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THE DISCRETE FRAGMENTATION EQUATION: SEMIGROUPS, COMPACTNESS AND ASYNCHRONOUS EXPONENTIAL GROWTH

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(Communicated by Reinhard Illner)

ABSTRACT. In this paper we present a class of fragmentation semigroups which are compact in a scale of spaces defined in terms of finite higher moments. We use this compactness result to analyse the long time behaviour of such semigroups and, in particular, to prove that they have the asynchronous growth property. We note that, despite compactness, this growth property is not automatic as the fragmentation semigroups are not irreducible.

1. **Introduction.** The process of cluster fragmentation occurs in many areas of pure and applied science such as depolymerisation, rock fracture and droplet breakup. When it is assumed that each cluster of size $n \in \mathbb{N}$ (an n-mer) in a system of particle clusters is composed of n identical fundamental units (monomers), then the mass of each cluster is simply a positive integer multiple of the mass of the monomer. By appropriate scaling, each monomer can be assumed to have unit mass. This leads to a discrete model of the fragmentation process, in which the evolution of clusters is described by

$$\frac{d}{dt} u_n(t) = -a_n u_n(t) + \sum_{k=n+1}^{\infty} b_{n,k} \, a_k \, u_k(t), \quad t > 0, \quad u_n(0) = \overset{\circ}{u}_n, \quad n = 1, 2, 3, \ldots. \tag{1}$$

In (1), $u_n(t)$ represents the concentration of n-mers at time t, $a_n \geq 0$ is the average break-up rate of an n-mer and $b_{n,k}$ is the average number of n-mers produced upon the break-up of a k-mer. Clearly we require $b_{n,k} = 0$ for all $n \geq k$. Moreover, for

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the total mass in the system to be a conserved quantity, the coefficients a_n and $b_{n,k}$ are constrained by the conditions

$$a_1 = 0 \qquad \text{and} \qquad \sum_{n=1}^{k-1} n b_{n,k} = k, \quad (k = 2, 3, \ldots). \tag{2}$$

A simple calculation then shows that, formally,

$$\frac{d}{dt} \sum_{n=1}^{\infty} n u_n(t) = 0. \tag{3}$$

In a number of previous rigorous mathematical investigations into (1), a typical strategy has been to consider finite-dimensional truncations to which standard methods from the theory of ordinary differential equations can be applied. This yields a sequence of solutions to the truncated equations and compactness arguments then establish that there is a subsequence that converges to a solution $\mathbf{u}(t) = (u_1(t), u_2(t), u_3(t), \ldots)$ of an integral version of the fragmentation equation. This approach has been used, for example, by Laurençot in [14] for the system (1), and by Ball & Carr [2] and da Costa [9] for the case of discrete binary fragmentation. In contrast to this truncation-limit procedure, an alternative method involving the theory of semigroups of operators has also been used in several papers [5, 6, 15, 18]. This operator-based approach enables existence and uniqueness results to be established for *strongly differentiable* (strict) solutions of the abstract Cauchy problem (ACP) associated with (1).

One very recent development in the application of semigroup techniques to fragmentation and coagulation equations has been the discovery that, in certain cases, the underlying semigroup can be shown to have additional desirable properties. For example, in [6], the ACP associated with (1) is posed in certain weighted Banach spaces (moment spaces) $X_p := \ell_p^1$ of sequences $\mathbf{f} = (f_n)_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} n^p |f_n| < \infty$. Sufficient conditions on the fragmentation coefficients $b_{n,k}$ are given under which the fragmentation semigroup is analytic in X_p provided that $p > 1$. Properties of analytic semigroups are then exploited to prove the existence and uniqueness of solutions to the related coagulation-fragmentation equation under less restrictive conditions on the coagulation coefficients than the usual boundedness required in other semigroup-based investigations.

Our aim in the current paper is to continue with this theme of establishing and utilising additional properties of semigroups associated with discrete fragmentation equations. More specifically, we concentrate on the compactness of the semigroups and show that this property can then be used to obtain information on the asymptotic behaviour of solutions. Not surprisingly, solutions ultimately reach a steady state in which only monomers are present in the system. When the semigroup is also known to be compact, we prove that this decay to steady state must occur in an exponential manner for a large class of initial data. We emphasize that this is not a standard application of say, [11, Theorem 3.5] or [8, Theorem 8.17], as the fragmentation semigroup is not irreducible. For the same reason, as the steady state of the fragmentation process is not strictly positive, powerful probabilistic methods based on the theory of Harris operators and the Fougel alternative, see [16, 17], also are not directly applicable.

2. **The Fragmentation Semigroup.** It is convenient at this stage to give a brief summary of some results proved recently in [5, 6, 15, 18] using the theory of sub-stochastic semigroups [3, Chapter 6]. We begin by expressing the initial-value problem (IVP) (1) more concisely as

$$\mathbf{u}'(t) = \mathcal{F}\mathbf{u}(t), \quad \mathbf{u}(0) = \overset{\circ}{\mathbf{u}}, \tag{4}$$

where $\mathbf{u}(t) = (u_n(t))_{n=1}^\infty$ is a time-dependent sequence whose components $u_n(t)$, $n = 1, 2, \ldots$, give the number of n-mers in the system and

$$\left(\mathcal{F}\mathbf{u}(t)\right)_{n=1}^\infty := \left(-a_n u_n(t) + \sum_{i=n+1}^\infty b_{n,i}\, a_i\, u_i(t)\right)_{n=1}^\infty.$$

To enable semigroup techniques to be applied, the IVP (4) must be expressed as an ACP in some Banach space X. Since mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the weighted ℓ_1 space given by $X_1 = \ell_1^1 := \{\mathbf{f} = (f_n)_{n=1}^\infty : \|\mathbf{f}\|_1 := \sum_{n=1}^\infty n\,|f_n| < \infty\}$. However, our aim here is not only to state conditions for the existence and uniqueness of physically relevant solutions, but also to determine the asymptotic behaviour of solutions. For this we require the semigroup associated with the ACP to have additional properties, such as analyticity and compactness. Although the arguments used to prove [18, Theorems 3.4 and 3.5] can easily be adapted to establish that these properties hold within the framework of the space X_1 for the specific case when

$$a_n = n - 1, \quad b_{n,k} = 2/(k-1),$$

the problem of determining more general sufficient conditions under which the semigroup is analytic and compact on X_1 remains open. For this reason we follow the approach used in [6] and introduce the following class of Banach spaces

$$X_p = \ell_p^1 := \{\mathbf{f} = (f_n)_{n=1}^\infty : \|\mathbf{f}\|_p := \sum_{n=1}^\infty n^p\,|f_n| < \infty\}, \ p \geq 1.$$

