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JN-METD2
A FORTRAN-IV PROGRAMME FOR SOLVING NEUTRON TRANSPORT PROBLEMS WITH ISOTROPIC SCATTERING IN MULTILAYER SLABS BY THE j_N METHOD

by

T. ASAOKA and E. CAGLIOTI BONANNI

1972

Joint Nuclear Research Centre
Ispra Establishment - Italy

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ABSTRACT

The mathematical formulae of the j_N method for the description of neutron transport in a multilayer slab system are summarized within the context of a multigroup model under the assumption that the scattering of neutrons is spherically symmetric in the laboratory system. A Fortran-IV computer programme JN-METD2 is described in detail for the use of accurately solving the transport problem according to these formulae. The computer code calculates the eigenvalue of the integral transport equation, the effective multiplication factor or the asymptotic decay constant of neutrons, as well as the eigenfunction, the space, angle and energy dependent flux distribution. In addition, it evaluates the first three time moments of the time-dependent flux resulting from a delta function boundary source with space, angle and energy variables.

KEYWORDS

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NEUTRON SPECTRA
MULTIPLICATION FACTORS
TIME DEPENDENCE
BUNDARY CONDITIONS
NEUTRON SOURCES
REACTORS

C O N T E N T S

JN-METD2, A FORTRAN-IV PROGRAMME FOR SOLVING NEUTRON TRANSPORT PROBLEMS WITH ISOTROPIC SCATTERING IN MULTILAYER SLABS BY THE j_N METHOD

1. Introduction

Under the assumption that the scattering of neutrons is spherically symmetric in the laboratory system, the newly developed j_N method has already yielded accurate solutions to space-energy time-dependent transport problems in bare spheres (ASAOKA, 1968-1) and space-angle energy-time dependent problems in homogeneous slabs (ASAOKA, 1968-2). The neutron flux for a stationary state has also been obtained as a simple limiting case of time-dependent problems. For dealing with these problems, a computer code JN-METD1 has been developed within the context of the multigroup and (up to) j_7 approximation (ASAOKA,1971).

As already shown by several authors, the approach can easily be extended to take into account anisotropic scattering of neutrons (KSCHWENDT, 1971) or to treat multilayer slab systems (MANGIAROTTI, 1971). For the description of time-dependent neutron transport in multilayer slabs with anisotropic scattering, a general formalism has been developed by the present authors (ASAOKA and CAGLIOTI, 1969 and 1972) and applied to an optimization study of moderators in pulsed reactors. Furthermore, the application of the method to convex geometries has been demonstrated for a homogeneous medium in which the neutron scattering is isotropic (HEMBD, 1970).

The present report is concerned with the computer code JN-METD2 designed to solve transport problems in multilayer slab systems with isotropic scattering of neutrons. By the use of a multigroup model and the j_N ($N \leq 7$) approximation, the computer code calculates:

(a) The space, angle and energy dependent neutron flux due to a stationary point-isotropic boundary source, as well as the first and second time moments of the time-dependent flux resulting from a point-isotropic delta function source on one boundary.

(b) The value of the effective multiplication factor k_{eff} of a multilayer slab reactor and the stationary flux distribution as a function of space, angle and energy.

(c) The asymptotic decay constant of the fundamental neutron distribution in a multilayer slab system.

2. Mathematical Formulae

Since a general formulation for time-dependent transport in multilayer slabs with anisotropic scattering has already been shown in a previous paper (ASAOKA and CAGLIOTI, 1972), we only summarize here the mathematical formulae for the description of neutron transport in a M-region slab within the context of a G-energy-group model and the j_N approximation (scattering being assumed spherically symmetric).

Let x be the space coordinate, μ the direction cosine of the neutron velocity, Σ_g^i and v_g the macroscopic total cross section of the i-th region (extending from $x=a_{i-1}$ to a_i) and the speed of neutrons in the g-th group, respectively, and $C_i(g'\to g)$ the mean number of secondary neutrons produced in the g-th group as a result of collisions in the g'-th group and i-th region. The number of the g-th group neutrons in the j-th region resulting from a point-isotropic delta function source $S_g(x,\mu,t)=2S_g\mu\,\delta(x)\delta(t)$ can be written as

$$
\begin{aligned}
v_g n_g^j(x,\mu,t) &= 2S_g\,\delta\!\left(t-\frac{x}{v_g\mu}\right)\exp\!\left[-\left(\sum_{k=1}^{j-1}\Sigma_g^k(a_k-a_{k-1})+\Sigma_g^j(x-a_{j-1})\right)/\mu\right]+\\
&+\sum_k \exp[\Sigma_1^1 v_1(s_k-1)t]\sum_{p=0}^{N}\left\{\sum_{i=1}^{j}F_p\!\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1v_1s_k;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\right)\times\right.\\
&\times B_p^i(g,s_k)+\left.\sum_{i=j+1}^{M}F_p\!\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1v_1s_k;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\right)B_p^i(g,s_k)\right\}+\\
&+\frac{1}{2\pi}\int_{-\infty}^{\infty}dy\,\exp[-(\Sigma v-iy)t]\sum_{p=0}^{N}\left\{\sum_{i=1}^{j}F_p\!\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1v_1-\Sigma v+iy;-\sum_{k=i}^{j}\alpha_g^k+\right.\right.\\
&+\left.\frac{\alpha_g^j+\alpha_g^i}{2}\right)b_p^i(g,\Sigma_1^1v_1-\Sigma v+iy)+\sum_{i=j+1}^{M}F_p\!\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1v_1-\Sigma v+iy;\sum_{k=j}^{i}\alpha_g^k-\right.\\
&-\left.\left.\frac{\alpha_g^i+\alpha_g^j}{2}\right)b_p^i(g,\Sigma_1^1v_1-\Sigma v+iy)\right\},
\end{aligned}
\tag{1}
$$

where $\alpha_g^k=P_g^k\Sigma_g^k(a_k-a_{k-1})$ $[P_g^k=1-(\Sigma_1^1v_1-s)/(\Sigma_g^kv_g)]$, Σv stands for the minimum value of $\Sigma_g^kv_g$ for all g and k, and

$$F_p(\alpha_g^i,\alpha_g^j,\xi,\mu,s;d)=\frac{1}{4\pi}P_g^j\int_{-\infty}^{\infty}dz\, j_p(\alpha_g^i z)\exp[i\alpha_g^i(1-2\xi)]\times$$
$$\times e^{idz}\int_0^{\infty}dt'\exp[-P_g^j(1-iz\mu)t']. \tag{2}$$

The function F_p is equal to $F_{p\ell}$ with $\ell=0$ evaluated previously (ASAOKA and CALGIOTI, 1972). The explicit expression for F_p in the j_7 approximation ($p \leq 7$) is given in the Appendix 1, Section 2. In addition, $s=\Sigma_1^1\nu_1 s_k$ and $B_p^i(g,s_k)$ in equation (1) are respectively a pole and the residue of $b_p^i(g,s)$ which satisfies the following linear equation:

$$\frac{1}{2g+1}b_g^j(g',s)=\sum_{g=1}^{G}c_j(g\to g')\alpha_g^j S_g C_{gb}\left(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\right)+$$
$$+\sum_{g=1}^{G}c_j(g\to g')\sum_{r=0}^{N}\Big[\sum_{i=1}^{j}J_{gr}\left(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\right)b_r^i(g,s)+$$
$$+\sum_{i=j+1}^{M}J_{gr}\left(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\right)b_r^i(g,s)\Big], \tag{3}$$

where

$$C_{gb}(\alpha_g^j,s;d,\alpha_g^1)=\frac{1}{\pi}\int_{-\infty}^{\infty}dz\, j_g(\alpha_g^j z)\, e^{idz}\exp(-i\alpha_g^1 z)\times$$
$$\times\int_0^{\infty}dt'\exp(-P_g^j t')\int_0^1 d\mu\,\mu\exp(iP_g^j z t'\mu), \tag{4}$$

$$J_{gr}(\alpha_g^j,\alpha_g^i,s;d)=\frac{1}{2\pi}\alpha_g^j\int_{-\infty}^{\infty}dz\, j_g(\alpha_g^j z)\, j_r(\alpha_g^i z)\, e^{idz}\times$$
$$\times\int_0^{\infty}dt'\exp(-P_g^j t')\int_{-1}^1 d\mu\exp(iP_g^j z t'\mu). \tag{5}$$

The explicit expressions for the integrals C_{gb} and J_{gr} are respectively shown in the Appendix 1, Sections 4 and 1.

For a stationary state, only one largest pole $s=\Sigma_1^1\nu_1$ of $b_g^j(g,s)$ is of importance. Hence, by multiplying $s-\Sigma_1^1\nu_1$ on both sides of equation (3) and taking the limit $s\to\Sigma_1^1\nu_1$, we get [assuming a boundary source $S_g(x,\mu)= = 2S_g\mu\delta(x)$]

$$\frac{1}{2\ell+1}B_\ell^j(g')=\sum_{g=1}^{G}c_j(g\to g')\alpha_g^j S_g C_{\ell\ell}\left(\frac{\alpha_g^j}{2},\Sigma_1^1 v_1;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\right)+$$
$$+\sum_{g=1}^{G}c_j(g\to g')\sum_{r=0}^{N}\Big[\sum_{i=1}^{j}J_{\ell r}\left(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},\Sigma_1^1 v_1;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\right)B_r^i(g)+$$
$$+\sum_{i=j+1}^{M}J_{\ell r}\left(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},\Sigma_1^1 v_1;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\right)B_r^i(g)\Big], \tag{6}$$

where $B_\ell^j(g)=\lim_{s\to\Sigma_1^1 v_1}(s-\Sigma_1^1 v_1)b_\ell^j(g,s)$. The stationary vector flux can thus be written as

$$v_g n_g^j(x,\mu)=2S_g \exp\Big[-\Big(\sum_{k=1}^{j-1}\Sigma_g^k(a_k-a_{k-1})+\Sigma_g^j(x-a_{j-1})\Big)/\mu\Big]+$$
$$+\sum_{p=0}^{N}\Big\{\sum_{i=1}^{j}F_p\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1 v_1;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\right)B_p^i(g)+$$
$$+\sum_{i=j+1}^{M}F_p\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,\Sigma_1^1 v_1;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\right)B_p^i(g)\Big\}. \tag{7}$$

Upon integrating equation (7) over μ from -1 to 1, the scalar flux is obtained in the form

$$v_g n_g^j(x)=2S_g E_2\Big[\sum_{k=1}^{j-1}\Sigma_g^k(a_k-a_{k-1})+\Sigma_g^j(x-a_{j-1})\Big]+$$
$$+\sum_{p=0}^{N}\Big\{\sum_{i=1}^{j}G_p\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}}-1,\Sigma_1^1 v_1;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\right)B_p^i(g)+$$
$$+\sum_{i=j+1}^{M}G_p\left(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},2\frac{x-a_{j-1}}{a_j-a_{j-1}}-1,\Sigma_1^1 v_1;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\right)B_p^i(g)\Big\}, \tag{8}$$

in which

$$G_p(\alpha_g^i,\alpha_g^j,\xi,s;d)=\int_{-1}^{1}d\mu\, F_p(\alpha_g^i,\alpha_g^j,(1+\xi)/2,\mu,s;d), \tag{9}$$

the expression for G_p being given in the Appendix 1, Section 3.

It is seen from equation (6) that the critical condition for a system without extraneous source S_g = 0 is to be obtained by solving the determinantal equation:

$$\left|\frac{\delta_{gg'}\delta_{\ell r}\delta_{ji}}{2\ell+1}-c_j(g\to g')J_{\ell r}\left(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s=\Sigma_1^1 v_1;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\right)\right|=0,$$
$$g,g'=1,2,\cdots,G;\quad \ell,r=0,1,\cdots,N;\quad j,i=1,2,\cdots,M. \tag{10}$$

In order to get the value of the effective multiplication factor k_{eff} for a given reactor, $C_j(g\to g')$ is divided into two parts. These are the scattering part $C_s^j(g\to g')=\Sigma_s^j(g\to g')/\Sigma_g^j$ and the fission part $C_f^j(g\to g')= = \chi_{g'}(\nu\Sigma_f)_g^j/\Sigma_g^j$ where χ_g stands for the proportion of fission neutrons born in the g-th group. By the use of this separation, the value of k_{eff} is obtained by solving equation (10) with

$$C_j(g\to g')=C_s^j(g\to g')+C_f^j(g\to g')/k_{eff}. \tag{11}$$

The ratios between $B_r^i(g)$'s can now be obtained, under the condition (10), from equation (6) with $S_g=0$ and $C_j(g\to g')$ given by equation (11) for calculating the flux distribution in a multilayer slab reactor according to equation (7) or (8) with $S_g=0$. In addition, equation (10) with $s=\Sigma_1^1 v_1 s_1$ instead of $\Sigma_1^1 v_1$ gives the asymptotic decay constant $\lambda = \Sigma_1^1 v_1(1-s_1)$ which governs the asymptotic behaviour of neutrons as $t\to\infty$ [see equation (1)] .

It is also easy to get the time moments of the time-dependent flux resulting from the incidence of an external delta function source on one boundary: $S_g(x,\mu,t)=2S_g\mu\delta(x)\delta(t)$. The first three time moments of the angular flux (1) are written as follows:

$$\int_0^\infty dt\, v_g n_g^j(x,\mu,t)=v_g n_g^j(x,\mu), \tag{12}$$

which is given by equation (7) with $B_p^i(g)=b_p^i(g,\Sigma_1^1 v_1)$ [compare equation (3) with (6)] .

$$\begin{aligned}\int_0^\infty dt\, t\, v_g n_g^j(x,\mu,t) &= 2S_g x \exp\Big[-\Big(\sum_{k=1}^{j-1}\Sigma_g^k(a_k-a_{k-1})+\Sigma_g^j(x-a_{j-1})\Big)/\mu\Big]/(v_g\mu)-\\ &-\frac{d}{ds}\Big[\sum_{p=0}^{N}\Big\{\sum_{i=1}^{j}F_p\Big(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\Big)b_p^i(g,s)+\\ &+\sum_{i=j+1}^{M}F_p\Big(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\Big)b_p^i(g,s)\Big\}\Big]_{s=\Sigma_1^1 v_1},\end{aligned} \tag{13}$$

$$\int_0^\infty dt\, t^2 v_g n_g^j(x,\mu,t) = 2S_g x^2 \exp\Big[-\Big(\sum_{k=1}^{j-1}\Sigma_g^k(a_k-a_{k-1})+\Sigma_g^j(x-a_{j-1})\Big)/\mu\Big]/(v_g\mu)^2 +$$
$$+\frac{d^2}{ds^2}\Big[\sum_{p=0}^{N}\Big\{\sum_{i=1}^{j} F_p\Big(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;-\sum_{k=i}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^i}{2}\Big)b_p^i(g,s)+$$
$$+\sum_{i=j+1}^{M} F_p\Big(\frac{\alpha_g^i}{2},\frac{\alpha_g^j}{2},\frac{x-a_{j-1}}{a_j-a_{j-1}},\mu,s;\sum_{k=j}^{i}\alpha_g^k-\frac{\alpha_g^i+\alpha_g^j}{2}\Big)b_p^i(g,s)\Big]_{s=\Sigma_1^1 v_1}. \tag{14}$$

According to equation (3), the first and second derivatives of $b_p^i(g,s)$ at $s=\Sigma_1^1 v_1$ are obtained by solving respectively the following equations:

$$(2\ell+1)\sum_{g=1}^{G}c_j(g\to g')\sum_{r=0}^{N}\sum_{i=1}^{M}J_{\ell r}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)\frac{d}{ds}b_r^i(g,s)-$$
$$-\frac{d}{ds}b_\ell^j(g',s) = -(2\ell+1)\sum_{g=1}^{G}c_j(g\to g')\Big[\alpha_g^j S_g\frac{d}{ds}C_{\ell\ell}\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+$$
$$+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\Big)+\alpha_g^j S_g C_{\ell\ell}\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\Big)/(\Sigma_g^j v_g P_g^j)+$$
$$+\sum_{r=0}^{N}\sum_{i=1}^{M}\frac{d}{ds}J_{\ell r}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)b_r^i(g,s)\Big], \tag{15}$$

$$(2\ell+1)\sum_{g=1}^{G}c_j(g\to g')\sum_{r=0}^{N}\sum_{i=1}^{M}J_{\ell r}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)\frac{d^2}{ds^2}b_r^i(g,s)-$$
$$-\frac{d^2}{ds^2}b_\ell^j(g',s) = -(2\ell+1)\sum_{g=1}^{G}c_j(g\to g')\Big\{\alpha_g^j S_g\frac{d^2}{ds^2}C_{\ell\ell}\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+$$
$$+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\Big)+2\alpha_g^j S_g\frac{d}{ds}C_{\ell\ell}\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\Big)/(\Sigma_g^j v_g P_g^j)+$$
$$\sum_{r=0}^{N}\Big[\sum_{i=1}^{M}2\frac{d}{ds}J_{\ell r}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)\frac{d}{ds}b_r^i(g,s)+\frac{d^2}{ds^2}J_{\ell r}\Big(\frac{\alpha_g^j}{2},$$
$$\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)b_r^i(g,s)\Big]\Big\}. \tag{16}$$

For a non-multiplying system in which there is no up-scattering of neutrons, equations (3) [or (6)] , (15) and (16) can be simplified to those which are solved in the same way as for a one-group model. For example, equation (3) can be reduced to

$$(2\ell+1)c_j(g'\to g')\sum_{r=0}^{N}\sum_{i=1}^{M}J_{\ell r}\Big(\frac{\alpha_{g'}^j}{2},\frac{\alpha_{g'}^i}{2},s;\mp\Big(\sum_k\alpha_{g'}^k-\frac{\alpha_{g'}^j+\alpha_{g'}^i}{2}\Big)\Big)b_r^i(g',s)-$$
$$-b_\ell^j(g',s)=-(2\ell+1)\Big[\sum_{g=1}^{g'}c_j(g\to g')\alpha_g^j S_g C_{\ell\ell}\Big(\frac{\alpha_g^j}{2},s;-\sum_{k=1}^{j}\alpha_g^k+\frac{\alpha_g^j+\alpha_g^1}{2},\frac{\alpha_g^1}{2}\Big)+$$
$$+\sum_{g=1}^{g'-1}c_j(g\to g')\sum_{r=0}^{N}\sum_{i=1}^{M}J_{\ell r}\Big(\frac{\alpha_g^j}{2},\frac{\alpha_g^i}{2},s;\mp\Big(\sum_k\alpha_g^k-\frac{\alpha_g^j+\alpha_g^i}{2}\Big)\Big)b_r^i(g,s)\Big]. \tag{17}$$

From equations (12)-(14), the first three time moments of the total flux can be obtained as follows:

$$\int_0^\infty dt\, v_g n_g^j(x,t) = v_g n_g^j(x) \qquad \text{given by equation (8),} \tag{18}$$

$$\begin{aligned}\int_0^\infty dt\, t\, v_g n_g^j(x,t) = {} & 2S_g x E_1\Big[\sum_{k=1}^{j-1} \Sigma_g^k (a_k - a_{k-1}) + \Sigma_g^j (x - a_{j-1})\Big]/v_g - \\ & - \frac{d}{ds}\Big[\sum_{p=0}^{N} \Big\{ \sum_{i=1}^{j} G_p\Big(\frac{\alpha_g^i}{2}, \frac{\alpha_g^j}{2}, 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, s; -\sum_{k=i}^{j} \alpha_g^k + \frac{\alpha_g^j + \alpha_g^i}{2}\Big) b_p^i(g,s) + \\ & + \sum_{i=j+1}^{M} G_p\Big(\frac{\alpha_g^i}{2}, \frac{\alpha_g^j}{2}, 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, s; \sum_{k=j}^{i} \alpha_g^k - \frac{\alpha_g^i + \alpha_g^j}{2}\Big) b_p^i(g,s) \Big\} \Big]_{s=\Sigma_1^1 v_1},\end{aligned} \tag{19}$$

$$\begin{aligned}\int_0^\infty dt\, t^2 v_g n_g^j(x,t) = {} & 2S_g x^2 \exp\Big[-\Big(\sum_{k=1}^{j-1} \Sigma_g^k (a_k - a_{k-1}) + \Sigma_g^j (x - a_{j-1})\Big)\Big] \times \\ & \times 1/\Big[v_g^2 \Big(\sum_{k=1}^{j-1} \Sigma_g^k (a_k - a_{k-1}) + \Sigma_g^j (x - a_{j-1})\Big)\Big] + \frac{d^2}{ds^2}\Big[\sum_{p=0}^{N} \Big\{ \sum_{i=1}^{j} G_p\Big(\frac{\alpha_g^i}{2}, \frac{\alpha_g^j}{2}, \\ & 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, s; -\sum_{k=i}^{j} \alpha_g^k + \frac{\alpha_g^j + \alpha_g^i}{2}\Big) b_p^i(g,s) + \sum_{i=j+1}^{M} G_p\Big(\frac{\alpha_g^i}{2}, \frac{\alpha_g^j}{2}, 2\frac{x-a_{j-1}}{a_j - a_{j-1}} - 1, \\ & s; \sum_{k=j}^{i} \alpha_g^k - \frac{\alpha_g^i + \alpha_g^j}{2}\Big) b_p^i(g,s) \Big\} \Big]_{s=\Sigma_1^1 v_1}.\end{aligned} \tag{20}$$

3. JN-METD2 Computer Code

3.1 Input data (see the Appendix 3)

After a title card with a 20A4 format, 16 integers are read with a 25I3 format. These input integers are defined as follows:

IIO	3, 5 or 7 for the j_3, j_5 or j_7 approximation (0 to stop the execution; see the Appendix 2, Section 1)
IIII	0 for solving a new problem or 1 for restarting a problem for which punched cards for the residues RES are available (see below)

NSOUCE	-1, 0 or 1 for the problem to obtain the asymptotic decay constant (NSPH=1 and LLL=0; if NSLOWD=1 the decay constant of neutrons belonging to the lowest energy-group being calculated), to compute the value of the effective multiplication factor (NSLOWD=0 and NSPH=1) or to deal with a subcritical system with an external source for obtaining the flux distribution
NSLOWD	1 for non-multiplying system without up-scattering of neutrons (0 otherwise)
IGRP	Total number of energy groups
IHT IHS JHL	Arrangement of reaction type of the cross section (XSEC) for the g-th group and i-th region; XSEC (1,g,i) = Σ_{ag}^{i} ,...., XSEC(IHT-1, g,i) = Σ_{tg}^{i} , XSEC (IHT$\geq$3,g,i) = Σ_{trg}^{i} , XSEC (IHT+1,g,i) = Σ^{i}(g+IHS-IHT$\rightarrow$g),...., XSEC (IHS-1,g,i) = Σ^{i}(g+1$\rightarrow$g), XSEC (IHS$\geq$IHT,g,i) = Σ^{i}(g$\rightarrow$g), XSEC (IHS+1,g,i) = Σ^{i}(g-1$\rightarrow$g),...., XSEC (JHL$\geq$IHS,g,i) = Σ^{i}(g-JHL+IHS$\rightarrow$g) [Σ_{trg} is used for Σ_{g} for taking into account the anisotropic scattering of neutrons and Σ_{tg} is for calculating c(g$\rightarrow$g') = Σ(g$\rightarrow$g')/Σ_{tg}.
NSPH	3 for obtaining the first and second time moments of the flux due to a δ(t) source in addition to the stationary flux (1 otherwise)
NNNN	Total number of homogeneous regions in the multilayer slab system
LLL	1 for computing the flux distribution (0 otherwise)
IAA	Total number of input cards for the present problem
NENRGY	Number of energy groups for which the flux distributions are to be calculated (see the array NGRUP mentioned below)
NTFLUX	1 for computing the total flux (0 otherwise)

NTSPAC	Number of space points at which the angular and/or total flux are to be obtained (see the array NSPACE mentioned below)
NANGL	Number of angular points at which the angular flux is to be calculated (total range of μ from -1 to 1 is divided into NANGL-1 to have an equal spacing, and only μ= 1 if NANGL=1)

Next, in the subroutine JNMETD, the following data depending on the input integer NSOUCE are read with 7F10.6 (energy-dependent quantities are ordered respectively by energy-group beginning with the first or highest group):

NSOUCE=1	SOCE; boundary source intensity S_g
	VG; speed of neutrons V_g
NSOUCE=0	CK1,CK2,EPSK; the first and second guess for the value of k_{eff} and the required relative accuracy when IIII=0 (CK1 = k_{eff} if IIII=1)
	VG; fission spectrum χ_g
NSOUCE=1	CK1,CK2,EPSK; the first and second guess for the asymptotic time constant $1-\lambda_1$ and the required relative accuracy
	VG; speed of neutrons V_g

The following data are then read with 7F10.6 (or 8F9.6, F8.5 for XSEC) in the order of space region, the total number of cards being NNNN* $\{$ 1+[(IGRP+6)/7]+ +[(JHL*IGRP+8)/9]+(if NSOUCE=0, [(IGRP+6)/7])$\}$:

A	Thickness of the region

BUCLG	Buckling for taking into account the finite extention of the system in y and z directions, $(B_y^2+B_z^2)_g$, ordered by group g
XSEC	Nuclear cross sections for all types of reactions arranged as mentioned above in the first group, then for those in the second group and so on
If NSOUCE=0, XFSEC	$(\nu\Sigma_f)_g$ order by g

For the case where IIII=1 (NSOUCE$\geq$0), a punched card dump with a (5D15.8) format for the residue (RES) $B_p^i(g)$ is then read in the same order as in the punched output or output print: For NSLOWD=1 (NSOUCE=1), it reads first $B_p^i(g)$'s , p=0,1,...,IIO, for i=1 and g=1, then those for i=2 and g=1 and so on until those for i=NNNN and g=1. These are followed, if any (NSPH=3), by $\frac{d}{ds}B_p^i(g,s)]_{s=\Sigma_1^i v_1}$'s , p=0-IIO, for i=1 and g=1, i=2 and g=1,..., i=NNNN and g=1, and then by $\frac{d^2}{ds^2}B_p^i(g,s)]_{s=\Sigma_1^i v_1}$'s . All these data are repeated for g=2, g=3,..., g=IGRP. For other cases (NSLOWD=0 and NSOUCE$\geq$0), it reads first $B_p^i(g)$'s , p=0-IIO, for g=1 and i=1, g=2 and i=1,..., g=IGRP and i=1, which are followed by those for i=2 and so on till i=NNNN. Then, if any, $\frac{d}{ds}B_p^i(g,s)]_{s=\Sigma_1^i v_1}$'s are read in the same order as $B_p^i(g)$'s and $\frac{d^2}{ds^2}B_p^i(g,s)]_{s=\Sigma_1^i v_1}$'s follow them. The total number of cards is therefore always NSPH*NNNN*IGRP*[(IIO+5)/5].

