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TRANSIENT HEAT CONDUCTION THROUGH HEAT PRODUCING LAYERS

by

H. WUNDT

1972

Joint Nuclear Research Centre
Ispra Establishment-Italy
Technology

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for every given set of geometrical and material data, i.e. independently of the proper temporal integration. The method is therefore particularly suitable for a code.

The advantage of this method over known codes with intermeshed direct integration of the partial differential equation is that it can be coupled with other systems of ordinary differential equations. This is extremely important for nuclear reactor dynamics, where this other system is the neutron kinetics and where temperatures feedback on reactivity. Moreover, if desired, accuracy can be improved by simply taking more series terms, without destroying the previously obtained results, in full contrast to usual difference techniques.

In this paper one- and two-layer problems are treated for plane, cylindrical and spherical geometries and for boundary conditions of the third kind.

A certain amount of mathematics is unavoidable. Heat engineers need only use the summarized set of solutions. On the other hand, mathematicians will find, as a by-product, a number of interesting summation-formulas in an Appendix.

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ABSTRACT

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MOTS-CLÉS

HEAT TRANSFER
TRANSIENTS
INTERFACES
HEAT GENERATION
GEOMETRY
TEMPERATURE
DISTRIBUTION
LAPLACE TRANSFORM
TRANSFER FUNCTIONS
INVERSE FOURIER TRANSFORMATION
BESSEL FUNCTIONS

Index *)

*) Manuscript received on March 14, 1972

Introduction

The problem of calculating transient temperature distributions in adjacent layers, one of which is heat producing, and which may have different material properties, is frequently encountered in technology. An example is the heat transfer through cladded nuclear fuel elements during load changes.

The physical behaviour of such a system is described by the well-known FOURIER equation. We are interested in two kinds of disturbances, namely in temporal changes of the heat source strength and of the ambient temperature with time. These two given time functions serve as inputs in our system, possibly simultaneously.

We assume a preponderant direction of heat conduction towards the cooling surface and neglect the heat conduction in the other two directions in the absence of essential temperature gradients. Under these circumstances, FOURIER's equation contains two independent variables, a spatial coordinate r, and the time t.

The solution of such a partial differential equation offers no difficulty in principle. Nevertheless, finding the analytical solution of the transient problem as a sum of eigenfunctions can become very tedious, if not impossible, for quite arbitrary input functions. On the other hand, a digital computer treatment by intermeshed RUNGE-KUTTA integrations in both variables is always possible but yields a particular solution only, without any information on the structure of general solutions. Finally, an analog computation is not possible without first eliminating one of the independent variables.

This last remark suggests the method to be developed in this paper. Normally, the whole two-dimensional temperature field need not be known, but only the time evolution of the temperature at some selected or space-averaged positions, i.e. a set of functions depending on the time only.

The partial differential equation is LAPLACE-transformed with respect to time. The remaining ordinary differential equation is solved with respect to the space coordinate, treating the LAPLACE variable s as a parameter in the complex domain. Then the space coordinate is eliminated by spatial averaging or by selecting certain points. The inverse transformation is performed by partial fraction development of the transfer functions acting on the two given inputs. The solution is therefore an infinite series of convolution integrals that converges quite rapidly in general.

Each convolution integral involves only one input function as a factor, so that only a few types exist. Following original ideas of PALINSKI [1], they may be generated by "transient temperature complement functions Γ" which obey simple first order ordinary linear differential equations each with an input function as source term, and with coefficients in ascending order derived from the "characteristic equation" of the problem.

Thus, the final form of the solution is the sum of the corresponding stationary solution and an infinite series of these Γ-functions with coefficients calculated from the previous partial fraction development. This form is particularly instructive as it clearly shows the actual transient deviations from the steady state. In every case, the series may be truncated after a few terms, depending on the input perturbations.

Moreover, the computation of these "conditional" solutions (conditional because the auxiliary differential equations for the Γ's must first be solved) can be programmed advantageously. This is why the general problem is treated for all three main geometries simultaneously.
Several non-dimensional parameters allow for the material properties and for the heat transfer coefficient at the boundary.

The structure of our solution is such that accuracy is easily improved by simply taking more summation terms without destroying the previous computation work, in full contrast to what happens when reducing the stepwidth in usual integration methods for boundary value problems.

The theoretical derivation of the final solution might appear rather complex. Nevertheless, all proofs and most intermediate steps may be forgotten afterwards by the user. All the necessary formulas can be digitally programmed so as to ensure convergence. Then, by simply choosing a few appropriate parameters, and by specifying the perturbation inputs, the answers are immediately obtainable.

To make the reader familiar with the procedure, we first treat the one-layer-problem in Part A. Moreover, this important case cannot simply be extracted from the subsequent two-layer solution. The reader can then follow the two-layer-problem in Part B, which is much more complex. We must moreover consider whether the "internal" layer (around the symmetry axis) or the "external" layer is heat producing. For convenience we treat these two problems simultaneously.

The sequence of operations is often very similar so that the explanatory text need not be repeated. The formulas are simply given in sequence with only the equation numbers as source references. We ask the reader to excuse this condensed notation.

Part A

TRANSIENT HEAT CONDUCTION THROUGH ONE HEAT PRODUCING LAYER

This part prepares for the study of the two-layer problem in Part B. The text in Part B is more concise, so in the interests of better understanding it is recommended that Part A should not be passed over.

1. Formulation of the problem

FOURIER's equation of heat conduction through a heat producing layer is

$$\frac{1}{a^2}\frac{\partial \Theta_k}{\partial t} = \frac{\partial^2 \Theta_k}{\partial r^2} + \frac{k-1}{r}\frac{\partial \Theta_k}{\partial r} + \frac{W(t)}{\lambda} \tag{1.1}$$

with $\Theta(r,t)$ - (required) temperature,
$W(t)$ - (given) heat source per unit time and volume,
λ - thermal conductivity,
ρc - heat capacity per unit volume,
$a^2 = \lambda/\rho c$ - thermal diffusivity.

N.B. In this article, W depends upon t but not on r (uniform heat source).

k is the "geometric index" which assumes the values 1,2 or 3 only, according to

$k = 1$ - plane geometry,
$k = 2$ - cylindrical geometry,
$k = 3$ - spherical geometry.

Let $r = 0$ be the symmetry axis of the layer, and $r = R$ its boundary. Then the boundary conditions are

at $r = 0$: Θ_k an even function , (1.2)

$$\text{at } r = R: \quad - \lambda \left.\frac{\partial \Theta_k}{\partial r}\right|_{r=R} = \alpha \left[\Theta_k(R,t) - \Theta_u(t)\right] \quad . \tag{1.3}$$

α is the heat transfer coefficient; $\Theta_u(t)$ is the given ambient temperature.

(1.3) is the boundary condition of the third kind. Nevertheless, the boundary condition of the first kind (given boundary temperature) is the special case $\alpha \to \infty$ of (1.3). Because of the non-vanishing temperature gradient at the boundary (the temperature would otherwise be identically constant), the square bracket in (1.3) must then be zero.

The boundary condition of the second kind (given boundary temperature gradient) is not treated here. This case is so simple that the solution can be found immediately in practice.

As the initial condition, we take the steady state temperature distribution, to be calculated later on.

The two "inputs" of the system are $W(t)$ and $\Theta_u(t)$; they may vary simultaneously. It does not matter that one comes in through the differential equation itself, the other through the boundary condition only.