In the sequel, when $Z \subseteq X_p$ is a given set of sequences, Z_+ will denote the subset of Z consisting of all nonnegative sequences in Z. Note that any $\mathbf{f} \in (X_p)_+$ will possess moments

$$M_r(\mathbf{f}) := \sum_{n=1}^\infty n^r f_n \tag{5}$$

of all orders $r \in [0, p]$. Although some of the results that we state, such as the existence of a strongly continuous semigroup of contractions associated with the ACP, hold in X_p for all $p \geq 1$, other important properties will require $p > 1$. For each $p \geq 1$, we define operators A_p and B_p in X_p by

$$A_p\,\mathbf{f} := (a_n f_n)_{n=1}^\infty, \quad D(A_p) := \{\mathbf{f} \in X_p : \sum_{n=1}^\infty n^p\, a_n|f_n| < \infty\}; \tag{6}$$

$$B_p\,\mathbf{f} := \left(\sum_{k=n+1}^\infty b_{n,k}\, a_k f_k\right)_{n=1}^\infty, \quad D(B_p) := D(A_p). \tag{7}$$

Throughout, we assume that the coefficients a_n and $b_{n,k}$ satisfy the mass conservation conditions in (2). A routine calculation [6, Equation (2.3)] then leads to

$$\|B_p\,\mathbf{f}\|_p \leq \|A_p\,\mathbf{f}\|_p, \quad \forall \mathbf{f} \in D(A_p),$$

showing that B_p is well defined on $D(A_p)$.

Theorem 2.1. *For each $p \geq 1$ and $k = 2, 3, \ldots$, let*

$$\Delta_k^{(p)} := k^p - \sum_{n=1}^{k-1} n^p \, b_{n,k}. \tag{8}$$

Then

: *(a) the closure $\overline{(-A_p + B_p, D(A_p))} =: (F_p, D(F_p))$ generates a positive semi-group of contractions $(S_{F_p}(t))_{t \geq 0}$ on X_p, which satisfies*

$$\frac{d}{dt} \| S_{F_p}(t) \overset{\circ}{\mathbf{u}} \|_p = -c_p \, (S_{F_p}(t) \overset{\circ}{\mathbf{u}}), \quad \overset{\circ}{\mathbf{u}} \in D(F_p)_+, \tag{9}$$

where

$$c_p \, (\mathbf{f}) := \sum_{k=2}^{\infty} a_k \Delta_k^{(p)} f_k, \quad \mathbf{f} \in D(F_p)_+; \tag{10}$$

: *(b) $F_p = -A_p + B_p$ whenever*

$$\liminf_{k \to \infty} \frac{\Delta_k^{(p)}}{k^p} > 0. \tag{11}$$

Proof. See [6, Theorem 2.1]. □

Note that $\Delta_k^{(p)} > 0$ for all $p > 1$ and $k = 2, 3, \ldots$, but $\Delta_k^{(1)} = 0$ for all k [6, Equation (2.4)] and so (11) is not satisfied when $p = 1$. However, if (11) holds for some $p_0 > 1$, then it holds for any $p \geq p_0$.

Remark 1. We note that for $p = 1$ the right hand side of (9) is zero. Thus, since the norm $\| \cdot \|_1$ of a nonnegative distribution gives the total mass of the ensemble, (9) expresses the principle of the conservation of mass.

It follows immediately from Theorem 2.1 that the ACP

$$\frac{d}{dt} \, \mathbf{u}(t) = F_p \, \mathbf{u}(t), \ t > 0; \quad \mathbf{u}(0) = \overset{\circ}{\mathbf{u}}, \tag{12}$$

has a unique, nonnegative strict solution $\mathbf{u} : [0, \infty) \to X_p$ given by $\mathbf{u}(t) = S_{F_p}(t) \overset{\circ}{\mathbf{u}}$ for each $\overset{\circ}{\mathbf{u}} \in D(F_p)_+$. Moreover, from [5, Lemma 1] and [6, Lemma 2.1], an explicit representation of the semigroup $(S_{F_p}(t))_{t \geq 0}$ on X_p is provided by the matrix function $t \to \mathcal{S}(t) = [s_{in}(t)]_{1 \leq i, n < \infty}$ where

$$s_{nn}(t) = e^{-a_n t}, \ n \geq 1, \quad s_{in}(t) = 0 \ \text{ whenever } n < i, \tag{13}$$

and

$$s_{in}(t) = a_n e^{-a_n t} \sum_{k=i}^{n-1} b_{k,n} \int_0^t s_{i,k}(\tau) e^{a_n \tau} \, d\tau, \quad i \geq 1, \ n \geq i+1. \tag{14}$$

From the definition of the spaces X_p, it is obvious that the natural embedding $J : X_p \to X_1$ defined by $J\mathbf{f} = \mathbf{f}$, $\mathbf{f} \in X_p$, is in $B(X_p, X_1)$ for each $p \geq 1$ with $\|\mathbf{f}\|_1 \leq \|\mathbf{f}\|_p \ \forall \mathbf{f} \in X_p$. Furthermore, arguing as in [15, Lemma 5.2], if we define the finite-rank operators $J_r \in B(X_p, X_1)$ by

$$J_r \mathbf{f} := \sum_{k=1}^{r} f_k, \quad \mathbf{f} \in X_p, \ r = 1, 2, \ldots,$$

then

$$\| J\mathbf{f} - J_r \mathbf{f} \|_1 \leq (r+1)^{1-p} \, \|\mathbf{f}\|_p \quad \forall \mathbf{f} \in X_p,$$

and so

$$\|J - J_r\| \le (r+1)^{1-p} \to 0 \text{ as } r \to \infty, \ \forall p > 1.$$

Thus, X_p is compactly embedded in X_1 for each $p > 1$ and, since $(F_p, D(F_p))$, $p > 1$, can be interpreted as a restriction of the operator $(F_1, D(F_1))$ in X_1, the solution of (12) will also be the unique strict solution $\mathbf{u} : [0, \infty) \to X_1$ of

$$\frac{d}{dt}\mathbf{u}(t) = F_1\,\mathbf{u}(t), \ t > 0; \quad \mathbf{u}(0) = \overset{\circ}{\mathbf{u}} \in D(F_p)_+. \tag{15}$$

