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COMMISSION OF THE EUROPEAN COMMUNITIES

TRANSIENT HEAT CONDUCTION THROUGH A CYLINDRICAL THREE-LAYER HOLLOW FUEL ELEMENT WITH INTERNAL AND EXTERNAL COOLING

by

H. WUNDT

1972

Joint Nuclear Research Centre
Ispra Establishment - Italy

Technology

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ABSTRACT

The method, developed in an earlier paper (EUR 4834), of solving Fourier's equation for locally selected temperatures by Laplace transform techniques is extended and applied to the case of a three layer hollow cylinder. The middle, heat-generating cylinder is canned internally and externally by non-producing layers resp., before the heat is carried away by convection, on both sides. The effect of small gaps between the layers is considered. Both coolant bulk temperatures as well as the heat source density are arbitrary input functions of the time. They may also be the outputs of some other programme (neutron kinetics equations, e.g.) so that the systems may be treated simultaneously.

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KEYWORDS

FUEL ELEMENTS
CYLINDERS
CONFIGURATION
HEAT TRANSFER
COOLING
TRANSIENTS
TIME DEPENDENCE
PROGRAMMING
FOURIER HEAT EQUATION
LAPLACE TRANSFORM

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INTRODUCTION *)

In the frame of reactor dynamics, the spatio-temporal temperature distribution in fuel elements must be followed up simultaneously with the evolution of the neutron flux, because temperature changes feedback on reactivity. It is therefore not sufficient to integrate the heat conduction equation for the respective configuration by any method which cannot be coupled with the integration procedure of the reactor-kinetic equations.

In most cases the point model is good enough for the neutron kinetics. This means that one has to deal a priori with ordinary differential equations. Against this, the transient heat conduction equation contains at least two independent variables, the time and (mostly) one space coordinate. However, of practical interest are only temperatures at some selected points and, for the feedback on the reactivity, averaged temperatures over homogeneous regions.

It is therefore obvious to develop a solution method of the heat conduction equation which reduces the partial differential equations to ordinary ones by appropriate elimination of the space variable. Such a reduced system is then suitable to be treated simultaneously with the reactor kinetic equations, by analog as well as by digital computers. The coupling with the kinetic equations is however not necessary. The method rather provides solutions also for the conduction equation separately, whereby the heat source density and the ambient temperature may be considered as free, i.e. not otherwise guided, perturbation inputs as functions of the time.

The method has been developed in an earlier report [1] for one- and two-layer problems in plane, cylindrical and spherical geometry simultaneously. In this paper, the method will be extended to a hollow-cylindrical fuel element, the heat producing layer of which is canned internally as well as externally by a further layer. The convective cooling is, from these canning layers, at the same time inwards and outwards.

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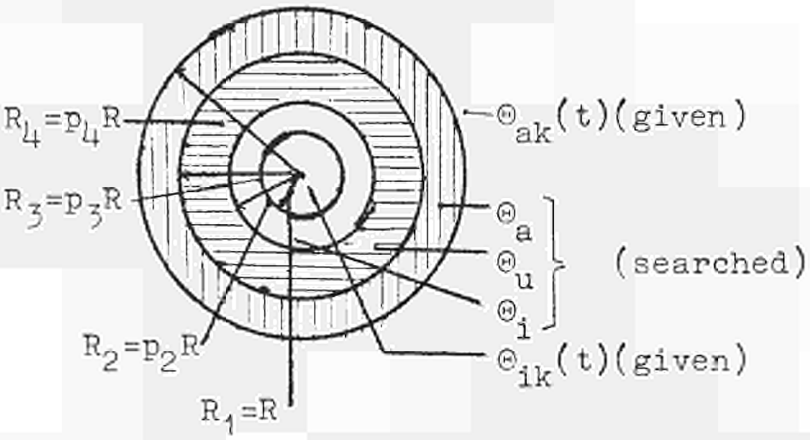
*) Manuscript received on May 23, 1972

It appears that the reasoning of the analysis can be completely taken over from [1]; no particularly new mathematical problems do occur. As however the number of boundary conditions increases with the number of layers considered, the numerical expense grows progressively.

For this reason, the formulation in this part is such that the numerical parameters should be specified in advance. The programmation does not result in a general code, as it is the case in [1], but rather in the treatment of a particular case.

An explanation of the different steps (and their why) is no longer provided here; this must be read in every case in [1].

1. Formulation of the Problem



$0 \leq r \leq R$	internal bore, flowed through by coolant,
$R \leq r \leq p_2R$	internal canning cylinder, index i,
$p_2R \leq r \leq p_3R$	fuel cylinder, with homogeneous heat source, index u,
$p_3R \leq r \leq p_4R$	external canning cylinder, index a,
$p_4R \leq r$	external space, convective cooling.

The dependent system variables are:

$\Theta_i(r,t)$	temperature of the internal cylinder,
$\Theta_u(r,t)$	temperature of the fuel cylinder,
$\Theta_a(r,t)$	temperature of the external cylinder.

The temperature of the coolant flowing inside is denoted by $\Theta_{ik}(t)$, that of the coolant flowing outside by $\Theta_{ak}(t)$. The uniform heat production in the fuel cylinder per unit time and volume is $W(t)$. These three time functions are arbitrary system inputs, or may possibly be furnished by other systems coupled with the present problem.

As the temperature gradients in axial and tangential directions are negligible against that in radial direction (towards the cooling surface), the space coordinates z and φ are disregarded.

The material values which may be different only from layer to layer are denoted by

$\tilde{\lambda}$ - thermal conductivity,
$\tilde{a}^2$ - thermal diffusivity,
ρc - specific heat per unit volume.

There are gaps between the layers i and u, and between u and a, respectively, the widths d_2 and d_3 of which are negligible against the layer thicknesses. The gap widths, as well as the material values, must be assumed to be constant in order to avoid non-linearities.

The FOURIER equations of the three layer temperatures are

$$\frac{1}{\tilde{a}_i^2}\frac{\partial\Theta_i}{\partial t} = \frac{\partial^2\Theta_i}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta_i}{\partial r} \quad , \tag{1.1}$$

$$\frac{1}{\tilde{a}_u^2}\frac{\partial\Theta_u}{\partial t} = \frac{\partial^2\Theta_u}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta_u}{\partial r} + \frac{W(t)}{\tilde{\lambda}_u} \quad , \tag{1.2}$$

$$\frac{1}{\tilde{a}_a^2}\frac{\partial\Theta_a}{\partial t} = \frac{\partial^2\Theta_a}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta_a}{\partial r} \quad . \tag{1.3}$$

The two boundary conditions across a gap between the media 1 and 2 are

$\lambda_1\frac{\partial\Theta_1}{\partial r} = \lambda_2\frac{\partial\Theta_2}{\partial r}$ (heat flux condition; gap without heat capacity) and

$$\Theta_1 = \Theta_2 + \Delta\Theta \ (\Delta\Theta = \text{temperature step across the gap}).$$

We assume neither radiation nor convection, but conduction only. The temperature gradient across the gap is then constant so that

$$\Delta\Theta = \frac{\partial\Theta_g}{\partial r} \cdot d \quad , \qquad \text{with } d = \text{gap width.}$$

In order to evaluate the gradient $\partial\Theta_g/\partial r$, we consider the heat flux condition at one of the surfaces (the other one leading to the same result):

$$\tilde{\lambda}_g \frac{\partial\Theta_g}{\partial r} = \tilde{\lambda}_2 \frac{\partial\Theta_2}{\partial r} \ ,$$

hence, by substitution

$$\Theta_1 = \Theta_2 + d . \frac{\tilde{\lambda}_2}{\tilde{\lambda}_g} \frac{\partial\Theta_2}{\partial r} \ .$$

The temperature Θ_1 at the gap surface is a linear combination of the temperature Θ_2 of the opposite surface and its inward gradient.

For the cooling surfaces R and p_4R (p_2, p_3, p_4 being the radii ratios with respect to the inner radius ($R_1 = R$), we assume the usual convective boundary conditions of the third kind with heat transfer coefficients α_i and α_a, respectively.

Paying attention to the correct signs, the following six boundary conditions may be established:

$$r = R: \ - \tilde{\lambda}_i \frac{\partial\Theta_i}{\partial r}\bigg|_{r=R} = \alpha_i [\Theta_{ik}(t) - \Theta_i(R,t)] \ , \qquad (1.4)$$

$$r = p_2R: \ \tilde{\lambda}_u \frac{\partial\Theta_u}{\partial r}\bigg|_{r=p_2R} = \tilde{\lambda}_i \frac{\partial\Theta_i}{\partial r}\bigg|_{r=p_2R} \ , \qquad (1.5)$$

$$\Theta_u(p_2R,t) = \Theta_i(p_2R,t) + d_2 \cdot \frac{\tilde{\lambda}_i}{\tilde{\lambda}_g} \left.\frac{\partial\Theta_i}{\partial r}\right|_{r=p_2R} , \tag{1.6}$$

$$r = p_3R: \quad \Theta_u(p_3R,t) = \Theta_a(p_3R,t) - d_3 \cdot \frac{\tilde{\lambda}_a}{\tilde{\lambda}_g} \left.\frac{\partial\Theta_a}{\partial r}\right|_{r=p_3R} , \tag{1.7}$$

$$\tilde{\lambda}_a \left.\frac{\partial\Theta_u}{\partial r}\right|_{r=p_3R} = \tilde{\lambda}_a \left.\frac{\partial\Theta_a}{\partial r}\right|_{r=p_3R} , \tag{1.8}$$

$$r = p_4R: + \tilde{\lambda}_a \left.\frac{\partial\Theta_a}{\partial r}\right|_{r=p_4R} = \alpha_a[\Theta_{ak}(t) - \Theta_a(p_4R,t)] . \tag{1.9}$$

The equations and the boundary conditions are LAPLACE-transformed with respect to the time. The initial state is supposed to be the stationary state (all temperatures being then only deviations from this state) so that initial values $\Theta(0)$ do not explicitly occur. With $\vartheta = \mathcal{L}\{\Theta\}$ and $\tilde{W} = \mathcal{L}\{W\}$ we have

$$\frac{s}{\tilde{a}_i^2}\vartheta_i = \frac{d^2\vartheta_i}{dr^2} + \frac{1}{r}\frac{d\vartheta_i}{dr} , \tag{1.10}$$

$$\frac{s}{\tilde{a}_u^2}\vartheta_u = \frac{d\vartheta_u}{dr^2} + \frac{1}{r}\frac{d\vartheta_u}{dr} + \frac{\Delta\tilde{W}(s)}{\tilde{\lambda}_u} , \tag{1.11}$$

$$\frac{s}{\tilde{a}_a^2}\vartheta_a = \frac{d\vartheta_a}{dr^2} + \frac{1}{r}\frac{d\vartheta_a}{dr} . \tag{1.12}$$

We now make the variable transformation

$$x = \frac{\sqrt{s}}{\tilde{a}_u} r, \quad \frac{d}{dx} = {}' \ , \quad \frac{d^2}{dx^2} = {}'' \ , \text{ and } X = \frac{\sqrt{s}}{\tilde{a}_u} R . \tag{1.13}$$

The coefficients are made dimensionless by defining

$$\left.\begin{aligned} &\frac{\tilde{a}_u^2}{\tilde{a}_i^2} = a_i^2, \quad \frac{\tilde{a}_u^2}{\tilde{a}_a^2} = a_a^2 , \quad \frac{\tilde{\lambda}_u}{\tilde{\lambda}_i} = \lambda_i , \quad \frac{\tilde{\lambda}_u}{\tilde{\lambda}_a} = \lambda_i , \quad \frac{\tilde{\lambda}_u}{\tilde{\lambda}_a} = \lambda_a , \\ &\beta_i = \frac{\tilde{\lambda}_i}{\tilde{\lambda}_g} \cdot \frac{d_2}{R} , \quad \beta_a = \frac{\tilde{\lambda}_a}{\tilde{\lambda}_g} \cdot \frac{d_3}{R}, \quad \alpha_i = \frac{\tilde{\lambda}_i}{\alpha_i R} , \quad \alpha_a = \frac{\tilde{\lambda}_a}{\alpha_a R} . \end{aligned}\right\} \tag{1.14}$$

