit orb taking
212 base point to pt.

  */

  pos := Position(orb, pt);
217 r := v[pos];
  u := Id(G);
  y := Id(G);
  while r ne 0 do
    y := r gt 0 select S[r]^-1 else S[-r];
222 u *:= y;
    pt := pt^y;
    pos := Position(orb, pt);
    r := v[pos];
  end while;
227 return u^-1;
end function;
```

Listing B.3: recfrmtdef.m

```
/*
2
File:                recfrmtdef.m


Last modified:       Wed, 10 Nov 2010 16:37:18 +0000
7

Author(s):           Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


12 Company:          University of Warwick <http://www.warwick.ac.
   uk>


Description:
```

```
17 The records used to represent PCBF-groups are defined here.

   Notes:

22 MAGMA V2.16-13.

   Copyright 2006-2010, University of Warwick. All rights reserved.

27 */

   // define record formats to store the relevant data

32
   PDF := recformat<
     G: GrpPerm, // the permutation group itself
     natset: SetIndx, // the natural set on which G acts
     B: SeqEnum, // base of gp
37   bo: SeqEnum, // basic orbits of gp
     S: SetIndx, // set of strong generators of gp
     sgindexlist: SeqEnum, // sgindexlist[i] is list of indices of
        strong gens lying in the i-th basic stabiliser G[i]
     sv: SeqEnum, // sv[i] is Schreier vector for basic orbit
        number i
     tail: SeqEnum,
42   tailinv: SeqEnum,
     utail: SeqEnum,
     utailinv: SeqEnum
     >;

47 PCBF := recformat<
     E, // master group, used for testing
     L, // normal subgroup, used for testing
     Q, // quotient E/L
     rho: Map, // natural homomorphism from E to Q, used for
        testing
52   GR: Rec, // record of type PDF
     strgenpreimgs: SetIndx, // preimages of strong generators
     strgeninvpreimgs: SetIndx, // preimages of inverses of strong
        generators
     N: Grp, // polycyclically presented group isomorphic to L
     phi: Map, // isomorphism from L onto N, used for testing
57   pcgens: SetIndx, // generators of polycyclic normal subgroup
     pcgenconjugates: SeqEnum,
     pcgenconjugatesinv: SeqEnum,
```

```
    taileltss: SeqEnum,
    taileltsinv: SeqEnum,
62  e: Tup, // identity element of this PCBF-group
    strgenpreimgsnf: SeqEnum,
    strgeninvpreimgsnf: SeqEnum,
    pcgensnf: SeqEnum, // sequence of elements (in normal form)
        that generate normal subgroup
    grpgens: SeqEnum, // sequence of elements (in normal form)
        that generate the entire group
67  ismaster: BoolElt, // true iff this record is created from a
        MAGMA-known group
    supergrp: Rec // parent group
    >;
```

Listing B.4: permdata.m

```
1  /*

   File:                 permdata.m


6  Last modified:        Fri, 07 Jan 2011 17:21:11 +0000


   Author(s):            Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>

11
   Company:              University of Warwick <http://www.warwick.ac.
       uk>


   Description:
16
   The methods in this file are used to collect the required data
       from the
   finite quotient of a polycyclic-by-finite group.


21 Notes:

   MAGMA V2.16-13.


26 Copyright 2006-2010, University of Warwick. All rights reserved.

   */


31 load "grputils.m";
```

```
load "permsv.m";
load "recfrmtdef.m";


forward StrongGenWord, WordPermutation;


GrpPermData := function(G, B, S, sv, bo)

  /*

  Arguments:

    G: Permutation group, Type(s): GrpPerm

    B: Base of G, Type(s): SeqEnum

    S: Strong generating set of G, Type(s): SetIndx

    sv: Seqeunce of Schreier vectors relative to (B, S),
    Type(s): SeqEnum

    bo: Sequence of basic orbits relative to B, Type(s):
    SeqEnum


  Parameters:


  Return Type(s):

    Rec


  Description:

    This function collects the data required from a permutation
    group that is to be subsequently viewed as a quotient of a
    PCBF-group.

    For some 1 <= i <= #B, some 1 <= j <= #bo[i], let u be the
    permutation defined by sv[i] which takes B[i] to bo[i][j].
    Let x_k be a strong generator belonging to S[i], and let
    h_1, h_2 be the permutations defined by sv[i] which take
    B[i] to the image of B[i] under u * x_k, u * (x_k)^-1
    respectively.

    Then
```

```
81        u * x_k = t_1 * h_1

       and

          u * (x_k)^-1 = t_2 * h_2
86
       for some t_1, t_2 in G[i + 1]. The elements tail[i][j][k]
       and tailinv[i][j][k] are lists of integers defining words
       over S, which represent t_1 and t_2 respectively. If G is a
       quotient group, E/N, of some larger group, E, then
91
          u * x_k = t_1 * h_1 * n_1

       and

96        u * (x_k)^-1 = t_2 * h_2 * n_2

       for some n_1, n_2 in N. The entries utail[i][j][k] and
       utailinv[i][j][k] act as pointers to the elements n_1 and
       n_2, a 0 value indicating that the normal subgroup element
101    is trivial.

       This function returns a record holding BSGS data for G
       together with the arrays tail, tailinv, utail and utailinv
       which are used in the construction of the PCBF-group of
106    which this group is a finite quotient.

     */

     GR := rec<PDF | >;
111  GR`G := G;
     GR`natset := GSet(G);
     GR`B := B;
     l := #GR`B;
     GR`S := S;
116  m := #S;
     GR`bo := bo;
     GR`sv := sv;
     bs := [Stabiliser(G, B[1 .. i - 1]) : i in [1 .. l]];
     sgindexlist:=[[1 .. m]];
121  for i in [2..l] do
       sgindexlist[i] := [k : k in [1 .. m] | S[k] in bs[i]];
     end for;
     GR`sgindexlist := sgindexlist;
     tail := [];
126  tailinv := [];
     utail := [];
```

```
utailinv := [];
gct := 0;
gctinv := 0;
for i in [1 .. l] do
  tail[i]:=[];
  tailinv[i] := [];
  utail[i]:= [];
  utailinv[i] := [];
  orb := GR`bo[i];
  v := sv[i];
  sg := sgindexlist[i];
  orbsize := #orb;
  for j in [1 .. orbsize] do
    tail[i][j]:= [];
    tailinv[i][j] := [];
    utail[i][j]:= [];
    utailinv[i][j] := [];
    pt := orb[j];
    nsg := #sg;
    for k in [1 .. nsg] do
      ipt := pt^S[sg[k]];
      pos := Position(orb, ipt);
      if v[pos] eq sg[k] or v[j] eq -sg[k] then
        // This image is a definition
        tail[i][j][k] := [Integers()|];
        utail[i][j][k] := 0;
      else
        tail[i][j][k] := StrongGenWord(GR, SVPermutationInv(G,
            S, v, orb, pt)*S[sg[k]]*SVPermutationInv(G, S, v,
           orb, ipt)^-1);
        gct +:= 1;
        utail[i][j][k] := gct;
      end if;
      ipt := pt^(S[sg[k]]^-1);
      pos := Position(orb, ipt);
      if v[pos] eq -sg[k] or v[j] eq sg[k] then
        // This image is a definition
        tailinv[i][j][k] := [Integers()|];
        utailinv[i][j][k] := 0;
      else
        tailinv[i][j][k] := StrongGenWord(GR, SVPermutationInv
           (G, S, v, orb, pt)*(S[sg[k]]^-1)*SVPermutationInv(G
           , S, v, orb, ipt)^-1);
        gctinv +:= 1;
        utailinv[i][j][k] := gctinv;
      end if;
    end for;
  end for;
```

```
    end for;
    GR`tail := tail;
    GR`utail := utail;
    GR`tailinv := tailinv;
176 GR`utailinv := utailinv;
    return GR;
end function;


181 StrongGenWord := function(GR, g)

    /*

    Arguments:
186
      GR: Record of type PDF, Type(s): Rec

      g: Element of GR`G, Type(s): GrpPermElt

191
    Parameters:


    Return Type(s):
196
      SeqEnum


    Description:
201
      Returns an integer sequence representing a word in the
      strong generators for g.

    */
206
    w := [];
    l := #GR`B;
    for i in [1 .. l] do
      ipt := GR`B[i]^g;
211   w := SVWordInv(GR`G, GR`S ,GR`sv[i], GR`bo[i], ipt) cat w;
      g *:= SVPermutationInv(GR`G, GR`S, GR`sv[i], GR`bo[i], ipt)
          ^-1;
    end for;
    return w;
end function;
216

StrongGenNormalForm := function(GR, g)
```

```
    /*
221
    Arguments:

      GR: Record of type PDF, Type(s): Rec

226   g: Element of GR`G, Type(s): GrpPermElt


    Parameters:

231
    Return Type(s):

      SeqEnum

236
    Description:

      Returns a 2-D integer sequence representing the normal form
      of g relative to the base and strong generating set held in
241   GR.

    */

    nf := [];
246 l := #GR`B;
    for i in [1 .. l] do
      ipt := GR`B[i]^g;
      nf[i] := SVWordInv(GR`G, GR`S ,GR`sv[i], GR`bo[i], ipt);
      g *:= WordPermutation(GR, nf[i])^-1;
251 end for;
    return nf;
  end function;


256 WordPermutation := function(GR, w)

    /*

    Arguments:
261
      GR: Record of type PDF, Type(s): Rec

      w: Integer sequence representing word over the strong
      generating set held in GR, Type(s): SeqEnum
266
```

```
  Parameters:


  Return Type(s):

    GrpPermElt


  Description:

    Returns the permutation of GR`G represented by the integer
    sequence w.

  */

  g := Id(GR`G);
  if IsEmpty(GR`S) then return g; end if;
  wordlength := #w;
  for i in [1 .. wordlength] do
    g *:= w[i] gt 0 select GR`S[w[i]] else (GR`S[-w[i]])^-1;
  end for;
  return g;
end function;
```

Listing B.5: pcbfconstruct.m

```
/*

File:                pcbfconstruct.m


Last modified:       Fri, 07 Jan 2011 17:21:11 +0000


Author(s):           Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


Company:             University of Warwick <http://www.warwick.ac.
    uk>


Description:

The methods in this file enable the user to:
1. Contruct records of type PCBF to represent polycyclic-by-
    finite group.
2. Access basic structural information about a polycyclic-by-
    finite group
```

```
20 represented in a record of type PCBF.

   Notes:

25 MAGMA V2.16-13.

   Copyright 2006-2010, University of Warwick. All rights reserved.

30 */

   load "permdata.m";

35
   PCBFMasterConstruct := function(E, L, Q, rho)

     /*

40   Arguments:

       E: Polycyclic-by-finite group, Type(s): Grp

       L: Polycyclic normal subgroup of E, Type(s): GrpPerm,
45     GrpMat

       Q: Quotient E/L (finite), Type(s): GrpPerm

       rho: Natural map E -> Q, Type(s): Map
50

     Parameters:


55   Return Type(s):

       Rec


60   Description:

       Returns a record representing a PCBF-group isomorphic to E,
       with normal polycyclic subgroup and quotient ismorphic to L
       and Q respectively.
65
     */
```

```
   EG := rec<PCBF | >;
   EG`E := E;
70 EG`L := L;
   EG`Q := Q;
   B := Base(Q);
   S := StrongGenerators(Q);
   ReduceGenerators(~Q); // a nonredundant set of strong generators is required
75 S := StrongGenerators(Q);
   bo := BasicOrbits(Q);
   sv := SchreierVectors(Q);
   GR := GrpPermData(Q, B, S, sv, bo);
   EG`GR := GR;
80 EG`rho := rho;
   strgenpreimgs := (GR`S)@@rho;
   EG`strgenpreimgs := strgenpreimgs;
   m := #strgenpreimgs;
   strgeninvpreimgs := {@@}; // this set is needed in the case where a strong generator has order 2
85 for i in [1 .. m] do
     Include(~strgeninvpreimgs, (Order(S[i]) eq 2) select strgenpreimgs[i] else strgenpreimgs[i]^-1);
   end for;
   EG`strgeninvpreimgs := strgeninvpreimgs;
   // the following only works if L is of the type GrpPerm, GrpMat, GrpAb or GrpGPC, phi : L -> N is an isomorphism onto the polycyclically presented group, N
90 if IsFinite(L) then
     N, phi := PCGroup(L);
   else
     N, phi := GPCGroup(L);
   end if;
95 pcgens := PCGenerators(N);
   r := #pcgens;
   pcgenconjugates := [];
   pcgenconjugatesinv := [];
   for i in [1 .. m] do
100  pcgenconjugates[i] := [N | ];
     pcgenconjugatesinv[i] := [N | ];
     for j in [1 .. r] do
       // computing conjugates of the j-th polycyclic generator with the ith preimage of the strong generating set of Q and its inverse
       yim := (pcgens[j])@@phi;
105    pcgenconjugates[i][j] := phi(yim^strgenpreimgs[i]); // stored as an element of N
       pcgenconjugatesinv[i][j] := phi(yim^strgeninvpreimgs[i]); // stored as an element of N
```

```
      end for;
    end for;
    l := #GR`B; // number of elements in the stored base of the
        quotient group
110 tail := GR`tail;
    tailinv := GR`tailinv;
    utail := GR`utail;
    utailinv := GR`utailinv;
    h := [];
115 g := [];
    tailelts := [N | ];
    taileltsinv := [N | ];
    for i in [1 .. l] do
      orb := GR`bo[i];
120   v := GR`sv[i];
      sg := GR`sgindexlist[i];
      for j in [1 .. #orb] do
        pt := orb[j];
        for k in [1 .. #sg] do
125       g := SVWordInv(Q, S, v, orb, pt);
          if utail[i][j][k] ne 0 then
            ipt := pt^S[sg[k]];
            h := SVWordInv(Q, S, v, orb, ipt);
            ux := WordMultiply(strgenpreimgs, strgeninvpreimgs, g)
                 * strgenpreimgs[sg[k]];
130         th := WordMultiply(strgenpreimgs, strgeninvpreimgs,
                tail[i][j][k]) * WordMultiply(strgenpreimgs,
                strgeninvpreimgs, h);
            tailelts[utail[i][j][k]] := phi((th^-1) * ux);
          end if;
          if utailinv[i][j][k] ne 0 then
            ipt := pt^(S[sg[k]]^-1);
135         h := SVWordInv(Q, S, v, orb, ipt);
            uxinv := WordMultiply(strgenpreimgs, strgeninvpreimgs,
                 g) * strgeninvpreimgs[sg[k]];
            th := WordMultiply(strgenpreimgs, strgeninvpreimgs,
                tailinv[i][j][k]) * WordMultiply(strgenpreimgs,
                strgeninvpreimgs, h);
            taileltsinv[utailinv[i][j][k]] := phi((th^-1) * uxinv)
                ;
          end if;
140     end for;
      end for;
    end for;
    EG`N := N;
    EG`pcgens := pcgens;
145 EG`pcgenconjugates := pcgenconjugates;
    EG`pcgenconjugatesinv := pcgenconjugatesinv;
```

```
  EG`tailelts := tailelts;
  EG`taileltsinv := taileltsinv;
  e := <StrongGenNormalForm(GR, Id(GR`G)), Id(N)>;
150  EG`e := e;
  EG`ismaster := true;
  EG`strgenpreimgsnf := [];
  EG`strgeninvpreimgsnf := [];
  for i in [1 .. #GR`S] do
155    EG`strgenpreimgsnf[i] := <StrongGenNormalForm(GR, S[i]), Id(
         N)>;
    EG`strgeninvpreimgsnf[i] := <StrongGenNormalForm(GR, S[i
         ]^-1), Id(N)>;
  end for;
  EG`pcgensnf := [];
  for i in [1 .. #pcgens] do
160    EG`pcgensnf[i] := <e[1], pcgens[i]>;
  end for;
  EG`phi := phi;
  EG`grpgens := EG`strgenpreimgsnf cat EG`pcgensnf;
  EG`supergrp := rec<PCBF | >;
165  return EG;
end function;


PCBFPrecomputedDataConstruct := function(GR, N, pcgens,
    pcgenconjugates, pcgenconjugatesinv, tailelts, taileltsinv)
170
  /*

  Arguments:

175    GR: Record containing permutation group data, Type(s): Rec

    N: Normal Subgroup, Type(s): GrpPC or GrpGPC

    pcgens: Generating sequence of N, Type(s): SeqEnum
180
    pcgenconjugates: 2-D sequence of conjugates, Type(s):
    SeqEnum

    pcgenconjugatesinv: 2-D sequence of conjugates, Type(s):
185    SeqEnum

    tailelts: Sequence of tail elements, Type(s): SeqEnum

    taileltsinv: Sequence of tail elements, Type(s): SeqEnum

190
```

```
    Parameters:


195 Return Type(s):

      Rec


200 Description:

      This function constructs a record representing a PCBF-group
      from the data provided. The exact specifications of the
      arguments are given below.
205
      The record, GR, is assumed to be output from the function
      GrpPermData. See the documentation in the function header
      for details.

