Amalgamation Classes of Directed Graphs in Model Theory and Infinite Permutation Groups

A thesis submitted to the School of Mathematics of the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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April 2012

Abstract

In this thesis we investigate several classes of directed graphs which have an amalgamation property.

The first class we look at is a variation on a class introduced by David Evans to answer a question of Peter M. Neumann. We show that there are continuum many primitive permutation groups of countable degree which have a finite suborbit paired with a suborbit of size $\aleph_0$. The results here indicate that there is no possibility of classifying the highly arc transitive primitive digraphs with a given isomorphism type of descendant set.

We then look at the model theoretic properties of stability, independence and two tree properties for the theory of a Fraïssé-type limit of one of the classes. We show that this limit is unstable, having the strict order property, the independence property, the tree property and the tree property of the second kind.

We next look at a class of undirected graphs obtained from a Hrushovski construction using a predimension and see that this can be viewed more naturally as the family of undirected reducts of a class of directed graphs. We then restrict this directed class by limiting the number of primitive extensions any given set can have and obtain an amalgamation lemma for the class. This directed version corresponds to imposing a bound on the multiplicity of minimally simply algebraic extensions from Hrushovski's construction of a strongly minimal set. We axiomatize the theory of the Fraïssé-type limit and show that it is stable and trivial. The reduct of this obtained by forgetting the direction on the edges is then considered and we finally look at stability in this setting, showing that, in contrast to the unrestricted case, the undirected reduct is strictly stable.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

In this thesis we investigate several classes of directed graphs (or digraphs) which have an amalgamation property. We start by considering a class of digraphs originally studied to answer a question posed by Peter M. Neumann in [20]. He asked whether n_1, n_2 could arise as the subdegrees of a primitive permutation group for $2 \leq n_1 < \aleph_0 \leq n_2$. This class of digraphs was first used in [13] to show that there is a primitive permutation group which has a finite suborbit paired with a suborbit of size κ, for every infinite cardinal κ (Corollary 2.10, [13]). Neumann then posed the question as to whether there were uncountably many of these permutation groups of countable degree. This question has been answered here as Theorem 2.0.2 using unbalanced digraphs. The proof uses a similar class of digraphs as in [13] and was suggested by David M. Evans.

The digraphs that we construct are highly arc transitive, that is the automorphism groups are transitive on the set of n-arcs for all finite n. In our examples, the descendant set of a vertex is a directed binary tree. Primitive highly arc transitive digraphs with finite out-valency are analyzed in detail in [1] and [2]. It is shown that the descendant set of a vertex is quite constrained in such a digraph, in particular up to isomorphism there are only countably many possibilities for the descendant set. Thus while results in [1] suggest that it may be possible to classify descendant sets of vertices in highly arc transitive primitive digraphs, the results here indicate that there is no possibility of classifying the ones with a given isomorphism type of descendant set.

We then explore the stability properties of these digraphs. We also look at the independence property, the tree property and the tree property of the second kind for these digraphs in order to understand more fully their model theoretic properties. We originally aimed to produce a stable theory, a model M and a type p such that the group of automorphisms induced on $p(M) = \{a \in M : M \models p(a)\}$ by $\mathrm{Aut}(M)$ is primitive with an unbalanced suborbit. We have been unsucessful in our attempts, however we explain our findings.

After this we look at a class obtained from a Hrushovski construction using a predimension and see that this can be viewed more naturally as a class of digraphs. We then restrict this class by limiting the number of primitive extensions any given set can have and obtain an amalgamation lemma for the class. We axiomatize this theory and consider the properties of completeness, stability and triviality. The reduct of this class obtained by forgetting the direction on the edges is then considered and we again look at stability in this setting.

We now summarize the contents of each of the chapters in this thesis.

In Chapter 2 we use digraphs with each vertex having two descendants and with some extra structure, to construct many primitive permutation groups with a finite suborbit paired with a suborbit of size $\aleph_0$. Firstly, we introduce some notation and then we use it to define continuum many classes of isomorphism types of digraphs with certain properties (Definition 2.1.6). We then show that these classes are amalgamation classes and that we can take a Fraïssé-type limit of each one. It is then shown that the automorphism group of this structure is primitive. Finally we show that the properties we used to define the classes give continuum many different examples of such permutation groups and thus prove Theorem 2.0.2. The material in this chapter has been published in [8].

In Chapter 3 we consider the stability of the types of digraphs constructed in the previous chapter. Primarily we examine the case where the only relation is the digraph relation. Here we find a formula which defines a partial order with infinite chains. This means that the theory we considered has the strict order property (Proposition 3.1.5) and so is unstable. We then attempt to further understand the classes we have produced by considering some model theoretic properties of them, showing that the theory does have the independence property, the tree property and the tree property of the second kind.

So the primitive structures we have produced are, perhaps rather surprisingly, quite bad from a model theoretic viewpoint. We therefore ask whether they can be seen as part of a better-behaved structure. Specifically, it would be interesting to know whether they can appear as the induced structure on a type in a stable structure. Thus we attempt to modify the conditions on the class in order to obtain a stable theory in which the digraphs used earlier are found on the set of realizations of some complete type. This would then give what we wanted as we would have a stable theory and the primitivity and unbalanced suborbit conditions would not be lost. We explain one attempt that was made at this and describe why it fails and the implications that this has for further variations of this theory.

Finally, in Chapter 4 we explore a connection between some directed graphs and Hrushovski constructions from [18]. The construction in [18] is usually seen as a two part process : a free amalgamation construction and then a more difficult amalgamation known generally as 'collapse'. In [18], the second part is required to obtain structures of finite Morley rank. In this chapter we first detail the construction of a Hrushovski class which can be viewed more naturally using ≤ 2-out digraphs and show how this relates to the first part of Hrushovski's construction from [18]. This process introduces two classes of graphs - digraphs with each vertex having at most two descendants, ≤ 2-*out digraphs* and the reduct of this obtained by removing the direction on the edges. We show that the reduct of the Fraïssé-type limit of the class of digraphs is isomorphic to the Fraïssé-type limit of the class of undirected graphs (as is also shown in [14]).

We then try to imitate the second part of Hrushovski's construction (the 'collapse') in the context of the directed graphs. We define minimal, primitive and regular extensions in the digraph setting and study the class of digraphs in which the number of primitive extensions is restricted. We prove amalgamation lemmas for the cases of minimal and regular extensions (Corollary 4.2.8 and 4.2.10 respectively) and then use these to produce an axiomatization of the theory. Algebraic closure in these structures is then considered which provides some insight into forking. With this we see that the theory is complete, stable and trivial. Finally, we look at the reduct of this theory obtained by forgetting the direction on the edges and show that it is strictly stable. So the process of 'collapse' does not commute with taking the reduct : the undirected reduct of the 'collapsed' digraph $\mathcal{N}_\nu$ is not the 'collapse' of the undirected graph which is given by Hrushovski's construction in [18].

It will be useful to first outline some background material on permutation groups, graph theory, stability theory, the independence and tree properties, Fraïssé limits and forking and dividing. References will also be given so that further information on each of the topics described can be found if desired. It will be assumed that basic model theoretic notions such as structures, models and theories are understood. Background model theory can be studied in many sources, including in [17] if needed.

1.2 Permutation Groups

We start with some general information about permutation groups. Further details on the concepts briefly introduced here can be found in [4] or [7]. Let Ω be an arbitrary non-empty set. A bijection of Ω onto itself is called a *permutation* of Ω and the set of all permutations forms a group with the binary operation being composition of maps. This group is called the *symmetric group* of Ω and is denoted by $Sym(\Omega)$. A *permutation group* G on Ω is a subgroup of $Sym(\Omega)$ and is often denoted by (G, Ω). The *degree* of a permutation group is $|\Omega|$.

An *isomorphism* between two structures A, B is a bijection $f : A \to B$ such that both f and its inverse f^{-1} are homomorphisms (structure preserving maps).

An *automorphism* is an isomorphism from a structure to itself. The set of all automorphisms of a structure M with the binary operation being composition of maps forms a group which is called the *automorphism group* and is denoted by $\mathrm{Aut}(M)$.

Now let G be a group and Ω a non-empty set. Assume that for all elements $\alpha \in \Omega$ and $x \in G$ we have defined an element of Ω which we will denote by $x\alpha$. Then this defines an *action* of G on Ω if $i\alpha = \alpha$ (where i is the identity element of G) and if $y(x\alpha) = (yx)\alpha$ for all $\alpha \in \Omega$ and for all $x, y \in G$. If G acts on Ω and $\alpha \in \Omega$ then $G\alpha = \{x\alpha : x \in G\}$, the set of elements of Ω that α gets sent to by the action of G is the *orbit* of α under G. If the action of G on Ω has only one orbit, so $G\alpha = \Omega$ for all $\alpha \in \Omega$ then G is said to act *transitively* on Ω. Equivalently, G acts transitively on Ω if for all $\alpha, \beta \in \Omega$ there exists $x \in G$ such that $x\alpha = \beta$.

For Δ a non-empty subset of Ω and for x in G let $x\Delta$ denote the set $\{x\alpha : \alpha \in \Delta\}$. Then Δ is called a *block* for G if for each element x in G either $x\Delta = \Delta$ or $x\Delta \cap \Delta = \emptyset$. Every action on Ω has Ω and the singletons $\{\alpha\}$ for $\alpha \in \Omega$ as blocks. These blocks are called *trivial blocks*. A group G acting transitively on Ω is *primitive* if G has no non-trivial blocks on Ω. Alternatively, in more model theoretic terms, a group G is primitive if there is no non-trivial $\mathrm{Aut}(G)$-invariant equivalence relation on G.

Let G be a group acting transitively on a set Ω and let $\alpha \in \Omega$. Define G_α to be the subgroup $\{g \in G : g\alpha = \alpha\}$ and call the G_α-orbits on Ω *suborbits* and the G-orbits on $\Omega^2 = \Omega \times \Omega$ *orbitals*. Then $\{\alpha\}$ is a trivial suborbit and $\{(\omega, \omega) : \omega \in \Omega\}$ is the trivial orbital.

Take $\Delta \subseteq \Omega^2$, a non-trivial orbital, and consider the corresponding digraph which is a directed graph with vertex set Ω and edge set Δ (this is called the *orbital digraph*). Then G acts as a group of automorphisms of this digraph and is transitive on vertices and on directed edges. If we now ignore the direction on the edges we obtain the *orbital graph* which has vertex set Ω and edge set $\{\{\beta, \gamma\} : (\beta, \gamma) \in \Delta\}$. The suborbit Γ corresponding to the orbital Δ is the set of out-vertices for α (vertices coming out of α, that is the set $\{\beta \in \Omega : (\alpha, \beta) \in \Delta\}$) in the orbital digraph with edge set Δ. The paired suborbit Γ^* is the set of in-vertices for α (i.e. $\{\beta \in \Omega : (\beta, \alpha) \in \Delta\}$). If the suborbit Γ and the paired suborbit Γ^* have different cardinalities then we say that the permutation group has an *unbalanced suborbit*. This shows the equivalence between orbital digraphs and permutation groups with an unbalanced suborbit. This correspondence is described in more detail in [13].

There is a very useful criterion for primitivity of actions using these orbital digraphs that was discovered by D.G. Higman in [16] and was stated in a different form in [13]. This restatement of the condition is the one that will be used later, so that is the one expressed in the following lemma. See Section 1.3 below for the definition of a connected graph.

Lemma 1.2.1. *([13], Lemma 1.1) The transitive permutation group (G, Ω) is primitive if and only if all its non-trivial orbital graphs are connected.*

1.3 Graph Theory

Since we are using directed graphs throughout it will be helpful to highlight some notions from graph theory. A more detailed account of these concepts can be found in [3] and [6].

A graph, G is usually denoted by $G(V, E)$ where V is the set of vertices and E is a set of subsets of size two from V, being seen as the set of edges of the graph. A *directed graph* is a graph that has a direction added to each edge, which means that a directed graph has an asymmetric relation.

A *path* in the graph $G(V, E)$ from vertex v_1 to vertex v_2 is a sequence of vertices starting with v_1 and ending with v_2 such that every vertex is joined by an edge to the vertex that follows it in the sequence. A graph is called *connected* if there is a path between every pair of vertices in the graph. A *bipartite graph* is a graph in which the set of vertices can be split into two disjoint parts such that each part has no edges between any of its elements. This can be represented by $G = G(V_1 \cup V_2, E)$. In a graph $G = G(V, E)$ the set of vertices that are joined to vertex v is $N(v)$, that is $N(v) = \{w : \{v, w\} \in E\}$ and similarly if X is a set of vertices then $N(X) = \{v : \{v, w\} \in E, w \in X\}$. The *degree* of a vertex v is $|N(v)|$. Let $G = G((V_1 \cup V_2), E)$ be a bipartite graph. A *matching* in G is a subset of E such that each vertex (in both V_1 and V_2) has degree at most one. Such a matching is called *perfect for* V_1 if every vertex in V_1 has degree exactly one (the vertices in V_2 will still have degree at most one). Figure 1 demonstrates some of these definitions.

The following theorem is needed in Chapter 4.

Theorem 1.3.1 (Hall's Marriage Theorem)**.** *Let $G = (V_1 \cup V_2, E)$ be a finite bipartite graph. Then G has a perfect matching for V_1 if and only if $|X| \leq |N(X)|$ for all $X \subseteq V_1$.*

This theorem is essentially Theorem 2.1.2 from [6] and three different proofs of

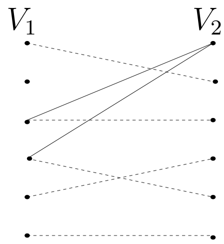

Figure 1: Example of a bipartite graph with a matching for V_1 (dotted lines), but not a perfect matching

it are given there. In [6] the author has restricted to the case where $|V_1| = |V_2|$ and his definition of matching is the same as perfect matching for V_1 used herein (as $|V_1| = |V_2|$ is assumed).

1.4 Stability

We will now introduce some of the definitions in stability theory which will be needed in later chapters. For a more comprehensive discussion of the topic see [19], [17] (includes the strict order property) or [21] (looks at stability using types).

Throughout, let L be a first-order language and let T be a complete L-theory. The order property can be defined in several ways; the definition of it given here is taken from [23]. An L-formula $\phi(\bar{x}, \bar{y})$ has the *order property* relative to the theory T if we can find $\bar{a}_n$ and $\bar{b}_m$ in a model M of T with $n, m \in \omega$ such that $\phi(\bar{a}_n, \bar{b}_m)$ is true if $n \leq m$ and false if $n > m$ (where overlines are used to indicate tuples). We can see that this means the formula $\varphi(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) = \phi(\bar{x}_1, \bar{y}_2)$ orders the infinite set $(\bar{a}_i\bar{b}_i : i < \omega)$. An L-formula $\phi(\bar{x}, \bar{y})$ has the *strict order property* if we can find $(\bar{b}_n : n \in \mathbb{N})$ in a model M of T such that the formula $(\exists \bar{x})(\neg\phi(\bar{x}, \bar{b}_n) \wedge \phi(\bar{x}, \bar{b}_m))$ is true if $n < m$ and false if $n > m$. In other words, T has the strict order property if there is a definable partial order on a subset of M^p for some $M \models T$ and $p \in \mathbb{N}$, which has infinite chains. It can be seen that if a theory T has the strict order property then there is a formula that has the order property relative to T. A *stable theory* is a theory in which no formula has the order property relative to the theory.

An alternative definition for a stable theory uses types. A (complete) *n-type* over the theory T (for $n \in \mathbb{N}$) is the set of L-formulas which is satisfied by some n-tuple in a model of the theory T. Equivalently, a complete n-type is a maximal set of formulas in n-variables consistent with T. A *partial n-type* is a non-maximal set of formulas in n-variables consistent with T. Now suppose that M is an L-structure and $A \subseteq M$. Then L_A is the language obtained by adding constant symbols to L for all elements of A. Let $Th_A(M)$ be the set of all L_A-sentences that are true in M. Then an *n-type over A* is a set of L_A-formulas in free variables $x_1, \ldots, x_n$ that is consistent with $Th_A(M)$.

The theory T is said to be *λ-stable* for the cardinal λ if for all $A \subseteq M \models T$ such that $|A| = \lambda$ and for all finite $n \geq 1$, $|S_n(A)| \leq \lambda$, where $S_n(A)$ is the set of n-types over A. The theory T is *stable* if it is λ-stable for some λ and it is *ω-stable* if it is $\aleph_0$-stable. The theory is *superstable* if it is λ-stable for all $\lambda \geq 2^{|T|}$ and it is *strictly stable* if it is λ-stable if and only if $\lambda^\omega = \lambda$. It can be shown that any ω-stable theory is superstable, and it is trivial to see that any superstable theory is stable. The statement " ω-stability implies superstability " can be found as (Proposition 5.28, [21]).

1.5 Independence and Tree Properties

We give the definitions of the independence property, the tree property and the tree property of the second kind along with references for further information.

The following definition was introduced by Shelah in [24]. An L-formula $\phi(\bar{x}, \bar{y})$ has the *independence property* with respect to the theory T if for each $n \in \omega$ there is a model M of T and sequences $(\bar{b}_i : i < n)$ and $(\bar{a}_w : w \subseteq \{0, \ldots, n-1\})$ from M such that $M \models \phi(\bar{a}_w, \bar{b}_i)$ if and only if $i \in w$. A theory T has the independence property if some L-formula $\phi(\bar{x}, \bar{y})$ has the independence property.

The following definition again due to Shelah is based on a series of definitions found in [26] (there is a part of the definition missing in the book), which can be referred to for details about theories without the tree property. Note that ${}^{\omega}\omega$ is the set of infinite sequences of natural numbers and ${}^{<\omega}\omega$ is the set of finite sequences of natural numbers. If $\xi \in {}^{\omega}\omega$ and $n < \omega$ then $\xi|n \in {}^{<\omega}\omega$ is the restriction of ξ to the first n terms. If $\nu \in {}^{<\omega}\omega$ then $\nu^\wedge i$ is the finite sequence consisting of ν and the extra term i. Now let $\phi(\bar{x}, \bar{y})$ be an L-formula. Then ϕ has the *tree property* with respect to the theory T if there exists $k < \omega$ and a collection $(\bar{a}_\nu : \nu \in {}^{<\omega}\omega)$ of tuples in a model of T such that for all $\xi \in {}^{\omega}\omega$ the set $\{\phi(\bar{x}, \bar{a}_{\xi|n}) : n < \omega\}$ is consistent with T and for all $\nu \in {}^{<\omega}\omega$ the set $\{\phi(\bar{x}, \bar{a}_{\nu^\wedge i}) : i < \omega\}$ is k-inconsistent with T (which means that any finite subset of $\{\phi(\bar{x}, \bar{a}_{\nu^\wedge i}) : i < \omega\}$ of size k is inconsistent with T). A theory T has the tree property if there is a formula which has the tree property with respect to T.

The following definition was also introduced by Shelah and is taken from [5], which contains several results regarding theories with NTP_2 (ie it does not have the tree property of the second kind). A theory T has TP_2 (the tree property of the second kind) if there exists a formula $\phi(x, y)$, a number $k < \omega$ and an array of elements $\langle a_i^j : i, j < \omega \rangle$ in a model of T such that :

1. every row is k-inconsistent (that is, for all $j < \omega$ and for all $i_1 < \ldots < i_k < \omega, \phi(x, a_{i_1}^j) \wedge \ldots \wedge \phi(x, a_{i_k}^j)$ is inconsistent with T), and
2. every vertical path is consistent (that is, for all $f : \omega \to \omega$, $\bigwedge_{j<\omega} \phi(x, a_{f(j)}^j)$ is consistent with T).

1.6 Fraïssé Limits

It will be useful to have an understanding of the construction method of Fraïssé limits. A brief explanation of a general version of the construction is given here and variations of the method are used as needed in Chapter 2. This will require some variation in usage of the terminology but we hope that this does not cause confusion. It is also recommended to investigate ([11], Section 2) for a more comprehensive treatment of the general version of the method and also for several useful examples.

Let L be a first-order language and let $\mathcal{C} = (\mathcal{C}, \leq)$ be a collection of countable L-structures with a distinguished notion of embeddings (denoted by $\leq$) which satisfies $A \leq A$ for all $A \in \mathcal{C}$ and $A \leq B \leq C$ implies $A \leq C$. We say that an L-structure A is in $\mathcal{C}$ to mean that A is isomorphic to an element of $\mathcal{C}$, since we only need to consider isomorphism types of elements of $\mathcal{C}$. Say that $\mathcal{C}$ is an *amalgamation class* if it has the following properties:

1. hereditary property : if $A \in \mathcal{C}$ and B is a $\leq$-substructure of A then $B \in \mathcal{C}$;

2. joint embedding property : if $A, B \in \mathcal{C}$ then there exists $C \in \mathcal{C}$ such that A, B are isomorphic to $\leq$-substructures of C;

3. amalgamation property : if $A, B_1, B_2 \in \mathcal{C}$ and $\alpha_i : A \to B_i$ are $\leq$-embeddings then there exists $C \in \mathcal{C}$ and $\leq$-embeddings $\beta_i : B_i \to C$ with $\beta_1\alpha_1 = \beta_2\alpha_2$.

In the original version of Fraïssé's Theorem the class $\mathcal{C}$ consists of finite structures and the distinguished embedding notion $\leq$ is just that of being a substructure.

Theorem 1.6.1. *Suppose that L is a first-order language and $\mathcal{C} = (\mathcal{C}, \leq)$ is an amalgamation class of finite L-structures. Suppose that $\mathcal{C}$ has countably many isomorphism types of structures. Then there exists a countable L-structure M and substructures $(A_i : i < \omega)$ in $\mathcal{C}$ such that:*

1. $A_0 \leq A_1 \leq A_2 \leq \ldots$ *and* $M = \bigcup_{i<\omega} A_i$

2. *if* $A \leq A_i$ *and* $A \leq B \in \mathcal{C}$ *then there is some* $j > i$ *and a* $\leq$*-embedding* $f : B \to A_j$ *such that* $f(a) = a$ *for all* $a \in A$ *(the extension property).*

Moreover, M is uniquely determined up to isomorphism by these conditions.

We refer to M in the above as the *Fraïssé limit* of $(\mathcal{C}, \leq)$. If for $A \in \mathcal{C}$ we write $A \leq M$ to mean $A \leq A_i$ for some $i < \omega$ then for $A, A' \leq M$, if $h : A \to A'$ is an isomorphism then h extends to an automorphism of M (which preserves $\leq$).

We will use a variation on this in Chapter 2 in which 'finite' is replaced by 'finitely generated' in a suitable sense.

1.7 Forking and Dividing

We use forking and dividing in Chapter 4 so we recall very briefly the notions of dividing and forking in stable (or simple) theories and also some of the basic properties of the resulting notion of independence (non-forking). A convenient reference for this material (presented in the way in which we shall use it) is Chapter 2 of [26].

Suppose T is a complete, stable theory and M is a large saturated model of T.

Definition 1.7.1. Consider a sequence a_i of elements of the model M indexed by a totally ordered set I. Say that this set is *indiscernible* if for every natural number n, whenever $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ are two strictly increasing n-tuples of I, the n-tuples $(a_{i_1}, \ldots a_{i_n})$ and $(a_{j_1}, \ldots, a_{j_n})$ have the same type.