3. **Convergence to Steady State.** On physical grounds we expect all solutions of the IVP (4) to decay to the equilibrium state

$$\widetilde{\mathbf{u}} := M_1(\overset{\circ}{\mathbf{u}})\mathbf{e}_1\,, \tag{16}$$

where $\mathbf{e}_1 = (1, 0, 0, \ldots)$ and $M_1(\overset{\circ}{\mathbf{u}})$, defined via (5), is the initial mass in the system. This was initially established in [7, Theorem 4.1] using a proof which is technically quite involved. An alternative proof, involving the theory of substochastic semigroups in the space X_1, can be found in [5, Section 2]. Our aim in this section is to show that the arguments used in [5] can easily be adapted to prove that the result holds in the space X_p for any $p \ge 1$.

Henceforth, we assume that

$$a_n > 0 \quad \forall\, n \ge 2. \tag{17}$$

As in [5], we examine a reduced version of (1) in which only the unknowns $u_2, u_3, \ldots$ feature. Let Y_p be the closed subspace of X_p defined by

$$Y_p := \{\mathbf{g} = (g_n)_{n=1}^{\infty} \in X_p : g_1 = 0\}. \tag{18}$$

Clearly we can decompose X_p into the direct sum

$$X_p = Y_p \oplus Z_p,$$

where $Z_p = \{\mathbf{h} = (h_n)_{n=1}^{\infty} \in X_p : h_n = 0 \ \forall\, n \ge 2\}$, and therefore we can write

$$\mathbf{f} = Q_p\,\mathbf{f} + (I - Q_p)\mathbf{f} \quad \forall \mathbf{f} \in X_p,$$

where Q_p is the projection from X_p onto Y_p defined by

$$Q_p\,\mathbf{f} := (0, f_2, f_3, \ldots), \ \mathbf{f} = (f_n)_{n=1}^{\infty} \in X_p. \tag{19}$$

Let $\mathcal{A}_p$ denote the restriction of A_p to $D(\mathcal{A}_p) := D(A_p) \cap Y_p$ and define $\mathcal{B}_p$ on $D(\mathcal{A}_p)$ by

$$(\mathcal{B}_p\,\mathbf{g})_n := \begin{cases} 0 & \text{for} \quad n = 1, \\ \sum\limits_{k=n+1}^{\infty} b_{n,k}\, a_k g_k & \text{for} \quad n \ge 2. \end{cases}$$

By definition, the ranges of $\mathcal{A}_p$ and $\mathcal{B}_p$ are both contained within Y_p. To show that the closure $\overline{(-\mathcal{A}_p + \mathcal{B}_p, D(\mathcal{A}_p))} =: (\mathcal{F}_p, D(\mathcal{F}_p))$ generates a substochastic semigroup $(S_{\mathcal{F}_p}t)_{t \ge 0}$ on Y_p, we adapt the Arlotti extension based arguments [3, Chapter 6] which were used for the case $p = 1$ in [5, Section 2]. First we note that, for

$\mathbf{g} \in D(\mathcal{A}_p)_+$, by changing the order of summation,

$$\sum_{n=1}^{\infty} n^p \left((-\mathcal{A}_p + \mathcal{B}_p) \, \mathbf{g} \right)_n$$

$$= -\sum_{n=2}^{\infty} n^p \, a_n g_n + \sum_{k=3}^{\infty} \left(\sum_{n=1}^{k-1} n^p \, b_{n,k} \right) a_k g_k - \sum_{k=3}^{\infty} b_{1,k} \, a_k g_k$$

$$= \quad -\sum_{k=2}^{\infty} a_k \Delta_k^{(p)} g_k - \sum_{k=2}^{\infty} b_{1,k} \, a_k g_k$$

$$= \quad - c_p(\mathbf{g}) - (\mathcal{B}_p \mathbf{g})_1 =: -\mathcal{C}_p(\mathbf{g}).$$

Moreover, using $a_1 = 0$ and again changing the order of summation,

$$\sum_{n=1}^{N} \left(-n^p \, a_n g_n + \sum_{k=n+1}^{\infty} n^p \, b_{n,k} a_k g_k \right)$$

$$= \quad -\sum_{n=2}^{N} n^p \, a_n g_n + \sum_{k=3}^{N+1} \left(a_k g_k \sum_{n=2}^{k-1} n^p \, b_{n,k} \right) + \sum_{n=2}^{N} \left(\sum_{k=N+2}^{\infty} n^p \, b_{n,k} \, a_k g_k \right)$$

$$= \quad -\sum_{n=2}^{N} a_n \Delta_n^{(p)} g_n - \sum_{n=2}^{N} b_{1,n} \, a_n g_n + a_{N+1} f_{N+1} \sum_{n=2}^{N} n^p \, b_{n,N+1}$$

$$+ \sum_{n=2}^{N} \left(\sum_{k=N+2}^{\infty} n^p \, b_{n,k} \, a_k g_k \right).$$

It follows that, for sequences $\mathbf{g}$ such that $-\mathcal{A}_p \mathbf{g} + \mathcal{B}_p \mathbf{g} \in Y_p$ and $-\mathcal{C}_p(\mathbf{g})$ exists,

$$\lim_{N \to \infty} \sum_{n=2}^{N} \left(-n^p a_n g_n + \sum_{k=n+1}^{\infty} n^p \, b_{n,k} \, a_k g_k \right) \geq -\mathcal{C}_p(\mathbf{g}),$$

and therefore, from [3, Theorem 6.22] (as used in [3, Section 7.4]), it follows that the operator $\mathcal{F}_p = \overline{(-\mathcal{A}_p + \mathcal{B}_p, D(\mathcal{A}_p))}$ has the desired property of being the infinitesimal generator of a substochastic semigroup $(S_{\mathcal{F}_p} t))_{t \geq 0}$ on Y_p. Furthermore,

$$\frac{d}{dt} M_p \left(S_{\mathcal{F}_p}(t) \overset{\circ}{\mathbf{v}} \right) = -\mathcal{C}_p \left(S_{\mathcal{F}_p}(t) \overset{\circ}{\mathbf{v}} \right), \quad \text{for } \overset{\circ}{\mathbf{v}} \in D(\mathcal{F}_p)_+. \tag{20}$$