210   The group, N, is the normal polycyclic subgroup of the
      polycyclic-by-finite group in question, of type either
      GrpPC or GrpGPC.

      The enumerated sequence pcgens contains the generating
215   sequence of N used to compute pcgenconjugates and
      pcgenconjugatesinv.

      Let S be the strong generating set held in the record GR.
      The 2-D sequence pcgenconjugates contains the image of each
220   element of pcgens under the conjugation action of each
      element of S. Specifically,

        pcgenconjugates[i][j] = pcgens[j]^S[i].

225   Similarly,

        pcgenconjugatesinv[i][j] = pcgens[j]^(S[i]^-1).

      The enumerated sequence tailelts contains the elements of N
230   that result when one attempts to right multiply (in the
      proposed polycyclic-by-finite group) a basic transversal
      element of the base and strong generating structure held in
      GR by a strong generator in S.

235   Let G be the permutation group held in GR, B its base and
      assume that B defines the stabiliser chain

        G[1] > G[2] > ... > G[m + 1] = 1.
```

```
    For 1 <= i <= #B, 1 <= j <= #GR`bo[i], let g be the
    permutation defined by sv[i] which takes B[i] to bo[i][j].
    Let x_k be a strong generator belonging to S[i], and let h
    be the permutation defined by sv[i] which take B[i] to the
    image of B[i] under g * x_k.

    Then

      g * x_k = t * h * n

    for some t in G[i + 1], n in N. The element, t, is encoded
    by the integer list GR`tail[i][j][k] as described in the
    function header of GrpPermData. The element, n, is stored
    in tailelts as follows:

      tailelts[GR`utail[i][j][k]] = n,

    where GR`utail is the sequence of integer pointers computed
    in GrpPermData. Likewise,

      tailelts inv[GR`utailinv[i][j][k]] = n',

    where

      g * (x_k)^-1 = t' * h' * n'

    for some t' in G[i + 1] (encoded by tailinv[i][j][k]), n'
    in N.

    A record representing a PCBF-group is returned.

    Given the complicated nature of the arguments to this
    function, the easiest course to follow may be to copy and
    modify the code in the function PCBFMasterConstruct to suit
    the particular example with which you wish to work.

*/

EG := rec<PCBF | >;
EG`Q := GR`G;
S := GR`S;
EG`GR := GR;
EG`N := N;
EG`pcgens := pcgens;
EG`pcgenconjugates := pcgenconjugates;
EG`pcgenconjugatesinv := pcgenconjugatesinv;
EG`tailelts := tailelts;
EG`taileltsinv := taileltsinv;
```

```
  e := <StrongGenNormalForm(GR, Id(GR`G)), Id(N)>;
  EG`e := e;
290  EG`ismaster := true;
  EG`strgenpreimgsnf := [];
  EG`strgeninvpreimgsnf := [];
  for i in [1 .. #GR`S] do
    EG`strgenpreimgsnf[i] := <StrongGenNormalForm(GR, S[i]), Id(
       N)>;
295    EG`strgeninvpreimgsnf[i] := <StrongGenNormalForm(GR, S[i
       ]^-1), Id(N)>;
  end for;
  EG`pcgensnf := [];
  for i in [1 .. #pcgens] do
    EG`pcgensnf[i] := <e[1], pcgens[i]>;
300  end for;
  EG`grpgens := EG`strgenpreimgsnf cat EG`pcgensnf;
  EG`supergrp := rec<PCBF | >;
  return EG;
end function;
305

PCBFModCohoConstruct := function(C, E, rho)

  /*
310
  Arguments:

    C: Cohomology Module, Type(s): ModCoho

315    E, rho : Result of call: E, rho := Extension(C, ?), Type(s):
    Grp, Hom


  Return Type(s):
320
    Rec


  Description:
325
    Returns a record representing a PCBF-group isomorphic to E,
    with normal abelian subgroup Module(C) and quotient
    ismorphic to Group(C).

330  */

  EG := rec<PCBF | >;
        EG`E := E;
```

```
        Q := Group(C);
335 EG`Q := Group(C);
    B := C`gr`b;
    S := C`gr`sg;
    bo := C`gr`bo;
    sv := C`gr`sv;
340 GR := GrpPermData(Q, B, S, sv, bo);
    EG`GR := GR;
        EG`rho := rho;
        M := Module(C);
        p := IsFinite(BaseRing(M)) select #BaseRing(M) else 0;
345     r := Dimension(M);
        assert Ngens(E) eq #S + r;
        L := sub<E | [E.i : i in [#S + 1 .. Ngens(E)]]>;
    EG`L := L;
        N := p ne 0 select AbelianGroup(GrpPC, [p : i in [1 .. r
           ]])
350         else AbelianGroup(GrpGPC, [p : i in [1 .. r]]);
    EG`N := N;
        phi := hom<L -> N | [N.i : i in [1 .. r]]>;
        strgenpreimgs := (GR`S)@@rho;
    m := #strgenpreimgs;
355     EG`strgenpreimgs := strgenpreimgs;
        strgeninvpreimgs := {@@}; // this set is needed in the
           case where a strong generator has order 2
        for i in [1 .. m] do
              Include(~strgeninvpreimgs, (Order(S[i]) eq 2)
                 select strgenpreimgs[i] else strgenpreimgs[i
                 ]^-1);
        end for;
360     EG`strgeninvpreimgs := strgeninvpreimgs;
    pcgens := PCGenerators(N);
    EG`pcgens := pcgens;
    pcgenconjugates := [];
    pcgenconjugatesinv := [];
365 NF := E`NormalForm;
    for i in [1 .. m] do
      pcgenconjugates[i] := [N | ];
      pcgenconjugatesinv[i] := [N | ];
      for j in [1 .. r] do
370     // computing conjugates of the jth polycyclic generator
           with the ith preimage of the strong generating set of Q
            and its inverse
        yim := (pcgens[j])@@phi;
        pcgenconjugates[i][j] := phi(NF(yim^strgenpreimgs[i])); //
            stored as an element of N
                        pcgenconjugatesinv[i][j] := phi(NF(yim^
                           strgeninvpreimgs[i])); // stored as
```

```
                                   an element of N
        end for;
375   end for;
      l := #GR`B; // number of elements in the stored base of the
          quotient group
      tail := GR`tail;
      tailinv := GR`tailinv;
      utail := GR`utail;
380   utailinv := GR`utailinv;
      h := [];
      g := [];
      w := [];
      tailelts := [N | ];
385   taileltsinv := [N | ];
      for i in [1 .. l] do
        orb := GR`bo[i];
        v := GR`sv[i];
        sg := GR`sgindexlist[i];
390     for j in [1 .. #orb] do
          pt := orb[j];
          for k in [1..#sg] do
            g := SVWordInv(Q, S, v, orb, pt);
            if utail[i][j][k] ne 0 then
395           ipt := pt^S[sg[k]];
              h := SVWordInv(Q, S, v, orb, ipt);
                                                    h := SVWordInv(Q, S, v,
                                                        orb, ipt);
                                                    ux := WordMultiply(
                                                        strgenpreimgs,
                                                        strgeninvpreimgs, g)
                                                        * strgenpreimgs[sg[k
                                                        ]];
                                                    th := WordMultiply(
                                                        strgenpreimgs,
                                                        strgeninvpreimgs,
                                                        tail[i][j][k]) *
                                                        WordMultiply(
                                                        strgenpreimgs,
                                                        strgeninvpreimgs, h);
400                                                 tailelts[utail[i][j][k]]
                                                        := phi(NF(th^-1 * ux
                                                        ));
            end if;
            if utailinv[i][j][k] ne 0 then
              ipt := pt^(S[sg[k]]^-1);
              h := SVWordInv(Q, S, v, orb, ipt);
405 uxinv := WordMultiply(strgenpreimgs, strgeninvpreimgs, g) *
      strgeninvpreimgs[sg[k]];
```

```
                                                    th := WordMultiply(
                                                       strgenpreimgs,
                                                       strgeninvpreimgs,
                                                       tailinv[i][j][k]) *
                                                       WordMultiply(
                                                       strgenpreimgs,
                                                       strgeninvpreimgs, h);
                                                    taileltsinv[utailinv[i][
                                                       j][k]] := phi(NF(th
                                                       ^-1 * uxinv));
          end if;
        end for;
410   end for;
    end for;
    EG`pcgenconjugates := pcgenconjugates;
    EG`pcgenconjugatesinv := pcgenconjugatesinv;
    EG`tailelts := tailelts;
415 EG`taileltsinv := taileltsinv;
    e := <StrongGenNormalForm(GR, Id(GR`G)), Id(N)>;
    EG`e := e;
    EG`ismaster := true;
    EG`strgenpreimgsnf := [];
420 EG`strgeninvpreimgsnf := [];
    for i in [1 .. #GR`S] do
      EG`strgenpreimgsnf[i] := <StrongGenNormalForm(GR, S[i]), Id(
         N)>;
                    EG`strgeninvpreimgsnf[i] := <StrongGenNormalForm
                       (GR, S[i]^-1), Id(N)>;
    end for;
425 EG`pcgensnf := [];
    for i in [1 .. #pcgens] do
      EG`pcgensnf[i] := <e[1], pcgens[i]>;
    end for;
    EG`phi := phi;
430 EG`grpgens := EG`strgenpreimgsnf cat EG`pcgensnf;
    EG`supergrp := rec<PCBF | >;
    return EG;
  end function;

435
  PCBFFPQuotConstruct := function(G, Q, rho, L, psi)

    /*

440 Arguments:

      G: Finitely presented group, Type(s): GrpFP
```

```
  Q: Quotient of E via rho, Type(s): GrpPerm
445
  rho: Natural map E -> Q, Type(s): Map

  L: Quotient of ker(rho) via psi Type(s): GrpPC, GrpGPC

450  psi: Natural map ker(rho) -> L, Type(s): Map


 Parameters:

455
 Return Type(s):

  Rec

460
 Description:

  Returns a record representing a PCBF-group isomorphic to
  the extension of L by Q.
465
 */

 N := L;
 // psi : ker(rho) -> L
470 EG := rec<PCBF | >;
 EG`E := G;
 EG`L := L;
 EG`Q := Q;
 B := Base(Q);
475 S := StrongGenerators(Q);
 ReduceGenerators(~Q); // a nonredundant set of strong
    generators is required
 S := StrongGenerators(Q);
 bo := BasicOrbits(Q);
 sv := SchreierVectors(Q);
480 GR := GrpPermData(Q, B, S, sv, bo);
 EG`GR := GR;
 strgenpreimgs := (GR`S)@@rho;
 EG`strgenpreimgs := strgenpreimgs;
 m := #strgenpreimgs;
485 strgeninvpreimgs := {@@}; // this set is needed in the case
    where a strong generator has order 2
 for i in [1 .. m] do
  Include(~strgeninvpreimgs, (Order(S[i]) eq 2) select
     strgenpreimgs[i] else strgenpreimgs[i]^-1);
 end for;
```

```
    EG`strgeninvpreimgs := strgeninvpreimgs;
490 pcgens := PCGenerators(N);
    r := #pcgens;
    pcgenconjugates := [];
    pcgenconjugatesinv := [];
    for i in [1 .. m] do
495   pcgenconjugates[i] := [N | ];
      pcgenconjugatesinv[i] := [N | ];
      for j in [1 .. r] do
        // computing conjugates of the j-th polycyclic generator
            with the ith preimage of the strong generating set of Q
             and its inverse
        yim := (pcgens[j])@@psi;
500     pcgenconjugates[i][j] := psi(yim^strgenpreimgs[i]); //
            stored as an element of N
        pcgenconjugatesinv[i][j] := psi(yim^strgeninvpreimgs[i]);
            // stored as an element of N
      end for;
    end for;
    l := #GR`B; // number of elements in the stored base of the
        quotient group
505 tail := GR`tail;
    tailinv := GR`tailinv;
    utail := GR`utail;
    utailinv := GR`utailinv;
    h := [];
510 g := [];
    tailelts := [N | ];
    taileltsinv := [N | ];
    for i in [1 .. l] do
      orb := GR`bo[i];
515   v := GR`sv[i];
      sg := GR`sgindexlist[i];
      for j in [1 .. #orb] do
        pt := orb[j];
        for k in [1 .. #sg] do
520       g := SVWordInv(Q, S, v, orb, pt);
          if utail[i][j][k] ne 0 then
            ipt := pt^S[sg[k]];
            h := SVWordInv(Q, S, v, orb, ipt);
            ux := WordMultiply(strgenpreimgs, strgeninvpreimgs, g)
                 * strgenpreimgs[sg[k]];
525         th := WordMultiply(strgenpreimgs, strgeninvpreimgs,
                tail[i][j][k]) * WordMultiply(strgenpreimgs,
                strgeninvpreimgs, h);
            tailelts[utail[i][j][k]] := psi((th^-1) * ux);
          end if;
          if utailinv[i][j][k] ne 0 then
```

```
            ipt := pt^(S[sg[k]]^-1);
530         h := SVWordInv(Q, S, v, orb, ipt);
            uxinv := WordMultiply(strgenpreimgs, strgeninvpreimgs,
                 g) * strgeninvpreimgs[sg[k]];
            th := WordMultiply(strgenpreimgs, strgeninvpreimgs,
               tailinv[i][j][k]) * WordMultiply(strgenpreimgs,
               strgeninvpreimgs, h);
            taileltsinv[utailinv[i][j][k]] := psi((th^-1) * uxinv)
               ;
          end if;
535     end for;
      end for;
    end for;
    EG`N := N;
    EG`pcgens := pcgens;
540 EG`pcgenconjugates := pcgenconjugates;
    EG`pcgenconjugatesinv := pcgenconjugatesinv;
    EG`tailelts := tailelts;
    EG`taileltsinv := taileltsinv;
    e := <StrongGenNormalForm(GR, Id(GR`G)), Id(N)>;
545 EG`e := e;
    EG`ismaster := true;
    EG`strgenpreimgsnf := [];
    EG`strgeninvpreimgsnf := [];
    for i in [1 .. #GR`S] do
550   EG`strgenpreimgsnf[i] := <StrongGenNormalForm(GR, S[i]), Id(
         N)>;
      EG`strgeninvpreimgsnf[i] := <StrongGenNormalForm(GR, S[i
         ]^-1), Id(N)>;
    end for;
    EG`pcgensnf := [];
    for i in [1 .. #pcgens] do
555   EG`pcgensnf[i] := <e[1], pcgens[i]>;
    end for;
    EG`phi := IdentityHomomorphism(L);
    EG`grpgens := EG`strgenpreimgsnf cat EG`pcgensnf;
    EG`supergrp := rec<PCBF | >;
560 return EG;
  end function;


  PCBFGenerators := function(EG)
565
    /*

    Arguments:

570   EG: PCBF-group, Type(s): Rec
```

```
    Parameters:

575
    Return Type(s):

      SeqEnum

580
    Description:

      Returns a sequence containing a full generating set for EG.

585 */

    return EG`grpgens;
end function;

590
PCBFNgens := function(EG)

    /*

595 Arguments:

      EG: PCBF-group, Type(s): Rec

    Parameters:
600

    Return Type(s):

      RngIntElt
605

    Description:

      Returns the number of defining generators for EG.
610
    */

    return #EG`grpgens;
end function;
615

PCBFNumberOfGenerators := function(EG)
```

```
    /*
620
    Arguments:

      EG: PCBF-group, Type(s): Rec

625 Parameters:


    Return Type(s):

630   RngIntElt


    Description:

635   Returns the number of defining generators for EG.

    */

    return PCBFNgens(EG);
640 end function;


PCBFQuotGrp := function(EG)

645 /*

    Arguments:

      EG: PCBF-group, Type(s): Rec
650

    Parameters:


655 Return Type(s):

      GrpPerm


660 Description:

      Returns a permutation group isomorphic to the quotient group
      used to construct EG.

665 */
```

```
  return EG`Q;
end function;

670
PCBFNormalSubgrp := function(EG)

  /*

675 Arguments:

    EG: PCBF-group, Type(s): Rec


680 Parameters:


  Return Type(s):

685   GrpPC or GrpGPC


  Description:

690   Returns a polycyclically presented group isomorphic to the
      normal subgroup group used to construct EG.

  */

695 return EG`N;
end function;


PCBFSuperGrp := function(EG)
700
  /*

  Arguments:

705   EG: PCBF-group, Type(s): Rec


  Parameters:

710
  Return Type(s):

    Rec
```

```
715
    Description:

      Returns the PCBF-group from which EG was primarily derived.