Definition 1.7.2. *([26], 2.2.1)*

1. A formula $\phi(\bar{x}, \bar{b})$ (with parameters $\bar{b}$) *divides* over a set A if there is a sequence $(\bar{b}_i : i < \omega)$ with $tp(\bar{b}_i/A) = tp(\bar{b}/A)$ which is indiscernible over A and such that $\bigwedge_{i<\omega} \phi(\bar{x}, \bar{b}_i)$ is inconsistent.

2. A formula $\varphi(\bar{x})$ (possibly with parameters) *forks* over a set A if there exist formulas $\phi_1(\bar{x}, \bar{c}_1), \ldots, \phi_r(\bar{x}, \bar{c}_r)$ such that $\vdash \varphi(\bar{x}) \to \bigvee_{i \leq r} \phi_i(\bar{x}, \bar{c}_i)$ and each $\phi_i(\bar{x}, \bar{c}_i)$ divides over A.

3. If $\bar{d}$ is a tuple and B is a set we say that $tp(\bar{d}/B)$ *divides* over A (respectively *forks* over A) if some $\phi(\bar{x}, \bar{b}) \in tp(\bar{d}/B)$ divides (respectively, forks) over A.

We write $\bar{d} \perp_A B$ to mean that $tp(\bar{d}/B)$ does not fork over A. In a stable theory $tp(\bar{d}/B)$ does not fork over A if and only if $tp(\bar{d}/B)$ does not divide over A.

For a subset D, the notion $D \mathop{\smile\!\!\!\!\!\!|}_A B$ means that $\bar{d} \mathop{\smile\!\!\!\!\!\!|}_A B$ for every tuple $\bar{d}$ from D. We now give some definitions and then some properties of the relation $\mathop{\smile\!\!\!\!\!\!|}$.

Definition 1.7.3. 1. The theory T is *simple* if no formula has the tree property in T. Note that any stable theory is simple.

2. An element a of the set A is *algebraic* over A if it satisfies a formula with parameters in A that is satisfiable by only finitely many elements.

3. The *algebraic closure* of the set A, denoted $acl(A)$ is the set of elements that are algebraic over A.

Theorem 1.7.4. *([26], Theorem 2.3.13) Suppose T is simple and $A \subseteq B \subseteq C$. Then:*

1. *Existence : For all c in the large saturated model M, $c \mathop{\smile\!\!\!\!\!\!|}_A \mathrm{acl}(A)$.*

2. *Extension : Every partial type over B which does not fork over A has a completion which does not fork over A.*

3. *Reflexivity : $B \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $B \subseteq \mathrm{acl}(A)$.*

4. *Monotonicity : If p and q are types with $p \vdash q$ and p does not fork over A, then q does not fork over A.*

5. *Finite Character : $D \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $\bar{d} \mathop{\smile\!\!\!\!\!\!|}_A B$ for every finite $\bar{d} \in D$.*

6. *Symmetry : $D \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $B \mathop{\smile\!\!\!\!\!\!|}_A D$.*

7. *Transitivity : $D \mathop{\smile\!\!\!\!\!\!|}_A C$ if and only if $D \mathop{\smile\!\!\!\!\!\!|}_A B$ and $D \mathop{\smile\!\!\!\!\!\!|}_B C$.*

8. *Local Character : For any $p \in S(A)$ there is $A_0 \subseteq A$ with $|A_0| \leq |T|$, such that p does not fork over A_0.*

From these we can deduce the following well known facts:

Corollary 1.7.5. *Suppose T is simple and $A \subseteq B \subseteq C$.*

1. *If $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $e \in \mathrm{acl}(cA)$ then $e \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*

2. *If $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$ then $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$.*

Proof. (Sketch)

1. This is done by 'forking calculus' using the properties in the above theorem. From the given $c \mathop{\smile\!\!\!\!\!\!\mid}_A B$ we get $c \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{b}$ for every finite $\bar{b} \in B$ by finite character. Symmetry then gives $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A c$ and using transitivity we see that $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} c$. Existence then gives $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} acl(Ac)$. We are given that $e \in \mathrm{acl}(cA)$, so we have $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} e$. Using $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A c$, $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{Ac} e$ and transitivity we get $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A ce$ and so $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_A e$. Finally, symmetry and finite character give $e \mathop{\smile\!\!\!\!\!\!\mid}_A B$ as required.

2. It is clear that $\mathrm{acl}(cA) \cap \mathrm{acl}(B) \supseteq \mathrm{acl}(A)$. So take $e \in \mathrm{acl}(cA) \cap \mathrm{acl}(B)$ and we need to show that $e \in \mathrm{acl}(A)$. By (1) with $e = c$ and $A = B$ we obtain $e \mathop{\smile\!\!\!\!\!\!\mid}_A e$ and so by reflexivity, $e \in \mathrm{acl}(A)$ as required.

□

Chapter 2

Constructing Continuum Many Examples

In this chapter we prove the following Theorem, which is an extension of ([13], Corollary 2.10). The proof of this theorem uses a Fraïssé-type construction on suitable amalgamation classes.

Theorem 2.0.1. *There are continuum many primitive permutation groups of countable degree which have a finite suborbit paired with a suborbit of size* $\aleph_0$.

As is explained in the introduction (and also in [13]), this follows from Theorem 2.0.2 below.

Theorem 2.0.2. *There are continuum many pairwise non-isomorphic countable directed graphs in which each vertex has finite out-valency and invalency* $\aleph_0$*, and whose automorphism group is primitive on vertices and transitive on directed edges.*

In this thesis we construct directed graphs with each vertex having out-valency two. This was done for simplicity and we could repeat all of the work with two replaced by any finite natural number.

2.1 Definition of the Amalgamation Classes

The first part of the proof of Theorem 2.0.2 is to construct classes of isomorphism types of L_I-structures (where L_I is a first-order language) and then to prove that these are amalgamation classes. The required digraphs will be obtained from Fraïssé-type limits of these classes. To start we investigate digraphs that have some extra structure. The digraphs will have no directed cycles and no multiple edges.

Notation 2.1.1. We use a binary relation R (which represents the digraph relation) and for each $n \in \mathbb{N}$, $n \geq 3$ we have an n-ary relation R_n. For $I \subseteq \mathbb{N} \backslash \{0, 1, 2\}$ let L_I denote the language $\{R\} \cup \{R_n : n \in I\}$. The case where $I = \emptyset$, so with no R_n relations is the arrangement considered in [13]. Let T denote the *rooted binary tree,* which is the directed graph with no undirected cycles such that each vertex has out-valency two and every vertex except for the root (which has no predecessor) has a unique predecessor.

Definition 2.1.2. If A is an L_I-structure and $a \in A$ then the set of *descendants* (or the *descendant set*) of a in A is the set of vertices (including a) in A that can be reached from a by an outward directed path, that is,

$$\{b : \exists n \in \mathbb{N} \exists a_1, \ldots, a_n \in A, (a, a_1), (a_1, a_2), \ldots, (a_n, b) \in R\}.$$

Denote this by $\mathrm{desc}^A(a)$ or simply by $\mathrm{desc}(a)$ if it is clear what structure we are working in. If X is a set of vertices in A then the set of descendants of X is $\mathrm{desc}^A(X) = \bigcup\{\mathrm{desc}^A(x) : x \in X\}$. If $X = \{x_1, \ldots, x_n\}$ write $\mathrm{desc}^A(x_1, \ldots, x_n)$ for $\bigcup_{i=1,\ldots,n} \mathrm{desc}^A(x_i)$. The set of *ancestors* of a vertex $a \in A$ is the set of vertices $\{x \in A : a \in \mathrm{desc}^A(x)\}$.

Definition 2.1.3. Let $A \subseteq B$ be L_I-structures. Say that A is *descendant closed* in B if for all $a \in A$, $\mathrm{desc}^A(a) = \mathrm{desc}^B(a)$. In this case write $A \leq B$.

Definition 2.1.4. Let A be an L_I-structure. Say that a set V of vertices of A is *finitely generated* if it is the union of the descendant sets of finitely many elements from A, that is if we have $V = \bigcup_{i=1}^n \mathrm{desc}^A(a_i)$ for some $a_i \in A$ and $n \in \mathbb{N}$.

Definition 2.1.5. Define $\mathrm{desc}(a) \leq^+ A$ for an L_I-structure A and for $a \in A$ to mean that $\mathrm{desc}(a) \cap \mathrm{desc}(b)$ is finitely generated for any $b \in A$, and if $\mathrm{desc}(b) \backslash \mathrm{desc}(a)$ is finite then $b \in \mathrm{desc}(a)$. More generally, for $B \subseteq A$ finitely generated say that $B \leq^+ A$ if for all $a \in A$, $\mathrm{desc}(a) \cap B$ is finitely generated and if $a \in A$ and $\mathrm{desc}(a) \backslash B$ is finite then $a \in B$. Note that $B \leq^+ A \implies B \leq A$.

This relation will introduce some control over the intersections of descendant sets of elements in the digraph that we are constructing (from the first part of the definition), by preventing the intersection of any two elements being too big. The second part of the definition is used to obtain primitivity. As an example, the relation does not hold if the descendant sets are as in Figure 2.

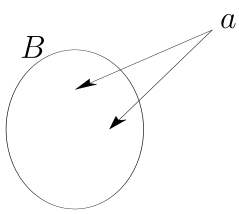

Figure 2: Example of a graph forbidden by the relation $\leq^+$ because $\mathrm{desc}(a) \backslash B$ is finite but $a \notin B$.

Definition 2.1.6. Let $(\mathcal{C}_I, \leq^+)$ consist of countable L_I-structures A such that R gives a digraph on A and the following conditions hold:

1. the descendant set of every element a in A forms a rooted binary tree with no other structure (so the only relations on $\mathrm{desc}(a)$ are from R, and the R_n are not involved);
2. we have $\mathrm{desc}(a) \leq^+ A$ for all $a \in A$;
3. A is finitely generated;
4. if $A \models R_n(a_1, \ldots, a_n)$ then $\mathrm{desc}(a_i) \cap \mathrm{desc}(a_j) = \emptyset$ for $i \neq j$, $a_1, \ldots a_n \in A$ have no common ancestor in A and $\mathrm{desc}(a_1, \ldots, a_n) \leq^+ A$;
5. the number of instances of the relations R_n on A is finite (meaning that there are only finitely many n for which there are any R_n relations, and for each n there are only finitely many $\bar{a}$ such that $R_n(\bar{a})$ holds).

We will now show that these classes of digraphs are amalgamation classes. The following definitions are based on those used in the method of constructing Fraïssé limits but are slightly different to the ones in the introduction since we are using a variation of the original construction.

Definition 2.1.7. The class $(\mathcal{C}_I, \leq^+)$ has the *hereditary property* if for all $A \in \mathcal{C}_I$, if B is a finitely generated descendant closed substructure of A then $B \in \mathcal{C}_I$.

Definition 2.1.8. The class $(\mathcal{C}_I, \leq^+)$ has the *amalgamation property* if whenever $A, B_1, B_2 \in \mathcal{C}_I$ and we have $\leq^+$-embeddings $f_i : A \to B_i$ then there is an L_I-structure $C \in \mathcal{C}_I$ and $\leq^+$-embeddings $g_i : B_i \to C$ such that $g_1 f_1 = g_2 f_2$. This definition is represented in Figure 3.

Definition 2.1.9. The class $(\mathcal{C}_I, \leq^+)$ is an *amalgamation class* if it has the hereditary property and the amalgamation property.

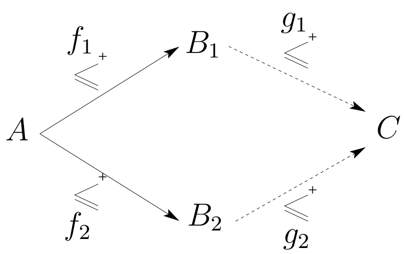

Figure 3: The amalgamation property

Therefore to check that the classes we have defined are amalgamation classes we need to check that they have the two required properties.

Proposition 2.1.10. *The class* $(\mathcal{C}_I, \leq^+)$ *has the hereditary property.*

Proof. Take any $A \in \mathcal{C}_I$ and let B be a finitely generated descendant closed substructure of A. We need to check that Conditions 1-5 in Definition 2.1.6 hold for B.

For any $b \in B$ we have $b \in A$ and hence $\mathrm{desc}(b)$ forms a rooted binary tree with no other structure. We are given that B is finitely generated, and therefore that Condition 3 for $B \in \mathcal{C}_I$ holds. Now let $b_1, b_2 \in B$, and note that we also thus have $b_1, b_2 \in A$. Then we have that $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2)$ is finitely generated and that $\mathrm{desc}(b_1) \backslash \mathrm{desc}(b_2)$ being finite implies $b_1 \in \mathrm{desc}(b_2)$, since $\mathrm{desc}(a) \leq^+ A$ for all $a \in A$. This shows that $\mathrm{desc}(b) \leq^+ B$ for all $b \in B$. Since $B \subseteq A$ and the number of instances of the R_n on A is finite, we must have the same for B. This then leaves only Condition 4 for $B \in \mathcal{C}_I$ to be checked. For this, let $b_1, \ldots, b_n \in B$ and assume that $B \models R_n(b_1, \ldots, b_n)$. Now $b_1, \ldots, b_n$ are also in A and $B \models R_n(b_1, \ldots, b_n)$ means $A \models R_n(b_1, \ldots, b_n)$ as well, therefore $\mathrm{desc}(b_i) \cap \mathrm{desc}(b_j) = \emptyset$ if $i \neq j$. We also see that $b_1, \ldots, b_n$ have no common ancestor in B because if they did then they would have a common ancestor in A, but this is impossible by $A \models R_n(b_1, \ldots, b_n)$. Finally, $\mathrm{desc}(b_1, \ldots, b_n) \leq^+ B$

because $\mathrm{desc}(b_1, \ldots, b_n) \leq^+ A$ and $B \subseteq A$. Therefore we have shown that all the necessary conditions hold for $B \in \mathcal{C}_I$. $\square$

To prove that the classes $(\mathcal{C}_I, \leq^+)$ have the amalgamation property we need the following two lemmas.

Lemma 2.1.11. *If $A, B, C \in \mathcal{C}_I$, $A, B \leq C$ and A, B are finitely generated, then $A \cap B$ is finitely generated.*

Proof. Since A and B are finitely generated we can write each of them as the descendant set of a finite number of elements of $\mathcal{C}_I$. So for some $m, n \in \mathbb{N}$ write $A = \mathrm{desc}^A(a_1, \ldots, a_n)$ and $B = \mathrm{desc}^B(b_1, \ldots, b_m)$. We also have $A, B \leq C$ and therefore

$$\mathrm{desc}^A(a_1, \ldots, a_n) = \mathrm{desc}^C(a_1, \ldots, a_n)$$

and

$$\mathrm{desc}^B(b_1, \ldots, b_m) = \mathrm{desc}^C(b_1, \ldots, b_m).$$

Then $A \cap B = \bigcup_{i,j} \left(\mathrm{desc}^C(a_i) \cap \mathrm{desc}^C(b_j)\right)$. Since $C \in \mathcal{C}_I$ and we have each $a_i, b_j \in C$ Condition 2 of Definition 2.1.6 says that $\mathrm{desc}^C(a_i) \cap \mathrm{desc}^C(b_j)$ is finitely generated for all i, j. Therefore $A \cap B$ is the union of finitely many finitely generated sets, and as such is finitely generated. $\square$

Lemma 2.1.12. *Let $X \subseteq Y$ and $Y \subseteq Z$ be in $\mathcal{C}_I$. If $X \leq^+ Y$ and $Y \leq^+ Z$ then $X \leq^+ Z$.*

Proof. Let $z \in Z$ and consider $\mathrm{desc}^Z(z) \cap X$. By definition of the relation $\leq^+$ we have $X \leq Y$ and $Y \leq Z$. This means that for $x \in X$,

$$\mathrm{desc}^X(x) = \mathrm{desc}^Y(x) = \mathrm{desc}^Z(x).$$

So X can be writen as $\mathrm{desc}^Z(x_1, \ldots, x_n)$ (because X is finitely generated as it is in $\mathcal{C}_I$). Hence by the argument used in the proof of Lemma 2.1.11 we have that $\mathrm{desc}^Z(z) \cap X$ is finitely generated.

Now assume $\mathrm{desc}^Z(z) \backslash \mathrm{desc}^Z(X)$ is finite. Then we see that $\mathrm{desc}^Z(z) \backslash \mathrm{desc}^Z(Y)$ must be finite since $X \subseteq Y$ and so

$$\left(\mathrm{desc}^Z(z) \backslash \mathrm{desc}^Z(Y)\right) \subseteq \left(\mathrm{desc}^Z(z) \backslash \mathrm{desc}^Z(X)\right).$$

Then, because $Y \leq^+ Z$ we get $z \in Y$. Therefore $z \in X$ as we have that for $y \in Y$, $\mathrm{desc}(y) \backslash \mathrm{desc}(X)$ being finite implies $y \in X$. Therefore we have shown that both properties required for $X \leq^+ Z$ hold. □

The following definition is adapted from ([25], Definition 2.9).

Definition 2.1.13. Let $A_0 \leq^+ A_i \in \mathcal{C}_I$ $(i = 1, 2)$ and $A_1 \cap A_2 = A_0$. Then the *free amalgam* of A_1 and A_2 over A_0 is the L_I-structure with underlying set $A_1 \cup A_2$, whose only relations are those induced from A_1 and A_2 (so there are no relations between elements of A_1 and elements of A_2). We denote it by $A_1 \coprod_{A_0} A_2$. A diagram of this structure is given in Figure 4.

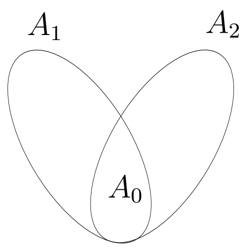

Figure 4: Free amalgam

Proposition 2.1.14. *The class $(\mathcal{C}_I, \leq^+)$ has the amalgamation property.*

Proof. Let $A, B_i \in \mathcal{C}_I$ for $i = 1, 2$ and assume we have $\leq^+$-embeddings $\alpha_i : A \to B_i$, so $\alpha_i(A) \leq^+ B_i$. Without loss of generality, $B_1 \cap B_2 = A$

and α_i is the identity on A, so $A \leq^+ B_i$. Then let C be the free amalgam $B_1 \coprod_A B_2$, so C is the digraph on the disjoint union $B_1 \cup B_2$ over A where the only relations are those induced from B_1 and B_2, that is there are no edges between an element of $B_1 \backslash A$ and an element of $B_2 \backslash A$ and $R_n(a_1, \ldots, a_n)$ does not hold if some of the a_i are in B_1 and others are in B_2.

Claim 1. We have $B_i \leq^+ C$.

Proof. Note that by the construction of C if $b \in B_i$ then $\mathrm{desc}(b) \subseteq B_i$. Let $b_1, b_2 \in C$. We need to show that $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2)$ is finitely generated and that if $\mathrm{desc}(b_2) \backslash \mathrm{desc}(b_1)$ is finite then $b_2 \in \mathrm{desc}(b_1)$. For this there are two cases to consider.

Case 1 Without loss of generality, $b_1, b_2 \in B_1$. Now $B_1 \in \mathcal{C}_I$ and therefore $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2)$ is finitely generated and $\mathrm{desc}(b_2) \backslash \mathrm{desc}(b_1)$ being finite implies that $b_2 \in \mathrm{desc}(b_1)$.

Case 2 Without loss, $b_1 \in B_1, b_2 \in B_2$. Assume that $\mathrm{desc}(b_2) \backslash \mathrm{desc}(b_1)$ is finite. Now $\mathrm{desc}(b_2) \cap \mathrm{desc}(b_1) \subseteq A$ due to the construction of C and so we have that $\mathrm{desc}(b_2) \backslash A$ is finite. We are given $A \leq^+ B_2$ and hence we get that $b_2 \in A$, so $b_2 \in B_1$ and we are in Case 1.

Hence we have shown that $\mathrm{desc}(b_2) \backslash \mathrm{desc}(b_1)$ being finite implies $b_2 \in \mathrm{desc}(b_1)$.

Now consider $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2)$. As above $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2) \subseteq A$ and we see that $\mathrm{desc}(b_i) \cap A$ (for $i = 1, 2$) is finitely generated by Lemma 2.1.11 (as $\mathrm{desc}(b_i), A, B_i \in \mathcal{C}_I$ and $\mathrm{desc}(b_i), A \leq B_i$ for each i). We can then use Lemma 2.1.11 in A to get that $\mathrm{desc}(b_1) \cap \mathrm{desc}(b_2)$ is finitely generated.

$\square$ Claim 1.

Claim 2. We have $C \in \mathcal{C}_I$.

Proof. We need to check that the five conditions in Definition 2.1.6 hold in this structure. Let $c \in C$. Then $c \in B_1$ or $c \in B_2$ since $C = B_1 \coprod_A B_2$ and each B_i is descendant closed in C. We have that for $b \in B_i$, $\mathrm{desc}^{B_i}(b)$ forms a rooted binary tree with no other structure because $B_i \in \mathcal{C}_I$. Hence $\mathrm{desc}^C(c)$ forms a rooted binary tree with no other structure. Using Claim 1 and Lemma 2.1.12 we see that $\mathrm{desc}(c) \leq^+ C$ since we have $\mathrm{desc}(c) \leq^+ B_i$ as $c \in B_i$. Given $C = B_1 \coprod_A B_2$, each B_i is finitely generated (because $B_i \in \mathcal{C}_I$) and the number of instances of the R_n relations on each B_i is finite, we have that C is finitely generated and that there are only finitely many occurrences of the R_n relations on C. Therefore conditions 1, 2, 3 and 5 hold for C to be in $\mathcal{C}_I$. Finally, for Condition 4, let $c_1, \ldots, c_n \in C$ and suppose that $C \models R_n(c_1, \ldots, c_n)$.

Case 1 All the c_j are in B_i for $i = 1$ or $i = 2$ - say they are in B_1, but they are not all in A. In this case $B_1 \models R_n(c_1, \ldots, c_n)$ and therefore, as $B_1 \in \mathcal{C}_I$ $\mathrm{desc}^{B_1}(c_i) \cap \mathrm{desc}^{B_1}(c_j) = \emptyset$ if $i \neq j$, $c_1, \ldots, c_n$ have no common ancestor in B_1 and $\mathrm{desc}(c_1, \ldots, c_n) \leq^+ B_1$. Since C is the free amalgamation of B_1 and B_2 over A, $c_1, \ldots, c_n$ then have no common ancestor in C, and as B_1 is descendant closed in C we have that $\mathrm{desc}^C(c_i) \cap \mathrm{desc}^C(c_j) = \emptyset$ if $i \neq j$. Also we have $\mathrm{desc}(c_1, \ldots, c_n) \leq^+ C$ by Lemma 2.1.12 since $\mathrm{desc}(c_1, \ldots, c_n) \leq^+ B_1$ and $B_1 \leq^+ C$. Hence Condition 4 holds in this case.