By integrating each side of (20) and then using density arguments, we obtain

$$M_p \left(S_{\mathcal{F}_p}(t) \overset{\circ}{\mathbf{v}} \right) = M_p \left(S_{\mathcal{F}_p}(0) \overset{\circ}{\mathbf{v}} \right) - \int_0^t \mathcal{C}_p \left(S_{\mathcal{F}_p}(s) \overset{\circ}{\mathbf{v}} \right) ds, \quad \forall \overset{\circ}{\mathbf{v}} \in (Y_p)_+. \tag{21}$$

For each fixed integer $N \geq 2$ and sequence $\mathbf{g} \in Y_p$, let $\mathcal{P}_N \mathbf{g}$ be defined by

$$(\mathcal{P}_N \mathbf{g})_{n=1}^{\infty} := \begin{cases} g_n & \text{for} \quad n \leq N, \\ 0 & \text{for} \quad n > N. \end{cases} \tag{22}$$

Clearly, $\mathcal{P}_N$ is a projection from Y_p onto the finite-dimensional closed subspace $\mathcal{P}_N(Y_p)$ of Y_p for each $p \geq 1$, and $\mathcal{P}_N \mathbf{g} \to \mathbf{g}$ in Y_p as $N \to \infty$ for any $\mathbf{g} \in Y_p$. Moreover, we can identify $\mathcal{P}_N(Y_p)$ with the space $Y_{p,N}$ of finite sequences $(g_n^N)_{n=2}^{N}$ equipped with the norm induced from Y_p. Similarly, we can identify the bounded

operators $\mathcal{A}_p \mathcal{P}_N$ and $\mathcal{B}_p \mathcal{P}_N$ with operators $\mathcal{A}_{p,N}$ and $\mathcal{B}_{p,N}$ on $Y_{p,N}$, and consider the finite-dimensional system of equations

$$\frac{d}{dt} v_{N,k}(t) = -\mathcal{A}_{p,N} v_{N,k}(t) + \mathcal{B}_{p,N} v_{N,k}(t), \ t > 0, \ v_{N,k}(0) = \overset{\circ}{v}_{N,k}, \ k = 2, 3, \dots N. \tag{23}$$

When $(\overset{\circ}{v}_{N,k})_{k=2}^N = (\delta_{k,n})_{k=2}^N$, where $\delta_{k,n}$ is the Kronecker delta, the solution of (23) is given by

$$\mathbf{v}_{N,n}(t) = (s_{2n}(t), \dots, s_{nn}(t), 0, \dots, 0),$$

where the components $s_{in}(t)$ are defined by (13) and (14). Hence, by linearity, the solution of (23) for a general $\overset{\circ}{\mathbf{v}}_N \in Y_{p,N}$ is

$$\mathbf{v}_N(t) = \mathcal{S}_N(t) \overset{\circ}{\mathbf{v}}_N, \quad \mathcal{S}_N(t) = [s_{in}(t)]_{2 \leq i, n \leq N}. \tag{24}$$

By uniqueness of solutions,

$$S_{\mathcal{F}_p}(t)(\mathcal{P}_N \mathbf{g}) = \widetilde{\mathcal{S}}_N(t)(\mathcal{P}_N \mathbf{g}), \quad \forall N \geq 2, \ \mathbf{g} \in Y_p$$

where, on the right-hand side, $\mathbf{g}$ is regarded as an infinite column vector and $\widetilde{S}_N(t)$ is the infinite matrix

$$\begin{pmatrix} s_{22}(t) & s_{23}(t) & \dots & s_{2N}(t) & 0 & \dots, \\ 0 & s_{33}(t) & \dots & s_{3N}(t) & 0 & \dots, \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & S_{NN}(t) & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

This leads to an explicit representation of the semigroup $(S_{\mathcal{F}_p}(t))_{t \geq 0}$ on Y_p being provided by the matrix function $t \to \widetilde{\mathcal{S}}(t) = [s_{in}(t)]_{2 \leq i, n < \infty}$. Note also that

$$\widetilde{\mathcal{S}}_N(t)(\mathcal{P}_N \mathbf{g}) \to S_{\mathcal{F}_p}(t)\mathbf{g} \ \text{in} \ Y_p \ \text{as} \ N \to \infty, \quad \forall \mathbf{g} \in (Y_p)_+, \tag{25}$$

where the convergence is monotonic in N for any t, and uniform in t on bounded time intervals.

Lemma 3.1. *Let $\alpha \geq 0$ and $\mathbf{f} \in (X_p)_+$ be fixed and let $Q_p \mathbf{f} \in Y_p$ be defined via (19). Then*

$$\lim_{t \to \infty} e^{\alpha t} \|S_{F_p}(t)\mathbf{f} - M_1(\mathbf{f})\mathbf{e}_1\|_p = 0 \Leftrightarrow \lim_{t \to \infty} e^{\alpha t} \|S_{\mathcal{F}_p}(t)Q_p \mathbf{f}\|_p = 0.$$

Proof. The proof relies on the simple observation that

$$S_{F_p}(t)\mathbf{f} = \mu_{\mathbf{f}}(t)\mathbf{e}_1 + S_{\mathcal{F}_p}(t)Q_p \mathbf{f}, \tag{26}$$

where

$$\mu_{\mathbf{f}}(t) = \sum_{k=1}^{\infty} s_{1k}(t) f_k.$$

It follows from (26) that

$$\|S_{F_p}(t)\mathbf{f} - M_1(\mathbf{f})\mathbf{e}_1\|_p = |\mu_{\mathbf{f}}(t) - M_1(\mathbf{f})| + \|S_{\mathcal{F}_p}(t)Q_p\mathbf{f}\|_p, \tag{27}$$

and therefore an exponential decay to 0 of the left-hand side implies the same for each term on the right-hand side and, in particular, the decay of $\|S_{\mathcal{F}_p}(t)Q_p \mathbf{f}\|_p$ to 0 at the same rate.