We transform the equation and the boundary conditions according to LAPLACE with respect to the time:

$$\frac{s}{a^2}\,\vartheta_k = \frac{d^2\vartheta_k}{dr^2} + \frac{k-1}{r}\,\frac{d\vartheta_k}{dr} + \frac{\Delta\tilde{W}(s)}{\lambda} \,, \tag{1.4}$$

$$\vartheta_k(0,s) \text{ even,} \tag{1.5}$$

$$- \lambda \left.\frac{d\vartheta_k}{dr}\right|_{r=R} = \alpha \left[\vartheta_k(R,s) - \vartheta_u(s)\right] . \tag{1.6}$$

ϑ is the transformed temperature, $\Delta\tilde{W}$ the transformed source, and s the complex LAPLACE variable which acts as a parameter only in

(1.4). The equation is ordinary with respect to r.

As no initial value Θ_k^o occurs in (1.4), the initial stationary temperature field is the reference distribution from which temperatures are counted, and $\Delta\tilde{W}$ is the excess above the initial stationary value.

We now make the usual variable transformation

$$x = \frac{\sqrt{s}}{a}\, r \text{ , and } X = \frac{\sqrt{s}}{a}\, R \text{ .} \tag{1.7}$$

The problem may be written:

$$\vartheta_k = \vartheta_k'' + \frac{k-1}{x}\,\vartheta_k' + \frac{P(s)}{s} \tag{1.8}$$

where the dash means derivation with respect to x, and

$$P(s) \equiv a^2\,\frac{\Delta\tilde{W}(s)}{\lambda} = \frac{\Delta\tilde{W}(s)}{\rho c} \text{ .} \tag{1.9}$$

If $q \equiv \frac{\lambda}{\alpha R}$ (dimensionless, reciprocal NUSSELT number), (1.10)

the boundary condition (1.6) at x = X becomes:

$$-\,qX\vartheta_k'(X,s) = \vartheta_k(X,s) - \vartheta_u(s) \text{ .} \tag{1.11}$$

The limit case $\alpha \to \infty$ (boundary condition of first kind) reduces to the case q = 0.

2. The solutions by means of "fundamental functions"

Let $F_k(x)$ and $\Phi_k(x)$ be pairs of (linearly independent) fundamental solutions of reduced eq. (1.8). Then the general solution is

$$\vartheta_k(x,s) = A_{k1}(s)F_k(x) + A_{k2}(s)\Phi_k(x) + \frac{P(s)}{s} \text{ .} \tag{2.1}$$

The functions F_k and Φ_k are taken as

$$\left.\begin{array}{ll} F_1(x) = \cosh x & \Phi_1(x) = \sinh x \quad {}^{*)} \\ F_2(x) = I_0(x) & \Phi_2(x) = K_0(x) \\ F_3(x) = \dfrac{\sinh x}{x} & \Phi_3(x) = \dfrac{\cosh x}{x} \end{array}\right\} \tag{2.2}$$

The F_k are even, the Φ_k are odd or asymmetric functions, respectively. Thus, because of condition (1.5), $A_{k2} = 0$.

The other boundary condition, (1.11), yields

$$A_{k1} = \frac{\vartheta_u(s) - \frac{P(s)}{s}}{qXF_k'(X) + F_k(X)} \tag{2.3}$$

so that the solution assumes the form

$$\vartheta_k(x,s) = \frac{F_k(x)}{\Delta_k(X)}\left[\vartheta_u(s) - \frac{P(s)}{s}\right] + \frac{P(s)}{s} \tag{2.4}$$

with

$$\Delta_k(X) = qXF_k'(X) + F_k(X) \; . \tag{2.5}$$

For reasons that become clear in Part B, we call this important quantity the system discriminant.

(2.4) still represents the complete temperature field in the complex domain. To eliminate the space coordinate x, we choose the following three temperatures of particular interest:

- the mean temperature $\bar{\vartheta}_k(s)$,
- the central temperature $\vartheta_k(0,s)$,
- the boundary temperature $\vartheta_k(X,s)$.

In the last two cases, $F_k(x)$ simply assumes the arguments 0 or X

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*) In Part B, different Φ_k are used for reasons explained below.

respectively, but $\bar{F}_k$ must be defined:

$$\bar{F}_k = \frac{k}{X^k} \int_0^X F_k(x)\, x^{k-1} dx \; . \tag{2.6}$$

For these and only for these fundamental functions:

$$\bar{F}_k = \frac{k}{X} F_k'(X) \; . \tag{2.7}$$

$\bar{F}_k$ is even about the origin.

Hence, the three selected temperatures are

$$\text{"mean"} \quad : \quad \bar{\vartheta}_k(s) = \frac{\frac{k}{X} F_k'(X)}{\Delta_k(X)} \left[\vartheta_u(s) - \frac{P(s)}{s} \right] + \frac{P(s)}{s} \; , \tag{2.8}$$

$$\text{"central"} \quad : \quad \vartheta_k(0,s) = \frac{1}{\Delta_k(X)} \left[\vartheta_u(s) - \frac{P(s)}{s} \right] + \frac{P(s)}{s} \; , \tag{2.9}$$

$$\text{"boundary"} \quad : \quad \vartheta_k(X,s) = \frac{F_k(X)}{\Delta_k(X)} \left[\vartheta_u(s) - \frac{P(s)}{s} \right] + \frac{P(s)}{s} \tag{2.10}$$

3. Partial fraction development of the transfer functions

The fractions involving $\Delta_k(X)$ in (2.8), (2.9) and (2.10) are the transfer functions acting on the input $\vartheta_u(s) - \frac{P(s)}{s}$, i.e. the transfer functions on $\vartheta_u(s)$ and on $\frac{P(s)}{s}$ differ only in sign. In Part B, entirely different transfer functions occur.

It is obvious that the inverse transforms of rather complex transfer functions can only occasionally be found in tables; moreover, there would be no general method to programme them. The straightforward method is to develop the transfer functions into partial fraction series which can be transformed back term by term.

It can easily be checked that all our transfer functions are meromorphic i.e. have only isolated and single poles in the finite domain. These poles are just the zeros of the denominator Δ_k, if the corresponding numerator does not vanish simultaneously. A common zero would eliminate the corresponding X-value as a pole, but this never occurs throughout this article.

This being assumed, and $N_k^{(\varepsilon)}$ being any numerator, the fraction series development may be written:

$$\frac{N_k^{(\varepsilon)}(X)}{\Delta_k(X)} = \sum_{n=1}^{\infty} \frac{\varepsilon_{kn}}{s + \frac{\sigma_{kn}^2}{\nu}} + \varepsilon_{ko}(s) \; . \tag{3.1}$$

where

$$\nu = \frac{R^2}{a^2} \quad \text{so that } X = \sqrt{\nu s} \; . \tag{3.2}$$

ν is the "time constant" of the system.

The poles s_{kn} of every transfer function are at $s = -\frac{\sigma_{kn}^2}{\nu}$, i.e. at $X = \sqrt{-\sigma_{kn}^2} = \pm i\sigma_{kn}$ $(n = 1, 2, \ldots \infty)$, because of their common denominator Δ_k. The coefficients ε_{kn} are the residues at these poles.

The notation σ_{kn}^2 has been chosen because the fundamental functions F_k have no poles on the real axis of X, but only at purely imaginary arguments.

Even if the sum in (3.1) converges, one cannot be sure that it truly represents the function at the left hand side. Only the singularities in each finite domain need be identical. The two functions can still differ by an integral (poleless) function $\varepsilon_{ko}(s)$. We expect this ε_{ko} to vanish in most of our cases, but it is difficult to prove this in general *). A solution obtained by omitting ε_{ko} provisionally must be checked to see if it still satisfies the differential equation as well as the boundary condi-

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*) A necessary but not sufficient condition is $\lim_{X\to\infty} \varepsilon_{ko}(X) = 0$.

tions. If not, as is sometimes the case in Part B, the necessary ε_{ko} can easily be evaluated from these checks.

