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COMMISSION OF THE EUROPEAN COMMUNITIES

ON THE SOLUTION OF PLANE STRESS PROBLEMS BY FINITE ELEMENTS COMPUTER PROGRAMS

by

G. GAGGERO and G. CANDOLFO

1973

Joint Nuclear Research Centre
Ispra Establishment — Italy

Scientific Data Processing Centre — CETIS

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D.G. XIII - C.I.D.
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ABSTRACT

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A comparison of different computer programs, when applied to a series of discrete models of a cantilever beam, is given in paragraph number 1.

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KEYWORDS

COMPUTER CODES
COST
PERFORMANCE
ITERATIVE METHODS
STRESSES
MATHEMATICAL MODELS

ON THE SOLUTION OF PLANE STRESS PROBLEMS BY FINITE ELEMENTS COMPUTER PROGRAMS *)

INTRODUCTION

The finite-element method is based on the idea of approximating a continuous field (e.g. displacement field in elastic continua) by a discrete model which consists of a set of values of the field (e.g. displacements) at a finite number of points (called nodes) and of piecewise approximations (shape functions) of the field over a finite number of subdomaines (called finite elements).

The local approximation of the field is uniquely defined in terms of the discrete values of the field at the nodes of each finite element.

The continuum with infinite degrees of freedom is thus represented by a discrete model with finite degrees of freedom.

It has been demonstrated that the behaviour of the discrete system converges to that of the continuous system, if certain completeness conditions are satisfied, when:

1) the number of finite elements is increased;
2) the size of finite elements is decreased.

It has also been recognized that a closer approximation of the real situation may be obtained not only by increasing the number of finite elements but by using more sofisticated types of finite elements (i.e. higher order approximations).

In practice, the accuracy of the results obtained by using computer programs based on the finite-element method is limited by the core storage capacity of the computer (i.e. by the number of finite elements), and/or by the cost of the computation (computer time increases with the number of finite elements and their complexity).

From these preliminary remarks it appears of primary importance the necessity of finding practical rules for an optimum (with respect to accuracy and cost) choice of the type and number of finite-elements to be used in constructing the descrete model.

On the other hand, once the descrete model has been set up, one has to face the problem of solving the resulting system of linear equations.

*) Manuscript received on November 14, 1972

Numerical analysis provides different techniques (e.g. iterative and direct methods) and it is important to assess that the obtained results are not affected, in the limits of the required accuracy, by the choice of the solution method.

The goal of the present study is to obtain information which facilitates the treating of practical problems with satisfactory results and low computer cost.

A comparison of different computer programs, when applied to a series of discrete models of a cantilever beam, is presented in paragraph 1.

In paragraph 2, the calculated results are also used to study the influence of the mesh orientation and refinement on the convergence of the discrete model to the actual situation. The finite element used in the discretization is the simple plane stress triangle.

In paragraph 3, different type of elements (triangle, triangle with midside nodes, quadrilateral and quadrilateral with midside nodes) are investigated and the obtained displacements and stresses are comparated.

Finally some considerations on the required computer time are presented in paragraph 4.

COMPARISON OF DIFFERENT COMPUTER PROGRAMS

At first one has undertaken a comparative study, for detecting the influence of the mesh refinement and the mesh orientation on the mathematical solution of the linear equations system, whose coefficients represent the terms of the stiffness matrix.

The following four programs have been considered:

1) Safe Plane /_A_7: iterative method (Gauss-Seidel, accelerated with successive over-relation or S.O.R.).
2) Safe - 2D /_B_7: direct method (tridiagonalization in blocks).
3) Zienkiewicz /_C_7: same method as Safe - 2D.
4) Bersafe /_D_7; direct method (front-solution).

By these programs some easy sample problems have been solved and the results compared.

The developed example is a cantilever beam shear loaded at the free end face.

Eight different types of subdivision in finite elements have been applied to this structure, changing the number of nodes and elements or simply the orientation of the last ones: for explanation see Fig. 1-1. This simple case has been chosen because the theoretical calculation of the deflection and stresses is easy to perform and sufficiently accurate and also in good agreement with the experiments.

In appendix I, theoretical formulae and results are reported relative to:

1) the deflection of the beam centre line;
2) the axial stress along the top edge of the beam, σ, and the shear stress, τ_{xy} relative to a generic beam section.

The deflection values computed by the above mentioned programs are nearly identical, differing only in the fourth significant digit (see Table I). Concerning the stress values there is a good agreement between the various programs.

TABLE I

Case	Safe 2D	Safe Plane	Zienkiewicz	Bersafe
1	0.05257	0.05254	0.05263	0.05265
2	0.05547	0.05544	0.05555	0.05555
3	0.05287	0.05282	0.05285	0.05284
4	0.05607	0.05602	0.05612	0.05604
5	0.06372	0.06363	0.06368	0.06370
6	0.06624	0.06617	0.06623	0.06620
7	0.07051	0.07040	0.07047	0.07050
8	0.07292	0.07283	0.07289	0.07290

a) Beam center line maximum computed deflection (theoretical value = = 0.0933 inch).

Case	1	2	3	4	5	6	7	8
Error %	44	40	43	39	32	29	24	22

b) Relative spread between theoretical and calculated maximum deflection values.

From these results it appears that, at least for small problems, the computed displacements are practically indipendent from the numerical method used to solve the resulting system of linear equations.

Furthermore it is clear that the difference, which exists between the programs which use a different numerical technique is of the same order of magnitude of the difference between programs using the same technique.

INFLUENCE OF THE ORIENTATION AND REFINEMENT OF THE MESH

The above calculated results are also used to study the influence of the mesh orientation and refinement on the convergence of the discrete model to the actual situation.

Fig. 1-2 represents the centre line deflections as obtained with the eight considered discretizations. With increasing number of nodes and elements, improved approximations are obtained, even if the deflection values still remain below of the theoretical one (dot and dash). In particular, the two kinds of lines, dashed (cases 1-3-5-7) and continuous (cases 2-4-6-8), refer to subdivisions with a different orientation of the elements: one can see that also this simple difference has an appreciable influence on the accuracy of the results.

For the eight different types of meshes considered a comparison has been also made, which regards the axial stress along the top edge of the beam (see Fig. 1-3). To do such a comparison the element stress values rather than the averaged nodal stress values have been considered because these values were directly available from Bersafe and Safe-2D programs only.

In Fig. 1-3 it is possible to see that a mesh refinement near the constrained face improves only the local stress values, but it does not transmit any influence towards the free end of the beam; or rather, while this mesh refinement improves the displacements along the whole beam, stresses are only locally affected.

Concerning the shear stress behaviour, the nodal stress values produced by Safe-2D program have been considered. To obtain the maximum number of points, the section X = 2 inch, in which the meshes are more dense has been considered.

The behaviour is not very satisfactory and expecially the type of element orientation which proved to be the most convenient for displacements now appears unsatisfactory for shear stresses. (see Fig. 1-4).

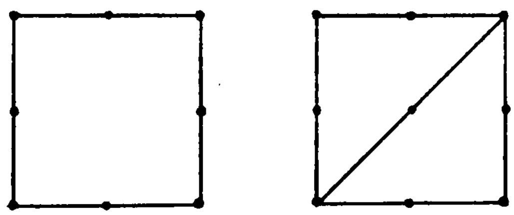
INFLUENCE OF THE TYPE OF ELEMENTS

The present study may be considered a supplement to the work reported in RD/B/N1848 by K. Fullard and T.K. Hellen /¯F_7. It has been already pointed out that not only the mesh refinement improves the results, but also the choice of an appropriate type of element, which is able to describe with sufficient accuracy the actual deformation field.

Fig. 1-5 illustrates this fact in a limit case. A discretization with only two elements of the EP16 type (quadrilateral element with mid-side nodes) gives a quite accurate beam center line deflection, even better than the one obtained by using a subdivision with 46 elements of the EP6 type (simple triangles).

This result may be generalized and one can say that the use of elements with more complex shape functions (e.g. with mid-side nodes) produces better results also in cases with a considerably lower number of nodes and elements.

Referring to the subdivisions of the Fig. 1-6, one can see that regarding the displacements (see Fig. 1-5), the following two element patterns are equivalent.



Concerning the axial stresses along the top edge of the beam (Fig. 1-7), those calculated by using quadrilateral elements with mid-side nodes are more accurate not withstanding one has only half the number of elements and one node less for each quadrilateral element.

In addition we must point out that both the discretization are less accurate near the constrained and the free beam faces than in the central portion of the beam. To obtain accurate results also in these two regions, it appears necessary to introduce a local mesh refinement.

A comparison based either on the deflection (Fig. 1-5) or on the axial stresses (Fig. 1-8) has been also made between the following types of elements:

1) EP6 (simple triangles)
2) FP8 (simple quadrilateral)

The adopted subdivisions are illustrated in Fig. 1-6 (cases 10-11). In these conditions the beam center line deflection is far away from the exact one although the deflection given by the FP8 elements is the less accurate. Regarding the stresses, one can see (Fig. 1-8) that while the EP8 elements give acceptable values, the EP6 elements give a considerable deviation which in some zones achieve the values of 40 or 50 %.

Using subdivisions with EP12 and EP16 elements (Fig. 1-6, cases 12-13) one obtains a very good approximation relative to beam deflection (see Fig. 1-5).

It is interesting to see how much are different the theoretical and calculated values, when are comparated the displacements or the stresses for the same type of element. This comparison is illustrated in Table II, for six different values of the X coordinate and for the above four cases 10 to 13.

TABLE II

	EP6		EP8		EP12		EP16	
x inch	Error on Deflect. %	Error on Stress σ_x %	Error on Deflect. %	Error on Stress σ_x %	Error on Deflect. %	Error on Stress σ_x %	Error on Deflect. %	Error on Stress σ_x %
2	+ 2.9.	- 42.	+ 50.	- 8.	+ 29.	+ 3.	+ 36.	- 3.
6	- 2.3.	- 44.	- 2.	- 8.	+ 5.	+ 1.	+ 5.	- 0.2
10	- 3.4.	- 44.	- 9.	- 8.	+ 0.5	+ 2.	+ 0.5	+ 0.1
14	- 38.	- 44.	- 10.	- 8.	+ 0.0	- 0.1	+ 0.2	+ 0.0
18	- 39.	- 44.	- 10.	- 8.	+ 0.5	- 1.	+ 0.6	- 0.4
22	- 40.	- 35.	- 10.	- 11.	+ 0.6	- 20.	- 0.6	- 46

Relative spreads between calculated and theoretical deflections and stresses.