Finally, if LLL=1, the following data are read with a 25I3 format in the subroutine FLUXCA:

NGRUP	Energy-group indices of NENRGY groups for which the flux distributions are to be calculated (in increasing order)
NSPACE	NNNN numbers of space points at which the flux is to be calculated. (The first integer is the number of space points for the first region, the second integer is for the second region and so on). If NSPACE(I)>1 the I-th region is divided into NSPACE(I)-1 to have an equal spacing and if NSPACE(I)=1 one space point is selected at the boundary between the I-th and (I+1)-th region.

3.2 Computer programme

The JN-METD2 package consists of 15 programmes: MAIN, JNMETD, FLUXCA, FCAL, FSCAL, FSCON, SGMOD, CCALC, DET, ITRTON, SOLEQ, GCAL, FMCAL, EP and VARIAC. In addition, the code makes use of the library subprogrammes, MAXO, DEXP, DLOG, DATAN, DSIN and DCOS.

Almost all subscript variables and their dimension informations are stored in a blank COMMON for the use of the adjustable dimensioning. The present size of the COMMON for subscript variables is 64 K bytes so that the programme requires the core storage less than 300 K bytes in the Fortran -IV, Version G on the IBM-360/65. For altering the dimension of the COMMON to fit core storage, the 12 statements should be adjusted. (All 15 programme decks are respectively numbered.) These are 5 cards in the MAIN programme: the 30th (dimension of ACOM), 31st (dimension of ICOM), 32nd (dimension of BCOM), 43rd (clear COMMON) and 132nd card (available $\lesseqgtr$ required storage?), and 7 COMMON statements (dimension of ACOM) which are the 20th card of JNMETD, 13th of FLUXCA, 14th of FCAL, 10th of CCALC, 5th of ITRTON, 12th of GCAL and 5th of EP.

In the MAIN programme, as can be seen from the flow chart shown in the Appendix 2, Section 1, sizes of the required arrays are computed based on input parameters and then first-word addresses are calculated for these arrays. The locations of these pointers and the associated arrays with their dummy dimensions are given in Table I which shows also the fact that the storage locations bigger than IA(38) are used in two different ways, once in JNMETD and then again in FLUXCA. The actual values of the integer variables specifying the sizes of arrays are summarized in Table II. The first-word addresses and the dimension informations are transferred through a call statement and a part of vector in the blank COMMON is treated as a multi-dimensional array in subprogrammes.

The subroutine JNMETD computes:

(a) The residue $B_g^j(g)$ according to equation (6) [or (17) for a non-multiplying system without up-scattering of neutrons] and if NSPH=3 $\frac{d}{ds}f_g^j(g,s)$ and $\frac{d^2}{ds^2}f_g^j(g,s)$ at $s=\Sigma_1^1 V_1$ by solving respectively equations (15) and (16) (or the corresponding equations for a non-multiplying system without up-scattering), for a multilayer slab system with a stationary (or for the derivatives, a delta function) point-isotropic boundary source (NSOUCE=1).

(b) The value of k_{eff} for a multilayer slab reactor (NSOUCE=0, NSLOWD=0 and NSPH=1) and if LLL>0 the ratios between $B_r^i(g)$'s from equations (10) and (6) with $S_g=0$ and $C_j(g\to g')$ given by equation (11).

(c) The asymptotic time constant $1-\lambda_1$ for obtaining the asymptotic decay constant $\Sigma_1^1 v_1(1-\lambda_1)$ for a multilayer slab (LLL=0 and NSPH=1) or if NSLOWD=1 the asymptotic decay constant of neutrons belonging to the lowest energy group.

For the problems (b) and (c), the values of $C_k(i\to j)$ are first modified according to the guess of k_{eff} or $1-\lambda_1$, and for the problem (c) the values of $\alpha_i^k = P_i^k \Sigma_i^k (a_k - a_{k-1})$ are calculated (see the flow diagram shown in the Appendix 2, Section 2). With these values of $C_k(i\to j)$ and α_i^k , the matrix elements for equation (6) or (17) [also for equations (15) and (16)] are then calculated by calling the subroutine FCAL $(\alpha_g^j, \alpha_g^i, d,$ IIO, MM, $\Sigma_g^j v_g P_g^j \times$ $\times 1/(\Sigma_g^i v_g P_g^i), \Sigma_g^j v_g P_g^j \frac{d}{d\lambda} d, 1/(\Sigma_g^j v_g P_g^j))$ [note that $\frac{d}{d\lambda}\alpha_g^k = \alpha_g^k/(\Sigma_g^k v_g P_g^k)$ and $\frac{d^2}{d\lambda^2}\alpha_g^k = 0$] which computes the value $(\pm i)P_g^j(\frac{d}{d\lambda})^n J_{qr}(\alpha_g^j, \alpha_g^i, \lambda; d)$ for n=0$\sim$MM+1 by the use of their explicit expressions shown in the Appendix 1, Section 1 [see formulae (A14)-(A18) . In the case where $\alpha_g^j + \alpha_g^i + |d| \leq 5$, it calls the subroutine FSCAL$(\alpha_g^j, \alpha_g^i, d, n-1,$ IIO, JII, $\Sigma_g^j v_g P_g^j/(\Sigma_g^i v_g P_g^i), \Sigma_g^j v_g P_g^j \frac{d}{d\lambda} d)$ in which the series expansion shown by the formulae (A23) and (A24) are used for the calculation depending on the values of parameters α_g^j, α_g^i and d (JII stands for the parameter range). For computing the first and second derivatives of J_{qr} with $q=1-7$ and $r=7$, the FSCAL calls another subroutine FSCON. In addition, the FCAL and FSCAL use the subroutine SGMOD (SSI,I,....) in which when I>0 $X_{\lambda+m,n}(\alpha_j, \alpha_i, d)$ is modified to $X_{\lambda+m+1,n+1}$ [see equation (A11) in the Appendix 1] , when I=0 the summation of (A16) is performed or when I<0 a multiplication is carried out for calculating the derivatives of J_{qr} by using their series expansions. The exponential integral $E_n(x)$ appeared on the right hand side of equation (A12) is evaluated by the function subprogramme EP(n,x) which comes from the subprogramme EP(n,x,b,....) in the computer code JN-METD1 (ASAOKA, 1971). At the end of the FCAL, the recurrence relation (A7) given in the Appendix 1 is adopted for computing the functions J_{qr} with q = 2-7 and r = 1-6 (and their derivatives if NSPH = 3) from the values of J_{0r} and J_{1r} with r = 0-7 and J_{q0} and J_{q7} with q = 2-7.

After having been obtained the matrix elements, the JNMETD calls, for the problems (b) and (c), the subroutine DET to evaluate the determinant (10) [with $\delta = \Sigma_1{}^1 \nu_1 \delta_1$ for the problem (c)] or the corresponding equation for the problem (c) with NSLOWD=1. The subroutine ITRTON is then used for iterating the process to make the value of the determinant zero until the relative difference between two successive values of k_{eff} or $1-\delta_1$ becomes smaller than EPSK. For the problem (b) with LLL>0, after obtained the converged value of k_{eff}, the ratios between the residues are calculated by evaluating the cofactors of the determinant by the use of the subroutine SOLEQ which solves a system of simultaneous linear equations.

For the problem (a), in addition to the matrix elements, the first term on the right hand side of equation (3) [and if NSPH=3, the right hand sides of equations (15) and (16)] or if NSLOWD=1 the right hand side of equation (17) [and the expressions corresponding respectively to the right hand sides of equations (15) and (16)] is evaluated with the help of the subroutine CCALC$(\alpha_g{}^j, \alpha_g{}^1-d, \mathrm{IIO}, \mathrm{MMMM}, \Sigma_g{}^j \nu_g P_g{}^j \frac{d}{d\delta}(\alpha_g{}^1-d), (\Sigma_g{}^j \nu_g P_g{}^j \frac{d}{d\delta})^2(\alpha_g{}^1-d))$. The CCALC computes $(-i)2\alpha_g{}^j P_g{}^j (\Sigma_g{}^j \nu_g P_g{}^j \frac{d}{d\delta})^{\mathrm{MMMM}+1} C_{gg}(\alpha_g{}^j, \delta; d, \alpha_g{}^1)$ with MMMM=1,0 or -1 by using their explicit expressions if $\alpha_g{}^j + \alpha_g{}^1 + |d| > 3$ or the series expansions otherwise. As is seen from the expression (A43) it uses the function subprogramme EP for evaluating E_1. The residues (and their derivatives, for NSPH=3) are thus obtain in the JNMETD by calling the SOLEQ to solve equation (6) [and equations (15) and (16)] if NSLOWD=0 or (17) [and the equations corresponding respectively to (15) and (16)] if NSLOWD=1.

The subroutine FLUXCA computes for NTFLUX>0 the total flux and/or for NANGL>0 the angular flux by using the values of the residues (or the ratios between them) obtained as mentioned above in the JNMETD. As is seen from the flow diagram of the FLUXCA shown in the Appendix 2, Section 3, after having calculated the angle points (the values of μ) at which the angular fluxes are to be computed if NANGL>0, the space points $(0 \leq \xi \leq 1)$ are determined in each region and the total fluxes are calculated at these points with the help of the subroutine GCAL$(\alpha_g{}^i, \alpha_g{}^j(2\xi-1)-d, \mathrm{IIO}, \mathrm{MMMM}, \Sigma_g{}^i \nu_g P_g{}^i \frac{d}{d\delta}[\alpha_g{}^j(2\xi-1)-d], (\Sigma_g{}^i \nu_g P_g{}^i \frac{d}{d\delta})^2[\alpha_g{}^j(2\xi-1)-d])$ if NTFLUX>0. The GCAL computes $(i)(\Sigma_g{}^i \nu_g P_g{}^i \frac{d}{d\delta})^{\mathrm{MMMM}+1} \times G_p(\alpha_g{}^i, \alpha_g{}^j, 2\xi-1, \delta; d)$ for MMMM=1, 0 or -1 [see equations (8) and (17)-(20)] by adopting the explicit expressions with the help of the function EP when

$\alpha_g^i + |\alpha_g^j(2\xi-1)-d| > 5$ or the series expansions otherwise (see the Appendix 1, Section 3). For NANGL>0, the FLUXCA calls the subroutine FMCAL $(\alpha_g^i, \alpha_g^j(2\xi-1)-d, \mu, \mathrm{IIO}, \mathrm{MMMM}, \Sigma_g^i v_g P_g^i \frac{d}{ds}[\alpha_g^j(2\xi-1)-d], (\Sigma_g^i v_g P_g^i \frac{d}{ds})^2[\alpha_g^j(2\xi-1)-d])$ which computes $(i)(\Sigma_g^i v_g P_g^i \frac{d}{ds})^{\mathrm{MMMM}+1} F_p(\alpha_g^i, \alpha_g^j, \xi, \mu, s; d)$ with MMMM=1, 0 or -1 for calculating the second term on the right hand side of equation (7), (13) or (14). The FMCAL uses the series expansions given in the Appendix 1, Section 2, if $(\alpha_g^i + |\alpha_g^j(2\xi-1)-d|)/|\mu| \leq 6$.

In the case where NSOUCE=1, the FLUXCA evaluates also the contribution of uncollided source neutrons to the total or/and angular flux according to the first term on the right hand side of equation (8), (19) or (20), with the help of the function EP, or/and equation (7), (13) or (14). If NSPH=3, the above-mentioned calculations are followed by the evaluation of the mean emission time $\bar{t}$ and the variance σ^2 of the time-dependent flux due to the delta function boundary source. For the angular flux, these are written as follows [see equations (12)-(14)]:

$$\bar{t}_g^j = \int_0^\infty dt\, t\, v_g n_g^j(x,\mu,t) \Big/ \int_0^\infty dt\, v_g n_g^j(x,\mu,t), \tag{21}$$

$$(\sigma_g^j)^2 = \int_0^\infty dt\, t^2 v_g n_g^j(x,\mu,t) \Big/ \int_0^\infty dt\, v_g n_g^j(x,\mu,t) - (\bar{t}_g^j)^2, \tag{22}$$

which are calculated in the subroutine VARIAC.

4. Remarks

Since we have already developed a general formulation of the j_N method for dealing with time-dependent transport in a multilayer slab system with anisotropic scattering of neutrons (ASAOKA and CAGLIOTI, 1969 and 1972), it is hoped that the present computer programme can easily be extended to treat anisotropic scattering as well as to obtain a detailed time evolution of neutrons. However, as having been seen in the Appendix 1, the analytical expressions for the functions appeared in the formulation are already rather complicated and hence the programming of the computer code needs care upon keeping always the rounding error reasonably small. In the present code JN-METD2, the functions are evaluated on the basis of either their explicit expressions or series expansions obtained under the assumption that the values of all arguments of the function are small. Therefore, in the case where the ratio between the arguments is very large, it is possible that the function is evaluated with a large rounding error. In such a case,it will be a crucial point for obtaining an accurate result which order of the j_N approximation should be applied to the calculation, because more complex functions are required for the higher order approximation. Generally speaking, the j_5 approximation gives a reasonably accurate result for almost all physical problems. It saves also execution time of the computation by about 30% compared to the j_7 calculation.

Typical running time on the IBM-360/65 is nearly 1.5 min. to obtain, in the context of a 7-group j_5 approximation, the total and angular flux of the lowest group neutrons at 3 angle and 6 space points in a 3-region slab with a stationary boundary source. However, the calculation of the time moments of the time-dependent flux requires a rather long time. A 7-group j_5 calculation takes about 10 min. to obtain the first three time moments of the 7th group angular flux, resulting from a delta function boundary source, at 3 angle and 6 space points in a 3-region slab. The j_3 approximation requires nearly 7 min. for solving this problem. All three sample problems shown in the Appendix 3 take only about 10 sec.

It remains to note that, in the present code, the introduction of the lateral buckling $(B_y^2+B_z^2)_g$ to account for the finite extension of a slab system in two directions leads to modify only the values of $C(g \to g')$ as if the absorption cross section increases by $(B_y^2+B_z^2)_g/(3\Sigma_{trg})$ but not the value of Σ_{trg} which replaces Σ_g to take into consideration the anisotropic scattering of neutrons. It will therefore be necessary in some cases to modify also Σ_{trg} to increase the value by $(B_y^2+B_z^2)_g/(3\Sigma_{trg})$, though the contribution of uncollided source neutrons to the flux is given always in terms of Σ_{tg} without the buckling correction.

Table I Location of the first elements of Real*8 (● Real*4 or ◎ Integer) arrays stored in the blank COMMON and their dimensions†

Location	Array name (dimension)		
IA(31)	ALPHA(IGRP,NNNN)		
IA(51)	XV(IA(11),NNNN)		
IA(32)	RES(IIO,IGRP,NNNN,IA(10))		
IA(33)	● A(NNNN)		
IA(34)	● SOCE(IA(16))		
IA(35)	● XSEC(JHL,IGRP,NNNN)		
IA(36)	● VG(IA(3))		
IA(37)	● XFSEC(IA(3),IA(4))		
IA(38)	ED(IGO,IA(1),IA(15))	IA(38)	X(NTSPAC)
IA(39)	E(IGO,IA(2))	IA(46)	● ANGL(NANGL)
IA(52)	E1(IGO,IA(14))	IA(47)	● TFLUX(IA(9),NENRGY,IA(13))
IA(40)	C1(IA(3),IA(3),IA(6))		
IA(53)	ALS(IGRP,IA(12))	IA(48)	● VFLUX(NANGL,NTSPAC,NENRGY, IA(10))
IA(41)	C2(IA(5),IA(5),IA(6))		
IA(42)	SC(IA(7),IA(8),IA(15))	IA(49)	◎ NGRUP(NENRGY)
IA(43)	● BUCLG(IGRP,NNNN)	IA(50)	◎ NSPACE(NNNN)
IA(44)	● CS(IGRP,IGRP,NNNN)		
IA(45)	● CF(IA(3),IA(3),IA(4))		

† IGO=IIO*IGRP*NNNN or IIO*NNNN for NSLOWD=0 or 1.

Table II Computed integers for specifying the array dimensions (LFF=IIO*IGRP*NNNN)

<table>
<tr><td>NSLOWD</td><td colspan="4">0</td><td colspan="3">1</td></tr>
<tr><td>NSOUCE</td><td>-1</td><td>0</td><td colspan="2">1</td><td>-1</td><td colspan="2">1</td></tr>
<tr><td>NSPH</td><td colspan="3">1</td><td>3</td><td colspan="2">1</td><td>3</td></tr>
<tr><td>IA(1)</td><td>LFF</td><td>LFF+1</td><td>0</td><td>LFF</td><td colspan="3">IIO*NNNN</td></tr>
<tr><td>IA(2)</td><td>LFF</td><td>⊕ LFF</td><td colspan="2">⊕ LFF+2</td><td>IIO*NNNN</td><td colspan="2">⊕ IIO*NNNN+2</td></tr>
<tr><td>IA(3)</td><td colspan="7">IGRP</td></tr>
<tr><td>IA(4)</td><td>0</td><td>NNNN</td><td colspan="2">0</td><td colspan="3">0</td></tr>
<tr><td>IA(5)</td><td>0</td><td>⊕ IGRP</td><td colspan="2">0</td><td colspan="3">0</td></tr>
<tr><td>IA(6)</td><td>NNNN</td><td>NNNN</td><td colspan="2">0</td><td>NNNN</td><td colspan="2">0</td></tr>
<tr><td>IA(7)</td><td colspan="3">0</td><td>LFF</td><td>0</td><td colspan="2">IIO*NNNN</td></tr>
<tr><td>IA(8)</td><td colspan="3">0</td><td>1</td><td>0</td><td colspan="2">IGRP</td></tr>
<tr><td>IA(9)</td><td colspan="7">▲ NTSPAC</td></tr>
<tr><td>IA(10)</td><td>0</td><td>● 1</td><td>1</td><td>3</td><td>0</td><td>1</td><td>3</td></tr>
<tr><td>IA(11)</td><td>IGRP</td><td>0</td><td>0</td><td>IGRP</td><td>IGRP</td><td>0</td><td>IGRP</td></tr>
<tr><td>IA(12)</td><td>NNNN</td><td>0</td><td>0</td><td>0</td><td>NNNN</td><td>0</td><td>0</td></tr>
<tr><td>IA(13)</td><td colspan="3">▲ 1</td><td>▲ 3</td><td colspan="2">▲ 1</td><td>▲ 3</td></tr>
<tr><td>IA(14)</td><td colspan="3">0</td><td>⊕ LFF</td><td colspan="2">0</td><td>⊕ IIO*NNNN</td></tr>
<tr><td>IA(15)</td><td>1</td><td>⊕ 1</td><td>0</td><td>⊕ 2</td><td>1</td><td>⊕ 1</td><td>⊕ 3</td></tr>
<tr><td>IA(16)</td><td colspan="2">0</td><td colspan="2">IGRP</td><td>0</td><td colspan="2">IGRP</td></tr>
</table>

⊕ Only if IIII=0, ● only if LLL>0 and ▲ only if NTSPAC>0.

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Appendix 1. Analytical Expressions of Functions

Since the general formulae of the functions appeared in the present formulation have been developed in a previous paper (ASAOKA and CAGLIOTI, 1972), we show here only the final expressions and then summarize the explicit expressions and series expansions for the functions in the solution of the j_7 approximation.

1. $J_{qr}(\alpha_j,\alpha_i,s;d)$

We consider here $J_{qr}(\alpha_j,\alpha_i,s;d)$ with $\alpha_j \geq \alpha_i > 0$ because

$$J_{qr}(\alpha_j,\alpha_i,s;d) = (\alpha_j/\alpha_i)\, J_{rq}(\alpha_i,\alpha_j,s;d).$$

The parameter range is divided into five:

(a) $-\alpha_j-\alpha_i-d>0$,

(b) $-\alpha_j-\alpha_i-d<0$ and $-\alpha_j+\alpha_i-d>0$,

(c) $-\alpha_j+\alpha_i-d<0$ and $\alpha_j-\alpha_i-d>0$,

(d) $\alpha_j-\alpha_i-d<0$ and $\alpha_j+\alpha_i-d>0$,

(e) $\alpha_j+\alpha_i-d<0$.

Since $J_{qr}(\alpha_j,\alpha_i,s;d)$ for the parameter range (d) or (e) is equal to $(-1)^{r+q}J_{qr}(\alpha_j,\alpha_i,s;-d)$ for the range (b) or (a), we show only the general expressions for the range (a), (b) and (c):

$$8(i)^{q+r}\frac{P_j}{\alpha_j}J_{qr}(\alpha_j,\alpha_i,s;d) = \sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^{s}\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!\,(r+s-n)!}{n!\,(q-n)!\,(s-n)!\,(r-s+n)!}\times$$

$$\times\begin{cases} [X_{sn}(\alpha_j,\alpha_i,d)+(-1)^r X_{sn}(\alpha_j,-\alpha_i,d)+(-1)^q X_{sn}(-\alpha_j,\alpha_i,d)+(-1)^{r+q}\times \\ \times X_{sn}(-\alpha_j,-\alpha_i,d)-Y_{1sn}(\alpha_j,\alpha_i,d)-(-1)^r Y_{1sn}(\alpha_j,-\alpha_i,d)-(-1)^q\times \\ \times Y_{1sn}(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_{1sn}(-\alpha_j,-\alpha_i,d)], \qquad \text{for (a)}, \qquad \text{(A1)} \\ [X_{sn}(-\alpha_j,-\alpha_i,-d)+(-1)^r X_{sn}(\alpha_j,-\alpha_i,d)+(-1)^q X_{sn}(-\alpha_j,\alpha_i,d)+ \\ +(-1)^{r+q}X_{sn}(-\alpha_j,-\alpha_i,d)+Y_{2sn}(\alpha_j,\alpha_i,d)-(-1)^r Y_{2sn}(\alpha_j,-\alpha_i,d)- \qquad \text{(A2)} \\ -(-1)^q Y_{2sn}(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_{2sn}(-\alpha_j,-\alpha_i,d)], \qquad \text{for (b)}, \end{cases}$$

$$\times\begin{cases}[X_{sn}(-\alpha_j,-\alpha_i,-d)+(-1)^r X_{sn}(-\alpha_j,\alpha_i,-d)+(-1)^q X_{sn}(-\alpha_j,\alpha_i,d)+ \\ +(-1)^{r+q}X_{sn}(-\alpha_j,-\alpha_i,d)+Y_{2sn}(\alpha_j,\alpha_i,d)+(-1)^r Y_{2sn}(\alpha_j,-\alpha_i,d)- \\ -(-1)^q Y_{2sn}(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_{2sn}(-\alpha_j,-\alpha_i,d)], \qquad \text{for (c)},\end{cases} \tag{A3}$$

where

$$X_{sn}(\alpha_j,\alpha_i,d)=\frac{1}{\alpha_j^{n+1}\alpha_i^{s-n+1}}\frac{2}{(s+2)!}[\exp(\alpha_j+\alpha_i+d)\sum_{u=1}^{s+2}(s+2-u)!\,(\alpha_j+\alpha_i+d)^{u-1}+$$
$$+(\alpha_j+\alpha_i+d)^{s+2}E_1(-(\alpha_j+\alpha_i+d))], \tag{A4}$$

$$Y_{1sn}(\alpha_j,\alpha_i,d)=\frac{1}{\alpha_j^{n+1}\alpha_i^{s-n+1}}\sum_{k=0}^{s+1}\frac{(\alpha_j+\alpha_i+d)^{s+1-k}}{(k+1)(s+1-k)!}, \tag{A5}$$

$$Y_{2sn}(\alpha_j,\alpha_i,d)=Y_{1sn}(\alpha_j,\alpha_i,d)-Y_{1sn}(-\alpha_j,-\alpha_i,-d). \tag{A6}$$

The functions which we have to evaluate in the j_7 approximation (which requires J_{qr}'s with q=0,1,...,7 and r=0,1,...,7) are J_{0r} and J_{1r} with r=0,1,...,7, and J_{q0} and J_{q7} with q=2,3,...,7, because, due to the recurrence formula of the spherical Bessel function, we have the following relation:

$$J_{q+1,r}(\alpha_j,\alpha_i,s;d)=\frac{\alpha_i}{\alpha_j}\frac{2q+1}{2r+1}[J_{q,r+1}+J_{q,r-1}]-J_{q-1,r}\,. \tag{A7}$$

The explicit expressions for the X_{sn} term of these functions, $g(i)^{q+r}(P_j/\alpha_j)\times J_{qr}(\alpha_j,\alpha_i,s;d)$'s, are written as follows [see equations (A1)-(A3)]:

$$\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!\,(r+s-n)!}{n!\,(q-n)!\,(s-n)!\,(r-s+n)!}X_{sn}(\alpha_j,\alpha_i,d)$$
$$=\sum_{n=0}^{q}\frac{(q+n)!}{n!\,(q-n)!}\sum_{s=n}^{r+n}\left(-\frac{1}{2}\right)^s\frac{(r+s-n)!}{(s-n)!\,(r-s+n)!}X_{sn}(\alpha_j,\alpha_i,d)$$
$$=\begin{cases}\displaystyle\sum_{s=0}^{r}\left(-\frac{1}{2}\right)^s\frac{(r+s)!}{s!\,(r-s)!}X_{s0}(\alpha_j,\alpha_i,d), & \text{for } q=0, \qquad \text{(A8)}\\ \displaystyle\sum_{s=0}^{q}\left(-\frac{1}{2}\right)^s\frac{(q+s)!}{s!\,(q-s)!}X_{ss}(\alpha_j,\alpha_i,d)=\sum_{s=0}^{q}\left(-\frac{1}{2}\right)^s\frac{(q+s)!}{s!\,(q-s)!}X_{s0}(\alpha_i,\alpha_j,d), & \text{for } r=0, \qquad \text{(A9)}\end{cases}$$

$$
=\begin{cases}
\sum_{s=0}^{r}\left(-\frac{1}{2}\right)^{s}\frac{(r+s)!}{s!(r-s)!}\left[X_{s0}(\alpha_j,\alpha_i,d)-X_{s+1,1}(\alpha_j,\alpha_i,d)\right], \quad \text{for } q=1, \\[2ex]
\sum_{s=0}^{7}\left(-\frac{1}{2}\right)^{s}\frac{(7+s)!}{s!(7-s)!}\sum_{n=0}^{q}(-1)^{n}\frac{(q+n)!}{(2n)!!\,(q-n)!}X_{s+n,n}(\alpha_j,\alpha_i,d),
\end{cases}
\qquad \text{(A10)}
$$

for r=7 (A11)