720 */

    return EG`ismaster select EG else EG`supergrp;
  end function;

725
  PCBFGrpOrder := function(EG)

    /*

730 Arguments:

      EG: PCBF-group, Type(s): Rec


735 Parameters:


    Return Type(s):

740   RngIntElt


    Description:

745   Returns the order of the PCBF-group EG.

    */

    return Order(PCBFQuotGrp(EG)) * Order(PCBFNormalSubgrp(EG));
750 end function;


  PCBFIsFinite := function(EG)

755 /*

    Arguments:

      EG: PCBF-group, Type(s): Rec
760

    Parameters:
```

```
765     Return Type(s):

           BoolElt

770     Description:

           Returns true if and only if EG is finite.

        */
775
        return IsFinite(PCBFGrpOrder(EG));
    end function;

780 PCBFIsTrivial := function(EG)

        /*

        Arguments:
785
          EG: PCBF-group, Type(s): Rec


        Parameters:
790

        Return Type(s):

           BoolElt
795

        Description:

           Returns true if and only if EG is trivial.
800
        */

        return PCBFGrpOrder(EG) eq 1;
    end function;
805

    PCBFIsSoluble := function(EG)

        /*
810
```

```
    Arguments:

      EG: PCBF-group, Type(s): Rec
815
    Parameters:


    Return Type(s):
820
      BoolElt


    Description:
825
      Returns true if and only if EG is soluble.

    */

830 return IsSoluble(PCBFQuotGrp(EG));
end function;


PCBFIsSolvable := function(EG)
835
    /*

    Arguments:

840   EG: PCBF-group, Type(s): Rec


    Parameters:

845
    Return Type(s):

      BoolElt

850
    Description:

      Returns true if and only if EG is soluble.

855 */

    return PCBFIsSoluble(EG);
end function;
```

```
860
PCBFCreateElement := function(EG, g, n)

  /*

865 Arguments:

    EG: PCBF-group, Type(s): Rec

    g: Element of EG`Q, Type(s): GrpPermElt
870
    n: Element of EG`N, Type(s): GrpPCElt or GrpGPCElt


  Parameters:
875

  Return Type(s):

    Tup
880

  Description:

    Returns an ordered pair representing the element, (g, n),
885 of EG.

  */

  GR := EG`GR;
890 return <StrongGenNormalForm(GR, g), n>;
end function;


PCBFIdentity := function(EG)
895
  /*

  Arguments:

900 EG: PCBF-group, Type(s): Rec


  Parameters:


905
  Return Type(s):
```

```
      Tup

910
    Description:

      Returns the identity element of EG.

915 */

   return EG`e;
end function;

920
PCBFId := function(EG)

   /*

925 Arguments:

      EG: PCBF-group, Type(s): Rec


930 Parameters:


    Return Type(s):

935   Tup


    Description:

940   Returns the identity element of EG.

   */

   return PCBFIdentity(EG);
945 end function;


PCBFrho := function(EG, ele)

950 /*

    Arguments:

      EG: PCBF-group, Type(s): Rec
```

```
955
      ele: Element of EG, Type(s): Tup


    Parameters:
960

    Return Type(s):

      GrpPermElt
965

    Description:

      Returns the image of the element ele of EG under the
970   natural map onto the quotient EG`Q.

    */

    word := [];
975 for i in [#ele[1] .. 1 by -1] do
      word cat:= ele[1][i]; // first entry specifies a word in the
          strong generators of the quotient group
    end for;
    return WordPermutation(EG`GR, word);
  end function;
980

  PCBFphi := function(EG, ele)

    /*
985
    Arguments:

      EG: PCBF-group, Type(s): Rec

990   ele: Element of EG, Type(s): Tup


    Parameters:

995
    Return Type(s):

      GrpPCElt or GrpGPCElt

1000
    Description:
```

```
      Returns the second component of the ordered pair, ele. This
      is NOT a homomorphism as no check is made to determine
1005  whether ele represents an element of the normal subgroup
      used in the construction of EG.

    */

1010 return ele[2];
end function;


PCBFphiinv := function(EG, n)
1015
    /*

    Arguments:

1020   EG: PCBF-group, Type(s): Rec

       ele: Element of EG`N, Type(s): GrpPCElt or GrpGPCElt


1025 Parameters:


    Return Type(s):

1030   Tup


    Description:

1035   Returns image in EG of the element, n, of EG`N.

    */

    return <PCBFId(EG)[1], n>;
1040 end function;


PCBFeq := function(EG, ele1, ele2)

1045 /*

    Arguments:

       EG: PCBF-group, Type(s): Rec
```

```
1050
        ele1: Element of EG, Type(s): Tup

        ele2: Element of EG, Type(s): Tup

1055
    Parameters:


    Return Type(s):
1060
        BoolElt


    Description:
1065
        Returns true if and only if ele1 and ele2 are equal in EG.

    */

1070  return PCBFrho(EG, ele1) eq PCBFrho(EG, ele2) and PCBFphi(EG,
        ele1) eq PCBFphi(EG, ele2);
end function;


PCBFne := function(EG, ele1, ele2)
1075
    /*

    Arguments:

1080    EG: PCBF-group, Type(s): Rec

        ele1: Element of EG, Type(s): Tup

        ele2: Element of EG, Type(s): Tup
1085

    Parameters:


1090  Return Type(s):

        BoolElt


1095  Description:
```

```
      Returns true if and only if ele1 and ele2 are not equal in
      EG.

1100   */

    return not PCBFeq(EG, ele1, ele2);
end function;

1105
PCBFIsIdentity := function(EG, ele)

   /*

1110   Arguments:

      EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup
1115

   Parameters:


1120   Return Type(s):

      BoolElt


1125   Description:

      Returns true if and only if ele is the identity element of
      EG.

1130   */

    return PCBFeq(EG, ele, PCBFId(EG));
end function;

1135
PCBFIsId := function(EG, ele)

   /*

1140   Arguments:

      EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup
```

```
1145

    Parameters:


1150 Return Type(s):

       BoolElt


1155 Description:

       Returns true if and only if ele is the identity element of
       EG.

1160 */

    return PCBFIsIdentity(EG, ele);
end function;


1165
PCBFRandom := function(EG)

    /*

1170 Arguments:

       EG: PCBF-group, Type(s): Rec


1175 Parameters:


    Return Type(s):

1180   Tup


    Description:

1185   Generates a pseudo-random element of EG.

    */

    g := Random(EG`Q);
1190 n := Random(EG`N);
    return PCBFCreateElement(EG, g, n);
end function;
```

```
1195 PCBFRandomElement := function(EG)

       /*

       Arguments:
1200
         EG: PCBF-group, Type(s): Rec


       Parameters:
1205

       Return Type(s):

         Tup
1210

       Description:

         Generates a pseudo-random element of EG.
1215
       */

       return PCBFRandom(EG);
     end function;
1220

     PCBFRandomSequence := function(EG, l)

       /*
1225
       Arguments:

         EG: PCBF-group, Type(s): Rec

1230
       Parameters:


       Return Type(s):
1235
         SeqEnum


       Description:
1240
```

```
    Generates a sequence (of length l) of pseudo-random
    elements of EG.

  */
1245
  randseq := [car<PowerSequence(PowerSequence(IntegerRing())),
     EG`N> |];
  for i in [1 .. l] do
    Append(~randseq, PCBFRandom(EG));
  end for;
1250 return randseq;
end function;


PCBFRepresentative := function(EG)
1255
  /*

  Arguments:

1260  EG: PCBF-group, Type(s): Rec


  Parameters:

1265
  Return Type(s):

    Tup

1270
  Description:

    Returns a representative element of EG.

1275 */

  return PCBFId(EG);
end function;

1280
PCBFRep := function(EG)

  /*

1285 Arguments:

    EG: PCBF-group, Type(s): Rec
```

```
    Parameters:


    Return Type(s):

      Tup


    Description:

      Returns a representative element of EG.

    */

    return PCBFRepresentative(EG);
end function;
```

Listing B.6: pcbfarithmetic.m

```
/*

File:                  pcbfarithmetic.m


Last modified:         Fri, 07 Jan 2011 17:21:11 +0000


Author(s):             Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


Company:               University of Warwick <http://www.warwick.ac.
    uk>


Description:

The methods in this file are used to perform element arithmetic
    in
polycyclic-by-finite groups represented by records of type PCBF.


Notes:

MAGMA V2.16-13.


Copyright 2006-2010, University of Warwick. All rights reserved.
```

```
*/

30
load "pcbfconstruct.m";

forward PCBFConjugateByStrongGen, PCBFConjugateByStrongGenWord;
35

PCBFMult := function(EG, ele1, ele2)

  /*
40
  Arguments:

    EG: PCBF-group, Type(s): Rec

45    ele1: Element of EG, Type(s): Tup

    ele2: Element of EG, Type(s): Tup


50  Parameters:


  Return Type(s):

55    Tup


  Description:

60    Computes the product, ele1 * ele2.

  */

  GR := EG`GR;
65  if PCBFrho(EG, ele2) eq Id(GR`G) then
    return <ele1[1], ele1[2] * ele2[2]>; //easy
  end if;
  u := [];
  l := #GR`B;
70  g := Id(GR`G);
  N := EG`N;
  rightword := [];
  for i in [#ele2[1] .. 1 by -1] do
    rightword cat:= ele2[1][i];
```

```
75    end for;
      leftword := [];
      nrightseq := [];
      nrightseq[1] := ele1[2];
      for i in [2 .. l] do
80      nrightseq[i] := Id(N);
      end for;
      for i in [1 .. l] do
        g := WordPermutation(GR, ele1[1][i]);
        pt := GR`B[i]^g;
85      while #rightword gt 0 do
          j := Position(GR`bo[i], pt);
          nshift := Id(N);
          if rightword[1] lt 0 then
            pt := GR`B[i]^(g * (GR`S[-rightword[1]])^-1);
90          k := Position(GR`sgindexlist[i], -rightword[1]);
            leftword cat:= GR`tailinv[i][j][k];
            if GR`utailinv[i][j][k] ne 0 then
              nshift := EG`taileltsinv[GR`utailinv[i][j][k]];
            end if;
95        else
            pt := GR`B[i]^(g * GR`S[rightword[1]]);
            k := Position(GR`sgindexlist[i], rightword[1]);
            leftword cat:= GR`tail[i][j][k];
            if GR`utail[i][j][k] ne 0 then
100           nshift := EG`tailelts[GR`utail[i][j][k]];
            end if;
          end if;
          nrightseq[i] := nshift * PCBFConjugateByStrongGen(EG,
             nrightseq[i], rightword[1]);
          g := SVPermutationInv(GR`G, GR`S, GR`sv[i], GR`bo[i], pt);
105       Remove(~rightword, 1);
        end while;
        u[i] := StrongGenWord(GR, g);
        rightword := leftword;
        leftword := [];
110   end for;
      n := Id(N);
      for i in [l .. 1 by -1] do
        n := PCBFConjugateByStrongGenWord(EG, n, u[i]) * nrightseq[i
           ];
      end for;
115   return <u, n * ele2[2]>;
    end function;


    PCBFConjugateByStrongGen := function(EG, n, i)
120
```

```
  /*

  Arguments:

125   EG: PCBF-group, Type(s): Rec

      n: Element of EG`N, Type(s): GrpPCElt or GrpGPCElt

      i: Integer representing a preimage of a strong generator of
130   the quotient group, EG`Q, Type(s): RngIntElt


  Parameters:

135
  Return Type(s):

      GrpPCElt or GrpGPCElt

140
  Description:

      Returns the conjugate of the element, n, by the element of
      EG represented by i.
145
  */

  N := EG`N;
  if n eq Id(N) then return n; end if;
150 w := Eltseq(n);
  ans := Id(N);
  for j in [1 .. #w] do
    ans *:= i gt 0 select EG`pcgenconjugates[i][j]^w[j] else EG`
        pcgenconjugatesinv[-i][j]^w[j];
  end for;
155 return ans;
end function;


PCBFConjugateByStrongGenWord := function(EG, n, word)
160
  /*

  Arguments:

165   EG: PCBF-group, Type(s): Rec

      n: Element of EG`N, Type(s): GrpPCElt or GrpGPCElt
```

```
      word: Sequence of integers representing a word in the
170   preimages of the strong generators of the quotient group,
      EG`Q, Type(s): SeqEnum


    Parameters:
175

    Return Type(s):

      GrpPCElt or GrpGPCElt
180

    Description:

      Returns the conjugate of the element n by the element of EG
185   represented by word.

    */

    ans := n;
190 for i in [1 .. #word] do
      ans := PCBFConjugateByStrongGen(EG, ans, word[i]);
    end for;
    return ans;
  end function;
195

  PCBFMultiply := function(EG, ele1, ele2)

    /*
200
    Arguments:

      EG: PCBF-group, Type(s): Rec

205   ele1: Element of EG, Type(s): Tup

      ele2: Element of EG, Type(s): Tup


210 Parameters:


    Return Type(s):

215   Tup
```

```
    Description:

220     Computes the product ele1 * ele2.

    */

    return PCBFMult(EG, ele1, ele2);
225 end function;


PCBFStar := function(EG, ele1, ele2)

230  /*

    Arguments:

        EG: PCBF-group, Type(s): Rec
235
        ele1: Element of EG, Type(s): Tup

        ele2: Element of EG, Type(s): Tup

240
    Parameters:


    Return Type(s):
245
        Tup


    Description:
250
        Computes the product ele1 * ele2.

    */

255  return PCBFMult(EG, ele1, ele2);
end function;


PCBFWordMultiply := function(EG, posimgs, negimgs, word)
260
    /*

    Arguments:
```

```
265       EG: PCBF-group, Type(s): Rec

          posimgs: Sequence of ordered pairs representing elements of
          EG, Type(s): SeqEnum

270       negimgs: Sequence of ordered pairs representing elements of
          EG, Type(s): SeqEnum

          word: Sequence of integers representing a word in the
          elements indexed by posimgs and negimgs, Type(s): SeqEnum
275

        Parameters:


280     Return Type(s):

          Tup


285     Description:

          Computes the product of the sequence of elements of EG
          represented by word.

290     */

        ans := PCBFId(EG);
        for i in [1 .. #word] do
          ans := PCBFMult(EG, ans, word[i] gt 0 select posimgs[word[i
              ]] else negimgs[-word[i]]);
295     end for;
        return ans;
      end function;


300 PCBFrhoinv := function(EG, g)

        /*

        Arguments:
305
          EG: PCBF-group, Type(s): Rec

          g: Element of EG`Q, Type(s): GrpPermElt

310
```

```
    Parameters:


    Return Type(s):
315
      Tup


    Description:
320
      Returns an inverse of the element, g, of EG`Q under the
      natural map from EG onto EG`Q.

    */
325
    word := StrongGenWord(EG`GR, g);
    return PCBFWordMultiply(EG, EG`strgenpreimgsnf, EG`
        strgeninvpreimgsnf, word);
end function;

330
PCBFrhoinvSeq := function(EG, elts)

    /*

335 Arguments:

      EG: PCBF-group, Type(s): Rec

      elts: sequence of elements of EG`Q, Type(s): SeqEnum
340

    Parameters:


345 Return Type(s):

      SeqEnum


350 Description:

      Returns a sequence of preimages in EG corresponding to the
      elements in elts.

355 */

    return [PCBFrhoinv(EG, elts[i]) : i in [1 .. #elts]];
```

```
end function;

360
PCBFInverse := function(EG, ele)

    /*

365 Arguments:

      EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup
370

    Parameters:


375 Return Type(s):

      Tup


380 Description:

      Computes ele^-1.

    */
385
    elt := PCBFrhoinv(EG, PCBFrho(EG, ele)^-1); // ele * elt is an
        element of the normal subgroup used to construct EG
    return PCBFMult(EG, elt, PCBFphiinv(EG, PCBFphi(EG, PCBFMult(
        EG, ele, elt))^-1));
end function;

390
PCBFInv := function(EG, ele)

    /*

395 Arguments:

      EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup
400

    Parameters:
```

```
405    Return Type(s):

         Tup


410    Description:

         Computes ele^-1.

       */
415
       return PCBFInverse(EG, ele);
    end function;


420 PCBFDivide := function(EG, ele1, ele2)

       /*

       Arguments:
425
         EG: PCBF-group, Type(s): Rec

         ele1: Element of EG, Type(s): Tup

430      ele2: Element of EG, Type(s): Tup


       Parameters:

435
       Return Type(s):

         Tup

440
       Description:

         Computes ele1 * (ele2)^-1.

445    */

       return PCBFMult(EG, ele1, PCBFInverse(EG, ele2));
    end function;

450
    PCBFSlash := function(EG, ele1, ele2)
```

```
    /*
455 Arguments:
      EG: PCBF-group, Type(s): Rec
      ele1: Element of EG, Type(s): Tup
460
      ele2: Element of EG, Type(s): Tup

    Parameters:
465

    Return Type(s):
      Tup
470

    Description:
      Computes ele1 * (ele2)^-1.
475
    */
    return PCBFDivide(EG, ele1, ele2);
end function;
480

PCBFConjugate := function(EG, ele1, ele2)
    /*
485
    Arguments:
      EG: PCBF-group, Type(s): Rec
490   ele1: Element of EG, Type(s): Tup
      ele2: Element of EG, Type(s): Tup

495 Parameters:

    Return Type(s):
```

```
500      Tup


      Description:

505      Computes ele1^ele2 = (ele2)^-1 * ele1 * ele2.

      */

      return PCBFMult(EG, PCBFMult(EG, PCBFInverse(EG, ele2), ele1),
          ele2);
510 end function;


    PCBFCommutator := function(EG, ele1, ele2)

515   /*

      Arguments:

        EG: PCBF-group, Type(s): Rec
520
         ele1: Element of EG, Type(s): Tup

         ele2: Element of EG, Type(s): Tup

525
      Parameters:


      Return Type(s):
530
         Tup


      Description:
535
         Computes (ele1, ele2) = ele1^-1 * (ele2)^-1 * ele1 * ele2.