It may be noted that deflections are in general determined with a greater accuracy than stresses by the displacement formulation of the finite element method.

For a more detailed analysis of the performances of the different element types, the following discretizations involving further mesh refinements were studied:

1) EP6; 864 Elements, 481 nodes (case 14)
2) EP8: 432 Elements, 481 nodes (case 15)
3) EP16;432 Elements,1393 nodes (case 16)

The diagram in Fig. 1-9 relative to the sections X = 2 inch and X = 12 inch, shows the axial stresses as a function of the Y coordinate.

The calculate results are very closed to the theoretical ones, in the EP16 case they are even coincident.

The only zone in which there is some deviation is the one near the top and bottom edges of the beam. For better understanding this fact let us consider the diagram in Fig. 1-10. This diagram reproduces the σ_{max} behaviour along the whole beam for cases 14 and 15 (EP6 and EP8 elments). Here the triangular and quadrilateral elements are compared taking the stresses values in each element before any averaging procedure.

One can see from the diagram, that the values relative to the section X = 12 inch are better than those of the section X = 2 inch.

This is due to the fact that such a section is near the constrained face of the beam and that it is affected by the approximative representation of the boundary conditions. Indeed such a deviation takes place also for the sections between X = 0 and X = 2,5 inches.

Concerning the case 16 there is an excellent agreement between theoretical and calculated values along the whole beam (Fig. 1-11). Also the discontinuity of the stress between adjacent elements is reduced to small values and goes to zero for X = 4 inch.

The above examples show, that with the exception of the zones near the faces of the beam, very satisfactory results may be obtained also with a limited mesh refinement or simply by using higher order elements.

Shear stresses have been analysed for the three types of subdivisions already mentioned, i.e. EP6, EP8, EP16. Only these cases have a sufficient number of points to describe a reasonable detailed behaviour of the shear stress.

One can see in Fig. 1-12, that at least for the section X = 12 inch, a good approximation, which in the EP16 case is a perfect coincidence, may be obtained. Besides the EP6 discretization gives better results than the EP8 case, except for the zones near the center line and the edges of the beam.

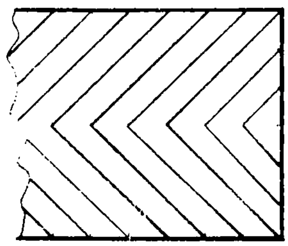
Indeed it must be noted that the shear stress does not remain strictly constant in every section as anticipated by the beam theory.

This is caused by the imperfect simulation of both the constraints and the loads.

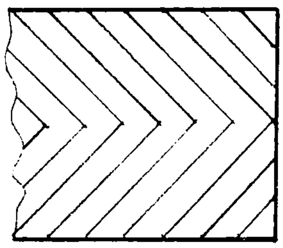
The shear stress becomes constant at a distance from the constrained face equal to the half height of the beam in the EP8 case and to a quater in the EP6 case. Near the loaded face of the beam the distance becomes about the half of the previous values.

Regarding the influence of the mesh orientation on the results it is possible to observe (see Fig. 1-12) that a triangular mesh of the herring-bone kind, orientated like in case 1: presents some points which do not respect the theoretical behaviour. More precisely, the shear stress curve shows a depression on the beam center line, while, near the edges of the beam, it mantains values far from zero.

On the contrary if one uses a mesh like in case 2 (i.e. orientated in the opposite sense): the maximum value is overextimated and the shear stress on the edge of the beam reaches a value which is less than half the one of the previous case.



case 1



case 2

On the other hand, one must consider that the first type of mesh gives more satisfactory deflection values than the second type.

For better understanding the behaviour of the shear stress in the neighbourhood of the constrained section one may observe the diagram in Fig. 1-13, related to cases 14-15.

One can see that in the constrained section the quadrilateral elements give values deviating from the theoretical ones. For X = 2 inch the curve begins to approximate considerably the theoretical shape. The triangular elements are less disturbed by the boundary, and for X = 2 inch the shear stress behaviour is realy satisfactory and is very close to the theoretical one.

In Fig. 1-4 the τ_{xy} distribution, calculated with the discretization of case 16, is shown for different sections along the beam axis.

For the constrained section, X = 0 inch the curve has the less satisfactory behaviour but on the section at X = 2 inch, the behaviour begins to be good. From X = 4 inch to X = 6 inch the τ_{xy} curve arrives to a definitive shape, which is maintained almost till the loaded section. For X = 22 inch the deviation starts to become significant, and for the loaded section X = 24 inch there are some little obscillations around the exact value.

CONSIDERATION ON THE REQUIRED COMPUTER TIME

The computer used for the above described calculations was an IBM 360/65 and all the values of time referred in this paragraph are relative to this machine.

Following the analysis on the reliability of the finite element approach we proceeded to examine the computer time necessary for the solution of the previously considered problems.

The Zienkiewicz program for a single run spent 17 seconds, the Safe-Plane program about 28 seconds and the Bersafe and the Safe-2D programs both 31 seconds.

One can conclude that the Zienkiewicz program is the faster one, but the other three are equivalent also with respect to computer time.

These four programs have very different capabilities concerning the type and size of the problems they can solve.

The limits with respect to the size of the problem are on the total number of nodes and elements. The Safe-2D and the Zienkiewicz programs have also limits on the number of elements for each partition and Bersafe has a limit on the matrix band-width. All the four programs can solve two dimensional cases with simple triangular elements. The Bersafe program can solve two and three-dimensional problems with various types of elements.

The required storage space on the IBM 360/65 computer is the following:

146 K BYTES for the Zienkiewicz
208 K BYTES for the Safe-Plane
324 K BYTES for the Safe-2D
352 K BYTES for the Bersafe

Running by the last program the four cases 10 to 13 of the Fig. 1-6, needed the following execution times:

Case	Element type	Number of elements	Number of nodes	Total time	Time to calculate stiffnes matrix and to solve the system
10	EP6	24	21	34 sec	4 sec
11	EP8	12	21	33 sec	5 sec
12	EP12	24	65	47 sec	11 sec
13	EP16	12	63	43 sec	9 sec

From this series of data we may **conclude** that the increase of the execution time is more a consequence of the number of nodes than of the use of higher order elements.

For better seeing the influence of these two different factors, it is useful to consider the three cases above analysed:

Case	Element type	Number of elements	Number of nodes	**Calculation time** (min)	Total time (min)
14	EP6	864	481	4.36	11.31
15	EP8	432	481	3.27	7.04
16	EP16	432	1393	24.15	34.47

From these values and considering also the numerical results, one can see that the EP8 element is the more convenient to solve plane problems. It gives fairly good values and the results which can be obtained by using the EP16 element does not justify the involved increase of execution time. **One can note that** for the problem solved the EP8 elements needs less time than the same EP 6.

APPENDIX I

For the theoretical calculation of the deflection relative to the centre line of a cantilever beam and of the axial and tangential stresses we have used the formula of the beam theory valid in the elastic field. The deflection equation is the following:

$$f = \left[\frac{1}{E\,I}\left(\frac{P\,\bar{x}^3}{6} - \frac{Pl^2\bar{x}}{2} + \frac{Pl^3}{3}\right)\right] + \frac{\beta\,(1-\bar{x})}{GA}\,P$$

where the first term between square brackets accounts for bending only, while the other one considers the shearing stresses. The orientation of the $\bar{X}$ axis must be taken like in **Fig. 1**, with the origin at the free end. The notations which appear in the equation have the following signification and value.

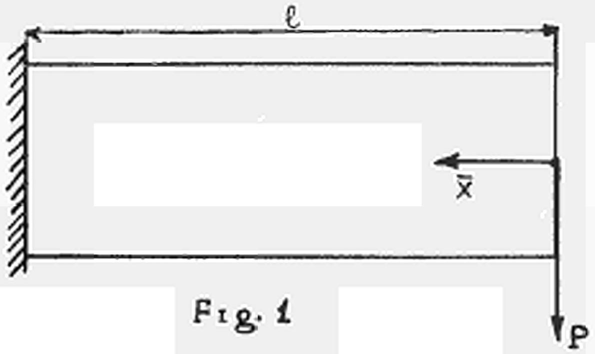


Fig. 1

Young's modulus; $E = 10^6$ lb in^{-2}.

Concentrated load: $P = 800$ lb

Beam lenght: $l = 24$ in

Momentum of Inertia: $I = 42{,}66$ in^4

Shear modulus; $G = \dfrac{E}{2(1+\nu)}$

Poisson's modulus: $\nu = 0.2$

Shear factor; $\beta = \dfrac{6}{5} = 1{,}2$

Cross-sectional area: $A = 8$ in^2

The calculated values are the following, ($x = 1 - \bar{X}$):

$f_{x=o} = 0.0000$in	$f_{x=12} = 0.0304$in
$f_{x=2} = 0.0014$in	$f_{x=16} = 0.0494$in
$f_{x=4} = 0.0044$in	$f_{x=20} = 0.0707$in
$f_{x=6} = 0.0081$in	$f_{x=24} = 0.0933$in
$f_{x=8} = 0.0148$in	

We did the same for the calculation of σ_{max}. The used formula is:

$$\sigma_{max} = \frac{4\,P.b}{I}\,(1 - \bar{x}) \qquad b = 1 \text{ inch}$$

The applied notation have the same signification and the same values of those used in the previous calculation. The obtained results are the following:

$\sigma_{x=0} = 1800$ lb/in^2

$\sigma_{x=2} = 1650$ lb/in^2

$\sigma_{x=4} = 1500$ lb/in^2

$\sigma_{x=6} = 1350$ lb/in^2

$\sigma_{x=8} = 1200$ lb/in^2

$\sigma_{x=10} = 1050$ lb/in^2

$\sigma_{x=12} = 900$ lb/in^2

$\sigma_{x=14} = 750$ lb/in^2

$\sigma_{x=16} = 600$ lb/in^2

$\sigma_{x=18} = 450$ lb/in^2

$\sigma_{x=20} = 300$ lb/in^2

$\sigma_{x=22} = 150$ lb/in^2

$\sigma_{x=24} = 0$ lb/in^2

One can see clearly that the behaviour is linear with the maximum value in the constrained section which decrease with decreasing abscissa and becomes zero for $\bar{X} = 0$ inch.