The expression on the right hand side of equation (A8) leads

$$
\alpha_j\alpha_i\sum_{s=0}^{r}\left(-\frac{1}{2}\right)^{s}\frac{(r+s)!}{s!(r-s)!}X_{s0}(\alpha_j,\alpha_i,d)=
$$

$$
\begin{cases}
X_1(x)e^{x}+x^2E_1(-x), & r=0,\\[1ex]
\left[X_1(x)-\frac{1}{3\alpha_i}X_2(x)\right]e^{x}+\left(1-\frac{1}{3\alpha_i}x\right)x^2E_1(-x), & r=1,\\[1ex]
\left[X_1(x)-\frac{1}{\alpha_i}X_2(x)+\frac{1}{4\alpha_i^2}X_3(x)\right]e^{x}+\left(1-\frac{1}{\alpha_i}x+\frac{1}{4\alpha_i^2}x^2\right)x^2E_1(-x), & r=2,\\[1ex]
\left[X_1(x)-\frac{2}{\alpha_i}X_2(x)+\frac{5}{4\alpha_i^2}X_3(x)-\frac{1}{4\alpha_i^3}X_4(x)\right]e^{x}+ & \\
\quad +\left(1-\frac{2}{\alpha_i}x+\frac{5}{4\alpha_i^2}x^2-\frac{1}{4\alpha_i^3}x^3\right)x^2E_1(-x), & r=3,\\[1ex]
\left[X_1(x)-\frac{10}{3\alpha_i}X_2(x)+\frac{15}{4\alpha_i^2}X_3(x)-\frac{7}{4\alpha_i^3}X_4(x)+\frac{7}{24\alpha_i^4}X_5(x)\right]e^{x}+ & \\
\quad +\left(1-\frac{10}{3\alpha_i}x+\frac{15}{4\alpha_i^2}x^2-\frac{7}{4\alpha_i^3}x^3+\frac{7}{24\alpha_i^4}x^4\right)x^2E_1(-x), & r=4,\\[1ex]
\left[X_1(x)-\frac{5}{\alpha_i}X_2(x)+\frac{35}{4\alpha_i^2}X_3(x)-\frac{7}{\alpha_i^3}X_4(x)+\frac{21}{8\alpha_i^4}X_5(x)-\frac{3}{8\alpha_i^5}X_6(x)\right]e^{x}+ & \\
\quad +\left(1-\frac{5}{\alpha_i}x+\frac{35}{4\alpha_i^2}x^2-\frac{7}{\alpha_i^3}x^3+\frac{21}{8\alpha_i^4}x^4-\frac{3}{8\alpha_i^5}x^5\right)x^2E_1(-x), & r=5,\\[1ex]
\left[X_1(x)-\frac{7}{\alpha_i}X_2(x)+\frac{35}{2\alpha_i^2}X_3(x)-\frac{21}{\alpha_i^3}X_4(x)+\frac{105}{8\alpha_i^4}X_5(x)-\frac{33}{8\alpha_i^5}X_6(x)+\right. & \\
\quad \left.+\frac{33}{64\alpha_i^6}X_7(x)\right]e^{x}+\left(1-\frac{7}{\alpha_i}x+\frac{35}{2\alpha_i^2}x^2-\frac{21}{\alpha_i^3}x^3+\frac{105}{8\alpha_i^4}x^4-\frac{33}{8\alpha_i^5}x^5+\right. & \\
\quad \left.+\frac{33}{64\alpha_i^6}x^6\right)x^2E_1(-x), & r=6,\\[1ex]
\left[X_1(x)-\frac{28}{3\alpha_i}X_2(x)+\frac{63}{2\alpha_i^2}X_3(x)-\frac{105}{2\alpha_i^3}X_4(x)+\frac{385}{8\alpha_i^4}X_5(x)-\frac{99}{4\alpha_i^5}X_6(x)+\right. & \\
\quad \left.+\frac{429}{64\alpha_i^6}X_7(x)-\frac{143}{192\alpha_i^7}X_8(x)\right]e^{x}+\left(1-\frac{28}{3\alpha_i}x+\frac{63}{2\alpha_i^2}x^2-\frac{105}{2\alpha_i^3}x^3+\right. & \\
\quad \left.+\frac{385}{8\alpha_i^4}x^4-\frac{99}{4\alpha_i^5}x^5+\frac{429}{64\alpha_i^6}x^6-\frac{143}{192\alpha_i^7}x^7\right)x^2E_1(-x), & r=7,
\end{cases}
\qquad \text{(A12)}
$$

where $X_m(x)=x X_{m-1}(x)+m!$, $X_1(x)=x+1$ [or $X_m(x)=\sum_{u=1}^{m+1}(m+1-u)!\,x^{u-1}$] and $x=\alpha_j+\alpha_i+d$. The $X_{s+1,1}$ term on the right hand side of equation (A10), $-\alpha_j\alpha_i\sum_{s=0}^{r}(-\frac{1}{2})^s\frac{(r+s)!}{s!(r-s)!}\times$ $\times X_{s+1,1}(\alpha_j,\alpha_i,d)$, r=0-7, can easily be written down by replacing $X_m(x)$ in equation (A12) and x^{m-1} in the coefficient of $x^2E_1(-x)$ by $-X_{m+1}(x)/[(m+2)\alpha_j]$ and $-x^m/[(m+2)\alpha_j]$, respectively. In the same manner, $(-1)^n\alpha_j\alpha_i\sum_{s=0}^{7}(-\frac{1}{2})^s\frac{(7+s)!}{s!(7-s)!}\times$ $\times X_{s+n,n}(\alpha_j,\alpha_i,d)$ of equation (A11) for n=2-7 can be obtained from the last equation of (A12) by replacing repeatedly $X_m(x)$ and x^{m-1} in the expression for $X_{s+n,n}$ by $-X_{m+1}(x)/[(m+2)\alpha_j]$ and $-x^m/[(m+2)\alpha_j]$, respectively, to get that for $X_{s+n+1,n+1}$ [compare (A11) with (A28)].

The extra terms consisting of Y_{1sn} or Y_{2sn} on the right hand side of equations (A1)-(A3) give the following expression:

$$\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!\,(r+s-n)!}{n!\,(q-n)!\,(s-n)!\,(r-s+n)!}\left[Y_{1sn}(\alpha_j,\alpha_i,d)+(-1)^r Y_{1sn}(\alpha_j,-\alpha_i,d)+\right.$$
$$\left.+(-1)^q Y_{1sn}(-\alpha_j,\alpha_i,d)+(-1)^{r+q}Y_{1sn}(-\alpha_j,-\alpha_i,d)\right]=0,\ \text{for all cases.} \tag{A13}$$

$$\alpha_j\alpha_i\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!\,(r+s-n)!}{n!\,(q-n)!\,(s-n)!\,(r-s+n)!}\left[Y_{2sn}(\alpha_j,\alpha_i,d)-(-1)^r Y_{2sn}(\alpha_j,-\alpha_i,d)-\right.$$
$$\left.-(-1)^q Y_{2sn}(-\alpha_j,\alpha_i,d)-(-1)^{r+q}Y_{2sn}(-\alpha_j,-\alpha_i,d)\right]=$$

$$=\begin{cases}
4(\alpha_j+\alpha_i+d), & q=0,\ r=0,\\
-\frac{4}{3\alpha_i}-\frac{2}{\alpha_i}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d), & q=0,\ r=1,\\
-\frac{4}{\alpha_i}+\frac{2}{\alpha_i^2}(\alpha_j+\alpha_i+d)[(\alpha_j-\alpha_i+d)(\alpha_j+d)+2], & q=0,\ r=2,\\
-\frac{8}{\alpha_i}-\frac{12}{\alpha_i^3}-\frac{1}{2\alpha_i^3}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[5(\alpha_j+d)^2-\alpha_i^2+20], & q=0,\ r=3,\\
-\frac{40}{3\alpha_i}-\frac{84}{\alpha_i^3}+\frac{1}{\alpha_i^4}(\alpha_j+\alpha_i+d)\{(\alpha_j-\alpha_i+d)(\alpha_j+d)[\frac{7}{2}(\alpha_j+d)^2-\frac{3}{2}\alpha_i^2+\frac{70}{3}]+ & \\
\qquad +\frac{40}{3}\alpha_i^2+84\}, & q=0,\ r=4,\\
-\frac{20}{\alpha_i}-\frac{336}{\alpha_i^3}-\frac{540}{\alpha_i^5}-\frac{1}{\alpha_i^5}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[\frac{21}{4}(\alpha_j+d)^4-\frac{7}{2}(\alpha_j+d)^2\alpha_i^2+ & \\
\qquad +\frac{1}{4}\alpha_i^4+\frac{105}{2}(\alpha_j+d)^2+\frac{35}{2}\alpha_i^2+378], & q=0,\ r=5,\\
-\frac{28}{\alpha_i}-\frac{1008}{\alpha_i^3}-\frac{5940}{\alpha_i^5}+\frac{1}{\alpha_i^6}(\alpha_j+\alpha_i+d)\{(\alpha_j-\alpha_i+d)(\alpha_j+d)[\frac{33}{4}(\alpha_j+d)^4-\frac{15}{2}(\alpha_j+d)^2\alpha_i^2+ & \\
\qquad +\frac{5}{4}\alpha_i^4+\frac{231}{2}(\alpha_j+d)^2+\frac{21}{2}\alpha_i^2+1386]+28\alpha_i^4+1008\alpha_i^2+5940\}, & q=0,\ r=6,
\end{cases}$$

$$\left\{ \begin{array}{l} -\frac{112}{3\alpha_i}-\frac{2520}{\alpha_i^3}-\frac{35640}{\alpha_i^5}-\frac{60060}{\alpha_i^7}-\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\left[\frac{429}{32}(\alpha_j+d)^6-\right. \\ \quad -\frac{495}{32}(\alpha_j+d)^4\alpha_i^2+\frac{135}{32}(\alpha_j+d)^2\alpha_i^4-\frac{5}{32}\alpha_i^6+\frac{1001}{4}(\alpha_j+d)^4-\frac{77}{2}(\alpha_j+d)^2\alpha_i^2+ \\ \quad \left.+\frac{161}{4}\alpha_i^4+\frac{9009}{2}(\alpha_j+d)^2+\frac{4851}{2}\alpha_i^2+38610\right], \qquad q=0,\ r=7. \end{array} \right. \tag{A14}$$

For q=1-7 and r=0, the expressions are the same as those for r=1-7 and q=0 shown in equation (A14) except for interchanging α_j with α_i (and vice versa). The expressions for q=1 and r=1-7 are obtained by taking respectively the sum of those for q=0 and r=1-7 and the following formulae for q=1 and r=1-7:

$$\begin{array}{ll} -\frac{4}{3\alpha_j}+\frac{1}{3\alpha_j\alpha_i}(\alpha_j+\alpha_i+d)[2(\alpha_j-\alpha_i+d)(\alpha_j+d)+4-4\alpha_i^2], & q=1,\ r=1, \\ -\frac{4}{3\alpha_j}-\frac{12}{5\alpha_j\alpha_i^2}-\frac{1}{2\alpha_j\alpha_i^2}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)[(\alpha_j+d)^2-\alpha_i^2+4], & q=1,\ r=2, \\ -\frac{4}{3\alpha_j}-\frac{12}{\alpha_j\alpha_i^2}+\frac{1}{\alpha_j\alpha_i^3}(\alpha_j+\alpha_i+d)\left\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\left[\frac{1}{2}(\alpha_j+d)^2-\frac{1}{2}\alpha_i^2+\frac{10}{3}\right]+\right. & \\ \qquad\qquad \left.+\frac{4}{3}\alpha_i^2+12\right\}, & q=1,\ r=3, \\ -\frac{4}{3\alpha_j}-\frac{36}{\alpha_j\alpha_i^2}-\frac{60}{\alpha_j\alpha_i^4}-\frac{1}{\alpha_j\alpha_i^4}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\left[\frac{7}{12}(\alpha_j+d)^4-\frac{2}{3}(\alpha_j+d)^2\alpha_i^2+\right. & \\ \quad \left.+\frac{1}{12}\alpha_i^4+\frac{35}{6}(\alpha_j+d)^2+\frac{5}{6}\alpha_i^2+42\right], & q=1,\ r=4, \\ -\frac{4}{3\alpha_j}-\frac{84}{\alpha_j\alpha_i^2}-\frac{540}{\alpha_j\alpha_i^4}+\frac{1}{\alpha_j\alpha_i^5}(\alpha_j+\alpha_i+d)\left\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\left[\frac{3}{4}(\alpha_j+d)^4-(\alpha_j+d)^2\alpha_i^2+\right.\right. & \\ \quad \left.\left.+\frac{1}{4}\alpha_i^4+\frac{21}{2}(\alpha_j+d)^2-\frac{7}{6}\alpha_i^2+126\right]+\frac{4}{3}\alpha_i^4+84\alpha_i^2+540\right\}, & q=1,\ r=5, \\ -\frac{4}{3\alpha_j}-\frac{168}{\alpha_j\alpha_i^2}-\frac{2700}{\alpha_j\alpha_i^4}-\frac{4620}{\alpha_j\alpha_i^6}-\frac{1}{\alpha_j\alpha_i^6}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\left[\frac{33}{32}(\alpha_j+d)^6-\frac{51}{32}(\alpha_j+d)^4\alpha_i^2+\right. & \\ \quad +\frac{19}{32}(\alpha_j+d)^2\alpha_i^4-\frac{1}{32}\alpha_i^6+\frac{77}{4}(\alpha_j+d)^4-7(\alpha_j+d)^2\alpha_i^2+\frac{7}{4}\alpha_i^4+\frac{693}{2}(\alpha_j+d)^2+ & \\ \quad \left.+\frac{315}{2}\alpha_i^2+2970\right], & q=1,\ r=6, \\ -\frac{4}{3\alpha_j}-\frac{1512}{5\alpha_j\alpha_i^2}-\frac{9900}{\alpha_j\alpha_i^4}-\frac{60060}{\alpha_j\alpha_i^6}+\frac{1}{\alpha_j\alpha_i^7}(\alpha_j+\alpha_i+d)\left\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\left[\frac{143}{96}(\alpha_j+d)^6-\right.\right. & \\ \quad -\frac{253}{96}(\alpha_j+d)^4\alpha_i^2+\frac{125}{96}(\alpha_j+d)^2\alpha_i^4-\frac{5}{32}\alpha_i^6+\frac{143}{4}(\alpha_j+d)^4-22(\alpha_j+d)^2\alpha_i^2+\frac{17}{4}\alpha_i^4+ & \\ \quad \left.\left.+\frac{9009}{10}(\alpha_j+d)^2+\frac{2079}{10}\alpha_i^2+12870\right]+\frac{4}{3}\alpha_i^6+\frac{1512}{5}\alpha_i^4+9900\alpha_i^2+60060\right\}, & q=1,\ r=7. \end{array} \tag{A15}$$

Similarly to equation (A11), the expressions for q=2-7 and r=7 can be written as

$$\text{Expression for } (q,7)=\sum_{n=0}^{q}\frac{(q+n)!}{(2n)!\,(q-n)!}\,(n,7)_1\,, \tag{A16}$$

in which $(0,7)_1$ is the expression for (0,7) given by the last equation of (A14), $(1,7)_1$ stands for the last formula of (A15) and $(n,7)_1$ for n=2-7 are

$$\alpha_j^2(2,7)_1=-\frac{336}{5\alpha_i}-\frac{5400}{\alpha_i^3}-\frac{83160}{\alpha_i^5}-\frac{147420}{\alpha_i^7}-\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\Big[\frac{143}{320}(\alpha_j+d)^8-$$
$$-\frac{11}{10}(\alpha_j+d)^6\alpha_i^2+\frac{139}{160}(\alpha_j+d)^4\alpha_i^4-\frac{9}{40}(\alpha_j+d)^2\alpha_i^6+\frac{3}{320}\alpha_i^8+\frac{429}{32}(\alpha_j+d)^6-\frac{495}{32}(\alpha_j+d)^4\alpha_i^2+$$
$$+\frac{135}{32}(\alpha_j+d)^2\alpha_i^4-\frac{5}{32}\alpha_i^6+\frac{9009}{20}(\alpha_j+d)^4-\frac{693}{10}(\alpha_j+d)^2\alpha_i^2+\frac{1449}{20}\alpha_i^4+\frac{19305}{2}(\alpha_j+d)^2+$$
$$+\frac{10395}{2}\alpha_i^2+90090\Big],$$

$$\alpha_j^3(3,7)_1=-12-\frac{3240}{\alpha_i^2}-\frac{115500}{\alpha_i^4}-\frac{737100}{\alpha_i^6}+\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)\Big\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\times$$
$$\times\Big[\frac{13}{64}(\alpha_j+d)^8-\frac{21}{32}(\alpha_j+d)^6\alpha_i^2+\frac{3}{4}(\alpha_j+d)^4\alpha_i^4-\frac{11}{32}(\alpha_j+d)^2\alpha_i^6+\frac{3}{64}\alpha_i^8+\frac{715}{96}(\alpha_j+d)^6-$$
$$-\frac{1265}{96}(\alpha_j+d)^4\alpha_i^2+\frac{625}{96}(\alpha_j+d)^2\alpha_i^4-\frac{25}{32}\alpha_i^6+\frac{1287}{4}(\alpha_j+d)^4-198(\alpha_j+d)^2\alpha_i^2+\frac{153}{4}\alpha_i^4+$$
$$+\frac{19305}{2}(\alpha_j+d)^2+\frac{4455}{2}\alpha_i^2+150150\Big]+12\alpha_i^6+3240\alpha_i^4+115500\alpha_i^2+737100\Big\},$$

$$\alpha_j^4(4,7)_1=-\frac{1680}{\alpha_i}-\frac{147000}{\alpha_i^3}-\frac{2381400}{\alpha_i^5}-\frac{4365900}{\alpha_i^7}-\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\times$$
$$\times\Big[\frac{91}{768}(\alpha_j+d)^{10}-\frac{371}{768}(\alpha_j+d)^8\alpha_i^2+\frac{287}{384}(\alpha_j+d)^6\alpha_i^4-\frac{203}{384}(\alpha_j+d)^4\alpha_i^6+\frac{119}{768}(\alpha_j+d)^2\alpha_i^8-$$
$$-\frac{7}{768}\alpha_i^{10}+\frac{1001}{192}(\alpha_j+d)^8-\frac{77}{6}(\alpha_j+d)^6\alpha_i^2+\frac{973}{96}(\alpha_j+d)^4\alpha_i^4-\frac{21}{8}(\alpha_j+d)^2\alpha_i^6+\frac{7}{64}\alpha_i^8+$$
$$+\frac{9009}{32}(\alpha_j+d)^6-\frac{10395}{32}(\alpha_j+d)^4\alpha_i^2+\frac{2835}{32}(\alpha_j+d)^2\alpha_i^4-\frac{105}{32}\alpha_i^6+\frac{45045}{4}(\alpha_j+d)^4-$$
$$-\frac{3465}{2}(\alpha_j+d)^2\alpha_i^2+\frac{7245}{4}\alpha_i^4+\frac{525525}{2}(\alpha_j+d)^2+\frac{282975}{2}\alpha_i^2-2579850\Big],$$

$$\alpha_j^5(5,7)_1=-540-\frac{158760}{\alpha_i^2}-\frac{5953500}{\alpha_i^4}-\frac{39293100}{\alpha_i^6}+\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)\Big\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\times$$

$$\times\Big[\frac{21}{256}((\alpha_j+d)^2-\alpha_i^2)^5+\frac{273}{64}(\alpha_j+d)^8-\frac{441}{32}(\alpha_j+d)^6\alpha_i^2+\frac{63}{4}(\alpha_j+d)^4\alpha_i^4-\frac{231}{32}(\alpha_j+d)^2\alpha_i^6+$$

$$+\frac{63}{64}\alpha_i^8+\frac{9009}{32}(\alpha_j+d)^6-\frac{15939}{32}(\alpha_j+d)^4\alpha_i^2+\frac{7875}{32}(\alpha_j+d)^2\alpha_i^4-\frac{945}{32}\alpha_i^6+\frac{57915}{4}(\alpha_j+d)^4-$$

$$-8910(\alpha_j+d)^2\alpha_i^2+\frac{6885}{4}\alpha_i^4+\frac{945945}{2}(\alpha_j+d)^2+\frac{218295}{2}\alpha_i^2+7739550\Big]+540\alpha_i^6+$$

$$+158760\alpha_i^4+5953500\alpha_i^2+39293100\Big\},$$

$$\alpha_j^6(6,7)_1=-\frac{129360}{\alpha_i}-\frac{11907000}{\alpha_i^3}-\frac{2593344600}{13\alpha_i^5}-\frac{374594220}{\alpha_i^7}-\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)\times$$

$$\times\Big[\frac{33}{512}((\alpha_j+d)^2-\alpha_i^2)^6+\frac{1001}{256}(\alpha_j+d)^{10}-\frac{4081}{256}(\alpha_j+d)^8\alpha_i^2+\frac{3157}{128}(\alpha_j+d)^6\alpha_i^4-$$

$$-\frac{2233}{128}(\alpha_j+d)^4\alpha_i^6+\frac{1309}{256}(\alpha_j+d)^2\alpha_i^8-\frac{77}{256}\alpha_i^{10}+\frac{99099}{320}(\alpha_j+d)^8-\frac{7623}{10}(\alpha_j+d)^6\alpha_i^2+$$

$$+\frac{96327}{160}(\alpha_j+d)^4\alpha_i^4-\frac{6237}{40}(\alpha_j+d)^2\alpha_i^6+\frac{2079}{320}\alpha_i^8+\frac{637065}{32}(\alpha_j+d)^6-\frac{735075}{32}(\alpha_j+d)^4\alpha_i^2+$$

$$+\frac{200475}{32}(\alpha_j+d)^2\alpha_i^4+\frac{7425}{32}\alpha_i^6+\frac{3468465}{4}(\alpha_j+d)^4-\frac{266805}{2}(\alpha_j+d)^2\alpha_i^2+\frac{557865}{4}\alpha_i^4+$$

$$+\frac{42567525}{2}(\alpha_j+d)^2+\frac{22920975}{2}\alpha_i^2+216112050\Big],$$

$$\alpha_j^7(7,7)_1=-60060-\frac{18574920}{\alpha_i^2}-\frac{720373500}{\alpha_i^4}-\frac{4869724860}{\alpha_i^6}+\frac{1}{\alpha_i^7}(\alpha_j+\alpha_i+d)\times$$

$$\times\Big\{(\alpha_j-\alpha_i+d)(\alpha_j+d)\Big[\frac{143}{2560}(\alpha_j+d)^{12}-\frac{253}{640}(\alpha_j+d)^{10}\alpha_i^2+\frac{3083}{2560}(\alpha_j+d)^8\alpha_i^4-$$

$$-\frac{493}{240}(\alpha_j+d)^6\alpha_i^6+\frac{16399}{7680}(\alpha_j+d)^4\alpha_i^8-\frac{2657}{1920}(\alpha_j+d)^2\alpha_i^{10}+\frac{4387}{7680}\alpha_i^{12}+\frac{1001}{256}((\alpha_j+d)^2-\alpha_i^2)^5+$$

$$+\frac{117117}{320}(\alpha_j+d)^8-\frac{189189}{160}(\alpha_j+d)^6\alpha_i^2+\frac{27027}{20}(\alpha_j+d)^4\alpha_i^4-\frac{99099}{160}(\alpha_j+d)^2\alpha_i^6+\frac{27027}{320}\alpha_i^8+$$

$$+\frac{920205}{32}(\alpha_j+d)^6-\frac{1628055}{32}(\alpha_j+d)^4\alpha_i^2+\frac{804375}{32}(\alpha_j+d)^2\alpha_i^4-\frac{96525}{32}\alpha_i^6+\frac{6441435}{4}(\alpha_j+d)^4-$$

$$-990990(\alpha_j+d)^2\alpha_i^2+\frac{765765}{4}\alpha_i^4+\frac{110675565}{2}(\alpha_j+d)^2+\frac{25540515}{2}\alpha_i^2+936485550\Big]+$$

$$+60060\alpha_i^6+18574920\alpha_i^4+720373500\alpha_i^2+4869724860-\frac{4}{15}\alpha_i^{14}\Big\}.$$

(A17)