Case 2 All of the c_j are in A. The only part which is different from Case 1 for this is checking that the c_j have no common ancestor in C. We know they have no common ancestor in B_1 from the above case. In this case we also have $B_2 \models R_n(c_1, \ldots, c_n)$ from $C \models R_n(c_1, \ldots, c_n)$ and so $c_1, \ldots, c_n$ have no common ancestor in B_2 either. So this gives c_j have no common ancestor in C.

Case 3 Some of the c_i are in $B_1 \backslash A$ and some are in $B_2 \backslash A$. In this case the definition of R_n in C gives $C \not\models R_n(c_1, \ldots, c_n)$.

We have therefore shown that all of the properties in Definition 2.1.6 hold and so we have $C \in \mathcal{C}_I$, hence we have proved the claim.

$\square$ Claim 2.

So there are $\leq^+$-embeddings $\beta_i : B_i \to C$ for $i = 1, 2$ with $\beta_1\alpha_1 = \beta_2\alpha_2$ (since $A \subseteq B_i$ and $B_i \subseteq C$ for $i = 1, 2$ take α_i, β_i to be identity maps). Hence $(\mathcal{C}_I, \leq^+)$ has the amalgamation property. $\square$

Propositions 2.1.10 and 2.1.14 give us that the classes defined in Definition 2.1.6 are amalgamation classes. The next step in the proof of Theorem 2.0.2 is to construct a Fraïssé-type limit of each of these amalgamation classes.

2.2 Fraïssé-type Limits

The next step is to construct a Fraïssé-type limit of each of these classes, and to do this we first need to know a countability condition.

Definition 2.2.1. A subset $D = (d_1, \ldots, d_n)$ of $A \in \mathcal{C}_I$ is *independent* if, for $i \neq j$, we have $\mathrm{desc}(d_i) \cap \mathrm{desc}(d_j) = \emptyset$.

Definition 2.2.2. Consider $A \in \mathcal{C}_I$ as an R-structure (that is without the R_n relations). Suppose $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ are independent subsets of $A \in \mathcal{C}_I$ and the rooted binary tree T respectively. We can see that $\mathrm{desc}(\bar{a}) \simeq \mathrm{desc}(\bar{b})$ by independence and because $\mathrm{desc}(a_i) \simeq T$ (as $A \in \mathcal{C}_I$) and $\mathrm{desc}(b_i) \simeq T$ for all $i \in \{1, \ldots, n\}$. Hence there is an isomorphism from $\mathrm{desc}(\bar{a})$ to $\mathrm{desc}(\bar{b})$ which takes a_i to b_i for $i \leq n$. Define the *free amalgam* of A and T over

$\bar{a}$ and $\bar{b}$ to be the digraph with vertex set the disjoint union of A and T over $\mathrm{desc}(\bar{a}) \simeq \mathrm{desc}(\bar{b})$, and edge set the union of the edge sets of A and T.

Notation 2.2.3. (This notation is taken from ([10], Definition 2.2)). Denote the free amalgam of A and T over $\bar{a}$ and $\bar{b}$ by $(A, \bar{a}) * (T, \bar{b})$.

Lemma 2.2.4. *Let $A \in \mathcal{C}_I$ and consider it as an R-structure (as in the above definition). The isomorphism type of $(A, \bar{a}) * (T, \bar{b})$ is independent of the choice of isomorphism from $\mathrm{desc}(\bar{a})$ to $\mathrm{desc}(\bar{b})$.*

Proof. Any automorphism of $\mathrm{desc}(\bar{b})$ which fixes each b_i can be extended to an automorphism of T. Therefore different isomorphisms from $\mathrm{desc}(\bar{a})$ to $\mathrm{desc}(\bar{b})$ give isomorphic free amalgams. Hence $(A, \bar{a}) * (T, \bar{b})$ has the same isomorphism type for any choice of isomorphism from $\mathrm{desc}(\bar{a})$ to $\mathrm{desc}(\bar{b})$. ☐

Definition 2.2.5. Let $A, B, B' \in \mathcal{C}_I$ and let $f : A \to B$ and $f' : A \to B'$ be $\leq^+$-embeddings. Then f is *isomorphic* to f' if there exists an isomorphism $h : B \to B'$ such that $f' = hf$. Equivalently, f, f' are isomorphic if the diagram in Figure 5 commutes.

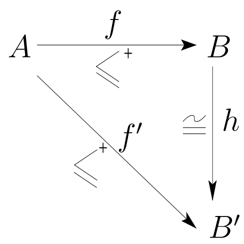

Figure 5: An isomorphism between f and f'

Proposition 2.2.6. *There are countably many isomorphism types of $\leq^+$-embeddings in the class $(\mathcal{C}_I, \leq^+)$.*

Proof. We will show that there are countably many isomorphism types of $\leq^+$-embeddings of elements of $\mathcal{C}_I$, if we ignore the R_n relations. Then Condition 5 of Definition 2.1.6 says there are only finitely many instances of the R_n relations on any element of $\mathcal{C}_I$. Placing finitely many instances of the R_n relations on each countable digraph is similar to choosing finitely many tuples from a countable set. So there are countably many arrangements of these finitely many R_n relations and hence we see that there are countably many isomorphism types of structures in $\mathcal{C}_I$. Therefore there are countably many isomorphism types of $\leq^+$-embeddings in $(\mathcal{C}_I, \leq^+)$.

We prove by induction on n that there are countably many n-generator structures in the class $\mathcal{C}_I$ (recall we are only considering the R relations and not the R_n relations for this part).

Base Step : Let $n = 1$, so $A = \operatorname{desc}(b_1)$ for some $b_1 \in \mathcal{C}_I$. Since $\operatorname{desc}(b_i) \simeq T$ for all $b_i \in \mathcal{C}_I$, there is only one 1-generator structure, A up to isomorphism.

Inductive Step : Let $B = \operatorname{desc}(b_1, \ldots, b_k, b_{k+1})$ be a $(k + 1)$-generator structure in $\mathcal{C}_I$ and let $A = \operatorname{desc}(b_1, \ldots, b_k)$. By Condition 2 of Definition 2.1.6, $A \cap \operatorname{desc}(b_{k+1})$ is finitely generated, for example by $X \subset A$. Take $|X|$ to be minimal, letting $X = \{c_1, \ldots, c_r\} \subseteq B$.

Claim 1. The set $\{c_1, \ldots, c_r\}$ is independent.

Proof. Since $\{c_1, \ldots, c_r\}$ is minimal, $c_i \notin \operatorname{desc}(c_j)$ for every $i, j \in \{1, \ldots r\}$, $i \neq j$. We know $\{c_1, \ldots, c_r\} \subseteq \operatorname{desc}(b_{k+1})$ and that this is isomorphic to the rooted binary tree, T. Suppose $\operatorname{desc}(c_i) \cap \operatorname{desc}(c_j) \neq \emptyset$ for some $i, j \in \{1, \ldots, r\}, i \neq j$. Then let $a \in \operatorname{desc}(c_i) \cap \operatorname{desc}(c_j)$. This gives an undirected cycle in $\operatorname{desc}(b_{k+1})$ as shown in Figure 6.

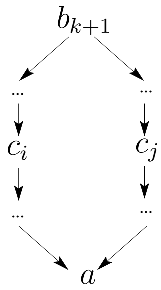

Figure 6: An undirected cycle

This contradicts $\mathrm{desc}(b_{k+1}) \simeq T$, and hence we have that $\{c_1, \ldots, c_r\}$ is independent.

□ Claim 1.

Claim 2. B is isomorphic to the free amalgam of A and $\mathrm{desc}(b_{k+1})$ over $\mathrm{desc}(X)$.

Proof. We have $\leq^+$-embeddings $f : \mathrm{desc}(X) \to A$ (since $\mathrm{desc}(X) \subseteq A$) and $g : \mathrm{desc}(X) \to \mathrm{desc}(b_{k+1})$ (since $\mathrm{desc}(X) \subseteq \mathrm{desc}(b_{k+1})$). We also have that $A \subset B$ and $\mathrm{desc}(b_{k+1}) \subset B$. Therefore $\mathrm{desc}(b_{k+1}) \cup A$ is the free amalgam of $\mathrm{desc}(b_{k+1})$ and A over $\mathrm{desc}(X)$ (since there are no edges between $\mathrm{desc}(b_{k+1}) \backslash \mathrm{desc}(X)$ and $A \backslash \mathrm{desc}(X)$) and it is contained in B. It is clear that these are the only elements in B, since

$$\begin{aligned} B = \mathrm{desc}(b_1, \ldots, b_{k+1}) &= \bigcup_{i=1}^{k+1} \mathrm{desc}(b_i) \\ &= \left(\bigcup_{i=1}^{k} \mathrm{desc}(b_i) \right) \cup \mathrm{desc}(b_{k+1}) \\ &= A \cup \mathrm{desc}(b_{k+1}) \end{aligned}$$

□ Claim 2.

By Lemma 2.2.4 we have that the isomorphism type of B is independent of the choice of automorphism of $\mathrm{desc}(X)$. We also have, from the inductive hypothesis that there are countably many possibilities for A and hence countably many possibilities for X. Therefore there are countably many possibilities for B. $\square$

Now that we have Proposition 2.2.6 we can construct the Fraïssé-type limits. We do this in the following theorem.

Theorem 2.2.7. *There is a countable L_I-structure N^I such that*

1. *N^I is the union of substructures $N_1^I \subseteq N_2^I \subseteq \ldots$ such that each $N_i^I \in \mathcal{C}_I$ ($i \in \mathbb{N}$) and $N_i^I \leq^+ N_{i+1}^I$ for all i,*
2. *(Extension Property) whenever $A \leq^+ N_i^I$ and $A \leq^+ B \in \mathcal{C}_I$ there is $s \geq i$ and a $\leq^+$-embedding $f : B \to N_s^I$ with $f|_A = id$.*

The extension property is represented diagramatically in Figure 7.

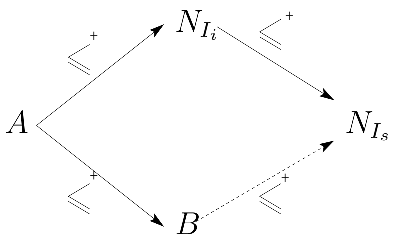

Figure 7: The extension property

Proof. To prove this we construct the N_i^I inductively, taking $N_1^I = \emptyset$ for example. For the purposes of the proof it will be useful to fix a bijection $\eta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the property that $\eta(a, b, c) \geq a, b, c$.

Suppose we have constructed $N_1^I \leq^+ \ldots \leq^+ N_i^I \in \mathcal{C}_I$. There are countably many finitely generated $\leq^+$-substructures of N_i^I - list these as $(A_j^i : j \in \mathbb{N})$.

For each A^i_j there are countably many isomorphism types of $\leq^+$-embeddings into elements of $\mathcal{C}_I$ - list these as $\theta^i_{jk} : A^i_j \to B_k$. Note that at stage i we will have done this for each N^I_m with $m \leq i$. The point is that the extension problem (as in Property 2) corresponding to θ^i_{jk} will be solved at stage $s = \eta(i, j, k) + 1$. So let $(i', j', k') = \eta^{-1}(i)$. We have $\theta^{i'}_{j'k'} : A^{i'}_{j'} \to B_{k'}$, $A^{i'}_{j'} \leq^+ N^I_{i'} \leq^+ N^I_i$. Then use the amalgamation property of $\mathcal{C}_I$ on $A^{i'}_{j'}$, $B_{k'}$ and N^I_i to get $N^I_{i+1} \in \mathcal{C}_I$ with $N^I_i \leq^+ N^I_{i+1}$ and $B_{k'} \leq^+ N^I_{i+1}$ such that the diagram commutes. We may assume that N^I_i is a substructure of N^I_{i+1} and then we have that $A^{i'}_{j'}$ is fixed pointwise.

Now let N^I be the union of these N^I_n. For the last part, take $A \leq^+ N^I_i$ such that $A \leq^+ B \in \mathcal{C}_I$. From the construction of N^I there will be an $s \geq i$ and a $\leq^+$-embedding from B to N^I_s, as required by Property 2. □

Remark 2.2.8. For any $i \in \mathbb{N}$ $N^I_i \leq^+ N^I$. To see this let $a \in N^I$, then $a \in N^I_j$ for some $j > i$ by the construction of N^I. Since $N^I_i \leq^+ N^I_j$ this gives $\mathrm{desc}(a) \cap N^I_i$ is finitely generated and if $\mathrm{desc}(a) \backslash N^I_i$ is finite then $a \in N^I_i$, that is $N^I_i \leq +N^I$.

Definition 2.2.9. The N^I defined in the above theorem is $\leq^+$-*homogeneous* if for any finitely generated $A_1, A_2 \leq^+ N^I$, any isomorphism $\theta : A_1 \to A_2$ can be extended to an automorphism of N^I

Corollary 2.2.10. *The N^I defined in the above theorem is unique up to isomorphism and satisfies $\leq^+$-homogeneity.*

Proof. This corollary follows from the following statement : If N^I_1, N^I_2 satisfy the properties for N^I in Theorem 2.2.7, $A_i \leq^+ N^I_i$ (for $i = 1, 2$) are finitely generated and $\theta : A_1 \to A_2$ is an isomorphism, then θ can be extended to an isomorphism $\tilde{\theta} : N_1^I \to N_2^I$. In particular, if we take $A_1 = A_2 = \emptyset$ then we obtain the uniqueness stated in Corollary 2.2.10, and taking $N_1^I = N_2^I$ gives the required $\leq^+$-homogeneity.

The proof of this statement is done using a 'back and forth' argument (which is possible because N_1^I and N_2^I are countable).

For the 'forth' step, let $b \in N_1^I$. We have to find $A_1' \leq^+ N_1^I$ and $A_2' \leq^+ N_2^I$ with $A_1 \subseteq A_1'$, $b \in A_1'$ and an isomorphism $\theta' : A_1' \to A_2'$ extending θ. By the first property for N_1^I there is $A_1' \leq^+ N_1^I$ with $A_1 \subseteq A_1'$ and $b \in A_1'$. For example, take A_1' to be some $N_j'^I$ containing b and all the generators of A_1. There are $\leq^+$-embeddings $f : A_1 \to A_1'$ and $\theta^{-1} : A_2 \to A_1$ and so by composition of maps we get a $\leq^+$-embedding $f \circ \theta^{-1} : A_2 \to A_1'$. We have a $\leq^+$-embedding $g : A_2 \to N_2^I$ and so the extension property for N_2^I gives a $\leq^+$-embedding $\theta' : A_1' \to N_2^I$ (as shown in the Figure 8). This embedding has the properties $\theta'(A_1') = A_2' \leq^+ N_2^I$ and $\theta'|A_1 = \theta$. This concludes the 'forth' direction of the proof. The 'back' direction is then symmetrical to this. $\square$

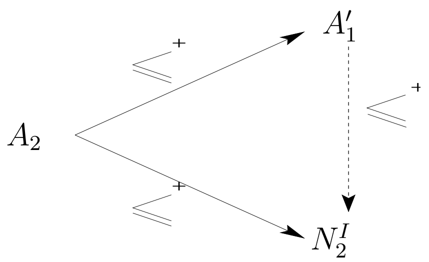

Figure 8: The extension property for A_2

Definition 2.2.11. Call the structure N^I defined above the *Fraïssé-type limit* of $(\mathcal{C}_I, \leq^+)$.

2.3 Primitivity

We now need to prove that the automorphism group of each of these Fraïssé-type limits is primitive. For this we use the criterion for primitivity given in the introduction (Lemma 1.2.1) and we will require two lemmas which will be presented below.

Definition 2.3.1. For $A \in \mathcal{C}_I$ and X a finite subset of A define the *closure* of X in A to be $\mathrm{cl}^A(X) = \{y \in A : \mathrm{desc}^A(y) \backslash \mathrm{desc}^A(X) \text{is finite}\}$.

Note 2.3.2.

1. If $X \subseteq Y \leq^+ A$ then $\mathrm{cl}^A(X) \subseteq Y$.
2. For all X, A, $\mathrm{cl}^A(X) \leq A$.
3. If we know that $\mathrm{cl}^A(X)$ is finitely generated then $\mathrm{cl}^A(X) \leq^+ A$ (knowing $\mathrm{cl}^A(X)$ to be finitely generated gives that $\mathrm{cl}^A(X) \cap \mathrm{desc}(a)$ for any $a \in A$ is finitely generated because this is the union of $\mathrm{desc}(c) \cap \mathrm{desc}(a)$ for each generator $c \in \mathrm{cl}^A(X)$ and we know each $\mathrm{desc}(c) \cap \mathrm{desc}(a)$ is finitely generated by Condition 2 of Definition 2.1.6). Therefore $\mathrm{cl}^A(X)$ is the smallest $\leq^+$ subset of A containing X.

Lemma 2.3.3. *Let $A \in \mathcal{C}_I$ and let X be a finite subset of A. Then the closure of X in A, $\mathrm{cl}^A(X)$ is finitely generated.*

Proof. As $A \in \mathcal{C}_I$ we know A is finitely generated. Let $a \in \mathrm{cl}^A(X) \backslash \mathrm{desc}(X)$, $X = \{x_1, \ldots, x_n\}$ and h be one of the generators of A such that $a \in \mathrm{desc}(h)$ (note there are only finitely many such h). This is shown in Figure 9.

Define $\mathrm{gen}(h, x_1)$ to be the vertices which are the generators of the intersection of $\mathrm{desc}(h)$ and $\mathrm{desc}(x_1)$. Note that this set is finite since the intersection of any two descendant sets of elements of A is finitely generated. Using this

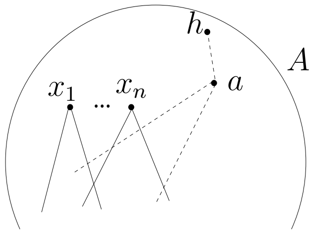

Figure 9: The arrangement needed to show that $\mathrm{cl}^A(X)$ is finitely generated

definition and knowing that $\mathrm{desc}(h)$ is a tree, we can see that $\mathrm{gen}(a, x_j) \subseteq \mathrm{gen}(h, x_j)$ for all j (if $b \in \mathrm{gen}(a, x_j)$ and $b \notin \mathrm{gen}(h, x_j)$ for some j then because $\mathrm{desc}(a) \cap \mathrm{desc}(x_j) \subseteq \mathrm{desc}(h) \cap \mathrm{desc}(x_j)$ there must be $c \in \mathrm{gen}(h, x_j)$ with $b \in \mathrm{desc}(c)$, which would give a cycle in $\mathrm{desc}(h)$). Therefore the number of vertices contained in a shortest path from h to a, $\mathrm{dist}(h, a)$ is at most

$$max\{\mathrm{dist}(h, z) : z \in \mathrm{gen}(h, x_j), j = 1, \ldots, n\}.$$

This says that a is closer to h than the furthest generator in the intersection of $\mathrm{desc}(h)$ and $\bigcup_{j=1}^{n} \mathrm{desc}(x_j)$. Since $\mathrm{gen}(h, X) = \bigcup_{j=1}^{n} \mathrm{gen}(h, x_j)$ is finite this means that there are only finitely many possibilities for a. With these finitely many possible a's and with the finite number of elements of X we get that $\mathrm{cl}^A(X)$ must be finitely generated. □

Note that we can define $\mathrm{cl}_{N^I}(X)$ for X a finite subset of N^I. Then by the above lemma it follows that $\mathrm{cl}_{N^I}(X) \leq^+ {N^I}_i$ for some i, and that it is finitely generated.

Lemma 2.3.4. *The Fraïssé-type limit N^I as a digraph with relation R is connected.*

Proof. Let $n_1, n_2 \in N^I$. If $\mathrm{desc}^{N^I}(n_1) \cap \mathrm{desc}^{N^I}(n_2) \neq \emptyset$ then there is a non-directed path from n_1 to n_2 going via this intersection. If the intersection

$\mathrm{desc}(n_1) \cap \mathrm{desc}(n_2) = \emptyset$ then we use the extension property of N^I. Let B, C be rooted binary trees with top vertices b, c respectively which intersect as shown in Figure 10. Let $n \in N^I$ be such that for $i = 1, 2$, $\mathrm{desc}(n) \cap \mathrm{desc}(n_i) = \emptyset$ and $\mathrm{desc}(n) \cup \mathrm{desc}(n_i)$ has no R_k relations on it for any $k \in I$. The extension property then gives that $\mathrm{desc}(n_1) \cup \mathrm{desc}(n_2)$ can be $\leq^+$-embedded into $B \cup C$ where n_1, n are two edges away from b and n_2, n are two edges away from c (as shown in Figure 10).

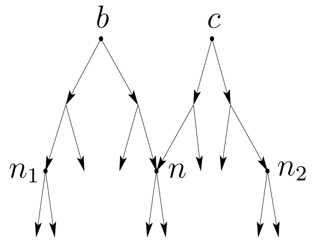

Figure 10: The arrangement needed to show N^I is connected

Therefore there is an undirected path of length at most eight from n_1 to n_2. Using this approach with $n_3 \in \mathrm{cl}_{N^I}(n_1)$ we can get a path of length at most 16 between any two vertices in N^I, which is a property which can be expressed by a first-order sentence. So we have shown that any model of $Th(N^I)$ is connected. □

Proposition 2.3.5. *The automorphism group* $\mathrm{Aut}(N^I)$ *is transitive on* N^I.

Proof. This follows due to $\leq^+$-homogeneity and by Conditions 1 and 2 in Definition 2.1.6. □

Proposition 2.3.6. *The automorphism group* $\mathrm{Aut}(N^I)$ *is primitive on* N^I.

Proof. This proof is similar to the proof of ([13], Theorem 2.9), though we have a different argument in case 3 below as the original argument appears to be somewhat inaccurate. From Lemma 1.2.1 we have that a transitive permutation group is primitive if and only if all of its non-trivial orbital graphs are connected. From the above proposition we have $\mathrm{Aut}(N^I)$ is transitive on N^I. So $\mathrm{Aut}(N^I)$ is a transitive permutation group and thus to prove this theorem we need to show that all of the non-trivial orbital graphs of $\mathrm{Aut}(N^I)$ are connected. To see that all of the non-trivial orbital graphs are connected we prove that if $a \neq b \in N^I$ then the orbital graph, G with vertex set the elements of N^I and edge set $\{\{fa, fb\} : f \in \mathrm{Aut}(N^I)\}$ is connected. As N^I is connected via R-edges by Lemma 2.3.4 it is enough to show that if $x, y \in N^I$ are such that (x, y) is an R-edge of N^I then x, y lie in the same connected component of G.

Without loss of generality, assume $x = a$ and let

$$H_1 = \mathrm{cl}_{N^I}(a, b) = \{n \in N^I : \mathrm{desc}(n) \backslash (\mathrm{desc}(a) \cup \mathrm{desc}(b)) \text{is finite}\},$$

which is the closure of a and b in N^I. We have that H_1 is finitely generated by Lemma 2.3.3 and $H_1 \leq^+ N^I$.