We calculated the tangential stresses also with the beam theory. They stay constant for each section of the beam, which means it is independent from the fixed section distance. It has a parabolic behaviour with respect to the beam height. The used formula for this theoretical calculation is:

$$\tau_{xy} = \frac{P}{8\,I}\,(h^2 - 4\,y^2)$$

One can see immediately that it is zero on the outline, indeed $h = 2y$. The found values are:

$\tau_{y=4.00} = 0.00$ lb in^{-2}

$\tau_{y=3.75} = 18.16$ lb in^{-2}

$\tau_{y=3.50} = 35.15$ lb in^{-2}

$\tau_{y=3.25} = 50.97$ lb in^{-2}

$\tau_{y=3.00} = 65.62$ lb in^{-2}

$\tau_{y=2.75} = 70.10$ lb in^{-2}

$\tau_{y=2.50} = 91.40$ lb in^{-2}

$\tau_{y=2.25} = 102.52$ lb in^{-2}

$\tau_{y=2.00} = 112.50$ lb in^{-2}

$\tau_{y=1.75} = 121.28$ lb in^{-2}

$\tau_{y=1.50} = 128.90$ lb in^{-2}

$\tau_{y=1.25} = 135.35$ lb in^{-2}

$\tau_{y=1.00} = 140.62$ lb in^{-2}

$\tau_{y=0.75} = 144.72$ lb in^{-2}

$\tau_{y=0.50} = 147.65$ lb in^{-2}

$\tau_{y=0.25} = 149.41$ lb in^{-2}

$\tau_{y=0.00} = 150.25$ lb in^{-2}

CONCLUDING REMARKS

The analysis performed on the accuracy of the computer programs based on the finite element method allows the following remarks:

A) The considered computer programs are equivalent in the limit of the ‘‘engineering accuracy’’.

B) All the programs converge to the ‘‘exact’’ solution when the number of finite elements used in the discrete models is increased.

C) However a satisfactory and cheaper (with respect to the computer time) solution may be obtained by the choice of elements of a suitable type.

ACKNOWLEDGMENTS

Authors acknowledge with thank Mr. A. Inzaghi for the valuable assistance in the use of all investigated computer programs.

REFERENCES

[A] D.C. CORNELL, Y.R. RASHID ''A Computer Program for the Stress Analysis and Design of Two-dimensional Composite Bodies'' Report GA - 7851 June 30, 1967.

[B] D.C. CORNELL, V.R. RASHID ''A Computer Program for the Stress Analysis of Plane and Axisymmetric Composite Structures'' Report GA - 9076 February 12, 1969.

[C] O.C. ZIENKIEWICZ ''The Finite Element Method in Structural and Continuum Mechanics'' London, Mc Grow-Hill 1967.

[D] T.K. HELLEN ''A Front-solution for Finite Element Techniques'' CEGB internal report.

[E] T.K. HELLEN ''Bersafe (phase 1), A Computer System for Stress Analysis'' Part. 1, Report RD/B/N1761.

[F] K. FULLARD and T.K. HELLEN ''An Investigation of the Accuracy of Plane Finite Elements in the Cantilever Beam Program'' Report RD/B/N/1848, November 1970.

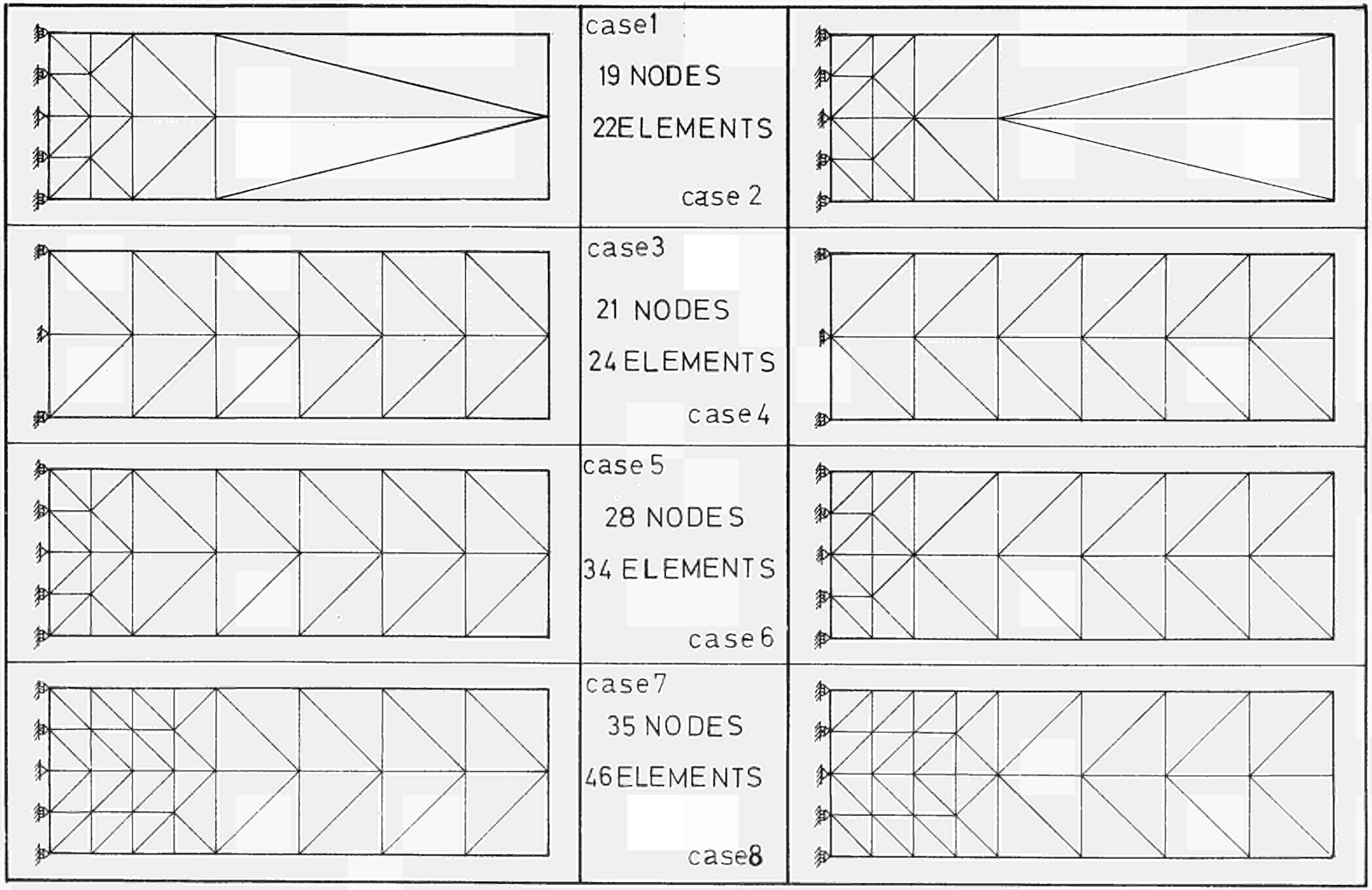
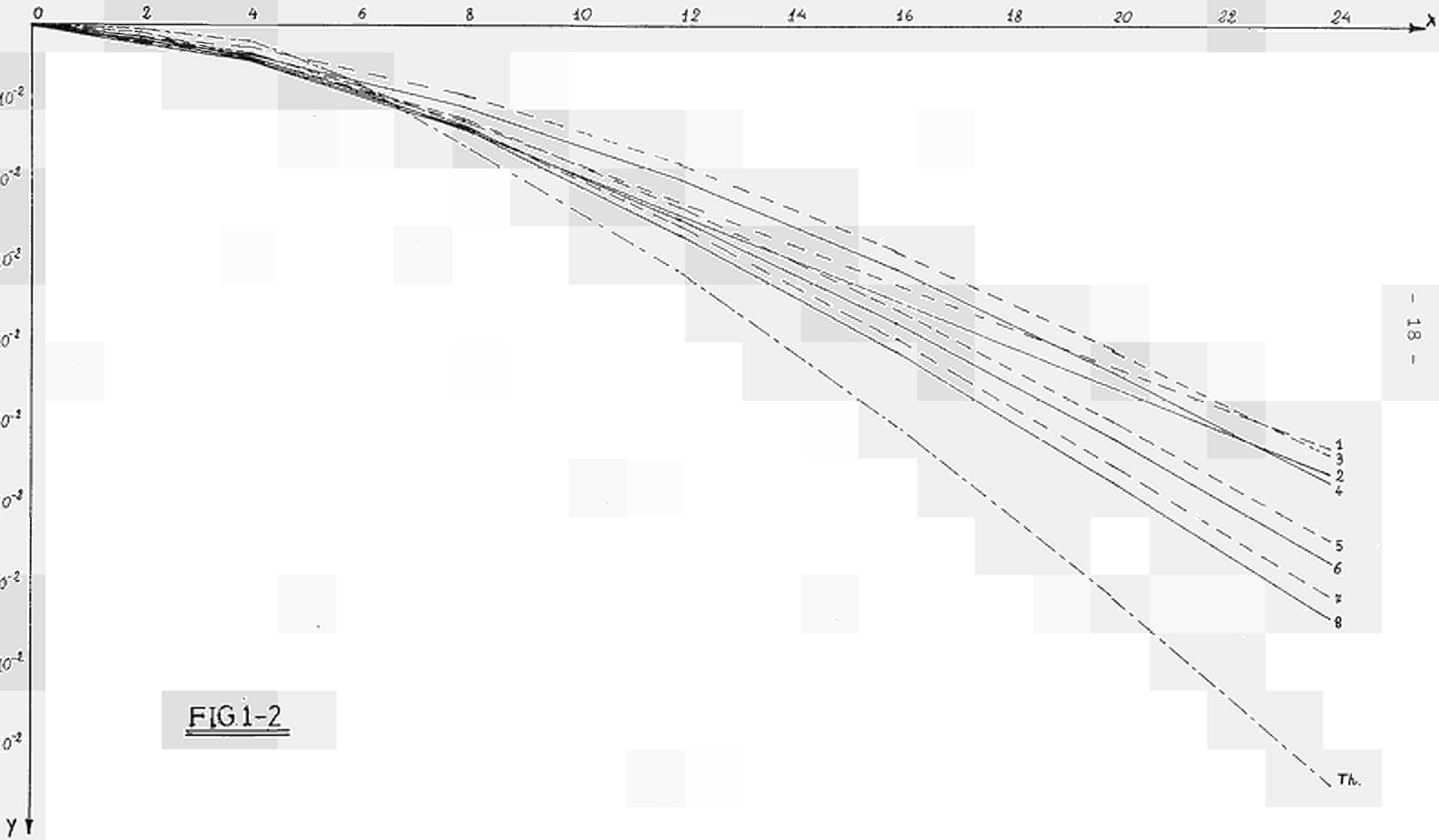

case1
19 NODES
22ELEMENTS
case 2
case3
21 NODES
24 ELEMENTS
case4
case 5
28 NODES
34 ELEMENTS
case 6
case7
35 NODES
46ELEMENTS
case8