The extra terms on the right hand side of equation (A3), $\alpha_j\alpha_i\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\sum_{n=s-r\geq 0}^{s\leq q}$ $\frac{(q+n)!\,(r+s-n)!}{n!\,(q-n)!\,(s-n)!\,(r-s+n)!}\left[Y_{2sn}(\alpha_j,\alpha_i,d)+(-1)^r Y_{2sn}(\alpha_j,-\alpha_i,d)-(-1)^q Y_{2sn}(-\alpha_j,\alpha_i,d)-\right.$ $\left.-(-1)^{r+q}Y_{2sn}(-\alpha_j,-\alpha_i,d)\right]$, are always equal to zero except for the cases where they give the following expressions:

$$8\alpha_i, \qquad q=r=0,$$

$$-8d\alpha_i/\alpha_j, \qquad q=1,\ r=0,$$

$$-4\alpha_i(\alpha_j^2-\alpha_i^2-3d^2-2)/\alpha_j^2, \qquad q=2,\ r=0,$$

$$4\alpha_i d(3\alpha_j^2-5\alpha_i^2-5d^2-10)/\alpha_j^3, \qquad q=3,\ r=0,$$

$$\alpha_i(3\alpha_j^4-10\alpha_j^2\alpha_i^2+7\alpha_i^4+35d^4-30\alpha_j^2d^2+70\alpha_i^2d^2-20\alpha_j^2+\tfrac{140}{3}\alpha_i^2+140d^2+168)/\alpha_j^4,$$
$$q=4,\ r=0,$$

$$-\alpha_i d(15\alpha_j^4-70\alpha_j^2\alpha_i^2+210\alpha_i^2d^2-70\alpha_j^2d^2+63d^4+63\alpha_i^4-140\alpha_j^2+420d^2+$$
$$+420\alpha_i^2+1512)/\alpha_j^5, \qquad q=5,\ r=0,$$

$$-\alpha_i\left[\tfrac{5}{2}\alpha_j^6-\tfrac{35}{2}\alpha_j^4\alpha_i^2+\tfrac{63}{2}\alpha_j^2\alpha_i^4-\tfrac{33}{2}\alpha_i^6-\tfrac{231}{2}d^6+\left(\tfrac{315}{2}\alpha_j^2-\tfrac{1155}{2}\alpha_i^2\right)d^4-\right.$$
$$-\left(\tfrac{105}{2}\alpha_j^4-315\alpha_j^2\alpha_i^2+\tfrac{693}{2}\alpha_i^4\right)d^2-35\alpha_j^4+210\alpha_j^2\alpha_i^2-231\alpha_i^4-1155d^4+$$
$$\left.+(630\alpha_j^2-2310\alpha_i^2)d^2+756\alpha_j^2-2772\alpha_i^2-8316d^2-11880\right]/\alpha_j^6, \qquad q=6,\ r=0,$$

$$\alpha_i d\left[\tfrac{35}{2}\alpha_j^6-\tfrac{315}{2}\alpha_j^4\alpha_i^2+\tfrac{693}{2}\alpha_j^2\alpha_i^4-\tfrac{429}{2}\alpha_i^6-\tfrac{429}{2}d^6+\left(\tfrac{693}{2}\alpha_j^2-\tfrac{3003}{2}\alpha_i^2\right)d^4-\right.$$
$$-\left(\tfrac{315}{2}\alpha_j^4-1155\alpha_j^2\alpha_i^2+\tfrac{3003}{2}\alpha_i^4\right)d^2-315\alpha_j^4+2310\alpha_j^2\alpha_i^2-3003\alpha_i^4-3003d^4+$$
$$\left.+(2310\alpha_j^2-10010\alpha_i^2)d^2+8316\alpha_j^2-36036\alpha_i^2-36036d^2-154440\right]/\alpha_j^7, \quad q=7,\ r=0,$$

$$-\tfrac{8}{3}\alpha_i^2/\alpha_j, \qquad q=r=1,$$

$$-\tfrac{8}{15}\alpha_i^8/\alpha_j^7, \qquad q=r=7. \tag{A18}$$

In the case where $\alpha_j+\alpha_i+|d|$ is small, we can obtain the following series expansion for the parameter range (a) [see equation (A1)]:

$$
\begin{aligned}
8(i)^{q+r}\frac{P_j}{\alpha_j}J_{qr}(\alpha_j,\alpha_i,s;d) = &\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\frac{2}{(s+2)!}\left(\sum_{u=1}^{s+2}\frac{1}{u}-\gamma\right)[Y_{3s}(\alpha_j,\alpha_i,d)+\\
&+(-1)^r Y_{3s}(\alpha_j,-\alpha_i,d)+(-1)^q Y_{3s}(-\alpha_j,\alpha_i,d)+(-1)^{r+q}Y_{3s}(-\alpha_j,-\alpha_i,d)]-\\
&-Y_4(\alpha_j,\alpha_i,d)\ln(-(\alpha_j+\alpha_i+d))-(-1)^r Y_4(\alpha_j,-\alpha_i,d)\ln(-(\alpha_j-\alpha_i+d))-\\
&-(-1)^q Y_4(-\alpha_j,\alpha_i,d)\ln(-(-\alpha_j+\alpha_i+d))-(-1)^{r+q}Y_4(-\alpha_j,-\alpha_i,d)\ln(-(-\alpha_j-\alpha_i+d))-\\
&-\sum_{m=2}^{\infty}\frac{1}{m-1}Y_{5m}(\alpha_j,\alpha_i,d)(\alpha_j+\alpha_i+d)^{m-1}-(-1)^r\sum_{m=2}^{\infty}\frac{1}{m-1}Y_{5m}(\alpha_j,-\alpha_i,d)(\alpha_j-\alpha_i+d)^{m-1}-\\
&-(-1)^q\sum_{m=2}^{\infty}\frac{1}{m-1}Y_{5m}(-\alpha_j,\alpha_i,d)(-\alpha_j+\alpha_i+d)^{m-1}-\\
&\qquad-(-1)^{r+q}\sum_{m=2}^{\infty}\frac{1}{m-1}Y_{5m}(-\alpha_j,-\alpha_i,d)(-\alpha_j-\alpha_i+d)^{m-1},
\end{aligned}
\tag{A19}
$$

where

$$
Y_{3s}(\alpha_j,\alpha_i,d)=(\alpha_j+\alpha_i+d)^{s+2}\sum_{n=s-r\geq 0}^{s\leq q}\frac{(q+n)!(r+s-n)!}{n!(q-n)!(s-n)!(r-s+n)!}\,\frac{1}{\alpha_j^{n+1}\alpha_i^{s-n+1}},
\tag{A20}
$$

$$
Y_4(\alpha_j,\alpha_i,d)=\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\frac{2}{(s+2)!}\,Y_{3s}(\alpha_j,\alpha_i,d),
\tag{A21}
$$

$$
Y_{5m}(\alpha_j,\alpha_i,d)=\sum_{s=0}^{q+r}\left(-\frac{1}{2}\right)^s\frac{2}{(s+m+1)!}\,Y_{3s}(\alpha_j,\alpha_i,d),
\tag{A22}
$$

and γ is the Euler-Mascheroni constant. For the parameter range (b), $(\alpha_j+\alpha_i+d)$'s, the argument of a logarithmic function multiplied by $Y_4(\alpha_j,\alpha_i,d)$ and the variable of a series expansion multiplied by $Y_{5m}(\alpha_j,\alpha_i,d)$, on the right hand side of equation (A19) are replaced by $-(\alpha_j+\alpha_i+d)$'s . For the range (c), in addition to these replacements, two $(\alpha_j-\alpha_i+d)$'s , the argument of a logarithmic function and the variable of a series expansion, are replaced by $-(\alpha_j-\alpha_i+d)$'s .

The summation of four Y_{3s} terms and four Y_4 terms with logarithmic functions on the right hand side of equation (A19) gives the following expressions:

$$12-8\gamma-\frac{1}{\alpha_j\alpha_i}(\alpha_j+\alpha_i+d)^2\ln(-(\alpha_j+\alpha_i+d))+\frac{1}{\alpha_j\alpha_i}(\alpha_j-\alpha_i+d)^2\ln(-(\alpha_j-\alpha_i+d))+$$

$$+\frac{1}{\alpha_j\alpha_i}(-\alpha_j+\alpha_i+d)^2\ln(-(-\alpha_j+\alpha_i+d))-\frac{1}{\alpha_j\alpha_i}(-\alpha_j-\alpha_i+d)^2\ln(-(-\alpha_j-\alpha_i+d)),\quad q=r=0,$$

$$-\frac{8}{3}\frac{d}{\alpha_i}-\frac{2}{3}\frac{\alpha_i}{\alpha_j}\ln\left[\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{(-\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]-\left(1-\frac{1}{3}\frac{\alpha_j^2}{\alpha_i^2}-\frac{d^2}{\alpha_i^2}\right)\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]-$$

$$-\frac{d}{\alpha_j}\left(1-\frac{\alpha_j^2}{\alpha_i^2}-\frac{1}{3}\frac{d^2}{\alpha_i^2}\right)\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\qquad q=0,\ r=1,$$

$$-\frac{5}{3}+\frac{\alpha_j^2}{\alpha_i^2}+3\frac{d^2}{\alpha_i^2}+\frac{\alpha_i}{\alpha_j}\left[-\frac{1}{4}+\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}-\frac{1}{4}\frac{\alpha_j^4}{\alpha_i^4}+\left(\frac{1}{2}-\frac{3}{2}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}-\frac{1}{4}\frac{d^4}{\alpha_i^4}\right]\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right]+\frac{d}{\alpha_i}\left(1-\frac{\alpha_j^2}{\alpha_i^2}-\frac{d^2}{\alpha_i^2}\right)\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right],$$

$$q=0,\ r=2,$$

$$\frac{d}{\alpha_i}\left(\frac{10}{3}-4\frac{\alpha_j^2}{\alpha_i^2}-4\frac{d^2}{\alpha_i^2}\right)+\left[\frac{1}{4}-\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1}{4}\frac{\alpha_j^4}{\alpha_i^4}-\left(\frac{3}{2}-\frac{5}{2}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}+\frac{5}{4}\frac{d^4}{\alpha_i^4}\right]\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{d}{\alpha_j}\left[d\longleftrightarrow\alpha_j\right]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],$$

$$q=0,\ r=3,$$

$$\frac{9}{10}-\frac{19}{9}\frac{\alpha_j^2}{\alpha_i^2}+\frac{7}{6}\frac{\alpha_j^4}{\alpha_i^4}-\left(\frac{19}{3}-\frac{35}{3}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}+\frac{35}{6}\frac{d^4}{\alpha_i^4}+\frac{d}{\alpha_i}\left[-\frac{3}{4}+\frac{5}{2}\frac{\alpha_j^2}{\alpha_i^2}-\frac{7}{4}\frac{\alpha_j^4}{\alpha_i^4}+\right.$$

$$\left.+\left(\frac{5}{2}-\frac{35}{6}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}-\frac{7}{4}\frac{d^4}{\alpha_i^4}\right]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_i}{\alpha_j}\left[\frac{1}{24}-\frac{3}{8}\frac{\alpha_j^2}{\alpha_i^2}+\frac{5}{8}\frac{\alpha_j^4}{\alpha_i^4}-\right.$$

$$\left.-\frac{7}{24}\frac{\alpha_j^6}{\alpha_i^6}-\left(\frac{3}{8}-\frac{15}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{35}{8}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}+\left(\frac{5}{8}-\frac{35}{8}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}-\frac{7}{24}\frac{d^6}{\alpha_i^6}\right]\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\qquad q=0,\ r=4,$$

$$\frac{d}{\alpha_i}\left[-\frac{49}{15}+12\frac{\alpha_j^2}{\alpha_i^2}-9\frac{\alpha_j^4}{\alpha_i^4}+\left(12-30\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}-9\frac{d^4}{\alpha_i^4}\right]+\left[-\frac{1}{8}+\frac{5}{8}\frac{\alpha_j^2}{\alpha_i^2}-\frac{7}{8}\frac{\alpha_j^4}{\alpha_i^4}+\frac{3}{8}\frac{\alpha_j^6}{\alpha_i^6}+\right.$$

$$\left.+\left(\frac{15}{8}-\frac{35}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{63}{8}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}-\left(\frac{35}{8}-\frac{105}{8}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}+\frac{21}{8}\frac{d^6}{\alpha_i^6}\right]\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{d}{\alpha_j}\left[d\longleftrightarrow\alpha_j\right]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\ q=0,\ r=5,$$

$$-\frac{919}{1680}+\frac{243}{80}\frac{\alpha_j^2}{\alpha_i^2}-\frac{73}{16}\frac{\alpha_j^4}{\alpha_i^4}+\frac{33}{16}\frac{\alpha_j^6}{\alpha_i^6}+\left(\frac{729}{80}-\frac{365}{8}\frac{\alpha_j^2}{\alpha_i^2}+\frac{693}{16}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}-\left(\frac{365}{16}-\frac{1155}{16}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}+$$

$$+\frac{231}{16}\frac{d^6}{\alpha_i^6}+\frac{d}{\alpha_i}\left[\frac{5}{8}-\frac{35}{8}\frac{\alpha_j^2}{\alpha_i^2}+\frac{63}{8}\frac{\alpha_j^4}{\alpha_i^4}-\frac{33}{8}\frac{\alpha_j^6}{\alpha_i^6}-\left(\frac{35}{8}-\frac{105}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{231}{8}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}+\right.$$

$$+\left(\frac{63}{8}-\frac{231}{8}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}-\frac{33}{8}\frac{d^6}{\alpha_i^6}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_i}{\alpha_j}\Big[-\frac{1}{64}+\frac{5}{16}\frac{\alpha_j^2}{\alpha_i^2}-$$

$$-\frac{35}{32}\frac{\alpha_j^4}{\alpha_i^4}+\frac{21}{16}\frac{\alpha_j^6}{\alpha_i^6}-\frac{33}{64}\frac{\alpha_j^8}{\alpha_i^8}+\left(\frac{5}{16}-\frac{105}{16}\frac{\alpha_j^2}{\alpha_i^2}+\frac{315}{16}\frac{\alpha_j^4}{\alpha_i^4}-\frac{231}{16}\frac{\alpha_j^6}{\alpha_i^6}\right)\frac{d^2}{\alpha_i^2}-\Big(\frac{35}{32}-\frac{315}{16}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{1155}{32}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^4}{\alpha_i^4}+\left(\frac{21}{16}-\frac{231}{16}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^6}{\alpha_i^6}-\frac{33}{64}\frac{d^8}{\alpha_i^8}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\quad g=0,\ \gamma=6,$$

$$\frac{d}{\alpha_i}\Big[\frac{849}{280}-\frac{1373}{60}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1045}{24}\frac{\alpha_j^4}{\alpha_i^4}-\frac{143}{6}\frac{\alpha_j^6}{\alpha_i^6}-\left(\frac{1373}{60}-\frac{5225}{36}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1001}{6}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}+\Big(\frac{1045}{24}-$$

$$-\frac{1001}{6}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^4}{\alpha_i^4}-\frac{143}{6}\frac{d^6}{\alpha_i^6}\Big]+\Big[\frac{5}{64}-\frac{35}{48}\frac{\alpha_j^2}{\alpha_i^2}+\frac{63}{32}\frac{\alpha_j^4}{\alpha_i^4}-\frac{33}{16}\frac{\alpha_j^6}{\alpha_i^6}+\frac{143}{192}\frac{\alpha_j^8}{\alpha_i^8}-\Big(\frac{35}{16}-\frac{315}{16}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{693}{16}\frac{\alpha_j^4}{\alpha_i^4}-\frac{429}{16}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^2}{\alpha_i^2}+\left(\frac{315}{32}-\frac{1155}{16}\frac{\alpha_j^2}{\alpha_i^2}+\frac{3003}{32}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^4}{\alpha_i^4}-\left(\frac{231}{16}-\frac{1001}{16}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^6}{\alpha_i^6}+$$

$$+\frac{429}{64}\frac{d^8}{\alpha_i^8}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{d}{\alpha_j}\left[d\leftrightarrow\alpha_j\right]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],$$

$$g=0,\ \gamma=7,$$

$$\frac{\alpha_i}{\alpha_j}\left(-1-\frac{\alpha_j^2}{\alpha_i^2}-\frac{1}{3}\frac{d^2}{\alpha_i^2}\right)+\frac{2}{3}\frac{\alpha_i d}{\alpha_j^2}\ln\left[\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{(-\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{2}{3}\frac{\alpha_j d}{\alpha_i^2}\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+$$

$$+\frac{\alpha_i^2}{\alpha_j^2}\Big[\frac{1}{4}-\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1}{4}\frac{\alpha_j^4}{\alpha_i^4}+\left(1+\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{2\alpha_i^2}-\frac{1}{12}\frac{d^4}{\alpha_i^4}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\quad g=\gamma=1,$$

$$\frac{d}{\alpha_j}\left(-\frac{7}{15}+\frac{11}{5}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1}{5}\frac{d^2}{\alpha_i^2}\right)+\frac{2}{15}\frac{\alpha_i^2}{\alpha_j^2}\ln\left[\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{(-\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_j}{\alpha_i}\left(\frac{1}{3}-\frac{1}{5}\frac{\alpha_j^2}{\alpha_i^2}-\frac{d^2}{\alpha_i^2}\right)\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{d\alpha_i}{\alpha_j^2}\Big[\frac{1}{4}+\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}-\frac{3}{4}\frac{\alpha_j^4}{\alpha_i^4}-\left(\frac{1}{6}+\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}+\frac{1}{20}\frac{d^4}{\alpha_i^4}\Big]\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\qquad g=1,\ \gamma=2,$$

$$\frac{\alpha_i}{\alpha_j}\Big[-\frac{1}{6}+\frac{11}{9}\frac{\alpha_j^2}{\alpha_i^2}-\frac{5}{6}\frac{\alpha_j^4}{\alpha_i^4}+\left(\frac{1}{3}-\frac{13}{3}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}-\frac{1}{6}\frac{d^4}{\alpha_i^4}\Big]+\frac{\alpha_j d}{\alpha_i^2}\left(-1+\frac{\alpha_j^2}{\alpha_i^2}+\frac{5}{3}\frac{d^2}{\alpha_i^2}\right)\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_i^2}{\alpha_j^2}\Big[\frac{1}{24}+\frac{1}{8}\frac{\alpha_j^2}{\alpha_i^2}-\frac{3}{8}\frac{\alpha_j^4}{\alpha_i^4}+\frac{5}{24}\frac{\alpha_j^6}{\alpha_i^6}-\left(\frac{1}{8}+\frac{3}{4}\frac{\alpha_j^2}{\alpha_i^2}-\frac{15}{8}\frac{\alpha_j^4}{\alpha_i^4}\right)\times$$

$$\times\frac{d^2}{\alpha_i^2}+\left(\frac{1}{8}+\frac{5}{8}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}-\frac{1}{24}\frac{d^6}{\alpha_i^6}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\qquad g=1,\ \gamma=3,$$

$$\frac{d}{\alpha_j}\Big[\frac{1}{6}-\frac{41}{9}\frac{\alpha_j^2}{\alpha_i^2}+\frac{29}{6}\frac{\alpha_j^4}{\alpha_i^4}-\left(\frac{1}{3}-\frac{25}{3}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^2}{\alpha_i^2}+\frac{1}{6}\frac{d^4}{\alpha_i^4}\Big]+\frac{\alpha_j}{\alpha_i}\Big[-\frac{1}{4}+\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}-\frac{1}{4}\frac{\alpha_j^4}{\alpha_i^4}+\left(\frac{5}{2}-\frac{7}{2}\frac{\alpha_j^2}{\alpha_i^2}\right)\times$$

$$\times\frac{d^2}{\alpha_i^2}-\frac{35}{12}\frac{d^4}{\alpha_i^4}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_i d}{\alpha_j^2}\Big[-\frac{1}{24}-\frac{3}{8}\frac{\alpha_j^2}{\alpha_i^2}+\frac{15}{8}\frac{\alpha_j^4}{\alpha_i^4}-\frac{35}{24}\frac{\alpha_j^6}{\alpha_i^6}+$$

$$+\left(\frac{1}{8}+\frac{5}{4}\frac{\alpha_j^2}{\alpha_i^2}-\frac{35}{8}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^2}{\alpha_i^2}-\left(\frac{1}{8}+\frac{7}{8}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^4}{\alpha_i^4}+\frac{1}{24}\frac{d^6}{\alpha_i^6}\Big]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],$$

$$g=1,\ \gamma=4,$$

$$\frac{\alpha_i}{\alpha_j}\Big[\frac{1}{48}-\frac{839}{720}\frac{\alpha_j^2}{\alpha_i^2}+\frac{119}{48}\frac{\alpha_j^4}{\alpha_i^4}-\frac{21}{16}\frac{\alpha_j^6}{\alpha_i^6}+\Big(-\frac{11}{48}+\frac{307}{24}\frac{\alpha_j^2}{\alpha_i^2}-\frac{297}{16}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^2}{\alpha_i^2}+\Big(\frac{19}{48}-\frac{255}{16}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^4}{\alpha_i^4}-$$

$$-\frac{3}{16}\frac{d^6}{\alpha_i^6}\Big]+\frac{\alpha_j d}{\alpha_i^2}\Big[\frac{5}{4}-\frac{7}{2}\frac{\alpha_j^2}{\alpha_i^2}+\frac{9}{4}\frac{\alpha_j^4}{\alpha_i^4}-\Big(\frac{35}{6}-\frac{21}{2}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^2}{\alpha_i^2}+\frac{21}{4}\frac{d^4}{\alpha_i^4}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Big]+$$

$$+\frac{\alpha_i^2}{\alpha_j^2}\Big[-\frac{1}{192}-\frac{1}{16}\frac{\alpha_j^2}{\alpha_i^2}+\frac{15}{32}\frac{\alpha_j^4}{\alpha_i^4}-\frac{35}{48}\frac{\alpha_j^6}{\alpha_i^6}+\frac{21}{64}\frac{\alpha_j^8}{\alpha_i^8}+\Big(\frac{1}{16}+\frac{15}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{105}{16}\frac{\alpha_j^4}{\alpha_i^4}+\frac{105}{16}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^2}{\alpha_i^2}-$$

$$-\Big(\frac{5}{32}+\frac{35}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{315}{32}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^4}{\alpha_i^4}+\Big(\frac{7}{48}+\frac{21}{16}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^6}{\alpha_i^6}-\frac{3}{64}\frac{d^8}{\alpha_i^8}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Big],$$

$$q=1,\ r=5,$$

$$\frac{d}{\alpha_j}\Big[-\frac{1}{16}+\frac{1543}{240}\frac{\alpha_j^2}{\alpha_i^2}-\frac{901}{48}\frac{\alpha_j^4}{\alpha_i^4}+\frac{605}{48}\frac{\alpha_j^6}{\alpha_i^6}+\Big(\frac{17}{48}-\frac{2315}{72}\frac{\alpha_j^2}{\alpha_i^2}+\frac{2849}{48}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{25}{48}-\frac{1463}{48}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^4}{\alpha_i^4}+$$

$$+\frac{11}{48}\frac{d^6}{\alpha_i^6}\Big]+\frac{\alpha_j}{\alpha_i}\Big[\frac{5}{24}-\frac{7}{8}\frac{\alpha_j^2}{\alpha_i^2}+\frac{9}{8}\frac{\alpha_j^4}{\alpha_i^4}-\frac{11}{24}\frac{\alpha_j^6}{\alpha_i^6}-\Big(\frac{35}{8}-\frac{63}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{99}{8}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^2}{\alpha_i^2}+\Big(\frac{105}{8}-\frac{231}{8}\frac{\alpha_j^2}{\alpha_i^2}\Big)\times$$

$$\times\frac{d^4}{\alpha_i^4}-\frac{77}{8}\frac{d^6}{\alpha_i^6}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Big]+\frac{\alpha_i d}{\alpha_j^2}\Big[\frac{1}{64}+\frac{5}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{105}{32}\frac{\alpha_j^4}{\alpha_i^4}+\frac{105}{16}\frac{\alpha_j^6}{\alpha_i^6}-$$

$$-\frac{231}{64}\frac{\alpha_j^8}{\alpha_i^8}-\Big(\frac{5}{48}+\frac{35}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{315}{16}\frac{\alpha_j^4}{\alpha_i^4}+\frac{385}{16}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^2}{\alpha_i^2}+\Big(\frac{7}{32}+\frac{63}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{693}{32}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^4}{\alpha_i^4}-$$

$$-\Big(\frac{3}{16}+\frac{33}{16}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^6}{\alpha_i^6}+\frac{11}{192}\frac{d^8}{\alpha_i^8}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Big],\qquad q=1,\ r=6,$$

$$\frac{\alpha_i}{\alpha_j}\Big[-\frac{1}{160}+\frac{297}{280}\frac{\alpha_j^2}{\alpha_i^2}-\frac{1877}{400}\frac{\alpha_j^4}{\alpha_i^4}+\frac{253}{40}\frac{\alpha_j^6}{\alpha_i^6}-\frac{429}{160}\frac{\alpha_j^8}{\alpha_i^8}+\Big(\frac{3}{20}-\frac{577}{24}\frac{\alpha_j^2}{\alpha_i^2}+\frac{5357}{60}\frac{\alpha_j^4}{\alpha_i^4}-\frac{8723}{120}\frac{\alpha_j^6}{\alpha_i^6}\Big)\times$$

$$\times\frac{d^2}{\alpha_i^2}-\Big(\frac{139}{240}-\frac{5489}{72}\frac{\alpha_j^2}{\alpha_i^2}+\frac{41041}{240}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^4}{\alpha_i^4}+\Big(\frac{11}{15}-\frac{7007}{120}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^6}{\alpha_i^6}-\frac{143}{480}\frac{d^8}{\alpha_i^8}\Big]+\frac{\alpha_j d}{\alpha_i^2}\times$$

$$\times\Big[-\frac{35}{24}+\frac{63}{8}\frac{\alpha_j^2}{\alpha_i^2}-\frac{99}{8}\frac{\alpha_j^4}{\alpha_i^4}+\frac{143}{24}\frac{\alpha_j^6}{\alpha_i^6}+\Big(\frac{105}{8}-\frac{231}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{429}{8}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{231}{8}-\frac{3003}{40}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^4}{\alpha_i^4}+$$

$$+\frac{143}{8}\frac{d^6}{\alpha_i^6}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Big]+\frac{\alpha_i^2}{\alpha_j^2}\Big[\frac{1}{640}+\frac{5}{128}\frac{\alpha_j^2}{\alpha_i^2}-\frac{35}{64}\frac{\alpha_j^4}{\alpha_i^4}+\frac{105}{64}\frac{\alpha_j^6}{\alpha_i^6}-$$

$$-\frac{231}{128}\frac{\alpha_j^8}{\alpha_i^8}+\frac{429}{640}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\Big(\frac{5}{128}+\frac{35}{32}\frac{\alpha_j^2}{\alpha_i^2}-\frac{945}{64}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1155}{32}\frac{\alpha_j^6}{\alpha_i^6}-\frac{3003}{128}\frac{\alpha_j^8}{\alpha_i^8}\Big)\frac{d^2}{\alpha_i^2}+\Big(\frac{35}{192}+\frac{315}{64}\frac{\alpha_j^2}{\alpha_i^2}-$$

$$-\frac{3465}{64}\frac{\alpha_j^4}{\alpha_i^4}+\frac{5005}{64}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^4}{\alpha_i^4}-\Big(\frac{21}{64}+\frac{231}{32}\frac{\alpha_j^2}{\alpha_i^2}-\frac{3003}{64}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^6}{\alpha_i^6}+\Big(\frac{33}{128}+\frac{429}{128}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^8}{\alpha_i^8}-$$

$$-\frac{143}{1920}\frac{d^{10}}{\alpha_i^{10}}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Big],\qquad q=1,\ r=7,$$

$$\frac{\alpha_i d}{\alpha_j^2}\Big[\frac{3}{160}+\frac{13}{80}\frac{\alpha_j^2}{\alpha_i^2}-\frac{3073}{300}\frac{\alpha_j^4}{\alpha_i^4}+\frac{6361}{240}\frac{\alpha_j^6}{\alpha_i^6}-\frac{7969}{480}\frac{\alpha_j^8}{\alpha_i^8}-\Big(\frac{11}{80}+\frac{79}{60}\frac{\alpha_j^2}{\alpha_i^2}-\frac{46813}{720}\frac{\alpha_j^4}{\alpha_i^4}+\frac{12181}{120}\frac{\alpha_j^6}{\alpha_i^6}\Big)\times$$

$$\times\frac{d^2}{\alpha_i^2}+\Big(\frac{3}{10}+\frac{41}{16}\frac{\alpha_j^2}{\alpha_i^2}-\frac{17927}{240}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^4}{\alpha_i^4}-\Big(\frac{21}{80}+\frac{169}{120}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^6}{\alpha_i^6}+\frac{13}{160}\frac{d^8}{\alpha_i^8}\Big]+\frac{\alpha_j^2}{\alpha_i^2}\Big[-\frac{7}{24}+$$

$$+\frac{9}{8}\frac{\alpha_j^2}{\alpha_i^2}-\frac{11}{8}\frac{\alpha_j^4}{\alpha_i^4}+\frac{13}{24}\frac{\alpha_j^6}{\alpha_i^6}+\Big(\frac{63}{8}-\frac{99}{4}\frac{\alpha_j^2}{\alpha_i^2}+\frac{143}{8}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{231}{8}-\frac{429}{8}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^4}{\alpha_i^4}+\frac{1001}{47}\frac{d^6}{\alpha_i^6}\Big]\times$$

$$\times\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Big]+\frac{\alpha_i^2 d}{\alpha_j^3}\Big[-\frac{3}{640}-\frac{5}{128}\frac{\alpha_j^2}{\alpha_i^2}-\frac{35}{64}\frac{\alpha_j^4}{\alpha_i^4}+\frac{315}{64}\frac{\alpha_j^6}{\alpha_i^6}-\frac{1155}{128}\frac{\alpha_j^8}{\alpha_i^8}+$$