With the abbreviation

$$P(s) = \tilde{a}_u^2 \frac{\Delta\tilde{W}(s)}{\tilde{\lambda}_u} = \frac{\Delta\tilde{W}(s)}{(\rho c)_u} \tag{1.15}$$

the FOURIER equations read

$$a_i^2 \vartheta_i = \vartheta_i'' + \frac{1}{x}\vartheta_i' \quad , \tag{1.16}$$

$$\vartheta_u = \vartheta_u'' + \frac{1}{x}\vartheta_u' + \frac{P(s)}{s} \quad , \tag{1.17}$$

$$a_a^2 \vartheta_a = \vartheta_a'' + \frac{1}{x}\vartheta_a' \quad , \tag{1.18}$$

with the boundary conditions

$$x = X : - q_i X \vartheta_i'(X,s) = \vartheta_{ik}(s) - \vartheta_i(X,s) \quad , \tag{1.19}$$

$$x = p_2 X : \lambda_i \vartheta_u'(p_2 X,s) = \vartheta_i'(p_2 X,s) \quad , \tag{1.20}$$

$$\vartheta_u(p_2 X,s) = \vartheta_i(p_2 X,s) + \beta_i X \vartheta_i'(p_2 X,s) \quad , \tag{1.21}$$

$$x = p_3 X : \vartheta_u(p_3 X,s) = \vartheta_a(p_3 X,s) - \beta_a X \vartheta_a'(p_3 X,s) \quad , \tag{1.22}$$

$$\lambda_a \vartheta_u'(p_3 X,s) = \vartheta_a'(p_3 X,s) \quad , \tag{1.23}$$

$$x = p_4 X : + q_a X \vartheta_a'(p_4 X,s) = \vartheta_{ak}(s) - \vartheta_a(p_4 X,s) \; . \tag{1.24}$$

2. The General Solution in the Complex Domain

The general solution of equations (1.16) to (1.18) is:

$$\vartheta_i(x,s) = I_1(s).F(a_i x) + I_2(s).\Phi(a_i x) \qquad 2.1)$$

$$\vartheta_u(x,s) = U_1(s).F(x) + U_2(s).\Phi(x) + \frac{P(s)}{s} \tag{2.2}$$

$$\vartheta_a(x.s) = A_1(s).F(a_a x) + A_2(s).\Phi(a_a x) \tag{2.3}$$

with coefficients I, U, A to determine from the boundary conditions. The functions F and Φ form a system of linearly independent fundamental solutions of reduced eq. (1.17).

$$F(x) = + I_0(x) \qquad \Phi(x) = - K_0(x) - i\frac{\pi}{2} I_0(x) \tag{2.4}$$

$$F'(x) = + I_1(x) \qquad \Phi'(x) = + K_1(x) - i\frac{\pi}{2} I_1(x) \; . \tag{2.5}$$

For the choice of these combinations of BESSEL functions, see [1]. Do not confuse the BESSEL function $I_1(x)$ with the above coefficient $I_1(s)$.

The coefficients a_i^2 and a_a^2 in (1.16) and (1.18), respectively, reappear in a simple way in the arguments of F and Φ (Eqs. 2.1 and 2.3).

All coefficients I_1 to A_2 from (2.1) to (2.3) depend on the parameter s only, not on x. When substituting the solutions (2.1) into the boundary conditions (1.19) to (1.24), we get the following system of linear equations in matrix notation:

$$\begin{pmatrix} -q_i X F'_{a_i} + F_{a_i} & -q_i X \Phi'_{a_i} + \Phi_{a_i} & 0 & 0 & 0 & 0 \\ -F'_{a_i p_2} & -\Phi'_{a_i p_2} & \lambda_i F'_{p_2} & \lambda_i \Phi'_{p_2} & 0 & 0 \\ -\beta_i X F'_{a_i p_2} - F_{a_i p_2} & -\beta_i X \Phi'_{a_i p_2} - \Phi_{a_i p_2} & F_{p_2} & \Phi_{p_2} & 0 & 0 \\ 0 & 0 & F_{p_3} & \Phi_{p_3} & +\beta_a X F'_{a_a p_3} - F_{a_a p_3} & +\beta_a X \Phi'_{a_a p_3} - \Phi_{a_a p_3} \\ 0 & 0 & \lambda_a F'_{p_3} & \lambda_a \Phi'_{p_3} & -F'_{a_a p_3} & -\Phi'_{a_a p_3} \\ 0 & 0 & 0 & 0 & +q_a X F'_{a_a p_4} + F_{a_a p_4} & +q_a X \Phi'_{a_a p_4} + \Phi_{a_a p_4} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ U_1 \\ U_2 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} +\vartheta_{ik}(s) \\ 0 \\ -\frac{P(s)}{s} \\ -\frac{P(s)}{s} \\ 0 \\ +\vartheta_{ak}(s) \end{pmatrix} \qquad (2.6)$$

The index notation is an abbreviation of the arguments to be applied, e.g., $F'_{a_a p_3} \equiv F'(a_a p_3 X)$. All matrix elements depend upon X only, but not directly on s. The s-dependency enters the system through the inhomogeneous terms ϑ_{ik}, ϑ_{ak}, and P, respectively, the true perturbation functions.

The determinant of the matrix shall be called $\Delta(X)$; it has the infinitely many single zeros X_n (n = 1,2,...). This function-theoretical behaviour is known from [1], no matter how large are the parameter values q, β, λ, a, p.

From here, the computational expense becomes apparent - continuous necessity to develop 6-row determinants. Therefore, we denote the matrix elements briefly with << ik >> (i = line index, k = column index).

E.g., the coefficient $I_1(s)$ results to be $I_1(s) = \frac{\Delta I_1}{\Delta}$, where the determinant ΔI_1 is built from Δ by exchanging (in this case) the first column by the vector of the inhomogeneous terms.

The tedious work to develop all these determinants is not reproduced; we give directly the results:

$$\Delta \doteq \begin{vmatrix} 11 & 12 & 0 & 0 & 0 & 0 \\ 21 & 22 & 23 & 24 & 0 & 0 \\ 31 & 32 & 33 & 34 & 0 & 0 \\ 0 & 0 & 43 & 44 & 45 & 46 \\ 0 & 0 & 53 & 54 & 55 & 56 \\ 0 & 0 & 0 & 0 & 65 & 66 \end{vmatrix}$$

$$= \begin{vmatrix} 11 & 12 & 0 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{vmatrix} \cdot \begin{vmatrix} 44 & 45 & 46 \\ 54 & 55 & 56 \\ 0 & 65 & 66 \end{vmatrix} - \begin{vmatrix} 11 & 12 & 0 \\ 21 & 22 & 24 \\ 31 & 32 & 34 \end{vmatrix} \cdot \begin{vmatrix} 43 & 45 & 46 \\ 53 & 55 & 56 \\ 0 & 65 & 66 \end{vmatrix} \qquad (2.7)$$

This relatively simple representation by products of 3-row determinants only is obviously due to the particular zero-element distribution in the main determinant only.

When developing the determinants ΔI_1 etc., we arrange the results according to the three inputs $\vartheta_{ak}(s)$, $\vartheta_{ak}(s)$, and $P(s)/s$, multiplied respectively with certain conglomerates $Z_{\cdot\cdot}(X)$, which, divided by $\Delta(X)$, are just the transfer functions to apply on these inputs.

$$\Delta I_1 = +\,\vartheta_{ik}(s)\cdot\underbrace{\left(\begin{vmatrix}22 & 23\\ 32 & 33\end{vmatrix}\cdot\begin{vmatrix}44 & 45 & 46\\ 54 & 55 & 56\\ 0 & 65 & 66\end{vmatrix} - \begin{vmatrix}22 & 24\\ 32 & 34\end{vmatrix}\cdot\begin{vmatrix}43 & 45 & 46\\ 53 & 55 & 56\\ 0 & 65 & 66\end{vmatrix}\right)}_{Z_{I_1 i}} +$$

$$+\,\frac{P(s)}{s}\cdot\underbrace{\left(-12\cdot\left[\left\{\begin{vmatrix}23 & 24\\ 43 & 44\end{vmatrix} - \begin{vmatrix}23 & 24\\ 33 & 44\end{vmatrix}\right\}\cdot\begin{vmatrix}55 & 56\\ 65 & 66\end{vmatrix} - \begin{vmatrix}23 & 24\\ 53 & 54\end{vmatrix}\cdot\begin{vmatrix}45 & 46\\ 65 & 66\end{vmatrix}\right]\right)}_{Z_{I_1 p}} +$$

$$+\,\vartheta_{ak}(s)\cdot\underbrace{\left(-12\cdot\begin{vmatrix}23 & 24\\ 33 & 34\end{vmatrix}\cdot\begin{vmatrix}45 & 46\\ 55 & 56\end{vmatrix}\right)}_{Z_{I_1 a}} \qquad (2.8)$$

Of course, none of these elements except "0", also not "12" as a factor, means the number itself, but the respective element to be identified from the main matrix in (2.6), with the aid of (2.7).

$$\Delta I_2 = +\,\vartheta_{ik}(s)\cdot\underbrace{\left(-\begin{vmatrix}21 & 23\\ 31 & 33\end{vmatrix}\cdot\begin{vmatrix}44 & 45 & 46\\ 54 & 55 & 56\\ 0 & 65 & 66\end{vmatrix} + \begin{vmatrix}21 & 24\\ 31 & 34\end{vmatrix}\cdot\begin{vmatrix}43 & 45 & 46\\ 53 & 55 & 56\\ 0 & 65 & 66\end{vmatrix}\right)}_{Z_{I_2 i}} +$$

$$+\,\frac{P(s)}{s}\cdot\underbrace{\left(+11\cdot\left[\left\{\begin{vmatrix}23 & 24\\ 43 & 44\end{vmatrix} - \begin{vmatrix}23 & 24\\ 33 & 34\end{vmatrix}\right\}\cdot\begin{vmatrix}55 & 56\\ 65 & 66\end{vmatrix} - \begin{vmatrix}23 & 24\\ 65 & 66\end{vmatrix}\cdot\begin{vmatrix}45 & 46\\ 65 & 66\end{vmatrix}\right]\right)}_{Z_{I_2 p}} +$$

$$+\ \vartheta_{ak}(s).\underbrace{\left(+\ 11\ .\begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} . \begin{vmatrix} 45 & 46 \\ 55 & 56 \end{vmatrix}\right)}_{Z_{I_2 a}} \qquad (2.9)$$

ΔI_2 results from ΔI_1 by replacing always the rear index 2 by 1 and by changing the sign in the respective terms.

$$\Delta U_1 = +\ \vartheta_{ik}(s).\underbrace{\left(\begin{vmatrix} 21 & 22 \\ 31 & 32 \end{vmatrix} . \begin{vmatrix} 44 & 45 & 46 \\ 54 & 55 & 56 \\ 0 & 65 & 66 \end{vmatrix}\right)}_{Z_{U_1 i}} +$$

$$+\ \frac{P(s)}{s}\ .\ \underbrace{\left(-\begin{vmatrix} 11 & 12 \\ 21 & 22 \end{vmatrix} . \begin{vmatrix} 44 & 45 & 46 \\ 54 & 55 & 56 \\ 0 & 65 & 66 \end{vmatrix} + \begin{vmatrix} 55 & 56 \\ 65 & 66 \end{vmatrix} . \begin{vmatrix} 11 & 12 & 0 \\ 21 & 22 & 24 \\ 31 & 32 & 34 \end{vmatrix}\right)}_{Z_{U_1 p}} +$$

$$+\ \vartheta_{ak}(s)\ .\ \underbrace{\left(-\begin{vmatrix} 45 & 46 \\ 55 & 56 \end{vmatrix} . \begin{vmatrix} 11 & 12 & 0 \\ 21 & 22 & 24 \\ 31 & 32 & 34 \end{vmatrix}\right)}_{Z_{U_1 a}} \qquad (2.10)$$

$$\Delta U_2 = +\ \vartheta_{ik}(s).\underbrace{\left(-\begin{vmatrix} 21 & 22 \\ 31 & 32 \end{vmatrix} . \begin{vmatrix} 43 & 45 & 46 \\ 53 & 55 & 56 \\ 0 & 65 & 66 \end{vmatrix}\right)}_{Z_{U_2 i}} +$$

$$+\frac{P(s)}{s}\cdot\underbrace{\left(\begin{vmatrix}11 & 12\\ 21 & 22\end{vmatrix}\cdot\begin{vmatrix}43 & 45 & 46\\ 53 & 55 & 56\\ 0 & 65 & 66\end{vmatrix}-\begin{vmatrix}55 & 56\\ 65 & 66\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 23\\ 31 & 32 & 33\end{vmatrix}\right)}_{Z_{U_2p}}+$$

$$+\,\vartheta_{ak}(s)\cdot\underbrace{\left(\begin{vmatrix}45 & 46\\ 55 & 56\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 23\\ 31 & 32 & 33\end{vmatrix}\right)}_{Z_{U_2a}}. \qquad (2.11)$$

ΔU_2 results from ΔU_1 by replacing the rear index 4 by 3 and by changing the sign in the respective terms.