      */

540   return PCBFMult(EG, PCBFConjugate(EG, PCBFInverse(EG, ele2),
         ele1), ele2);
    end function;


    function PCBFSeqCommutator(EG, elts)
545
```

```
    /*

    Arguments:

550   EG: PCBF-group, Type(s): Rec

      elts: Sequence of elements of EG, Type(s): SeqEnum


555 Parameters:


    Return Type(s):

560   Tup


    Description:

565   Computes the (left-normed) commutator of the elements in
         elts.

    */

    n := #elts;
570 if n eq 2 then
      // base case n = 2
      return PCBFCommutator(EG, elts[1], elts[2]);
    end if;
    // recurse here
575 return PCBFCommutator(EG, PCBFSeqCommutator(EG, elts[1 .. n -
       1]), elts[n]);
  end function;


  PCBFPower := function(EG, ele, exp)
580
    /*

    Arguments:

585   EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup

      exp: Integer, Type(s): RngIntElt

590
```

```
    Parameters:

595 Return Type(s):

      Tup

600 Description:

      Computes ele^exp.

    */
605
    if exp lt 0 then
      ele := PCBFInverse(EG, ele);
      exp := -exp;
    end if;
610 ans := PCBFIdentity(EG);
    while exp gt 0 do
      if IsOdd(exp) then
        ans := PCBFMult(EG, ans, ele);
      end if;
615   exp := exp div 2;
      ele := PCBFMult(EG, ele, ele);
    end while;
    return ans;
  end function;
620

  PCBFEleOrder := function(EG, ele)

    /*
625
    Arguments:

      EG: PCBF-group, Type(s): Rec

630   ele: Element of EG, Type(s): Tup


    Parameters:

635
    Return Type(s):

      RngIntElt
```

```
640
    Description:

      Computes the order of ele.

645 */

    quotord := Order(PCBFrho(EG, ele));
    return quotord * Order(PCBFphi(EG, PCBFPower(EG, ele, quotord)
      ));
end function;
```

Listing B.7: pcbfcattrans.m

```
1 /*

  File:                pcbfcattrans.m


6 Last modified:       Fri, 07 Jan 2011 17:21:11 +0000


  Author(s):           Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>

11
  Company:             University of Warwick <http://www.warwick.ac.
    uk>


  Description:
16
  The methods in this file are used to rewrite polycyclic-by-
    finite groups
  represented by records of type PCBF as GrpPC or GrpGPC if such a
  representation is possible.


21
  Notes:

  MAGMA V2.16-13.


26
  Copyright 2006-2010, University of Warwick. All rights reserved.

  */


31
  load "pcbfarithmetic.m";
```

```
PCBFPolycyclicGroup := function(EG)
36
  /*

  Arguments:

41    EG: PCBF-group, Type(s): Rec


  Parameters:


46
  Return Type(s):

    GrpPC or GrpGPC, Tup


51
  Description:

    Returns a group of type GrpPC or GrpGPC isomorphic to EG.
    Data used to compute the isomorphism and its inverse is
56  returned as a second argument.

  */

  // more rigorous testing to be done!
61 Q, kappa := PCGroup(PCBFQuotGrp(EG)); // kappa: EG`Q --> Q
  N := PCBFNormalSubgrp(EG); // N is of type GrpPC or GrpGPC
  isgrppc := Type(N) eq GrpPC;
  a := PCGenerators(Q);
  b := PCGenerators(N);
66 pcgenpreimgs := [PCBFrhoinv(EG, a[i]@@kappa) : i in [1 .. #a]]
      cat [PCBFphiinv(EG, b[i]) : i in [1 .. #b]];
  pclength := #a + #b;
  F := FreeGroup(pclength);
  c := {@ F.i : i in [1 .. pclength] @}; // rank is finite so
      this is ok
  powerrelations := []; // power relations for new polycyclic
      presentation
71 conjugaterelations := []; // conjugate relations for new
      polycyclic presentation
  p := [];
  if Order(Q) gt 1 then
    p := PCPrimes(Q);
  end if;
76 m := isgrppc select PCPrimes(N) else PCExponents(N);
  m := p cat m;
```

```
for j in [1 .. #a] do
  // the power and conjugate relations for the pc-generators
     of the permutation group are computed in this loop
  // isomorphic copies of the polycyclic relations for the
     normal subgroup are added to R, along with the additional
      conjugation relations
81  // power relations
  rel := c[j]^m[j] = Id(F);
  rhs := PCBFPower(EG, pcgenpreimgs[j], m[j]);
  for i in [j + 1 .. #a] do
    alpha := Eltseq(kappa(PCBFrho(EG, rhs)))[i];
86    rel := LHS(rel) = RHS(rel) * c[i]^alpha;
    rhs := PCBFMult(EG, PCBFPower(EG, pcgenpreimgs[i], -alpha)
       , rhs);
  end for;
  eleseq := Eltseq(PCBFphi(EG, rhs));
  for i in [Max(j, #a) + 1 .. pclength] do
91    rel := LHS(rel) = RHS(rel) * c[i]^eleseq[i - #a];
  end for;
  Append(~powerrelations, rel);
  // conjugate relations
  for i in [1 .. j - 1] do
96    rel := c[j]^c[i] = Id(F);
    rhs := PCBFConjugate(EG, pcgenpreimgs[j], pcgenpreimgs[i])
       ;
    for k in [i + 1 .. #a] do
      alpha := Eltseq(kappa(PCBFrho(EG, rhs)))[k];
      rel := LHS(rel) = RHS(rel) * c[k]^alpha;
101      rhs := PCBFMult(EG, PCBFPower(EG, pcgenpreimgs[k], -
         alpha), rhs);
    end for;
    eleseq := Eltseq(PCBFphi(EG, rhs));
    for k in [Max(i, #a) + 1 .. pclength] do
      rel := LHS(rel) = RHS(rel) * c[k]^eleseq[k - #a];
106    end for;
    Append(~conjugaterelations, rel);
  end for;
end for;
for j in [#a + 1 .. pclength] do
111  // the power and conjugate relations for the normal subgroup
      generators are computed in this loop
  // isomorphic copies of the polycyclic relations for the
     normal subgroup are added to R, along with the additional
      conjugation relations
  // power relations
  if m[j] ne 0 then
    rel := c[j]^m[j] = Id(F);
116    eleseq := Eltseq(b[j - #a]^m[j]);
```

```
        for i in [j + 1 .. pclength] do
          rel := LHS(rel) = RHS(rel) * c[i]^eleseq[i - #a];
        end for;
        Append(~powerrelations, rel);
121   end if;
      for i in [1 .. #a] do
        rel := c[j]^c[i] = Id(F);
        rhs := PCBFConjugate(EG, pcgenpreimgs[j], pcgenpreimgs[i])
          ;
        for k in [i + 1 .. #a] do
126       alpha := Eltseq(kappa(PCBFrho(EG, rhs)))[k];
          rel := LHS(rel) = RHS(rel) * c[k]^alpha;
          rhs := PCBFMult(EG, PCBFPower(EG, pcgenpreimgs[k], -
             alpha), rhs);
        end for;
        eleseq := Eltseq(PCBFphi(EG, rhs));
131     for k in [#a + 1 .. pclength] do
          rel := LHS(rel) = RHS(rel) * c[k]^eleseq[k - #a];
        end for;
        Append(~conjugaterelations, rel);
      end for;
136   for i in [#a + 1 .. j - 1] do
        rel := c[j]^c[i] = Id(F);
        eleseq := Eltseq(b[j - #a]^b[i - #a]);
        for k in [i + 1 .. pclength] do
          rel := LHS(rel) = RHS(rel) * c[k]^eleseq[k - #a];
141     end for;
        Append(~conjugaterelations, rel);
        if m[i] eq 0 then
          rel := c[j]^(c[i]^-1) = Id(F);
          eleseq := Eltseq(b[j - #a]^(b[i - #a]^-1));
146       for k in [i + 1 .. pclength] do
            rel := LHS(rel) = RHS(rel) * c[k]^eleseq[k - #a];
          end for;
          Append(~conjugaterelations, rel);
        end if;
151   end for;
    end for;
    R := powerrelations cat conjugaterelations;
    if isgrppc then
      H, _ := quo<GrpPC : F | R>;
156 else
      H, _ := quo<GrpGPC : F | R >;
    end if;
    return H, <kappa, pcgenpreimgs>; // the isomorphism kappa is
        needed to compute the isomorphisms between EG and the new
        polycyclic group
end function;
```

```
161

PCBFPolycycliciso := function(EG, P, pcisodata, ele)

  /*
166
  Arguments:

    EG: PCBF-group, Type(s): Rec

171    P: Polycyclic group isomorphic to EG, Type(s): GrpPC or
    GrpGPC

    pcisodata: Data to compute isomorphism, Type(s): Tup

176    ele: Element of EG, Type(s): Tup


  Parameters:

181
  Return Type(s):

    GrpPCElt or GrpGPCElt

186
  Description:

    Computes the image of ele (in the polycyclic group
    isomorphic to EG) as defined by pcisodata.
191
  */

  kappa := pcisodata[1];
  Q := Codomain(kappa);
196  pcgenpreimgs := pcisodata[2];
  alpha := [];
  t := 0;
  if Order(Q) gt 1 then
    t := #Eltseq(Id(Q));
201  end if;
  for i in [1 .. t] do
    alpha[i] := Eltseq(kappa(PCBFrho(EG, ele)))[i];
    ele := PCBFMult(EG, PCBFPower(EG, pcgenpreimgs[i], -alpha[i
        ]), ele);
  end for;
206  alpha cat:= Eltseq(PCBFphi(EG, ele));
  return P ! alpha;
```

```
end function;

211 PCBFPolycycliclisoSeq := function(EG, P, pcisodata, elts)

    /*

    Arguments:
216
      EG: PCBF-group, Type(s): Rec

      P: Polycyclic group isomorphic to EG, Type(s): GrpPC or
      GrpGPC
221
      pcisodata: Data to compute isomorphism, Type(s): Tup

      elts: Sequence of elements of EG, Type(s): SeqEnum

226
    Parameters:


    Return Type(s):
231
      SeqEnum


    Description:
236
      Returns a sequence of the respective images (in P) of each
      element of elts.

    */
241
    return [PCBFPolycycliciso(EG, P, pcisodata, elts[i]) : i in [1
        .. #elts]];
end function;


246 PCBFPolycyclicisoinv := function(EG, pcisodata, y)

    /*

    Arguments:
251
      EG: PCBF-group, Type(s): Rec

      pcisodata: Data to compute isomorphism, Type(s): Tup
```

```
256      y: Element of identified polycyclic group isomorphic to EG,
         Type(s): GrpPCElt or GrpGPCElt


      Parameters:
261

      Return Type(s):

         Tup
266

      Description:

         Computes the image of y in EG as defined by pcisodata.
271
      */

      ans := PCBFId(EG);
      alpha := Eltseq(y);
276   for i in [1 .. #alpha] do
         ans := PCBFMult(EG, ans, PCBFPower(EG, pcisodata[2][i],
            alpha[i]));
      end for;
      return ans;
   end function;
281

   PCBFPolycyclicisoinvSeq := function(EG, pcisodata, pcelts)

      /*
286
      Arguments:

         EG: PCBF-group, Type(s): Rec

291      pcisodata: Data to compute isomorphism, Type(s): Tup

         y: Sequence of elements of the polycyclic group isomorphic
         to EG, Type(s): SeqEnum


296
      Parameters:


      Return Type(s):
301
```

```
    SeqEnum

  Description:
306
    Returns a sequence of the respective images (in EG) of each
    element of y.

  */
311
  return [PCBFPolycyclicisoinv(EG, pcisodata, pcelts[i]) : i in
      [1 .. #pcelts]];
end function;
```

Listing B.8: pcbfschreiersims.m

```
 /*
2
 File:               pcbfschreiersims.m


 Last modified:      Sun, 06 Feb 2011 22:14:13 +0000
7

 Author(s):          Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


12 Company:          University of Warwick <http://www.warwick.ac.
    uk>


 Description:

17 The methods in this file allow one to work with the permutation
    action
 (induced by the quotient group) of a polycyclic-by-finite group
    represented
 by a record of type PCBF.


22 Notes:

 MAGMA V2.16-13.


27 Copyright 2006-2010, University of Warwick. All rights reserved.

 */
```

```
32 load "pcbfcattrans.m";

   PCBFQuotAction := function(EG, alpha, ele)

37   /*

     Arguments:

       EG: PCBF-group, Type(s): Rec
42
       alpha: Element of the set upon which EG`Q acts, Type(s):
       Elt

       ele: Element of EG, Type(s): Tup
47

     Parameters:


52   Return Type(s):

       Elt


57   Description:

       Computes the action of the first component of ele on the
       point alpha.

62   */

     word := [];
     S := EG`GR`S;
     for i in [#ele[1] .. 1 by -1] do
67     word cat:= ele[1][i];
     end for;
     gamma := alpha;
     for i in [1 .. #word] do
       gamma := word[i] gt 0 select gamma^(S[word[i]]) else gamma
           ^((S[-word[i]])^-1);
72   end for;
     return gamma;
   end function;


77 PCBFQuotOrbit := function(EG, alpha, X)
```

```
/*

Arguments:
82
  EG: PCBF-group, Type(s): Rec

  alpha: Element of the set upon which EG`Q acts, Type(s):
  Elt
87
  X: Sequence of ordered pairs representing elements of EG,
  Type(s): SeqEnum


92 Parameters:


Return Type(s):

97  SeqEnum, SeqEnum


Description:

102  Computes the orbit of the point alpha under the action of
       X    . The function returns the orbit and a Schreier
       vector
  corresponding to the orbit with indexing relative to the
  set X.

107 */

r := #X;
      v := [0];
orb := [alpha];
112 orbsize := 1;
c := 1;
while c le orbsize do
  pt := orb[c];
  for i in [1 .. r] do
117   ppt := PCBFQuotAction(EG, pt, X[i]);
    if ppt notin orb then
      Append(~orb, ppt);
      orbsize +:= 1;
      v[orbsize] := i;
122   end if;
  end for;
  c +:= 1;
```

```
    end while;
    return v, orb;
127 end function;


PCBFQuotSVWordInv := function(EG, Ypos, Yneg, v, orb, pt)

132   /*

      Arguments:

        EG: PCBF-group, Type(s): Rec
137
        Ypos: Sequence of ordered pairs representing elements of
        EG, Type(s): SeqEnum

        Yneg: Sequence of ordered pairs representing elements of
142     EG, Type(s): SeqEnum

        v: Sequence of integers representing a Schreier vector
        (with indexing relative to Ypos and Yneg), Type(s): SeqEnum

147     orb: Sequence containing exactly the orbit of orb[1] under
        Y, Type(s): SeqEnum

        pt: Element of the set upon which EG`Q acts, Type(s): Elt

152
      Parameters:


      Return Type(s):
157
        SeqEnum


      Description:
162
        Returns inverse of what PCBFQuotSVWord should return, as an
        integer sequence i.e. a word in strong generators taking
        base point to pt as defined by the Schreier vector v.

167   */

      pos := Position(orb, pt);
      r := v[pos];
      w := [];
172   while r ne 0 do
```

```
      Append(~w, -r);
      y := r gt 0 select PCBFInverse(EG, Ypos[r]) else Yneg[-r];
      pt := PCBFQuotAction(EG, pt, y);
      pos := Position(orb, pt);
177   r := v[pos];
    end while;
    return WordInverse(w);
  end function;

182
  PCBFQuotSVElement := function(EG, Ypos, Yneg, v, orb, pt)

    /*

187 Arguments:

      EG: PCBF-group, Type(s): Rec

      Ypos: Sequence of ordered pairs representing elements of
192   EG, Type(s): SeqEnum

      Yneg: Sequence of ordered pairs representing elements of
      EG, Type(s): SeqEnum

197   v: Sequence of integers representing a Schreier vector
      (with indexing relative to Ypos and Yneg), Type(s): SeqEnum

      orb: Sequence containing exactly the orbit of orb[1] under
      Y, Type(s): SeqEnum
202
      pt: Element of the set upon which EG`Q acts, Type(s): Elt


    Parameters:
207

    Return Type(s):

      Tup
212

    Description:

      Returns the element of EG in normal form taking pt to base
217   point as defined by the Schreier vector v.