$$+\frac{3003}{640}\frac{\alpha_j^{10}}{\alpha_i^{10}}+\Big(\frac{5}{128}+\frac{35}{96}\frac{\alpha_j^2}{\alpha_i^2}+\frac{315}{64}\frac{\alpha_j^4}{\alpha_i^4}-\frac{1155}{32}\frac{\alpha_j^6}{\alpha_i^6}+\frac{5005}{128}\frac{\alpha_j^8}{\alpha_i^8}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{7}{64}+\frac{63}{64}\frac{\alpha_j^2}{\alpha_i^2}+\frac{693}{64}\frac{\alpha_j^4}{\alpha_i^4}-$$

$$-\frac{3003}{64}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^4}{\alpha_i^4}+\Big(\frac{9}{64}+\frac{33}{32}\frac{\alpha_j^2}{\alpha_i^2}+\frac{429}{64}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^6}{\alpha_i^6}-\Big(\frac{11}{128}+\frac{143}{384}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^8}{\alpha_i^8}+\frac{13}{640}\frac{d^{10}}{\alpha_i^{10}}\Big]\times$$

$$\times\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Big],\qquad q=2,\ r=7,$$

$$\frac{\alpha_i^3}{\alpha_j^3}\Big[\frac{1}{384}+\frac{59}{576}\frac{\alpha_j^2}{\alpha_i^2}+\frac{41}{960}\frac{\alpha_j^4}{\alpha_i^4}-\frac{2513}{1600}\frac{\alpha_j^6}{\alpha_i^6}+\frac{5929}{1920}\frac{\alpha_j^8}{\alpha_i^8}-\frac{1001}{640}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\Big(\frac{17}{384}+\frac{113}{480}\frac{\alpha_j^2}{\alpha_i^2}+\frac{357}{320}\frac{\alpha_j^4}{\alpha_i^4}-$$

$$-\frac{4797}{160}\frac{\alpha_j^6}{\alpha_i^6}+\frac{22711}{640}\frac{\alpha_j^8}{\alpha_i^8}\Big)\frac{d^2}{\alpha_i^2}+\Big(\frac{29}{192}+\frac{273}{320}\frac{\alpha_j^2}{\alpha_i^2}+\frac{719}{192}\frac{\alpha_j^4}{\alpha_i^4}-\frac{55159}{960}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^4}{\alpha_i^4}-\Big(\frac{41}{192}+$$

$$+\frac{1499}{1440}\frac{\alpha_j^2}{\alpha_i^2}+\frac{2821}{960}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^6}{\alpha_i^6}+\Big(\frac{53}{384}+\frac{793}{1920}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^8}{\alpha_i^8}-\frac{13}{384}\frac{d^{10}}{\alpha_i^{10}}\Big]+\frac{\alpha_j^3 d}{\alpha_i^4}\Big[\frac{9}{4}-\frac{11}{2}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{13}{4}\frac{\alpha_j^4}{\alpha_i^4}-\Big(\frac{33}{2}-\frac{143}{6}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^2}{\alpha_i^2}+\frac{429}{20}\frac{d^4}{\alpha_i^4}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Big]+\frac{\alpha_i^4}{\alpha_j^4}\Big[-\frac{1}{1536}-$$

$$-\frac{3}{1280}\frac{\alpha_j^2}{\alpha_i^2}-\frac{5}{512}\frac{\alpha_j^4}{\alpha_i^4}-\frac{35}{384}\frac{\alpha_j^6}{\alpha_i^6}+\frac{315}{512}\frac{\alpha_j^8}{\alpha_i^8}-\frac{231}{256}\frac{\alpha_j^{10}}{\alpha_i^{10}}+\frac{1001}{256}\frac{\alpha_j^{12}}{\alpha_i^{12}}+\Big(\frac{3}{256}+\frac{15}{256}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{35}{128}\frac{\alpha_j^4}{\alpha_i^4}+\frac{315}{128}\frac{\alpha_j^6}{\alpha_i^6}-\frac{3465}{256}\frac{\alpha_j^8}{\alpha_i^8}+\frac{3003}{256}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{25}{512}+\frac{35}{128}\frac{\alpha_j^2}{\alpha_i^2}+\frac{315}{256}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1155}{128}\frac{\alpha_j^6}{\alpha_i^6}-$$

$$-\frac{15015}{512}\frac{\alpha_j^8}{\alpha_i^8}\Big)\frac{d^4}{\alpha_i^4}+\Big(\frac{35}{384}+\frac{63}{128}\frac{\alpha_j^2}{\alpha_i^2}+\frac{231}{128}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1001}{128}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^6}{\alpha_i^6}-\Big(\frac{45}{512}+\frac{99}{256}\frac{\alpha_j^2}{\alpha_i^2}+\frac{429}{512}\frac{\alpha_j^4}{\alpha_i^4}\Big)\times$$

$$\times\frac{d^8}{\alpha_i^8}+\Big(\frac{11}{256}+\frac{143}{1280}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^{10}}{\alpha_i^{10}}-\frac{13}{1536}\frac{d^{12}}{\alpha_i^{12}}\Big]\ln\Big[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Big],\quad q=3,\ r=7,$$

$$\frac{\alpha_i^3 d}{\alpha_j^4}\Big[-\frac{7}{384}-\frac{61}{1152}\frac{\alpha_j^2}{\alpha_i^2}-\frac{131}{960}\frac{\alpha_j^4}{\alpha_i^4}-\frac{133}{320}\frac{\alpha_j^6}{\alpha_i^6}+\frac{2773}{384}\frac{\alpha_j^8}{\alpha_i^8}-\frac{873}{128}\frac{\alpha_j^{10}}{\alpha_i^{10}}+\Big(\frac{35}{384}+\frac{11}{32}\frac{\alpha_j^2}{\alpha_i^2}+\frac{941}{960}\frac{\alpha_j^4}{\alpha_i^4}+$$

$$+\frac{281}{96}\frac{\alpha_j^6}{\alpha_i^6}-\frac{3551}{128}\frac{\alpha_j^8}{\alpha_i^8}\Big)\frac{d^2}{\alpha_i^2}-\Big(\frac{35}{192}+\frac{137}{192}\frac{\alpha_j^2}{\alpha_i^2}+\frac{341}{192}\frac{\alpha_j^4}{\alpha_i^4}+\frac{679}{192}\frac{\alpha_j^6}{\alpha_i^6}\Big)\frac{d^4}{\alpha_i^4}+\Big(\frac{35}{192}+\frac{175}{288}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{179}{192}\frac{\alpha_j^4}{\alpha_i^4}\Big)\frac{d^6}{\alpha_i^6}-\Big(\frac{35}{384}+\frac{71}{384}\frac{\alpha_j^2}{\alpha_i^2}\Big)\frac{d^8}{\alpha_i^8}+\frac{7}{384}\frac{d^{10}}{\alpha_i^{10}}\Big]+\frac{\alpha_j^4}{\alpha_i^4}\Big[\frac{1}{4}-\frac{1}{2}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1}{4}\frac{\alpha_j^4}{\alpha_i^4}-$$

$$-\Bigl(\frac{11}{2}-\frac{13}{2}\frac{\alpha_j^2}{\alpha_i^2}\Bigr)\frac{d^2}{\alpha_i^2}+\frac{143}{12}\frac{d^4}{\alpha_i^4}\Bigr]\ln\Bigl[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Bigr]+\frac{\alpha_i^4 d}{\alpha_j^5}\Bigl[\frac{7}{1536}+\frac{3}{256}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{15}{512}\frac{\alpha_j^4}{\alpha_i^4}+\frac{35}{384}\frac{\alpha_j^6}{\alpha_i^6}+\frac{315}{512}\frac{\alpha_j^8}{\alpha_i^8}-\frac{693}{256}\frac{\alpha_j^{10}}{\alpha_i^{10}}+\frac{1001}{512}\frac{\alpha_j^{12}}{\alpha_i^{12}}-\Bigl(\frac{7}{256}+\frac{25}{256}\frac{\alpha_j^2}{\alpha_i^2}+\frac{35}{128}\frac{\alpha_j^4}{\alpha_i^4}+\frac{105}{128}\frac{\alpha_j^6}{\alpha_i^6}+$$

$$+\frac{1155}{256}\frac{\alpha_j^8}{\alpha_i^8}-\frac{3003}{256}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Bigr)\frac{d^2}{\alpha_i^2}+\Bigl(\frac{35}{512}+\frac{35}{128}\frac{\alpha_j^2}{\alpha_i^2}+\frac{189}{256}\frac{\alpha_j^4}{\alpha_i^4}+\frac{231}{128}\frac{\alpha_j^6}{\alpha_i^6}+\frac{3003}{512}\frac{\alpha_j^8}{\alpha_i^8}\Bigr)\frac{d^4}{\alpha_i^4}-$$

$$-\Bigl(\frac{35}{384}+\frac{45}{128}\frac{\alpha_j^2}{\alpha_i^2}+\frac{99}{128}\frac{\alpha_j^4}{\alpha_i^4}+\frac{143}{128}\frac{\alpha_j^6}{\alpha_i^6}\Bigr)\frac{d^6}{\alpha_i^6}+\Bigl(\frac{35}{512}+\frac{55}{256}\frac{\alpha_j^2}{\alpha_i^2}+\frac{143}{512}\frac{\alpha_j^4}{\alpha_i^4}\Bigr)\frac{d^8}{\alpha_i^8}-\Bigl(\frac{7}{256}+$$

$$+\frac{13}{256}\frac{\alpha_j^2}{\alpha_i^2}\Bigr)\frac{d^{10}}{\alpha_i^{10}}+\frac{7}{1536}\frac{d^{12}}{\alpha_i^{12}}\Bigr]\ln\Bigl[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Bigr],\qquad g=4,\ \gamma=7,$$

$$\frac{\alpha_i^5}{\alpha_j^5}\Bigl[-\frac{3}{256}-\frac{5}{384}\frac{\alpha_j^2}{\alpha_i^2}-\frac{197}{11520}\frac{\alpha_j^4}{\alpha_i^4}-\frac{181}{6720}\frac{\alpha_j^6}{\alpha_i^6}-\frac{73}{1280}\frac{\alpha_j^8}{\alpha_i^8}+\frac{275}{384}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\frac{143}{256}\frac{\alpha_j^{12}}{\alpha_i^{12}}+\Bigl(\frac{9}{128}+\frac{61}{384}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{287}{960}\frac{\alpha_j^4}{\alpha_i^4}+\frac{183}{320}\frac{\alpha_j^6}{\alpha_i^6}+\frac{165}{128}\frac{\alpha_j^8}{\alpha_i^8}-\frac{1061}{128}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Bigr)\frac{d^2}{\alpha_i^2}-\Bigl(\frac{45}{256}+\frac{97}{192}\frac{\alpha_j^2}{\alpha_i^2}+\frac{639}{640}\frac{\alpha_j^4}{\alpha_i^4}+\frac{325}{192}\frac{\alpha_j^6}{\alpha_i^6}+$$

$$+\frac{2063}{768}\frac{\alpha_j^8}{\alpha_i^8}\Bigr)\frac{d^4}{\alpha_i^4}+\Bigl(\frac{15}{64}+\frac{133}{192}\frac{\alpha_j^2}{\alpha_i^2}+\frac{673}{576}\frac{\alpha_j^4}{\alpha_i^4}+\frac{235}{192}\frac{\alpha_j^6}{\alpha_i^6}\Bigr)\frac{d^6}{\alpha_i^6}-\Bigl(\frac{45}{256}+\frac{169}{384}\frac{\alpha_j^2}{\alpha_i^2}+\frac{347}{768}\frac{\alpha_j^4}{\alpha_i^4}\Bigr)\frac{d^8}{\alpha_i^8}+$$

$$+\Bigl(\frac{9}{128}+\frac{41}{384}\frac{\alpha_j^2}{\alpha_i^2}\Bigr)\frac{d^{10}}{\alpha_i^{10}}-\frac{3}{256}\frac{d^{12}}{\alpha_i^{12}}\Bigr]+\frac{\alpha_j^5 d}{\alpha_i^6}\Bigl(-1+\frac{\alpha_j^2}{\alpha_i^2}+\frac{13}{3}\frac{d^2}{\alpha_i^2}\Bigr)\ln\Bigl[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\Bigr]+$$

$$+\frac{\alpha_i^6}{\alpha_j^6}\Bigl[\frac{3}{1024}+\frac{7}{3072}\frac{\alpha_j^2}{\alpha_i^2}+\frac{3}{1024}\frac{\alpha_j^4}{\alpha_i^4}+\frac{5}{1024}\frac{\alpha_j^6}{\alpha_i^6}+\frac{35}{3072}\frac{\alpha_j^8}{\alpha_i^8}+\frac{63}{1024}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\frac{231}{1024}\frac{\alpha_j^{12}}{\alpha_i^{12}}+\frac{143}{1024}\frac{\alpha_j^{14}}{\alpha_i^{14}}-$$

$$-\Bigl(\frac{21}{1024}+\frac{21}{512}\frac{\alpha_j^2}{\alpha_i^2}+\frac{75}{1024}\frac{\alpha_j^4}{\alpha_i^4}+\frac{35}{256}\frac{\alpha_j^6}{\alpha_i^6}+\frac{315}{1024}\frac{\alpha_j^8}{\alpha_i^8}+\frac{693}{512}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\frac{3003}{1024}\frac{\alpha_j^{12}}{\alpha_i^{12}}\Bigr)\frac{d^2}{\alpha_i^2}+\Bigl(\frac{63}{1024}+$$

$$+\frac{175}{1024}\frac{\alpha_j^2}{\alpha_i^2}+\frac{175}{512}\frac{\alpha_j^4}{\alpha_i^4}+\frac{315}{512}\frac{\alpha_j^6}{\alpha_i^6}+\frac{1155}{1024}\frac{\alpha_j^8}{\alpha_i^8}+\frac{3003}{1024}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Bigr)\frac{d^4}{\alpha_i^4}-\Bigl(\frac{105}{1024}+\frac{245}{768}\frac{\alpha_j^2}{\alpha_i^2}+\frac{315}{512}\frac{\alpha_j^4}{\alpha_i^4}+$$

$$+\frac{231}{256}\frac{\alpha_j^6}{\alpha_i^6}+\frac{1001}{1024}\frac{\alpha_j^8}{\alpha_i^8}\Bigr)\frac{d^6}{\alpha_i^6}+\Bigl(\frac{105}{1024}+\frac{315}{1024}\frac{\alpha_j^2}{\alpha_i^2}+\frac{495}{1024}\frac{\alpha_j^4}{\alpha_i^4}+\frac{429}{1024}\frac{\alpha_j^6}{\alpha_i^6}\Bigr)\frac{d^8}{\alpha_i^8}-\Bigl(\frac{63}{1024}+\frac{77}{512}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{143}{1024}\frac{\alpha_j^4}{\alpha_i^4}\Bigr)\frac{d^{10}}{\alpha_i^{10}}+\Bigl(\frac{21}{1024}+\frac{91}{3072}\frac{\alpha_j^2}{\alpha_i^2}\Bigr)\frac{d^{12}}{\alpha_i^{12}}-\frac{3}{1024}\frac{d^{14}}{\alpha_i^{14}}\Bigr]\times\ln\Bigl[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\Bigr],$$

$$g=5,\ \gamma=7,$$

$$\frac{\alpha_i^5 d}{\alpha_j^6}\Bigl[\frac{4387}{49920}+\frac{131}{1560}\frac{\alpha_j^2}{\alpha_i^2}+\frac{4699}{49920}\frac{\alpha_j^4}{\alpha_i^4}+\frac{5189}{43680}\frac{\alpha_j^6}{\alpha_i^6}+\frac{8521}{49920}\frac{\alpha_j^8}{\alpha_i^8}+\frac{1889}{6240}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\frac{5411}{3840}\frac{\alpha_j^{12}}{\alpha_i^{12}}-\Bigl(\frac{2657}{12480}+$$

$$+\frac{5831}{14976}\frac{\alpha_j^2}{\alpha_i^2}+\frac{877}{1560}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1871}{2496}\frac{\alpha_j^6}{\alpha_i^6}+\frac{7309}{7488}\frac{\alpha_j^8}{\alpha_i^8}+\frac{2533}{1920}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Bigr)\frac{d^2}{\alpha_i^2}+\Bigl(\frac{16399}{49920}+\frac{4721}{6240}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{46809}{41600}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1331}{1040}\frac{\alpha_j^6}{\alpha_i^6}+\frac{3943}{3840}\frac{\alpha_j^8}{\alpha_i^8}\Bigr)\frac{d^4}{\alpha_i^4}-\Bigl(\frac{493}{1560}+\frac{3101}{4160}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1953}{2080}\frac{\alpha_j^4}{\alpha_i^4}+\frac{623}{960}\frac{\alpha_j^6}{\alpha_i^6}\Bigr)\frac{d^6}{\alpha_i^6}+$$

$$+\left(\frac{3083}{16640}+\frac{461}{1248}\frac{\alpha_j^2}{\alpha_i^2}+\frac{361}{1280}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^8}{\alpha_i^8}-\left(\frac{253}{4160}+\frac{47}{640}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^{10}}{\alpha_i^{10}}+\frac{11}{1280}\frac{d^{12}}{\alpha_i^{12}}\Bigg]-\frac{2}{195}\frac{\alpha_i^7}{\alpha_j^7}\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{(-\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{\alpha_j^6}{\alpha_i^6}\left(-\frac{1}{13}+\frac{1}{15}\frac{\alpha_j^2}{\alpha_i^2}+\frac{d^2}{\alpha_i^2}\right)\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+$$

$$+\frac{\alpha_i^6 d}{\alpha_j^7}\Bigg[-\frac{33}{1024}-\frac{21}{1024}\frac{\alpha_j^2}{\alpha_i^2}-\frac{21}{1024}\frac{\alpha_j^4}{\alpha_i^4}-\frac{25}{1024}\frac{\alpha_j^6}{\alpha_i^6}-\frac{35}{1024}\frac{\alpha_j^8}{\alpha_i^8}-\frac{63}{1024}\frac{\alpha_j^{10}}{\alpha_i^{10}}-\frac{231}{1024}\frac{\alpha_j^{12}}{\alpha_i^{12}}+\frac{429}{1024}\frac{\alpha_j^{14}}{\alpha_i^{14}}+$$

$$+\left(\frac{77}{1024}+\frac{63}{512}\frac{\alpha_j^2}{\alpha_i^2}+\frac{175}{1024}\frac{\alpha_j^4}{\alpha_i^4}+\frac{175}{768}\frac{\alpha_j^6}{\alpha_i^6}+\frac{315}{1024}\frac{\alpha_j^8}{\alpha_i^8}+\frac{231}{512}\frac{\alpha_j^{10}}{\alpha_i^{10}}+\frac{1001}{1024}\frac{\alpha_j^{12}}{\alpha_i^{12}}\right)\frac{d^2}{\alpha_i^2}-\Bigg(\frac{693}{5120}+$$

$$+\frac{315}{1024}\frac{\alpha_j^2}{\alpha_i^2}+\frac{245}{512}\frac{\alpha_j^4}{\alpha_i^4}+\frac{315}{512}\frac{\alpha_j^6}{\alpha_i^6}+\frac{693}{1024}\frac{\alpha_j^8}{\alpha_i^8}+\frac{3003}{5120}\frac{\alpha_j^{10}}{\alpha_i^{10}}\Bigg)\frac{d^4}{\alpha_i^4}+\Bigg(\frac{165}{1024}+\frac{105}{256}\frac{\alpha_j^2}{\alpha_i^2}+\frac{315}{512}\frac{\alpha_j^4}{\alpha_i^4}+$$

$$+\frac{165}{256}\frac{\alpha_j^6}{\alpha_i^6}+\frac{429}{1024}\frac{\alpha_j^8}{\alpha_i^8}\Bigg)\frac{d^6}{\alpha_i^6}-\left(\frac{385}{3072}+\frac{315}{1024}\frac{\alpha_j^2}{\alpha_i^2}+\frac{385}{1024}\frac{\alpha_j^4}{\alpha_i^4}+\frac{715}{3072}\frac{\alpha_j^6}{\alpha_i^6}\right)\frac{d^8}{\alpha_i^8}+\Bigg(\frac{63}{1024}+$$

$$+\frac{63}{512}\frac{\alpha_j^2}{\alpha_i^2}+\frac{91}{1024}\frac{\alpha_j^4}{\alpha_i^4}\Bigg)\frac{d^{10}}{\alpha_i^{10}}-\left(\frac{231}{13312}+\frac{21}{1024}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^{12}}{\alpha_i^{12}}+\frac{11}{5120}\frac{d^{14}}{\alpha_i^{14}}\Bigg]\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],$$

$$q=6,\ r=7,$$

$$\frac{\alpha_i^7}{\alpha_j^7}\Bigg[-\frac{429}{4096}+\frac{121}{4096}\frac{\alpha_j^2}{\alpha_i^2}+\frac{431}{20480}\frac{\alpha_j^4}{\alpha_i^4}+\frac{2643}{143360}\frac{\alpha_j^6}{\alpha_i^6}-\left(\frac{25147}{61440}+\frac{10511}{30720}\frac{\alpha_j^2}{\alpha_i^2}+\frac{19573}{61440}\frac{\alpha_j^4}{\alpha_i^4}+\frac{33583}{215040}\frac{\alpha_j^6}{\alpha_i^6}\right)\times$$

$$\times\frac{d^2}{\alpha_i^2}+\left(\frac{34913}{61440}+\frac{32107}{36864}\frac{\alpha_j^2}{\alpha_i^2}+\frac{31009}{30720}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^4}{\alpha_i^4}-\left(\frac{37159}{61440}+\frac{17797}{15360}\frac{\alpha_j^2}{\alpha_i^2}+\frac{70259}{102400}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^6}{\alpha_i^6}+$$

$$+\left(\frac{27191}{61440}+\frac{17739}{20480}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^8}{\alpha_i^8}-\left(\frac{4283}{20480}+\frac{2123}{3072}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^{10}}{\alpha_i^{10}}+\frac{1177}{20480}\frac{d^{12}}{\alpha_i^{12}}-\frac{143}{40960}\frac{d^{14}}{\alpha_i^{14}}\Bigg]+$$

$$+\frac{\alpha_j^7}{\alpha_i^7}[\alpha_i\longleftrightarrow\alpha_j]+\frac{2}{15}\frac{\alpha_i^7 d}{\alpha_j^8}\ln\left[\frac{(\alpha_j+\alpha_i+d)(\alpha_j-\alpha_i+d)}{(-\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]+\frac{2}{15}\frac{\alpha_j^7 d}{\alpha_i^8}\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j+\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j-\alpha_i+d)}\right]$$

$$+\Bigg\{\frac{\alpha_i^8}{\alpha_j^8}\Bigg[\frac{429}{16384}-\frac{33}{2048}\frac{\alpha_j^2}{\alpha_i^2}-\frac{21}{4096}\frac{\alpha_j^4}{\alpha_i^4}-\frac{7}{2048}\frac{\alpha_j^6}{\alpha_i^6}-\frac{25}{16384}\frac{\alpha_j^8}{\alpha_i^8}+\Bigg(\frac{429}{2048}+\frac{231}{2048}\frac{\alpha_j^2}{\alpha_i^2}+\frac{189}{2048}\frac{\alpha_j^4}{\alpha_i^4}+$$

$$+\frac{175}{2048}\frac{\alpha_j^6}{\alpha_i^6}\Bigg)\frac{d^2}{\alpha_i^2}-\left(\frac{1001}{4096}+\frac{693}{2048}\frac{\alpha_j^2}{\alpha_i^2}+\frac{1575}{4096}\frac{\alpha_j^4}{\alpha_i^4}+\frac{1225}{6144}\frac{\alpha_j^6}{\alpha_i^6}\right)\frac{d^4}{\alpha_i^4}+\Bigg(\frac{3003}{10240}+\frac{1155}{2048}\frac{\alpha_j^2}{\alpha_i^2}+$$

$$+\frac{735}{1024}\frac{\alpha_j^4}{\alpha_i^4}\Bigg)\frac{d^6}{\alpha_i^6}-\left(\frac{2145}{8192}+\frac{1155}{2048}\frac{\alpha_j^2}{\alpha_i^2}+\frac{2835}{8192}\frac{\alpha_j^4}{\alpha_i^4}\right)\frac{d^8}{\alpha_i^8}+\left(\frac{1001}{6144}+\frac{693}{2048}\frac{\alpha_j^2}{\alpha_i^2}\right)\frac{d^{10}}{\alpha_i^{10}}-\Bigg(\frac{273}{4096}+$$

$$+\frac{231}{4096}\frac{\alpha_j^2}{\alpha_i^2}\Bigg)\frac{d^{12}}{\alpha_i^{12}}+\frac{33}{2048}\frac{d^{14}}{\alpha_i^{14}}-\frac{143}{163840}\frac{d^{16}}{\alpha_i^{16}}\Bigg]+\frac{\alpha_j^8}{\alpha_i^8}[\alpha_i\longleftrightarrow\alpha_j]\Bigg\}\times$$

$$\times\ln\left[\frac{(\alpha_j+\alpha_i+d)(-\alpha_j-\alpha_i+d)}{(\alpha_j-\alpha_i+d)(-\alpha_j+\alpha_i+d)}\right],\qquad q=r=7,$$

(A23)

where $[a \longleftrightarrow b]$ stands for the fact that the expression is the same as shown just before except for interchanging a with b (and vice versa). In addition, the expressions for q=1-7 and r=0 are respectively the same as those for q=0 and r=1-7 except for interchanging α_j with α_i .