$$\Delta A_1 = +\vartheta_{ik}(s).\underbrace{\left(+66\cdot\begin{vmatrix}43 & 44\\ 53 & 54\end{vmatrix}\cdot\begin{vmatrix}21 & 22\\ 31 & 32\end{vmatrix}\right)}_{Z_{A_1i}}+$$

$$+\frac{P(s)}{s}\cdot\underbrace{\left(+66\cdot\left[\left\{\begin{vmatrix}33 & 34\\ 53 & 54\end{vmatrix}-\begin{vmatrix}43 & 44\\ 53 & 54\end{vmatrix}\right\}\cdot\begin{vmatrix}11 & 12\\ 21 & 22\end{vmatrix}-\begin{vmatrix}23 & 24\\ 53 & 54\end{vmatrix}\cdot\begin{vmatrix}11 & 12\\ 31 & 32\end{vmatrix}\right]\right)}_{Z_{A_1p}}+$$

$$+\vartheta_{ak}(s).\underbrace{\left(\begin{vmatrix}44 & 46\\ 54 & 56\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 23\\ 31 & 32 & 33\end{vmatrix}+\begin{vmatrix}43 & 46\\ 53 & 56\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 24\\ 31 & 32 & 34\end{vmatrix}\right)}_{Z_{A_1a}} \qquad (2.12),$$

$$\Delta A_2 = +\vartheta_{ik}(s).\underbrace{\left(-65\cdot\begin{vmatrix}43 & 44\\ 53 & 54\end{vmatrix}\cdot\begin{vmatrix}21 & 22\\ 31 & 32\end{vmatrix}\right)}_{Z_{A_2i}}+$$

$$+\frac{P(s)}{s}\cdot\underbrace{\left(-65.\left[\left\{\begin{vmatrix}33 & 34\\ 53 & 54\end{vmatrix}-\begin{vmatrix}43 & 44\\ 53 & 54\end{vmatrix}\right\}\cdot\begin{vmatrix}11 & 12\\ 21 & 22\end{vmatrix}-\begin{vmatrix}23 & 24\\ 53 & 54\end{vmatrix}\cdot\begin{vmatrix}11 & 22\\ 31 & 32\end{vmatrix}\right]\right)}_{Z_{A_2p}}+$$

$$+\vartheta_{ak}(s)\cdot\underbrace{\left(\begin{vmatrix}44 & 45\\ 54 & 55\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 23\\ 31 & 32 & 33\end{vmatrix}-\begin{vmatrix}43 & 45\\ 53 & 55\end{vmatrix}\cdot\begin{vmatrix}11 & 12 & 0\\ 21 & 22 & 24\\ 31 & 32 & 34\end{vmatrix}\right)}_{Z_{A_2a}} \qquad (2.13)$$

ΔA_2 results from ΔA_1 by replacing the rear index 6 by 5 and by changing the sign in the respective terms.

Moreover, ΔI_1 changes into ΔA_2, ΔI_2 into ΔA_1, and ΔU_1 into ΔU_2, by replacing $+\vartheta_{ik}(s)$ by $+\vartheta_{ak}(s)$ and $\frac{P(s)}{s}$ by itself, and by changing all determinant elements into their complement numbers to 7. Possible line exchanges always cancel.

The determinant development has been stopped at this stage, because the further treatment must anyhow be performed by a computer. For this purpose, the presentation is sufficiently lucid.

3. The Selected Temperatures

The following temperatures are selected to be treated further:

Internal cylinder	- internal surface	:	$\vartheta_i(X,s)$
" "	- mean temperature	:	$\overline{\vartheta}_i(s)$
" "	- external surface	:	$\vartheta_i(p_2X,s)$
Fuel cylinder	- internal surface	:	$\vartheta_u(p_2X,s)$
" "	- mean temperature	:	$\overline{\vartheta}_u(s)$
" "	- external surface	:	$\vartheta_u(p_3X,s)$
External cylinder	- internal surface	:	$\vartheta_a(p_3X,s)$
" "	- mean temperature	:	$\overline{\overline{\vartheta}}_a(s)$
" "	- external surface	:	$\vartheta_a(p_4X,s)$

It would be interesting to know also the maximum temperature of the heat producing middle cylinder. This computation is however not feasible, as the position of the maximum is not known but should be evaluated from $d\vartheta_u/dx = 0$. The locus of x_{max} depends therefore on s (i.e. on the time) that makes the inverse transformation impossible.

The needed averaging rules are (see [I]):

internal cylinder: $$\bar{F}=\frac{2}{X^2(p_2^2-1)}\int_X^{p_2X} F(a_i x)x\,dx=\frac{2}{a_iX(p_2^2-1)}(p_2F'_{a_ip_2}-F'_{a_i}) \quad (3.1)$$

middle cylinder: $$\bar{\bar{F}}=\frac{2}{X^2(p_3^2-p_2^2)}\int_{p_2X}^{p_3X} F(x)x\,dx=\frac{2}{X(p_3^2-p_2^2)}(p_3F'_{p_3}-p_2F'_{p_2}) \quad (3.2)$$

external cylinder: $$\bar{\bar{\bar{F}}}=\frac{2}{X^3(p_4^2-p_3^2)}\int_{p_3X}^{p_4X} F(a_a x)x\,dx=\frac{2}{a_aX(p_4^2-p_3^2)}(p_4F'_{a_ap_4}-p_3F'_{a_ap_3}) \quad (3.3)$$

and correspondingly for $\bar{\Phi}$, $\bar{\bar{\Phi}}$, $\bar{\bar{\bar{\Phi}}}$.

For the functions G and ψ, to be defined in chapter 5, one has

$$\bar{G} = \frac{-2}{a_i\sigma(p_2^2-1)}(p_2G'_{a_ip_2}-G'_{a_i}) \quad ,$$

and so on (always with minus sign).

4. Solution for the Selected Temperatures in the Complex Domain

When writing down the solutions (2.1) to (2.3) for the selected temperatures and arranging the coefficients according to the three input functions $\vartheta_{ik}(s)$, $\vartheta_{ak}(s)$ and $P(s)/s$, one obtains by using the notation Z_{I_1i}, ..., defined in chapter 2:

$$\vartheta_i(X,s) = \frac{1}{\Delta}\Big[(Z_{I_1 i}\cdot F_{ai} + Z_{I_1 i}\cdot\Phi_{ai})\,\vartheta_{ik}(s) + (Z_{I_1 p}\cdot F_{ai} + Z_{I_2 p}\cdot\Phi_{ai})\frac{P(s)}{s} + (Z_{I_1 a}\cdot F_{ai} + Z_{I_2 a}\cdot\Phi_{ai})\,\vartheta_{ak}(s)\Big] \quad (4.1)$$

$$\bar{\vartheta}_i(s) = \frac{1}{\Delta}\Big[(Z_{I_1 i}\cdot\bar{F} + Z_{I_2 i}\cdot\bar{\Phi})\,\vartheta_{ik}(s) + (Z_{I_1 p}\cdot\bar{F} + Z_{I_2 p}\cdot\bar{\Phi})\frac{P(s)}{s} + (Z_{I_1 a}\bar{F} + Z_{I_2 a}\cdot\bar{\Phi})\,\vartheta_{ak}(s)\Big] \quad (4.2)$$

$$\vartheta_i(p_2X,s) = \frac{1}{\Delta}\Big[(Z_{I_1 i}\cdot F_{aip_2} + Z_{I_2 i}\cdot\Phi_{aip_2})\,\vartheta_{ik}(s) + (Z_{I_1 p}\cdot F_{aip_2} + Z_{I_2 p}\cdot\Phi_{aip_2})\frac{P(s)}{s} + (Z_{I_1 a}\cdot F_{aip_2} + Z_{I_2 a}\cdot\Phi_{aip_2})\,\vartheta_{ak}(s)\Big] \quad (4.3)$$

$$\vartheta_u(p_2X,s) = \frac{1}{\Delta}\Big[(Z_{U_1 i}\cdot F_{p_2} + Z_{U_2 i}\cdot\Phi_{p_2})\,\vartheta_{ik}(s) + (Z_{U_1 p}\cdot F_{p_2} + Z_{U_2 p}\cdot\Phi_{p_2})\frac{P(s)}{s} + (Z_{U_1 a}\cdot F_{p_2} + Z_{U_2 a}\cdot\Phi_{p_2})\,\vartheta_{ak}(s)\Big] + \frac{P(s)}{s} \quad (4.4)$$

$$\bar{\bar{\vartheta}}_u(s) = \frac{1}{\Delta}\Big[(Z_{U_1 i}\cdot\bar{\bar{F}} + Z_{U_2 i}\cdot\bar{\bar{\Phi}})\,\vartheta_{ik}(s) + (Z_{U_1 p}\cdot\bar{\bar{F}} + Z_{U_2 p}\cdot\bar{\bar{\Phi}})\frac{P(s)}{s} + (Z_{U_1 a}\cdot\bar{\bar{F}} + Z_{U_2 a}\cdot\bar{\bar{\Phi}})\,\vartheta_{ak}(s)\Big] + \frac{P(s)}{s} \quad (4.5)$$

$$\vartheta_u(p_3X,s) = \frac{1}{\Delta}\Big[(Z_{U_1 i}\cdot F_{p_3} + Z_{U_2 i}\cdot\Phi_{p_3})\,\vartheta_{ik}(s) + (Z_{U_1 p}\cdot F_{p_3} + Z_{U_2 p}\cdot\Phi_{p_3})\frac{P(s)}{s} + (Z_{U_1 a}\cdot F_{p_3} + Z_{U_2 a}\cdot\Phi_{p_3})\,\vartheta_{ak}(s)\Big] + \frac{P(s)}{s} \quad (4.6)$$

$$\vartheta_a(p_3X,s) = \frac{1}{\Delta}\Big[(Z_{A_1 i}\cdot F_{a_a p_3} + Z_{A_2 i}\cdot\Phi_{a_a p_3})\,\vartheta_{ik}(s) + (Z_{A_1 p}\cdot F_{a_a p_3} + Z_{A_2 p}\cdot\Phi_{a_a p_3})\frac{P(s)}{s} + (Z_{A_1 a}\cdot F_{a_a p_3} + Z_{A_2 a}\cdot\Phi_{a_a p_3})\,\vartheta_{ak}(s)\Big] \quad (4.7)$$

$$\bar{\bar{\bar{\vartheta}}}_a(s) = \frac{1}{\Delta}\Big[(Z_{A_1 i}\cdot\bar{\bar{\bar{F}}} + Z_{A_2 i}\cdot\bar{\bar{\bar{\Phi}}})\,\vartheta_{ik}(s) + (Z_{A_1 p}\cdot\bar{\bar{\bar{F}}} + Z_{A_2 p}\cdot\bar{\bar{\bar{\Phi}}})\frac{P(s)}{s} + (Z_{A_2 a}\cdot\bar{\bar{\bar{F}}} + Z_{A_2 a}\cdot\bar{\bar{\bar{\Phi}}})\,\vartheta_{ak}(s)\Big] \quad (4.8)$$

$$\vartheta_a(p_4X,s) = \frac{1}{\Delta}\Big[(Z_{A_1 i}\cdot F_{a_a p_4} + Z_{A_2 i}\cdot\Phi_{a_a p_4})\,\vartheta_{ik}(s) + (Z_{A\ p}\cdot F_{a_a p_4} + Z_{A_2 p}\cdot\Phi_{a_a p_4})\frac{P(s)}{s} + (Z_{A_1 a}\cdot F_{a_a p_4} + Z_{A_2 a}\cdot\Phi_{a_a p_4})\,\vartheta_{ak}(s)\Big] \quad (4.9)$$

The expressions in curved brackets, divided by Δ, are the transfer functions acting on the inputs $\vartheta_{ik}(s)$, $\frac{P(s)}{s}$, or $\vartheta_{ak}(s)$, respectively.

These transfer functions are now developed into partial fraction series, whereby it is known that only single poles occur, because Δ has only single zeros. Numerator and denominator have never common zeros.