In order to evaluate the σ_{kn} we must solve the transcendental equation

$$\Delta_k(\pm i\sigma) = \pm\, qi\sigma F_k'(\pm i\sigma) + F_k(\pm i\sigma) = 0 \quad . \tag{3.3}$$

Now, for purely imaginary arguments:

$$\left.\begin{aligned} F_1(\pm ix) &= \cosh ix = \cos x \equiv G_1(x) \\ F_2(\ ix) &= I_0(ix) = J_0(x) \equiv G_2(x) \\ F_3(\pm ix) &= \frac{\sinh ix}{ix} = \frac{\sin x}{x} \equiv G_3(x) \end{aligned}\right\} \tag{3.4}$$

$$\left.\begin{aligned} F_1'(\pm ix) &= \pm \sinh ix = \pm\, i \sin x \equiv \mp\, i\, G_1'(x) \\ F_2'(\pm ix) &= \pm\, I_1(ix) = \pm\, i\, J_1(x) \equiv \mp\, i\, G_2'(x) \\ F_3'(\pm ix) &= \pm\left(\frac{\cosh ix}{ix} - \frac{\sinh ix}{(ix)^2}\right) = \pm\, i\left(-\frac{\cos x}{x} + \frac{\sin x}{x^2}\right) \equiv \\ &\equiv \mp\, i\, G_3'(x) \end{aligned}\right\} \tag{3.5}$$

These formulas define the so-called "modified" functions $G_k(x)$. The relationships can be summarized

$$F_k(\pm ix) = G_k(x) \ ; \quad F_k'(\pm ix) = \mp\, i\, G_k'(x) \ . \tag{3.6}$$

Similar relationships play an important role in Part B.

It is now possible to write the "discriminant equation" (3.3) in the convenient form

$$\boxed{\Delta_k(\pm i\sigma) = q\sigma G_k'(\sigma) + G_k(\sigma) = 0} \ . \Longrightarrow \sigma_{kn} \tag{3.7}$$

Note that the double sign drops out when the modified functions are used. Since both $\sigma G_k'(\sigma)$ and $G_k(\sigma)$ are even functions, $-\sigma_{kn}$ is also a root, if $+\sigma_{kn}$ is a root. Both $\pm\sigma_{kn}$ yield the same pole s_{kn} of the transfer function as only σ_{kn}^2 appears.

For convenience, we give this equation explicitly for the three geometries:

k = 1 (plane geometry)

$$- q \, \sigma \sin \sigma + \cos \sigma = 0 \tag{3.8}$$

k = 2 (cylindrical geometry)

$$- q \, \sigma \, J_1(\sigma) + J_0(\sigma) = 0 \tag{3.9}$$

k = 3 (spherical geometry)

$$- q \, \sigma\left(- \frac{\cos \sigma}{\sigma} + \frac{\sin \sigma}{\sigma^2}\right) + \frac{\sin \sigma}{\sigma} = + q \cos \sigma + (1-q) \frac{\sin \sigma}{\sigma} = 0 \, . \tag{3.10}$$

In all three cases $\sigma = 0$ is not a root.
In order to calculate the residues ε_{kn} at these poles σ_{kn}, we proceed as follows:

Let $N(s)/\Delta(s)$ be any meromorphic function, the poles of which lie at s_n:

$$\frac{N(s)}{\Delta(s)} = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{s-s_n} + \varepsilon_0(s). \tag{3.11}$$

We multiply both sides by $s-s_1$ and note that $\frac{\Delta(s_1)}{N(s_1)} = 0$; hence

$$\frac{1}{\dfrac{\frac{\Delta(s)}{N(s)} - \frac{\Delta(s_1)}{N(s_1)}}{s-s_1}} = \varepsilon_1 + \sum_{n=2}^{\infty} \varepsilon_n \frac{s-s_1}{s-s_n} + \varepsilon_0(s) \cdot (s-s_1) \, . \tag{3.12}$$

In the limit $s \to s_1$, the denominator at the left hand side becomes

comes

$$\frac{d}{ds}\left.\frac{\Delta(s)}{N(s)}\right|_{s=s_1} \quad ; \text{ hence}$$

$$\varepsilon_1 = \frac{1}{\dfrac{d}{ds}\left.\dfrac{\Delta(s)}{N(s)}\right|_{s=s_1}}, \quad \text{and similarly} \quad \varepsilon_n = \frac{1}{\dfrac{d}{ds}\left.\dfrac{\Delta(s)}{N(s)}\right|_{s=s_n}}. \tag{3.13}$$

The derivative is in general

$$\frac{d}{ds}\frac{\Delta(s)}{N(s)} = \frac{\dfrac{d\Delta(s)}{ds}}{N(s)} - \frac{\Delta(s)\,\dfrac{dN(s)}{ds}}{N^2(s)}, \tag{3.14}$$

but as $\Delta(s_1) = 0$,

$$\frac{d}{ds}\left.\frac{\Delta(s)}{N(s)}\right|_{s=s_1} = \left.\frac{\dfrac{d\Delta(s)}{ds}}{N(s)}\right|_{s=s_1}. \tag{3.15}$$

It is essential that $N(s_1) \neq 0$.

Hence

$$\varepsilon_n = \left.\frac{N(s)}{\dfrac{d}{ds}\Delta(s)}\right|_{s=s_n}, \tag{3.16}$$

even if there is an additional integral function $\varepsilon_0(s)$.

As

$$\frac{d\Delta_k}{ds} = \frac{d\Delta_k}{dX}\cdot\frac{dX}{ds} = \frac{d\Delta_k}{dX}\cdot\frac{d}{ds}\sqrt{\nu s} = \frac{d\Delta_k}{dX}\cdot\frac{\nu}{2X}, \tag{3.17}$$

by substituting from (2.5), we get

$$\frac{d\Delta_k}{ds} = \frac{\nu}{2X}\left[(q+1)\,F_k'(X) + qXF_k''(X)\right]. \tag{3.18}$$

As $F_k(X)$ satisfies the reduced differential equation (1.8) at $x = X$

$$F_k''(X) = F_k(X) - \frac{k-1}{X} F_k'(X) \ . \tag{3.19}$$

Thus

$$\frac{d\Delta_k}{ds} = \frac{\nu}{2}\left[\frac{1 + q(2-k)}{X} F_k'(X) + qF_k(X) \right] \ , \tag{3.20}$$

and

$$\left.\frac{d\Delta_k}{ds}\right|_{\substack{s=-\frac{\sigma_{kn}^2}{\nu} \\ (X=\pm i\sigma_{kn})}} = \frac{\nu}{2}\left[- \frac{1 + q(2-k)}{\sigma_{kn}} G_k'(\sigma_{kn}) + qG_k(\sigma_{kn}) \right] . \tag{3.21}$$

By considering the discriminant equation (3.7), this can further be simplified:

$$\left.\frac{d\Delta_k}{ds}\right|_{s=-\frac{\sigma_{kn}^2}{\nu}} = - \frac{\nu}{2} \frac{G_k'(\sigma_{kn})}{\sigma_{kn}} \left[1 + q(2-k) + q^2\sigma^2_{kn}\right] \equiv \Delta_{kn}' \ . \tag{3.22}$$

N.B. This expression has no double sign.

The development coefficients for the various transfer functions in (2.8), (2.9) and (2.10) can now be evaluated by the rule (3.16).