The coefficient of the series expansion on the right hand side of (A19), $Y_{5m}(\alpha_j,\alpha_i,d)/(\alpha_j+\alpha_i+d)^2$, is written as follows (by the use of abbreviation $x=\alpha_j+\alpha_i+d$):

$$\frac{2}{(m+1)!}\,, \qquad q=r=0,$$

$$\frac{2}{(m+1)!}-\frac{2}{(m+2)!}\frac{x}{\alpha_i}\,, \qquad q=0,\ r=1,$$

$$\frac{2}{(m+1)!}-\frac{6}{(m+2)!}\frac{x}{\alpha_i}+\frac{6}{(m+3)!}\frac{x^2}{\alpha_i^2}\,, \qquad q=0,\ r=2,$$

$$\frac{2}{(m+1)!}-\frac{12}{(m+2)!}\frac{x}{\alpha_i}+\frac{30}{(m+3)!}\frac{x^2}{\alpha_i^2}-\frac{30}{(m+4)!}\frac{x^3}{\alpha_i^3}\,, \qquad q=0,\ r=3,$$

$$\frac{2}{(m+1)!}-\frac{20}{(m+2)!}\frac{x}{\alpha_i}+\frac{90}{(m+3)!}\frac{x^2}{\alpha_i^2}-\frac{210}{(m+4)!}\frac{x^3}{\alpha_i^3}+\frac{210}{(m+5)!}\frac{x^4}{\alpha_i^4}\,, \qquad q=0,\ r=4,$$

$$\frac{2}{(m+1)!}-\frac{30}{(m+2)!}\frac{x}{\alpha_i}+\frac{210}{(m+3)!}\frac{x^2}{\alpha_i^2}-\frac{840}{(m+4)!}\frac{x^3}{\alpha_i^3}+\frac{1890}{(m+5)!}\frac{x^4}{\alpha_i^4}-\frac{1890}{(m+6)!}\frac{x^5}{\alpha_i^5}\,, \qquad q=0,\ r=5,$$

$$\frac{2}{(m+1)!}-\frac{42}{(m+2)!}\frac{x}{\alpha_i}+\frac{420}{(m+3)!}\frac{x^2}{\alpha_i^2}-\frac{2520}{(m+4)!}\frac{x^3}{\alpha_i^3}+\frac{9450}{(m+5)!}\frac{x^4}{\alpha_i^4}-\frac{20790}{(m+6)!}\frac{x^5}{\alpha_i^5}+\frac{20790}{(m+7)!}\frac{x^6}{\alpha_i^6}\,,$$
$$q=0,\ r=6,$$

$$\frac{2}{(m+1)!}-\frac{56}{(m+2)!}\frac{x}{\alpha_i}+\frac{756}{(m+3)!}\frac{x^2}{\alpha_i^2}-\frac{6300}{(m+4)!}\frac{x^3}{\alpha_i^3}+\frac{34650}{(m+5)!}\frac{x^4}{\alpha_i^4}-\frac{124740}{(m+6)!}\frac{x^5}{\alpha_i^5}+$$
$$+\frac{270270}{(m+7)!}\frac{x^6}{\alpha_i^6}-\frac{270270}{(m+8)!}\frac{x^7}{\alpha_i^7}\,, \qquad q=0,\ r=7.$$

(A24)

The coefficients for q=1-7 and r=0 can be obtained respectively from the above-mentioned formulae for q=0 and r=1-7 by replacing α_i by α_j . The expressions for q=1 and r=1-7 are respectively equal to the formulae (A24) for q=0 and r=1-7 plus those obtained by replacing (m+p)! in the expression for q=0 and r=1-7 by (m+p+1)! and by multiplying them by $-x/\alpha_j$. Furthermore, the coefficients for q=2-7 and r=7 can be written by equation (A16) where the expression $(n,7)_1$ is obtained by replacing (m+p)! in $(n-1,7)_1$ by (m+p+1)! and by multiplying the resulting formula by $-(2n-1)x/\alpha_j$, $(0,7)_1$ being the coefficient for q=0 and r=7 shown at the end of (A24).

2. $F_p(\alpha_i, \alpha_j, \xi, \mu, s; d)$

We have obtained the following general expressions for three different parameter ranges $[\beta = \alpha_j(2\xi - 1) - d]$:

(a) $\beta - \alpha_i > 0$;

$$4\alpha_i F_p(\alpha_i, \alpha_j, \xi, \mu, s; d) = \begin{cases} Z_p(\alpha_i, \beta, \mu) - (-1)^p Z_p(-\alpha_i, \beta, \mu), & \mu > 0, \\ 0, & \mu < 0, \end{cases} \tag{A25}$$

(b) $\beta - \alpha_i < 0$ and $\beta + \alpha_i > 0$;

$$4\alpha_i F_p = \begin{cases} Z_p(\alpha_i, \beta, \mu) + T_p(\alpha_i, \beta, \mu), & \mu > 0, \\ (-1)^{p+1}[Z_p(-\alpha_i, \beta, \mu) + T_p(-\alpha_i, \beta, \mu)], & \mu < 0, \end{cases} \tag{A26}$$

(c) $\beta + \alpha_i < 0$;

$$4\alpha_i F_p = \begin{cases} 0, & \mu > 0, \\ -Z_p(\alpha_i, \beta, \mu) + (-1)^p Z_p(-\alpha_i, \beta, \mu), & \mu < 0, \end{cases} \tag{A27}$$

where

$$Z_p(\alpha, \beta, \mu) = (i)^{p+2} \sum_{n=0}^{p} \frac{(p+n)!}{(p-n)!\,(2n)!!} \left(\frac{\mu}{\alpha}\right)^n \exp\left(-\frac{\alpha+\beta}{\mu}\right), \tag{A28}$$

$$T_p(\alpha, \beta, \mu) = (i)^p \sum_{r=0}^{[p/2]} (-1)^r \left(\frac{\mu}{\alpha}\right)^{p-2r} \frac{(2p-2r-1)!!}{(2r)!!} \sum_{m=0}^{p-2r} \frac{1}{m!} \left(-\frac{\beta}{\mu}\right)^m. \tag{A29}$$

The explicit expressions for $Z_p(\alpha, \beta, \mu) \exp[(\alpha+\beta)/\mu]$ with p = 0-7 are as follows:

$$-1, \qquad p=0,$$

$$-i\,(1+\mu/\alpha), \qquad p=1,$$

$$1 + 3\frac{\mu}{\alpha} + 3\left(\frac{\mu}{\alpha}\right)^2, \qquad p=2,$$

$$i\left[1 + 6\frac{\mu}{\alpha} + 15\left(\frac{\mu}{\alpha}\right)^2 + 15\left(\frac{\mu}{\alpha}\right)^3\right], \qquad p=3,$$

$$-\left[1 + 10\frac{\mu}{\alpha} + 45\left(\frac{\mu}{\alpha}\right)^2 + 105\left(\frac{\mu}{\alpha}\right)^3 + 105\left(\frac{\mu}{\alpha}\right)^3\right], \qquad p=4,$$

$$-i\left[1 + 15\frac{\mu}{\alpha} + 105\left(\frac{\mu}{\alpha}\right)^2 + 420\left(\frac{\mu}{\alpha}\right)^3 + 945\left(\frac{\mu}{\alpha}\right)^4 + 945\left(\frac{\mu}{\alpha}\right)^5\right], \qquad p=5,$$

$$1+21\frac{\mu}{\alpha}+210\left(\frac{\mu}{\alpha}\right)^2+1260\left(\frac{\mu}{\alpha}\right)^3+4725\left(\frac{\mu}{\alpha}\right)^4+10395\left(\frac{\mu}{\alpha}\right)^5+10395\left(\frac{\mu}{\alpha}\right)^6, \quad p=6,$$

$$i\left[1+28\frac{\mu}{\alpha}+378\left(\frac{\mu}{\alpha}\right)^2+3150\left(\frac{\mu}{\alpha}\right)^3+17325\left(\frac{\mu}{\alpha}\right)^4+62370\left(\frac{\mu}{\alpha}\right)^5+135135\left(\frac{\mu}{\alpha}\right)^6+\right.$$

$$\left.+135135\left(\frac{\mu}{\alpha}\right)^7\right], \qquad p=7.$$

(A30)

The expressions for $T_p(\alpha,\beta,\mu)$ with p=0-7 are

$$1, \qquad p=0,$$

$$-i\left(\frac{\beta}{\alpha}-\frac{\mu}{\alpha}\right), \qquad p=1,$$

$$\frac{1}{2}-\frac{3}{2}\left(\frac{\beta}{\alpha}\right)^2+3\frac{\beta}{\alpha}\frac{\mu}{\alpha}-3\left(\frac{\mu}{\alpha}\right)^2, \qquad p=2,$$

$$-i\left[\frac{\beta}{\alpha}\left(\frac{3}{2}-\frac{5}{2}\left(\frac{\beta}{\alpha}\right)^2\right)-\left(\frac{3}{2}-\frac{15}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\frac{\mu}{\alpha}-15\frac{\beta}{\alpha}\left(\frac{\mu}{\alpha}\right)^2+15\left(\frac{\mu}{\alpha}\right)^3\right], \qquad p=3,$$

$$\frac{3}{8}-\frac{15}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{35}{8}\left(\frac{\beta}{\alpha}\right)^4+\frac{\beta}{\alpha}\left(\frac{15}{2}-\frac{35}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\frac{\mu}{\alpha}-\left(\frac{15}{2}-\frac{105}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^2-$$

$$-105\frac{\beta}{\alpha}\left(\frac{\mu}{\alpha}\right)^3+105\left(\frac{\mu}{\alpha}\right)^4, \qquad p=4,$$

$$-i\left[\frac{\beta}{\alpha}\left(\frac{15}{8}-\frac{35}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{63}{8}\left(\frac{\beta}{\alpha}\right)^4\right)-\left(\frac{15}{8}-\frac{105}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{315}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{\mu}{\alpha}-\frac{\beta}{\alpha}\left(\frac{105}{2}-\frac{315}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\times\right.$$

$$\left.\times\left(\frac{\mu}{\alpha}\right)^2+\left(\frac{105}{2}-\frac{945}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^3+945\frac{\beta}{\alpha}\left(\frac{\mu}{\alpha}\right)^4-945\left(\frac{\mu}{\alpha}\right)^5\right], \qquad p=5,$$

$$\frac{5}{16}-\frac{105}{16}\left(\frac{\beta}{\alpha}\right)^2+\frac{315}{16}\left(\frac{\beta}{\alpha}\right)^4-\frac{231}{16}\left(\frac{\beta}{\alpha}\right)^6+\frac{\beta}{\alpha}\left(\frac{105}{8}-\frac{315}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{693}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{\mu}{\alpha}-\left(\frac{105}{8}-\frac{945}{4}\left(\frac{\beta}{\alpha}\right)^2+\right.$$

$$\left.+\frac{3465}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\left(\frac{\mu}{\alpha}\right)^2-\frac{\beta}{\alpha}\left(\frac{945}{2}-\frac{3465}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^3+\left(\frac{945}{2}-\frac{10395}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^4+$$

$$+10395\frac{\beta}{\alpha}\left(\frac{\mu}{\alpha}\right)^5-10395\left(\frac{\mu}{\alpha}\right)^6, \qquad p=6,$$

$$-i\left[\frac{\beta}{\alpha}\left(\frac{35}{16}-\frac{315}{16}\left(\frac{\beta}{\alpha}\right)^2+\frac{693}{16}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{16}\left(\frac{\beta}{\alpha}\right)^6\right)-\left(\frac{35}{16}-\frac{945}{16}\left(\frac{\beta}{\alpha}\right)^2+\frac{3465}{16}\left(\frac{\beta}{\alpha}\right)^4-\frac{3003}{16}\left(\frac{\beta}{\alpha}\right)^6\right)\times\right.$$

$$\times\frac{\mu}{\alpha}-\frac{\beta}{\alpha}\left(\frac{945}{8}-\frac{3465}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{9009}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\left(\frac{\mu}{\alpha}\right)^2+\left(\frac{945}{8}-\frac{10395}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{45045}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\left(\frac{\mu}{\alpha}\right)^3+$$

$$+\frac{\beta}{\alpha}\left(\frac{10395}{2}-\frac{45045}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^4-\left(\frac{10395}{2}-\frac{135135}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\left(\frac{\mu}{\alpha}\right)^5-135135\frac{\beta}{\alpha}\left(\frac{\mu}{\alpha}\right)^6+$$

$$\left.+135135\left(\frac{\mu}{\alpha}\right)^7\right], \qquad p=7.$$

(A31)

In the case where the value $(\alpha+|\beta|)/|\mu|$ is small,

$$
Z_p(\alpha,\beta,\mu)+T_p(\alpha,\beta,\mu)=
$$

$$
\begin{aligned}
&-\sum_{q=1}^{\infty}\frac{1}{q!}\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=0,\\
&-i\sum_{q=1}^{\infty}\frac{1}{(q+1)!}\left(q-\frac{\beta}{\alpha}\right)\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=1,\\
&\sum_{q=1}^{\infty}\frac{1}{(q+2)!}\left[q^2-3\frac{\beta}{\alpha}q-1+3\left(\frac{\beta}{\alpha}\right)^2\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=2,\\
&i\sum_{q=1}^{\infty}\frac{1}{(q+3)!}\left[q^3-6\frac{\beta}{\alpha}q^2-\left(4-15\left(\frac{\beta}{\alpha}\right)^2\right)+\frac{\beta}{\alpha}\left(9-15\left(\frac{\beta}{\alpha}\right)^2\right)\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=3,\\
&-\sum_{q=1}^{\infty}\frac{1}{(q+4)!}\left[q^4-10\frac{\beta}{\alpha}q^3-\left(10-45\left(\frac{\beta}{\alpha}\right)^2\right)q^2+\frac{\beta}{\alpha}\left(55-105\left(\frac{\beta}{\alpha}\right)^2\right)q+\right.\\
&\qquad\left.+9-90\left(\frac{\beta}{\alpha}\right)^2+105\left(\frac{\beta}{\alpha}\right)^4\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=4,\\
&-i\sum_{q=1}^{\infty}\frac{1}{(q+5)!}\left[q^5-15\frac{\beta}{\alpha}q^4-\left(20-105\left(\frac{\beta}{\alpha}\right)^2\right)q^3+\left(195-420\left(\frac{\beta}{\alpha}\right)^2\right)q^2+\left(64-735\left(\frac{\beta}{\alpha}\right)^2+\right.\right.\\
&\qquad\left.\left.+945\left(\frac{\beta}{\alpha}\right)^4\right)q-\frac{\beta}{\alpha}\left(225-1050\left(\frac{\beta}{\alpha}\right)^2+945\left(\frac{\beta}{\alpha}\right)^4\right)\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=5,\\
&\sum_{q=1}^{\infty}\frac{1}{(q+6)!}\left[q^6-21\frac{\beta}{\alpha}q^5-\left(35-210\left(\frac{\beta}{\alpha}\right)^2\right)q^4+\frac{\beta}{\alpha}\left(525-1260\left(\frac{\beta}{\alpha}\right)^2\right)q^3+\left(259-\right.\right.\\
&\qquad\left.-3360\left(\frac{\beta}{\alpha}\right)^2+4725\left(\frac{\beta}{\alpha}\right)^4\right)q^2-\frac{\beta}{\alpha}\left(2079-10710\left(\frac{\beta}{\alpha}\right)^2+10395\left(\frac{\beta}{\alpha}\right)^4\right)q-\\
&\qquad\left.-225+4725\left(\frac{\beta}{\alpha}\right)^2-14175\left(\frac{\beta}{\alpha}\right)^4+10395\left(\frac{\beta}{\alpha}\right)^6\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=6,\\
&i\sum_{q=1}^{\infty}\frac{1}{(q+7)!}\left[q^7-28\frac{\beta}{\alpha}q^6-\left(56-378\left(\frac{\beta}{\alpha}\right)^2\right)q^5+\frac{\beta}{\alpha}\left(1190-3150\left(\frac{\beta}{\alpha}\right)^2\right)q^4+\left(784-\right.\right.\\
&\qquad\left.-11340\left(\frac{\beta}{\alpha}\right)^2+17325\left(\frac{\beta}{\alpha}\right)^4\right)q^3-\frac{\beta}{\alpha}\left(10612-59850\left(\frac{\beta}{\alpha}\right)^2+62370\left(\frac{\beta}{\alpha}\right)^4\right)q^2-\\
&\qquad-\left(2304-53487\left(\frac{\beta}{\alpha}\right)^2+173250\left(\frac{\beta}{\alpha}\right)^4-135135\left(\frac{\beta}{\alpha}\right)^6\right)q+\frac{\beta}{\alpha}\left(11025-99225\left(\frac{\beta}{\alpha}\right)^2+\right.\\
&\qquad\left.\left.+218295\left(\frac{\beta}{\alpha}\right)^4-135135\left(\frac{\beta}{\alpha}\right)^6\right)\right]\left(-\frac{\alpha+\beta}{\mu}\right)^q, && p=7.
\end{aligned}
\tag{A32}
$$

The series expansions for the expressions (A25) and (A27) can be obtained by regarding the formula (A32) as the series expansion for the function $Z_p(\alpha,\beta,\mu)$.

3. $G_p(\alpha_i,\alpha_j,\xi,s;d)$

From the expression for $F_p(\alpha_i,\alpha_j,\xi,\mu,s;d)$ shown in the Section 2, we get

(a) $\beta-\alpha_i>0$;

$$4\alpha_i G_p(\alpha_i,\alpha_j,\xi,s;d)=U_p(\alpha_i,\beta)-(-1)^p U_p(-\alpha_i,\beta), \tag{A33}$$

(b) $\beta-\alpha_i<0$ and $\beta+\alpha_i>0$;

$$4\alpha_i G_p=U_p(\alpha_i,\beta)+(-1)^p U_p(\alpha_i,-\beta)+V_p(\alpha_i,\beta), \tag{A34}$$

(c) $\beta+\alpha_i<0$;

$$4\alpha_i G_p=(-1)^p U_p(\alpha_i,-\beta)-U_p(-\alpha_i,-\beta), \tag{A35}$$

where

$$U_p(\alpha,\beta)=(i)^{p+2}\sum_{n=0}^{p}\frac{(2p-n)!}{n!\,(p-n)!\,(p-n+1)!\,(2\alpha)^{p-n}}\Big[\sum_{m=1}^{p-n+1}(p-n+1-m)!\,(-(\alpha+\beta))^{m-1}\times$$
$$\times e^{-(\alpha+\beta)}+(-(\alpha+\beta))^{p-n+1}E_1(\alpha+\beta)\Big], \tag{A36}$$

$$V_p(\alpha,\beta)=(i)^p\sum_{r=0}^{[p/2]}(-1)^r\frac{(2p-2r-1)!!}{(2r)!!\,(\alpha)^{p-2r}}\,2\sum_{q=0}^{[p/2]-r}\frac{(-\beta)^{2q+p-2[p/2]}}{(2q+p-2[p/2])!\,(2[p/2]-2r-2q+1)}. \tag{A37}$$

The explicit expressions for $U_p(\alpha,\beta)$ and $V_p(\alpha,\beta)$, p=0-7, are as follows:

$$U_0(\alpha,\beta)=-e^{-(\alpha+\beta)}+(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_1=-\frac{i}{2}\Big(1-\frac{\beta}{\alpha}+\frac{1}{\alpha}\Big)e^{-(\alpha+\beta)}+\frac{i}{2}\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_2=-\Big[\frac{\beta}{2\alpha}\Big(1-\frac{\beta}{\alpha}\Big)-\Big(1-\frac{\beta}{2\alpha}\Big)\frac{1}{\alpha}-\frac{1}{\alpha^2}\Big]e^{-(\alpha+\beta)}+\frac{\beta}{2\alpha}\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_3=-i\Big[\Big(\frac{1}{8}-\frac{5}{8}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)-\Big(\frac{9}{8}-\frac{5}{4}\frac{\beta}{\alpha}+\frac{5}{8}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\frac{1}{\alpha}-\Big(\frac{15}{4}-\frac{5}{4}\frac{\beta}{\alpha}\Big)\frac{1}{\alpha^2}-\frac{15}{4\alpha^3}\Big]e^{-(\alpha+\beta)}+$$
$$+i\Big(\frac{1}{8}-\frac{5}{8}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_4=-\Big[\frac{\beta}{8\alpha}\Big(3-7\Big(\frac{\beta}{\alpha}\Big)^2\Big)\Big(1-\frac{\beta}{\alpha}\Big)+\Big(1-\frac{11}{8}\frac{\beta}{\alpha}+\frac{7}{4}\Big(\frac{\beta}{\alpha}\Big)^2-\frac{7}{8}\Big(\frac{\beta}{\alpha}\Big)^3\Big)\frac{1}{\alpha}+\Big(8-\frac{21}{4}\frac{\beta}{\alpha}+\frac{7}{4}\Big(\frac{\beta}{\alpha}\Big)^2\Big)\frac{1}{\alpha^2}+$$

$$+\left(21-\frac{21}{4}\frac{\beta}{\alpha}\right)\frac{1}{\alpha^3}+\frac{21}{\alpha^4}\Big]e^{-(\alpha+\beta)}+\frac{\beta}{8\alpha}\left(3-7\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_5=-i\Big[\left(\frac{1}{16}-\frac{7}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{21}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)+\left(\frac{15}{16}-\frac{7}{8}\frac{\beta}{\alpha}+\frac{7}{4}\left(\frac{\beta}{\alpha}\right)^2-\frac{21}{8}\left(\frac{\beta}{\alpha}\right)^3+\frac{21}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{1}{\alpha}+$$

$$+\left(\frac{105}{8}-\frac{91}{8}\frac{\beta}{\alpha}+\frac{63}{8}\left(\frac{\beta}{\alpha}\right)^2-\frac{21}{8}\left(\frac{\beta}{\alpha}\right)^3\right)\frac{1}{\alpha^2}+\left(\frac{525}{8}-\frac{63}{2}\frac{\beta}{\alpha}+\frac{63}{8}\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^3}+\left(\frac{315}{2}-\frac{63}{2}\frac{\beta}{\alpha}\right)\frac{1}{\alpha^4}+$$

$$+\frac{315}{2\alpha^5}\Big]e^{-(\alpha+\beta)}+i\left(\frac{1}{16}-\frac{7}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{21}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_6=-\Big[\frac{\beta}{\alpha}\left(\frac{5}{16}-\frac{15}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)-\left(1-\frac{11}{16}\frac{\beta}{\alpha}+\frac{3}{8}\left(\frac{\beta}{\alpha}\right)^2-\frac{9}{4}\left(\frac{\beta}{\alpha}\right)^3+\frac{33}{8}\left(\frac{\beta}{\alpha}\right)^4-\frac{33}{16}\left(\frac{\beta}{\alpha}\right)^5\right)\frac{1}{\alpha}-$$

$$-\left(19-\frac{141}{8}\frac{\beta}{\alpha}+\frac{135}{8}\left(\frac{\beta}{\alpha}\right)^2-\frac{99}{8}\left(\frac{\beta}{\alpha}\right)^3+\frac{33}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{1}{\alpha^2}-\left(153-\frac{801}{8}\frac{\beta}{\alpha}+\frac{99}{2}\left(\frac{\beta}{\alpha}\right)^2-\frac{99}{8}\left(\frac{\beta}{\alpha}\right)^3\right)\frac{1}{\alpha^3}-$$

$$-\left(648-\frac{495}{2}\frac{\beta}{\alpha}+\frac{99}{2}\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^4}-\left(1485-\frac{495}{2}\frac{\beta}{\alpha}\right)\frac{1}{\alpha^5}-\frac{1485}{\alpha^6}\Big]e^{-(\alpha+\beta)}+\frac{\beta}{\alpha}\left(\frac{5}{16}-\frac{15}{8}\left(\frac{\beta}{\alpha}\right)^2+\right.$$

$$\left.+\frac{33}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)E_1(\alpha+\beta),$$

$$U_7=-i\Big[\left(\frac{5}{128}-\frac{135}{128}\left(\frac{\beta}{\alpha}\right)^2+\frac{495}{128}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{128}\left(\frac{\beta}{\alpha}\right)^6\right)\left(1-\frac{\beta}{\alpha}\right)-\left(\frac{133}{128}-\frac{69}{64}\frac{\beta}{\alpha}+\frac{3}{128}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{32}\left(\frac{\beta}{\alpha}\right)^3+\right.$$

$$\left.+\frac{363}{128}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{64}\left(\frac{\beta}{\alpha}\right)^5+\frac{429}{128}\left(\frac{\beta}{\alpha}\right)^6\right)\frac{1}{\alpha}-\left(\frac{1659}{64}-\frac{1521}{64}\frac{\beta}{\alpha}+\frac{759}{32}\left(\frac{\beta}{\alpha}\right)^2-\frac{825}{32}\left(\frac{\beta}{\alpha}\right)^3+\frac{1287}{64}\left(\frac{\beta}{\alpha}\right)^4-\right.$$

$$\left.-\frac{429}{64}\left(\frac{\beta}{\alpha}\right)^5\right)\frac{1}{\alpha^2}-\left(\frac{19215}{64}-\frac{3663}{16}\frac{\beta}{\alpha}+\frac{5049}{32}\left(\frac{\beta}{\alpha}\right)^2-\frac{1287}{16}\left(\frac{\beta}{\alpha}\right)^3+\frac{1287}{64}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{1}{\alpha^3}-\left(\frac{31185}{16}-\right.$$

$$\left.-\frac{16533}{16}\frac{\beta}{\alpha}+\frac{6435}{16}\left(\frac{\beta}{\alpha}\right)^2-\frac{1287}{16}\left(\frac{\beta}{\alpha}\right)^3\right)\frac{1}{\alpha^4}-\left(\frac{121275}{16}-\frac{19305}{8}\frac{\beta}{\alpha}+\frac{6435}{16}\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^5}-$$

$$-\left(\frac{135135}{8}-\frac{19305}{8}\frac{\beta}{\alpha}\right)\frac{1}{\alpha^6}-\frac{135135}{8\alpha^7}\Big]e^{-(\alpha+\beta)}+\frac{i}{128}\left(5-135\left(\frac{\beta}{\alpha}\right)^2+495\left(\frac{\beta}{\alpha}\right)^4-429\left(\frac{\beta}{\alpha}\right)^6\right)\times$$

$$\times\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)E_1(\alpha+\beta),$$

(A38)

$$V_0(\alpha,\beta)=2,$$

$$V_1=-2i\beta/\alpha,$$

$$V_2=1-3\left(\frac{\beta}{\alpha}\right)^2-\frac{2}{\alpha^2},$$

$$V_3=-i\frac{\beta}{\alpha}\left[3-5\left(\frac{\beta}{\alpha}\right)^2-\frac{10}{\alpha^2}\right],$$

$$V_4=\frac{3}{4}-\frac{15}{2}\left(\frac{\beta}{\alpha}\right)^2+\frac{35}{4}\left(\frac{\beta}{\alpha}\right)^4-\left(5-35\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^2}+\frac{42}{\alpha^4},$$

$$V_5=-i\frac{\beta}{\alpha}\left[\frac{15}{4}-\frac{35}{2}\left(\frac{\beta}{\alpha}\right)^2+\frac{63}{4}\left(\frac{\beta}{\alpha}\right)^4-\left(35-105\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^2}+\frac{378}{\alpha^4}\right],$$

$$V_6=\frac{5}{8}-\frac{105}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{315}{8}\left(\frac{\beta}{\alpha}\right)^4-\frac{231}{8}\left(\frac{\beta}{\alpha}\right)^6-\left(\frac{35}{4}-\frac{315}{2}\left(\frac{\beta}{\alpha}\right)^2+\frac{1155}{4}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{1}{\alpha^2}+$$
$$+\left(189-2079\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^4}-\frac{2970}{\alpha^6},$$
$$V_7=-i\frac{\beta}{\alpha}\left[\frac{35}{8}-\frac{315}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{693}{8}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{8}\left(\frac{\beta}{\alpha}\right)^6-\left(\frac{315}{4}-\frac{1155}{2}\left(\frac{\beta}{\alpha}\right)^2+\frac{3003}{4}\left(\frac{\beta}{\alpha}\right)^4\right)\frac{1}{\alpha^2}+\right.$$
$$\left.+\left(2079-9009\left(\frac{\beta}{\alpha}\right)^2\right)\frac{1}{\alpha^4}-\frac{38610}{\alpha^6}\right]. \qquad \text{(A39)}$$