After multiplication with s, one puts:

$$s\vartheta_i(X,s) = \Big(a_{io}+\sum\frac{a_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(a_{po}+\sum\frac{a_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(a_{ao}+\sum\frac{a_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.10}$$

$$s\overline{\vartheta_i}(s) = \Big(b_{io}+\sum\frac{b_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(b_{po}+\sum\frac{b_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(b_{ao}+\sum\frac{b_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.11}$$

$$s\vartheta_i(p_2X,s) = \Big(c_{io}+\sum\frac{c_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(c_{po}+\sum\frac{c_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(c_{ao}+\sum\frac{c_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.12}$$

$$s\vartheta_u(p_2X,s) = \Big(d_{io}+\sum\frac{d_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(d_{po}+\sum\frac{d_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(d_{ao}+\sum\frac{d_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.13}$$

$$s\overset{=}{\vartheta}_u(s) = \Big(e_{io}+\sum\frac{e_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(e_{po}+\sum\frac{e_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(e_{ao}+\sum\frac{e_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.14}$$

$$s\vartheta_u(p_3X,s) = \Big(f_{io}+\sum\frac{f_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(f_{po}+\sum\frac{f_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(f_{ao}+\sum\frac{f_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.15}$$

$$s\vartheta_a(p_3X,s) = \Big(g_{io}+\sum\frac{g_{in}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ik}(s) + \Big(g_{po}+\sum\frac{h_{pn}}{s+\frac{\sigma^2_n}{\nu}}\Big)P(s) + \Big(g_{ao}+\sum\frac{h_{an}}{s+\frac{\sigma^2_n}{\nu}}\Big)s\vartheta_{ak}(s) \tag{4.16}$$

$$s\bar{\bar{\vartheta}}_a(s) = (h_{io}+\sum\frac{h_{in}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})s\vartheta_{ik}(s) + (h_{po}+\sum\frac{h_{pn}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})P(s) + (h_{ao}+\sum\frac{h_{an}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})s\vartheta_{ak}(s) \quad (4.17)$$

$$s\vartheta_a(p_4X,s) = (j_{io}+\sum\frac{j_{in}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})s\vartheta_{ik}(s) + (j_{po}+\sum\frac{j_{pn}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})P(s) + (j_{ao}+\sum\frac{j_{an}}{\frac{\sigma^2}{s+\frac{n}{\nu}}})s\vartheta_{ak}(s) \quad (4.18)$$

where $\nu \equiv \frac{R^2}{\tilde{a}_u^2}$ is the time constant to be chosen because of $X = \frac{\sqrt{s}}{\tilde{a}_u} R$, $\sigma = \pm\sqrt{\nu s}$, $X = \pm i\sigma$. All summations go from n = 1 to infinity.

5. The Modified Functions and the Discriminant Equation

As the zeros of F and of G lie all on the imaginary axis, we pass over to the modified functions G and ψ, whose zeros are all real. So instead of considering e.g. the zeros of $F(X) = F(i\sigma)$, we consider the zeros of $G(\sigma)$ that is more simple.

The original and the modified functions must satisfy the functional relations (see [1]):

$F(+i\sigma) = + G(\sigma)$	$F(-i\sigma) = + G(\sigma)$	(5.1)
$F'(+i\sigma) = - iG'(\sigma)$	$F'(-i\sigma) = + iG'(\sigma)$	(5.2)
$\Phi(+i\sigma) = + \psi'(\sigma)$	$\Phi(-i\sigma) = + \psi(\sigma) - i\pi G(\sigma)$	(5.3)
$\Phi'(+i\sigma) = - i\psi'(\sigma)$	$\Phi'(-i\sigma) = - i\psi'(\sigma) + \pi G'(\sigma)$	(5.4)

so that the modified functions are simply (as desired):

$G(\sigma) = + J_o(\sigma)$	$\psi(\sigma) = + \frac{\pi}{2} Y_o(\sigma)$	(5.5)
$G'(\sigma) = - J_1(\sigma)$	$\psi'(\sigma) = - \frac{\pi}{2} Y_1(\sigma)$.	(5.6)

By substituting into (2.6) $X = +i\sigma$, we get (for argument $-i\sigma$, the additional terms with G and G' in (5.3) and (5.4) create an additional separable determinant which however vanishes so that no new zeros occur for $X = -i\sigma$):

$$\Delta(\sigma) = -\begin{vmatrix} -q_i\sigma G'_{a_i}+G_{a_i} & -q_i\sigma\psi'_{a_i}+\psi_{a_i} & 0 & 0 & 0 & 0 \\ -G'_{a_i p_2} & -\psi'_{a_i p_2} & \lambda_i G'_{p_2} & \lambda_i \psi'_{p_2} & 0 & 0 \\ -\beta_i\sigma G'_{a_i p_2}-G_{a_i p_2} & -\beta_i\sigma\psi'_{a_i p_2}-\psi_{a_i p_2} & G_{p_2} & \psi_{p_2} & 0 & 0 \\ 0 & 0 & G_{p_3} & \psi_{p_3} & +\beta_a\sigma G'_{a_a p_3}-G_{a_a p_3} & +\beta_a\sigma\psi'_{a_a p_3}-\psi_{a_a p_3} \\ 0 & 0 & \lambda_a G'_{p_3} & \lambda_a\psi'_{p_3} & -G'_{a_a p_3} & -\psi'_{a_a p_3} \\ 0 & 0 & 0 & 0 & +\sigma_a\sigma G'_{a_a p_4}+G_{a_a p_4} & +\sigma_a\sigma\psi'_{a_a p_4}+\psi_{a_a p_4} \end{vmatrix} \stackrel{!}{=} 0$$

*) **) (5.7)

*) N.B.

From each of the lines 2 and 5, a factor of -i has been extracted, giving together the minus sign before the determinant.

**) The indices mean anew the arguments to apply, e.g. $G'_{a_a p_3} \equiv G'(a_a p_3 \sigma)$.

This "discriminant equation", which may be developed just according to the general rule of (2.7), has the infinitely many single zeros σ_n $(n=1,\dots,\infty)$.

They must be determined for the given set of parameters q, β, a, λ, p by computer. All elements are then known numerical values if the function values of the G, G', ψ, ψ' are furnished from elsewhere or are directly furnished by the computer through a BESSEL function subroutine.

$\sigma = 0$ is in no case a zero. There is the special (pathological) case $q_i = q_a = 0$; $a_i = a_a = 1$; $\lambda_i = \lambda_a = 1$; $\beta_i = \beta_a = 0$. Here, factors $G_{p_2}\psi'_{p_2} - G'_{p_2}\psi_{p_2}$ (or with p_3 or p_4, resp.) occur which apparently give further zeros. However, these expressions are the WRONSKIan determinants which are zeroless in the finite domain; they have the values $\frac{1}{p_2\sigma}$, $\frac{1}{p_3\sigma}$, $\frac{1}{p_4\sigma}$, respectively.

6. Computation of the Residues

The coefficients of the partial fraction series (4.10) to (4.18) are the residues at the poles of the respective transfer functions.

Generally, we compute these residues according to the rule

$$a_{in} = \left. \frac{Z_{I_1 i}\cdot F_{ai} + Z_{I_2 i}\cdot \Phi_{ai}}{\frac{d\Delta}{ds}} \right|_{s=-\frac{\sigma_n^2}{\nu}} , \tag{6.1}$$

and correspondingly also the other coefficients $a_{rn}, \dots$, etc., by comparison of the respective terms from (4.1) to (4.5) with (4.10) to (4.18).

The derivative of the system determinant (5.7) is, because of $\sigma = \pm\, i\sqrt{\nu s}$:

$$\frac{d\Delta(\sigma)}{ds} = \frac{d\Delta}{d\sigma}\frac{d\sigma}{ds} = \pm i \frac{d\Delta}{d\sigma}\frac{d}{ds}\sqrt{\nu s} = \pm i \frac{d\Delta}{ds}\frac{\sqrt{\nu}}{2\sqrt{s}} = \pm\frac{d\Delta}{d\sigma}\frac{\nu}{2\sqrt{\nu s}} = -\frac{d\Delta}{d\sigma}\frac{\nu}{2.\pm\sqrt{\nu s}}$$

$$= -\frac{d\Delta}{d\sigma}\frac{\nu}{2\sigma} \quad \text{(no sign ambiguity)},$$

hence

$$\left.\frac{d\Delta}{ds}\right|_{s=-\frac{\sigma_n^2}{\nu}} = -\frac{\nu}{2\sigma_n}\left.\frac{d\Delta(\sigma)}{d\sigma}\right|_{\sigma=\sigma_n} \equiv \Delta'_n$$

A determinant is differentiated with respect to a parameter (here σ) by successively replacing the first column by its derivate (the vector of the differentiated elements) and evaluating the so resulting determinant, then by doing so for the second column, etc., and by finally adding all results. This work can automatically be performed by the computer as soon as the parameters are numerically fixed. Otherwise the preparative work in a multidimensional parameter space would be quite boundless.

When differentiating the elements of (5.7), the occurring second derivatives can be eliminated by means of the differential equations for G and ψ, e.g. $G''_{a_i p_2} = -G_{a_i p_2} - \frac{1}{a_i p_2 \sigma} G'_{a_i p_2}$, etc.

Thus, the differentiated column vectors are:

$$\left.\begin{array}{ll}
\underline{k=1} & \underline{k=2} \\[4pt]
\begin{pmatrix}
(-q_i+a_i+1)G'_{a_i}+a_i\sigma G_a \\
+\frac{1}{\sigma}G'_{a_ip_2}+a\;p_2G_{a_ip_2} \\
(-\beta_i-a_ip_2+1)G'_{a_ip_2}\;+a_ip_2\sigma G_{a_ip_2} \\
0 \\
0 \\
0
\end{pmatrix}
&
\begin{pmatrix}
\text{idem as for } k=1, \\
\text{but } G \text{ replaced by } \psi \\
\\
\\
\\
\\
\end{pmatrix}
\\[12pt]
\underline{k=3} & \underline{k=4} \\[4pt]
\begin{pmatrix}
0 \\
-\lambda_i\frac{1}{\sigma}G'_{p_2}-\lambda_ip_2G_{p_2} \\
+p_2G'_{p_2} \\
+p_3G'_{p_3} \\
-\lambda_a\frac{1}{\sigma}G'_{p_3}-\lambda_ap_3G_{p_3} \\
0
\end{pmatrix}
&
\begin{pmatrix}
\text{idem as for } k=3, \\
\text{but } G \text{ replaced by } \psi \\
\\
\\
\\
\\
\end{pmatrix}
\\[12pt]
\underline{k=5} & \underline{k=6} \\[4pt]
\begin{pmatrix}
0 \\
0 \\
0 \\
(+\beta_a-a_ap_3-1)G'_{a_ap_3}-a_ap_3\sigma G_{a_ap_3} \\
+\frac{1}{\sigma}G'_{a_ap_3}+a_ap_3G_{a_ap_3} \\
(+q_a+a_ap_4-1)G'_{a_ap_4}-a_ap_4\sigma G_{a_ap_4}
\end{pmatrix}
&
\begin{pmatrix}
\text{idem as for } k=5, \\
\text{but } G \text{ replaced by } \psi \\
\\
\\
\\
\\
\end{pmatrix}
\end{array}\right\} \quad (6.2)$$

For σ, one must always put σ_n, also in the arguments of G and ψ; i.e. all elements are numerically known values.

The terms containing $\frac{1}{\sigma_n}$ are proportional to the corresponding elements in Δ. Therefore, the factor pertaining to $\frac{1}{\sigma_n}$ in the development of Δ'_n is just $-\Delta(\sigma_n)$ which vanishes. One can thus omit all terms with $\frac{1}{\sigma_n}$ and write more simply:

$$k = 1: \quad \begin{pmatrix} (-q_i+a_i+1)G'_{a_i} + a_i\sigma G_{a_i} \\ +a_i p_2 G_{a_i p_2} \\ (-\beta_i - a_i p_2 + 1)G'_{a_i p_2} + a_i p_2 \sigma G_{a_i p_2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad k = 2: \quad \begin{pmatrix} \text{idem as for } k = 1, \\ \text{but } G \text{ replaced by } \psi \end{pmatrix}$$

$$k = 3: \quad \begin{pmatrix} 0 \\ -\lambda_i p_2 G_{p_2} \\ +p_2 G'_{p_2} \\ +p_3 G'_{p_3} \\ -\lambda_a p_3 G_{p_3} \\ 0 \end{pmatrix} \qquad k = 4: \quad \begin{pmatrix} \text{idem as for } k = 3, \\ \text{but } G \text{ replaced by } \psi \end{pmatrix} \qquad (6.3)$$

$$k = 5: \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ (+\beta_a - a_a p_3 - 1)G'_{a_a p_3} - a_a p_3 \sigma G_{a_a p_3} \\ +a_a p_3 G_{a_a p_3} \\ (+q_a + a_a p_4 - 1)G'_{a_a p_4} - a_a p_4 \sigma G_{a_a p_4} \end{pmatrix} \qquad k = 6: \quad \begin{pmatrix} \text{idem as for } k = 5, \\ \text{but } G \text{ replaced by } \psi \end{pmatrix}$$

In order to pass over in the transfer function numerators from the functions F and Φ to the modified functions G and ψ, and at the same time in the arguments from X to σ, we give the behaviour for all matrix elements of (2.6) or (2.7) in the following list:

$1k(X) \rightarrow 1k(\sigma)$

$2k(X) \rightarrow - i.2k(\sigma)$

$3k(X) \rightarrow 3k(\sigma)$

$4k(X) \rightarrow 4k(\sigma)$

$5k(X) \rightarrow - i.5k(\sigma)$

$6k(X) \rightarrow 6k(\sigma)$.