FIG 1-1


0
2
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6
8
10
12
14
16
18
20
22
24
x
1.10⁻²
2.10⁻²
3.10⁻²
4.10⁻²
5.10⁻²
6.10⁻²
7.10⁻²
8.10⁻²
9.10⁻²
y
1
3
2
4
5
6
7
8
Th.

FIG.1-2

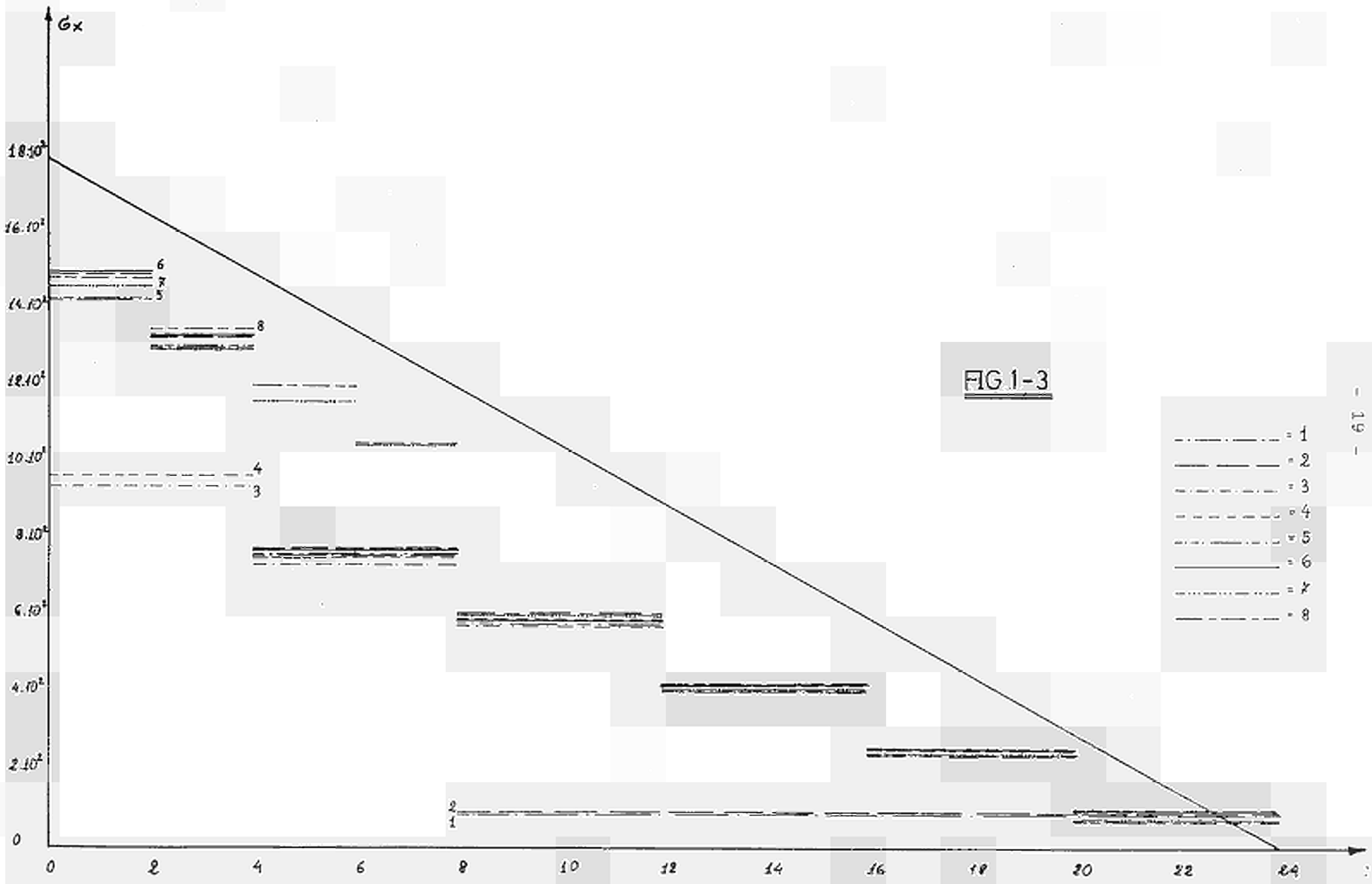

Gx
18.10²
16.10²
14.10²
12.10²
10.10²
8.10²
6.10²
4.10²
2.10²
0
0
2
4
6
8
10
12
14
16
18
20
22
24
6
7
5
8
4
3
2
1
= 1
= 2
= 3
= 4
= 5
= 6
= 7
= 8

FIG 1-3

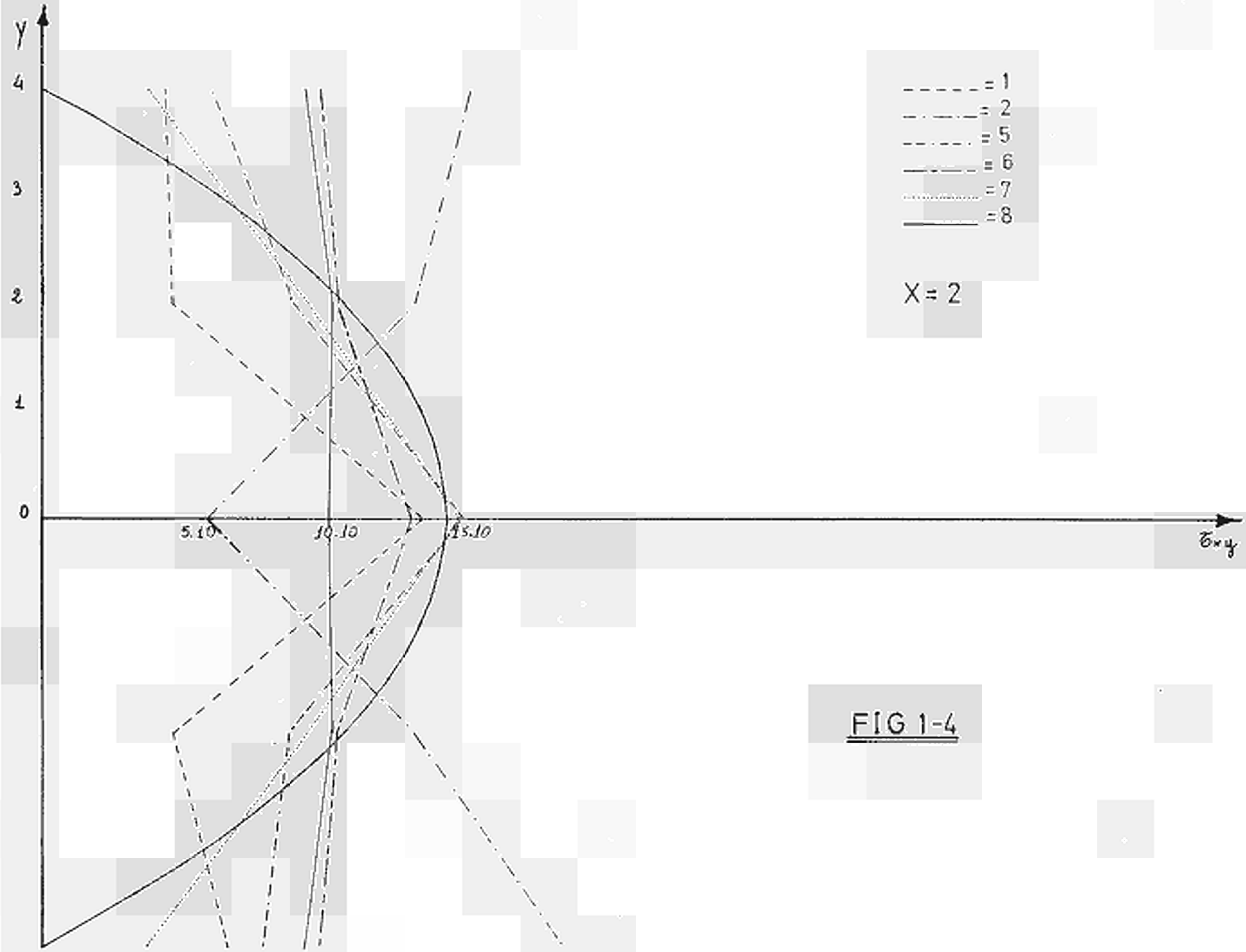
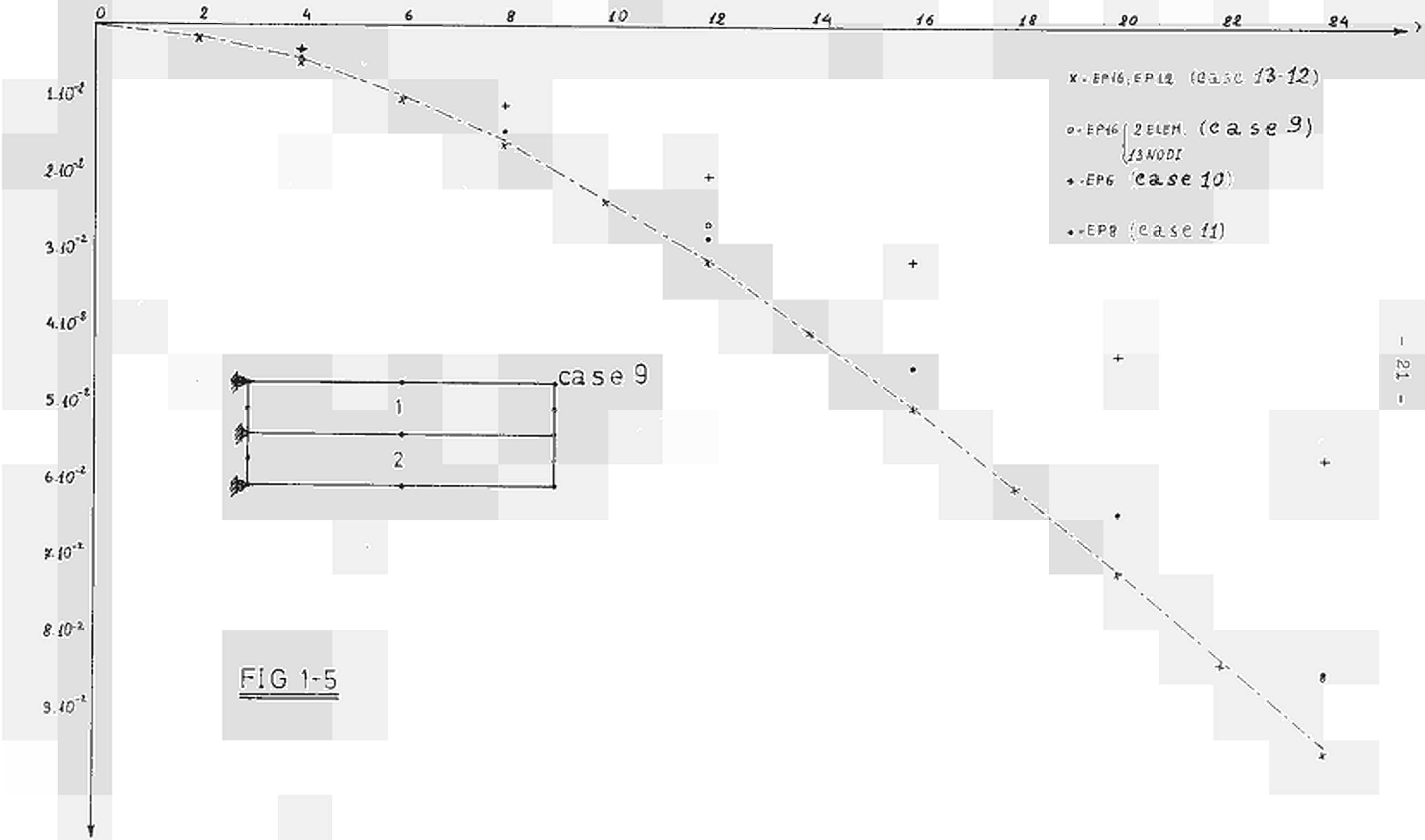