Case 1 Suppose $\mathrm{desc}(a) \cap \mathrm{desc}(b) = \emptyset$. Let H_2 be a copy of H_1 with $a' \in H_2$ corresponding to $a \in H_1$. Recalling that y is an out-vertex of a, identify $\mathrm{desc}_{H_1}(y)$ with $\mathrm{desc}_{H_2}(b)$, and take the free amalgam $H_{1,2}$ of H_1 and H_2 over $\mathrm{desc}_{H_1}(y)$ (this means not adding any new R or R_n relations). Then we have an L_I-structure with $\mathrm{desc}(a') \cap \mathrm{desc}(b) = \emptyset$. We see from the construction of $H_{1,2}$ that $\mathrm{desc}(a') \cup \mathrm{desc}(b) \leq^+ H_{1,2}$ (by construction there are no elements h in $H_{1,2}$ with $\mathrm{desc}(h) \backslash \mathrm{desc}(a') \cup \mathrm{desc}(b)$ finite) so we can adjoin a finite set X of new vertices to $H_{1,2}$ to obtain an L_I-structure $P \supseteq H_{1,2}$ in which

$$H_3 = \mathrm{cl}^P(a', b) = \mathrm{desc}(a', b) \cup X$$

is isomorphic to H_1 (via an isomorphism taking a' to a and b to b, and not adding any new R_n relations). So P is the union of H_1, H_2 and H_3, and we have $H_1 \cap H_3 = \mathrm{desc}(b)$, $H_2 \cap H_3 = \mathrm{desc}(a')$ and $H_1 \cap H_2 = \mathrm{desc}(y)$ as is shown in Figure 11. Moreover, any edge (and any R_n relation) is contained entirely within some H_i.

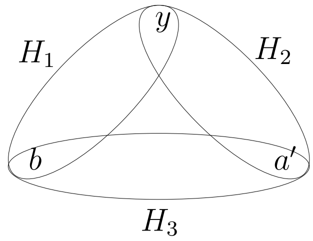

Figure 11: The arrangement of the H_i's

Claim. $P \in \mathcal{C}_I$.

Proof. It is clear that P is finitely generated due to its construction, therefore Condition 3 of Definition 2.1.6 holds. It is also clear that Condition 1 holds, since for every $p \in P$, $\mathrm{desc}(p)$ is contained in some H_i (as $H_i \leq P$) and so is isomorphic to T. For Condition 2 note that each H_i is descendant closed in P. We see that $\mathrm{desc}(y) \leq^+ H_1$ since $\mathrm{desc}(b) \leq^+ N^I$ and also $\mathrm{desc}(y) \leq^+ H_2$. Then use the amalgamation property to see that $H_1 \leq^+ H_{1,2}$. We then get $\mathrm{desc}(b) \cup \mathrm{desc}(a') \leq^+ H_{1,2}$. Using the amalgamation property again, this time with $\mathrm{desc}(b) \cup \mathrm{desc}(a') \subseteq H_3$ and $\mathrm{desc}(b) \cup \mathrm{desc}(a')$ (as shown in Figure 12) we get that $H_3 \leq^+ P$ as P is the free amalgam of $H_{1,2}$ and H_3 over $\mathrm{desc}(a') \cup \mathrm{desc}(b)$.

This argument can be seen to be symmetrical in $1, 2, 3$ (where $H_{i,j}$ is the union of H_i and H_j: note that these are freely amalgamated over their intersection in P). So we have $H_i \leq^+ P$ for $i = 1, 2, 3$. Then this gives us $\mathrm{desc}(p) \leq^+ P$ for every $p \in P$, that is Condition 2 holds.

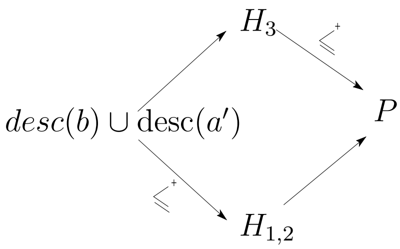

Figure 12: The amalgamation property

Since the only occurrences of the R_n relations are in H_1, H_2 or H_3 disjointly and the number of instances of the R_n relations in each case is finite, the number of instances of the relations R_n in P must also be finite. Therefore we have that Condition 5 for $p \in \mathcal{C}_I$ holds.

Finally, we need to check that Condition 4 holds, so assume $P \models R_n(p_1, \ldots, p_n)$ for $p_1, \ldots, p_n \in P$. By the construction of P this implies that $p_1, \ldots, p_n \in H_i$ for some i since this is the only way that $R_n(p_1, \ldots, p_n)$ can be true. Therefore $H_i \models R_n(p_1, \ldots, p_n)$ and so the p_j are independent, $\mathrm{desc}(p_1, \ldots, p_n) \leq^+ H_i$ and $p_1, \ldots, p_n$ have no common ancestor in H_i, since each $H_i \in \mathcal{C}_I$. By Lemma 2.1.12 we see that $\mathrm{desc}(p_1, \ldots, p_n) \leq^+ P$ (since $\mathrm{desc}(p_1, \ldots, p_n) \leq^+ H_i$ and $H_i \leq^+ P$). Each H_i is descendant closed since it is the closure of given elements. So suppose for a contradiction that $p_1, \ldots, p_n$ have a common ancestor, say $q \in P$. Then q must be in H_j for some $j \neq i$ (we know $p_1, \ldots, p_n$ have no common ancestor in H_i from above). However H_i and H_j are freely amalgamated over their intersection and as this is the descendant set of a single point, not all of $p_1, \ldots, p_n$ are in the intersection. As $H_i \cap H_j \leq P$ we have $q \notin H_i \cap H_j$. However this contradicts the freeness of the amalgam. Therefore Condition 4 holds and so we have $P \in \mathcal{C}_I$.

$\square$ Claim.

Now we use the extension property on $H_1 \leq^+ P$ and $H_1 \leq^+ N^I$ (as shown in Figure 13) to get that there is a copy P' of P over H_1 with $P' \leq^+ N^I$.

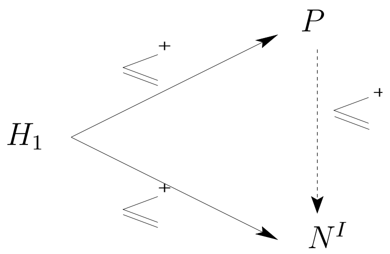

Figure 13: The extension property

Since $\mathrm{desc}(a) \leq^+ P$, there is a $\leq^+$-embedding $\phi : P \rightarrow N^I$ which keeps $\mathrm{desc}(a)$ fixed. Any isomorphism between finitely generated $\leq^+$-substructures of N^I can be extended to an automorphism of N^I. So if E denotes an edge in the orbital graph G then we have $aE\phi(b)E\phi(a')Ey$. Hence $x = a$ and y are at distance at most three in the orbital graph, and so x and y are in the same connected component of G.

Case 2 Suppose that $b \in \mathrm{desc}(a)$. In this case let b_0 denote the predecessor of b in $\mathrm{desc}(a)$, so (b_0, b) is an R-edge in N^I. Then let $b_1 \in \mathrm{desc}(a)$ be the other out-vertex of b_0. Then there is an automorphism of N^I fixing a and interchanging b and b_1, so b and b_1 are connected in the orbital graph G. We have that $\mathrm{desc}(b) \cap \mathrm{desc}(b_1) = \emptyset$ and hence Case 1 gives that the orbital graph with $\{b, b_1\}$ as an edge is connected. Therefore the orbital graph G is also connected.

Case 3 Suppose that $\mathrm{desc}(b) \backslash \mathrm{desc}(a)$ and $\mathrm{desc}(a) \backslash \mathrm{desc}(b)$ are infinite. In this case let $x_1, \ldots, x_r$ be a minimal generating set for $\mathrm{desc}(a) \cap \mathrm{desc}(b)$. Therefore $\mathrm{desc}(x_i) \cap \mathrm{desc}(x_j) = \emptyset$ for $i \neq j$ and we prove that the orbital graph G is connected in this case by induction on r, taking $r = 0$ as the base case (which is given by case 1 above). We can assume that x_r is at maximal distance from a amongst the x_i. Let z be the immediate predecessor of x_r in $\mathrm{desc}(a)$. Note that $z \notin \mathrm{desc}(a) \cap \mathrm{desc}(b)$ by minimality of the generating set. As $\mathrm{desc}(a) \cap \mathrm{desc}(b) \leq^+ \mathrm{desc}(a)$, not all of the successors of z lie in $\mathrm{desc}(a) \cap \mathrm{desc}(b)$. So we can choose x'_r to be one of its successors which is not amongst $x_1, \ldots, x_r$. The distance of x'_r from a in $\mathrm{desc}(a)$ is no smaller than the distance of that of any of the x_i. Thus $x_1, \ldots, x_{r-1}, x'_r$ are independent and $\mathrm{desc}(x_1, \ldots x_{r-1}, x'_r) \leq^+ \mathrm{desc}(a)$.

By a free amalgam and the extension property there is $b_1 \in N^I$ such that $\mathrm{desc}(b_1) \cap \mathrm{cl}(a, b) = \mathrm{desc}(x_1, \ldots, x_{r-1}, x'_r)$ and there exists an isomorphism $f : \mathrm{cl}(a, b) \to \mathrm{cl}(a, b_1)$ with $f(a, b, x_1, \ldots, x_{r-1}, x_r) = (a, b_1, \ldots, x_{r-1}, x'_r)$. By $\leq^+$- homogeneity, this extends to an automorphism of N^I. Therefore b and b_1 are in the same connected component of the orbital graph G. However we have that $\mathrm{desc}(b) \cap \mathrm{desc}(b_1) = \mathrm{desc}(x_1, \ldots, x_{r-1})$, and by the inductive hypothesis the orbital graph with $\{b, b_1\}$ as an edge is connected. Thus G is connected.

These are all of the possibilities since $a, b \in \mathcal{C}$ and therefore they satisfy Condition 2 which prohibits, for example $desc(a) \backslash \mathrm{desc}(b)$ finite and non-empty. Therefore we have shown that the automorphism group $\mathrm{Aut}(N^I)$ is primitive on N^I. $\square$

2.4 Non-isomorphism

Finally, we show that the continuum many structures that we have produced actually give different digraphs. To do this let $N^I|_R$ be the underlying digraph of N^I (that is, take the reduct of N^I obtained by forgetting the R_n relations).

Proposition 2.4.1. *Let n be a natural number. Then $n \in I$ if and only if there exist $a_1, \ldots, a_n \in N^I|_R$ with the following properties:*

1. $\mathrm{desc}(a_i) \cap \mathrm{desc}(a_j) = \emptyset$ *if* $i \neq j$ *and* $\mathrm{desc}(a_1, \ldots, a_n) = A \leq^+ N^I|_R$,
2. $a_1, \ldots, a_n$ *have no common ancestor in* $N^I|_R$,
3. *every finite subset X of A with $\mathrm{cl}^A(X) \neq A$ has a common ancestor in* $N^I|_R$.

Proof. First suppose that $n \in I$. Take A to be the graph generated by vertices $a_1, \ldots, a_n$ (the descendant set of each of these elements is a binary tree) such that $\mathrm{desc}(a_i) \cap \mathrm{desc}(a_j) = \emptyset$ if $i \neq j$ and with $A \models R_n(a_1, \ldots a_n)$ (and there are no other R_k relations on A for any k). Then $A \in \mathcal{C}_I$ by construction and so we may assume $A \leq^+ N^I|_R$. Hence A satisfies Conditions 1 and 2. Now let X be a finite subset of A with $\mathrm{cl}^A(X) \neq A$. We can assume that $\mathrm{desc}(X) \leq^+ A$ and that $X = max(\mathrm{cl}^A(X)) = \{x_1, \ldots, x_r\}$ (i.e. that X contains only the elements needed to generate $\mathrm{cl}^A(X)$). Note that the x_i are thus independent and therefore $\mathrm{cl}^A(X) = \mathrm{desc}(x_1) \cup \ldots \cup \mathrm{desc}(x_r)$. Let B be a rooted binary tree with root b and let d be a distance from b such that there are at least $2r$ elements at that distance. Let $b_1, \ldots, b_r \in B$ be independent, at distance at least d from b and such that each pair b_i, b_j has no immediate common ancestor. Then there is an isomorphism between $\bigcup \mathrm{desc}(x_i)$ and $\bigcup \mathrm{desc}(b_i)$ for $i = 1, \ldots, r$. A possible choice of the b_i is shown in Figure 14.

So $\mathrm{desc}(X)$ is isomorphic to $\mathrm{desc}(b_1, \ldots, b_r)$, $\mathrm{desc}(b_i) \cap \mathrm{desc}(b_j) = \emptyset$ for $i \neq j$, $b_1, \ldots, b_r$ have common ancestor b and $\mathrm{desc}(b_1, \ldots, b_r) \leq^+ \mathrm{desc}(b)$. Now use the amalgamation property to obtain $A' = A \coprod_{\mathrm{desc}(X)} \mathrm{desc}(b)$ with $A' \leq^+ N^I|_R$. This then gives a common ancestor for the elements of X in $N^I|_R$, which

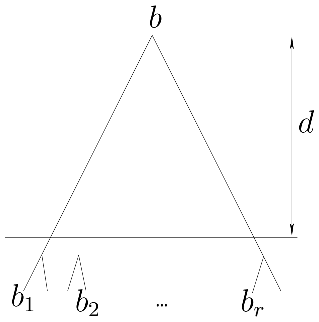

Figure 14: A possible arrangement of the b_i in B

shows that Condition 3 also holds for A.

Now suppose that A is as given, we have properties $1, 2, 3$ and, for a contradiction $n \notin I$. Then there is no relationship between the points of A except digraph relations. To see this let $a'_1, \ldots, a'_k \in A$ for $k \neq n$ and suppose $A \models R_k(a'_1, \ldots, a'_k)$. Then by Condition 4 of Definition 2.1.6 we must have $a'_1, \ldots, a'_k$ independent and $\mathrm{desc}(a'_1, \ldots, a'_k) \leq^+ A$. As $k \neq n$ we have $\mathrm{cl}^A(a'_1, \ldots, a'_k) \neq A$. Hence, by Condition 3 $a'_1, \ldots, a'_k$ have a common ancestor in $N^I|_R$, and this contradicts $A \models R_k(a'_1, \ldots, a'_k)$.

Now use the extension property with the embeddings $f : A \to B$ and $g : A \to N^I|_R$ to get an embedding from B into $N^I|_R$, where B is a graph in which $a_1, \ldots, a_n$ have a common ancestor (as Figure 15 indicates). This gives a common ancestor of $a_1, \ldots, a_n$ in $N^I|_R$, which contradicts the properties of A. □

Proposition 2.4.2. *If $I \neq J$ then the digraphs $N^I|_R$ and $N^J|_R$ are not isomorphic.*

Proof. This follows from Proposition 2.4.1. For example, let $n \in I$, $n \notin J$. Then we can find $a_1, \ldots, a_n \in N^I|_R$ so that the conditions in Proposition 2.4.1

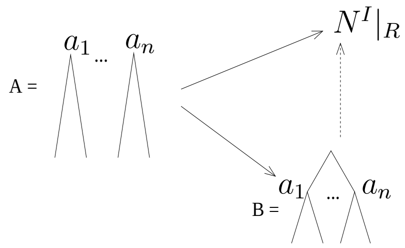

Figure 15: The extension property for A

are satisfied. However there are no $b_1, \ldots, b_n$ in $N^J|_R$ with these conditions. Therefore the digraphs $N^I|_R$ and $N^J|_R$ cannot be isomorphic. □

Remark 2.4.3. We have now constructed continuum many permutation groups $G_I = \mathrm{Aut}(N^I|_R)$ of countable degree and amongst these there are continuum many non-isomorphic orbital digraphs (the $N^I|_R$). There is a possibility that G_I has an orbital digraph which is isomorphic to $N^J|_R$ for some $J \neq I$. However, each G_I has only countably many orbital digraphs so there is a subset of $\{G_I : I \subseteq \mathbb{N}\}$ of size continuum such that no two graphs in this subset are isomorphic as permutation groups.

We have now shown that there are continuum many different primitive permutation groups with an unbalanced suborbit, that is we have proved Theorem 2.0.2. Having done this we decided to investigate their stability and other model theoretic properties.

Chapter 3

Stability, Independence and Tree Properties

In this chapter we consider the stability of the digraphs constructed in Chapter 2 and then briefly the independence property, the tree property and the tree property of the second kind. For simplicity, we consider the case where $I = \emptyset$, so there are only digraph relations. Denote the class of isomorphism types by $(\mathcal{C}_0, \leq^+)$ and let N_0 be the Fraïssé-type limit of $(\mathcal{C}_0, \leq^+)$, as constructed in the earlier chapter. This is the original version of the construction of these digraphs as seen in [13]. We find that $Th(N_0)$ is unstable and, as detailed in the introduction we would like to find a stable theory T, a model $M \models T$ and a type p such that the group of automorphisms induced on the set $p(M) = \{a \in M : M \models p(a)\}$ by $\mathrm{Aut}(M)$ is primitive with an unbalanced suborbit. Thus for the final part of this chapter we consider a variation of the digraphs we have been considering to see if we can find what we are looking for.

3.1 Stability

We first consider the stability of these digraphs, finding that $Th(N_0)$ is unstable.

Definition 3.1.1. Let the binary relation $S(a, b)$ be the formula

$$(\exists c)(\exists c_1)(\exists c_2)((c_1 \neq c_2) \wedge (cRc_1 \wedge cRc_2 \wedge c_1Ra \wedge c_2Rb))$$

where we write $a'Rb'$ for $(a', b') \in R$.

This relation is shown diagramatically in Figure 16.

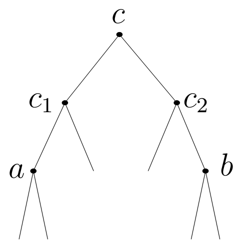

Figure 16: A diagram of the relation $S(a, b)$

Lemma 3.1.2. *Let* $a_1, a_2 \in N_0$. *Then we have* $\mathrm{desc}(a_1) \cap \mathrm{desc}(a_2) = \emptyset$ *and* $\mathrm{desc}(a_1) \cup \mathrm{desc}(a_2) \leq^+ N_0$ *if and only if* $N_0 \models S(a_1, a_2)$.

Proof. Let $A = \mathrm{desc}(a_1) \cap \mathrm{desc}(a_2)$ and $B = \mathrm{desc}(a_1) \cup \mathrm{desc}(a_2)$. First suppose that $N_0 \models S(a_1, a_2)$. Then the result is clear - for c from Definition 3.1.1, $\mathrm{desc}(c) \simeq T$ since $c \in \mathcal{C}_0$ and hence $A = \emptyset$. Also $B \leq^+ \mathrm{desc}(c)$ and $\mathrm{desc}(c) \leq^+ N_0$ and so by Lemma 2.1.12 $B \leq^+ N_0$.

Now suppose that $A = \emptyset$ and that $B \leq^+ N_0$. Then there is a $\leq^+$-embedding

$f : B \to \mathrm{desc}(c)$ obtained by sending $\mathrm{desc}(a_1)$ and $\mathrm{desc}(a_2)$ to the appropriate places in $\mathrm{desc}(c)$. Hence we have $c \in N_0$ which witnesses $S(a_1, a_2)$. $\square$

Definition 3.1.3. Define $\phi(a, b)$ to be $\forall x(S(a, x) \to S(b, x))$.

Lemma 3.1.4. *We have* $N_0 \models \phi(a,b)$ *if and only if* $\operatorname{desc}(b) \subseteq \operatorname{desc}(a)$.

Proof. Assume $\operatorname{desc}(b) \subseteq \operatorname{desc}(a)$, and hence $b \in \operatorname{desc}(a)$. If $N_0 \models S(a,x)$, then $\operatorname{desc}(a) \cap \operatorname{desc}(x) = \emptyset$ and $\operatorname{desc}(a) \cup \operatorname{desc}(x) \leq^+ N_0$ by Lemma 3.1.2. Since $b \in \operatorname{desc}(a)$ we then have $\operatorname{desc}(b) \cap \operatorname{desc}(x) = \emptyset$. We also know $\operatorname{desc}(b) \leq^+ \operatorname{desc}(a)$ and so

$$\operatorname{desc}(b) \cup \operatorname{desc}(x) \leq^+ \operatorname{desc}(a) \cup \operatorname{desc}(x) \leq^+ N_0,$$

(the first of these is by inspection of $\operatorname{desc}(a) \cup \operatorname{desc}(x)$) hence Lemma 2.1.12 gives $\operatorname{desc}(b) \cup \operatorname{desc}(x) \leq^+ N_0$. Therefore $N_0 \models S(b,x)$ and so $N_0 \models \phi(a,b)$.

Now suppose that $\operatorname{desc}(b) \not\subseteq \operatorname{desc}(a)$ and therefore $b \notin \operatorname{desc}(a)$. Since $a, b \in N_0$, $\operatorname{desc}(a) \cap \operatorname{desc}(b)$ is finitely generated.

Claim. There is $c \in \operatorname{desc}(b)$ such that $\operatorname{desc}(a) \cup \operatorname{desc}(c) \leq^+ \operatorname{desc}(a) \cup \operatorname{desc}(b)$ and $\operatorname{desc}(a) \cap \operatorname{desc}(c) = \emptyset$.

Proof. Let X be the set of generators of $\operatorname{desc}(a) \cap \operatorname{desc}(b)$. Since X is a finite set let x be the element of X which is furthest away from b. Let w be the predecessor of x in $\operatorname{desc}(b)$ and y the other out-vertex of w. We are able to choose $y \notin \operatorname{desc}(a)$ for otherwise we have $w \in \operatorname{desc}(a) \cap \operatorname{desc}(b)$ by $\operatorname{desc}(a) \cap \operatorname{desc}(b) \leq^+ \operatorname{desc}(a)$. Taking x at maximal distance away from b guarantees that $\operatorname{desc}(y) \cap \operatorname{desc}(X) = \emptyset$. Then choose c to be one of the out-vertices of y. This ensures that $\operatorname{desc}(X) \cap \operatorname{desc}(c) = \emptyset$ and so $\operatorname{desc}(a) \cap \operatorname{desc}(c) = \emptyset$. Also $\operatorname{desc}(a) \cup \operatorname{desc}(c) \leq^+ \operatorname{desc}(a) \cup \operatorname{desc}(b)$ because $\operatorname{desc}(X) \leq^+ \operatorname{desc}(b)$.

$\square$ Claim.

Now let d be the root of a rooted binary tree such that $\operatorname{desc}(c) \subset \operatorname{desc}(d)$. (An example of this arrangement is shown in Figure 17). Then by construction we have that $\mathrm{desc}(c) \leq^+ \mathrm{desc}(d)$. We also have $\mathrm{desc}(c) \leq^+ \mathrm{desc}(a) \cup \mathrm{desc}(b)$ and so by the amalgamation property of $(\mathcal{C}_0, \leq^+)$ the free amalgam, D of $\mathrm{desc}(a) \cup \mathrm{desc}(b)$ and $\mathrm{desc}(d)$ over $\mathrm{desc}(c)$ is in $\mathcal{C}_0$ and $\mathrm{desc}(a) \cup \mathrm{desc}(b) \leq^+ D$. By the extension property we may assume that $D \leq^+ N_0$. Therefore we have $\mathrm{desc}(d) \cap \mathrm{desc}(a) = \emptyset$ and $\mathrm{desc}(d) \cup \mathrm{desc}(a) \leq^+ N_0$, and hence by Lemma 3.1.2 $N_0 \models S(d, a)$. However $\mathrm{desc}(b) \cap \mathrm{desc}(d) \neq \emptyset$ and so $N_0 \not\models S(d, b)$. Therefore $N_0 \not\models \phi(a, b)$. □

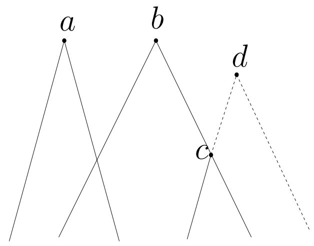

Figure 17: The arrangement for the proof of Lemma 3.1.4

Proposition 3.1.5. *$Th(N_0)$ has the strict order property.*

Proof. Lemma 3.1.4 shows that ϕ defines a partial order on $Th(N_0)$ which has infinite chains. This means that $Th(N_0)$ has the strict order property. □

We have therefore shown that $Th(N_0)$ is unstable.