    */
```

```
    pos := Position(orb, pt);
222 r := v[pos];
    u := PCBFId(EG);
    y := PCBFId(EG);
    while r ne 0 do
      y := r gt 0 select PCBFInverse(EG, Ypos[r]) else Yneg[-r];
227   u := PCBFMult(EG, u, y);
      pt := PCBFQuotAction(EG, pt, y);
      pos := Position(orb, pt);
      r := v[pos];
    end while;
232 return u;
end function;


PCBFQuotSVElementInv := function(EG, Ypos, Yneg, v, orb, pt)
237
    /*

    Arguments:

242   EG: PCBF-group, Type(s): Rec

      Ypos: Sequence of ordered pairs representing elements of
      EG, Type(s): SeqEnum

247   Yneg: Sequence of ordered pairs representing elements of
      EG, Type(s): SeqEnum

      v: Sequence of integers representing a Schreier vector
      (with indexing relative to Ypos and Yneg), Type(s): SeqEnum
252
      orb: Sequence containing exactly the orbit of orb[1] under
      Y, Type(s): SeqEnum

      pt: Element of the set upon which EG`Q acts, Type(s): Elt
257


    Parameters:


262 Return Type(s):

      Tup


267 Description:
```

```
      Returns the element of EG in normal form taking base point
      to pt as defined by the Schreier vector v.

272   */

    return PCBFInverse(EG, PCBFQuotSVElement(EG, Ypos, Yneg, v,
        orb, pt));
end function;

277
PCBFQuotStrip := function(EG, ele, B, S, Sinv, sv, bo)

    /*

282   Arguments:

      EG: PCBF-group, Type(s): Rec

      ele: Element of EG, Type(s): Tup
287
      B: Base of EG`Q, Type(s): SeqEnum

      S: 2-D sequence of ordered pairs representing elements of
      EG, Type(s): SeqEnum
292
      Sinv: 2-D sequence of ordered pairs representing elements
      of EG, Type(s): SeqEnum

      sv: Sequence of integer sequences each representing a
297   Schreier vector (with indexing relative to the sequences of
      S and Sinv), Type(s): SeqEnum

      bo: Sequence containing exactly the orbits of each base
      point under S, Type(s): SeqEnum
302

    Parameters:


307   Return Type(s):

      Tup, RngIntElt


312   Description:

      Strips the element ele as if it were an element of the
      quotient permutation group using the "base" B and "strong
```

```
      generating set" represented by the indexed set S.
317
   */

   // bo[i] is the i-th basic orbit, and sv[i] is the Schreier
      vector corresponding to this orbit and the generating set
      specified by S[i]
   h := ele;
322 l := #B;
   for i in [1 .. l] do
     // h fixes base points B[1], B[2], ..., B[i -1]
     pt := PCBFQuotAction(EG, B[i], h);
     if pt notin bo[i] then
327    return h, i;
     end if;
     u := PCBFQuotSVElement(EG, S[i], Sinv[i], sv[i], bo[i], pt);
     h := PCBFMult(EG, h, u);
   end for;
332 return h, l + 1;
end function;


PCBFDepth := function(EG, B, ele)
337
   /*

   Arguments:

342  EG: PCBF-group, Type(s): Rec

     B: Sequence of elements of EG`GR`natset, Type(s): SeqEnum

     ele: Element of EG, Type(s): Tup
347

   Parameters:


352 Return Type(s):

     RngIntElt


357 Description:

     Returns the integer k such that ele fixes B[1], B[2], ...,
     B[k - 1] and moves B[k].
```

```
362     */

        for k in [1 .. #B] do
          if PCBFQuotAction(EG, B[k], ele) ne B[k] then return k; end
              if;
        end for;
367     return #B + 1;
      end function;


      PCBFFindNewBasePoint := function(EG, Omega, ele)
372
        /*

        Arguments:

377       EG: PCBF-group, Type(s): Rec

          Omega: Non-empty sequence of elements of EG`GR`natset
          containing a base for EG`GR`G, Type(s): SeqEnum

382       ele: Element of EG with non-trivial first component,
          Type(s): Tup


        Parameters:
387

        Return Type(s):

          Elt
392

        Description:

          Returns a point in Omega that is moved by ele.
397
        */

        i := 1;
        while PCBFQuotAction(EG, Omega[i], ele) eq Omega[i] do
402       i +:= 1;
        end while;
        return Omega[i];
      end function;

407
      PCBFStandardSchreierSims := function(EG, gens)
```

```
    /*
412 Arguments:

      EG: PCBF-group, Type(s): Rec

      gens: Sequence of ordered pairs representing elements of
417   EG, Type(s): SeqEnum


    Parameters:

422
    Return Type(s):

      SeqEnum, SetIndx, SeqEnum

427
    Description:

      Returns a "base" and "strong generating set" for the
      subgroup of EG generated by gens, together with a
432   generating sequence for the intersection of this subgroup
      with the image of EG`N in EG.

    */

437 B := []; // new base
    Omega := EG`GR`B;
    G := EG`GR`G;
    k := 0;
    S := [];
442 Sinv := [];
    bo := [];
    sv := [];
    strippedgens := [];
    mixedgens := [];
447 for i in [1 .. #gens] do
      if PCBFrho(EG, gens[i]) ne Id(G) then
        Append(~mixedgens, gens[i]);
      elif not PCBFIsId(EG, gens[i]) then
        Append(~strippedgens, gens[i]);
452   end if;
    end for;
    strippedgensinv := [PCBFInv(EG, x) : x in strippedgens];
    depth := [];
    for i in [1 .. #mixedgens] do
```

```
457     depth[i] := PCBFDepth(EG, B, mixedgens[i]);
        if depth[i] eq k + 1 then
          Append(~B, PCBFFindNewBasePoint(EG, Omega, mixedgens[i]));
          k +:= 1;
        end if;
462   end for;
      for i in [1 .. k] do
        S[i] := [mixedgens[j] : j in [1 .. #mixedgens] | depth[j] ge
            i];
        Sinv[i] := [PCBFInv(EG, x) : x in S[i]];
        S[i] cat:= strippedgens;
467     Sinv[i] cat:= strippedgensinv;
        sv[i], bo[i] := PCBFQuotOrbit(EG, B[i], S[i]);
      end for;
      NH := sub<EG`N | [PCBFphi(EG, x) : x in strippedgens]>;
      i := k;
472   uptodate := true;
      while i ge 1 do
        for pt in bo[i] do
          u1 := PCBFQuotSVElementInv(EG, S[i], Sinv[i], sv[i], bo[i
              ], pt);
          for x in S[i] do
477         pt2 := PCBFQuotAction(EG, pt, x);
            u2 := PCBFQuotSVElement(EG, S[i], Sinv[i], sv[i], bo[i],
                pt2);
            ele := PCBFMult(EG, PCBFMult(EG, u1, x), u2);
            if PCBFIsId(EG, ele) then continue x; end if;
            uptodate := true;
482         ele, j := PCBFQuotStrip(EG, ele, B, S, Sinv, sv, bo);
            if PCBFrho(EG, ele) ne Id(G) then
              // new strong generator at level j
              uptodate := false;
              if j gt k then
487             // extend base
                Append(~B, PCBFFindNewBasePoint(EG, Omega, ele));
                k +:= 1;
                S[k] := strippedgens; // maintain inclusion
                Sinv[k] := strippedgensinv; // maintain inclusion
492           end if;
            elif PCBFphi(EG, ele) notin NH then
              uptodate := false;
              Append(~strippedgens, ele);
              Append(~strippedgensinv, PCBFInv(EG, ele));
497           NH := sub<EG`N | NH, PCBFphi(EG, ele)>;
              j := k;
            end if;
            if not uptodate then
              for t in [i + 1 .. j] do
```

```
502              Append(~S[t], ele);
                 Append(~Sinv[t], PCBFInv(EG, ele));
                 sv[t], bo[t] := PCBFQuotOrbit(EG, B[t], S[t]);
               end for;
               i := j + 1;
507            break pt;
             end if;
           end for;
         end for;
         i -:= 1;
512    end while;
       SGS := {@@};
       for i in [1 .. #S] do
         for j in [1 .. #S[i]] do
           if PCBFrho(EG, S[i][j]) ne Id(G) then
517          Include(~SGS, S[i][j]);
           end if;
         end for;
       end for;
       return B, SGS, Generators(NH);
522  end function;


     PCBFNCLSchreierSims := function(EG, gens)

527    /*

       Arguments:

         EG: PCBF-group, Type(s): Rec
532
         gens: Sequence of ordered pairs representing elements of
         EG, Type(s): SeqEnum


537    Parameters:


       Return Type(s):

542      SetIndx, SeqEnum


       Description:

547      Returns a "base" and "strong generating set" for the
         subgroup of EG generated by gens, together with a
         generating sequence for the intersection of this subgroup
```

```
     with the image of EG`N in EG.

552  */

     B := []; // new base
     Omega := EG`GR`B;
     G := EG`GR`G;
557  k := 0;
     S := [];
     Sinv := [];
     bo := [];
     sv := [];
562  depth := [];
     strippedgens := [];
     mixedgens := [];
     for i in [1 .. #gens] do
       if PCBFrho(EG, gens[i]) ne Id(G) then
567      Append(~mixedgens, gens[i]);
       elif not PCBFIsId(EG, gens[i]) then
         Append(~strippedgens, gens[i]);
       end if;
     end for;
572  depth := [];
     for i in [1 .. #mixedgens] do
       depth[i] := PCBFDepth(EG, B, mixedgens[i]);
       if depth[i] eq k + 1 then
         Append(~B, PCBFFindNewBasePoint(EG, Omega, mixedgens[i]));
577      k +:= 1;
       end if;
     end for;
     for i in [1 .. k] do
       S[i] := [mixedgens[j] : j in [1 .. #mixedgens] | depth[j] ge
           i];
582    Sinv[i] := [PCBFInv(EG, x) : x in S[i]];
       sv[i], bo[i] := PCBFQuotOrbit(EG, B[i], S[i]);
     end for;
     i := k;
     while i ge 1 do
587    for pt in bo[i] do
         u1 := PCBFQuotSVElementInv(EG, S[i], Sinv[i], sv[i], bo[i
             ], pt);
         for x in S[i] do
           pt2 := PCBFQuotAction(EG, pt, x);
           u2 := PCBFQuotSVElement(EG, S[i], Sinv[i], sv[i], bo[i],
               pt2);
592        ele := PCBFMult(EG, PCBFMult(EG, u1, x), u2);
           if PCBFIsId(EG, ele) then continue x; end if;
           ele, j := PCBFQuotStrip(EG, ele, B, S, Sinv, sv, bo);
```

```
          if PCBFrho(EG, ele) ne Id(G) then
            // new strong generator at level j
597         if j gt k then
              // extend base
              Append(~B, PCBFFindNewBasePoint(EG, Omega, ele));
              k +:= 1;
              S[k] := [];
602           Sinv[k] := [];
            end if;
            for t in [i + 1 .. j] do
              Append(~S[t], ele);
              Append(~Sinv[t], PCBFInv(EG, ele));
607           sv[t], bo[t] := PCBFQuotOrbit(EG, B[t], S[t]);
            end for;
            i := j + 1;
            break pt;
          elif not PCBFIsId(EG, ele) then
612         Include(~strippedgens, ele);
          end if;
        end for;
      end for;
      i -:= 1;
617 end while;
    SGS := {@@};
    for i in [1 .. #S] do
      for j in [1 .. #S[i]] do
        Include(~SGS, S[i][j]);
622   end for;
    end for;
    // find normal closure of strippedgens in the subgroup
       generated by gens
    NHgens := [PCBFphi(EG, x) : x in strippedgens];
    NH := sub<EG`N | NHgens>;
627 closurefound := false;
    while not closurefound do
      for a in NHgens do
        for ele in mixedgens do
          apreimg := PCBFphiinv(EG, a);
632       newconj := PCBFphi(EG, PCBFConjugate(EG, apreimg, ele));
          if newconj notin NH then
            NH := sub<EG`N | NH, newconj>;
          end if;
        end for;
637   end for;
      closurefound := true;
      NHgens := Generators(NH);
      // test to see if normal closure has been found
      for a in NHgens do
```

```
642         for ele in mixedgens do
              apreimg := PCBFphiinv(EG, a);
              newconj := PCBFphi(EG, PCBFConjugate(EG, apreimg, ele));
              if newconj notin NH then
                // conjugates still missing
647             closurefound := false;
                break a;
              end if;
            end for;
          end for;
652     end while;
        return B, SGS, NHgens;
      end function;


657 PCBFNCLKeepBaseSchreierSims := function(EG, gens)

      /*

      Arguments:
662
        EG: PCBF-group, Type(s): Rec

        gens: Sequence of ordered pairs representing elements of
        EG, Type(s): SeqEnum
667

      Parameters:


672   Return Type(s):

        SetIndx, SeqEnum


677   Description:

        Returns a "base" and "strong generating set" for the
        subgroup of EG generated by gens, together with a
        generating sequence for the intersection of this subgroup
682     with the image of EG`N in EG. The base returned is the same
        as that for EG`GR`G.

      */

687   B := EG`GR`B; // base to be used in Schreier--Sims procedure
      G := EG`GR`G;
```

```
    k := #B; // this does not change as it is known that no
        element of EG (with nontrivial first entry) can fix all of
        the elements of B
    S := [];
    Sinv := [];
692 bo := [];
    sv := [];
    strippedgens := [];
    mixedgens := [];
    depth := [];
697 for i in [1 .. #gens] do
      if PCBFrho(EG, gens[i]) ne Id(G) then
        Append(~mixedgens, gens[i]);
        depth[#mixedgens] := PCBFDepth(EG, B, gens[i]);
      elif not PCBFIsId(EG, gens[i]) then
702     Append(~strippedgens, gens[i]);
      end if;
    end for;
    for i in [1 .. k] do
      S[i] := [mixedgens[j] : j in [1 .. #mixedgens] | depth[j] ge
          i];
707   Sinv[i] := [PCBFInv(EG, x) : x in S[i]];
      sv[i], bo[i] := PCBFQuotOrbit(EG, B[i], S[i]);
    end for;
    i := k;
    while i ge 1 do
712   for pt in bo[i] do
        u1 := PCBFInverse(EG, PCBFQuotSVElement(EG, S[i], Sinv[i],
            sv[i], bo[i], pt));
        for x in S[i] do
          pt2 := PCBFQuotAction(EG, pt, x);
          u2 := PCBFQuotSVElement(EG, S[i], Sinv[i], sv[i], bo[i],
              pt2);
717       ele := PCBFMult(EG, PCBFMult(EG, u1, x), u2);
          if PCBFIsId(EG, ele) then continue x; end if;
          ele, j := PCBFQuotStrip(EG, ele, B, S, Sinv, sv, bo);
          if j le k then
            // new strong generator ele at level j
722         for t in [i + 1 .. j] do
              Append(~S[t], ele);
              Append(~Sinv[t], PCBFInv(EG, ele));
              sv[t], bo[t] := PCBFQuotOrbit(EG, B[t], S[t]);
            end for;
727         i := j + 1;
            break pt;
          elif not PCBFIsId(EG, ele) then
            Include(~strippedgens, ele);
          end if;
```

```
732         end for;
          end for;
          i -:= 1;
        end while;
        SGS := {@@};
737     for i in [1 .. #S] do
          for j in [1 .. #S[i]] do
            Include(~SGS, S[i][j]);
          end for;
        end for;
742     NHgens := [PCBFphi(EG, x) : x in strippedgens];
        NH := sub<EG`N | NHgens>;
        // find normal closure of NH in subgroup generated by gens
        closurefound := false;
        while not closurefound do
747       for a in NHgens do
            for ele in mixedgens do
              apreimg := PCBFphiinv(EG, a);
              newconj := PCBFphi(EG, PCBFConjugate(EG, apreimg, ele));
              if newconj notin NH then
752             NH := sub<EG`N | NH, newconj>;
              end if;
            end for;
          end for;
          closurefound := true;
757       NHgens := Generators(NH);
          // test to see if normal closure has been found
          for a in NHgens do
            for ele in mixedgens do
              apreimg := PCBFphiinv(EG, a);
762           newconj := PCBFphi(EG, PCBFConjugate(EG, apreimg, ele));
              if newconj notin NH then
                // conjugates still missing
                closurefound := false;
                break a;
767           end if;
            end for;
          end for;
        end while;
        return B, SGS, NHgens;
772 end function;


    PCBFsubBasicOrbits := function(EG, B, S)

777     /*

        Arguments:
```

```
      EG: PCBF-group , Type(s): Rec
782
      B: Base of the corresponding quotient group of the subgroup
      of EG generated by S, Type(s): SeqEnum

      S: Sequence of ordered pairs representing elements of EG,
787   Type(s): SetIndx


    Parameters:

792
    Return Type(s):

      SeqEnum, SeqEnum

797
    Description:

      Computes the basic orbits and Schreier vectors
      (relative to B and S) of the corresponding quotient group
802   of the subgroup of EG generated by S. This function is no
      longer used.