FIG 1-4



0
2
4
6
8
10
12
14
16
18
20
22
24
1.10^-2
2.10^-2
3.10^-2
4.10^-2
5.10^-2
6.10^-2
7.10^-2
8.10^-2
9.10^-2
x - EP16, EP12 (case 13-12)
o - EP16 2 ELEM. 13 NODI (case 9)
+ - EP6 (case 10)
• - EP8 (case 11)
case 9
1
2

FIG 1-5

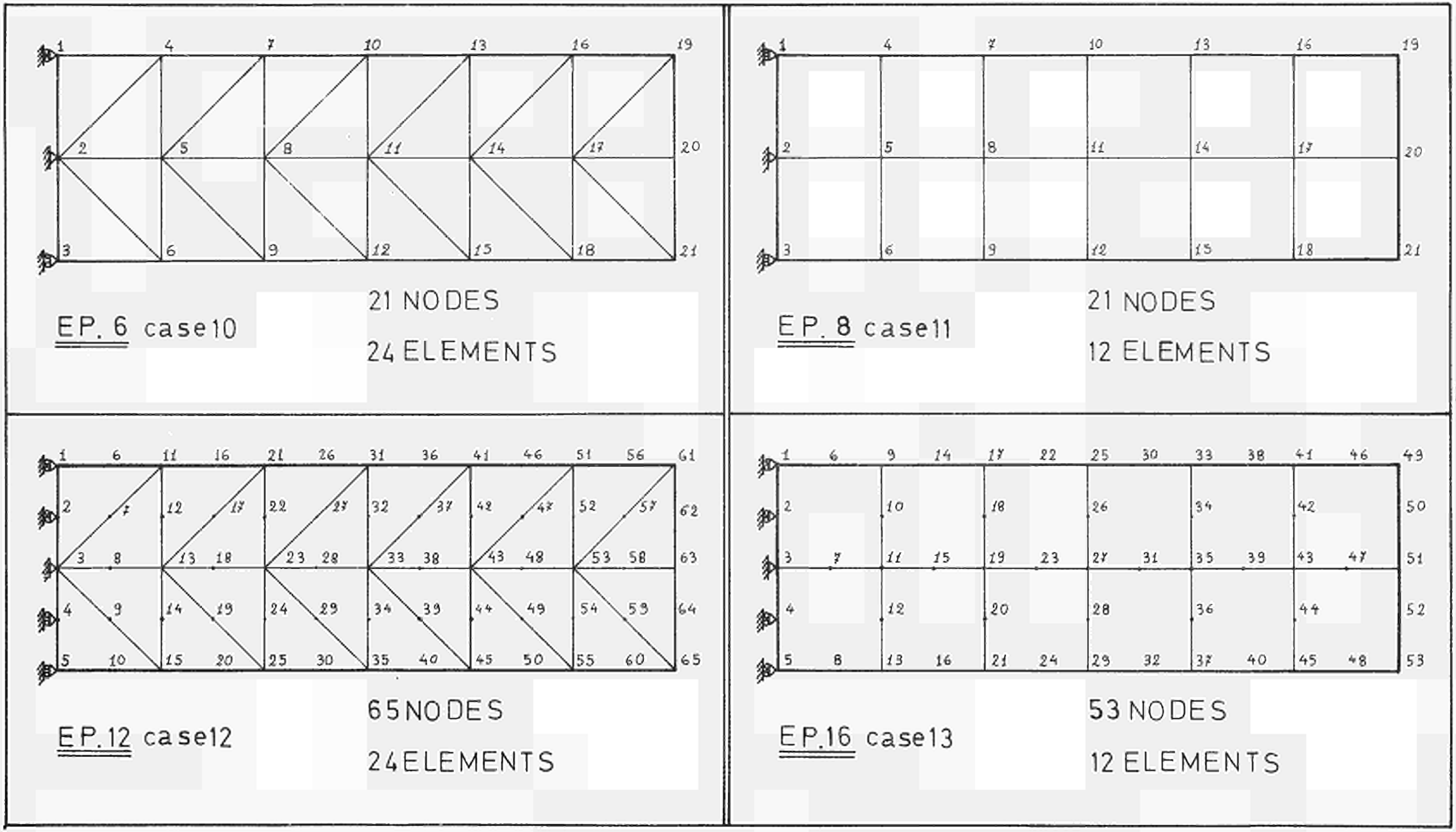

EP. 6 case10
21 NODES
24 ELEMENTS
EP. 8 case11
21 NODES
12 ELEMENTS
EP. 12 case12
65 NODES
24 ELEMENTS
EP. 16 case13
53 NODES
12 ELEMENTS