The coefficients for the mean temperature are

$$a_{kn} = \left. \frac{\frac{k}{X} F_k'(X)}{\frac{d}{ds} \Delta_k(X)} \right|_{X=\pm i\sigma_{kn}} =$$

$$= \frac{- \frac{k}{\sigma_{kn}} G_k'(\sigma_{kn})}{-\frac{\nu}{2} \frac{G_k'(\sigma_{kn})}{\sigma_{kn}} \left[1 + q(2-k) + q^2\sigma_{kn}^2\right]} =$$

$$= + \frac{2k}{\nu} \frac{1}{1 + q(2-k) + q^2\sigma_{kn}^2} = - \frac{k}{\sigma_{kn}} \frac{G_k'(\sigma_{kn})}{\Delta_{kn}'} . \qquad (3.23)$$

For the central temperature, we have simply

$$e_{kn} = \frac{1}{\frac{d}{ds} \Delta_k(X)} \Bigg|_{X=\pm i\sigma_{kn}} \qquad ^{*)}$$

$$= - \frac{2}{\nu} \cdot \frac{\sigma_{kn}}{G_k'(\sigma_{kn})} \cdot \frac{1}{1 + q(2-k) + q^2\sigma_{kn}^2} = \frac{1}{\Delta_{kn}'} , \qquad (3.24)$$

and the boundary temperature yields

$$j_{kn} = \frac{F_k(X)}{\frac{d}{ds} \Delta_k(X)} \Bigg|_{X=\pm i\sigma_{kn}} \qquad ^{*)}$$

$$= -\frac{2}{\nu} \cdot \frac{\sigma_{kn}}{G_k'(\sigma_{kn})} \cdot \frac{G_k(\sigma_{kn})}{1 + q(2-k) + q^2\sigma_{kn}^2} = \frac{G_k(\sigma_{kn})}{\Delta_{kn}'}$$

$$= +\frac{2}{\nu} \frac{q\sigma_{kn}^2}{1 + q(2-k) + q^2\sigma_{kn}^2} \quad \text{(by applying 3.7)} . \qquad (3.25)$$

4. Inverse transformation for the selected temperatures

We begin with the mean temperature. Eq. (2.8) may be written

$$s\,\bar{\vartheta}_k(s) = \sum \frac{a_{kn}}{s + \frac{\sigma_{kn}^2}{\nu}} [s\vartheta_u(s) - P(s)] + P(s) . \qquad (4.1)$$

N.B. Henceforward, summations go from n = 1 to n infinite. The omission of $a_{ko}(s)$ must eventually be justified.

According to known rules, the inverse transform of (4.1) is:

$$\dot{\bar{\Theta}}_k(t) = \left[\sum a_{kn}\, e^{-\frac{\sigma_{kn}^2}{\nu} t} \right] * \left[\dot{\Theta}_u(t) - \frac{\Delta W(t)}{\rho c} \right] + \frac{\Delta W(t)}{\rho c} \qquad (4.2)$$

*) The notation e_{kn} and j_{kn} for the coefficients corresponds to the systematic notation of Part B, chapter 8.

where * is the convolution integral operator. Explicitly, one has

$$\dot{\Theta}_k(t) = \sum a_{kn}\, e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t \left[\dot{\Theta}_u(\tau) - \frac{\Delta W(\tau)}{\rho c}\right] e^{+\frac{\sigma_{kn}^2}{\nu}\tau}\, d\tau + \frac{\Delta W(t)}{\rho c} \,. \tag{4.3}$$

Now we define "transient temperature complement functions":

$$e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t \dot{\Theta}_u(\tau)\, e^{+\frac{\sigma_{kn}^2}{\nu}\tau}\, d\tau \equiv \Gamma_{akn}(t) \quad , \; ^{*)} \tag{4.4}$$

$$e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t \frac{\Delta W(\tau)}{\rho c}\, e^{+\frac{\sigma_{kn}^2}{\nu}\tau}\, d\tau \equiv \Gamma_{bkn}(t) \quad , \; ^{*)} \tag{4.5}$$

so that

$$\dot{\Theta}_k(t) = \sum a_{kn}\, [\Gamma_{akn}(t) - \Gamma_{bkn}(t)] + \frac{\Delta W(t)}{\rho c} \,. \tag{4.6}$$

It is easy to see that the transient complement functions obey the equations

$$\boxed{\begin{aligned} \dot{\Gamma}_{akn}(t) + \frac{\sigma_{kn}^2}{\nu}\,\Gamma_{akn}(t) &= \dot{\Theta}_u(t) \qquad (4.7)\\ \dot{\Gamma}_{bkn}(t) + \frac{\sigma_{kn}^2}{\nu}\,\Gamma_{bkn}(t) &= \frac{\Delta W(t)}{\rho c} \qquad (4.8)\end{aligned}}$$

with initial conditions $\Gamma_{akn}(0) = 0$ and $\Gamma_{bkn}(0) = 0$.

These equations may be solved formally for $\Gamma_{akn}(t)$ and $\Gamma_{bkn}(t)$:

$$\Gamma_{akn}(t) = \frac{\nu}{\sigma_{kn}^2}\,[\dot{\Theta}_u(t) - \dot{\Gamma}_{akn}(t)] \quad , \tag{4.9}$$

$$\Gamma_{bkn}(t) = \frac{\nu}{\sigma_{kn}^2}\left[\frac{\Delta W(t)}{\rho c} - \dot{\Gamma}_{bkn}(t)\right] . \tag{4.10}$$

*) Obviously, the splitting of the convolution integral into two separate functions Γ is not essential but accords with the systematic notation of Part B.

Substitution into (4.6) yields:

$$\dot{\Theta}_k(t) = \sum \frac{\nu}{\sigma^2_{kn}} a_{kn} \left[\dot{\Theta}_u(t) - \frac{\Delta W(t)}{\rho c} - \dot{\Gamma}_{akn}(t) + \dot{\Gamma}_{bkn}(t)\right] + \frac{\Delta W(t)}{\rho c} \; . \tag{4.11}$$

As the first two terms in the parenthesis have no index n, these can be taken outside the summation. We therefore need the value of $\sum \frac{\nu}{\sigma^2_{kn}} a_{kn}$; which, by (3.1), can be found from the limits of the transfer functions for $s \to 0$, or $X \to 0$. This corresponds to the asymptotic state $t \to \infty$.

The best way to obtain the limits is to consider the first terms of the TAYLOR expansions of the functions $F_k(X)$ and $F'_k(X)$. Besides $F_k(0) = 1$, one finds $F'_k(+0) \approx + \frac{X}{k}$.

Therefore

$$\lim_{X \to 0} \Delta_k(X) \approx \frac{q}{k} X^2 + 1 \to + 1 \qquad \text{(cf. 2.5)} \; , \tag{4.12}$$

$$\lim_{X \to 0} \bar{F}_k(X) = + 1 \qquad \text{(cf. 2.7)} \; . \tag{4.13}$$

All three transfer functions in (2.8), (2.9) and (2.10) respectively, tend to unity as $s \to 0$.
But so must the partial fraction's sum for $s \to 0$, so that

$$\lim_{s \to 0} \sum \frac{a_{kn}}{s + \frac{\sigma^2_{kn}}{\nu}} = \nu \sum \frac{a_{kn}}{\sigma^2_{kn}} = + 1 \; , \tag{4.14}$$

$$\lim_{s \to 0} \sum \frac{e_{kn}}{s + \frac{\sigma^2_{kn}}{\nu}} = \nu \sum \frac{e_{kn}}{\sigma^2_{kn}} = + 1 \; , \tag{4.15}$$

$$\lim_{s \to 0} \sum \frac{j_{kn}}{s + \frac{\sigma^2_{kn}}{\nu}} = \nu \sum \frac{j_{kn}}{\sigma^2_{kn}} = + 1 \; . \; ^{*)} \tag{4.16}$$

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*) An important result for this summation is given in Appendix III for the case q = 0.

We emphasize that these expressions lead to many interesting sum formulas in special cases given in Appendix III. In some cases, known sum formulas are obtained for a specific q in the discriminant equation for the σ_{kn}.
If the sum in (4.14) is indeed unity, the troublesome additive integral function $a_{ko}(s)$ (cf. 3.1) is zero.