Only the second and fifth line furnish a factor -i, each, so that

$\Delta(X) \rightarrow - \Delta(\sigma)$ (see footnote to 5.7) ,

$Z..(X) \rightarrow - Z..(\sigma)$ (for all Z..) ,

and hence

$$\frac{Z..(X)}{\Delta(X)} \rightarrow + \frac{Z..(\sigma)}{\Delta(\sigma)} \quad \text{(no change).}$$

Furthermore:

$$\begin{array}{llll}
F(X) \rightarrow + G(\sigma) & & \Phi(X) \rightarrow + \psi(\sigma) \\
F'(X) \rightarrow - iG'(\sigma) & & \Phi'(X) \rightarrow - i\psi'(\sigma) \\
\overline{F}(X) \rightarrow - \overline{G}(\sigma) & & \overline{\Phi}(X) \rightarrow - \overline{\psi}(\sigma) \\
\overline{\overline{F}}(X) \rightarrow - \overline{\overline{G}}(\sigma) & & \overline{\overline{\Phi}}(X) \rightarrow - \overline{\overline{\psi}}(\sigma) \\
\overline{\overline{\overline{F}}}(X) \rightarrow - \overline{\overline{\overline{G}}}(\sigma) & & \overline{\overline{\overline{\Phi}}}(X) \rightarrow - \overline{\overline{\overline{\psi}}}(\sigma) .
\end{array}$$

By considering all these rules, one is now prepared to transpose the transfer functions into the modified function notation with argument σ_n, and gets for the residues:

$$\left.\begin{aligned}
a_{in} &= - \frac{Z_{I_1 i}(\sigma_n).G(a_i\sigma_n) + Z_{I_2}(\sigma_n).\psi(a_i\sigma_n)}{\Delta'_n} \\
a_{pn} &= - \frac{Z_{I_1 p}(\sigma_n).G(a_i\sigma_n) + Z_{I_2 p}(\sigma_n).\psi(a_i\sigma_n)}{\Delta'_n} \\
a_{an} &= - \frac{Z_{I_1 a}(\sigma_n).G(a_i\sigma_n) + Z_{I_2 a}(\sigma_n).\psi(a_i\sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.4)$$

$$\left.\begin{aligned}
b_{in} &= + \frac{Z_{I_1 i}(\sigma_n).\bar{G}(\sigma_n) + Z_{I_2 i}(\sigma_n).\bar{\psi}(\sigma_n)}{\Delta'_n} \\
b_{pn} &= + \frac{Z_{I_1 p}(\sigma_n).\bar{G}(\sigma_n) + Z_{I_2 p}(\sigma_n).\bar{\psi}(\sigma_n)}{\Delta'_n} \\
b_{an} &= + \frac{Z_{I_1 a}(\sigma_n).\bar{G}(\sigma_n) + Z_{I_2 a}(\sigma_n).\bar{\psi}(\sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.5)$$

$$\left.\begin{aligned}
c_{in} &= - \frac{Z_{I_1 i}(\sigma_n).G(a_i p_2 \sigma_n) + Z_{I_2 i}(\sigma_n).\psi(a_i p_2 \sigma_n)}{\Delta'_n} \\
c_{pn} &= - \frac{Z_{I_1 p}(\sigma_n).G(a_i p_2 \sigma_n) + Z_{I_2 p}(\sigma_n).\psi(a_i p_2 \sigma_n)}{\Delta'_n} \\
c_{an} &= - \frac{Z_{I_1 a}(\sigma_n).G(a_i p_2 \sigma_n) + Z_{I_2 a}(\sigma_n).\psi(a_i p_2 \sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.6)$$

$$\left.\begin{aligned}
d_{in} &= - \frac{Z_{U_1 i}(\sigma_n).G(p_2 \sigma_n) + Z_{U_2 i}(\sigma_n).\psi(p_2 \sigma_n)}{\Delta'_n} \\
d_{pn} &= - \frac{Z_{U_1 p}(\sigma_n).G(p_2 \sigma_n) + Z_{U_2 p}(\sigma_n).\psi(p_2 \sigma_n)}{\Delta'_n} \\
d_{an} &= - \frac{Z_{U_1 a}(\sigma_n).G(p_2 \sigma_n) + Z_{U_2 a}(\sigma_n).\psi(p_2 \sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.7)$$

$$\left.\begin{aligned}
e_{in} &= + \frac{Z_{U_1 i}(\sigma_n).\bar{\bar{G}}(\sigma_n) + Z_{U_2 i}(\sigma_n).\bar{\bar{\psi}}(\sigma_n)}{\Delta'_n} \\
e_{pn} &= + \frac{Z_{U_1 p}(\sigma_n).\bar{\bar{G}}(\sigma_n) + Z_{U_2 p}(\sigma_n).\bar{\bar{\psi}}(\sigma_n)}{\Delta'_n} \\
e_{an} &= + \frac{Z_{U_1 a}(\sigma_n).\bar{\bar{G}}(\sigma_n) + Z_{U_2 a}(\sigma_n).\bar{\bar{\psi}}(\sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.8)$$

$$\left.\begin{aligned}
f_{in} &= - \frac{Z_{U_1 i}(\sigma_n).G(p_3 \sigma_n) + Z_{U_2 i}(\sigma_n).\psi(p_3 \sigma_n)}{\Delta'_n} \\
f_{pn} &= - \frac{Z_{U_1 p}(\sigma_n).G(p_3 \sigma_n) + Z_{U_2 p}(\sigma_n).\psi(p_3 \sigma_n)}{\Delta'_n} \\
f_{an} &= - \frac{Z_{U_1 a}(\sigma_n).G(p_3 \sigma_n) + Z_{U_2 a}(\sigma_n).\psi(p_3 \sigma_n)}{\Delta'_n}
\end{aligned}\right\} \quad (6.9)$$

$$
\left.\begin{aligned}
g_{in} &= - \frac{Z_{A_1 i}(\sigma_n) . G(a_a p_3 \sigma_n) + Z_{A_1 i}(\sigma_n) . \psi(a_a p_3 \sigma_n)}{\Delta'_n} \\
g_{pn} &= - \frac{Z_{A_1 p}(\sigma_n) . G(a_a p_3 \sigma_n) + Z_{A_2 p}(\sigma_n) . \psi(a_a p_3 \sigma_n)}{\Delta'_n} \\
g_{an} &= - \frac{Z_{A_1 a}(\sigma_n) . G(a_a p_3 \sigma_n) + Z_{A_2 a}(\sigma_n) . \psi(a_a p_3 \sigma_n)}{\Delta'_n}
\end{aligned}\right\} \qquad (6.10)
$$

$$
\left.\begin{aligned}
h_{in} &= + \frac{Z_{A_1 i}(\sigma_n) . \overset{\equiv}{G}(\sigma_n) + Z_{A_2 i}(\sigma_n) . \overset{\equiv}{\psi}(\sigma_n)}{\Delta'_n} \\
h_{pn} &= + \frac{Z_{A_1 p}(\sigma_n) . \overset{\equiv}{G}(\sigma_n) + Z_{A_2 p}(\sigma_n) . \overset{\equiv}{\psi}(\sigma_n)}{\Delta'_n} \\
h_{an} &= + \frac{Z_{A_1 a}(\sigma_n) . \overset{\equiv}{G}(\sigma_n) + Z_{A_2 a}(\sigma_n) . \overset{\equiv}{\psi}(\sigma_n)}{\Delta'_n}
\end{aligned}\right\} \qquad (6.11)
$$

$$
\left.\begin{aligned}
j_{in} &= - \frac{Z_{A_1 i}(\sigma_n) . G(a_a p_4 \sigma_n) + Z_{A_2 i}(\sigma_n) . \psi(a_a p_4 \sigma_n)}{\Delta'_n} \\
j_{pn} &= - \frac{Z_{A_1 p}(\sigma_n) . G(a_a p_4 \sigma_n) + Z_{A_2 p}(\sigma_n) . \psi(a_a p_4 \sigma_n)}{\Delta'_n} \\
j_{an} &= - \frac{Z_{A_1 a}(\sigma_n) . G(a_a p_4 \sigma_n) + Z_{A_2 a}(\sigma_n) . \psi(a_a p_4 \sigma_n)}{\Delta'_n}
\end{aligned}\right\} \qquad (6.12)
$$

7. The Inverse Transformation

When transforming back the equations (4.10) to (4.18), one obtains

$$
\begin{aligned}
\dot{\Theta}_i(X,t) = {} & a_{io}\dot{\Theta}_{ik}(t) + \sum a_{in}\, e^{-\frac{\sigma_n^2}{\nu}t} \int_0^t \dot{\Theta}_{ik}(\tau) e^{+\frac{\sigma_n^2}{\nu}\tau} d\tau + \\
& + a_{po}\frac{\Delta W(t)}{(\rho c)_u} + \sum a_{pn}\, e^{-\frac{\sigma_n^2}{\nu}t} \int_0^t \frac{\Delta W(\tau)}{(\rho c)_u} e^{+\frac{\sigma_n^2}{\nu}\tau} d\tau + \\
& + a_{ao}\dot{\Theta}_{ak}(t) + \sum a_{an}\, e^{-\frac{\sigma_n^2}{\nu}t} \int_0^t \dot{\Theta}_{ak}(t)\, e^{+\frac{\sigma_n^2}{\nu}\tau} d\tau
\end{aligned} \qquad (7.1)
$$

etc.,where the convolution integrals are called:

$$e^{-\frac{\sigma_n^2}{\nu}t}\int_0^t \dot{\Theta}_{ik}(\tau)\, e^{+\frac{\sigma_n^2}{\nu}\tau}\, d\tau \equiv \Gamma_{in}(t), \tag{7.2}$$

$$e^{-\frac{\sigma_n^2}{\nu}t}\int_0^t \frac{\Delta W(\tau)}{(\rho c)_u}\, e^{+\frac{\sigma_n^2}{\nu}\tau}\, d\tau \equiv \Gamma_{pn}(t)\ , \tag{7.3}$$

$$e^{-\frac{\sigma_n^2}{\nu}t}\int_0^t \dot{\Theta}_{ak}(\tau)\, e^{+\frac{\sigma_n^2}{\nu}\tau}\, d\tau \equiv \Gamma_{an}(t)\ . \tag{7.4}$$

Obviously, the so-defined "transient complement functions" obey the differential equations:

$$\dot{\Gamma}_{in}(t) + \frac{\sigma_n^2}{\nu}\Gamma_{in}(t) = \dot{\Theta}_{ik}(t) \quad ; \ \Gamma_{in}(0) = 0 \tag{7.5}$$

$$\dot{\Gamma}_{pn}(t) + \frac{\sigma_n^2}{\nu}\Gamma_{pn}(t) = \frac{\Delta W(t)}{(\rho c)_u} \quad ; \ \Gamma_{pn}(0) = 0 \tag{7.6}$$

$$\dot{\Gamma}_{an}(t) + \frac{\sigma_n^2}{\nu}\Gamma_{an}(t) = \dot{\Theta}_{ak}(t) \quad ; \ \Gamma_{an}(0) = 0\ , \tag{7.7}$$

with the pseudo-solutions (needed later on):

$$\Gamma_{in}(t) = \frac{\nu}{\sigma_n^2}\,[\dot{\Theta}_{ik}(t) - \dot{\Gamma}_{in}(t)]\ , \tag{7.8}$$

$$\Gamma_{pn}(t) = \frac{\nu}{\sigma_n^2}\left[\frac{\Delta W(t)}{(\rho c)_u} - \dot{\Gamma}_{pn}(t)\right]\ , \tag{7.9}$$

$$\Gamma_{an}(t) = \frac{\nu}{\sigma_n^2}\,[\dot{\Theta}_{ak}(t) - \dot{\Gamma}_{an}(t)]\ . \tag{7.10}$$

Note that only these three functions Γ can occur, and no other ones.