3.2 Independence Property

We then decided to understand our original theory further and so considered whether or not the theory $Th(N_0)$ has the independence property. The following definition will be used in the proof.

Definition 3.2.1. Recall that two elements a_1, a_2 of N_0 are called independent if $\mathrm{desc}(a_1) \cap \mathrm{desc}(a_2) = \emptyset$. Let A be a rooted directed binary tree with root a. Say that an element $b \in A$ is on *level i* of A if there is a directed path of length i from a to b. Define a *levelled independent set* in A to be a set of independent elements a_i of A (for $1 \leq i < \omega$) where each a_i is on level i of A. Figure 18 gives an illustration of a levelled independent set in a rooted directed binary tree.

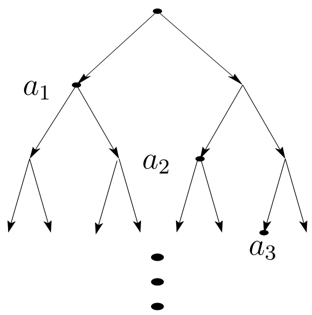

Figure 18: A levelled independent set

The following lemma shows that $Th(N_0)$ does have the independence property.

Lemma 3.2.2. $Th(N_0)$ *has the independence property.*

Proof. Consider the model N_0 of $Th(N_0)$ and let $\psi(x, y)$ be the formula that says '$\mathrm{desc}(x) \cap \mathrm{desc}(y) = \emptyset$', which is definable by Lemma 3.1.4 by the formula $\nexists z(\phi(x, z) \wedge \phi(y, z))$.

Claim. The formula $\psi(x, y)$ has the independence property with respect to the theory $Th(N_0)$.

Proof. For $n \in \mathbb{N}$ fix $b_1, \ldots, b_n$ independent elements of N_0 (so $\mathrm{desc}(b_i) \cap \mathrm{desc}(b_j) = \emptyset$ if $i \neq j$) with $\bigcup \mathrm{desc}(b_i) \leq^+ N_0$. Let $w \subseteq \{1, \ldots, n\}$. We show that there is $a_w \in N_0$ such that $\mathrm{desc}(a_w) \cap \mathrm{desc}(b_i) = \emptyset$ if and only if $i \in w$. For example, let C be a rooted binary tree with root c. Let $z_1, \ldots, z_{n+1}$ be a levelled independent set in C and let $u_1, \ldots, u_n$ be a levelled independent set in $\operatorname{desc}(z_{n+1})$. A possible configuration of this is shown in Figure 19.

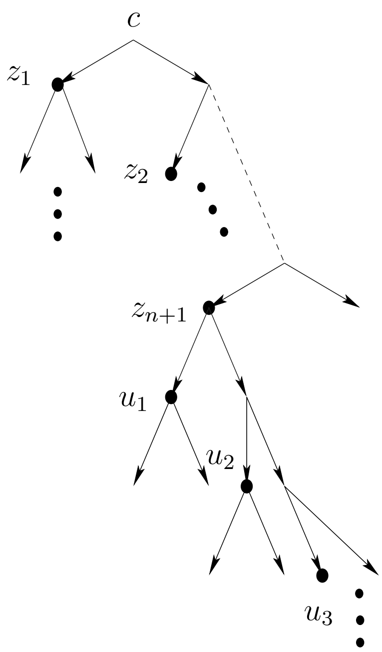

Figure 19: An arrangement for the proof of Lemma 3.2.2

Then there is an embedding of $\operatorname{desc}(b_1, \ldots, b_n)$ into C taking b_i to z_i if $i \in w$ and b_j to u_j if $j \notin w$. By the extension property (using $\operatorname{desc}(b_1, \ldots, b_n) \leq^+ N_0$ and $\operatorname{desc}(b_1, \ldots, b_n) \leq^+ C$) we can assume that this arrangement is in N_0. Then take a_w to be the vertex z_{n+1} and we get that $\operatorname{desc}(a_w) \cap \operatorname{desc}(b_i) = \emptyset$ if and only if $i \in w$ which is the required condition for a_w.

This arrangement can be repeated for any a_w. Hence the formula $\psi(x, y)$ has the independence property with respect to the theory $Th(N_0)$.

□ Claim.

We have shown that there is a formula which has the independence property with regard to the theory $Th(N_0)$ and therefore $Th(N_0)$ has the independence property. □

3.3 Tree Properties

In this section we show that the theory $Th(N_0)$ has the tree property and the tree property of the second kind.

Lemma 3.3.1. *The theory $Th(N_0)$ has the tree property.*

Proof. To prove this we give a formula $\phi(x, y)$, some $k < \omega$ and $(a_\nu : \nu \in {}^{<\omega}\omega)$ such that for all $\xi \in {}^{\omega}\omega$ the set $\{\phi(x, a_{\xi|n}) : n < \omega\}$ is consistent and for all $\nu \in {}^{<\omega}\omega$ the set $\{\phi(x, a_{\nu^\wedge i} : i < \omega\}$ is k-inconsistent. Let $\phi(x, y)$ be the formula such that $N_0 \models \phi(x, y)$ if and only if $x \in \mathrm{desc}(y)$.

Claim. The formula ϕ has the tree property with $k = 2$.

Proof. To prove the claim we find $(a_\nu : \nu \in {}^{<\omega}\omega)$ in N_0 such that for all $\nu \in {}^{<\omega}\omega$ the set $\{\phi(x, a_{\nu^\wedge i}) : i < \omega\}$ is 2-inconsistent, that is there does not exist an x with $x \in \mathrm{desc}(a_{\nu^\wedge i}) \cap \mathrm{desc}(a_{\nu^\wedge j})$ for any distinct $i, j < \omega$. We also require that for all $\xi \in {}^{\omega}\omega$ the set $\{\phi(x, a_{\xi|n}) : n < \omega\}$ is consistent, that is there exists an $x \in N_0$ with $x \in \cap \mathrm{desc}(a_{\xi|n})$ for $n < \omega$.

Let A be a rooted binary tree in N_0 with root a_0 and define the set $\{a_i : 1 \leq i < \omega\}$ to be a levelled independent set in A. By the definition of a levelled independent set there are as many elements as we need and the independence gives us the required conditions on the descendant sets up to this point. Now define the set $\{a_{ij} : j < \omega\}$ for each fixed i to be a levelled independent set in the respective $\mathrm{desc}(a_i)$, the set $\{a_{ijk} : k < \omega\}$ for each fixed ij to be a levelled independent set in the respective $\mathrm{desc}(a_{ij})$, and so on (with $a_{i0} = a_i$ etc). Figure 20 gives an illustration of this configuration.

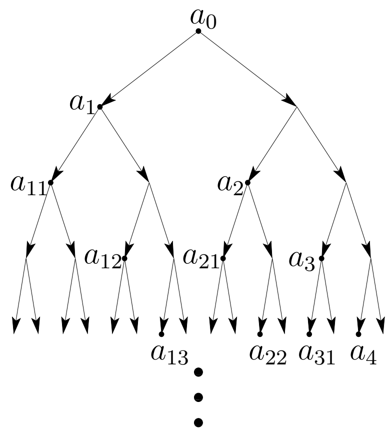

Figure 20: An illustration to show ϕ has the tree property

By construction $\{\phi(x, a_{\xi|n}) : n < \omega\}$ is consistent because $\bigcap_{n \leq m} \mathrm{desc}(a_{\xi|n}) \neq \emptyset$ for any $m < \omega$. This means that the consistency condition for the tree property holds. Also, there does not exist an x with $x \in \mathrm{desc}(a_{\nu^\wedge i}) \cap \mathrm{desc}(a_{\nu^\wedge j})$ for any distinct $i, j < \omega$ due to the independence of the levelled independent sets. Hence for all $\nu \in {}^{<\omega}\omega$ $\phi(x, a_{\nu^\wedge i}) \wedge \phi(x, a_{\nu^\wedge j})$ is inconsistent.

$\square$ Claim.

We have therefore found a formula with the tree property with respect to $Th(N_0)$ and hence $Th(N_0)$ has the tree property. $\square$

We now show that the theory $Th(N_0)$ has the tree property of the second kind (TP_2).

Lemma 3.3.2. *The theory $Th(N_0)$ has TP_2.*

Proof. Again, let $\phi(x, y)$ be the formula such that $N_0 \models \phi(x, y)$ if and only if

$x \in \mathrm{desc}(y)$. To prove that $Th(N_0)$ has TP_2 we find an array $\langle a_i^j : i, j < \omega \rangle$ in N_0 and $k < \omega$ such that for all $j < \omega$ and for all $i_0 < i_1 < \ldots < i_k < \omega$, $\phi(x, a_{i_0}^j) \wedge \phi(x, a_{i_1}^j) \wedge \ldots \wedge \phi(x, a_{i_k}^j)$ is inconsistent, and also for all $f : \omega \to \omega$, $\bigwedge_{j<\omega} \phi(x, a_{f(j)}^j)$ is consistent.

We define the array $\langle a_i^j : i, j < \omega \rangle$ with the required properties as follows.

Let a_i^1 for each $i < \omega$ be the root of a rooted directed binary tree in N_0 such that the a_i^1 are independent. This gives $\bigwedge_{i<\omega} \phi(x, a_i^1)$ is inconsistent: the formulas are 2-inconsistent as $\mathrm{desc}(a_i^1) \cap \mathrm{desc}(x, a_j^1) = \emptyset$ for all $i \neq j$ by the definition of independent elements. This arrangement is shown in Figure 21.

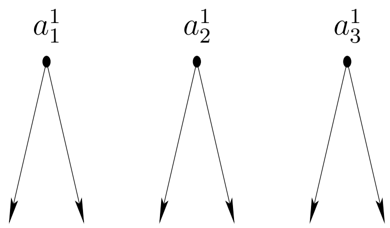

Figure 21: The first part of the array a_i^j

Let a_i^2 for each $i < \omega$ be the root of a rooted directed binary tree with $a_i^2 \notin \mathrm{desc}(a_k^1)$ for any k. Let $b_1^i, b_2^i, \ldots$ be a levelled independent set in $\mathrm{desc}(a_i^2)$ for each i. The tree $\mathrm{desc}(a_i^2)$ intersects $\mathrm{desc}(a_1^1)$ with generator b_1^i on level i of $\mathrm{desc}(a_1^1)$, $\mathrm{desc}(a_2^1)$ with generator b_2^i on level i of $\mathrm{desc}(a_2^1)$ and so on, such that the set of generators of the intersections of $\mathrm{desc}(a_i^2)$ and each $\mathrm{desc}(a_k^1)$ is a levelled independent set in $\mathrm{desc}(a_k^1)$. Notice that $\bigwedge_{i<\omega} \phi(x, a_i^2)$ is 2-inconsistent as there are no elements in $\mathrm{desc}(a_i^2) \cap \mathrm{desc}(a_j^2)$ for any $i \neq j$ by definition. Also $\mathrm{desc}(a_j^1) \cap \mathrm{desc}(a_k^2) \neq \emptyset$ for any j, k so $\bigwedge_{n<\omega, m=1,2} \phi(x, a_m^n)$ is consistent. This means that the required properties are satisfied up to this point. See Figure 22 for an example of this arrangement.

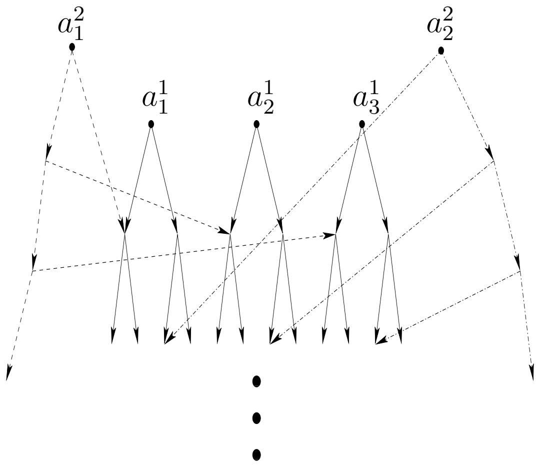

Figure 22: The second part of the array a_i^j

Let a_i^3 for each $i < \omega$ be the root of a rooted directed binary tree with $a_i^3 \notin \operatorname{desc}(a_k^j)$ for any k and $j < 3$. Let $c_1^i, c_2^i, \ldots$ be a levelled independent set in $\operatorname{desc}(a_i^3)$ for each i. The tree $\operatorname{desc}(a_i^3)$ intersects $\operatorname{desc}(a_1^1)$ with generator c_1^i on level i of $\operatorname{desc}(a_1^1) \cap \operatorname{desc}(a_1^2)$, $\operatorname{desc}(a_2^1)$ with generator c_2^i on level i of $\operatorname{desc}(a_2^1) \cap \operatorname{desc}(a_2^2)$, and so on, such that the set of generators of the intersections of $\operatorname{desc}(a_i^3)$ and each $\operatorname{desc}(a_k^1)$ is a levelled independent set in $\operatorname{desc}(a_k^1) \cap \operatorname{desc}(a_k^2)$. Again note that the required properties are satified up to this point since $\operatorname{desc}(a_i^3) \cap \operatorname{desc}(a_j^3) = \emptyset$ for all $i \neq j$, $\operatorname{desc}(a_j^1) \cap \operatorname{desc}(a_k^3) \neq \emptyset$ and $\operatorname{desc}(a_j^2) \cap \operatorname{desc}(a_k^3) \neq \emptyset$ for all j, k.

We can continue in this way defining the a_i^j so that for all j the intersection $\operatorname{desc}(a_i^j) \cap \operatorname{desc}(a_k^j) = \emptyset$ for all $i \neq k$, and with $\operatorname{desc}(a_i^j)$ intersecting every $\operatorname{desc}(a_k^l)$ whenever $i \neq k$. This means we have defined an array $\langle a_i^j : i, j < \omega \rangle$ such that for all $j < \omega$ and for all $i_1 < \ldots < i_k < \omega$, $\phi(x, a_{i_1}^j) \wedge \ldots \wedge \phi(x, a_{i_k}^j)$ is inconsistent, and also for all $f : \omega \to \omega$, $\bigwedge_{j<\omega} \phi(x, a_{f(j)}^j)$ is consistent as required. $\square$

3.4 Variation

As stated earlier, we would like to find a stable theory T, a model $M \models T$ and a type p such that the group of automorphisms induced on the set $p(M) = \{a \in M : M \models p(a)\}$ by $\mathrm{Aut}(M)$ is primitive with an unbalanced suborbit. Therefore we try to find a stable theory where we can find digraphs such as the ones constructed earlier on the set of realizations of some complete type of the theory. Note that we do not necessarily have the strict order property from the previous work since $p(M)$ may not be definable.

Definition 3.4.1. Let $\mathcal{D}_0$ be the class of finite digraphs (with digraph relation R) where each vertex has at most two direct descendants and which forbids directed cycles and subgraphs of the form $a_1 R a_2 R \dots R a_n, b_1 R b_2 R \dots R b_m$ where $a_1 = b_1$ and $a_n = b_m$ (i.e. we are forbidding cycles of the form shown in Figure 23).

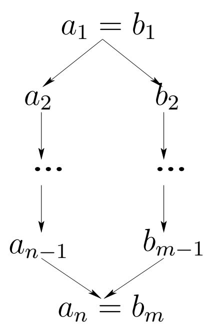

Figure 23: A forbidden cycle

Definition 3.4.2. Let $A \in \mathcal{D}_0$, $a \in A$ and $n \in \mathbb{N}$. An *n-path* from a is a sequence $a = a_0, a_1, \dots, a_n$ of elements of A where either

1. $a_i R a_{i+1}$, or

2. a_i is terminal (i.e. it has no descendants) in A and $a_{i+1} = a_i$.

In this case write $a_n \in \mathrm{desc}(a)^n$. Equivalently, $a_n \in \mathrm{desc}(a)^n$ if there is either a path of length n from a to a_n or a_n is a terminal vertex and there is a path of length m from a to a_n where $m \leq n$.

Definition 3.4.3. Define a binary relation $\sim$ for elements $a, b \in A \in \mathcal{D}_0$ by $a \sim b$ if for some n either $\mathrm{desc}(a)^n \subseteq \mathrm{desc}(a) \cap \mathrm{desc}(b)$ or $\mathrm{desc}(b)^n \subseteq \mathrm{desc}(a) \cap \mathrm{desc}(b)$.

Definition 3.4.4. For $X, A \in \mathcal{D}_0$ write $X \leq^+ A$ if X is descendant closed in A and for $a \in A$ if $\mathrm{desc}(a)^n \subseteq X$ then $a \in X$. For $a \in A$, $\mathrm{desc}(a) \leq^+ A$ means that if $a \sim b$ for some $b \in A$ then either $a \in \mathrm{desc}(b)$ or $b \in \mathrm{desc}(a)$.

Notation 3.4.5. This relation $\leq^+$ is similar to the relation defined in Chapter 2 hence we use the same symbol, however it is not identical since in this case we are dealing only with finite graphs and previously we were considering infinite graphs.

Definition 3.4.6. Define the class $(\mathcal{D}_1, \leq^+)$ as the set $\{A \in \mathcal{D}_0 : \mathrm{desc}(a) \leq^+ A$ for all $a \in A\}$.

We want to axiomatize the Fraïssé-type limit of the class $(\mathcal{D}_1, \leq^+)$ and have a full amalgamation property for the class. This would enable us to use a 'back and forth' argument to prove completeness of the axiomatization. The following lemma is a weaker version of the amalgamation property than is required, since in this case we need $X \leq^+ A$ and not just $X \subseteq A$. Then we explain that this amalgamation property is not enough to be able to axiomatize $Th(M)$ where M is the Fraïssé-type limit of $(\mathcal{D}_1, \leq^+)$.

Lemma 3.4.7. *Let $A, X, Y \in \mathcal{D}_1$. If $X \leq^+ Y$ and $X \leq^+ A$, then the free amalgam Z of Y and A over X is in $\mathcal{D}_1$ and $A, Y \leq^+ Z$.*

Proof. Firstly, as $A, X, Y \in \mathcal{D}_1$ and $Z = Y \coprod_X A$ we can see that $Z \in \mathcal{D}_0$

(no edges are added to those in A, Y when the free amalgam is taken so no forbidden cycles are created). So to ensure that $Z \in \mathcal{D}_1$ we need to take an element $z_1 \in Z$ and check that $\mathrm{desc}(z_1) \leq^+ Z$. For this take $z_2 \in Z$ such that $z_1 \sim z_2$. This gives us three cases.

Case 1 Suppose that $z_1, z_2 \in A$. In this case we see that either $z_1 \in \mathrm{desc}(z_2)$ or $z_2 \in \mathrm{desc}(z_1)$ trivially since $A \in \mathcal{D}_1$ and $X \leq^+ A$.

Case 2 Suppose that $z_1, z_2 \in Y$. Again it is trivial to see that $z_1 \in \mathrm{desc}(z_2)$ or $z_2 \in \mathrm{desc}(z_1)$ as $Y \in \mathcal{D}_1$ and $X \leq^+ Y$.

Case 3 Suppose, without loss of generality that $z_1 \in A \backslash X$ and $z_2 \in Y \backslash X$. Since $z_1 \sim z_2$ we know $\mathrm{desc}(z_i)^n \subseteq \mathrm{desc}(z_1) \cap \mathrm{desc}(z_2)$ for $i = 1$ or $i = 2$ and for some $n \in \mathbb{N}$. We also know that $\mathrm{desc}(z_1) \cap \mathrm{desc}(z_2) \subseteq X$. Due to these facts and due to $X \leq^+ A, Y$ we have $z_i \in X$ for $i = 1$ or $i = 2$, which means we are back to one of the two cases above.

Therefore we have seen that $Z \in \mathcal{D}_1$.

Now we must prove $A, Y \leq^+ Z$. Since the arrangement is symmetrical we only need to show this for A and then the proof for Y will be identical. We see that A is descendant closed in Z since $X \leq^+ A, Y$ and since $Z = A \coprod_X Y$. Let $z \in Z$ and suppose that $\mathrm{desc}(z)^n \subseteq A$ for some $n \in \mathbb{N}$. We need to show that if this is the case then we actually have $z \in A$. Therefore we need to take $z \in Y \backslash X$ for this to be non-trivial. As in the proof of the first part, $\mathrm{desc}(z)^n \subseteq A$ requires that $\mathrm{desc}(z)^n \subseteq X$. Then, as $X \leq^+ Y$ this means z has to be in X and so it is in A as required. $\square$

Using this lemma we can take a Fraïssé-type limit M_1 of $(\mathcal{D}_1, \leq^+)$ which has the $\leq^+$-extension property (that is, if $X \in \mathcal{D}_1$ and $X \leq^+ A$ then there is a $\leq^+$-embedding $f : A \to \mathcal{D}_1$). However, the condition $X \leq^+ A$ cannot be expressed in a first order way. Hence we cannot see how to axiomatize M_1. We would need to have $X \subseteq A$ and not be restricted to $X \leq^+ A$ in the amalgamation lemma above. However Lemma 3.4.7 with the condition $X \leq^+ A$ replaced by $X \subseteq A$ does not hold. We can see that this is the case by checking the conditions for Z to be in $\mathcal{D}_1$.

Let $a, b \in Z$ be such that $a \sim b$. We want to show that either $a \in \mathrm{desc}(b)$ or $b \in \mathrm{desc}(a)$. If we take $a, b \in Y \backslash X$ then we can get $a \not\sim b$ in Y, but $a \sim b$ in Z. Figure 24 shows an arrangement where this is the case (note that the dotted lines for some edges in the diagram have no significance - they are there only to add clarity). This means that it is impossible to force $a \in \mathrm{desc}(b)$ or $b \in \mathrm{desc}(a)$ as we need for $Z \in \mathcal{D}_1$.

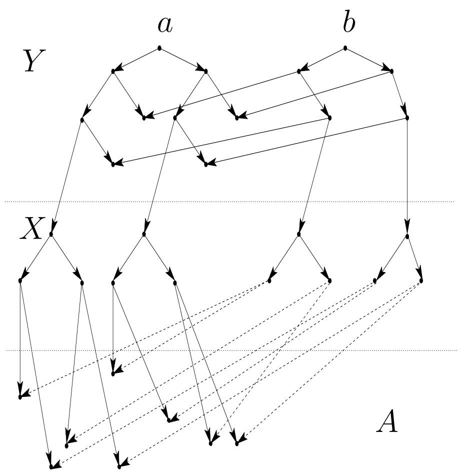

Figure 24: One case where $a \not\sim b$ in Y but $a \sim b$ in Z

Note that this situation does not occur if X is descendant closed in Z. However we can not restrict to this case for the same reason as we cannot restrict to $X \leq^+ A$. This problem will also occur in any other theory where we have

$X_1, X_2 \in X$ with $\mathrm{desc}(X_1)^n, \mathrm{desc}(X_2)^n \subseteq (\mathrm{desc}(X_1) \cap \mathrm{desc}(X_2)) \backslash X$.