    */

807 l := #B;
    if l eq 0 then return [], []; end if;
    m := #S;
    X := Isetseq(S);
    rhoX := [PCBFrho(EG, X[i]) : i in [1 .. m]];
812 Gsub := sub<EG`GR`G | rhoX>;
    sv := [];
    bo := [];
    bs := [Stabiliser(Gsub, B[1 .. i - 1]) : i in [1 .. l]];
    sv[1], bo[1] := PCBFQuotOrbit(EG, B[1], X);
817 for i in [2 .. l] do
      indexmap := [k : k in [1 .. m] | rhoX[k] in bs[i]];
      sv[i], bo[i] := PCBFQuotOrbit(EG, B[i], X[indexmap]);
      for j in [2 .. #sv[i]] do
        sv[i][j] := indexmap[sv[i][j]];
822   end for;
    end for;
    return sv, bo;
  end function;
```

Listing B.9: pcbfsubgroup.m

```
/*

File:                 pcbfsubgroup.m

Last modified:        Sun, 06 Feb 2011 22:14:13 +0000


Author(s):            Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


Company:              University of Warwick <http://www.warwick.ac.
    uk>


Description:

The methods in this file are used to construct subgroups of
polycyclic-by-finite groups represented by records of type PCBF.


Notes:

MAGMA V2.16-13.


Copyright 2006-2010, University of Warwick. All rights reserved.

*/


BATCHSIZE := 10;


load "pcbfschreiersims.m";


forward PCBFiotaSeq, PCBFiota, PCBFiotainv,
    PCBFCleanGeneratingSet, PCBFsubConstruct;


function PCBFsub(EG, gens)

  /*

  Arguments:

    EG: PCBF-group, Type(s): Rec
```

```
    gens: Sequence of ordered pairs representing elements of
    EG, Type(s): SeqEnum or SetIndx
50

  Parameters:


55 Return Type(s):

    Rec


60 Description:

    Returns a PCBF record representing the subgroup of EG
    generated by the elements in gens.

65 */

  if not EG`ismaster then
    // subgroups are only constructed as subgroups of a master
       group
    gens := PCBFiotaSeq(EG, gens);
70  EG := EG`supergrp; // master group
    return PCBFsub(EG, gens);
  end if;
  gens := PCBFCleanGeneratingSet(EG, gens);
  B, SGS, LHgens := PCBFNCLSchreierSims(EG, gens);
75 sgseq := Isetseq(SGS);
  sgseq, pointer := PCBFCleanGeneratingSet(EG, sgseq);
  if pointer le #sgseq then
    LHgens cat:= sgseq[pointer .. #sgseq];
  end if;
80 strgenpreimgs := {@@}; // indexed set to correspond with
     definiton
  S := {@@}; // indexed set to correspond with record definition
  for i in [1 .. pointer - 1] do
    Include(~strgenpreimgs, sgseq[i]); // preimages for strong
       generators in transversal
    Include(~S, PCBFrho(EG, sgseq[i])); // strong generators for
        permutation bit of subgroup
85 end for;
  QH := sub<EG`Q | S>;
  sv, bo := BasicOrbitsSV(QH, B, S);
  GR := GrpPermData(QH, B, S, sv, bo);
  LH := sub<EG`N | LHgens>;
90 return PCBFsubConstruct(EG, LH, GR, strgenpreimgs);
```

```
end function;

PCBFCleanGeneratingSet := function(EG, gens)
95
  /*

  Arguments:

100    EG: PCBF-group, Type(s): Rec

    gens: Sequence of ordered pairs representing elements of
    EG, Type(s): SeqEnum

105
  Parameters:


  Return Type(s):
110
    SeqEnum, RngIntElt

  Description:

115    Returns a sequence which contains no redundant generators
    and generates the same subgroup in EG as gens.
  */

  gensl := [car<PowerSequence(PowerSequence(IntegerRing())), EG`
     N> | ]; // the mixed generators
120  gensr := [car<PowerSequence(PowerSequence(IntegerRing())), EG`
     N> | ]; // these elements map to the trivial element in the
       quotient group
  for i in [1 .. #gens] do
    // nonredundant generating set created from gens in one pass
    if PCBFrho(EG, gens[i]) eq Id(EG`GR`G) then
      // gens[i] maps to the identity in the quotient group
125      if gens[i][2] ne Id(EG`N) then
        // add nontrivial generator to gensr
        Include(~gensr, gens[i]);
      end if;
    else
130      // check to see whether gens[i] has the same image in the
          quotient group as any previously added generator
      j := 1;
      while j le #gensl do
        if gens[i][1] eq gensl[j][1] then
```

```
            ele := PCBFMult(EG, gens[i], PCBFInverse(EG, gensl[j])
               );
135         // ele maps to the identity in the quotient group
               hence it can be added to gensr
            if ele[2] ne Id(EG`N) then
              // add nontrivial generator to gensr
              Include(~gensr, ele);
            end if;
140         break; // throw gens[i] away
          else
            // continue checking
            j +:= 1;
          end if;
145     end while;
        if j gt #gensl then Append(~gensl, gens[i]); end if; //
           genuine new generator
      end if;
    end for;
    return gensl cat gensr, #gensl + 1;
150 end function;


PCBFsubConstruct := function(EG, LH, GR, strgenpreimgs)

155 /*

    Arguments:

      EG: PCBF-group, Type(s): Rec
160
      LH: Intersection of the subgroup of EG generated by S and
      the image of EG`N in EG, Type(s): GrpPC or GrpGPC

      GR: Record holding data for corresponding quotient group of
165   the subgroup of EG to be constructed as a permutation
      group, Type(s): Rec

      strgenpreimgs: Indexed set containing preimages (in EG) of
      the strong generators for the corresponding quotient group
170   of the subgroup of EG to be constructed, Type(s): SetIndx

    Parameters:


175 Return Type(s):

      Rec
```

```
180   Description:

        Mimics the construction in PCBFMasterConstruct to construct
        a PCBF record representing the required subgroup of EG.

185   */

      EGsub := rec<PCBF | >;
      EGsub`E := EG`E;
      EGsub`GR := GR;
190   EGsub`Q := GR`G;
      EGsub`N := LH;
      EGsub`supergrp := EG;
      EGsub`strgenpreimgs := strgenpreimgs;
      m := #strgenpreimgs;
195   strgeninvpreimgs := {@@};
      for i in [1 .. m] do
        Include(~strgeninvpreimgs, (PCBFEleOrder(EG, strgenpreimgs[i
           ]) eq 2) select strgenpreimgs[i] else PCBFInv(EG,
           strgenpreimgs[i]));
      end for;
      EGsub`strgeninvpreimgs := strgeninvpreimgs;
200   e := <StrongGenNormalForm(GR, Id(GR`G)), Id(LH)>;
      EGsub`e := e;
      pcgens := PCGenerators(LH);
      r := #pcgens;
      pcgenconjugates := [];
205   pcgenconjugatesinv := [];
      for i in [1 .. m] do
        pcgenconjugates[i] := [LH | ];
        pcgenconjugatesinv[i] := [LH | ];
        xim := strgenpreimgs[i];
210     for j in [1 .. r] do
          // computing conjugates of the jth polycyclic generator
             with the ith preimage of the strong generating set of Q
              and its inverse
          yim := PCBFphiinv(EG, EG`N ! pcgens[j]);
          pcgenconjugates[i][j] := LH ! PCBFConjugate(EG, yim,
             strgenpreimgs[i])[2]; // stored as an element of LH
          pcgenconjugatesinv[i][j] := LH ! PCBFConjugate(EG, yim,
             strgeninvpreimgs[i])[2]; // stored as an element of LH
215     end for;
      end for;
      l := #GR`B; // number of elements in the stored base of the
         quotient group
      S := GR`S;
      tail := GR`tail;
```

```
220     tailinv := GR`tailinv;
        utail := GR`utail;
        utailinv := GR`utailinv;
        h := [];
        g := [];
225     w := [];
        tailelts := [LH | ];
        taileltsinv := [LH | ];
        for i in [1 .. l] do
          orb := GR`bo[i];
230       v := GR`sv[i];
          sg := GR`sgindexlist[i];
          for j in [1 .. #orb] do
            pt := orb[j];
            for k in [1 .. #sg] do
235           g := SVWordInv(GR`G, S, v, orb, pt);
              if utail[i][j][k] ne 0 then
                ipt := pt^S[sg[k]];
                h := SVWordInv(GR`G, S, v, orb, ipt);
                ux := PCBFMult(EG, PCBFWordMultiply(EG, strgenpreimgs,
                     strgeninvpreimgs, g), strgenpreimgs[sg[k]]);
240             th := PCBFMult(EG, PCBFWordMultiply(EG, strgenpreimgs,
                     strgeninvpreimgs, tail[i][j][k]), PCBFWordMultiply
                    (EG, strgenpreimgs, strgeninvpreimgs, h));
                tailelts[utail[i][j][k]] := LH ! PCBFphi(EG, PCBFMult(
                    EG, PCBFInv(EG, th), ux));
              end if;
              if utailinv[i][j][k] ne 0 then
                ipt := pt^(S[sg[k]]^-1);
245             h := SVWordInv(GR`G, S, v, orb, ipt);
                uxinv := PCBFMult(EG, PCBFWordMultiply(EG,
                    strgenpreimgs, strgeninvpreimgs, g),
                    strgeninvpreimgs[sg[k]]);
                th := PCBFMult(EG, PCBFWordMultiply(EG, strgenpreimgs,
                     strgeninvpreimgs, tailinv[i][j][k]),
                    PCBFWordMultiply(EG, strgenpreimgs,
                    strgeninvpreimgs, h));
                taileltsinv[utailinv[i][j][k]] := LH ! PCBFphi(EG,
                    PCBFMult(EG, PCBFInv(EG, th), uxinv));
              end if;
250         end for;
          end for;
        end for;
        EGsub`pcgens := pcgens;
        EGsub`pcgenconjugates := pcgenconjugates;
255     EGsub`pcgenconjugatesinv := pcgenconjugatesinv;
        EGsub`tailelts := tailelts;
        EGsub`taileltsinv := taileltsinv;
```

```
  EGsub`ismaster := false;
  EGsub`pcgensnf := [];
260  for i in [1 .. #pcgens] do
    EGsub`pcgensnf[i] := <e[1], pcgens[i]>;
  end for;
  EGsub`strgenpreimgsnf := [];
  EGsub`strgeninvpreimgsnf := [];
265  for i in [1 .. #strgenpreimgs] do
    _, EGsub`strgenpreimgsnf[i] := PCBFiotainv(EGsub,
        strgenpreimgs[i]);
    _, EGsub`strgeninvpreimgsnf[i] := PCBFiotainv(EGsub,
        strgeninvpreimgs[i]);
  end for;
  EGsub`grpgens := EGsub`strgenpreimgsnf cat EGsub`pcgensnf;
270  return EGsub;
end function;


PCBFiotainv := function(EGsub, ele)
275
  /*

  Arguments:

280    EGsub: PCBF-group, Type(s): Rec

    ele: Ordered pair representing an element of
    EGsub`supergrp, Type(s): Tup


285
  Parameters:


  Return Type(s):
290
    BoolElt, Tup


  Description:
295
    Tests to see if ele is an element of EGsub and if it is,
    returns true and the normal form of ele in the subgroup.
    ele is assumed to be a member of the super group of EGsub.
    Some of this code overlaps with the PCBFQuotStrip function.
300    Do NOT use this method directly. Use PCBFCoerce instead.

  */
```

```
    if EGsub`ismaster then return true, ele; end if;
305 EG := PCBFSuperGrp(EGsub);
    h := ele; // element of EG
    B := EGsub`GR`B;
    strgenpreimgs := EGsub`strgenpreimgs;
    strgeninvpreimgs := EGsub`strgeninvpreimgs;
310 bo := EGsub`GR`bo;
    sv := EGsub`GR`sv;
    l := #B;
    u := [];
    for i in [1 .. l] do
315   // strip-like process
      // h fixes base points B[1], B[2],..., B[i -1]
      pt := PCBFQuotAction(EG, B[i], h);
      if pt notin bo[i] then
        return false, <>;
320   end if;
      u[i] := PCBFQuotSVWordInv(EG, strgenpreimgs,
          strgeninvpreimgs, sv[i], bo[i], pt);
      h := PCBFMult(EG, h, PCBFInv(EG, PCBFWordMultiply(EG,
          strgenpreimgs, strgeninvpreimgs, u[i])));
    end for;
    // if ele is an element of EGsub then its first component is u
325 h := ele;
    word := [];
    for i in [1 .. l] do
      word cat:= WordInverse(u[i]);
    end for;
330 // premultiply h by the inverse of the element encoded by u to
        find the component belonging to EG`N
    h := PCBFMult(EG, PCBFWordMultiply(EG, strgenpreimgs,
       strgeninvpreimgs, word), h);
    if PCBFrho(EG, h) ne Id(EG`GR`G) then return false, <>; end if
       ;
    n := PCBFphi(EG, h);
    if n notin EGsub`N then return false, <>; end if;
335 return true, <u, EGsub`N ! n>;
  end function;


  PCBFiota := function(EG, ele)
340
    /*

    Arguments:

345   EG: PCBF-group, Type(s): Rec
```

```
      ele: Ordered pair representing an element of EG, Type(s):
      Tup

350
    Parameters:


    Return Type(s):
355
      Tup


    Description:
360
      Returns the image of ele in the super group to which it
      belongs. Do NOT use this method directly. Use PCBFCoerce
      instead.

365 */

    if EG`ismaster then return ele; end if;
    EGsupergrp := PCBFSuperGrp(EG);
    word := [];
370 for i in [#ele[1] .. 1 by -1] do
      word cat:= ele[1][i];
    end for;
    return PCBFMult(EGsupergrp, PCBFWordMultiply(EGsupergrp, EG`
        strgenpreimgs, EG`strgeninvpreimgs, word), <PCBFId(
        EGsupergrp)[1], EGsupergrp`N ! ele[2]>);
end function;
375

PCBFiotaSeq := function(EG, elts)

    /*
380
    Arguments:

      EG: PCBF-group, Type(s): Rec

385   elts: Sequence of ordered pairs representing elements of
      EG, Type(s): Seqenum


    Parameters:
390

    Return Type(s):
```

```
      SeqEnum
395

    Description:

      Returns a sequence of the respective images (in the super
400   group to which they belong) of each element of elts.

    */

    return [PCBFiota(EG, elts[i]) : i in [1 .. #elts]];
405 end function;


PCBFCoerce := function(EG1, ele, EG2)

410   /*

    Arguments:

      EG1: PCBF-group, Type(s): Rec
415
      ele: Ordered pair representing an element of EG1, Type(s):
      Tup

      EG2: PCBF-group, Type(s): Rec
420

    Parameters:


425 Return Type(s):

      BoolElt, Tup


430 Description:

      Coerces the element, ele, of EG1 into EG2.

    */
435
    return PCBFiotainv(EG2, PCBFiota(EG1, ele));
end function;


440 PCBFin := function(EG1, ele, EG2)
```

```
/*

Arguments:
445
  EG1: PCBF-group, Type(s): Rec

  ele: Ordered pair representing an element of EG1, Type(s):
  Tup
450
  EG2: PCBF-group, Type(s): Rec


Parameters:
455

Return Type(s):

  BoolElt
460

Description:

  Returns true if and only if the element, ele, of EG1 is an
465 element of EG2.

*/

ans, _ := PCBFCoerce(EG1, ele, EG2);
470 return ans;
end function;


PCBFnotin := function(EG1, ele, EG2)
475
/*

Arguments:

480 EG1: PCBF-group, Type(s): Rec

  ele: Ordered pair representing an element of EG1, Type(s):
  Tup

485 EG2: PCBF-group, Type(s): Rec


Parameters:
```

```
490
    Return Type(s):

      BoolElt

495
    Description:

      Returns true if and only if the element, ele, of EG1 is not
      an element of EG2.
500
    */

    return not PCBFin(EG1, ele, EG2);
  end function;
505

  PCBFQuotSubrhoinv := function(EG, H)

    /*
510
    Arguments:

      EG: PCBF group, Type(s): Rec

515   H: Subgroup of EG`Q, Type(s): GrpPerm


    Parameters:

520
    Return Type(s):

      Rec

525
    Description:

      Returns the preimage of the subgroup, H, of EG`Q in EG.