FIG 1-6

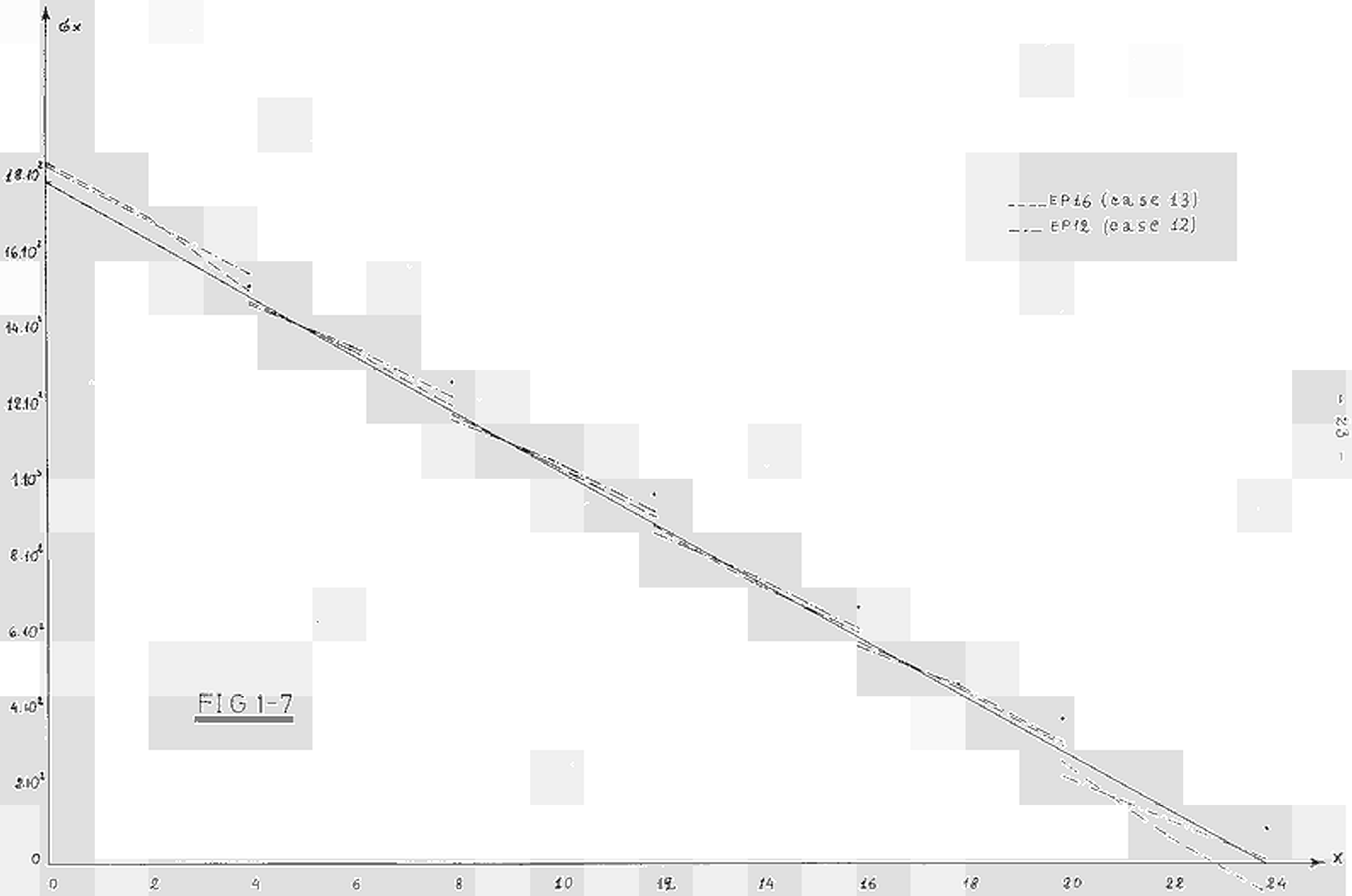
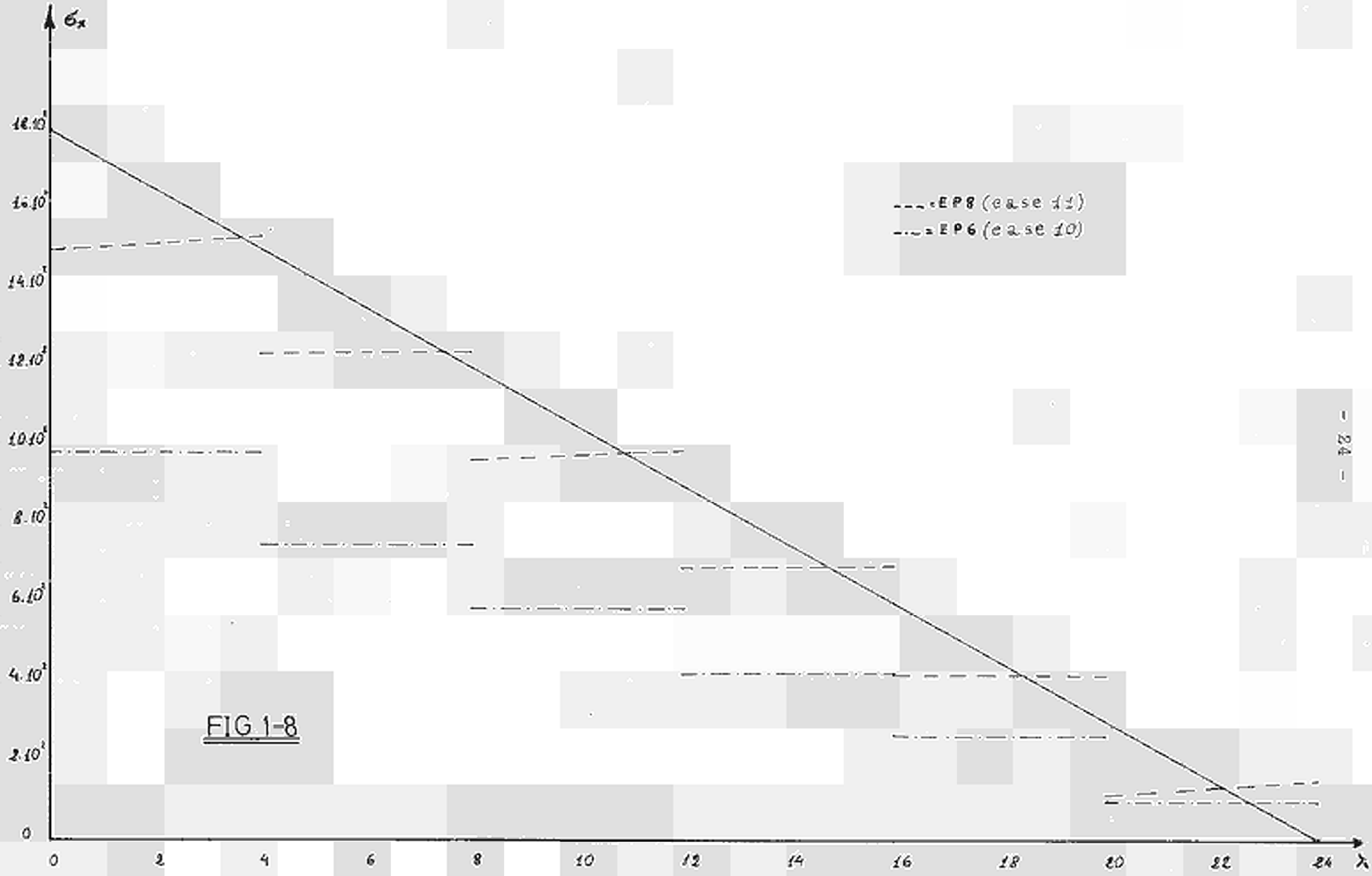

σx
18.10²
16.10²
14.10²
12.10²
1.10³
8.10²
6.10²
4.10²
2.10²
0
0
2
4
6
8
10
12
14
16
18
20
22
24
x
EP16 (case 13)
EP12 (case 12)

FIG 1-7



σ_x
18.10²
16.10²
14.10²
12.10²
10.10²
8.10²
6.10²
4.10²
2.10²
0
0
2
4
6
8
10
12
14
16
18
20
22
24
λ
EP8 (case 11)
EP6 (case 10)

FIG 1-8

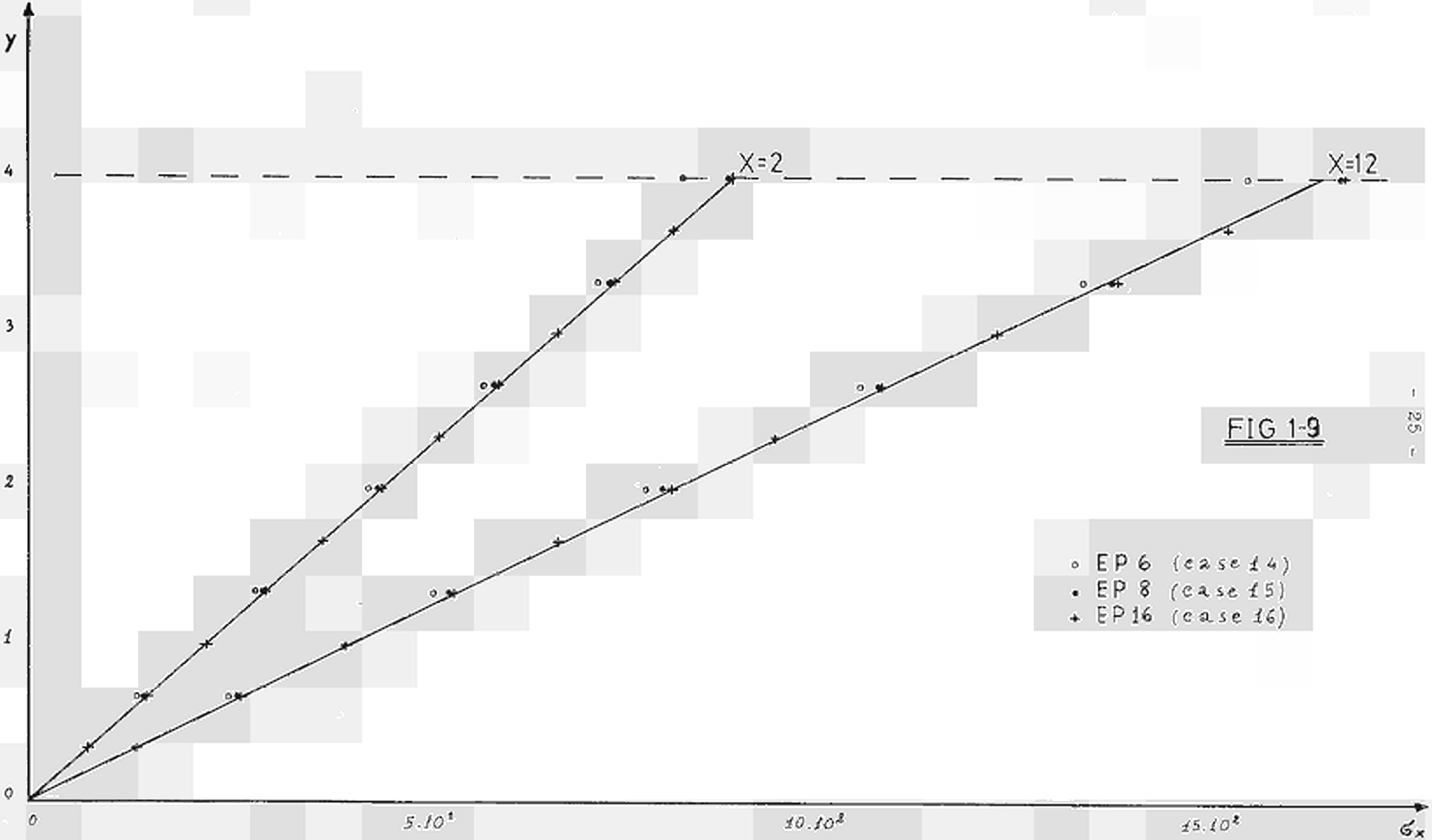

y
4
3
2
1
0
0
5.10²
10.10²
15.10²
σx
X=2
X=12
○ EP 6 (case 14)
• EP 8 (case 15)
+ EP 16 (case 16)