(4.14) is now substituted into (4.11) to yield:

$$\dot{\bar{\Theta}}_k(t) = \dot{\Theta}_u(t) - \nu \sum \frac{a_{kn}}{\sigma_{kn}^2} [\dot{\Gamma}_{akn}(t) - \dot{\Gamma}_{bkn}(t)] . \tag{4.17}$$

On integration:

$$\bar{\Theta}_k(t) = \Theta_u(t) - \nu \sum \frac{a_{kn}}{\sigma_{kn}^2} [\Gamma_{akn}(t) - \Gamma_{bkn}(t)] . \tag{4.18}$$

The constant of integration is zero, since $\bar{\Theta}_k(0) = \Theta_u(0) = 0$, as the temperatures are the excesses over the initial steady values.

Substitution from (4.9) and (4.10):

$$\bar{\Theta}_k(t) = \Theta_u(t) - \nu^2 \sum \frac{a_{kn}}{\sigma_{kn}^4} \left[\dot{\Theta}_u(t) - \frac{\Delta W(t)}{\rho c} - \dot{\Gamma}_{akn}(t) + \dot{\Gamma}_{bkn}(t)\right] . \tag{4.19}$$

If we omit the time derivatives and take constant W_o, (4.19) reduces to the stationary solution. We must therefore evaluate the stationary solution of (1.1) directly to determine the value of $\sum \frac{a_{kn}}{\sigma_{kn}^4}$.

This elementary computation shows that

$$\Theta_{\kappa,stat}(r) = \Theta_u - \frac{W_o r^2}{2k\lambda} + \frac{W_o R^2}{k\lambda} (q + \frac{1}{2}) . \tag{4.20}$$

The average temperature is according to rule (2.6):

$$\bar{\Theta}_{k,stat} = \Theta_u + \frac{W_o R^2}{k\lambda} \left(q + \frac{1}{k+2}\right) . \tag{4.21}$$

As

$$\nu^2 \frac{W_0}{\rho c} = \nu \frac{W_0 R^2}{\lambda} , \tag{4.22}$$

the stationary part of (4.19) is

$$\bar{\Theta}_{k,stat} = \Theta_u + \frac{W_0 R^2}{\lambda} \cdot \nu \sum \frac{a_{kn}}{\sigma_{kn}^4} . \tag{4.23}$$

Comparing of (4.21) and (4.23) yields

$$\nu \sum \frac{a_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left(q + \frac{1}{k+2}\right) . \tag{4.24}$$

This type of formula is very interesting, too; examples are given in Appendix III.

Substituting from (4.24) into (4.19) gives the final form of solution

$$\boxed{\bar{\Theta}_k(t) = \Theta_u(t) + \frac{\nu}{k}\left(q+\frac{1}{k+2}\right) \cdot \left[\frac{W(t)}{\rho c} - \dot{\Theta}_u(t)\right] + \nu^2 \sum \frac{a_{kn}}{\sigma_{kn}^4}\left[\dot{\Gamma}_{akn}(t) - \dot{\Gamma}_{bkn}(t)\right]} \tag{4.25}$$

This form is most instructive; the stationary solution is shown explicitly, and the terms in the summation give the transient complements. The only integrations still to perform are those of eq. (4.7), (4.8), to generate the Γ-functions. It is to be noted that $\Theta_u(t)$ in the first term comes in through the boundary and initial conditions, whereas $-\dot{\Theta}_u(t)$ represents a perturbation input equivalent to $W(t)/\rho c$.

The "central" and "boundary" temperatures are treated correspondingly, with the coefficients a_{kn} replaced by the e_{kn} and j_{kn} respectively. (cf. 3.23, 3.24, 3.25).

We can therefore omit the whole analysis up eq. (4.19) and put

$$\Theta_k(0,t) = \Theta_u(t) - \nu^2 \sum \frac{e_{kn}}{\sigma_{kn}^4} \left[\dot{\Theta}_u(t) - \frac{W(t)}{\rho c} - \dot{\Gamma}_{akn}(t) + \dot{\Gamma}_{bkn}(t)\right], \tag{4.26}$$

$$\Theta_K(R,t) = \Theta_u(t) - \nu^2 \sum \frac{j_{Kn}}{\sigma_{Kn}^4} \left[\dot{\Theta}_u(t) - \frac{W(t)}{\rho c} - \dot{\Gamma}_{akn}(t) + \dot{\Gamma}_{bkn}(t) \right]. \tag{4.27}$$

Note in particular that no new functions Γ appear, because the inputs on which the new transfer functions operate, are still the same, and eqs. (4.7) and (4.8) still apply.

However, the subsequent analysis differs. The direct stationary solutions for the central and the boundary temperatures are respectively:

$$\Theta_{k,stat}(0) = \Theta_u + \frac{W_0 R^2}{k\lambda} \left(q + \frac{1}{2}\right), \tag{4.28}$$

$$\Theta_{k,stat}(R) = \Theta_u + \frac{W_0 R^2}{k\lambda} \cdot q \quad . \tag{4.29}$$

A comparison with the respective stationary parts of (4.26) and (4.27) yields:

$$\nu \sum \frac{e_{Kn}}{\sigma_{Kn}^4} = \frac{1}{k} \left(q + \frac{1}{2}\right), \tag{4.30}$$

$$\nu \sum \frac{j_{Kn}}{\sigma_{Kn}^4} = \frac{q}{k} \quad . \tag{4.31}$$

Hence the final form of the solutions is:

$$\boxed{\Theta_K(0,t) = \Theta_u(t) + \frac{\nu}{k}\left(q + \frac{1}{2}\right)\left[\frac{W(t)}{\rho c} - \dot{\Theta}_u(t)\right] + \nu^2 \sum \frac{e_{Kn}}{\sigma_{Kn}^4} \left[\dot{\Gamma}_{akn}(t) - \dot{\Gamma}_{bkn}(t)\right]}, \tag{4.32}$$

$$\boxed{\Theta_K(R,t) = \Theta_u(t) + \frac{\nu q}{k}\left[\frac{W(t)}{\rho c} - \dot{\Theta}_u(t)\right] + \nu^2 \sum \frac{j_{Kn}}{\sigma_{Kn}^4} \left[\dot{\Gamma}_{akn}(t) - \dot{\Gamma}_{bkn}(t)\right]}. \tag{4.33}$$

In practice, the sums in all three solutions (4.25), (4.32) and (4.33) must be truncated, according to the desired accuracy. This means that whenever an improved accuracy is needed, the previous computations remain valid and utilizable. In contrast, the intermeshed direct integration of FOURIER's equation must be fully repeated when the stepwidth is reduced to improve accuracy.

Part B

TRANSIENT HEAT CONDUCTION THROUGH TWO ADJACENT LAYERS ONE OF WHICH IS HEAT PRODUCING

Though the reasoning in Part B is just the same as in A, a thorough treatment of new specific difficulties is necessary at some points.
We must distinguish between two problems:

(I) internal layer with heat source / external layer without heat source,
(II) internal layer without heat source / external layer with heat source.

5. Formulation of the problem

We now assign quantities with an upper index "$^{(1)}$" to the first problem and those with upper index "$^{(2)}$" to the second. Quantities without such an index pertain to both problems.

We treat both problems simultaneously as they show many common features.