Question 3.4.8. *Is it possible to use these methods to find a stable theory with the properties that we were looking for?*

Chapter 4

Directed Graphs and Hrushovski Constructions

In this chapter we describe the connection between a Hrushovski construction using a predimension (as set out in [18]) and general digraphs where each vertex has at most two out-vertices (note that these digraphs are allowed to have directed circuits unlike the ones in the preceeding chapters). We show that the Fraïssé-type limit of the Hrushovski class $(\mathcal{C}, \leq)$ and the reduct of the Fraïssé-type limit of the class $(\mathcal{D}, \sqsubseteq)$ of finite ≤ 2-oriented digraphs obtained by forgetting the directions on the edges are isomorphic. This is done in a more general form in [14]. We then define minimal, primitive and regular extensions which correspond roughly with simply algebraic and minimally simply algebraic extensions in [18]. With these definitions we define a new class of digraphs in which the number of primitive extensions is limited. We prove an amalgamation lemma for this class which is similar to Lemma 3 (Algebraic amalgamation) in [18]. Using this result we axiomatize the theory of these directed graphs. We then consider algebraic closure in these structures which gives us some information about forking and allows us to show that the theory is complete, stable and trivial. We finally look at the undirected reduct of the Fraïssé-type limit of this directed class and show that although it is a proper reduct, it is strictly stable.

4.1 Comparing a Hrushovski Construction and Directed Graphs

We begin this section by defining the predimension from which the Hrushovski class is obtained. We then define a class of finite digraphs and show that the reduct of the Fraïssé-type limit of this class and that of the Hrushovski class are isomorphic.

Definition 4.1.1. Take a language, $\mathcal{L}$ with a binary relation, E. Let T be the theory of graphs in this language. For a finite graph A, $|E[A]|$ is the number of edges in the graph A and $|A|$ is the number of vertices. Define a *predimension* for this theory to be $\delta(A) = 2|A| - |E[A]|$. Then the class $\mathcal{C}$ is the class of finite models A of T where $\delta(A') \geq 0$ for all $A' \subseteq A$.

Definition 4.1.2. If $D \in \mathcal{C}$ and $A \subseteq D$ write $A \leq D$ if $\delta(D') \geq \delta(A)$ whenever $A \subseteq D' \subseteq D$. Hence for all $D \in \mathcal{C}$ we have $\emptyset \leq D$.

The following lemma is essentially the same as ([18], Lemma 1) but the definition of δ that we are using is slightly different so we give the proof here.

Lemma 4.1.3. *Let $A \leq B \in \mathcal{C}$ be $\mathcal{L}$-structures and $X \subseteq B$. Then we have $\delta(A \cap X) \leq \delta(X)$.*

Proof. From $A \leq B$ and $A \subseteq A \cup X \subseteq B$ we have $\delta(A \cup X) \geq \delta(A)$. Now

$$\begin{aligned} \delta(A \cup X) &= 2|A \cup X| - |E[A \cup X]| \\ &\leq 2(|A| + |X| - |A \cap X|) - (|E[A]| + |E[X]| - |E[A \cap X]|). \end{aligned}$$

Putting this together with $\delta(A) = 2|A| - |E[A]|$ and $\delta(A \cup X) \geq \delta(A)$ gives

$$2(|A| + |X| - |A \cap X|) - |E[A]| - |E[X]| + |E[A \cap X]| \geq 2|A| - |E[A]|,$$

and then rearranging gives

$$2|X| - |E[X]| \geq 2|A \cap X| - |E[A \cap X]|,$$

which is

$$\delta(X) \geq \delta(A \cap X).$$

□

Definition 4.1.4. Let A be an undirected graph. A *2-orientation* of A is a directed graph A' with the same vertex and edge sets as A where the edges are given directions in such a way that each vertex in A' has at most two out-vertices. If such a 2-orientation of A exists then say that A can be *2-oriented*.

Lemma 4.1.5. *If A is a finite graph then $A \in \mathcal{C}$ if and only if there is a 2-orientation of A.*

Note that this means any graph in $\mathcal{C}$ can be represented as a digraph with at most two out-vertices for each vertex.

Proof. $\Leftarrow$ Assume that A can be 2-oriented. Then there is a directed graph A' with the same vertex and edge sets as A where the edges are given directions in such a way that each vertex has at most two out-vertices. So there are no more than two edges coming away from each vertex and since this is a finite graph this gives that there are at most $2|A|$ edges in the graph, hence $\delta(A) = 2|A| - |E[A]| \geq 0$. The same is true for all induced subgraphs of A, therefore $\delta(A') \geq 0$ for all $A' \subseteq A$.

$\Rightarrow$ Next we use Hall's Marriage Theorem to show that every graph in $\mathcal{C}$ can be 2-oriented. To use Hall's Marriage Theorem we construct a bipartite graph. For the first part take $E[A]$ (so the vertices in this part of the graph are the edges from the original graph). Then take two copies of the set of vertices of A, $V(A)$ and $V(A)'$ to form the second part of the bipartite graph needed. So the graph has $E[A]$ for one part and $V(A) \cup V(A)'$ for the other and has edges from vertex $\{v_i, v_j\}$ in $E[A]$ to vertices v_i and v_j in $V(A)$ and to vertices v_i' and v_j' in $V(A)'$. This arrangement is shown in Figure 25. If we can obtain a perfect matching for $E[A]$ in this graph then if the matching chooses v_i or v_i' to be joined to $\{v_i, v_j\}$ then make v_i the initial vertex of the directed edge (v_i, v_j) in the oriented graph. Since we have two copies of $V(A)$ each v_i could then be the initial vertex of at most two edges, and since the matching is perfect for $E[A]$ every edge will be directed. Thus we will be able to orient every graph in $\mathcal{C}$ as required.

Let $X \subseteq E[A]$ and check to see whether there are at least $|X|$ elements in $N(X)$, the set of vertices in the bipartite graph that are joined to elements of X. Define the graph $Y \subseteq A$ to be the induced subgraph of A with vertex set $\{v_i : \exists v_j \{v_i, v_j\} \in X\}$. Then $|N(X)| = 2|Y|$ and also $|X| \leq |E[Y]|$. As $Y \subseteq A$ and from the definition of $A \in \mathcal{C}$ we have $\delta(Y) = 2|Y| - |E[Y]| \geq 0$. Therefore we have

$$|X| \leq |E[Y]| \leq 2|Y| = |N(X)|,$$

and hence by Hall's Marriage Theorem we get the matching as required. $\square$

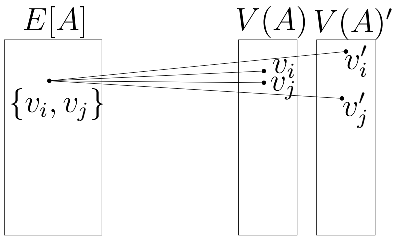

Figure 25: The bipartite graph used in Hall's Marriage Theorem in the proof of Lemma 4.1.5

Definition 4.1.6. Let $B \subseteq A$ be finite graphs which can be 2-oriented. Then we say there is a *2-orientation of A in which B is closed* if A has a 2-orientation A' in which there are no edges pointing out of B' (using the notation from the previous chapters, this is equivalent to $\mathrm{desc}^{A'}(B') = B'$). An example of this definition is shown in Figure 26).

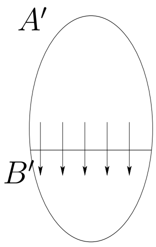

Figure 26: An orientation of A in which B is closed

The following lemma is a generalization of Lemma 4.1.5 and is from ([14], Lemma 1.5) with $r = 2$.

Lemma 4.1.7. *If A is a finite graph and $B \subseteq A$ then $\emptyset \leq B \leq A$ if and only if there is a 2-orientation of A in which B is closed.*

Proof. $\Leftarrow$ Suppose that there is a 2-orientation of A in which B is closed. So there is a 2-orientation of A and hence by Lemma 4.1.5 $A \in \mathcal{C}$ and so we have

$\delta(A) = 2|A| - |E[A]| \geq 0$. The existence of a 2-orientation of A in which B is closed means that there is a 2-orientation of B and so we have $\delta(B) \geq 0$ again by Lemma 4.1.5, which gives $\emptyset \leq B$ as stated in Definition 4.1.2. We see that $2|A \backslash B| \geq |E[A] \backslash E[B]|$ (each edge in $E[A] \backslash E[B]$ has at least one of its end vertices in $A \backslash B$) which gives $\delta(A) \geq \delta(B)$. The same argument works for any subgraph A' of A containing B (since A' also has a 2-orientation in which B is closed, taking the same orientation as for A) and hence we have $B \leq A$ as required.

$\Rightarrow$ For this direction we again use Hall's Marriage Theorem. Assume that A is a finite graph, $B \subseteq A$ and $\emptyset \leq B \leq A$. Take $E[A] \backslash E[B]$ for one part of the bipartite graph, and take two copies of $V(A) \backslash V(B)$ (called $V(A) \backslash V(B)$ and $(V(A) \backslash V(B))'$) for the other part. If $v_i, v_j \notin B$, put edges between $\{v_i, v_j\}$ in $E[A] \backslash E[B]$ and v_i, v_j in $V(A) \backslash V(B)$ and v_i', v_j' in $(V(A) \backslash V(B))'$. If $v_m \in B$ and $v_n \notin B$ then put edges between $\{v_m, v_n\}$ and v_n, v_n'. This then gives the bipartite graph that we use in Hall's Marriage Theorem and it is shown in Figure 27. Take $X \subseteq E[A] \backslash E[B]$ and define $Y \subseteq A$ to be the induced graph on the set of vertices $\{v_i : \exists v_j, \{v_i, v_j\} \in X\}$. Then $|N(X)| = 2|Y \backslash B|$ and $|X| \leq |E[Y] \backslash E[B]|$. We have assumed $B \leq A$ and so we have $\delta(B \cap Y) \leq \delta(Y)$ by Lemma 4.1.3, that is

$$2|B \cap Y| - |E[B \cap Y]| \leq 2|Y| - |E[Y]|.$$

We can split the vertices of Y into two sets so $|Y| = |Y \backslash B| + |Y \cap B|$. This and rearranging gives

$$\begin{aligned} 2|B \cap Y| - |E[B \cap Y]| &\leq 2|Y \backslash B| + 2|Y \cap B| - |E[Y]| \\ |E[Y]| - |E[B \cap Y]| &\leq 2|Y \backslash B|. \end{aligned}$$

It is clear that $|E[Y \backslash B]| \leq |E[Y]| - |E[Y \cap B]|$. Putting everything together then gives

$$|X| \leq |E[Y \backslash B]| \leq 2|Y \backslash B| = |N(X)|.$$

Thus Hall's Marriage Theorem gives a perfect matching for $E[A] \backslash E[B]$ in this graph. This gives a 2-orientation of $E[A] \backslash E[B]$ where $v_i \in B$ is not the first vertex of any edge. Since $\emptyset \leq B$ we can use Lemma 4.1.5 to give a 2-orientation of B, and putting this together with the orientation above we have a 2-orientation of A in which B is closed. $\square$

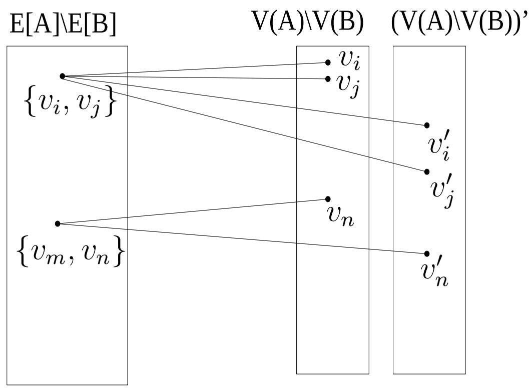

Figure 27: The bipartite graph used in Hall's Marriage Theorem in the proof of Lemma 4.1.7

Now we have shown that Hrushovski's class $(\mathcal{C}, \leq)$ arises by considering the rather more natural class of finite ≤ 2-out digraphs.

For the last part of this section we are going to consider the Fraïssé-type limit of two classes of graphs. The first of these classes is $(\mathcal{C}, \leq)$, as we considered above. The other is $(\mathcal{D}, \sqsubseteq)$, the class of finite ≤ 2-out digraphs (where each vertex has at most two out-vertices) without self loops where $A \sqsubseteq B$ means that A is closed in B. This definition of $\sqsubseteq$ is equivalent to that given in ([14], Definition 1.4). We will require the following amalgamation lemma for the class $(\mathcal{D}, \sqsubseteq)$.

Lemma 4.1.8. *Let $A, B_1, B_2 \in \mathcal{D}$ such that $A \subseteq B_1$ and $A \sqsubseteq B_2$. Then the free amalgam $C = B_1 \coprod_A B_2$ of B_1 and B_2 over A is in $\mathcal{D}$ and $B_1 \sqsubseteq C$.*

Proof. Since $B_1, B_2 \in \mathcal{D}$ and C is the free amalgam (i.e. there are no edges other than those fully contained in B_1 or fully contained in B_2) it is clear that C is a finite graph where each vertex has at most two out-vertices, that is $C \in \mathcal{D}$. From $A \sqsubseteq B_2$ and the definition of the free amalgam we get $B_1 \sqsubseteq C$, since if this was not the case then it would contradict $A \sqsubseteq B_2$. $\square$

By Lemma 4.1.7 we see that taking the reduct of the class $(\mathcal{D}, \sqsubseteq)$ where we forget the direction on the edges gives the class $(\mathcal{C}, \leq)$. We would therefore like to know whether the reduct of the Fraïssé-type limit of $(\mathcal{D}, \sqsubseteq)$ is isomorphic to the Fraïssé-type limit of $(\mathcal{C}, \leq)$.

Notation 4.1.9. Denote the Fraïssé-type limit of $(\mathcal{C}, \leq)$ by M and the Fraïssé-type limit of $(\mathcal{D}, \sqsubseteq)$ by N. Call the reduct of N obtained by forgetting the direction on the edges M'.

Remark 4.1.10. The Fraïssé-type limit N has directed cycles and each element has infinite in-degree and out-degree at most 2.

We are going to prove the following theorem.

Theorem 4.1.11. *M is isomorphic to M'*

To prove this we prove the two lemmas below.

Lemma 4.1.12. *Let $X \sqsubseteq Y \in \mathcal{D}$ and X' be a reorientation of X such that $X' \in \mathcal{D}$. If Y' is the result of replacing X by X' in Y then $Y' \in \mathcal{D}$ and $X' \sqsubseteq Y'$.*

Proof. If $X \sqsubseteq Y$ then X is closed in Y. Reorienting X does not affect the edges which come from Y into X. Therefore Y' is given a 2-orientation in which X', the reoriented X is closed by giving $Y' \backslash X'$ the same orientation as $Y \backslash X$. □

Definition 4.1.13. Let $A \subseteq B \in \mathcal{D}$. Then the *closure* of A in B is the set $\mathrm{cl}_B(A)$ of $b \in B$ such that b can be reached from some $a \in A$ by an outward directed path (so $\mathrm{cl}_B(A) = \mathrm{desc}_B(A)$ in the notation of Chapter 2).

Lemma 4.1.14. *If A is a finite subset of M' and $A \leq B \in \mathcal{C}$ then there is an embedding $f : B \to M'$ with $f(B) \leq M'$ and $f|A = id$, i.e. M' has the extension property.*

Proof. Take an orientation A' of A in $\mathcal{D}$ and let $A_1 = \mathrm{cl}_N(A')$, which is finite because A is finite and due to the definition of the Fraïssé-type limit N. As $A \leq B$ Lemma 4.1.12 then says this A' can be extended to an orientation B' of B in which A' is closed. Figure 28 shows this arrangement. The class $(\mathcal{D}, \sqsubseteq)$ has full amalgamation (as proved in Lemma 4.1.8), so the free amalgam C' of B' and A_1 over A' is in $\mathcal{D}$ and $A_1 \sqsubseteq C'$. Now use the $\sqsubseteq$-extension property in N to obtain a $\sqsubseteq$-embedding of B' into N. Finally, forget the directions on the edges to get the required result. □

Lemma 4.1.14 shows that M is isomorphic to M' (using uniqueness of the Fraïssé-type limit from Lemma 1.6.1) and so proves Theorem 4.1.11.

It has been shown, for example in [14], that the limit structure N is stable, but not superstable. It is also possible to check that this is a trivial theory (for the definition of a trivial theory see Definition 4.3.5) and that the reduct described here is non-trivial (see [9] where several model theoretic properties of these

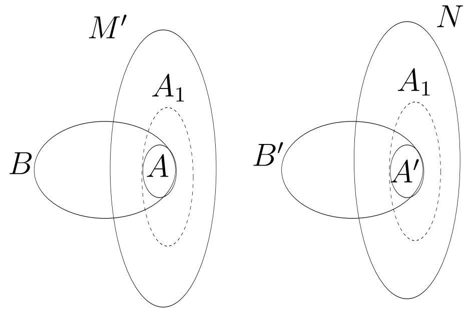

Figure 28: The arrangement for the proof of Lemma 4.1.14

theories are considered).

4.2 Limiting Primitive Extensions

We now define minimal, primitive and regular extensions and consider the class of directed graphs where the number of primitive extensions is limited. We then prove amalgamation lemmas and axiomatize the theory.

Definition 4.2.1. Let the class $(\bar{\mathcal{D}}, \sqsubseteq)$ be the class of possibly infinite ≤ 2-out digraphs. Let $A \subseteq B$ be directed graphs in $\bar{\mathcal{D}}$. Then A is *descendant closed* in B, $A \sqsubseteq B$ if $\mathrm{cl}_B(A) \subseteq A$ (with the definition of cl from Definition 4.1.13 in the previous section). Note that we had the same definition in Chapter 2 but here we are using different notation to fit in with other notation that we need in this section. A vertex a in A is called *full* in A if it has two out-vertices in A.

Definition 4.2.2. For $A \subseteq B \in \bar{\mathcal{D}}$ (possibly infinite), the extension $A \sqsubseteq B$ is *minimal* if $B \backslash A$ is finite and whenever $y \in B \backslash A$ then y is full in B and $B \backslash A \subseteq \mathrm{cl}_B(y)$ (an example is shown in Figure 29). The extension $A \sqsubseteq B$ is *primitive* if it is minimal and every $a \in A$ is an out-vertex of some vertex in $B \backslash A$ (an example is shown in Figure 30). The extension $A \sqsubseteq B$ is *regular* if there is no $B' \subseteq B$ with $A \sqsubset B'$ ($A \neq B'$) minimal.

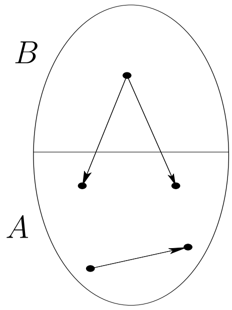

Figure 29: An example of a minimal extension

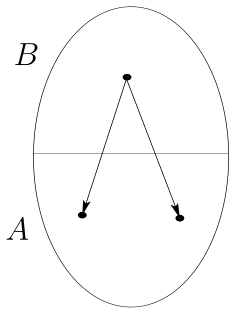

Figure 30: An example of a primitive extension

Remark 4.2.3. These definitions are motivated by Hrushovski's definition of a minimally simply algebraic extension in [18]. In fact it can be shown (see [14]) that if the extension $A \sqsubseteq B \in \mathcal{D}$ is minimal in the above sense then the undirected reduct $A^- \leq B^- \in \mathcal{C}$ is simply algebraic in Hrushovski's sense. Moreover, if $A \sqsubseteq B$ is primitive then $A^- \leq B^-$ is minimally simply algebraic. Thus in what follows, controlling the multiplicities of primitive extensions corresponds to imposing a bound μ on the multiplicity of minimally simply algebraic extensions in Hrushovski's construction of a strongly minimal set in [18].

Definition 4.2.4. Let $A \sqsubseteq B \in \mathcal{D}$ be a primitive extension and $\nu(B/A)$ be a function from isomorphism types of primitive extensions to the natural numbers. Let $\mathcal{D}_\nu$ be the class of finite directed graphs C in $\mathcal{D}$, such that if

$A_0 \sqsubseteq B_i \sqsubseteq C$ for $i = 1, \ldots r$ with $B_i \cap B_j = A_0$ for all $i \neq j$ and there is a primitive extension $A \sqsubseteq B$ and isomorphisms $f_i : B_i \to B$ with $f_i(A_0) = A$ then $r \leq \nu(B/A)$. This means that there are at most $\nu(B/A)$ primitive extensions isomorphic to each primitive extension $A \sqsubseteq B$ over A in $\mathcal{C}$. Let $\bar{\mathcal{D}}_\nu$ be the class of directed graphs as above, but allowing infinite graphs.

Remark 4.2.5. Note that if $A \sqsubseteq B$ is minimal then there is a unique $A_0 \subseteq A$ such that $A_0 \sqsubseteq A_0 \cup (B \backslash A)$ is primitive. Call this A_0 the *base* of $A \sqsubseteq B$. In this case, B is the free amalgam of $A_0 \cup (B \backslash A)$ and A over A_0. An example of the base of a minimal extension is shown in Figure 31.

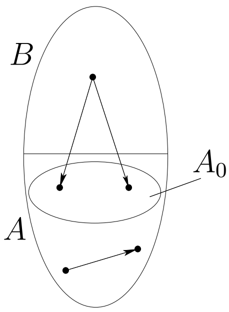

Figure 31: An example to illustrate the base of a minimal extension

Lemma 4.2.6. *Suppose $A \sqsubseteq B$ and $Y \subseteq B$ is such that $Y \cap A \sqsubseteq Y$ is primitive. Then $Y \cup A \sqsubseteq B$ and $Y \cup A$ is the free amalgam of A and Y over $Y \cap A$.*

Proof. By definition of primitivity, $y \in Y \backslash A$ is full in Y and hence $\mathrm{cl}_B(y) \subseteq Y \cup A$. The lemma is then immediate. □

Lemma 4.2.7 (Amalgamation Lemma)**.** *Suppose $C \in \bar{\mathcal{D}}_\nu$, $A \subseteq C$ is finite and $A \sqsubseteq B \in \bar{\mathcal{D}}_\nu$. Let F be the free amalgam $F = B \coprod_A C$. If F is not in $\bar{\mathcal{D}}_\nu$ then there is a primitive extension $X \sqsubseteq Y$ with $X \subseteq A$ and there are $r = \nu(Y/X) + 1$ copies $Y_1, \ldots, Y_r$ of Y over X inside F such that*

1. *$Y_1 \not\subseteq C$ and $Y_2 \not\subseteq B$, and*

2. $A \cup Y_1$ *is the free amalgam of* A *and* Y_1 *over* X *and* $A \cup Y_1 \sqsubseteq B$.