To prove the result in the opposite direction, we again use (26), together with the hypothesis, to obtain

$$\|e^{\alpha t}S_{F_p}(t)\mathbf{f} - e^{\alpha t}\mu_{\mathbf{f}}(t)\mathbf{e}_1\|_p = e^{\alpha t}\|S_{\mathcal{F}_p}(t)Q_p\mathbf{f}\|_p \to 0 \text{ as } t \to \infty.$$

Now, by $\|\cdot\|_p \geq \|\cdot\|_1$ and the reverse triangle inequality

$$
\begin{aligned}
\left\|e^{\alpha t}S_{F_p}(t)\mathbf{f} - e^{\alpha t}\mu_{\mathbf{f}}(t)\mathbf{e}_1\right\|_p &\geq \left\|e^{\alpha t}S_{F_p}(t)\mathbf{f} - e^{\alpha t}\mu_{\mathbf{f}}(t)\mathbf{e}_1\right\|_1 \\
&\geq \left|\|e^{\alpha t}S_{F_p}(t)\mathbf{f}\|_1 - \|e^{\alpha t}\mu_{\mathbf{f}}(t)\mathbf{e}_1\|_1\right| \\
&= e^{\alpha t}|M_1(\mathbf{f}) - \mu_{\mathbf{f}}(t)|.
\end{aligned}
$$

Consequently,

$$e^{\alpha t}|M_1(\mathbf{f}) - \mu_{\mathbf{f}}(t)| \to 0 \text{ as } t \to \infty,$$

and the result follows from (27). $\qquad\square$

We can now prove the main result of this section.

Theorem 3.2. *For any $p \geq 1$ we have*

$$\lim_{t \to \infty} \|S_{F_p}(t)\overset{\circ}{\mathbf{u}} - M_1(\overset{\circ}{\mathbf{u}})\mathbf{e}_1\|_p = 0$$

for any $\overset{\circ}{\mathbf{u}} \in (X_p)_+$ if and only if $a_n \neq 0$ for all $n \geq 2$.

Proof. Let $\overset{\circ}{\mathbf{u}} \in (X_p)_+$ and consider $Q_p\overset{\circ}{\mathbf{u}} \in (Y_p)_+$. From (24), the solution of the truncated system (23), with initial conditions $\overset{\circ}{v}_k = (\mathcal{P}_N Q_p \overset{\circ}{\mathbf{u}})_k$, $k = 2, 3, \ldots, N$, may be expressed in terms of an $(N-1) \times (N-1)$ matrix, $\mathcal{S}_N(t)$, whose only eigenvalues are exponentials with negative exponents. Consequently, if, for ease of notation, we denote $\widetilde{\mathcal{S}}_N(t)(\mathcal{P}_N Q_p \overset{\circ}{\mathbf{u}})$ and $S_{\mathcal{F}_p}(t)(Q_p\overset{\circ}{\mathbf{u}})$ by $\mathbf{v}_N(t)$ and $\mathbf{v}(t)$ respectively, and define

$$M_{p,N}(t) := \sum_{k=1}^{N} k^p v_{k,N}(t) = M_p(\mathbf{v}_N(t)),$$

then

$$\lim_{t \to \infty} M_{p,N}(t) = 0.$$

On the other hand, from (21),

$$M_{p,N}(t) = M_{p,N}(0) - \int_0^t \mathcal{C}_p(\mathbf{v}_N(s))\, ds, \qquad (28)$$

and therefore

$$\lim_{t \to \infty} \int_0^t \mathcal{C}_p(\mathbf{v}_N(s))\, ds = M_{p,N}(0).$$

Since $\mathbf{v}_N(t)$ increases monotonically to $\mathbf{v}(t)$ and $\mathcal{C}_p$ is nonnegative, we obtain from (21),

$$M_p(\mathbf{v}(t)) = M_p(\mathbf{v}(0)) - \int_0^t \mathcal{C}_p(\mathbf{v}(s))\, ds \leq M_p(\mathbf{v}(0)) - \int_0^t \mathcal{C}_p(\mathbf{v}_N(s))\, ds.$$

Now, for any $\varepsilon > 0$, we can find N such that $|M_p(\mathbf{v}(0) - M_{p,N}(0)| \leq \varepsilon$. Hence

$$0 \leq \varlimsup_{t \to \infty} M_p(\mathbf{v}(t)) \leq \varepsilon + \lim_{t \to \infty}\left|M_{p,N}(0) - \int_0^t \mathcal{C}_p(\mathbf{v}_N(s))\, ds\right| = \varepsilon.$$

Since ε is arbitrary, $\lim_{t \to \infty} M_p(\mathbf{v}(t)) = 0$, and, on applying Lemma 3.1 with $\alpha = 0$, we deduce that

$$\lim_{t \to \infty} \|S_{F_p}(t)\overset{\circ}{\mathbf{u}} - M_1(\overset{\circ}{\mathbf{u}})\mathbf{e}_1\| = 0.$$

Conversely, if $a_N = 0$ for some $N \geq 2$ and we take $\mathring{\mathbf{u}} = (\delta_{Nj})_{j=1}^{\infty}$, then the solution $\mathbf{u}(t)$ will have $u_N(t) = 1$ for all t, showing that convergence to $\mathbf{e}_1$ cannot occur. $\quad\square$
Our aim in the next section is to show that the decay to equilibrium is exponentially fast for a large class of fragmentation models.

4. **Compactness of the Fragmentation Semigroup and AEG.** We know from Section 2 that F_p is the generator of a semigroup of contractions on X_p and therefore the resolvent $R(\lambda, F_p)$ exists as a bounded operator on X_p for each $\lambda > 0$. If we interpret $\mathbf{f} \in X_p$ as an infinite column vector, then it can be shown [6, Lemma 2.2] that $R(\lambda, F_p)$ is the realization on X_p of a matrix function

$$\mathcal{R}(\lambda, \mathcal{F}) = [r_{i,n}(\lambda)]_{1 \leq i, n < \infty}, \tag{29}$$

where

$$r_{n,n}(\lambda) = 1/(\lambda + a_n) \text{ for } n \geq 1, \quad r_{i,n}(\lambda) = 0 \text{ for } n < i, \tag{30}$$

and

$$r_{i,n}(\lambda) = \frac{a_n}{\lambda + a_n} \sum_{k=i}^{n-1} r_{i,k}(\lambda) b_{k,n}, \quad n \geq i+1, \ i \in \mathbb{N}. \tag{31}$$

Moreover if there exist a positive sequence $(\phi_k)_{k=1}^{\infty}$ and a positive constant C such that

$$\phi_k b_{k,n} \leq C \sum_{i=1}^{k} i b_{i,n}, \ 1 \leq k \leq n-1, \text{ for any } n \geq n_0, \tag{32}$$

where $n_0 \geq 2$ is some fixed positive integer, then it can be shown [6, Lemma 3.1] that

$$|r_{k,n}(\lambda)| \leq \frac{Cn}{|\lambda + a_k| \, \phi_k} \quad \forall n > k. \tag{33}$$

This matrix representation of the resolvent, together with the estimates (33), can be exploited to establish the following analyticity and compactness results.