The lower indices i or e mean internal or external layer, respectively.
The general FOURIER equation (1.1) is then split up as follows:

Problem (I):

$$\frac{1}{a_i^2}\frac{\partial\Theta_{ik}^{(1)}}{\partial t} = \frac{\partial^2\Theta_{ik}^{(1)}}{\partial r^2} + \frac{k-1}{r}\frac{\partial\Theta_{ik}^{(1)}}{\partial r} + \frac{W(t)}{\lambda_i}\,, \tag{5.1}$$

$$\frac{1}{a_e^2}\frac{\partial\Theta_{ek}^{(1)}}{\partial t} = \frac{\partial^2\Theta_{ek}^{(1)}}{\partial r^2} + \frac{k-1}{r}\frac{\partial\Theta_{ek}^{(1)}}{\partial r}\,, \tag{5.2}$$

and

Problem (II):

$$\frac{1}{a_i^2}\frac{\partial\Theta_{ik}^{(2)}}{\partial t} = \frac{\partial^2\Theta_{ik}^{(2)}}{\partial r^2} + \frac{k-1}{r}\frac{\partial\Theta_{ik}^{(2)}}{\partial r} , \tag{5.3}$$

$$\frac{1}{a_e^2}\frac{\partial\Theta_{ek}^{(2)}}{\partial t} = \frac{\partial^2\Theta_{ek}^{(2)}}{\partial r^2} + \frac{k-1}{r}\frac{\partial\Theta_{ek}^{(2)}}{\partial r} + \frac{W(t)}{\lambda_e} . \tag{5.4}$$

In order to formulate the boundary conditions, we define

$$R_i = R \qquad R_e = pR \qquad \text{so that } p = \frac{R_e}{R_i} \tag{5.5}$$

where R_i and R_e are the "radii" of the layers, both measured from the system axis. Thus, $p \geqq 1$; e.g. $p = 2$ means equal layer thicknesses.

Then the boundary conditions, applying to both problems, are:

at $r = 0$: $\quad \Theta_{ik}$ an even function (5.6)

at $r = R$: $\quad \Theta_{ik}(R,t) = \Theta_{ek}(R,t)$ (5.7)

$$\lambda_i \left.\frac{\partial\Theta_{ik}}{\partial r}\right|_{r=R} = \lambda_e \left.\frac{\partial\Theta_{ek}}{\partial r}\right|_{r=R} \tag{5.8}$$

at $r = pR$: $\quad - \lambda_e \left.\frac{\partial\Theta_{ek}}{\partial r}\right|_{r=pR} = \alpha\,[\Theta_{ek}(pR,t) - \Theta_u(t)]$. (5.9)

The initial conditions are: start from the steady state.

With the ratios

$$a^2 = \frac{a_i^2}{a_e^2} = \frac{\lambda_i}{\lambda_e}\cdot\frac{(\rho c)_e}{(\rho c)_i} \quad \text{and } \lambda = \frac{\lambda_i}{\lambda_e} , \quad {}^{*)} \tag{5.10}$$

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*) Do not confuse these dimensionless quantities with a^2 and λ of Part A: the latter material properties correspond to a_i and λ_i.

the LAPLACE-transformed equations are (cf. 1.8):

Problem (I):

$$\vartheta_{ik}^{(1)} = \vartheta_{ik}^{(1)\prime\prime} + \frac{k-1}{x}\,\vartheta_{ik}^{(1)\prime} + \frac{P(s)}{s}\,, \tag{5.11}$$

$$a^2\vartheta_{ek}^{(1)} = \vartheta_{ek}^{(1)\prime\prime} + \frac{k-1}{x}\,\vartheta_{ek}^{(1)\prime}, \tag{5.12}$$

and

Problem (II):

$$\vartheta_{ik}^{(2)} = \vartheta_{ik}^{(2)\prime\prime} + \frac{k-1}{x}\,\vartheta_{ik}^{(2)\prime}\,, \tag{5.13}$$

$$a^2\vartheta_{ek}^{(2)} = \vartheta_{ek}^{(2)\prime\prime} + \frac{k-1}{x}\,\vartheta_{ek}^{(2)\prime} + \lambda\frac{P(s)}{s}\,, \tag{5.14}$$

where the new independent variable x (cf. 1.7) involves a_i:

$$x = \frac{\sqrt{s}}{a_i}\,r\;, \text{ and } X = \frac{\sqrt{s}}{a_i}\,R\;. \tag{5.15}$$

The transformed common boundary conditions are (cf. 1.11):

$$\text{at } x = 0: \qquad \vartheta_{ik} \text{ an even function} \tag{5.16}$$

$$\text{at } X = X: \qquad \vartheta_{ik}(X,s) = \vartheta_{ek}(X,s) \tag{5.17}$$

$$\lambda\vartheta_{ik}'(X,s) = \vartheta_{ek}'(X,s) \tag{5.18}$$

$$\text{at } x = pX: \qquad -\,qX\,\vartheta_{ek}'(pX,s) = \vartheta_{ek}(pX,s) - \vartheta_u(s) \tag{5.19}$$

The new q involves λ_e, but the <u>internal</u> radius:

$$q = \frac{\lambda_e}{\alpha R}\;. \tag{5.20}$$

As in Part A, $q \to 0$ corresponds to the boundary condition of the first kind $\alpha \to \infty$.

6. The solution by means of "fundamental functions"

The solutions of the two-layer problem involve of course just the same fundamental functions we encountered in Part A. The only difference is that the ϑ_{ek} (eqs. 5.12 and 5.14) have the argument ax instead of simply x, as can easily be checked.

Thus, the general solutions are:

Problem (I):

$$\vartheta_{ik}^{(1)}(x,s) = I_k^{(1)}(s)\, F_k(x) + \frac{P(s)}{s}\,, \qquad (6.1)$$

$$\vartheta_{ek}^{(1)}(x,s) = E_{k1}^{(1)}(s)\, F_k(ax) + E_{k2}^{(1)}(s)\, \Phi_k(ax) \qquad (6.2)$$

Problem (II):

$$\vartheta_{ik}^{(2)}(x,s) = I_k^{(2)}(s)\, F_k(x)\,, \qquad (6.3)$$

$$\vartheta_{ek}^{(2)}(x,s) = E_{k1}^{(2)}(s)\, F_k(ax) + E_{k2}^{(2)}(s)\, \Phi_k(ax) + \lambda\frac{P(s)}{s}\,. \qquad (6.4)$$

The boundary condition (5.16) has already been considered by omitting any Φ_k in (6.1) and (6.3).

The coefficients I_k, E_{k1} and E_{k2} must be determined from the remaining three boundary conditions. The corresponding system of three linear algebraic equations reads in matrix notation:

$$\begin{pmatrix} -F & +F_a & +\Phi_a \\ -\lambda F' & +F'_a & +\Phi'_a \\ 0 & +qXF'_{ap}+F_{ap} & +qX\Phi'_{ap}+\Phi_{ap} \end{pmatrix} \begin{pmatrix} I \\ E_1 \\ E_2 \end{pmatrix} = \underbrace{\begin{pmatrix} +\frac{P(s)}{s} \\ 0 \\ +\vartheta_u(s) \end{pmatrix}}_{\text{Prob.(I)}}, \text{or} = \underbrace{\begin{pmatrix} -\lambda\frac{P(s)}{s} \\ 0 \\ +\vartheta_u(s)-\lambda\frac{P(s)}{s} \end{pmatrix}}_{\text{Prob.(II)}}, \text{resp} \qquad (6.5)$$

Only the inhomogeneous parts differ for the two problems.

We have introduced the following condensed notation: The geometric index k is omitted whenever possible without confusion. The arguments of F and Φ are written as subscripts omitting X, e.g. $F_k(apX) \equiv F_{ap}$, and so on.