Proof. If F is not in $\bar{\mathcal{D}}\nu$ then by Lemma 4.1.8 there is a primitive extension $X \sqsubseteq Y$ with $X \subseteq F$ and there are $r = \nu(Y/X) + 1$ copies $Y_1, \ldots, Y_r$ of Y over X in F. Note that $C \sqsubseteq F$ because $A \sqsubseteq B$.

Claim $X \subseteq A$.

Proof. We prove this by eliminating the other possibilities for X.

Case 1 $X \cap (B \backslash A) \neq \emptyset$. Since $B \in \bar{\mathcal{D}}_\nu$ some Y_i is not completely contained in B. There can not be an element $y \in (Y_i \backslash X) \cap (B \backslash A)$ by the definition of a primitive extension: let $z \in Y_i \backslash B$ then $Y_i \backslash X \subseteq \mathrm{cl}^{\sqsubseteq}{}_{Y_i}(z)$ contradicts $A \sqsubseteq B$ as $y \in \mathrm{cl}^{\sqsubseteq}{}_{Y_i}(z)$. By the primitivity of the extension every $x \in X$ is an out-vertex of some $y \in Y_i \backslash X$. However we have just shown $Y_i \backslash X \subseteq C$ and this again contradicts $A \sqsubseteq B$. Therefore we can not have this case.

Case 2 $X \cap (C \backslash A) \neq \emptyset$. We see that there is some i such that $Y_i \nsubseteq C$ by $C \in \bar{\mathcal{D}}_\nu$. Due to the closure property of primitivity and $A \sqsubseteq B$, we have $(Y_i \backslash X) \cap A = \emptyset$ (because with $z \in (Y_i \backslash X) \cap A$ the primitivity condition $Y_i \backslash X \subseteq \mathrm{cl}^{\sqsubseteq}{}_{Y_i}(z)$, knowing that $Y_i \backslash C \neq \emptyset$ contradicts $A \sqsubseteq B$). We also have $(Y_i \backslash X) \cap (C \backslash A) = \emptyset$ because the closure condition of primitivity would require a relation between an element of $C \backslash A$ and an element of $B \backslash A$ which is forbidden by the definition of a free amalgam. By primitivity every $x \in X$ and in particular every $x \in X \cap (C \backslash A)$ is an out-vertex of some vertex in $Y_i \backslash X$. However this too gives a relation between an element in $B \backslash A$ and an element in $C \backslash A$ which is forbidden by the definition of a free amalgam. So this case is not valid either.

The only other option is that $X \subseteq A$ and since neither of the above cases can occur, we must have this situation.

□ Claim.

Therefore we now have that there is a primitive extension $X \subseteq Y$ in F with $X \subseteq A$ and r copies $Y_1 \ldots, Y_r$ of Y over X in F. Since $X \subseteq A$ it is clear that, without loss of generality $Y_1 \not\subseteq C$ and $Y_2 \not\subseteq B$. It follows that $Y_1 \subseteq B$ and $Y_2 \subseteq C$. Also $A \cup Y_1$ is the free amalgam of A and Y_1 over X and $A \cap Y_1 \sqsubseteq B$ by Lemma 4.2.6. □

Corollary 4.2.8. *Suppose $C \in \bar{\mathcal{D}}_\nu$, $A \subseteq C$ is finite and $A \sqsubseteq B \in \bar{\mathcal{D}}_\nu$ is a minimal extension with base X. Let F be the free amalgam $F = B \coprod_A C$. Then one of the following occurs:*

1. *$F \in \bar{\mathcal{D}}_\nu$,*

2. *there exists an embedding $\alpha : B \to C$ over A,*

3. *there is a primitive extension $X \sqsubseteq Y$ in F and $k = \nu(Y/X)$ copies $Y_1, \ldots, Y_k$ of Y over X in C such that for each $i \leq k$ either*

    (a) there is $a \in A$ with some element of $Y_i \backslash X$ as one of its out-vertices, or

    (b) $(Y_i \backslash X) \cap A \neq \emptyset$.

Proof. Assume that $F \notin \bar{\mathcal{D}}_\nu$. Then by Lemma 4.2.7 there is a primitive extension $W \sqsubseteq Z$ with $W \subseteq A$ and $r = \nu(Z/W)+1$ copies $Z_1, \ldots, Z_r$ of Z over W in F such that $Z_1 \not\subseteq C$, $Z_2 \not\subseteq B$, $A \cup Z_1$ is the free amalgam of A and Z_1 over W and $A \cup Z_1 \sqsubseteq B$. By the minimality of $A \sqsubseteq B$ and the uniqueness of the base X we have $W = X$ and $Z = Z_1 = X \cup (B \backslash A)$. We now let $Y_i = Z_{i+1}$ for $i \leq r$

and $Y_{k+1} = Z_1$ (note this is simply renaming) which gives $\nu(Z/W) = \nu(Y/X)$ and $r = k + 1$. So we have $\nu(Y/X) + 1$ copies $Y_1, \ldots, Y_{k+1}$ of the primitive extension $X \sqsubseteq Y$ over X in F. Since $Y = Y_{k+1} = X \cup (B \backslash A)$ we have $\nu(Y/X)$ copies $Y_1, \ldots Y_k$ of the primitive extension $X \sqsubseteq Y$ over X in C. Now assume that 3 does not hold, hence there is an $i \leq k$ such that both (a) and (b) do not happen. So we have some i with $(Y_i \backslash X) \cap A = \emptyset$ and $A \cup Y_i = A \coprod_X Y_i$. Therefore $A \cup Y_i$ and B are isomorphic over A. Thus there is an embedding $\alpha : B \to C$ over A, that is 2 holds. □

Note 4.2.9. If $A \sqsubseteq B$ from the above corollary is a primitive extension (that is $X = A$) then Conditions 3(a) and 3(b) do not hold since $A \sqsubseteq Y$. Therefore if $A \sqsubseteq B$ is a primitive extension then either the free amalgam $F \in \bar{\mathcal{D}}_\nu$ or there exists an embedding $\alpha : B \to C$ over A.

Corollary 4.2.10. *Suppose $C \in \bar{\mathcal{D}}_\nu$, $A \subseteq C$ is finite and $A \sqsubseteq B \in \bar{\mathcal{D}}_\nu$ is regular. Let F be the free amalgam $F = B \coprod_A C$. Then $F \in \bar{\mathcal{D}}_\nu$.*

Proof. Since $A \sqsubseteq B$ is regular there does not exist $B' \subseteq B$ with $A \sqsubset B'$ ($A \neq B'$) minimal. Therefore there is no primitive extension $X \sqsubseteq Y$ in F with $X \subseteq A$ and $Y \not\subseteq C$. Hence Lemma 4.2.7 gives $F \in \bar{\mathcal{D}}_\nu$. □

Corollary 4.2.11. *The class $(\mathcal{D}_\nu, \sqsubseteq)$ has the amalgamation property.*

Proof. We need to show that for $A, B, C \in \mathcal{D}_\nu$ if $A \sqsubseteq B$ and $A \sqsubseteq C$ then we can amalgamate to a structure $E \in \mathcal{D}_\nu$ with $B, C \sqsubseteq E$. Arguing by induction on $|B \backslash A|$, we may assume that the extension $A \sqsubseteq B$ is either minimal or regular. If it is regular then Corollary 4.2.10 shows that we can take E to be the free amalgam of B and C over A. So now suppose that $A \sqsubseteq B$ is minimal and the free amalgam F is not in $\mathcal{D}_\nu$. As $A \sqsubseteq C$, Case 3 in Corollary 4.2.8

does not occur and so Case 2 of Corollary 4.2.8 must occur. This gives what we need. $\square$

Thus $(\mathcal{D}_\nu, \sqsubseteq)$ is an amalgamation class in the sense of Section 1.6 and we denote by $\mathcal{N}_\nu$ the Fraïssé-type limit of this class (as in Theorem 1.6.1). We would now like to axiomatize the theory $Th(\mathcal{N}_\nu)$.

Definition 4.2.12. For a set A of vertices in a directed graph let $A^{\rightarrow}$ be the set of immediate out-vertices of A, that is the set of elements reached from any element in A by a directed edge.

In the following $\Delta_A(\bar{x})$ denotes the basic diagram of A (that is the quantifier free formula which specifies that the $\bar{x}$ have the same isomorphism type as A) and $\Delta_{A,B}(\bar{x}, \bar{y})$ is the basic diagram of B with the variables arranged so that $\Delta_{A,B}(\bar{x}, \bar{y})$ implies $\Delta_A(\bar{x})$.

Suppose $A \sqsubseteq B$ is a finite minimal extension which has base $X \subseteq A$. Then let $Y = X \cup (B \backslash A)$ and $r = \nu(Y \backslash X)$ and define $\theta_{A,B}$ (using Corollary 4.2.8) as follows

$$\forall \bar{a} \Delta_A(\bar{a}) \rightarrow \exists \bar{y} \Delta_{A,B}(\bar{a}\bar{y}) \vee \left(\exists \bar{x} \Delta_X(\bar{x}) \wedge \exists \bar{y}_1, \ldots, \bar{y}_r \Delta_{X,Y}(\bar{x}\bar{y}_i) \wedge \bigwedge_i \varphi(\bar{a}\bar{y}_i) \right)$$

where $\varphi(\bar{a}\bar{y}_i)$ says that there is an element of the tuple $\bar{y}_i$ which is in A or there is an element of the tuple $\bar{y}_i$ which is in $A^{\rightarrow}$ (as in Case 3 of Corollary 4.2.8).

For $A \sqsubseteq B$ a finite regular extension define $\tau_{A,B}$ (using Corollary 4.2.10) to be

$$\forall \bar{a} \Delta_A(\bar{a}) \rightarrow \exists \bar{y} \Delta_{A,B}(\bar{a}\bar{y}) \wedge \phi(\bar{a}\bar{y})$$

where ϕ says that out-vertices of $\bar{y}$ are only in $\bar{a}\bar{y}$, which can be written in a first order way.

Let Σ be the collection of all $\theta_{A,B}$ and $\tau_{A,B}$.

Lemma 4.2.13. *We have* $\mathcal{N}_\nu \models \Sigma$.

Proof. Let $A \subseteq \mathcal{N}_\nu$ be finite, $A \sqsubseteq B \in \mathcal{D}_\nu$ and $C = \mathrm{cl}^{\sqsubseteq}(A)$. Note that C is finite. To show that $\mathcal{N}_\nu \models \Sigma$ it is enough to consider $A \sqsubseteq B$ being either a minimal extension or a regular extension and proving that $\mathcal{N}_\nu \models \theta_{A,B}$ or $\mathcal{N}_\nu \models \tau_{A,B}$ in the respective cases.

Case 1 $A \sqsubseteq B$ is a minimal extension. Let the extension have base X and let $Y = X \cup (B \backslash A)$. Assume that $\bar{a}$ in $\mathcal{N}_\nu$ is such that $\Delta_A(\bar{a})$ holds. By Corollary 4.2.8 we have that either the free amalgam F (of B and C over A) is in $\mathcal{D}_\nu$, or there is an embedding from B into C over A, or there are $\nu(Y/X)$ copies $Y_1, \ldots, Y_r$ of Y over X in C such that for each i one of two conditions hold. In the first case ($F \in \bar{\mathcal{D}}_\nu$) we have $C \sqsubseteq F$ (by definition of the free amalgam) and $C \sqsubseteq \mathcal{N}_\nu$ (since C is the closure of A in $\mathcal{N}_\nu$) so we can use the extension property of the Fraïssé-type limit to get an embedding of F into $\mathcal{N}_\nu$. This clearly gives us an embedding of B into $\mathcal{N}_\nu$ over A. In the second case we have an embedding of B into $C \in \mathcal{N}_\nu$ and so we have an embedding of B into $\mathcal{N}_\nu$ over A. Thus, in either of the first two cases we have an embedding of B into $\mathcal{N}_\nu$ over A, that is $\Delta_{A,B}(\bar{a}\bar{y})$ holds for some $\bar{y} \in \mathcal{N}_\nu$ and therefore $\mathcal{N}_\nu \models \theta_{A,B}$. In the third case we have $Y_1, \ldots, Y_r$ such that for each i either some $a \in A$ has an element of $Y_i \backslash X$ as one of its out-vertices or $(Y_i \backslash X) \cap A \neq \emptyset$. This means we have the second part of the conclusion to $\theta_{A,B}$ and so again $\mathcal{N}_\nu \models \theta_{A,B}$.

Case 2 $A \sqsubseteq B$ is a regular extension. In this case, Corollary 4.2.10 gives that the free amalgam of B and C over A is in $\bar{\mathcal{D}}_\nu$. As in a part of the previous case, we have $C \sqsubseteq F$ and $C \in \mathcal{N}_\nu$ so using the extension property of the Fraïssé-type limit gives an embedding of F into $\mathcal{N}_\nu$ such that $F \sqsubseteq \mathcal{N}_\nu$. In particular there is an embedding of B into $\mathcal{N}_\nu$ over A, so there is therefore some $\bar{y} \in \mathcal{N}_\nu$ such that $\Delta_{A,B}(\bar{a}\bar{y})$ holds. We also have to check that $\phi(\bar{a}\bar{y})$ holds, that means out-vertices of $\bar{y}$ are only in $\bar{a}\bar{y}$. Since we have $F \sqsubseteq \mathcal{N}_\nu$ and F is a free amalgam (so there are no edges between $B\backslash A$ and $C\backslash A$) it is clear that out-vertices of elements of $B\backslash A$ are within B. That gives $\phi(\bar{a}\bar{y})$ as required. Hence $\mathcal{N}_\nu \models \tau_{A,B}$. □

Definition 4.2.14. For a vertex x in a directed graph D let $x^{\Rightarrow}$ be the descendants of x in D, that is the set of vertices in D that can be reached from x by an outward directed path. Let $x^{\Rightarrow_n}$ for $n \in \mathbb{N}$ be the set of vertices in D that can be reached from x by an outward directed path of length at most n.

Notation 4.2.15. Let T be the theory of ≤ 2-oriented digraphs in the language L which has a binary relation symbol $R(x, y)$ for 'there is a directed edge from x to y'. Let T_ν be T together with the axioms Σ.

Lemma 4.2.16. *Let $N \models T_\nu$ be ω_1-saturated (N is not required to have cardinality $\aleph_1$). If $C \sqsubseteq N$ is finitely generated and $C \sqsubseteq D \in \bar{\mathcal{D}}_\nu$ is finitely generated then there is a $\sqsubseteq$-embedding $f : D \to N$ over C.*

Proof. Without loss of generality, we may assume $D = d^{\Rightarrow} \cup C$ for some d. For every $n \in \mathbb{N}$ let $D_n = d^{\Rightarrow_n}$ and $C_n = D_n \cap C$. We first prove by induction on $|D_n\backslash C_n|$ that we have an embedding $f : D_n \to N$ over C_n with $(C \cup f(D_n)) \sqsubseteq N$ and $f(D_n) \cap C = C_n$. The base case $|D_n\backslash C_n| = 0$ is trivial.

For the inductive step, first consider the case of $C_n \sqsubseteq D_n$ being a regular extension. Let $\bar{c}$ be the generators of C and let A be $\bar{c}^{\Rightarrow_m}$ for some m such that $C_n \subseteq A$. Let $B = D_n \coprod_{C_n} A$. Then $A \sqsubseteq B$ is a regular extension (if it were not regular then $C_n \sqsubseteq D_n$ could not be a regular extension). By the axiom $\tau_{A,B}$ there is an embedding $g : B \to N$ over A such that out-vertices of $g(B)\backslash C_n$ are only in $g(B) \cup C_n$. This gives $\mathrm{cl}^{\sqsubseteq}(g(B)) = \mathrm{cl}^{\sqsubseteq}(A) \cup g(B) = C \cup g(B)$. We can thus find an embedding $f : D_n \to N$ over C_n with $\mathrm{cl}^{\sqsubseteq}(f(D_n)) = \mathrm{cl}^{\sqsubseteq}(C_n) \cup f(D_n)$ (which means $(C \cup f(D_n)) \sqsubseteq N$) and $f(D_n) \cap C = C_n$, as required.

Now if $C_n \sqsubseteq D_n$ is not a regular extension then there is some $D'_n \subseteq D_n$ such that $C_n \sqsubseteq D'_n$ is a minimal extension. By the inductive hypothesis we may assume that $D'_n = D_n$. Let $X \subseteq C_n$ be the base of this extension, so $X \sqsubseteq X \cup (D_n \backslash C_n) = Y$ is primitive. Let $A \subseteq C$ be such that any copy Y_i ($i \leq \nu(Y/X)$) of Y over X in C is contained in A. By minimality, $Y \backslash X$ is finite and so each copy Y_i is finite. Hence we can choose A to be finite. Let $B = D_n \coprod_{C_n} A$. Then $A \sqsubseteq B$ is a minimal extension because $C_n \sqsubseteq D_n$ is a minimal extension. There is then some $X' \subseteq A$ such that $X' \sqsubseteq X' \cup (B \backslash A)$ is a primitive extension, and since this base is unique $X' = X$. Since $N \models \Sigma$ we use the axiom $\theta_{A,B}$ to give that either

1. there is an embedding of B into N over A, or
2. there are $r = \nu(Y'/X')$ copies $Y'_1, \ldots, Y'_r$ of Y' over X' in N such that for each i
   (a) $Y'_i \backslash X' \cap A \neq \emptyset$ or
   (b) $A^{\to} \cap (Y'_i \backslash X') \neq \emptyset$.

Claim Case 2 cannot occur.

To see this assume that there are $Y'_1, \ldots, Y'_r$ copies of Y' over X' in N. Since

$C \in \bar{\mathcal{D}}_\nu$ there must be at least one of these copies of Y' not contained completely in C. Let $Y'_1, \ldots, Y'_k$ (for $k < r$) be the copies of Y' contained completely in C (and so in A), and $Y'_{k+1}, \ldots, Y'_r$ be the other copies. Since $C \sqsubseteq N$ and $Y'_i \backslash X' \subseteq \mathrm{cl}^{\sqsubseteq}(y'_i)$ for any $y'_i \in Y'_i \backslash X$ by minimality, we get that for any $i \in \{k+1, \ldots, r\}$ the intersection $(Y'_i \backslash X') \cap C$ is empty (hence $(Y'_i \backslash X') \cap A = \emptyset$). Also $C \sqsubseteq N$ shows that $A^{\rightarrow} \cap (Y'_i \backslash X') = \emptyset$ for any $i \in \{k+1, \ldots, r\}$. Thus (a) and (b) do not hold for any of the copies $Y'_{k+1}, \ldots, Y'_r$ and hence case 2 cannot occur.

Therefore case 1 must occur and there is an embedding $h : B \rightarrow N$ over A. Since $A \sqsubseteq B$ is a minimal extension, the image $h(B)$ of B in N must also be a minimal extension of A. Therefore $\mathrm{cl}^{\sqsubseteq}(h(B)) = h(B) \cup \mathrm{cl}^{\sqsubseteq}(A) = h(B) \cup C$. We also have $h(B) \cap C = A$ since if this were not the case there would be some $x \in B \backslash A$ with $h(x) \in C$. This would then force $h(B) \subseteq C$ as $C \sqsubseteq N$ and $B \backslash A \subseteq \mathrm{cl}^{\sqsubseteq}(x)$. However, that would contradict the choice of A. Hence there is an embedding $f : D_n \rightarrow N$ over C_n with $\mathrm{cl}^{\sqsubseteq}(f(D_n)) = \mathrm{cl}^{\sqsubseteq}(C_n) \cup f(D_n)$ (which means that $(C \cup f(D_n)) \sqsubseteq N$) and $f(D_n) \cap C = C_n$.

We therefore have an embedding f from D_n into N over C_n for every n such that $(C \cup f(D_n)) \sqsubseteq N$ and $f(D_n) \cap C = C_n$.

Using the saturation of N and compactness we see that we have a $\sqsubseteq$-embedding from D into N over C. □

Finally in the axiomatization of $Th(\mathcal{N}_\nu)$ we look at types in this theory and use this to show that the theory is complete.

Lemma 4.2.17. *Let $M, N \models T_\nu$ and let $\bar{a}$ and $\bar{b}$ be n-tuples in M and N respectively. Then $\bar{a}$ and $\bar{b}$ have the same type if and only if the map $\bar{a} \mapsto \bar{b}$*

extends to an isomorphism between $\mathrm{cl}^{\sqsubseteq}{}_M(\bar{a})$ *and* $\mathrm{cl}^{\sqsubseteq}{}_N(\bar{b})$.

Proof. If the types of $\bar{a}$ and $\bar{b}$ are the same then there is clearly an isomorphism between their closures. For the converse, it is enough to show that if M, N are ω_1-saturated models of T_ν then the set of isomorphisms between closures of finite subsets of M, N is a back and forth system. This follows from Lemma 4.2.16. Let S be the set of partial isomorphisms between finitely generated substructures of M and N, let $\bar{a}, \bar{b}$ generate isomorphic substructures and f : $\mathrm{cl}^{\sqsubseteq}(\bar{a}) \rightarrow \mathrm{cl}^{\sqsubseteq}(\bar{b})$ be an isomorphism. Thus $f \in S$. Take $c \in M$ and let $A = \mathrm{cl}^{\sqsubseteq}{}_M(\bar{a})$ and $B = \mathrm{cl}^{\sqsubseteq}{}_N(\bar{b})$. So f gives a $\sqsubseteq$-embedding from B into $A \cup \mathrm{cl}^{\sqsubseteq}{}_M(c)$. Hence, by Lemma 4.2.16 we can find a $d \in N$ such that $A \cup \mathrm{cl}^{\sqsubseteq}{}_M(c)$ is isomorphic to $B \cup \mathrm{cl}^{\sqsubseteq}{}_N(d)$, extending the isomorphism f. This completes the 'forth' direction, and the 'back' direction is similar. ☐

Lemma 4.2.18. *The theory* T_ν *is complete.*

Proof. This follows from Lemma 4.2.17 with $\bar{a}, \bar{b}$ empty tuples. ☐

Theorem 4.2.19. *The theory* T_ν *axiomatizes* $Th(\mathcal{N}_\nu)$.

Proof. This follows from Lemmas 4.2.13, 4.2.16 and 4.2.18. ☐

4.3 Stability and Triviality

We now consider two further model theoretic properties, showing that the theory T_ν defined above is stable and trivial.

Theorem 4.3.1. *The theory* T_ν *is stable.*

Proof. Suppose N is a highly saturated model of T_ν. If $B \sqsubseteq N$ is small (which means of cardinality less than the degree of saturation) of size λ and $\bar{a}$ is a tuple of elements of N then, by the above lemma $\mathrm{tp}(\bar{a}/B)$ is determined by $\mathrm{cl}^{\sqsubseteq}(\bar{a}B)$. However $\mathrm{cl}^{\sqsubseteq}(\bar{a}B)$ is the free amalgam of $\mathrm{cl}^{\sqsubseteq}(\bar{a})$ and B over their intersection. Note that $\mathrm{cl}^{\sqsubseteq}(\bar{a})$ is countable and so the number of possibilities for the intersection is $|B|^{\aleph_0}$. The number of possibilities for the isomorphism type of $\mathrm{cl}^{\sqsubseteq}(\bar{a})$ over $\mathrm{cl}^{\sqsubseteq}(\bar{a}) \cap B$ is at most $2^{\aleph_0}$. Hence the number of n-types over B is at most $max(2^{\aleph_0}, |B|^{\aleph_0}) = |B|^{\aleph_0} = \lambda^{\aleph_0}$. Thus the theory is λ-stable for $\lambda = \lambda^{\aleph_0}$, which means it is stable. $\square$

We now give a description of algebraic closure in a model $N \models T_\nu$ and then look briefly at forking in the theory T_ν.