FIG 1-9

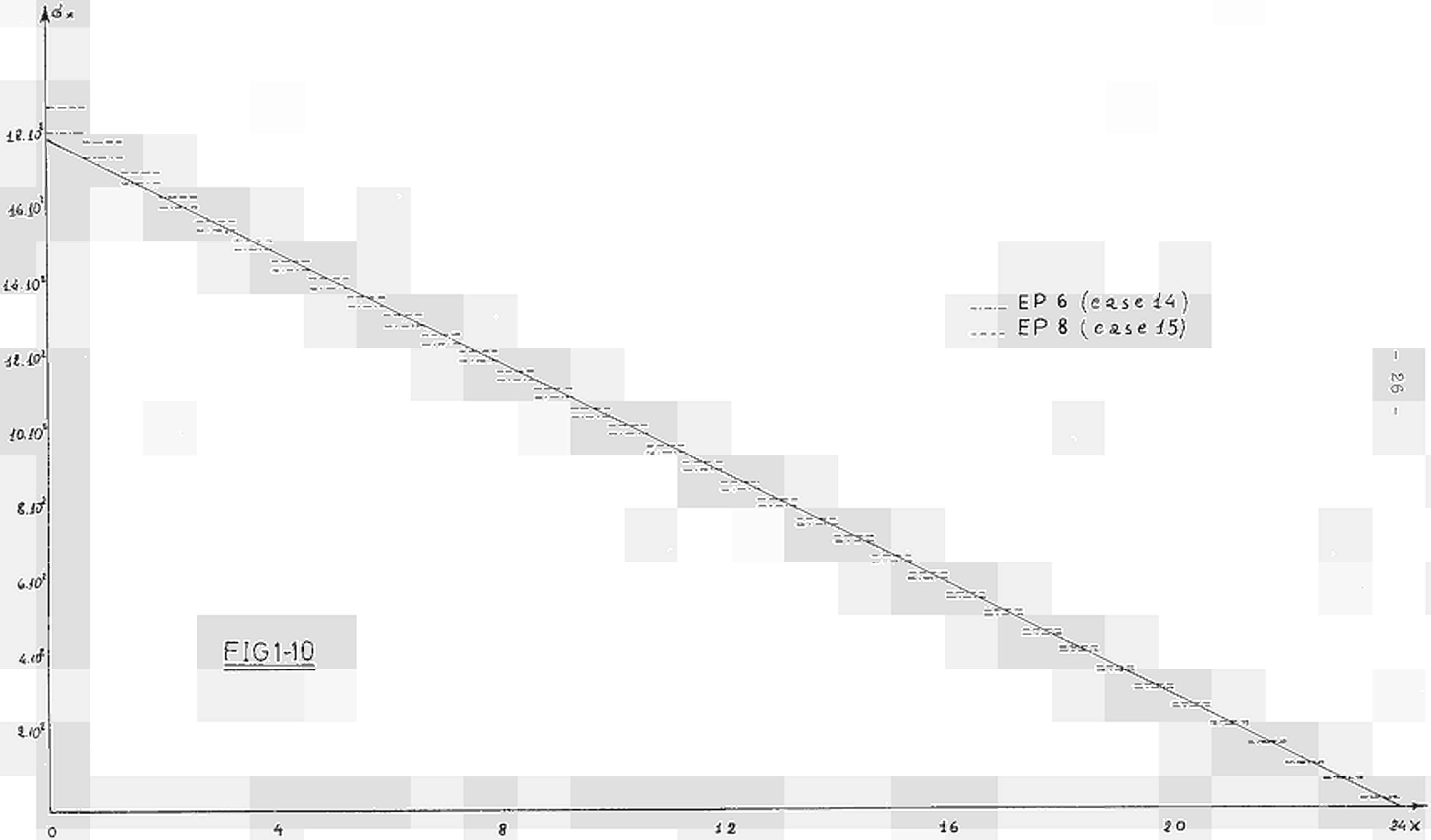
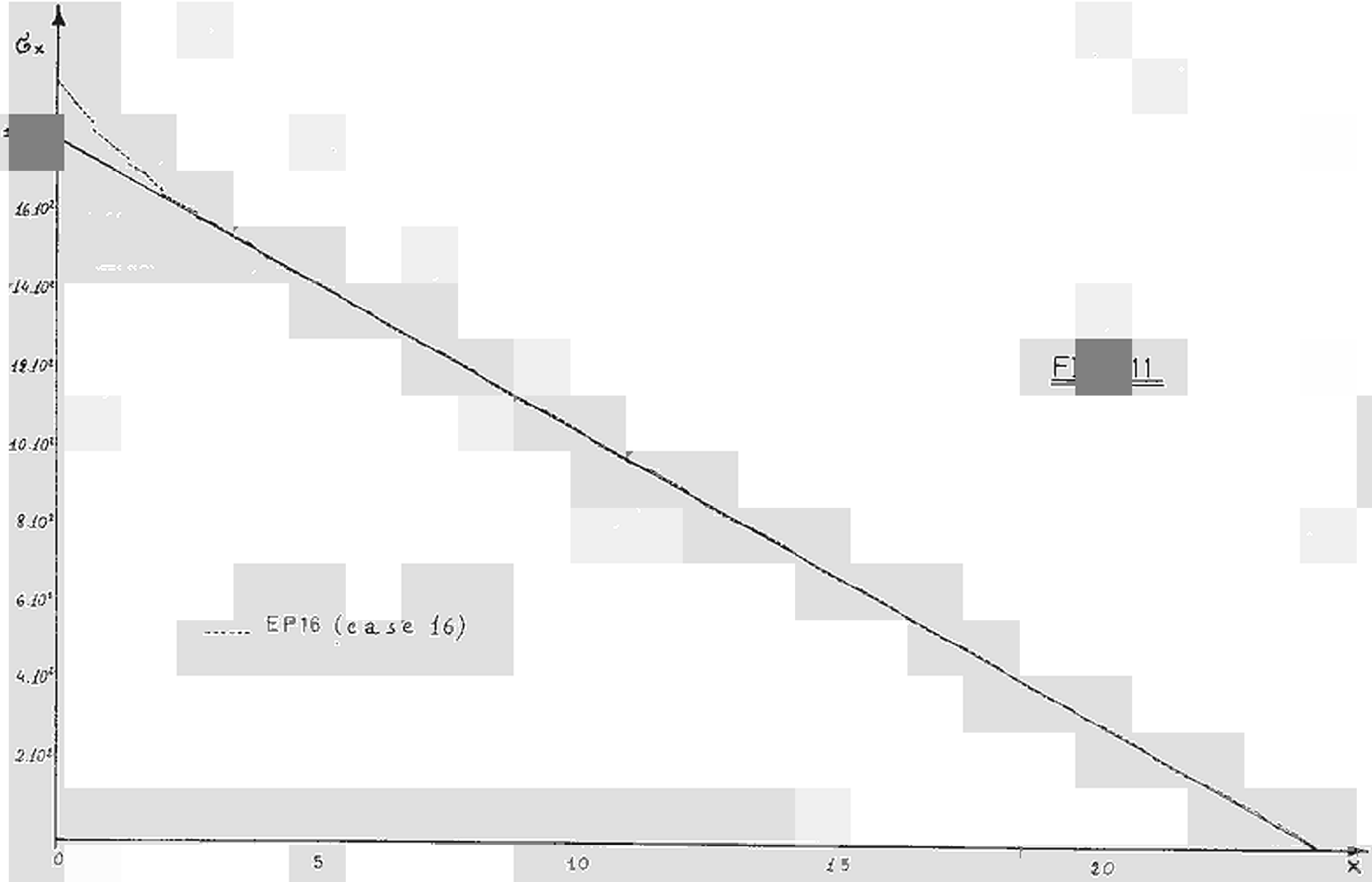

σx
18.10²
16.10²
14.10²
12.10²
10.10²
8.10²
6.10²
4.10²
2.10²
0
4
8
12
16
20
24 X
EP 6 (case 14)
EP 8 (case 15)

FIG1-10



σ_x
16.10²
14.10²
12.10²
10.10²
8.10²
6.10²
4.10²
2.10²
0
5
10
15
20
X
...... EP16 (case 16)