530 */

    Hgens := Setseq(Generators(H));
    gens := PCBFrhoinvSeq(EG, [EG`Q ! Hgens[i] : i in [1 .. #Hgens
        ]]) cat EG`pcgensnf;
    return PCBFsub(EG, gens);
535 end function;
```

```
PCBFSolubleRadical := function(EG)
540   /*
      Arguments:
        EG: PCBF group, Type(s): Rec
545
      Parameters:

550   Return Type(s):
        Rec

555   Description:
        Returns the soluble radical of EG.
      */
560
      return PCBFQuotSubrhoinv(EG, SolubleRadical(EG`Q));
end function;

565 PCBFSolvableRadical := function(EG)
      /*
      Arguments:
570
        EG: PCBF group, Type(s): Rec

      Parameters:
575
      Return Type(s):
        Rec
580
      Description:
```

```
      Returns the soluble radical of EG.
585
    */

    return PCBFSolubleRadical(EG);
  end function;
590

  PCBFncl := function(EG, gens)

    /*
595
    Arguments:

      EG: PCBF-group, Type(s): Rec

600   gens: Sequence of ordered pairs representing elements of
      EG, Type(s): SeqEnum or SetIndx


    Parameters:
605

    Return Type(s):

      Rec
610
    Description:

      Returns a PCBF record representing the normal closure of
      the subgroup of EG generated by the elements in gens.
615
    */

    closurefound := false;
    G := EG;
620 X := PCBFGenerators(G);
    Y := gens;
    H := PCBFsub(G, Y);
    while not closurefound do
      for i in [1 .. BATCHSIZE] do
625     g := PCBFRandom(G);
        _, h := PCBFCoerce(H, PCBFRandom(H), G);
        gen := PCBFConjugate(G, h, g);
        if PCBFnotin(G, gen, H) then
          Append(~Y, gen);
630       H := PCBFsub(G, Y);
        end if;
```

```
    end for;
    closurefound := true;
    for x in X do
635   for y in Y do
        if PCBFnotin(G, PCBFConjugate(G, y, x), H) then
          closurefound := false;
          break x;
        end if;
640   end for;
    end for;
  end while;
  return H;
end function;
645

PCBFSubgroupNormalClosure := function(G, H)

  /*
650
  Arguments:

    G: PCBF-group, Type(s): Rec

655   H: Subgroup of G, Type(s): Rec


  Parameters:

660
  Return Type(s):

    Rec

665
  Description:

    Returns a PCBF record representing the normal closure of
    the subgroup H in G.
670
  */

  h := PCBFGenerators(H);
  elts := [];
675 for i in [1 .. #h] do
    _, elts[i] := PCBFCoerce(H, h[i], G);
  end for;
  return PCBFncl(G, elts);
end function;
```

```
680

PCBFSubgroupConjugate := function(G, H, g)

  /*
685
  Arguments:

    G: PCBF-group, Type(s): Rec

690 H: Subgroup of G, Type(s): Rec

    g: Element of G, Type(s): Tup


695 Parameters:


  Return Type(s):

700 Rec


  Description:

705 Returns a PCBF record representing the conjugate of the
    subgroup H in G.

  */

710 h := PCBFGenerators(H);
  elts := [];
  for i in [1 .. #h] do
    _, elts[i] := PCBFCoerce(H, h[i], G);
  end for;
715 return PCBFsub(G, [PCBFConjugate(G, elts[i] ,g) : i in [1 .. #
      elts]]);
end function;
```

Listing B.10: pcbfderived.m

```
/*

File:             pcbfderived.m
4

Last modified:    Sun, 06 Feb 2011 22:14:13 +0000
```

```
9  Author(s):            Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>

   Company:              University of Warwick <http://www.warwick.ac.
       uk>

14
   Description:

   The methods in this file are used to construct derived subgroups
        and series
   of polycyclic-by-finite groups represented by records of type
       PCBF.
19

   Notes:

   MAGMA V2.16-13.
24

   Copyright 2006-2010, University of Warwick. All rights reserved.

   */
29

   load "pcbfsubgroup.m";


34 PCBFCommutatorSubgroup := function(H, K)

     /*

     Arguments:
39
         H: PCBF-group, Type(s): Rec

         K: PCBF-group, Type(s): Rec

44
     Parameters:


     Return Type(s):
49
         Rec


     Description:
```

```
54
      Returns a PCBF record representing the commutator subgroup
      of H and K. The PCBF-groups H and K are assumed to be
      subgroups of some common PCBF-group. Do NOT use this
      function to compute derived subgroups. The function
59    PCBFDerivedGroup is more efficient.

    */

    EG := PCBFSuperGrp(H);
64  X := PCBFGenerators(H);
    Y := PCBFGenerators(K);
    L := PCBFsub(EG, PCBFiotaSeq(H, X) cat PCBFiotaSeq(K, Y));
    // [H, K] is the normal closure in L of the subgroup generated
        by [x, y] with x in X, y in Y
    C := [car<PowerSequence(PowerSequence(IntegerRing())), L`N>
       |];
69  for i in [1 .. #X] do
      _, x := PCBFCoerce(H, X[i], L);
      for j in [1 .. #Y] do
        _, y := PCBFCoerce(K, Y[j], L);
        Append(~C, PCBFCommutator(L, x, y));
74    end for;
    end for;
    return PCBFncl(L, C);
  end function;

79
  PCBFDerivedGroup := function(G)

    /*

84  Arguments:

      G: PCBF-group, Type(s): Rec


89  Parameters:


    Return Type(s):

94    Rec


    Description:

99    Returns a PCBF record representing the derived subgroup of
```

```
      G.

    */

104 X := PCBFGenerators(G);
    // G' is the normal closure in G of the subgroup generated by
        (x1, x2) with x1, x2 in X
    return PCBFncl(G, [PCBFCommutator(G, x1, x2) : x1 in X, x2 in
        X]);
end function;

109
PCBFDerivedSubgroup := function(G)

    /*

114 Arguments:

      G: PCBF-group, Type(s): Rec

    Parameters:
119

    Return Type(s):

      Rec
124
    Description:

      Returns a PCBF record representing the derived subgroup of
      G.
129
    */

    return PCBFDerivedGroup(G);
end function;
134

PCBFDerivedSeries := function(G)

    /*
139
    Arguments:

      G: PCBF-group, Type(s): Rec

144
    Parameters:
```

```
    Return Type(s):

      SeqEnum


    Description:

      Returns a sequence containing the derived series of G.

    */

    DS := [G];
    H := PCBFDerivedGroup(G);
    i := 1;
    while Order(PCBFQuotGrp(H)) lt Order(PCBFQuotGrp(DS[i])) or
        PCBFNormalSubgrp(H) ne PCBFNormalSubgrp(DS[i]) do
      Append(~DS, H);
      H := PCBFDerivedGroup(H);
      i +:= 1;
    end while;
    return DS;
end function;


PCBFLowerCentralSeries := function(G)

    /*

    Arguments:

      G: PCBF-group, Type(s): Rec


    Parameters:


    Return Type(s):

      SeqEnum


    Description:

      Returns a sequence containing the lower central series of
      G.
```

```
    */
194
    LCS := [G];
    H := PCBFDerivedGroup(G);
    i := 1;
    while Order(PCBFQuotGrp(H)) lt Order(PCBFQuotGrp(LCS[i])) or
        PCBFNormalSubgrp(H) ne PCBFNormalSubgrp(LCS[i]) do
199     Append(~LCS, H);
        H := PCBFCommutatorSubgroup(G, H);
        i +:= 1;
    end while;
    return LCS;
204 end function;
```

Listing B.11: pcbfsylow.m

```
1 /*

  File:              pcbfsylow.m


6 Last modified:      Sun, 06 Feb 2011 22:14:13 +0000


  Author(s):         Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>

11
  Company:           University of Warwick <http://www.warwick.ac.
      uk>


  Description:
16
  The methods in this file are used to construct Sylow subgroups
      of (finite)
  polycyclic-by-finite groups represented by records of type PCBF.


21 Notes:

  MAGMA V2.16-13.


26 Copyright 2006-2010, University of Warwick. All rights reserved.

  */


31 load "pcbfderived.m";
```

```
PCBFSylow := function(G, p)
36  /*
    Arguments:
      G: PCBF-group, Type(s): Rec
41
      p: Prime integer, Type(s): RngIntElt

    Parameters:
46
    Return Type(s):
      Rec
51
    Description:
      Returns a Sylow p-subgroup of G.
56
    */
    H := PCBFQuotSubrhoinv(G, Sylow(G`Q, p)); // the Sylow p-
        subgroups of G are exactly the Sylow p-subgroups of H
    K, pcisodata := PCBFPolycyclicGroup(H);
61  sylowgensK := Setseq(Generators(Sylow(K, p)));
    sylowgensG := [car<PowerSequence(PowerSequence(IntegerRing()))
        , G`N> |];
    for i in [1 .. #sylowgensK] do
      // map back into K, then into H then into G
      _, sylowgensG[i] := PCBFCoerce(H, PCBFPolycyclicisoinv(H,
          pcisodata, K ! sylowgensK[i]), G);
66  end for;
    return PCBFsub(G, sylowgensG);
  end function;

71 PCBFSylowSubgroup := function(G, p)
    /*
    Arguments:
76
```

```
      G: PCBF-group, Type(s): Rec
      p: Prime integer, Type(s): RngIntElt
81
    Parameters:

    Return Type(s):
86
      Rec

    Description:
91
      Returns a Sylow p-subgroup of G.
    */
96  return PCBFSylow(G, p);
end function;

PCBFSylowConjugatingElement := function(G, P1, P2)
101
    /*
    Arguments:
106   G: PCBF-group, Type(s): Rec
      P1: Sylow p-subgroup of G, Type(s): Rec
      P2: Sylow p-subgroup of G, Type(s): Rec
111
    Parameters:

116 Return Type(s):
      Tup

121 Description:
      Returns an element of G conjugating P1 into P2.
```

```
  */
126
  // incomplete
  Q := G`GR`G;
  rhoP1 := PCBFQuotGrp(P1);
  rhoP2 := PCBFQuotGrp(P2);
131 P2N := PCBFQuotSubrhoinv(G, rhoP2); // P2N is finite and
      soluble
  _, z := IsConjugate(Q, rhoP1, rhoP2);
  g := PCBFrhoinv(G, z);
  P1 := PCBFSubgroupConjugate(G, P1, g); // P1 is now a Sylow p-
      subgroup of P2N
  // do the work using power-conjugate presentations
136 L, isodata := PCBFPolycyclicGroup(P2N);
  gens := [];
  for x in PCBFGenerators(P1) do
    _, ele := PCBFCoerce(P1, x, P2N);
    Append(~gens, PCBFPolycycliciso(P2N, L, isodata, ele));
141 end for;
  LP1 := sub<L | gens>; // image of P1 in L
  gens := [];
  for x in PCBFGenerators(P2) do
    _, ele := PCBFCoerce(P2, x, P2N);
146   Append(~gens, PCBFPolycycliciso(P2N, L, isodata, ele));
  end for;
  LP2 := sub<L | gens>; // image of P2 in L
  _, z := IsConjugate(L, LP1, LP2);
  _, y := PCBFCoerce(P2N, PCBFPolycyclicisoinv(P2N, isodata, z),
      G);
151 return PCBFMult(G, g, y);
end function;
```

Listing B.12: pcbfcentre.m

```
/*

3 File:               pcbfcentre.m


  Last modified:      Sun, 06 Feb 2011 22:14:13 +0000

8
  Author(s):          Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


  Company:            University of Warwick <http://www.warwick.ac.
    uk>
13
```

```
Description:

The methods in this file are used to compute the centre of a
18 polycyclic-by-finite group represented by records of type PCBF.


Notes:

23 MAGMA V2.16-13.


Copyright 2006-2010, University of Warwick. All rights reserved.

28 */


load "pcbfsylow.m";

33
PCBFCentre := function(G)

   /*

38   Arguments:

      G: PCBF-group, Type(s): Rec


43   Parameters:


     Return Type(s):

48      Rec


     Description:

53      Returns the centre of G.

     */

     S := PCBFSolubleRadical(G);
58   K, pcisodata := PCBFPolycyclicGroup(S); // pcisodata encodes
        an isomorphism from S onto P
      C := Centre(K);
      if IsTrivial(C) then return PCBFsub(G, []); end if;
```

```
centregensG := [car<PowerSequence(PowerSequence(IntegerRing())
    ), G`N> |];
  if PCBFIsSoluble(G) then
  // C is the centre of G
  centregensC := Setseq(Generators(C));
  for i in [1 .. #centregensC] do
    // map back into K, then into S and then into G
    _, centregensG[i] := PCBFCoerce(S, PCBFPolycyclicisoinv(S,
        pcisodata, K ! centregensC[i]), G);
  end for;
  return PCBFsub(G, centregensG);
end if;
  AC, kappa := AbelianGroup(C); // kappa: C -> AC
  invars := Invariants(AC);
  r := #invars;
  Z := Integers();
  mats := [MatrixAlgebra(Z, r) | ];
  I := IdentityMatrix(Z, r);
  O := ZeroMatrix(Z, r, r);
// for each generator of G, calculate the matrix specifying
   conjugation action of that generator on AC, and subtract
   the identity matrix
subjgens := PCBFGenerators(G);
matseq := [];
  for i in [1 .. #subjgens] do
  for j in [1 .. r] do
    _, g := PCBFCoerce(S, PCBFPolycyclicisoinv(S, pcisodata, K
       ! (AC.j @@ kappa)), G);
    _, s := PCBFCoerce(G, PCBFConjugate(G, g, subjgens[i]), S)
       ;
    matseq[j] := Eltseq((C ! PCBFPolycycliciso(S, K, pcisodata
       , s)) @ kappa);
  end for;
  mat := Matrix(matseq) - I;
  if mat ne O then Append(~mats, mat); end if;
end for;
if #mats eq 0 then
  // C is the centre of G
  centregensC := Setseq(Generators(C));
  for i in [1 .. #centregensC] do
    // map back into K, then into S then into G
    _, centregensG[i] := PCBFCoerce(S, PCBFPolycyclicisoinv(S,
        pcisodata, K ! centregensC[i]), G);
  end for;
  return PCBFsub(G, centregensG);
end if;
// the elements in the centre of G are basically those in the
   nullspace of each of the matrices. But the matrices are
```

```
      over the integers, and solutions modulo invars[i] are
      required. To handle this, extra rows are adjoined to the
      matrices
    invarmat := O;
    for i in [1 .. r] do invarmat[i][i] := invars[i]; end for;
    modmats := [];
103 for i in [1 .. #mats] do
        modmats[i] := VerticalJoin([mats[i]] cat [O : j in [1 ..
            i - 1]] cat [invarmat] cat [O : j in [i+1 .. #mats
            ]]);
  end for;
  // now the centre corresponds to the intersection of the
      nullspaces of the matrices - equivalently, the nullspace of
       their horizontal join
  nsmat := NullspaceMatrix(HorizontalJoin(modmats));
108 centregensC := [C | ];
  for i in [1 .. Nrows(nsmat)] do
    a := AC ! Eltseq(ExtractBlock(nsmat, i, 1, 1, r));
    if a ne AC ! 0 then
      Append(~centregensC, a @@ kappa);
113 end if;
  end for;
  for i in [1 .. #centregensC] do
    // map back into K, then into S and then into G
    _, centregensG[i] := PCBFCoerce(S, PCBFPolycyclicisoinv(S,
       pcisodata, K ! centregensC[i]), G);
118 end for;
  return PCBFsub(G, centregensG);
end function;


123 PCBFCenter := function(G)

  /*

  Arguments:

128
    G: PCBF-group, Type(s): Rec


  Parameters:

133

  Return Type(s):

    Rec

138
```

```
  Description:

    Returns the centre of G.
143
  */

  return PCBFCentre(G);
end function;
```

Listing B.13: pcbfcentraliser.m

```
/*
2
File:                    pcbfcentraliser-test.m


Last modified:           Sun, 06 Feb 2011 22:14:13 +0000
7

Author(s):               Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


12 Company:              University of Warwick <http://www.warwick.ac.
    uk>


Description:

17 The methods in this file are used to compute the centraliser of
    an element
belonging to a polycyclic-by-finite group represented by records
     of type
PCBF.


22 Notes:

MAGMA V2.16-13.