Hence, one has:

$$\dot{\Theta}_i(X,t) = \sum a_{in}\Gamma_{in}(t)+\sum a_{pn}\Gamma_{pn}(t)+\sum a_{an}\Gamma_{an}(t)+a_{io}\dot{\Theta}_{ik}(t)+a_{po}\frac{\Delta W(t)}{(\rho c)_u} +a_{ao}\dot{\Theta}_{ak}(t) \qquad (7.11)$$

$$\dot{\bar{\Theta}}_i(t) = \sum b_{in}\Gamma_{in}(t)+\sum b_{pn}\Gamma_{pn}(t)+\sum b_{an}\Gamma_{an}(t)+b_{io}\dot{\Theta}_{ik}(t)+b_{po}\frac{\Delta W(t)}{(\rho c)_u} +b_{ao}\dot{\Theta}_{ak}(t) \qquad (7.12)$$

$$\dot{\Theta}_i(p_2X,t) = \sum c_{in}\Gamma_{in}(t)+\sum c_{pn}\Gamma_{pn}(t)+\sum c_{an}\Gamma_{an}(t)+c_{io}\dot{\Theta}_{ik}(t)+c_{po}\frac{\Delta W(t)}{(\rho c)_u} +c_{ao}\dot{\Theta}_{ak}(t) \qquad (7.13)$$

$$\dot{\Theta}_u(p_2X,t) = \sum d_{in}\Gamma_{in}(t)+\sum d_{pn}\Gamma_{pn}(t)+\sum d_{an}\Gamma_{an}(t)+d_{io}\dot{\Theta}_{ik}(t)+(d_{po}+1)\frac{\Delta W(t)}{(\rho c)_u} + d_{ao}\dot{\Theta}_{ak}(t) \qquad (7.14)$$

$$\dot{\bar{\bar{\Theta}}}_u(t) = \sum e_{in}\Gamma_{in}(t)+\sum e_{pn}\Gamma_{pn}(t)+\sum e_{an}\Gamma_{an}(t)+e_{io}\dot{\Theta}_{ik}(t)+(e_{po}+1)\frac{\Delta W(t)}{(\rho c)_u} + e_{ao}\dot{\Theta}_{ak}(t) \qquad (7.15)$$

$$\dot{\Theta}_u(p_3X,t) = \sum f_{in}\Gamma_{in}(t)+\sum f_{pn}\Gamma_{pn}(t)+\sum f_{an}\Gamma_{an}(t)+f_{io}\dot{\Theta}_{ik}(t)+(f_{po}+1)\frac{\Delta W(t)}{(\rho c)_u} + f_{ao}\dot{\Theta}_{ak}(t) \qquad (7.16)$$

$$\dot{\Theta}_a(p_3X,t) = \sum g_{in}\Gamma_{in}(t)+\sum g_{pn}\Gamma_{pn}(t)+\sum g_{an}\Gamma_{an}(t)+g_{io}\dot{\Theta}_{ik}(t)+g_{po}\frac{\Delta W(t)}{(\rho c)_u} + g_{ao}\dot{\Theta}_{ak}(t) \qquad (7.17)$$

$$\dot{\bar{\bar{\Theta}}}_a(t) = \sum h_{in}\Gamma_{in}(t)+\sum h_{pn}\Gamma_{pn}(t)+\sum h_{an}\Gamma_{an}(t)+h_{io}\dot{\Theta}_{ik}(t)+h_{po}\frac{\Delta W(t)}{(\rho c)_u} + h_{ao}\dot{\Theta}_{ak}(t) \qquad (7.18)$$

$$\dot{\Theta}_a(p_4X,t) = \sum j_{in}\Gamma_{in}(t)+\sum j_{pn}\Gamma_{pn}(t)+\sum j_{an}\Gamma_{an}(t)+j_{io}\dot{\Theta}_{ik}(t)+j_{po}\frac{\Delta W(t)}{(\rho c)_u} + j_{ao}\dot{\Theta}_{ak}(t) \qquad (7.19)$$

Here, we substitute the expressions (7.8) to (7.10):

$$\dot{\Theta}_i(X,t) = \nu\sum \frac{a_{in}}{\sigma_n^2}\,[\dot{\Theta}_{ik}-\dot{\Gamma}_{in}] + \nu\sum \frac{a_{pn}}{\sigma_n^2}\left[\frac{\Delta W}{\rho c} -\dot{\Gamma}_{pn}\right] + \nu\sum \frac{a_{an}}{\sigma_n^2}\,[\dot{\Theta}_{ak}-\dot{\Gamma}_{an}]+a_{io}\dot{\Theta}_{ik} + a_{po}\frac{\Delta W}{\rho c} + a_{ao}\dot{\Theta}_{ak} \qquad (7.20)$$

etc.

When integrating once these equations to obtain the temperatures themselves, it is not allowed to get integrals over $\frac{\Delta W(t)}{(\rho c)_u}$, because such expressions would not fulfill FOURIER's equation. Therefore, certain sums must vanish, e.g. $(a_{po}+\nu\sum\frac{a_{pn}}{\sigma_n^2}) = 0$, which are to be enumerated at once.

$$\Theta_i(X,t) = (a_{io}+\nu\sum\frac{a_{in}}{\sigma_n^2})\Theta_{ik}(t)+(a_{ao}+\nu\sum\frac{a_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[a_{in}\Gamma_{in}(t)+a_{pn}\Gamma_{pn}(t)+a_{an}\Gamma_{an}(t)]$$

$$\text{with } a_{po}+\nu\sum\frac{a_{pn}}{\sigma_n^2} = 0, \qquad (7.21)$$

$$\bar{\Theta}_i(t) = (b_{io}+\nu\sum\frac{b_{in}}{\sigma_n^2})\Theta_{ik}(t)+(b_{ao}+\nu\sum\frac{b_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[b_{in}\Gamma_{in}(t)+b_{pn}\Gamma_{pn}(t)+b_{an}\Gamma_{an}(t)$$

$$\text{with } b_{po}+\nu\sum\frac{b_{pn}}{\sigma_n^2} = 0, \qquad (7.22)$$

$$\Theta_i(p_2X,t) = (c_{io}+\nu\sum\frac{c_{in}}{\sigma_n^2})\Theta_{ik}(t)+(c_{ao}+\nu\sum\frac{c_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[c_{in}\Gamma_{in}(t)+c_{pn}\Gamma_{pn}(t)+c_{an}\Gamma_{an}(t)]$$

$$\text{with } c_{po}+\nu\sum\frac{c_{pn}}{\sigma_n^2} = 0, \qquad (7.23)$$

$$\Theta_u(p_2X,t) = (d_{io}+\nu\sum\frac{d_{in}}{\sigma_n^2})\Theta_{ik}(t)+(d_{ao}+\nu\sum\frac{d_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[d_{in}\Gamma_{in}(t)+d_{pn}\Gamma_{pn}(t)+d_{an}\Gamma_{an}(t)]$$

$$\text{with } d_{po}+1+\nu\sum\frac{d_{pn}}{\sigma_n^2} = 0, \qquad (7.24)$$

$$\bar{\bar{\Theta}}_u(t) = (e_{io}+\nu\sum\frac{e_{in}}{\sigma_n^2})\Theta_{ik}(t)+(e_{ao}+\nu\sum\frac{e_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[e_{in}\Gamma_{in}(t)+e_{pn}\Gamma_{pn}(t)+e_{an}(t)]$$

$$\text{with } e_{po}+1+\nu\sum\frac{e_{pn}}{\sigma_n^2} = 0, \qquad (7.25)$$

$$\Theta_u(p_3X,t) = (f_{io}+\nu\sum\frac{f_{in}}{\sigma_n^2})\Theta_{ik}(t)+(f_{ao}+\nu\sum\frac{f_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[f_{in}\Gamma_{in}(t)+f_{pn}\Gamma_{pn}(t)+f_{an}\Gamma_{an}(t)]$$

$$\text{with } f_{po}+1+\nu\sum\frac{f_{pn}}{\sigma_n^2} = 0 \ , \qquad (7.26)$$

$$\Theta_a(p_3X,t) = (g_{io}+\nu\sum\frac{g_{in}}{\sigma_n^2})\Theta_{ik}(t)+(g_{ao}+\nu\sum\frac{g_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[g_{in}\Gamma_{in}(t)+g_{pn}\Gamma_{pn}(t)+g_{an}\Gamma_{an}(t)]$$

$$\text{with } g_{po}+\nu\sum\frac{g_{pn}}{\sigma_n^2} = 0 \ , \qquad (7.27)$$

$$\overset{\equiv}{\Theta}_a(t) = (h_{io}+\nu\sum\frac{h_{in}}{\sigma_n^2})\Theta_{ik}(t)+(h_{ao}+\nu\sum\frac{h_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[h_{in}\Gamma_{in}(t)+h_{pn}\Gamma_{pn}(t)+h_{an}\Gamma_{an}(t)]$$

$$\text{with } h_{po}+\nu\sum\frac{h_{pn}}{\sigma_n^2} = 0 \ , \qquad (7.28)$$

$$\Theta_a(p_4X,t) = (j_{io}+\nu\sum\frac{j_{in}}{\sigma_n^2})\Theta_{ik}(t)+(j_{ao}+\nu\sum\frac{j_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu\sum\frac{1}{\sigma_n^2}[j_{in}\Gamma_{in}(t)+j_{pn}\Gamma_{pn}(t)+j_{an}\Gamma_{an}(t)]$$

$$\text{with } j_{po}+\nu\sum\frac{j_{pn}}{\sigma_n^2} = 0 \ . \qquad (7.29)$$

We substitute once more the expressions (7.8) to (7.10):

$$\Theta_i(X,t) = (a_{io}+\nu\sum\frac{a_{in}}{\sigma_n^2})\Theta_{ik}(t)+(a_{ao}+\nu\sum\frac{a_{an}}{\sigma_n^2})\Theta_{ak}(t)-\nu^2\sum\frac{1}{\sigma_n^4}[a_{ik}(\dot{\Theta}_{ik}-\dot{\Gamma}_{in})+a_{pn}\left(\frac{\Delta W}{\rho c}-\dot{\Gamma}_{pn}\right)+a_{an}(\dot{\Theta}_{ak}-\dot{\Gamma}_{an})]$$

$$(7.30)$$

Therefore, by writing directly R instead of X in the temperatures' argument and by replacing $\nu^2 \frac{\Delta W(t)}{(\rho c)_u}$ by $\nu\frac{\Delta W(t)R^2}{\tilde{\lambda}_u}$:

$$\Theta_i(R,t) = (a_{io}+\nu\sum\frac{a_{in}}{\sigma_n^2})\Theta_{ik}(t) - \nu\sum\frac{a_{pn}}{\sigma_n^4}\frac{\Delta W(t)R^2}{\tilde{\lambda}_u} + (a_{ao}+\nu\sum\frac{a_{an}}{\sigma_n^2})\Theta_{ak}(t)+$$

$$+\nu^2\sum\frac{1}{\sigma_n^4}\Big\{a_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)] + a_{pn}\dot{\Gamma}_{pn}(t) +$$

$$+ a_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\Big\} \qquad (7.31)$$

etc., by simply exchanging a by b, ..., for the respective other temperatures.

When taking the initial values $\Theta_{ik,o}, W_o$, and $\Theta_{ak,o}$ for the inputs, the first line of expression (7.31) represents just the stationary solution $\Theta_{i,stat}(R)$, and correspondingly for the other temperatures, because all dotted quantities of the second line vanish for this state.

As the coefficients $a_{po}, \ldots, j_{po}$ are still free, the conditions mentioned with (7.21) to (7.29) can always be fulfilled. It is however not necessary to compute them because they do not appear afterwards.

The factors of $\Theta_{ik}(t)$, namely $(a_{io}+\nu\sum\frac{a_{in}}{\sigma_n^2}), \ldots, (j_{io}+\nu\sum\frac{j_{in}}{\sigma_n^2})$, and those of $\Theta_{ak}(t)$, namely $(a_{ao}+\nu\sum\frac{a_{an}}{\sigma_n^2}), \ldots, (j_{ao}+\nu\sum\frac{j_{an}}{\sigma_n^2})$ are obtained by comparison with the stationary solution, to be computed in the next chapter directly. The $a_{io}, \ldots, j_{io}$ and $a_{ao}, \ldots, j_{ao}$ being still free, this evaluation is always possible without contradiction.

However, the factors of $\frac{WR^2}{\tilde{\lambda}_u}$, namely $-\nu\sum\frac{a_{pn}}{\sigma_n^4}, \ldots, -\nu\sum\frac{j_{pn}}{\sigma_n^4}$, must also correspond with those of the stationary solution.

We have here no degree of freedom, as the single coefficients $a_{pn}, \ldots, j_{pn}$ as well as their infinite sums are known. This leads to certain summation formulas which may be checked numerically.

8. The Stationary Solution

In this chapter, we omit the subscript "stat"; the temperatures in question are always the stationary ones.