We may assume that the model $N \models T_\nu$ is sufficiently saturated. It is clear that if $X \subseteq N$ then $\mathrm{acl}(X) \sqsubseteq N$. Suppose $A \sqsubseteq N$ and $B \subseteq N$ is such that $A \cap B \sqsubseteq B$ is a primitive extension. Then $A \cup B \sqsubseteq N$ (by the definition of primitivity) and $B \subseteq \mathrm{acl}(A)$ (by the ν-function). Let $\tilde{A}$ be the closure of A under the operation of taking primitive extensions. We claim that $\tilde{A} = \mathrm{acl}(A)$. The previous observation gives that $\tilde{A} \subseteq \mathrm{acl}(A)$. On the other hand, if $b \in N \backslash \tilde{A}$ and $B = \mathrm{cl}(\tilde{A} \cup \{b\}) = \tilde{A} \cup \mathrm{cl}(b)$, then $\tilde{A} \sqsubseteq B$ is a regular extension and it follows (by free amalgamation as in Lemma 4.3.2 below) that there are infinitely many copies of B over $\tilde{A}$ in N. So, in particular $b \notin \mathrm{acl}(\tilde{A})$. This establishes the claim.

Lemma 4.3.2. *Let* $N \models T_\nu$, $A \sqsubseteq B \sqsubseteq N$ *and* $c \in N$. *Suppose that we have* $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$. *Then* $\mathrm{tp}(c/\mathrm{acl}(B))$ *does not divide over* $\mathrm{acl}(A)$.

Proof. Let $\tilde{A} = \mathrm{acl}(A)$, $\tilde{B} = \mathrm{acl}(B)$ and $\tilde{C} = \mathrm{acl}(cA)$. Note that for any set

X of N we have $\mathrm{cl}^{\sqsubseteq}(X) \subseteq acl(X)$ and so these sets ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$) are descendant closed in N. By the hypothesis $\tilde{C}$ and $\tilde{B}$ are freely amalgamated over $\tilde{A}$. Since $\tilde{C}, \tilde{B} \sqsubseteq N$ we get $\tilde{C} \cup \tilde{B} \sqsubseteq N$. Now suppose $(\tilde{B}_i : i < \omega)$ is a sequence of translates of $\tilde{B}$ over $\tilde{A}$ and let $X = \mathrm{acl}(\bigcup \tilde{B}_i)$. Since $\tilde{A}$ is $\sqsubseteq$-embedded in X and also in $\tilde{C}$ we can form the free amalgam F of $\tilde{C}$ and X over $\tilde{A}$ with $\sqsubseteq$-embeddings $f : \tilde{C} \to F$ and $g : X \to F$. As $\tilde{A}$ is algebraically closed in $\tilde{C}$, $\tilde{A} \sqsubseteq \tilde{C}$ is a regular extension. Hence we get that $F \in \bar{\mathcal{D}}_\nu$ by Lemma 4.2.7 and we can assume that N is sufficiently saturated so that $F \sqsubseteq N$. Let $f(\tilde{C}) = \tilde{C}'$ and note that by construction we have $\mathrm{tp}(\tilde{C}'/\tilde{A}) = \mathrm{tp}(\tilde{C}/\tilde{A})$. Also by the construction we have that for every i, $\tilde{C}' \cup \tilde{B}_i \sqsubseteq F$, hence $\tilde{C}' \cup \tilde{B}_i \sqsubseteq N$ and $\tilde{C}' \cup \tilde{B}_i$ is the free amalgam of $\tilde{C}'$ and $\tilde{B}_i$ over $\tilde{A}$. Lemma 4.2.17 says that types are determined purely by descendant closure and hence we see $\mathrm{tp}(\tilde{C}'\tilde{B}_i) = \mathrm{tp}(\tilde{C}\tilde{B})$ for all i. This gives $\mathrm{tp}(c/\mathrm{acl}(B))$ does not divide over $\mathrm{acl}(A)$. ☐

Corollary 4.3.3. *Let $A \sqsubseteq B \sqsubseteq N \models T_\nu$ and $c \in N$. Then $c \downarrow_A B$ if and only if* $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$.

Proof. $\Rightarrow$ Since N is a stable structure, this is given by Corollary 1.7.5.

$\Leftarrow$ Since the theory is stable, forking and dividing coincide and hence this is given by Lemma 4.3.2 above. ☐

We note the following additional property of algebraic closure.

Lemma 4.3.4. *For $A, B \sqsubseteq N \models T_\nu$ we have* $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \mathrm{acl}(A \cap B)$.

Proof. Firstly, it is clear that $\mathrm{acl}(A \cap B) \subseteq \mathrm{acl}(A) \cap \mathrm{acl}(B)$. So consider some $x \in \mathrm{acl}(A) \cap \mathrm{acl}(B)$, $x \notin A \cap B$ (if $x \in A \cap B$ then $x \in \mathrm{acl}(A \cap B)$ trivially). If, without loss of generality $x \in A \setminus B$ then $x \in \mathrm{acl}(B)$ gives a sequence of minimal extensions $B \sqsubseteq B_1$, $B_1 \sqsubseteq B_2$, ..., $B_{n-1} \sqsubseteq B_n$ such that $x \in B_n$ where n is the least such. Note that $B \sqsubseteq N$ so $x \notin \mathrm{cl}^{\sqsubseteq}(B)$ as $x \notin B$. We defined B_n so that $x \in B_n$ and $x \notin B_{n-1}$. The minimality of this extension thus gives $B_n \backslash B_{n-1} \subseteq \mathrm{cl}^{\sqsubseteq}{}_{B_n}(x)$ which together with $x \in A$ and $A \sqsubseteq N$ gives $B_n \backslash A = \emptyset$. This means that $B_j \subseteq A$ for all j and hence the sequence of minimal extensions is over $A \cap B$, that is $x \in \mathrm{acl}(A \cap B)$.

So we can now assume that $x \notin A$ and $x \notin B$. Hence there are sequences of minimal extensions $A \sqsubseteq A_1$, $A_1 \sqsubseteq A_2$, ..., $A_{n-1} \sqsubseteq A_n$ and $B \sqsubseteq B_1$, $B_1 \sqsubseteq B_2$, ..., $B_{m-1} \sqsubseteq B_m$ such that $x \in A_n$ and $x \in B_m$ and n, m are the least such. By the fullness property of minimal extensions, each of the two out-vertices of x is in A_n and is also in B_m. More specifically, each is in $A_n \backslash A_{n-1}$, $A_i \backslash A_{i-1}$ for some $i < n$ or A, and is in $B_m \backslash B_{m-1}$, $B_j \backslash B_{j-1}$ for some $j < m$ or B. Let us now consider the possibilities for one of these out-vertices, call it y within A_n.

If $y \in A_n \backslash A_{n-1}$ then we can repeat the above conditions for the two out-vertices of y due to the minimality of the extension. By definition of minimal extensions $A_n \backslash A_{n-1}$ is finite and so we can find a descendant of x which is not in $A_n \backslash A_{n-1}$. Thus we can assume without loss of generality that $y \notin A_n \backslash A_{n-1}$.

Now consider the possibility of $y \in A_i \backslash A_{i-1}$ for some $i < n$. Repeating the argument from above gives that we will eventually (possibly after passing through several of the A_i) come to a descendant of x which lies in A. Therefore let z be a descendant of x in A such that it is the out-vertex of an element which is not in A. We can assume that $z \notin A \cap B$ (if there is no such z then we have $\mathrm{cl}^{\sqsubseteq}(x) \cap A \subseteq A \cap B$, meaning that the sequence of minimal extensions $A \sqsubseteq A_1$, $A_1 \sqsubseteq A_2$, ..., $A_{n-1} \sqsubseteq A_n$ is over $A \cap B$, hence $x \in \mathrm{acl}(A \cap B)$ and we are done). We now consider this z with regards to the B_j extensions. Since

z is a descendant of x and the extensions are all minimal the fullness condition means we must have $z \in B$ or $z \in B_j \backslash B$ for some $j \leq m$. We have assumed that $z \notin B$ so we have $z \in B_j \backslash B$ for some $j \leq m$, which we shall say is the least such j. Using the closure property of the minimality definition we see $B_j \backslash B_{j-1} \subseteq \mathrm{cl}^{\sqsubseteq}(z)$. Recalling $A \sqsubseteq N$ then gives $B_j \backslash B_{j-1} \subseteq \mathrm{cl}^{\sqsubseteq}(z) \subseteq A$ and hence $B_j \subseteq A$. Since B_j is an extension over B contained in A it must be an extension over $A \cap B$. Thus the whole chain of B_k extensions is over $A \cap B$ giving $x \in \mathrm{acl}(A \cap B)$ as required. □

Note that in general $\mathrm{acl}(A) \cup \mathrm{acl}(B) = \mathrm{acl}(A \cup B)$ is not true for all $A, B \sqsubseteq N$. For example if A and B are disjoint sets then there could be a primitive extension over $A \cup B$ consisting of a single vertex with an edge to some vertex in A and an edge to some vertex in B. This is not a primitive extension over either A or B, hence $\mathrm{acl}(A) \cup \mathrm{acl}(B) \neq \mathrm{acl}(A \cup B)$ in this case.

Definition 4.3.5. A complete stable theory T is *trivial* if whenever a, b, c are tuples of elements from a model of T and A is a set of parameters, then a, b, c being pairwise independent over A implies that $a \mathop{\smile\!\!\!\!\!\mid}_A b, c$.

Lemma 4.3.6. *The theory T_ν is trivial.*

Proof. From Corollary 4.3.3 we can see that a, b, c being pairwise independent over A is equivalent to the properties

$$\mathrm{acl}(aA) \cap \mathrm{acl}(bA) = \mathrm{acl}(A),$$

$$\mathrm{acl}(bA) \cap \mathrm{acl}(cA) = \mathrm{acl}(A)$$

and

$$\mathrm{acl}(aA) \cap \mathrm{acl}(cA) = \mathrm{acl}(A).$$

Now consider $\mathrm{acl}(aA) \cap \mathrm{acl}(bcA)$. This can be rewritten

$$\mathrm{acl}(aA) \cap \mathrm{acl}(bcA) = \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA)) \cap \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(bcA)).$$

By Lemma 4.3.4

$$\begin{aligned} \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA)) \cap \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(bcA)) &= \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bcA)) \\ &= \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA) \cap (\mathrm{cl}^{\sqsubseteq}(bA \cup cA))). \end{aligned}$$

Since the descendant closure is disintegrated we get

$$\begin{aligned} \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA) \cap (\mathrm{cl}^{\sqsubseteq}(bA \cup cA))) &= \mathrm{acl}(\mathrm{cl}^{\sqsubseteq}(aA) \cap (\mathrm{cl}^{\sqsubseteq}(bA) \cup \mathrm{cl}^{\sqsubseteq}(cA))) \\ &= \mathrm{acl}((\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bA)) \cup (\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(cA))). \end{aligned}$$

It is clear that

$$\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bA) \subseteq \mathrm{acl}(aA) \cap \mathrm{acl}(bA)$$

and by the hypothesis we have

$$\mathrm{acl}(aA) \cap \mathrm{acl}(bA) = \mathrm{acl}(A).$$

Therefore

$$\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bA) \subseteq \mathrm{acl}(A)$$

and similarly we see that

$$\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(cA) \subseteq \mathrm{acl}(A).$$

Hence

$$(\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bA)) \cup (\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(cA)) \subseteq \mathrm{acl}(A)$$

which gives

$$\mathrm{acl}((\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(bA)) \cup (\mathrm{cl}^{\sqsubseteq}(aA) \cap \mathrm{cl}^{\sqsubseteq}(cA))) \subseteq \mathrm{acl}(A),$$

that is

$$\operatorname{acl}(aA) \cap \operatorname{acl}(bcA) \subseteq \operatorname{acl}(A).$$

The other inclusion is trivial and so we have $\operatorname{acl}(aA) \cap \operatorname{acl}(bcA) = \operatorname{acl}(A)$ which, by Corollary 4.3.3 is equivalent to $a \mathop{\smile\!\!\!\!\!\!\,|}_A b, c$. □

4.4 Non-superstability

In what follows N_ν is a large saturated model of T_ν. We now consider the reduct M_ν of N_ν which is obtained by disregarding the directions on the edges. Note that M_ν is saturated as it is the reduct of a saturated model. We show that $Th(M_\nu)$ is strictly stable (recall this means λ-stable if and only if $\lambda^\omega = \lambda$), which is in contrast to the reduct in Section 4.1.

Definition 4.4.1. Define a *directed triad* to be a triple a, b, c in N_ν such that there are directed edges from a to b and from a to c and no other edges. If we disregard the direction on the edges of a directed triad then we will call it an *undirected triad*.

Note 4.4.2. *A directed triad is made up of the primitive extension* $\{b, c\} \sqsubseteq \{a, b, c\}$.

Lemma 4.4.3. *Let* $\nu(P/Q) = 1$ *for the primitive extension* $Q \sqsubseteq P$ *where* $P = \{p, q, r\}$, $Q = \{q, r\}$ *and* p, q, r *is a directed triad. Then for any two elements* e, f *in* N_ν *there must be an element* d *in* N_ν *such that* d, e, f *is a directed triad.*

Proof. Recall that $\theta_{A,B}$ for a minimal extension $A \sqsubseteq B$ with base $X \subseteq A$ and $Y = X \cup (B \backslash A)$ is defined as

$$\forall \bar{a} \Delta_A(\bar{a}) \to \exists \bar{y} \Delta_{A,B}(\bar{a}\bar{y}) \vee \left(\exists \bar{x} \Delta_X(\bar{x}) \wedge \exists \bar{y}_1, \ldots, \bar{y}_r \Delta_{X,Y}(\bar{x}\bar{y}_i) \wedge \bigwedge_i \varphi(\bar{a}\bar{y}_i) \right)$$

where $\varphi(\bar{a}\bar{y}_i)$ says that there is an element of the tuple $\bar{y}_i$ which is in A or there is an element of the tuple $\bar{y}_i$ which is in $A^{\rightarrow}$. In this case $A = \{e, f\}$, $B = \{d, e, f\}$, $X = A$ and $Y = B$. The axiom therefore says that for any set isomorphic to A there is an extension of A isomorphic to $A \sqsubseteq B$ or there is an element y in N_ν such that y, e, f is a directed triad and either $y \in A$ or $y \in A^{\rightarrow}$. As we have $X = A$ Note 4.2.9 says that the second option cannot occur and so there must be an extension of A isomorphic to $A \sqsubseteq B$. That is, there is a d in N_ν such that d, e, f is a directed triad. □

Lemma 4.4.4. *Let $\nu(P/Q) = 1$ for the primitive extension $Q \sqsubseteq P$ where $P = \{p, q, r\}$, $Q = \{q, r\}$ and p, q, r is a directed triad. Let T be the rooted directed binary tree and let $A = \{a \in N_\nu : \mathrm{desc}(a) \simeq T\}$. Then for any $\alpha \in \mathrm{Aut}(M_\nu)$ we have $\alpha(A) \subseteq A$ (that is, the directions of binary trees in N_ν are preserved by automorphisms of M_ν).*

Proof. Let a, a_0, a_1 be a directed triad in a rooted binary tree in N_ν, so $a, a_0, a_1 \in A$ and consider the result of applying the automorphism α. Since there is a unique path of length two between a_0 and a_1 there must also be a unique path of length two between $\alpha(a_0)$ and $\alpha(a_1)$. Automorphisms preserve relations and so this unique path passes through the vertex $b = \alpha(a)$. By Lemma 4.4.3 there is a b' in N_ν such that $b', \alpha(a_0), \alpha(a_1)$ is a directed triad. Therefore we must have that $b' = b$ and thus $\alpha(a), \alpha(a_0), \alpha(a_1)$ is a directed triad. This means that the direction on a directed triad in a rooted binary tree in N_ν is preserved by any automorphism of M_ν, and so $\alpha(A) \subseteq A$. □

Lemma 4.4.5. *Let $\nu(P/Q) = 1$ for the primitive extension $Q \sqsubseteq P$ where $P = \{p, q, r\}$, $Q = \{q, r\}$ and p, q, r is a directed triad. Then the reduct M_ν is strictly stable.*

Proof. Note that M_ν is stable because it is the reduct of N_ν which is stable. Let $A = \{a \in N_\nu : \operatorname{desc}(a) \simeq T\}$ where T is the rooted binary tree. By Lemma 4.4.4, $\alpha(A) \subseteq A$ for any $\alpha \in \operatorname{Aut}(M_\nu)$. We know that M_ν is saturated so we now consider the number of 1-types over a set of size λ that are realized in A. We show that for each infinite cardinal λ there is a set $C \subseteq A$ with $|C| = \lambda$ and $\lambda^{\aleph_0}$ 1-types over C in M_ν. To do this let $(a_i : i < \lambda)$ be independent elements of N_ν with $a_i \in A$ and let $C = \operatorname{cl}(\bigcup a_i)$. Let $I : \omega \to \lambda$ be any countable increasing sequence of elements of λ. Then there is $b_I \in A$ with the following properties:

1. $\operatorname{desc}(b_I) \cap \operatorname{desc}(a_i) \neq \emptyset$ if and only if i is in the image of I,
2. if $i = I(n)$ then $\operatorname{desc}(b_I) \cap \operatorname{desc}(a_i) = \operatorname{desc}(c_{I,n})$ where $c_{I,n}$ is in the nth level of $\operatorname{desc}(a_i)$.

It is clear that if $I \neq I'$ then no $\alpha \in Aut(M_\nu/C)$ can have $\alpha(b_I) = b_{I'}$. Thus the number of 1-types over C in M_ν is at least $\lambda^{\aleph_0}$ (the number of such functions I). □

Now let us consider the general case when $\nu(P/Q) = n$ for the primitive extension $Q \sqsubseteq P$ where $P = \{p, q, r\}$, $Q = \{q, r\}$ and p, q, r is a directed triad. We get lemmas that generalise Lemmas 4.4.3, 4.4.4 and 4.4.5 as follows.

Lemma 4.4.6. *For any two elements e, f in N_ν there are exactly n elements in*

N_ν,$d_1, \ldots, d_n$ such that d_i, e, f is a directed triad for each i.

Proof. We prove by induction that for $m \leq n$ there are $d_1, \ldots, d_m \in N_\nu$ such that d_i, e, f is a directed triad for each i. To do this we consider the axiom $\theta_{A,B}$ for appropriate A, B.

The base step is where $m = 1$ and this is given by the proof of Lemma 4.4.3.

For the inductive step assume that the statement is true for $m - 1$, so there are $d_1, \ldots, d_{m-1}$ in N_ν such that d_i, e, f is a directed triad for $i = 1, \ldots, m - 1$. Now let $A = \{d_1, \ldots, d_{m-1}, e, f\}$ and $B = \{d_1, \ldots, d_m, e, f\}$, so $A \sqsubseteq B$ is a minimal extension with base $X = \{e, f\}$ and let $Y = \{d_m, e, f\}$. The axiom $\theta_{A,B}$ then says that for every set isomorphic to A there is an extension of A isomorphic to $A \sqsubseteq B$ unless there are $y_1, \ldots, y_m$ (with $y_i \neq y_j$ for all $i \neq j$) in N_ν such that y_i, e, f is a directed triad (that is $\{y_i, e, f\}$ is isomorphic to Y) such that either $y_i \in A$ or $y_i \in A^\rightarrow$ for every i. If there is an extension of every set isomorphic to A isomorphic to $A \sqsubseteq B$ then we are done, so assume that this is not the case. Therefore there are $y_1, \ldots, y_m$ such that y_i, e, f is a directed triad and either $y_i \in A$ or $y_i \in A^\rightarrow$ for every i. First let us consider $y_i \in A$. If $y_i = e$ or $y_i = f$ for any i then this creates a self loop at e or f respectively which is not allowed. Therefore if $y_i \in A$ then $y_i = d_j$ for some j. The m y_i's are all unique and there are only $m - 1$ d_j's so there must be at least one y_i, y_m say such that $y_m \notin A$. Therefore we must have $y_m \in A^\rightarrow$. The d_j are full, that is they can not have any descendants other than e and f. Hence we must have that y_m is a descendant of e or f. However this gives $\{e, f\} \not\sqsubseteq \{y_m, e, f\}$ which contradicts $\{y_m, e, f\}$ isomorphic to Y. So we must have an extension isomorphic to $A \sqsubseteq B$, that is for any two elements e, f in N_ν there are m elements $d_1, \ldots, d_m$ in N_ν such that d_i, e, f is a directed triad for each i. □

Lemma 4.4.7. *Let* $A = \{a \in N_\nu : \mathrm{desc}(a) \simeq T\}$ *where* T *is the rooted directed binary tree. Then for any* $\alpha \in \mathrm{Aut}(M_\nu)$ $\alpha(A) \subseteq A$ *(that is, the directions of the edges in* T *in* N_ν *are preserved by automorphisms of* M_ν*).*

Proof. The proof follows that from Lemma 4.4.4. Let $\nu(P/Q) = n$ for $P = \{p, q, r\}$, $Q = \{q, r\}$ and p, q, r a directed triad. Let a_1, b, c be a directed triad in A. Then by Lemma 4.4.6 there are $a_2, \ldots, a_n$ in N_ν such that a_i, b, c is a directed triad for all i. Now consider the result of applying the automorphism α. Since there are exactly n paths of length two between b and c there must also be exactly n paths of length two between $\alpha(b)$ and $\alpha(c)$. Automorphisms preserve relations and so these paths pass through the vertices $d_i = \alpha(a_i)$ for $i = 1, \ldots, n$. Using Lemma 4.4.6 again we see there are d'_i for $i = 1, \ldots, n$ in N_ν such that $d'_i, \alpha(b), \alpha(c)$ are directed triads. Therefore we must have that $\{d'_i\} = \{d_i\}$ and thus $\alpha(a_i), \alpha(b), \alpha(c)$ for $i = 1, \ldots, n$ are directed triads. This means that the directions of the edges of T in N_ν are preserved by any automorphism of M_ν, and so $\alpha(A) \subseteq A$. □

Lemma 4.4.8. *Let* $\nu(P/Q) = n$ *for the primitive extension* $Q \sqsubseteq P$ *where* $P = \{p, q, r\}$, $Q = \{q, r\}$ *and* p, q, r *is a directed triad. Then the theory of the reduct* M_ν *is strictly stable.*

Proof. The proof is exactly the same as the proof of Lemma 4.4.5. □

Hence we have shown that the theory of M_ν is not superstable. This means that the undirected reduct of the 'collapsed' digraph $\mathcal{N}_\nu$ is not the 'collapse' of the undirected graph which is given by Hrushovski's construction in [18].

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