The solution of (6.5) is:

$$I^{(1)} = \frac{1}{\Delta}\left\{[F_a\Phi_a' - F_a'\Phi_a]\vartheta_u(s) + [F_a'(qX\Phi_{ap}' + \Phi_{ap}) - \Phi_a'(qXF_{ap}' + F_{ap})]\,\frac{P(s)}{s}\right\} \quad (6.6)$$

$$E_1^{(1)} = \frac{1}{\Delta}\left\{[F\Phi_a' - \lambda F'\Phi_a]\vartheta_u(s) + [\lambda F'(qX\Phi_{ap}' + \Phi_{ap})]\,\frac{P(s)}{s}\right\} \quad (6.7)$$

$$E_2^{(1)} = \frac{1}{\Delta}\left\{-[FF_a' - \lambda F'F_a]\vartheta_u(s) - [\lambda F'(qXF_{ap}' + F_{ap})]\,\frac{P(s)}{s}\right\} \quad (6.8)$$

$$I^{(2)} = \frac{1}{\Delta}\left\{[F_a\Phi_a' - F_a'\Phi_a]\left(\vartheta_u(s) - \lambda\frac{P(s)}{s}\right) - [F_a'(qX\Phi_{ap}' + \Phi_{ap}) - \Phi_a'(qXF_{ap}' + F_{ap})]\lambda\frac{P(s)}{s}\right\} \quad (6.9)$$

$$E_1^{(2)} = \frac{1}{\Delta}\left\{[F\Phi_a' - \lambda F'\Phi_a]\left(\vartheta_u(s) - \lambda\frac{P(s)}{s}\right) - [\lambda F'(qX\Phi_{ap}' + \Phi_{ap})]\,\lambda\,\frac{P(s)}{s}\right\} \quad (6.10)$$

$$E_2^{(2)} = \frac{1}{\Delta}\left\{-[FF_a' - \lambda F'F_a]\left(\vartheta_u(s) - \frac{P(s)}{s}\right) + [\lambda F'(qXF_{ap}' + F_{ap})]\,\lambda\frac{P(s)}{s}\right\} \quad (6.11)$$

with the common system determinant

$$\Delta = (F\Phi_a' - \lambda F'\Phi_a)(qXF_{ap}' + F_{ap}) - (FF_a' - \lambda F'F_a)(qX\Phi_{ap}' + \Phi_{ap}) \,. \quad (6.12)$$

It corresponds to the quantity Δ of (2.5).

If one considers two inputs $+\ \vartheta_u(s)$ and $+\ \frac{P(s)}{s}$ for problem (I), and inputs $+\ \vartheta_u(s) - \lambda\frac{P(s)}{s}$ and $-\ \lambda\frac{P(s)}{s}$ for problem (II), the transfer functions (expressions in square brackets divided by Δ) are identical.

We must now consider the "fundamental functions" F_k and Φ_k and their "modified" functions G_k and ψ_k. The G_k have already been defined in chapter 3; the ψ_k should be related to the Φ_k as the G_k are to the F_k.

We call the fundamental functions with purely imaginary argument the "modified" functions, thus (cf. also 3.4):

$$F_k(\pm ix) = + G_k(x), \qquad \Phi_k(\pm ix) = \pm \psi_k(x) \quad ^{*)} \tag{6.13}$$

The G_k and ψ_k are a fundamental system for the "modified" equation

$$\vartheta_k = - \left(\vartheta_k'' + \frac{k-1}{x} \vartheta_k' \right) , \tag{6.14}$$

i.e. x is replaced by ix in the homogeneous part of eq. (5.11) and (5.13).

It follows immediately that

$$G_k(-ix) = F_k(x), \quad \text{and} \quad \psi_k(-ix) = \Phi_k(x) \ . \tag{6.15}$$

In order to choose our functions suitably, the following two properties of the G_k and ψ_k must be postulated:

- both functions G_k and ψ_k are real for real arguments,
- the zeros of G_k and ψ_k, lie on the respective real axes exclusively.

The reason for this is obvious from the analysis in chapter 3, and a discriminant equation like eq. (3.7) must be found involving the G_k and ψ_k. We therefore choose the following modified functions

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*) Except for the special case $\Phi_2(-ix) = + \psi_2(x) - i\pi J_0(x)$. The effect of the extra term is considered in chapter 7.

k	$G_k(x)$	$\psi_k(x)$
1	$+\cos x$	$+\sin x$
2	$+J_0(x)$	$+\frac{\pi}{2}Y_0(x)$ *)
3	$+\frac{\sin x}{x}$	$-\frac{\cos x}{x}$ *)

(6.16)

Hence, by applying (6.15), the original fundamental functions F_k and Φ_k are respectively:

k	$F_k(x)$	$\Phi_k(x)$
1	$+\cosh x$	$-i\sinh x$
2	$+I_0(x)$	$-K_0(x)-i\frac{\pi}{2}I_0(x)$ **)
3	$+\frac{\sinh x}{x}$	$-i\frac{\cosh x}{x}$

(6.17)

The frequently needed first derivatives are:

k	$F'_k(x)$	$\Phi'_k(x)$
1	$+\sinh x$	$-i\cosh x$
2	$+I_1(x)$	$+K_1(x)-i\frac{\pi}{2}I_1(x)$
3	$+\frac{\cosh x}{x}-\frac{\sinh x}{x^2}$	$-i\left(\frac{\sinh x}{x}-\frac{\cosh x}{x^2}\right)$

k	$G'_k(x)$	$\psi'_k(x)$
1	$-\sin x$	$+\cos x$
2	$-J_1(x)$	$-\frac{\pi}{2}Y_1(x)$
3	$+\frac{\cos x}{x}-\frac{\sin x}{x^2}$	$+\frac{\sin x}{x}+\frac{\cos x}{x^2}$

(6.18)

*) The coefficients of the ψ_k are chosen to ensure similar func-

The functions $F_k(x)$ and $G_k(x)$ are the same as in Part A. The complex-valued Φ_k are however unexpected and are unavoidable. The functions Φ_1 and Φ_3 are odd, whereas $\Phi_2(-x)$ is not the negative of $\Phi_2(x)$ but its conjugate-complex value. As $Y_0(x)$ is the only zero'th order BESSEL function which is singular in the origin and has only real zeros, the choice of $\Phi_2(x)$ is enforced by relation (6.15). One can no longer take $K_0(x)$ as the second fundamental solution.

Finally, we compile the functional relationships between all these functions (cf. also Appendix I):

$$\left.\begin{aligned} F_k(\pm ix) &= + G_k(x) \quad , & G_k(-ix) &= + F_k(x) \quad , \\ F'_k(\pm ix) &= \mp i\, G'_k(x) \; , & G'_k(-ix) &= + i\, F'_k(x) \quad , \\ \Phi_k(\pm ix) &= \pm \psi_k(x) \quad , & \psi_k(ix) &= + \Phi_k(x) \quad , \\ \Phi'_k(\pm ix) &= - i\, \psi'_k(x) \quad , & \psi'_k(-ix) &= + i\, \Phi'_k(x) \quad , \end{aligned}\right\} \qquad (6.19)$$

except for

$$\Phi_2(-ix) = + \psi_2(x) - i\pi J_0(x) \;, \; \Phi'_2(-ix) = + i\, \psi'_2(x) - \pi J_1(x) \; .$$

The relations between pairs $F_k \leftrightarrow G_k$ and $\Phi_k \leftrightarrow \psi_k$ respectively, are thus almost equivalent.

In order to eliminate the variable x from the general solutions (6.1) to (6.4), we choose the following particular temperatures for both problems:

- the internal layer averaged temperature $\bar{\vartheta}_{\underline{\underline{i}}k}(s)$,
- the external layer averaged temperature $\vartheta_{ek}(s)$,
- the central temperature $\vartheta_{ik}(0,s)$,
- the interface temperature $\vartheta_{ik}(X,s) = \vartheta_{ek}(X,s)$,
- the boundary temperature $\vartheta_{ek}(pX,s)$.

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tion behaviour at the origin; see Appendix II.