27 Copyright 2006-2010, University of Warwick. All rights reserved.

*/


32 load "pcbfcentre.m";


PCBFCentraliser := function(G, g)
```

```
37    /*

      Arguments:

        G: PCBF-group, Type(s): Rec
42
        g: Element of G, Type(s): Tup


      Parameters:
47

      Return Type(s):

        Rec
52

      Description:

        Returns the centraliser of g in G.
57
      */

      if PCBFeq(G, g, PCBFId(G)) then return G; end if; // cheap
          check for g = PCBFId(G)
      Q := G`GR`G;
62    c := PCBFrhoinvSeq(G, Setseq(Generators(Centraliser(Q, PCBFrho
          (G, g))))) cat G`pcgensnf; // generators of inverse image
          of centraliser of g in the quotient
      C := PCBFsub(G, c);
      N := ElementaryAbelianSeriesCanonical(G`N); // the canonical
          version of the elementary abelian series function is used
          as a sequence of normal subgroups is required
      t := #N;
      // the centraliser of g in G/N[1] has been computed, so it's
          time to move downward through the elementary abelian series
67    for i in [1 .. t - 1] do
        M, phi := GModule(N[1], N[i], N[i + 1]); // phi : N[i] --> M
        F := Field(M);
        d := Dimension(M);
        A := GL(d + 1, F);
72      I := Id(A);
        bas := [PCBFphiinv(G, (M.k)@@phi) : k in [1 .. d]];
        cimgs := [A | ];
        for j in [1 .. #c] do
          y := [];
77        for k in [1 .. d] do
```

```
        // computing k-th row of the image of c[i] under the
           affine map
        y[k] := Eltseq((PCBFphi(G, PCBFConjugate(G, bas[k], c[j
           ])))@phi) cat [Zero(F)];
      end for;
      y[d + 1] := Eltseq((PCBFphi(G, PCBFCommutator(G, g, c[j]))
         )@phi) cat [Id(F)];
82    cimgs[j] := (A ! Matrix(y)); // these images should really
          be used to define a homomorphism, but homomorphisms
         involving PCBF groups have not yet been implemented
         internally in MAGMA
    end for;
    defcimgs := [A | ];
    cnew := [car<PowerSequence(PowerSequence(IntegerRing())), G`
       N> |];
    indexmap := [];
87  // clean generating set
    for j in [1 .. #cimgs] do
      if cimgs[j] eq I then
        Append(~cnew, c[j]);
        continue;
92    end if;
      for k in [1 .. #defcimgs] do
        if cimgs[j] eq defcimgs[k] then
          Append(~cnew, PCBFMult(G, c[indexmap[k]], PCBFInverse(
             G, c[j])));
          continue j;
97      end if;
      end for;
      // cimgs[j] is a genuine new generator
      Append(~defcimgs, cimgs[j]);
      Append(~indexmap, j); // this sequence is needed to
         correspond the position of cimgs[j] in defcimgs to j
102 end for;
    if defcimgs eq [] then continue; end if;
    B := sub<A | defcimgs>; // range of alpha
    // compute stabliser in B of (0, ..., 0, 1)
    V := VectorSpace(A);
107 trickvec := V.(d + 1);
    S := Stabiliser(B, trickvec);
    s := Setseq(Generators(S));
    // map S back into G
    // the code that follows is slow and should be rewritten
       when homomorphims involving PCBF-groups become available
112 L, psi := FPGroup(B);
    // to map S back into G, the preimage kernel of psi in G is
       needed
    rels := Relations(L);
```

```
    kerpsipreimggens := [];
    for j in [1 .. #rels] do
117   ele := PCBFId(G);
      word := Eltseq(LHS(rels[j])*(RHS(rels[j])^-1));
      // multiply word in G
      for k in [1 .. #word] do
        ele := PCBFMult(G, ele, word[k] lt 0 select PCBFInverse(
          G, c[indexmap[-word[k]]]) else c[indexmap[word[k]]]);
122   end for;
      _, kerpsipreimggens[j] := PCBFCoerce(G, ele, C);
    end for;
    kerpsipreimg := PCBFncl(C, kerpsipreimggens);
    kerpsipreimggens := PCBFGenerators(kerpsipreimg);
127 offset := #cnew;
    for j in [1 .. #kerpsipreimggens] do
      _, cnew[offset + j] := PCBFCoerce(C, kerpsipreimggens[j],
        G);
    end for;
    for j in [1 .. #s] do
132   ele := PCBFId(G);
      word := Eltseq(s[j]@@psi);
      // multiply word in G
      for k in [1 .. #word] do
        ele := PCBFMult(G, ele, word[k] lt 0 select PCBFInverse(
          G, c[indexmap[-word[k]]]) else c[indexmap[word[k]]]);
137   end for;
      Append(~cnew, ele);
    end for;
    c := cnew;
    C := PCBFsub(G, c);
142 end for;
  return C;
end function;


147 PCBFCentralizer := function(G, g)

  /*

  Arguments:
152
    G: PCBF-group, Type(s): Rec

    g: Element of G, Type(s): Tup

157
  Parameters:
```

```
    Return Type(s):
162
      Rec


    Description:
167
      Returns the centraliser of g in G.

    */

172 return PCBFCentraliser(G, g);
end function;
```

Listing B.14: pcbfconjugacy.m

```
/*
2
File:                 pcbfconjugacy.m


Last modified:        Sun, 06 Feb 2011 22:14:13 +0000
7

Author(s):            Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>


12 Company:           University of Warwick <http://www.warwick.ac.
    uk>


Description:

17 The methods in this file are used to test conjugacy of elements
    belonging
to a polycyclic-by-finite group represented by records of type
    PCBF.


Notes:
22
MAGMA V2.16-13.


Copyright 2006-2010, University of Warwick. All rights reserved.
27
*/
```

```
load "pcbfcentraliser.m";
32
PCBFIsConjugate := function(G, g, h)

  /*
37
  Arguments:

    G: PCBF-group, Type(s): Rec

42  g: Element of G, Type(s): Tup

    h: Element of G, Type(s): Tup


47  Parameters:


  Return Type(s):

52  BoolElt, Rec


  Description:

57  Returns true, and an element conjugating g into h, if g and
    h are conjugate in G, false otherwise.

  */

62 if PCBFeq(G, g, h) then return true, PCBFId(G); end if; //
      cheap check for g = h
  Q := G`GR`G;
  N := ElementaryAbelianSeriesCanonical(G`N); // the canonical
      version of the elementary abelian series function is used
      as a sequence of normal subgroups is required
  t := #N;
  gbar := PCBFrho(G, g);
67 hbar := PCBFrho(G, h);
  isconj, zbar := IsConjugate(Q, gbar, hbar);
  if not isconj then return false, _; end if; // g and h aren't
      even conjugate in the quotient Q
  z := PCBFrhoinv(G, zbar); // preimage of conjugating element
  h := PCBFConjugate(G, h, PCBFInv(G, z)); // h is now equal to
      g in the quotient G/N
```

```
72    c := PCBFrhoinvSeq(G, Setseq(Generators(Centraliser(Q, PCBFrho
         (G, g))))) cat G`pcgensnf; // generators of inverse image
         of centraliser of g in the quotient
      C := PCBFsub(G, c);
      // the centraliser of g in G/N[1] has been computed, so it's
         time to move downward through the elementary abelian series
      for i in [1 .. t - 1] do
        ele := PCBFId(G);
77      M, phi := GModule(N[1], N[i], N[i + 1]); // phi : N[i] --> M
        F := Field(M);
        d := Dimension(M);
        A := GL(d + 1, F);
        I := Id(A);
82      bas := [PCBFphiinv(G, (M.k)@@phi) : k in [1 .. d]];
        cimgs := [A | ];
        for j in [1 .. #c] do
          y := [];
          for k in [1 .. d] do
87          // computing k-th row of the image of c[i] under the
               affine map
            y[k] := Eltseq((PCBFphi(G, PCBFConjugate(G, bas[k], c[j
               ])))@phi) cat [Zero(F)];
          end for;
          y[d + 1] := Eltseq((PCBFphi(G, PCBFCommutator(G, g, c[j]))
             )@phi) cat [Id(F)];
          cimgs[j] := (A ! Matrix(y)); // these images should really
              be used to define a homomorphism, but homomorphisms
             involving PCBF groups have not yet been implemented
             internally in MAGMA
92      end for;
        defcimgs := [A | ];
        cnew := [car<PowerSequence(PowerSequence(IntegerRing())), G`
           N> |];
        indexmap := [];
        // clean generating set
97      for j in [1 .. #cimgs] do
          if cimgs[j] eq I then
            Append(~cnew, c[j]);
            continue;
          end if;
102       for k in [1 .. #defcimgs] do
            if cimgs[j] eq defcimgs[k] then
              Append(~cnew, PCBFMult(G, c[indexmap[k]], PCBFInverse(
                 G, c[j])));
              continue j;
            end if;
107       end for;
          // cimgs[j] is a genuine new generator
```

```
      Append(~defcimgs, cimgs[j]);
      Append(~indexmap, j); // this sequence is needed to
         correspond the position of cimgs[j] in defcimgs to j
    end for;
112 if defcimgs eq [] then
      // if defcimgs is empty, then the new centraliser is equal
          to the old centraliser
      // if g and h are conjugate, then they should already be
         equal in N[i + 1]
      if PCBFphi(G, PCBFMult(G, PCBFInv(G, g), h)) notin N[i +
         1] then
        return false, _;
117   end if;
      continue;
    end if;
    B := sub<A | defcimgs>; // range of alpha
    L, psi := FPGroup(B); // image of B as a finitely presented
       group
122 // compute stabliser in B of (0, ..., 0, 1)
    V := VectorSpace(A);
    orbvec := V.(d + 1);
    S := Stabiliser(B, orbvec);
    O := Orbit(B, orbvec);
127 tau, P := OrbitAction(B, O); // P is a permutation group
       which gives the action of B on O, tau : B --> P
    testvec := V ! (Eltseq((PCBFphi(G, PCBFMult(G, PCBFInv(G, g)
       , h)))@phi) cat [Id(F)]); // this vector will be tested
       for membership in the orbit O
    if testvec notin O then
      return false, _;
    end if;
132 isconj, zperm := IsConjugate(P, tau(orbvec), tau(testvec));
    if not isconj then
      return false, _;
    end if;
    ele := PCBFId(G);
137 word := Eltseq((zperm@@tau)@@psi);
    // find preimage in C of (local) conjugating element
    for k in [1 .. #word] do
      ele := PCBFMult(G, ele, word[k] lt 0 select PCBFInverse(G,
          c[indexmap[-word[k]]]) else c[indexmap[word[k]]]);
    end for;
142 h := PCBFConjugate(G, h, PCBFInv(G, ele)); // h is now equal
        to g in the quotient G/N[i + 1]
    z := PCBFMult(G, ele, z); // modify (global) conjugating
       element
    // map S back into G
```

```
    // the code that follows is slow and should be rewritten
       when homomorphims involving PCBF-groups are implemented
    // to map S back into G, the preimage of the kernel of psi
       in G is needed
147 rels := Relations(L);
    kerpsipreimggens := [];
    for j in [1 .. #rels] do
      ele := PCBFId(G);
      word := Eltseq(LHS(rels[j]) * (RHS(rels[j])^-1));
152   // multiply word in G
      for k in [1 .. #word] do
        ele := PCBFMult(G, ele, word[k] lt 0 select PCBFInverse(
           G, c[indexmap[-word[k]]]) else c[indexmap[word[k]]]);
      end for;
      _, kerpsipreimggens[j] := PCBFCoerce(G, ele, C);
157 end for;
    kerpsipreimg := PCBFncl(C, kerpsipreimggens);
    kerpsipreimggens := PCBFGenerators(kerpsipreimg);
    offset := #cnew;
    for j in [1 .. #kerpsipreimggens] do
162   _, cnew[offset + j] := PCBFCoerce(C, kerpsipreimggens[j],
         G);
    end for;
    s := Setseq(Generators(S));
    for j in [1 .. #s] do
      ele := PCBFId(G);
167   word := Eltseq(s[j]@@psi);
      // multiply word in G
      for k in [1 .. #word] do
        ele := PCBFMult(G, ele, word[k] lt 0 select PCBFInverse(
           G, c[indexmap[-word[k]]]) else c[indexmap[word[k]]]);
      end for;
172   Append(~cnew, ele);
    end for;
    c := cnew;
    C := PCBFsub(G, c);
  end for;
177 return true, z;
end function;
```

Listing B.15: pcbfmain.m

```
/*
2
File:                pcbfmain.m


Last modified:       Sun, 06 Feb 2011 22:14:13 +0000
7
```

```
Author(s):             Shavak Sinanan <S.K.Sinanan@warwick.ac.uk>

12 Company:            University of Warwick <http://www.warwick.ac.
   uk>

Description:

17 Root file for the PCBF package.

Notes:

22 MAGMA V2.16-13.

Copyright 2006-2010, University of Warwick. All rights reserved.

27 */

load "pcbfconjugacy.m";
load "pcbfexamples.m";
32

PCBFNormalForm := function(EG, e)

  /*
37
  Arguments:

    EG: PCBF-group, Type(s): Rec

42    e: element of original polycyclic-by-finite group, Type(s):
      GrpElt

  Parameters:
47

  Return Type(s):

    GrpPermElt
52

  Description:
```

```
      Computes the normal form of the element y in the PCBF-group
57    EG. USED FOR TESTING ONLY.

    */

    g := EG`rho(e);
62  word := StrongGenWord(EG`GR, g);
    return <StrongGenNormalForm(GR, g), EG`phi(WordMultiply(EG`
        strgenpreimgs, EG`strgeninvpreimgs, word)^-1 * e)>;
  end function;


67 function PCBFRevert(EG, ele)

    /*

    Arguments:
72
      EG: PCBF-group, Type(s): Rec

      ele: ordered pair representing an element of the PCBF-group
      EG, Type(s): Tup
77

    Parameters:


82  Return Type(s):

      GrpElt


87  Description:

      Returns the element of the original master group
      represented by ele. USED FOR TESTING ONLY.

92  */

    if not EG`ismaster then
      // recurse if the given group is not a master group
      return PCBFRevert(EG`supergrp, PCBFiota(EG, ele));
97  end if;
    phi := EG`phi;
    word := [];
    for i in [#ele[1] .. 1 by -1] do
      word cat:= ele[1][i];
```

```
102   end for;
      ans := WordMultiply(EG`strgenpreimgs, EG`strgeninvpreimgs,
        word);
      ans *:= ele[2]@@phi;
      return ans;
    end function;
107

    function PCBFRevertSeq(EG, elts)

      /*
112
      Arguments:

        EG: PCBF-group, Type(s): Rec

117     elts: sequence of ordered pairs representing elements of
        the PCBF-group EG, Type(s): Tup


      Parameters:
122

      Return Type(s):

        SeqEnum
127

      Description:

        Returns a sequence of elements of the original master group
132     corresponding to elts. USED FOR TESTING ONLY.

      */

      return [PCBFRevert(EG, elts[i]) : i in [1 .. #elts]];
137 end function;


    function PCBFRevertSub(EG)

142   /*

      Arguments:

        EG: PCBF-group, Type(s): Rec

147
```

```
    Parameters:

152 Return Type(s):

       Grp

157 Description:

       Returns a subgroup of EG`E isomorphic to EG.
       USED FOR TESTING ONLY.

162 */

    return sub<EG`E | PCBFRevertSeq(EG, PCBFGenerators(EG))>;
end function;
```

Bibliography

Baumslag, Gilbert, Frank B. Cannonito, Derek J. S. Robsinson and Dan Segal (1991), 'The Algorithmic Theory of Polycyclic-by-Finite Groups', *Journal of Algebra* **142**, 118–149.

Bosma, Wieb, John Cannon and Catherine Playoust (1997), 'The Magma algebra system. I. The user language.', *J. Symbolic Comput.* **24(3-4)**, 235–265.

Chandler, Bruce and Wilhelm Magnus (1982), *The History of Combinatorial Group Theory: A Case Study in the History of Ideas*, Springer.

Dixon, J. D. (1982), 'Exact solution of linear equations using p-adic expansion', *Numerische Mathematik* **40**, 137–141.

Eick, Bettina (2001), Algorithms for Polycyclic Groups, Habilitationsschrift, Universität Kassel.

Haramoto, H. and M. Matsumoto (2009), 'A p-adic algorithm for computing the inverse of integer matrices', *Journal of Computational and Applied Mathematics* **225**, 320–322.

Holt, Derek F., Bettina Eick and Eamonn A. O'Brien (2005), *Handbook of Computational Group Theory*, Discrete Mathematics and its Applications, Chapman & Hall/CRC.

James, Gordon and Martin Liebeck (2001), *Representations and Characters of Groups*, Cambridge University Press.

Kleidman, Peter and Martin Liebeck (1990), *The Subgroup Structure of the Finite Classical Groups*, Vol. 129 of *London Math. Soc. Lecture Note Series*, Cambridge University Press.

Robsinson, Derek J. S. (1996), *A Course in the Theory of Groups*, Second edn, Springer.

Rotman, Joseph J. (1995), *An Introduction to the Theory of Groups*, Fourth edn, Springer.

Rotman, Joseph J. (2002), *Advanced Modern Algebra*, Prentice-Hall, Pearson Education.

Segal, Dan (1983), *Polycyclic Groups*, Cambridge University Press.

Seress, Ákos (2003), *Permutation Group Algorithms*, Vol. 152 of *Cambridge Tracts in Mathematics*, Cambridge University Press.

Sims, Charles C. (1970), Computational methods in the study of permutation groups, *in* J.Leech, ed., 'Computational Problems in Abstract Algebra', Pergamon Press, Oxford, pp. 169–183.

Sims, Charles C. (1971), Computation with permutation groups, *in* 'Proc. Second Symposium on Symbolic and Algebraic Manipulation', ACM Press, New York, pp. 23–28.

Sims, Charles C. (1994), *Computation with finitely presented groups*, Vol. 48 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press.