Theorem 4.1. *Let the coefficients $b_{k,n}$ satisfy (11) for some $p_0 > 1$. Then $(S_{F_p}(t))_{t \geq 0}$ is an analytic semigroup on X_p for each $p > p_0$.*

Proof. Originally, the result was proved in [6, Theorem 3.1] under the additional assumption that the coefficients $b_{k,n}$ satisfy (32) for some sequence $(\phi_k)_{k=1}^{\infty}$ with a growth rate of at least k^2 as $k \to \infty$. However, it is known that $-A_p$ is resolvent positive and generates an analytic semigroup. Also, B_p is positive on $D(A_p)$ and, under assumption (11), $-A_p + B_p$ is resolvent positive. Therefore the analyticity of $(S_{F_p}(t))_{t \geq 0}$ can be obtained directly from [1, Theorem 1.1]. $\quad\square$

Theorem 4.2. *Let $a_k \to \infty$ and let $(\phi_k)_{k=1}^{\infty}$ be a positive sequence such that (32) is satisfied and*

$$\left(\frac{1}{a_k \phi_k} \right)_{k=2}^{\infty} \in X_1.$$

Then $R(\lambda, F_p)$ is a compact operator on X_p for any $\lambda > 0$ and $p \geq 1$.

Proof. For each fixed $n \in \mathbb{N}$, let P_n be defined on X_p by

$$P_n \mathbf{f} = (f_1, f_2, \ldots, f_n, 0, \ldots) \tag{34}$$

and let $\mathcal{R}_n(\lambda) = P_n R(\lambda, F_p)$. Then

$$(R(\lambda, F_p)\mathbf{f} - \mathcal{R}_n(\lambda)\mathbf{f})_i = \begin{cases} 0 & \text{for} \quad 1 \leq i \leq n, \\ \displaystyle\sum_{k=i}^{\infty} r_{i,k}(\lambda) f_k & \text{for} \quad i \geq n+1. \end{cases}$$

Taking $\lambda > 0$, and noting that, from (33),

$$r_{i,k}(\lambda) \leq \frac{Ck}{(\lambda + a_i)\phi_i} \leq \frac{Ck}{a_i\phi_i}, \quad k > i,$$

we obtain

$$
\begin{aligned}
\|R(\lambda, F_p)\mathbf{f} - \mathcal{R}_n(\lambda)\mathbf{f}\|_p &\leq \sum_{i=n+1}^{\infty} i^p \, r_{i,i}(\lambda)|f_i| + \sum_{i=n+1}^{\infty} i^p \left(\sum_{k=i+1}^{\infty} r_{i,k}(\lambda)|f_k| \right) \\
&\leq (\sup_{i\geq n+1} a_i^{-1})\|\mathbf{f}\|_p + \sum_{k=n+2}^{\infty} k^p |f_k| \left(\frac{1}{k^p} \sum_{i=n+1}^{k-1} i^p \, r_{i,k}(\lambda) \right) \\
&\leq (\sup_{i\geq n+1} a_i^{-1})\|\mathbf{f}\|_p + C \sum_{k=n+2}^{\infty} k^p |f_k| \left(\frac{1}{k^{p-1}} \sum_{i=n+1}^{k-1} \frac{i^p}{a_i\phi_i} \right) \\
&\leq \omega_n\|\mathbf{f}\|_p,
\end{aligned}
$$

where

$$\omega_n := \left(\sup_{i\geq n+1} a_i^{-1} + C \sum_{i=n+1}^{\infty} \frac{i}{a_i\phi_i} \right).$$

Hence, if $a_i \to \infty$ as $i \to \infty$ and $(i(a_i\phi_i)^{-1})_{i\geq 2}$ is summable, then $\omega_n \to 0$ as $n \to \infty$ and we see that $\mathcal{R}_n(\lambda)$ converges to $R(\lambda, F_p)$ in the uniform operator topology as $n \to \infty$. As each $\mathcal{R}_n(\lambda)$ is a finite rank operator, it follows that $R(\lambda, F_p)$ is compact. $\qquad \square$

We note that if both Theorems 4.1 and 4.2 can be applied with some $p_0 > 1$, then for $p \geq p_0$, $(S_{F_p}(t))_{t\geq 0}$ is an analytic semigroup on X_p, whose generator F_p has a compact resolvent $R(\lambda, F_p)$ for all $\lambda > 0$. As any analytic semigroup is immediately norm continuous, it follows from [10, Theorem II.4.29] that $S_{F_p}(t)$ is compact on X_p for each $t > 0$; i.e. $(S_{F_p}(t))_{t\geq 0}$ is immediately compact on X_p. A discussion of specific classes of coefficients a_n and $b_{k,n}$ satisfying the assumptions of both theorems is given in Remark 2.

The relevance of the previous result to determining the asymptotic behaviour of solutions of the discrete fragmentation equation will become apparent after the following theorem.

Theorem 4.3. *If the fragmentation semigroup $(S_{F_p}(t))_{t\geq 0}$ is immediately compact on X_p, then it has the asynchronous exponential growth property (AEG) on X_p; that is, there exists $\alpha > 0$ such that for any $\overset{\circ}{\mathbf{u}} \in X_p$*

$$\|S_{F_p}(t)\overset{\circ}{\mathbf{u}} - M_1(\overset{\circ}{\mathbf{u}})\mathbf{e}_1\| \leq Ke^{-\alpha t}, \tag{35}$$

for some $K > 0$.