The stationary part of the basic equations (1.1) to (1.3) is

$$\frac{d^2\Theta_i}{dr^2} + \frac{1}{r}\frac{d\Theta_i}{dr} = 0 \quad , \tag{8.1}$$

$$\frac{d^2\Theta_u}{dr^2} + \frac{1}{r}\frac{d\Theta_u}{dr} + \frac{W}{\tilde{\lambda}_u} = 0 \quad , \tag{8.2}$$

$$\frac{d^2\Theta_a}{dr^2} + \frac{1}{r}\frac{d\Theta_a}{dr} = 0 \quad , \tag{8.3}$$

with the boundary conditions (cf. 1.4 to 1.9 or 1.19 to 1.24):

$$r = R: \quad -q_i R \left.\frac{d\Theta_i}{dr}\right|_{r=R} = \Theta_{ik} - \Theta_i(R) \quad , \tag{8.4}$$

$$r = p_2R: \quad \lambda_i \left.\frac{d\Theta_u}{dr}\right|_{r=p_2R} = \left.\frac{d\Theta_i}{dr}\right|_{r=p_2R} \quad , \tag{8.5}$$

$$\Theta_u(p_2R) = \Theta_i(p_2R) + \beta_i R \left.\frac{d\Theta_i}{dr}\right|_{r=p_2R} \quad , \tag{8.6}$$

$$r = p_3R: \quad \Theta_u(p_3R) = \Theta_a(p_3R) - \beta_a R \left.\frac{d\Theta_a}{dr}\right|_{r=p_3R} \quad , \tag{8.7}$$

$$\lambda_a \left.\frac{d\Theta_u}{dr}\right|_{r=p_3R} = \left.\frac{d\Theta_a}{dr}\right|_{r=p_3R} \quad , \tag{8.8}$$

$$r = p_4R: \quad + q_a R \left.\frac{d\Theta_a}{dr}\right|_{r=p_4R} = \Theta_{ak} - \Theta_a(p_4R) \quad . \tag{8.9}$$

We introduce the dimensionless variable $y = \frac{r}{R}$ and denote $\frac{d}{dy} = {}'$, $\frac{d^2}{dy^2} = {}''$:

$$\Theta_i'' + \frac{1}{y}\Theta_i' = 0 \quad , \tag{8.10}$$

$$\Theta_u'' + \frac{1}{y}\Theta_u' + \frac{WR^2}{\tilde{\lambda}_u} = 0 \quad , \tag{8.11}$$

$$\Theta_a'' + \frac{1}{y}\Theta_a' = 0 \quad , \tag{8.12}$$

with the boundary conditions:

$$y = 1: \quad -q_i\Theta_i'(t) = \Theta_{ik} - \Theta_i(1) \quad , \tag{8.13}$$

$$y = p_2: \quad \lambda_i\Theta_u'(p_2) = \Theta_i'(p_2) \quad , \tag{8.14}$$

$$\Theta_u(p_2) = \Theta_i(p_2) + \beta_i\Theta_i'(p_2) \quad , \tag{8.15}$$

$$y = p_3: \quad \Theta_u(p_3) = \Theta_a(p_3) - \beta_a\Theta_a'(p_3) \quad , \tag{8.16}$$

$$\lambda_a\Theta_u'(p_3) = \Theta_a'(p_3) \quad , \tag{8.17}$$

$$y = p_4: \quad +q_a\Theta_a'(p_4) = \Theta_{ak} - \Theta_a(p_4) \; . \tag{8.18}$$

The general form of the solutions being known, we try, with unknown coefficients A_i, A_u, A_a, B_i, B_u, B_a:

$$\Theta_i(y) = A_i \ln y + B_i \quad , \tag{8.19}$$

$$\Theta_u(y) = A_u \ln y + B_u - \frac{WR^2}{4\tilde{\lambda}_u} y^2 \quad , \tag{8.20}$$

$$\Theta_a(y) = A_a \ln y + B_a \quad , \tag{8.21}$$

with the derivatives

$$\Theta_i'(y) = \frac{A_i}{y} \quad , \quad \Theta_i''(y) = -\frac{A_i}{y^2} \quad , \tag{8.22}$$

$$\Theta_u'(y) = \frac{A_u}{y} - \frac{WR^2}{2\tilde{\lambda}_u} y \quad , \quad \Theta_u''(y) = -\frac{A_u}{y^2} - \frac{WR^2}{2\tilde{\lambda}_u} \quad , \tag{8.23}$$

$$\Theta_a'(y) = \frac{A_a}{y} \quad , \quad \Theta_a''(y) = -\frac{A_a}{y^2} \quad . \tag{8.24}$$

Hence, the selected stationary temperatures assume the following form:

$$\Theta_i(R) = + B_i \quad , \tag{8.25}$$

$$\overline{\Theta}_i = + A_i\left(\frac{p_2^2 \ln p_2}{p_2^2-1} - \frac{1}{2}\right) + B_i \quad , \tag{8.26}$$

$$\Theta_i(p_2R) = + A_i \ln p_2 + B_i \quad , \tag{8.27}$$

$$\Theta_u(p_2R) = + A_u \ln p_2 + B_u - \frac{WR^2}{4\tilde{\lambda}_u} p_2^2 \quad , \tag{8.28}$$

$$\overline{\overline{\Theta}}_u = + A_u\left(\frac{p_3^2 \ln p_3 - p_2^2 \ln p_2}{p_3^2 - p_2^2} - \frac{1}{2}\right) + B_u - \frac{WR^2}{4\tilde{\lambda}_u}\,\frac{p_3^2-p_2^2}{2} \quad , \tag{8.29}$$

$$\Theta_u(p_3R) = + A_u \ln p_3 + B_u - \frac{WR^2}{4\tilde{\lambda}_u} p_3^2 \quad , \tag{8.30}$$

$$\Theta_a(p_3R) = + A_a \ln p_3 + B_a \quad , \tag{8.31}$$

$$\overline{\overline{\Theta}}_a = + A_a\left(\frac{p_4^2 \ln p_4 - p_3^2 \ln p_3}{p_4^2 - p_3^2} - \frac{1}{2}\right) + B_a \quad , \tag{8.32}$$

$$\Theta_a(p_4R) = + A_a \ln p_4 + B_a \quad . \tag{8.33}$$

In this case, also the maximum temperature is computable, but it is not very useful because the comparison with the non-stationary solution is not possible:

$$\Theta_{u,max} = + A_u \ln p_{max} + B_u - \frac{WR^2}{4\tilde{\lambda}_u} p_{max}^2 \tag{8.34}$$

$$\text{with } p_{max} = + \sqrt{\frac{A_u}{2\,\frac{WR^2}{4\tilde{\lambda}_u}}} \quad , \tag{8.35}$$

so that

$$\Theta_{u,max} = + A_u(\ln p_{max} - \tfrac{1}{2}) + B_u \quad . \tag{8.36}$$

The boundary conditions give now the following linear equation system to determine the six coefficients A_i, A_u, A_a, B_i, B_u, and B_a:

$$- q_i A_i = \Theta_{ik} - B_i \tag{8.37}$$

$$+ \lambda_i \left(\frac{A_u}{p_2} - \frac{WR^2}{2\tilde{\lambda}_u} p_2 \right) = + \frac{A_i}{p_2} \tag{8.38}$$

$$+ A_u \ln p_2 + B_u - \frac{WR^2}{4\tilde{\lambda}_u} p_2^2 = + A_i \ln p_2 + B_i + \beta_i \frac{A_i}{p_2} \tag{8.39}$$

$$+ A_u \ln p_3 + B_u - \frac{WR^2}{4\tilde{\lambda}_u} p_3^2 = + A_a \ln p_3 + B_a - \beta_a \frac{A_a}{p_3} \tag{8.40}$$

$$+ \lambda_a \left(\frac{A_u}{p_3} - \frac{WR^2}{2\tilde{\lambda}_u} p_3 \right) = + \frac{A_a}{p_3} \tag{8.41}$$

$$+ q_a \frac{A_a}{p_4} = \Theta_{ak} - A_a \ln p_4 - B_a \tag{8.42}$$

or in matrix notation:

$$
\begin{pmatrix}
-q_i & +1 & 0 & 0 & 0 & 0 \\
-1 & 0 & +\lambda_i & 0 & 0 & 0 \\
-\frac{\beta_i}{p_2}-\ln p_2 & -1 & +\ln p_2 & +1 & 0 & 0 \\
0 & 0 & -\ln p_3 & +1 & +\frac{\beta_a}{p_3}-\ln p_3 & -1 \\
0 & 0 & +\lambda_a & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & +\frac{q_a}{p_4}+\ln p_4 & +1
\end{pmatrix}
\begin{pmatrix} A_i \\ B_i \\ A_u \\ B_u \\ A_a \\ B_a \end{pmatrix}
=
\begin{pmatrix}
\Theta_{ik} \\
\frac{\lambda_i p_2^2}{2}\cdot\frac{WR^2}{\tilde{\lambda}_u} \\
\frac{p_2^2}{4}\cdot\frac{WR^2}{\tilde{\lambda}_u} \\
\frac{p_3^2}{4}\cdot\frac{WR^2}{\tilde{\lambda}_u} \\
\frac{\lambda_a p_3^2}{2}\cdot\frac{WR^2}{\tilde{\lambda}_u} \\
\Theta_{ak}
\end{pmatrix}
\qquad (8.43)
$$

Omitting all intermediate calculations, we get for the system determinant:

$$
\Delta = \lambda_i\left(\frac{\beta_i}{p_2} + \ln p_2 + q_i\right) + \lambda_a\left(\frac{\beta_a}{p_3} - \ln p_3 + \frac{q_a}{p_4} + \ln p_4\right) + \left(\ln p_3 - \ln p_2\right) , \qquad (8.44)
$$

and for the coefficients the lengthy expressions:

$$
A_i = \frac{1}{\Delta}\left\{-\lambda_i\Theta_{ik} + \frac{\lambda_i}{4}\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{\tilde{\lambda}_u} + \lambda_i\Theta_{ak}\right\} \qquad (8.45)
$$

$$
B_i = \frac{1}{\Delta}\left\{\underbrace{\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2\right)+\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{p_a}{q_4}+\ln p_4\right)+(\ln p_3-\ln p_2)\right]}_{=\ \Delta-\lambda_i q_i}\Theta_{ik} + \right.
$$

$$
\left. +\frac{\lambda_i q_i}{4}\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{p_a}{q_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{\tilde{\lambda}_u} + \lambda_i q_i\Theta_{ak}\right\} \qquad (8.46).
$$

$$A_u=\frac{1}{\Delta}\left\{-\Theta_{ik}+\frac{1}{4}\left[(p_3^2-p_2^2)+2\lambda_i p_2^2\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)+2\lambda_a p_3^2\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)\right]\frac{WR^2}{\tilde{\lambda}_u}+\Theta_{ak}\right\} \tag{8.47}$$

$$\begin{aligned}B_u=\frac{1}{\Delta}\Bigg\{&\left[\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+\ln p_3\right]\Theta_{ik}+\frac{1}{4}\Bigg[p_2^2\left(-2\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)+1\right)\left(\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+\ln p_3\right)+\\&+p_3^2\left(\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-\ln p_2\right)\left(2\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+1\right)\Bigg]\frac{WR^2}{\tilde{\lambda}_u}+\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-\ln p_2\right]\Theta_{ak}\Bigg\}\end{aligned} \tag{8.48}$$

$$A_a=\frac{1}{\Delta}\left\{-\lambda_a\Theta_{ik}+\frac{\lambda_a}{4}\left[-2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)+(p_3^2-p_2^2)-2p_3^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{\tilde{\lambda}_u}+\lambda_a\Theta_{ak}\right\} \tag{8.49}$$

$$\begin{aligned}B_a=\frac{1}{\Delta}\Bigg\{&\lambda_a\left(\frac{q_a}{p_4}+\ln p_4\right)\Theta_{ik}+\frac{\lambda_a}{4}\left(\frac{q_a}{p_4}+\ln p_4\right)\left[2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-(p_3^2-p_2^2)+2p_3^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{\tilde{\lambda}_u}+\\&+\underbrace{\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)+\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3\right)+(\ln p_3-\ln p_2)\right]}_{=\,\Delta-\lambda_a\left(\frac{q_a}{p_4}+\ln p_4\right)}\Theta_{ak}\Bigg\}\end{aligned} \tag{8.50}$$

With given parameters, all these coefficients are of course simple numerical values.