For small values of $\alpha+|\beta|$, we can obtain the following series expansions for $U_p(\alpha,\beta)$ with p=0-7:

$$U_0(\alpha,\beta)=\alpha(1-\gamma)-(\alpha+\beta)\ln(\alpha+\beta)-\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+1)!}(\alpha+\beta)^{m+1},$$
$$U_1=-i\frac{\beta}{2}-\frac{i}{2}\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+2)!}\left(m+1-\frac{\beta}{\alpha}\right)(\alpha+\beta)^{m+1},$$
$$U_2=\alpha\left(\frac{1}{3}-\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^2\right)-\frac{\beta}{2\alpha}\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+3)!}\left[m^2+\left(2-3\frac{\beta}{\alpha}\right)m-\right.$$
$$\left.-3\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)\right](\alpha+\beta)^{m+1},$$
$$U_3=-i\beta\left(\frac{13}{24}-\frac{5}{8}\left(\frac{\beta}{\alpha}\right)^2\right)-\frac{i}{8}\left(1-5\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+4)!}\times$$
$$\times\left[m^3+\left(3-6\frac{\beta}{\alpha}\right)m^2-\left(1+12\frac{\beta}{\alpha}-15\left(\frac{\beta}{\alpha}\right)^2\right)m-\left(3-15\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)\right](\alpha+\beta)^{m+1},$$
$$U_4=\alpha\left(\frac{2}{15}-\frac{23}{24}\left(\frac{\beta}{\alpha}\right)^2+\frac{7}{8}\left(\frac{\beta}{\alpha}\right)^4\right)-\frac{\beta}{\alpha}\left(\frac{3}{8}-\frac{7}{8}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)-\sum_{m=1}^{\infty}(-1)^m\times$$
$$\times\frac{1}{m(m+5)!}\left[m^4+\left(4-10\frac{\beta}{\alpha}\right)m^3-\left(4+30\frac{\beta}{\alpha}-45\left(\frac{\beta}{\alpha}\right)^2\right)m^2-\left(16-25\frac{\beta}{\alpha}-90\left(\frac{\beta}{\alpha}\right)^2+105\left(\frac{\beta}{\alpha}\right)^3\right)\times\right.$$
$$\left.\times m+\frac{\beta}{\alpha}\left(45-105\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)\right](\alpha+\beta)^{m+1},$$
$$U_5=-i\beta\left(\frac{113}{240}-\frac{7}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{21}{16}\left(\frac{\beta}{\alpha}\right)^4\right)-\frac{i}{16}\left(1-14\left(\frac{\beta}{\alpha}\right)^2+21\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)-$$
$$-i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+6)!}\left[m^5+\left(5-15\frac{\beta}{\alpha}\right)m^4-\left(10+60\frac{\beta}{\alpha}-105\left(\frac{\beta}{\alpha}\right)^2\right)m^3-\left(50-105\frac{\beta}{\alpha}-\right.\right.$$
$$\left.-315\left(\frac{\beta}{\alpha}\right)^2+420\left(\frac{\beta}{\alpha}\right)^3\right)m^2+\left(9+330\frac{\beta}{\alpha}-420\left(\frac{\beta}{\alpha}\right)^2-840\left(\frac{\beta}{\alpha}\right)^3+945\left(\frac{\beta}{\alpha}\right)^4\right)m+$$
$$\left.+45\left(1-14\left(\frac{\beta}{\alpha}\right)^2+21\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)\right](\alpha+\beta)^{m+1},$$
$$U_6=\alpha\left(\frac{8}{105}-\frac{103}{80}\left(\frac{\beta}{\alpha}\right)^2+\frac{13}{4}\left(\frac{\beta}{\alpha}\right)^4-\frac{33}{16}\left(\frac{\beta}{\alpha}\right)^6\right)-\frac{\beta}{\alpha}\left(\frac{5}{16}-\frac{15}{8}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{16}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\times$$
$$\times\ln(\alpha+\beta)+\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+7)!}\left[m^6+\left(6-21\frac{\beta}{\alpha}\right)m^5-\left(20+105\frac{\beta}{\alpha}-210\left(\frac{\beta}{\alpha}\right)^2\right)m^4-\right.$$
$$-\left(120-315\frac{\beta}{\alpha}-840\left(\frac{\beta}{\alpha}\right)^2+1260\left(\frac{\beta}{\alpha}\right)^3\right)m^3+\left(64+1365\frac{\beta}{\alpha}-2100\left(\frac{\beta}{\alpha}\right)^2-3780\left(\frac{\beta}{\alpha}\right)^3+\right.$$

$$+4725\left(\frac{\beta}{\alpha}\right)^4\Big)m^2+\Big(384-609\frac{\beta}{\alpha}-5880\left(\frac{\beta}{\alpha}\right)^2+6930\left(\frac{\beta}{\alpha}\right)^3+9450\left(\frac{\beta}{\alpha}\right)^4-10395\left(\frac{\beta}{\alpha}\right)^5\Big)m-$$

$$-\frac{\beta}{\alpha}\Big(1575-9450\left(\frac{\beta}{\alpha}\right)^2+10395\left(\frac{\beta}{\alpha}\right)^4\Big)\left(1-\frac{\beta}{\alpha}\right)\Big](\alpha+\beta)^{m+1},$$

$$U_7=-i\beta\left(\frac{1873}{4480}-\frac{2039}{640}\left(\frac{\beta}{\alpha}\right)^2+\frac{781}{128}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{128}\left(\frac{\beta}{\alpha}\right)^6\right)-\frac{i}{128}\left(5-135\left(\frac{\beta}{\alpha}\right)^2+495\left(\frac{\beta}{\alpha}\right)^4-429\left(\frac{\beta}{\alpha}\right)^6\right)\times$$

$$\times\left(1-\frac{\beta}{\alpha}\right)(\alpha+\beta)\ln(\alpha+\beta)+i\sum_{m=1}^{\infty}(-1)^m\frac{1}{m(m+8)!}\Big[m^7+\left(7-28\frac{\beta}{\alpha}\right)m^6-\Big(35+168\frac{\beta}{\alpha}-$$

$$-378\left(\frac{\beta}{\alpha}\right)^2\Big)m^5-\Big(245-770\frac{\beta}{\alpha}-1890\left(\frac{\beta}{\alpha}\right)^2+3150\left(\frac{\beta}{\alpha}\right)^3\Big)m^4+\Big(259+4200\frac{\beta}{\alpha}-7560\left(\frac{\beta}{\alpha}\right)^2-$$

$$-12600\left(\frac{\beta}{\alpha}\right)^3+17325\left(\frac{\beta}{\alpha}\right)^4\Big)m^3+\Big(1813-3892\frac{\beta}{\alpha}-30240\left(\frac{\beta}{\alpha}\right)^2+40950\left(\frac{\beta}{\alpha}\right)^3+51975\left(\frac{\beta}{\alpha}\right)^4-$$

$$-62370\left(\frac{\beta}{\alpha}\right)^5\Big)m^2-\Big(225+2079\frac{\beta}{\alpha}-21357\left(\frac{\beta}{\alpha}\right)^2-107100\left(\frac{\beta}{\alpha}\right)^3+121275\left(\frac{\beta}{\alpha}\right)^4+124740\left(\frac{\beta}{\alpha}\right)^5-$$

$$-135135\left(\frac{\beta}{\alpha}\right)^6\Big)m-315\left(5-135\left(\frac{\beta}{\alpha}\right)^2+495\left(\frac{\beta}{\alpha}\right)^4-429\left(\frac{\beta}{\alpha}\right)^6\right)\left(1-\frac{\beta}{\alpha}\right)\Big](\alpha+\beta)^{m+1}. \qquad \text{(A40)}$$

The series expansion for the expression (A34) can easily be obtained from (A40) by regarding $V_p(\alpha,\beta)=0$ for all p.

4. $C_{g\ell}(\alpha_j, s; d, \alpha_1)$

Since

$$C_{g\ell}(\alpha_j, s; d, \alpha_1)=(4/P_j)\int_0^1 d\mu\,\mu\, F_g(\alpha_j,\alpha_1,1,\mu;d),$$

the use of the expression for F_g gives the following from $(\beta=\alpha_1-d)$:

(a) $\alpha_1-\alpha_j>d$;

$$\alpha_j P_j C_{g\ell}(\alpha_j, s; d, \alpha_1)=W_g(\alpha_j,\beta)-(-1)^g W_g(-\alpha_j,\beta), \qquad \text{(A41)}$$

(b) $\alpha_1+\alpha_j>d>\alpha_1-\alpha_j$;

$$\alpha_j P_j C_{g\ell}=W_g(\alpha_j,\beta)+S_g(\alpha_j,\beta), \qquad \text{(A42)}$$

(c) $d>\alpha_1+\alpha_j$; $\qquad \alpha_j P_j C_{g\ell}=0$,

where

$$W_g(\alpha,\beta)=(i)^{g+2}\sum_{m=0}^{g}\frac{(2g-m)!}{m!\,(g-m)!\,(g-m+2)!\,(2\alpha)^{g-m}}\Big[\sum_{p=1}^{g-m+2}(g-m+2-p)!\,(-(\alpha+\beta))^{p-1}\times$$

$$\times\exp(-(\alpha+\beta))+(-(\alpha+\beta))^{g-m+2}E_1(\alpha+\beta)\Big], \qquad \text{(A43)}$$

$$S_q(\alpha,\beta)=(i)^q\sum_{r=0}^{[q/2]}(-1)^r\frac{(2q-2r-1)!!}{(2r)!!\,(\alpha)^{q-2r}}\sum_{m=0}^{q-2r}\frac{1}{m!\,(q-m+2-2r)}(-\beta)^m. \tag{A44}$$

The expression $2W_r(\alpha,\beta)/(i)^{r+2}$ is the same as that shown in (A12) with $x=-(\alpha+\beta)$ and $\alpha_i=-\alpha$. The explicit expressions for $S_q(\alpha,\beta)$ with q=0-7 are

$$S_0(\alpha,\beta)=1/2,$$

$$S_1=\frac{i}{\alpha}\left(\frac{1}{3}-\frac{\beta}{2}\right),$$

$$S_2=\frac{1}{4}-\frac{1}{\alpha^2}\left(\frac{3}{4}-\beta+\frac{3}{4}\beta^2\right),$$

$$S_3=\frac{i}{\alpha}\left(\frac{1}{2}-\frac{3}{4}\beta\right)-\frac{i}{\alpha^3}\left(3-\frac{15}{4}\beta+\frac{5}{2}\beta^2-\frac{5}{4}\beta^3\right),$$

$$S_4=\frac{3}{16}-\frac{1}{\alpha^2}\left(\frac{15}{8}-\frac{5}{2}\beta+\frac{15}{8}\beta^2\right)+\frac{1}{\alpha^4}\left(\frac{35}{2}-21\beta+\frac{105}{8}\beta^2-\frac{35}{6}\beta^3+\frac{35}{16}\beta^4\right),$$

$$S_5=\frac{i}{\alpha}\left(\frac{5}{8}-\frac{15}{16}\beta\right)-\frac{i}{\alpha^3}\left(\frac{21}{2}-\frac{105}{8}\beta+\frac{35}{4}\beta^2-\frac{35}{8}\beta^3\right)+$$
$$+\frac{i}{\alpha^5}\left(135-\frac{315}{2}\beta+\frac{189}{2}\beta^2-\frac{315}{8}\beta^3+\frac{105}{8}\beta^4-\frac{63}{16}\beta^5\right),$$

$$S_6=\frac{5}{32}-\frac{1}{\alpha^2}\left(\frac{105}{32}-\frac{35}{8}\beta+\frac{105}{32}\beta^2\right)+\frac{1}{\alpha^4}\left(\frac{315}{4}-\frac{189}{2}\beta+\frac{945}{16}\beta^2-\frac{105}{4}\beta^3+\frac{315}{32}\beta^4\right)-$$
$$-\frac{1}{\alpha^6}\left(\frac{10395}{8}-1485\beta+\frac{3465}{4}\beta^2-\frac{693}{2}\beta^3+\frac{3465}{32}\beta^4-\frac{231}{8}\beta^5+\frac{231}{32}\beta^6\right),$$

$$S_7=\frac{i}{\alpha}\left(\frac{35}{48}-\frac{35}{32}\beta\right)-\frac{i}{\alpha^3}\left(\frac{189}{8}-\frac{945}{32}\beta+\frac{315}{16}\beta^2-\frac{315}{32}\beta^3\right)+\frac{i}{\alpha^5}\left(\frac{1485}{2}-\frac{3465}{4}\beta+\frac{2079}{4}\beta^2-\right.$$
$$\left.-\frac{3465}{16}\beta^3+\frac{1155}{16}\beta^4-\frac{693}{32}\beta^5\right)-\frac{i}{\alpha^7}\left(15015-\frac{135135}{8}\beta+\frac{19305}{2}\beta^2-\frac{15015}{4}\beta^3+\right.$$
$$\left.+\frac{9009}{8}\beta^4-\frac{9009}{32}\beta^5+\frac{1001}{16}\beta^6-\frac{429}{32}\beta^7\right). \tag{A45}$$

For small values of $\alpha+|\beta|$, the series expansions for $2[W_q(\alpha,\beta)+S_q(\alpha,\beta)]\times$ $\times 1/[(i)^q(\alpha+\beta)]$ with q=0-7 can be obtained as follows:

$$2+(\alpha+\beta)[\gamma+\ln(\alpha+\beta)-\frac{3}{2}]-\sum_{n=2}^{\infty}\frac{2}{(n-1)(n+1)!}(-(\alpha+\beta))^n, \qquad q=0,$$

$$1-\frac{\beta}{\alpha}+(\alpha+\beta)\left[\left(1-\frac{1}{3}\left(1+\frac{\beta}{\alpha}\right)\right)(\gamma+\ln(\alpha+\beta))-\frac{3}{2}+\frac{11}{18}\left(1+\frac{\beta}{\alpha}\right)\right]-$$
$$-\sum_{n=2}^{\infty}\frac{2}{(n-1)(n+2)!}\left[n+2-\left(1+\frac{\beta}{\alpha}\right)\right](-(\alpha+\beta))^n, \qquad q=1,$$

$$-\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)+\frac{1}{4}\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)(\gamma+\ln(\alpha+\beta))-\left\{\sum_{n=2}^{\infty}\frac{1}{4(n-1)(n-1)!}\left(1-\frac{\beta}{\alpha}\right)^2-\right.$$

$$-\sum_{n=1}^{\infty}\left[f_1(n)-\left(1+\frac{\beta}{\alpha}\right)f_2(n)+\frac{1}{4}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)\right]\Big\}(-(\alpha+\beta))^n, \qquad q=2,$$

$$-\left(\frac{1}{4}-\frac{5}{4}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)-\frac{1}{4}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)(\gamma+\ln(\alpha+\beta))+\Big\{\sum_{n=2}^{\infty}\frac{1}{4(n-1)(n-1)!}\frac{\beta}{\alpha}\left(1-\frac{\beta}{\alpha}\right)^2+$$
$$+\sum_{n=1}^{\infty}\left[f_1(n)-2\left(1+\frac{\beta}{\alpha}\right)f_2(n)+\frac{5}{4}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)-\frac{1}{4}\left(1+\frac{\beta}{\alpha}\right)^3 f_4(n)\right]\Big\}(-(\alpha+\beta))^n, \qquad q=3,$$

$$\frac{\beta}{\alpha}\left(\frac{3}{4}-\frac{7}{4}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)-\left(\frac{1}{24}-\frac{7}{24}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)(\gamma+\ln(\alpha+\beta))+\Big\{\sum_{n=2}^{\infty}\frac{1}{(n-1)(n-1)!}\times$$
$$\times\left(\frac{1}{24}-\frac{7}{24}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)^2+\sum_{n=1}^{\infty}\left[f_1(n)-\frac{10}{3}\left(1+\frac{\beta}{\alpha}\right)f_2(n)+\frac{15}{4}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)-\right.$$
$$\left.-\frac{7}{4}\left(1+\frac{\beta}{\alpha}\right)^3 f_4(n)+\frac{7}{24}\left(1+\frac{\beta}{\alpha}\right)^4 f_5(n)\right]\Big\}(-(\alpha+\beta))^n, \qquad q=4,$$

$$\left(\frac{1}{8}-\frac{7}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{21}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha}\left(\frac{1}{8}-\frac{3}{8}\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)(\gamma+\ln(\alpha+\beta))-\Big\{\sum_{n=2}^{\infty}\frac{1}{8(n-1)(n-1)!}\times$$
$$\times\frac{\beta}{\alpha}\left(1-3\left(\frac{\beta}{\alpha}\right)^2\right)\left(1-\frac{\beta}{\alpha}\right)^2-\sum_{n=1}^{\infty}\left[f_1(n)-5\left(1+\frac{\beta}{\alpha}\right)f_2(n)+\frac{35}{4}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)-7\left(1+\frac{\beta}{\alpha}\right)^3 f_4(n)+\right.$$
$$\left.+\frac{21}{8}\left(1+\frac{\beta}{\alpha}\right)^4 f_5(n)-\frac{3}{8}\left(1+\frac{\beta}{\alpha}\right)^5 f_6(n)\right]\Big\}(-(\alpha+\beta))^n, \qquad q=5,$$

$$-\frac{\beta}{\alpha}\left(\frac{5}{8}-\frac{15}{4}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{8}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)+\left(\frac{1}{64}-\frac{9}{32}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{64}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)(\gamma+\ln(\alpha+\beta))-$$
$$-\Big\{\sum_{n=2}^{\infty}\frac{1}{(n-1)(n-1)!}\left(\frac{1}{64}-\frac{9}{32}\left(\frac{\beta}{\alpha}\right)^2+\frac{33}{64}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)^2-\sum_{n=1}^{\infty}\left[f_1(n)-7\left(1+\frac{\beta}{\alpha}\right)f_2(n)-\frac{35}{2}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)-\right.$$
$$\left.-21\left(1+\frac{\beta}{\alpha}\right)^3 f_4(n)+\frac{105}{8}\left(1+\frac{\beta}{\alpha}\right)^4 f_5(n)-\frac{33}{8}\left(1+\frac{\beta}{\alpha}\right)^5 f_6(n)+\frac{33}{64}\left(1+\frac{\beta}{\alpha}\right)^6 f_7(n)\right]\Big\}(-(\alpha+\beta))^n,$$
$$q=6,$$

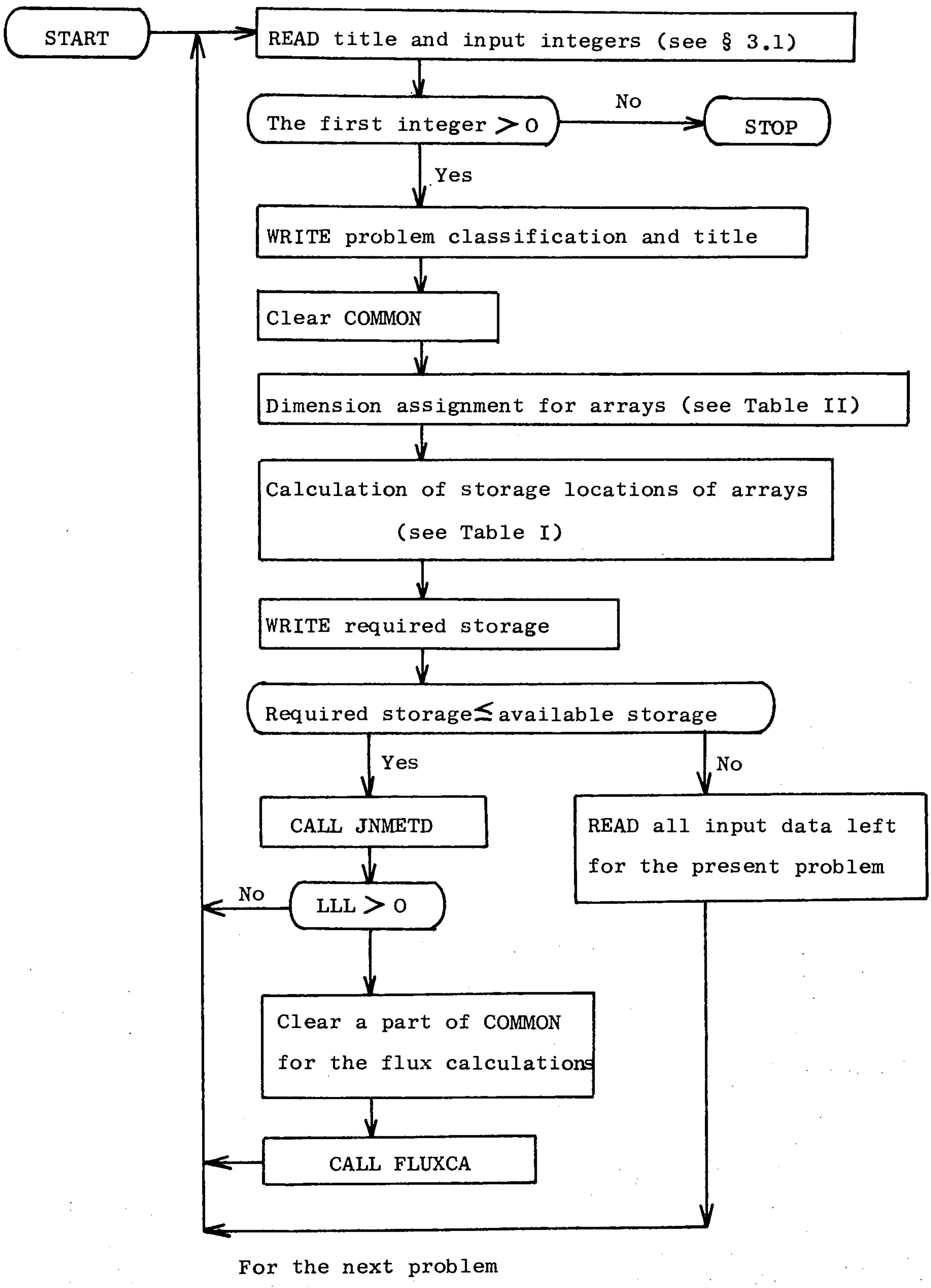
$$-\left(\frac{5}{64}-\frac{135}{64}\left(\frac{\beta}{\alpha}\right)^2+\frac{495}{64}\left(\frac{\beta}{\alpha}\right)^4-\frac{429}{64}\left(\frac{\beta}{\alpha}\right)^6\right)\left(1-\frac{\beta}{\alpha}\right)-\frac{\beta}{\alpha}\left(\frac{5}{64}-\frac{55}{96}\left(\frac{\beta}{\alpha}\right)^2+\frac{143}{192}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)^2(\alpha+\beta)\times$$
$$\times(\gamma+\ln(\alpha+\beta))+\Big\{\sum_{n=2}^{\infty}\frac{1}{(n-1)(n-1)!}\frac{\beta}{\alpha}\left(\frac{5}{64}-\frac{55}{96}\left(\frac{\beta}{\alpha}\right)^2+\frac{143}{192}\left(\frac{\beta}{\alpha}\right)^4\right)\left(1-\frac{\beta}{\alpha}\right)^2+\sum_{n=1}^{\infty}\left[f_1(n)-\right.$$
$$-\frac{28}{3}\left(1+\frac{\beta}{\alpha}\right)f_2(n)+\frac{63}{2}\left(1+\frac{\beta}{\alpha}\right)^2 f_3(n)-\frac{105}{2}\left(1+\frac{\beta}{\alpha}\right)^3 f_4(n)+\frac{385}{8}\left(1+\frac{\beta}{\alpha}\right)^4 f_5(n)-\frac{99}{4}\left(1+\frac{\beta}{\alpha}\right)^5 f_6(n)+$$
$$\left.+\frac{429}{64}\left(1+\frac{\beta}{\alpha}\right)^6 f_7(n)-\frac{143}{192}\left(1+\frac{\beta}{\alpha}\right)^7 f_8(n)\right]\Big\}(-(\alpha+\beta))^n, \qquad \text{(A46)}$$

where $f_m(n)=f_{m-1}(n)+m!/(n+m)!$ and $f_1(n)=(n+2)/(n+1)!$.

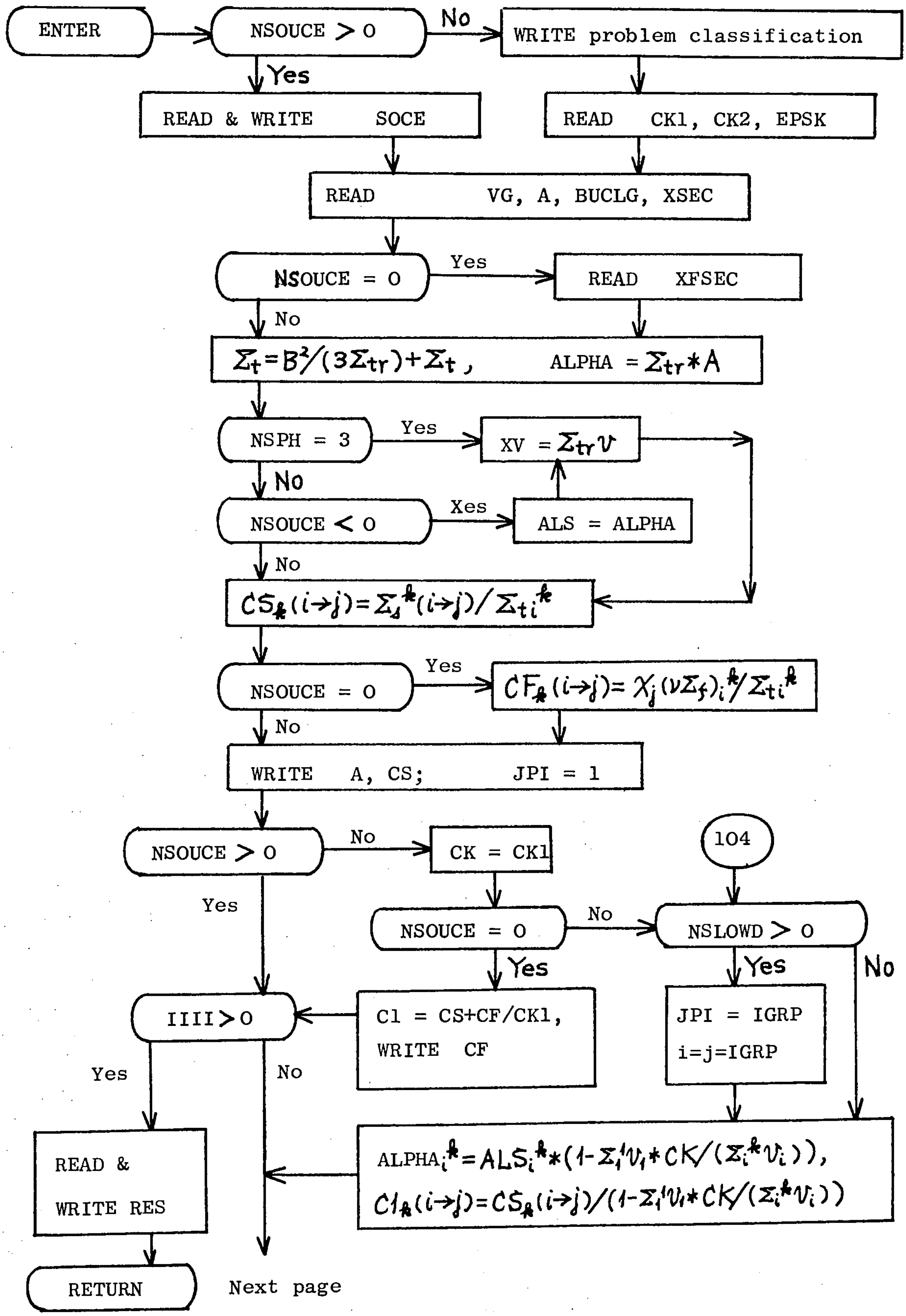
The series expansion for the expression (A41) can be obtained by regarding the formulae (A46) as the series expansions for $2W_q(\alpha,\beta)/[(i)^q(\alpha+\beta)]$ with q=0-7.