Proof. The assumption that $(S_{F_p}(t))_{t\geq 0}$ is immediately compact implies that F_p has compact resolvent [10, Theorem II.4.29]. Hence, from [10, Corollary V3.2], $\sigma(F_p)$ is at most countable and consists only of poles of $R(\cdot, F_p)$ of finite algebraic multiplicity. Since $F_p \mathbf{e}_1 = -A_p \mathbf{e}_1 + B_p \mathbf{e}_1 = 0 \mathbf{e}_1$, we deduce that 0 is an isolated eigenvalue of F_p of finite algebraic multiplicity (i.e. a pole of $R(\cdot, F_p)$). Moreover, $(S_{F_p}(t))_{t\geq 0}$ is a positive semigroup with essential growth bound $\omega_{\text{ess}}(F_p) = -\infty$ and so it follows from [4, Theorems 48 and 49] that the peripheral spectrum is finite and additively cyclic and so consists of the single point $s(F_p)$, where $s(F_p)$ denotes the spectral bound of F_p. In this case, the spectral bound coincides with the growth

bound $\omega_0(F_p)$ of the semigroup and so the peripheral spectrum consists only of 0, showing that $\lambda_1 = 0$ is the dominant eigenvalue of F_p; i.e. $\operatorname{Re}\lambda_n < 0$ for all other eigenvalues, $\lambda_n, n = 2, 3, \ldots,$ of F_p. By [10, Corollary V.3.2], we can write

$$S_{F_p}(t) = S_{F_p}^{(1)}(t) + R^{(1)}(t)$$

where, for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\|R^{(1)}(t)\| \leq M_\varepsilon e^{(\varepsilon + \operatorname{Re}\lambda_2)t}, \quad t \geq 0, \tag{36}$$

λ_2 being the eigenvalue which, after $\lambda_1 = 0$, has the next largest real part. The operator $S_{F_p}^{(1)}(t)$ has finite rank, and is given by

$$S_{F_p}^{(1)}(t) = \left(e^{\lambda_1 t} \sum_{j=0}^{k_1-1} \frac{t^j}{j!} \right) P^{(1)} = \left(\sum_{j=0}^{k_1-1} \frac{t^j}{j!} \right) P^{(1)}, \tag{37}$$

where k_1 is the order of the pole $\lambda_1 = 0$ and $P^{(1)}$, the corresponding residue, is the spectral projection onto a finite-dimensional subspace of X_p whose dimension is given by the algebraic multiplicity, m_a, of the eigenvalue 0. We shall prove that $k_1 = m_a = 1$.

By choosing ε sufficiently small, we can write (36) as

$$\|R^{(1)}(t)\| \leq M_\alpha e^{-\alpha t} \quad t \geq 0, \tag{38}$$

where $\alpha > 0$. Now $(S_{F_p}^{(1)}(t))_{t \geq 0}$ is a finite-dimensional semigroup generated by an operator which has only 0 as a spectral value. Consequently, both $(S_{F_p}^{(1)}(t))_{t \geq 0}$ and $(S_{F_p}(t))_{t \geq 0}$ are bounded and therefore we must have $k_1 = 1$, as otherwise an initial condition $\overset{\circ}{\mathbf{u}}$ can be found leading to a solution $S_{F_p}(t)\overset{\circ}{\mathbf{u}}$ with polynomial growth. To establish that $m_a = 1$, we examine the adjoint operator F_p^*. For each $p \geq 1$, we can, under identification, regard the dual space of X_p as

$$X_p^* = \{\mathbf{f}^* : \|\mathbf{f}^*\|_{X_p^*} := \sup_{k \geq 1} k^{-p} |f_k^*| < \infty\},$$

in which case the action of $f^* \in X_p^*$ on $f \in X_p$ is given by

$$\langle f^*, f \rangle := \sum_{i=1}^{\infty} f_i^* f_i.$$

For $\mathbf{f} \in X_p$ and suitably restricted $\mathbf{f}_p^* \in X_p^*$, routine calculations show that, for each fixed $N \in \mathbb{N}$,

$$\langle \mathbf{f}^*, F_p P_N \mathbf{f} \rangle = \langle \mathbf{f}^*, (A_p + B_p) P_N \mathbf{f} \rangle = \sum_{j=2}^{N} f_j a_j \left(-f_j^* + \sum_{i=1}^{j-1} b_{i,j} f_i^* \right), \tag{39}$$

where P_N is the projection operator on X_p defined by (34). Motivated by this, we consider the operator $\mathcal{F}_p^*$ defined by

$$(\mathcal{F}_p^* f^*)_1 := 0, \quad (\mathcal{F}_p^* f^*)_j := a_j \left(-f_j^* + \sum_{i=1}^{j-1} b_{i,j} f_i^* \right), \; j = 2, 3, \ldots, \tag{40}$$

with domain

$$D(\mathcal{F}_p^*) := \left\{ \mathbf{f}^* \in X_p^* : \sup_{j \geq 2} j^{-p} a_j \left| \left(-f_j^* + \sum_{i=1}^{j-1} b_{i,j} f_i^* \right) \right| < \infty \right\}. \tag{41}$$

In terms of $\mathcal{F}_p^*$, we have

$$\langle \mathbf{f}^*, F_p\, P_N \mathbf{f} \rangle = \langle \mathcal{F}_p^*\, \mathbf{f}^*, P_N \mathbf{f} \rangle, \ \forall \mathbf{f}^* \in D(\mathcal{F}_p^*), \ N = 1, 2, \ldots \tag{42}$$

If we consider initially the case when $\mathbf{f} \in D(A_p)$, then $P_N \mathbf{f} \to \mathbf{f}$ and $A_p\, P_N \mathbf{f} \to A_p\, \mathbf{f}$ in X_p as $N \to \infty$. Moreover, since $\|B_p\, \mathbf{f}\|_p \le \|A_p\, \mathbf{f}\|_p$ for all $\mathbf{f} \in D(A_p)$, it follows from (42) that

$$\langle \mathbf{f}^*, (A_p + B_p)\mathbf{f} \rangle = \langle \mathcal{F}_p^*\, \mathbf{f}^*, \mathbf{f} \rangle, \ \forall \mathbf{f} \in D(A_p), \ \mathbf{f}^* \in D(\mathcal{F}_p^*),$$

and, since F_p is the closure of $(A_p + B_p, D(A_p))$, we obtain

$$\langle \mathbf{f}^*, F_p\, \mathbf{f} \rangle = \langle \mathcal{F}_p^*\, \mathbf{f}^*, \mathbf{f} \rangle, \ \forall \mathbf{f} \in D(F_p), \ \mathbf{f}^* \in D(\mathcal{F}_p^*).$$