FI[illegible]11

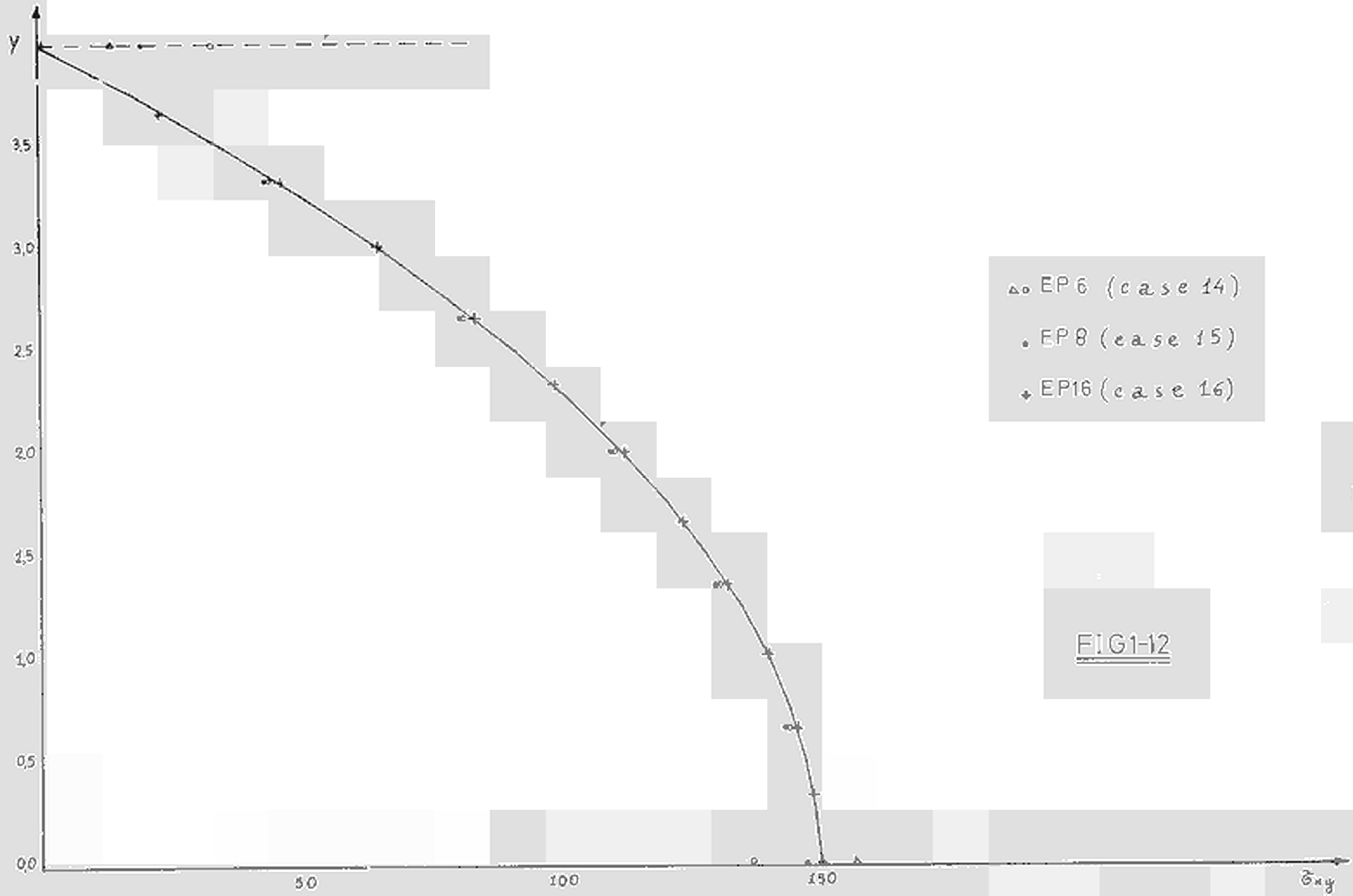

y
3,5
3,0
2,5
2,0
1,5
1,0
0,5
0,0
50
100
150
τxy
EP 6 (case 14)
EP 8 (case 15)
EP16 (case 16)

FIG1-12

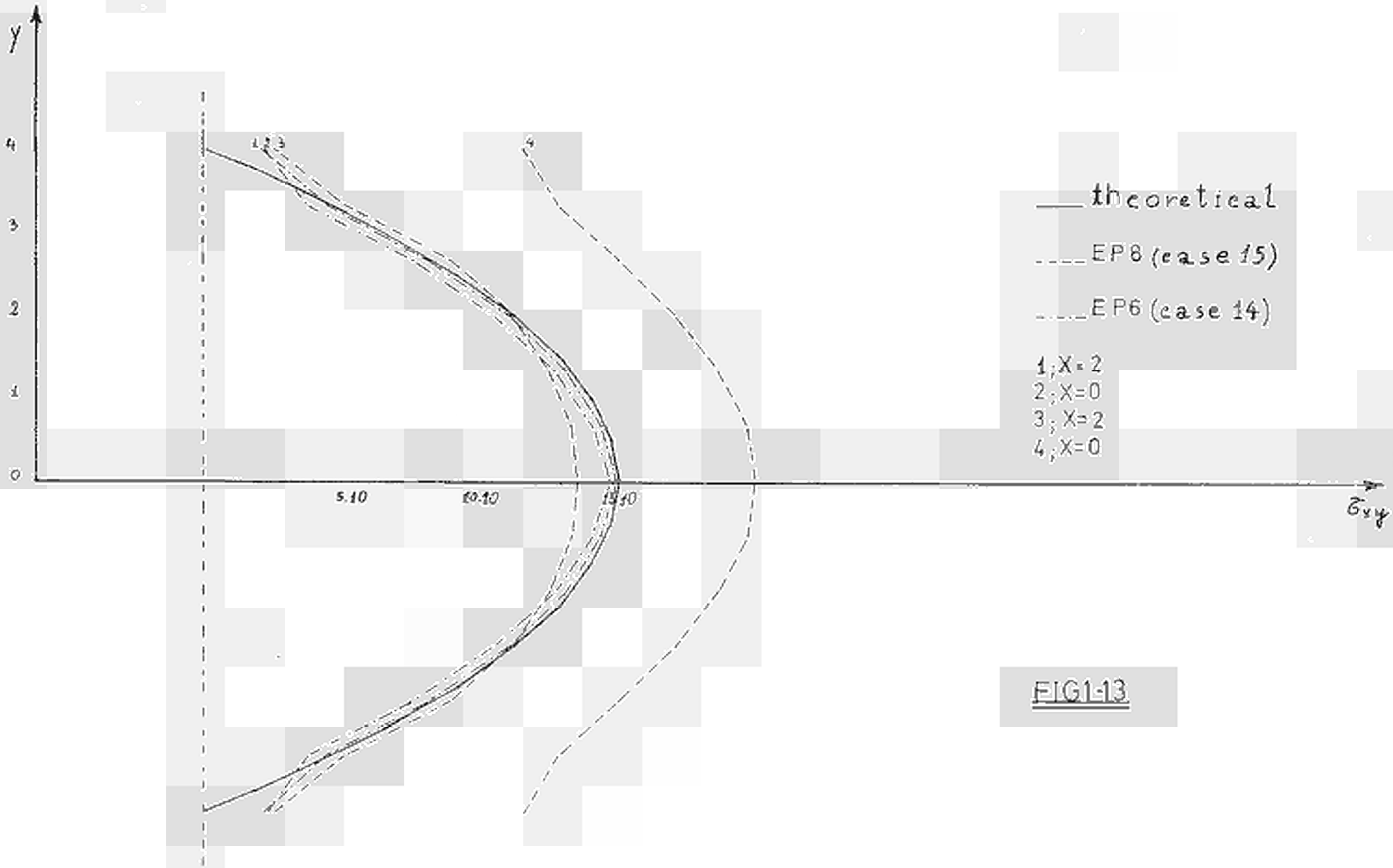

y
4
3
2
1
0
1,2,3
4
5.10
10.10
15.10
σ_{xy}
theoretical
EP8 (case 15)
EP6 (case 14)
1 ; X = 2
2 ; X = 0
3 ; X = 2
4 ; X = 0

FIG1-13

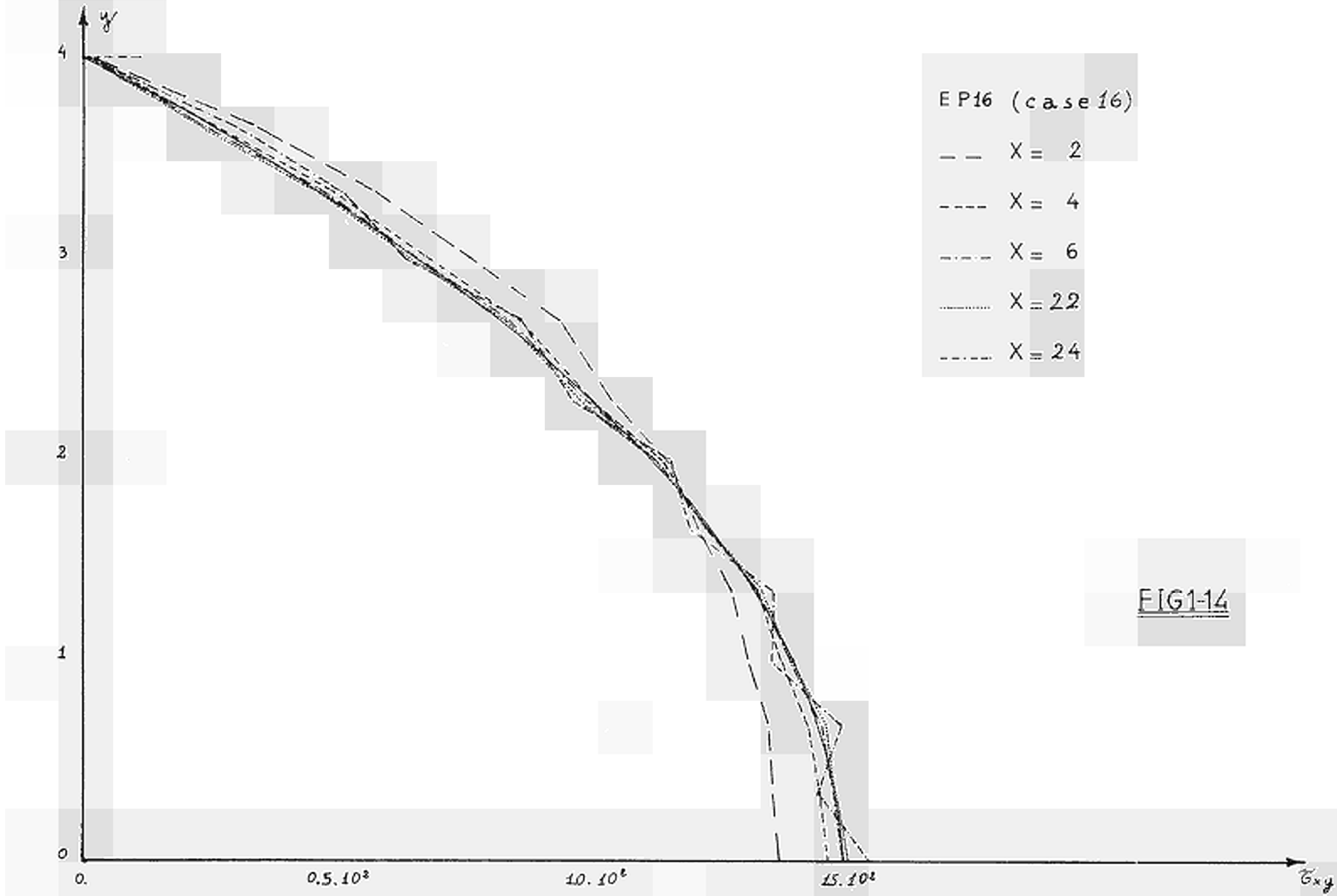


FIG 1-14

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Alfred Nobel

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