Appendix 2. Flow Diagrams of Computer Programmes

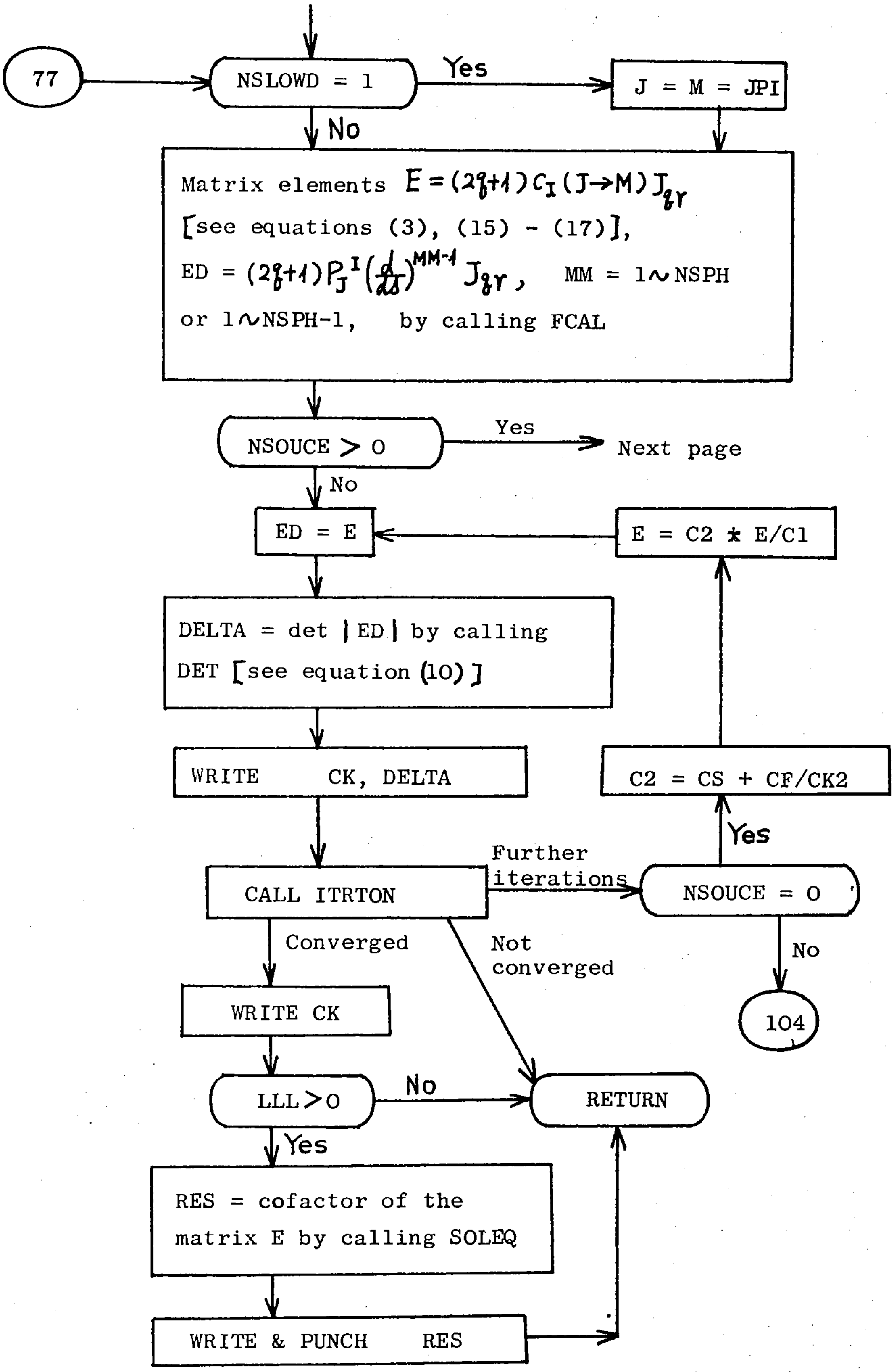
1. MAIN

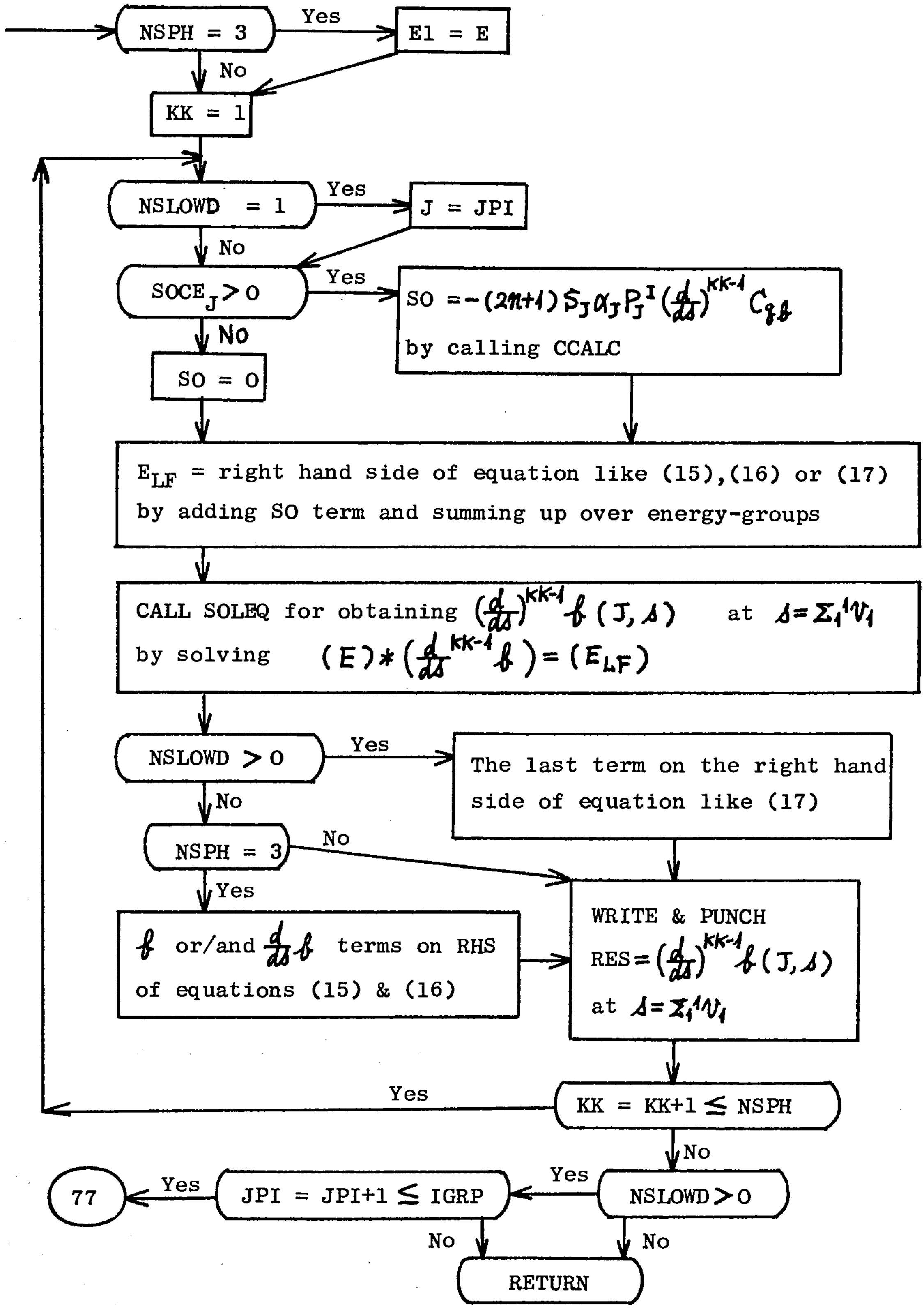


2. JNMETD

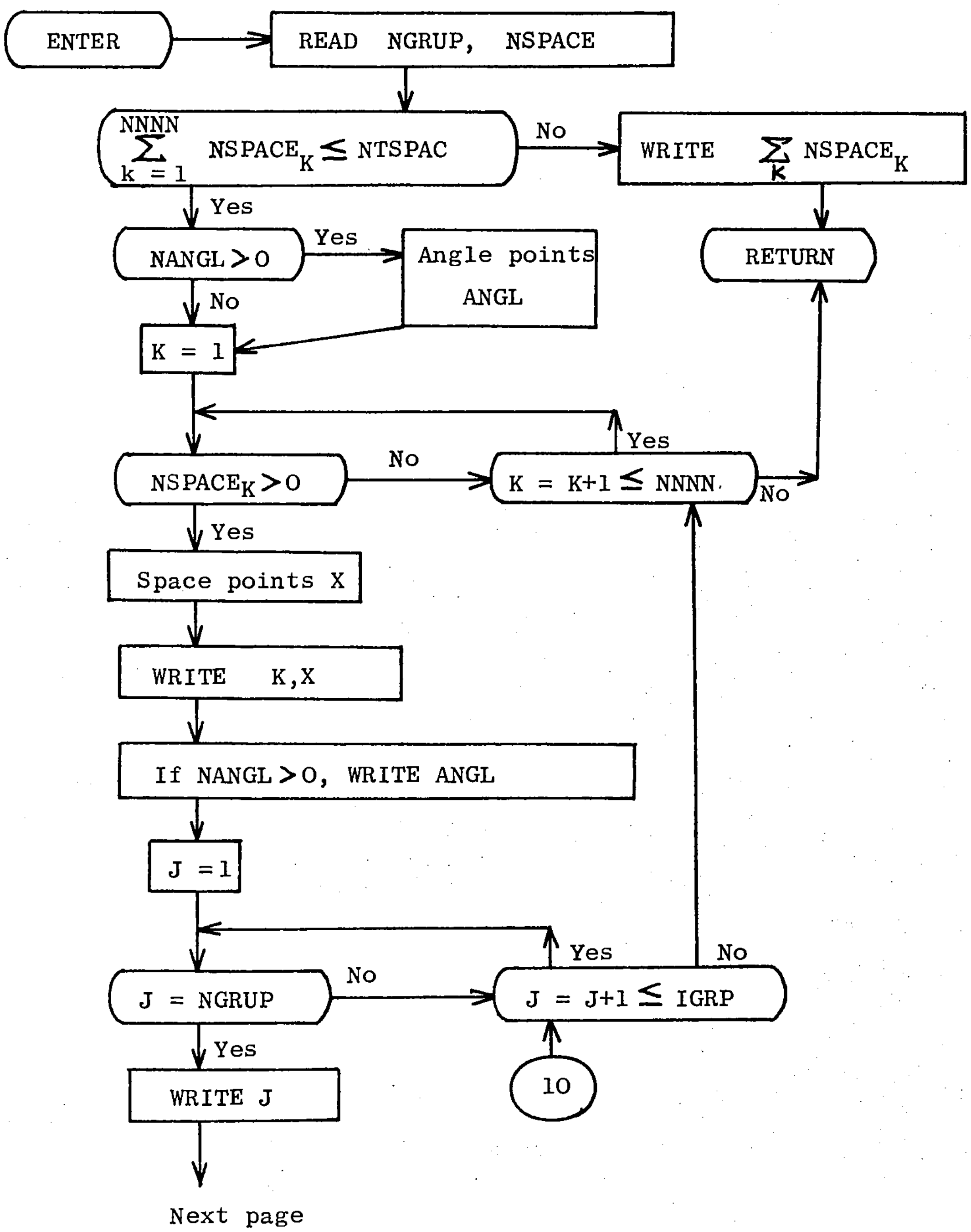


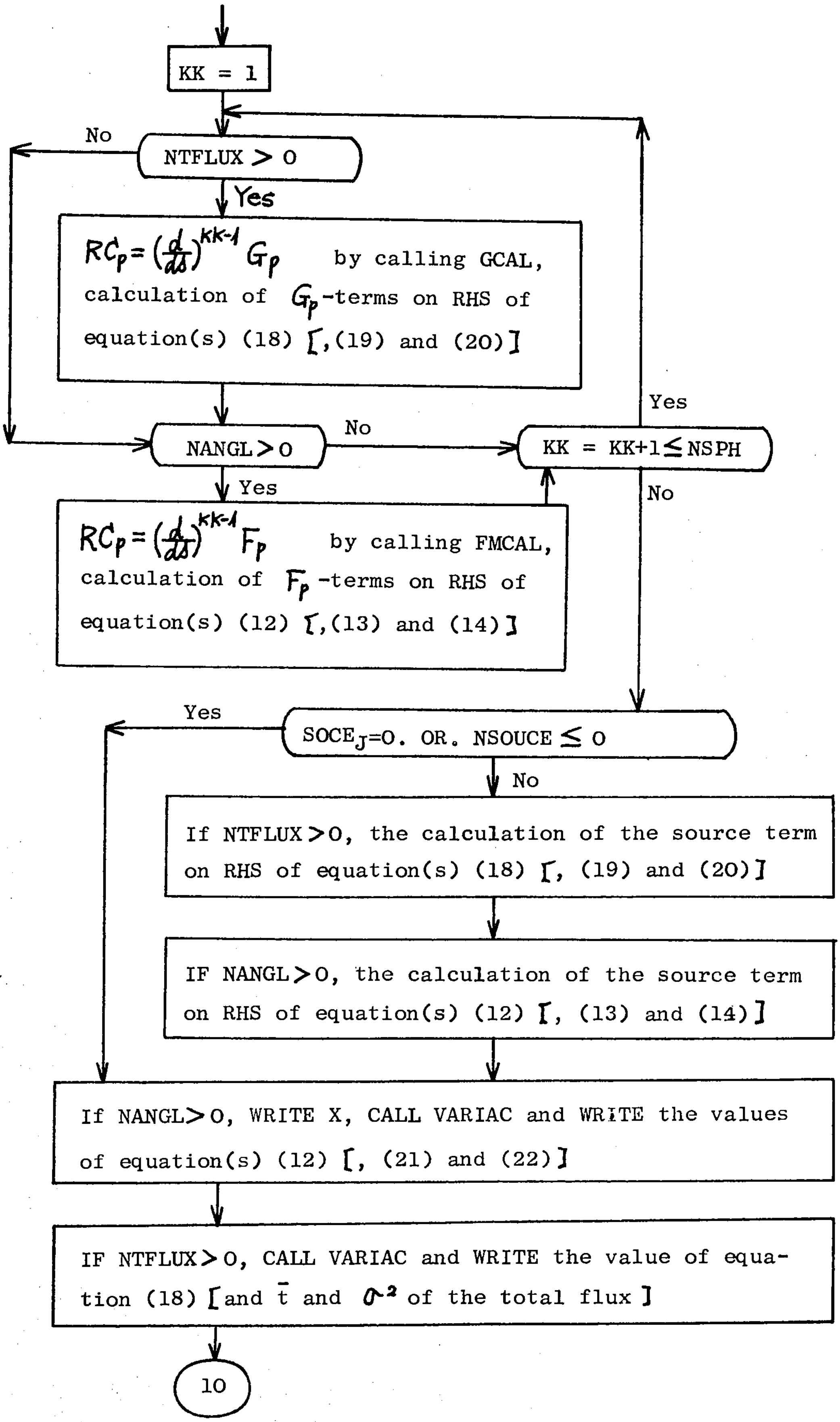
Next page


77
NSLOWD = 1
Yes
J = M = JPI
No
Matrix elements $E = (2\ell+1)C_I(J \to M) J_{\ell r}$
[see equations (3), (15) - (17)],
ED = $(2\ell+1)P_J^I\left(\frac{d}{dz}\right)^{MM-1} J_{\ell r}$, MM = 1∼NSPH
or 1∼NSPH-1, by calling FCAL
NSOUCE > 0
Yes
Next page
No
ED = E
E = C2 * E/C1
DELTA = det |ED| by calling
DET [see equation (10)]
WRITE CK, DELTA
C2 = CS + CF/CK2
Yes
Further iterations
CALL ITRTON
NSOUCE = 0
Converged
Not converged
No
WRITE CK
104
LLL>0
No
RETURN
Yes
RES = cofactor of the
matrix E by calling SOLEQ
WRITE & PUNCH RES


NSPH = 3
Yes
E1 = E
No
KK = 1
NSLOWD = 1
Yes
J = JPI
No
$SOCE_J > 0$
Yes
$SO = -(2n+1) S_J \alpha_J P_J^I (\frac{d}{ds})^{KK-1} C_{gb}$
by calling CCALC
No
SO = 0
E_{LF} = right hand side of equation like (15),(16) or (17)
by adding SO term and summing up over energy-groups
CALL SOLEQ for obtaining $(\frac{d}{ds})^{KK-1} b(J,s)$ at $s=\Sigma_1^1 V_1$
by solving $(E)*(\frac{d}{ds}^{KK-1} b) = (E_{LF})$
NSLOWD > 0
Yes
The last term on the right hand side of equation like (17)
No
NSPH = 3
No
Yes
b or/and $\frac{d}{ds}b$ terms on RHS of equations (15) & (16)
WRITE & PUNCH
$RES = (\frac{d}{ds})^{KK-1} b(J,s)$
at $s=\Sigma_1^1 V_1$
Yes
KK = KK+1 ≤ NSPH
No
77
Yes
JPI = JPI+1 ≤ IGRP
Yes
NSLOWD > 0
No
No
RETURN

3. FLUXCA


ENTER
READ NGRUP, NSPACE
NNNN
k = 1
NSPACE_K ≤ NTSPAC
No
WRITE Σ NSPACE_K
K
Yes
NANGL > 0
Yes
Angle points
ANGL
RETURN
No
K = 1
Yes
No
NSPACE_K > 0
K = K+1 ≤ NNNN
No
Yes
Space points X
WRITE K,X
If NANGL > 0, WRITE ANGL
J = 1
Yes
No
No
J = NGRUP
J = J+1 ≤ IGRP
Yes
WRITE J
10
Next page


KK = 1
NTFLUX > 0
No
Yes
$RC_p = \left(\frac{d}{ds}\right)^{KK-1} G_p$ by calling GCAL, calculation of G_p-terms on RHS of equation(s) (18) [,(19) and (20)]
NANGL > 0
No
Yes
KK = KK+1 ≤ NSPH
Yes
No
$RC_p = \left(\frac{d}{ds}\right)^{KK-1} F_p$ by calling FMCAL, calculation of F_p-terms on RHS of equation(s) (12) [,(13) and (14)]
SOCE$_J$=0. OR. NSOUCE ≤ 0
Yes
No
If NTFLUX > 0, the calculation of the source term on RHS of equation(s) (18) [, (19) and (20)]
IF NANGL > 0, the calculation of the source term on RHS of equation(s) (12) [, (13) and (14)]
If NANGL > 0, WRITE X, CALL VARIAC and WRITE the values of equation(s) (12) [, (21) and (22)]
IF NTFLUX > 0, CALL VARIAC and WRITE the value of equation (18) [and $\bar{t}$ and σ^2 of the total flux]
10

1. CALCULATION OF THE VALUE OF k_{eff}, ANGULAR AND TOTAL FLUX IN THE SECOND SLAB OF A 2-REGION SLAB REACTOR BY THE USE OF A 1 ENERGY-GROUP MODEL AND THE j_5 APPROXIMATION

```
                                     20                  40                  60                  80
                       I--------I---------I---------I---------I---------I---------I---------I--------I
Title                   TEST CASE 1      1-GROUP  (IIII=C,NSOUCE=C,NSLOWD=0)
                         5             1  3  4  4  1  2  1 14  1  1  6  3
k_eff guess            1.1       1.2       .00001
X                      1.
           {           .5
1st slab   {
           {           .1       1.       1.       .9
           {           .8
           {           .5
2nd slab   {
           {           .1       1.       1.       .9
           {           .8
                         1
1st group flux at           6
6 space points
                       I--------I---------I---------I---------I---------I---------I---------I--------I
                                     20                  40                  60                  80
```

2. CALCULATION OF THE FUNDAMENTAL DECAY CONSTANT OF A 1-REGION POLYETHYLENE SLAB BY THE USE OF A 7-GROUP MODEL AND THE j_7 APPROXIMATION

	1		20		40		60		80
Title	TEST CASE 2	POLYETHYLENE	(7-GROUP,IIII=0,NSOUCE=-1,NSLOWD=1)						
	7 -1 1 7 3 4 9 1 1 13								
Guess $1-\delta_1$	.000036	.000039	.000001						
Velocity	285.	171.2	82.24	18.45	2.118	.2402	.024484		
Thickness	7.								
Buckling	.03	.048	.048	.026	.015	.007	.007		
Cross section		.0970411	.0970411	-.0425304					
		.154504	.154504	-.0423623	.104429				
		.313406	.313406	.0641604	.177199	.0319487			
		.659001	.659001	.2904332	.241389	.0193013	.0031948		
	.00031222	.8652964	.8652964	.6005498	.3670025	.0078566	.000366		
	.0024435	.918663	.918883	.6059567	.2626097	.0015653			
	.024043	3.16585	3.16585	3.141807	.31048285	.0018247			

20 40 60 80

3. CALCULATION OF THE 7TH GROUP STATIONARY ANGULAR FLUX IN A 1-REGION WATER SLAB DUE TO A POINT-ISOTROPIC BOUNDARY SOURCE AND THE 1ST AND 2ND TIME MOMENT OF THE FLUX RESULTING FROM THE $\delta(t)$-SOURCE BY THE USE OF A 7-GROUP MODEL AND THE j_5 APPROXIMATION (USING THE PREVIOUSLY OBTAINED CARDS FOR THE RESIDUES)

```
                                20                  40                  60                  80
              I--------I--------I---------I---------I---------I---------I---------I---------I
Title          TEST CASE 3   WATER  (7-GROUP,IIII=1,NSOUCE=1,NSLOWC=1,NSPH=3)
                5  1  1  1  7  3  4  9  3  1  1 57  1      6  3
Source        .10757    .36278    .50403    .02559    .00003
Velocity      285.      171.2     82.24     18.45     2.118     .2402     .024484
Thickness     4.
Buckling      .009      .012      .016      .018      .019      .019      .021
             .00133721.08457866.08457866-.0305552
                       .1277501 .1277501 -.0377562.08437803
                       .2766694 .2766694 .06970756.14904233.02674421
Cross section          .51975166.51975166.21350756.20038489.0161576 .00267442
             .00026137.70096002.70096002.48070889.30493378.00657692.00030642
             .00204551.74581862.74581862.48544689.21846222.00131032
             .01947    2.11599  2.11599  2.09652  .25832622.00152754
             -0.10909201D-01 0.45695801D-02 0.12926113D-02-0.39436361D-03-0.18505205D-03
              0.88904446D-04
              0.12910933D-03 0.83106157D-04 0.37664988D-04-0.15238151D-04-0.63397172D-05
              0.35434387D-05
             -0.28421043D-05-0.36969023D-05-0.39032254D-06 0.55992029D-07-0.26803924D-08
             -0.51706190D-08
             -0.16677956D-01 0.13059422D-01 0.43515942D-02-0.16074608D-02-0.68481564D-03
Residues      0.37208512D-03
```

(to be continued)

Residues

```
 0.48269297D-03 0.16334716D-03 0.16234126D-03-0.72526986D-04-0.32907333D-04
 0.17619953D-04
-0.19397711D-04-0.16838857D-04-0.36648367D-05 0.46603169D-06 0.19090463D-08
-0.21212131D-07
 0.34092106D 00-0.21248054D 00-0.63002660D-01 0.23286401D-01 0.92503666D-02
-0.53463441D-02
-0.89247004D-02-0.18090460D-02-0.27848352D-02 0.11268896D-02 0.55565370D-03
-0.26589017D-03
 0.47438904D-03 0.20967298D-03 0.10353187D-03-0.65956566D-05-0.21421851D-05
-0.80625378D-06
 0.65451502D 00-0.41327290D 00-0.53662441D-01 0.20149547D-01-0.14563671D-03
-0.39893874D-02
-0.48022809D-01 0.38982833D-02-0.14804995D-01 0.37914989D-02 0.19454722D-02
-0.57094964D-03
 0.87922999D-02 0.74472006D-05 0.23553678D-02-0.12371200D-03-0.76581096D-04
-0.13140976D-04
 0.73456505D 00-0.24011050D 00 0.22150039D 00-0.61402081D-01-0.39744136D-01
 0.10303566D-01
-0.63552054D 00 0.65111414D-01-0.25645473D 00 0.27272359D-01 0.93633419D-02
 0.16550522D-03
 0.15049764D 01-0.62027031D-01 0.58199240D 00-0.23271933D-01-0.13440584D-01
-0.83535525D-03
 0.42520784D 00-0.73203951D-01 0.19152415D 00-0.32793456D-01-0.15719207D-01
 0.22180937D-02
-0.29711416D 01 0.18522176D 00-0.12976076D 01 0.83874297D-01 0.17868972D-01
 0.35062908D-02
 0.55712041D 02-0.14629734D 01 0.22269807D 02-0.57448172D 00-0.35237243D 00
-0.26708183D-01
 0.28750003D 01-0.12364112D 00 0.20435656D 01-0.84890122D-01 0.11532206D-02
-0.36587037D-02
-0.12114751D 04 0.14517595D 02-0.90100098D 03 0.10746213D 02-0.23658464D 02
 0.99815878D 00
 0.10402283D 07-0.37103109D 04 0.77770955D 06-0.27784134D 04 0.23311126D C5
-0.28543614D 03
```

7th group flux 7

at 6 space points 6

20 40 60 80

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Alfred Nobel

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