This shows that $\mathcal{F}_p^* \subseteq F_p^*$. To establish the reverse inclusion, we use the fact that, if $\mathbf{f}^* \in D(F_p^*)$, then, on replacing $P_N \mathbf{f}$ by $\mathbf{e}_N$, (39) becomes

$$(F_p^*\, \mathbf{f}^*)_N$$

$$= \langle F_p^*\, \mathbf{f}^*, \mathbf{e}_N \rangle = \langle \mathbf{f}^*, (A_p + B_p)\mathbf{e}_N \rangle = a_N \left(-f_N^* + \sum_{i=1}^{N-1} b_{i,N} f_i^* \right), \ N = 2, 3, \ldots$$

Since

$$|\langle F_p^*\, \mathbf{f}^*, \mathbf{e}_N \rangle| \le \|F_p^*\, \mathbf{f}^*\|_{X_p^*} \|\mathbf{e}_N\|_p = \|F_p^*\, \mathbf{f}^*\|_{X_p^*} N^p,$$

we deduce that $\mathbf{f}^* \in D(\mathcal{F}_p)$.

Now suppose that $\mathbf{e}^*$ is an eigenvector of F_p^* corresponding to the eigenvalue 0 and, without loss of generality, set $e_1^* = 1$. Since $a_j > 0$ for all $j \ge 2$, we must have

$$e_j^* = \sum_{i=1}^{j-1} b_{i,j} e_i^*, \quad j = 2, 3, \ldots$$

Hence

$$e_2^* = b_{1,2} e_1^* = b_{1,2} = 2, \quad e_3^* = b_{1,3} e_1^* + b_{2,3} e_2^* = b_{1,3} + 2b_{2,3} = 3,$$

where we have used the mass conservation condition (2). An inductive argument leads to $e_n^* = n$ for all $n \in \mathbb{N}$ and therefore the geometric multiplicity of $\lambda_1 = 0$ for F_p^* is at most 1. Hence, by [13, Remark 6.23], $\lambda_1 = 0$ is a simple dominating eigenvalue of F_p with corresponding eigenvector $\mathbf{e}_1$. The one-dimensional projection operator $P^{(1)}$ will therefore take the form

$$P^{(1)}\mathbf{f} = C_{\mathbf{f}} \mathbf{e}_1,$$

where $C_{\mathbf{f}}$ is a constant which depends on $\mathbf{f}$. However, $P^{(1)}$ is also given by

$$P^{(1)}\mathbf{f} = \frac{1}{2\pi i} \int_{\Gamma_1} R(\lambda, F_p)\mathbf{f}\, d\lambda$$

where Γ_1 is a circle, centred at 0, with sufficiently small radius. By invariance, $P^{(1)}(P_N \mathbf{f})$ is the spectral projection for the corresponding truncated N-dimensional problem, and by standard linear algebra [12, pp.42–43], this leads to

$$P^{(1)}(P_N \mathbf{f}) = \left(\sum_{k=1}^{N} k f_k \right) \mathbf{e}_1.$$

On letting $N \to \infty$, we obtain

$$S_{F_p}^{(1)}(t)\mathbf{f} = P^{(1)}\mathbf{f} = \left(\sum_{k=1}^{\infty} k f_k\right) \mathbf{e}_1.$$

$\square$

Remark 2. An extensive discussion of coefficients $b_{k,n}$ satisfying (11) and (32) is given in the Appendix of [6]. Here, for the sake of completeness , we summarize the main results. First we note that (32) is trivially satisfied for $\phi_k = k$ for $k = 1, 2, \dots$. In many models of fragmentation processes, it is often assumed that a fragmenting parent particle will always split into exactly two daughter particles; for example, see [2, 9]. In such binary processes, fragmentation can be characterized by a symmetric infinite matrix $(\psi_{i,j})_{i,j\geq 1}$, where, in our notation,

$$b_{j,n} = \frac{\psi_{j,n-j}}{a_n}, \quad a_n = \frac{1}{2}\sum_{j=1}^{n-1} \psi_{j,n-j}, \qquad n \geq 2, 1 \leq j \leq n - 1. \tag{43}$$

In the study of degradation of polymers common forms for $\psi_{i,j}$ are

$$\psi_{i,j} = (i + j)^{\beta} \quad \text{or} \quad \psi_{i,j} = (ij)^{\beta}, \qquad \beta > -1. \tag{44}$$

It follows, [6, Proposition 6.2], that if the coefficients a_n and $b_{k,n}$ are given by (43), where $\psi_{i,j}$ takes either of the forms in (44), then (11) and (32), with $\phi_k = k^2$, are satisfied. Consequently, in both cases identified in (44), the corresponding binary fragmentation semigroups will have the AEG property provided

$$a_n \sim n^{\delta} \text{ for some } \delta > 0 \text{ and large } n. \tag{45}$$

In the more general multiple fragmentation case, it is physically realistic to assume that the coefficients $b_{k,n}$ are non-increasing with respect to k for any n. This assumption also leads, [6, Proposition 6.1], to (11) and (32) being satisfied with $\phi_k = k^2$ and so once again the fragmentation semigroup will have the AEG property whenever (45) is satisfied.

On the other hand, let us consider a binary fragmentation process defined by

$$b_{1,2} = 2, \quad \text{and} \quad b_{1,n} = b_{n-1,n} = 1, \quad b_{i,n} = 0, \quad n \geq 2, 2 \leq i \leq n - 2. \tag{46}$$

Then $\Delta_n^{(p)} = n^p - (1 + (n-1)^p) = o(n^p)$ and so (11) is not satisfied. Indeed, [6, Proposition 6.3], for the fragmentation rates defined by $a_1 = 0$ and $a_n = n$ for $n \geq 2$, we have $(F_p, D(F_p)) = \overline{(-A_p + B_p, D(A_p))} \neq (-A_p + B_p, D(A_p))$ and $\{S_{F_p}(t)\}_{t\geq 0}$ is not analytic.

From the physical point of view, it seems that a fragmentation process generates an analytic and compact semigroup if the distribution of daughter particles is uniform or shifted towards smaller particles so that there is no domination of large particles as in the last example. Indeed, one can then think that the former are close to finite dimensional, and thus regular, fragmentation processes.

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