When substituting the coefficients into the formulas (8.25) to (8.33), one gets (for Δ, see 8.44):

$$\begin{aligned}\Theta_i(R)=&\left(1-\frac{\lambda_i q_i}{\Delta}\right)\Theta_{ik}+\frac{\lambda_i q_i}{\Delta}\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u}+\\&+\frac{\lambda_i q_i}{\Delta}\Theta_{ak}\end{aligned} \tag{8.51}$$

$$\overline{\Theta}_i = \left[1-\frac{\lambda_i}{\Delta}\left(\frac{p_2^2\ln p_2}{p_2^2-1}-\frac{1}{2}+q_i\right)\right]\Theta_{ik} +$$

$$+\frac{\lambda_i}{\Delta}\left(\frac{p_2^2\ln p_2}{p_2^2-1}-\frac{1}{2}+q_i\right)\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+\frac{\lambda_i}{\Delta}\left(\frac{p_2^2\ln p_2}{p_2^2-1}-\frac{1}{2}+q_i\right)\Theta_{ak} \qquad (8.52)$$

$$\Theta_i(p_2R) = \left[1-\frac{\lambda_i}{\Delta}(\ln p_2+q_i)\right]\Theta_{ik} +$$

$$+\frac{\lambda_i}{\Delta}(\ln p_2+q_i)\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+\frac{\lambda_i}{\Delta}(\ln p_2+q_i)\Theta_{ak} \qquad (8.53)$$

$$\Theta_u(p_2R) = \frac{1}{\Delta}\left[\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(\ln p_3-\ln p_2)\right]\Theta_{ik} +$$

$$+\frac{\lambda_i}{\Delta}\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+\frac{\lambda_i}{\Delta}\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\Theta_{ak} \qquad (8.54)$$

$$\bar{\bar{\Theta}}_u = \frac{1}{\Delta}\bigg[\lambda_a\Big(\frac{\beta_a}{p_3} - \ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big) + \ln p_3 - \frac{p_3^2 \ln p_3 - p_2^2 \ln p_2}{p_3^2 - p_2^2} + \frac{1}{2}\bigg]\Theta_{ik} +$$

$$+\frac{1}{\Delta}\bigg[\lambda_i\Big(\frac{\beta_i}{p_2}+\ln p_2 + q_i\Big)\Big(2p_2^4 \frac{\ln p_3 - \ln p_2}{p_3^2-p_2^2} - p_2^2 + \frac{p_3^2-p_2^2}{2}\Big) + \lambda_a\Big(\frac{\beta_a}{p_3} - \ln p_3 + \frac{q_a}{p_4}+\ln p_4\Big)\Big(2p_3^4\frac{\ln p_3-\ln p_2}{p_3^2-p_2^2} - p_3^2 - \frac{p_3^2-p_2^2}{2}\Big) +$$

$$+ 2\lambda_i\lambda_a(p_3^2-p_2^2)\Big(\frac{\beta_i}{p_2}+\ln p_2+q_i\Big)\Big(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\Big) + \frac{(\ln p_3-\ln p_2)(p_2^2+p_3^2)-(p_3^2-p_2^2)}{2}\bigg]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+ \frac{1}{\Delta}\bigg[\lambda_i\Big(\frac{\beta_i}{p_2}+\ln p_2+q_i\Big) - \ln p_2 + \frac{p_3^2\ln p_3 - p_2^2\ln p_2}{p_3^2-p_2^2} - \frac{1}{2}\bigg]\Theta_{ak} \qquad (8.55)$$

$$\Theta_u(p_3R) = \frac{\lambda_a}{\Delta}\Big(\frac{\beta_a}{p_3} - \ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big)\Theta_{ik} +$$

$$+ \frac{\lambda_a}{\Delta}\Big(\frac{\beta_a}{p_3} - \ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big)\bigg[2\lambda_i(p_3^2-p_2^2)\Big(\frac{\beta_i}{p_2}+\ln p_2+q_i\Big) - (p_3^2-p_2^2) + 2p_3^2(\ln p_3 - \ln p_2)\bigg]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+ \frac{1}{\Delta}\bigg[\lambda_i\Big(\frac{\beta_i}{p_2}+\ln p_2+q_i\Big) + (\ln p_3 - \ln p_2)\bigg]\Theta_{ak} \qquad (8.56)$$

$$\Theta_a(p_3R) = \frac{\lambda_a}{\Delta}\Big(-\ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big)\Theta_{ik} +$$

$$+ \frac{\lambda_a}{\Delta}\Big(-\ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big)\bigg[2\lambda_i(p_3^2-p_2^2)\Big(\frac{\beta_i}{p_2}+\ln p_2+q_i\Big) - (p_3^2-p_2^2) + 2p_3^2(\ln p_3 - \ln p_2)\bigg]\frac{WR^2}{4\tilde{\lambda}_u} +$$

$$+ \bigg[1 - \frac{\lambda_a}{\Delta}\Big(-\ln p_3 + \frac{q_a}{p_4} + \ln p_4\Big)\bigg]\Theta_{ak} \qquad (8.57)$$

$$\bar{\bar{\Theta}}_a=\frac{\lambda_a}{\Delta}\left(-\frac{p_4^2\ln p_4-p_3^2\ln p_3}{p_4^2-p_3^2}+\frac{1}{2}+\frac{q_a}{p_4}+\ln p_4\right)\Theta_{ik}+$$

$$+\frac{\lambda_a}{\Delta}\left(-\frac{p_4^2\ln p_4-p_3^2\ln p_3}{p_4^2-p_3^2}+\frac{1}{2}+\frac{q_a}{p_4}+\ln p_4\right)\left[2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-(p_3^2-p_2^2)+2p_3^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u}+$$

$$+\left[1-\frac{\lambda_a}{\Delta}\left(-\frac{p_4^2\ln p_4-p_3^2\ln p_3}{p_4^2-p_3^2}+\frac{1}{2}+\frac{q_a}{p_4}+\ln p_4\right)\right]\Theta_{ak} \qquad (8.58)$$

$$\Theta_a(p_4R)=\frac{\lambda_a\frac{q_a}{p_4}}{\Delta}\Theta_{ik}+\frac{\lambda_a\frac{q_a}{p_4}}{\Delta}\left[2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-(p_3^2-p_2^2)+2p_3^2(\ln p_3-\ln p_2)\right]\frac{WR^2}{4\tilde{\lambda}_u}+\left(1-\frac{\lambda_a\frac{q_a}{p_4}}{\Delta}\right)\Theta_{ak}\,.$$

$$(8.59)$$

9. The final presentation of the solutions

By substituting the stationary solutions (8.51) to (8.59) - the input being again time-dependent - into (7.31) etc., one gets the final presentation of the searched solutions:

$$\Theta_i(R,t)=\Theta_{ik}(t)+\frac{\lambda_i q_i}{\Delta}\left\{\Theta_{ak}(t)-\Theta_{ik}(t)+\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{W(t)R^2}{4\tilde{\lambda}_u}\right\}+$$

$$+\nu^2\sum\frac{1}{\sigma_n^4}\left\{a_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+a_{pn}\dot{\Gamma}_{pn}(t)+a_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\} \qquad (9.1)$$

$$\bar{\Theta}_i(t)=\Theta_{ik}(t)+\frac{\lambda_i}{\Delta}\left(\frac{p_2^2\ln p_2}{p_2^2-1}-\frac{1}{2}+q_i\right)\left\{\text{curved bracket equal to the first one in }\Theta_i(R,t)\right\}+$$

$$+\nu^2\sum\frac{1}{\sigma_n^4}\left\{b_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+b_{pn}\dot{\Gamma}_{pn}(t)+b_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\} \qquad (9.2)$$

$$\Theta_i(p_2R,t)=\Theta_{ik}(t)+\frac{\lambda_i}{\Delta}(\ln p_2+q_i)\left\{\text{curved bracket equal to the first one in } \Theta_i(R,t)\right\}+$$

$$+\ \nu^2\sum\frac{1}{\delta_n^4}\left\{c_{in}[\dot\Gamma_{in}(t)-\dot\Theta_{ik}(t)]+c_{pn}\dot\Gamma_{pn}(t)+c_{an}[\dot\Gamma_{an}(t)-\dot\Theta_{ak}(t)]\right\} \qquad (9.3)$$

$$\Theta_u(p_3R,t)=\frac{1}{\Delta}\left[\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(\ln p_3-\ln p_2)\right]\Theta_{ik}(t)+$$

$$+\ \frac{\lambda_i}{\Delta}\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\left[2\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+(p_3^2-p_2^2)-2p_2^2(\ln p_3-\ln p_2)\right]\frac{W(t)R^2}{4\tilde\lambda_u}+$$

$$+\frac{\lambda_i}{\Delta}\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\Theta_{ak}(t)\ +\ \nu^2\sum\frac{1}{\delta_n^4}\left\{d_{in}[\dot\Gamma_{in}(t)-\dot\Theta_{ik}(t)]+d_{pn}\dot\Gamma_{pn}(t)+d_{an}[\dot\Gamma_{an}(t)-\dot\Theta_{ak}(t)]\right\}$$

$$(9.4)$$

$$\bar{\bar{\Theta}}_u(t)=\frac{1}{\Delta}\left[\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+\ln p_3-\frac{p_3^2\ln p_3-p_2^2\ln p_2}{p_3^2-p_2^2}+\frac{1}{2}\right]\Theta_{ik}(t)\ +$$

$$+\ \frac{1}{\Delta}\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-\ln p_2+\frac{p_3^2\ln p_3-p_2^2\ln p_2}{p_3^2-p_2^2}-\frac{1}{2}\right]\Theta_{ak}(t)\ +$$

$$+\ \frac{1}{\Delta}\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\left(2p_2^4\,\frac{\ln p_3-\ln p_2}{p_3^2-p_2^2}-p_2^2+\frac{p_3^2-p_2^2}{2}\right)+\lambda_a\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)\left(2p_3^4\,\frac{\ln p_3-\ln p_2}{p_3^2-p_2^2}-p_3^2-\frac{p_3^2-p_2^2}{2}\right)\right.+$$

$$\left.+\ 2\lambda_i\lambda_a(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)+\frac{(\ln p_3-\ln p_2)(p_2^2+p_3^2)-(p_3^2-p_2^2)}{2}\right]\frac{W(t)R^2}{4\tilde\lambda_u}\ +$$

$$+\ \nu^2\sum\frac{1}{\delta_n^4}\left\{e_{in}[\dot\Gamma_{in}(t)-\dot\Theta_{ik}(t)]+e_{pn}\dot\Gamma_{pn}(t)+e_{an}[\dot\Gamma_{an}(t)-\dot\Theta_{ak}(t)]\right\} \qquad (9.5)$$

$$\Theta_u(p_3R,t)=\frac{1}{\Delta}\left[\lambda_i\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)+(\ln p_3-\ln p_2)\right]\Theta_{ak}(t)\;+$$

$$+\;\frac{\lambda_a}{\Delta}\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)\left[2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-(p_3^2-p_2^2)+2p_3^2(\ln p_3-\ln p_2)\right]\frac{W(t)R^2}{4\tilde{\lambda}_u}\;+$$

$$+\;\frac{\lambda_a}{\Delta}\left(\frac{\beta_a}{p_3}-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)\Theta_{ik}(t)\;+\;\nu^2\sum\frac{1}{\sigma_n^4}\left\{f_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+f_{pn}\dot{\Gamma}_{pn}(t)+f_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\}$$

(9.6)

$$\Theta_a(p_3R,t)=\Theta_{ak}(t)+\frac{\lambda_a}{\Delta}\left(-\ln p_3+\frac{q_a}{p_4}+\ln p_4\right)\left\{\Theta_{ik}(t)-\Theta_{ak}(t)+\left[2\lambda_i(p_3^2-p_2^2)\left(\frac{\beta_i}{p_2}+\ln p_2+q_i\right)-(p_3^2-p_2^2)+2p_3^2(\ln p_3-\ln p_2)\right]\frac{W(t)R^2}{4\tilde{\lambda}_u}\;+\right.$$

$$\left.+\;\nu^2\sum\frac{1}{\sigma_n^4}\left\{g_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+g_{pn}\dot{\Gamma}_{pn}(t)+g_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\}\right\} \qquad (9.7)$$

$$\overset{\equiv}{\Theta}_a(t)=\Theta_{ak}(t)+\frac{\lambda_a}{\Delta}\left(-\frac{p_4^2\ln p_4-p_3^2\ln p_3}{p_4^2-p_3^2}+\frac{1}{2}+\ln p_4\right)\left\{\text{curved bracket equal to the first one in } \Theta_a(p_3R,t)\right\}\;+$$

$$+\;\nu^2\sum\frac{1}{\sigma_n^4}\left\{h_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+h_{pn}\dot{\Gamma}_{pn}(t)+h_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\} \qquad (9.8)$$

$$\Theta_a(p_4R,t)=\Theta_{ak}(t)+\frac{\lambda_a\frac{q_a}{p_4}}{\Delta}\left\{\text{curved bracket equal to the first one in } \Theta_a(p_3R,t)\right\}\;+$$

$$+\;\nu^2\sum\frac{1}{\sigma_n^4}\left\{j_{in}[\dot{\Gamma}_{in}(t)-\dot{\Theta}_{ik}(t)]+j_{pn}\dot{\Gamma}_{pn}(t)+j_{an}[\dot{\Gamma}_{an}(t)-\dot{\Theta}_{ak}(t)]\right\} \qquad (9.9)$$

Δ is here always defined by (8.44) and must not be confused with the Δ_n in (2.7), (5.7), and in Chapter 6.

Reference

[1] WUNDT, H., Transient Heat Conduction Through Heat Producing Layers, EUR 4834(e), 1972.

CORRIGENDUM

EUR 4862 e

TRANSIENT HEAT CONDUCTION THROUGH A CYLINDRICAL THREE-LAYER HOLLOW FUEL ELEMENT WITH INTERNAL AND EXTERNAL COOLING by H. Wundt

In the ABSTRACT please read EUR 4818 instead of EUR 4834

This corrigendum cancels and replaces the one previously distributed

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