**) See Appendix I on BESSEL functions.

We define the averages for the internal layer:

$$\bar{F}_k = \frac{k}{X^k}\int_0^X F_k(x)\, x^{k-1}\, dx = \frac{k}{X}\, F_k'(X)\ , \tag{6.20}$$

$$\bar{\Phi}_k = \frac{k}{X^k}\int_0^X \Phi_k(x)\, x^{k-1}\, dx = \frac{k}{X}\, \Phi_k'(X) + k\left(\frac{i}{X}\right)^k \tag{6.21}$$

(not needed in the subsequent analysis),

and for the external layer:

$$\bar{\bar{F}}_k = \frac{k}{X^k(p^k-1)}\int_X^{pX} F_k(ax)x^{k-1}dx = \frac{k}{aX}\,\frac{p^{k-1}F_k(apX) - F_k(aX)}{p^k-1}\ , \tag{6.22}$$

$$\bar{\bar{\Phi}}_k = \frac{k}{X^k(p^k-1)}\int_X^{pX} \Phi_k(ax)x^{k-1}dx = \frac{k}{aX}\,\frac{p^{k-1}\Phi_k'(apX) - \Phi_k'(aX)}{p^k-1}\ . \tag{6.23}$$

The F_k and Φ_k behave similarly with respect to the averaging procedure for the external layer. Note that the right hand sides of the above equations are valid for our fundamental functions only.

In dealing with the transfer functions occurring in equations (6.6) to (6.11) for the coefficients I, E_1 and E_2, we denote the square bracket numerators by:

$$Z_{I1k}(X) = F_a\Phi_a' - F_a'\Phi_a\ , \tag{6.24}$$

$$Z_{I2k}(X) = F_a'(qX\Phi_{ap}' + \Phi_{ap}) - \Phi_a'(qXF_{ap}' + F_{ap})\ , \tag{6.25}$$

$$Z_{E_1 1k}(X) = F\Phi_a' - \lambda F'\Phi_a\ , \tag{6.26}$$

$$Z_{E_2 2k}(X) = \lambda F'(qX\Phi_{ap}' + \Phi_{ap})\ , \tag{6.27}$$

$$Z_{E_2 1k}(X) = -\ FF_a' + \lambda F'F_a\ , \tag{6.28}$$

$$Z_{E_2 2k}(X) = -\ \lambda F'(qXF_{ap}' + F_{ap})\ , \tag{6.29}$$

so that

$$I_k^{(1)}(X) = \frac{1}{\Delta_k}\left[Z_{I1k}(X)\vartheta_u(s) + Z_{I2k}(X)\frac{P(s)}{s}\right], \tag{6.30}$$

$$E_{k1}^{(1)}(X) = \frac{1}{\Delta_k}\left[Z_{E_1 1k}(X)\vartheta_u(s) + Z_{E_1 2k}(X)\frac{P(s)}{s}\right], \tag{6.31}$$

$$E_{k2}^{(1)}(X) = \frac{1}{\Delta_k}\left[Z_{E_2 1k}(X)\vartheta_u(s) + Z_{E_2 2k}(X)\frac{P(s)}{s}\right], \tag{6.32}$$

and

$$I_k^{(2)}(X) = \frac{1}{\Delta_k}\left[Z_{I1k}(X)\left(\vartheta_u(s) - \lambda\frac{P(s)}{s}\right) - Z_{I2k}(X)\,\lambda\frac{P(s)}{s}\right], \tag{6.33}$$

$$E_{k1}^{(2)}(X) = \frac{1}{\Delta_k}\left[Z_{E_1 1k}(X)\left(\vartheta_u(s) - \lambda\frac{P(s)}{s}\right) - Z_{E_1 2k}(X)\lambda\frac{P(s)}{s}\right], \tag{6.34}$$

$$E_{k2}^{(2)}(X) = \frac{1}{\Delta_k}\left[Z_{E_2 1k}(X)\left(\vartheta_u(s) - \lambda\frac{P(s)}{s}\right) - Z_{E_2 2k}(X)\lambda\frac{P(s)}{s}\right]. \tag{6.35}$$

On substituting these expressions into the general solutions (6.1) to (6.4), we get the following "selected" temperatures, now independent of the space coordinate x.

<u>Problem (I):</u>

"mean"

$$\overline{\vartheta}_{ik}^{(1)}(s) = \frac{Z_{I1k}\cdot\overline{F}}{\Delta_k}\vartheta_u(s) + \frac{Z_{I2k}\cdot\overline{F}_k}{\Delta_k}\frac{P(s)}{s} + \frac{P(s)}{s} \tag{6.36}$$

$$\overline{\overline{\vartheta}}_{ek}^{(1)}(s) = \frac{Z_{E_1 1k}\cdot F_k + Z_{E_2 1k}\cdot\overline{\overline{\Phi}}_k}{\Delta_k}\vartheta_u(s) + \frac{Z_{E_1 2k}\cdot F_k + Z_{E_2 2k}\cdot\Phi_k}{\Delta_k}\frac{P(s)}{s} \tag{6.37}$$

"central"

$$\vartheta_{ik}^{(1)}(0,s) = \frac{Z_{I1k}}{\Delta_k}\vartheta_u(s) + \frac{Z_{I2k}}{\Delta_k}\frac{P(s)}{s} + \frac{P(s)}{s} \tag{6.38}$$

"interface"

$$\vartheta_{ik}^{(1)}(X,s) = \frac{Z_{I1k}\cdot F}{\Delta_k}\vartheta_u(s) + \frac{Z_{I2k}\cdot F}{\Delta_k}\frac{P(s)}{s} + \frac{P(s)}{s} \tag{6.39}$$

"boundary" $$\vartheta_{ek}(pX,s) = \frac{Z_{E_1 1k}\cdot F_{ap} + Z_{E_2 1k}\cdot \Phi_{ap}}{\Delta_k}\,\vartheta_u(s) + \frac{Z_{E_1 2k}\cdot F_{ap} + Z_{E_2 2k}\cdot \Phi_{ap}}{\Delta_k}\,\frac{P(s)}{s} \tag{6.40}$$

Problem (II):

"mean"
$$\bar{\vartheta}^{(2)}_{ik}(s) = \frac{Z_{I1k}\cdot \bar{F}_k}{\Delta_k}\left(\vartheta_u(s)-\lambda\frac{P(s)}{s}\right) - \frac{Z_{I2k}\cdot \bar{F}_k}{\Delta_k}\cdot \lambda\frac{P(s)}{s} \tag{6.41}$$

$$\bar{\bar{\vartheta}}_{ek}(s) = \frac{Z_{E_1 1k}\cdot \bar{\bar{F}}_k + Z_{E_2 1k}\cdot \bar{\bar{\Phi}}_k}{\Delta_k}\left(\vartheta_u(s)-\lambda\frac{P(s)}{s}\right) - \frac{Z_{E_1 2k}\cdot \bar{\bar{F}}_k + Z_{E_2 2k}\cdot \bar{\bar{\Phi}}_k}{\Delta_k}\,\lambda\frac{P(s)}{s} + \lambda\frac{P(s)}{s} \tag{6.42}$$

"central" $$\vartheta^{(2)}_{ik}(0,s) = \frac{Z_{I1k}}{\Delta_k}\left(\vartheta_u(s)-\lambda\frac{P(s)}{s}\right) - \frac{Z_{I2k}}{\Delta_k}\cdot \lambda\frac{P(s)}{s} \tag{6.43}$$

"interface" $$\vartheta^{(2)}_{ik}(X,s) = \frac{Z_{I1k}\cdot F}{\Delta_k}\left(\vartheta_u(s)-\lambda\frac{P(s)}{s}\right) - \frac{Z_{I2k}\cdot F}{\Delta_k}\cdot \lambda\frac{P(s)}{s} \tag{6.44}$$

"boundary" $$\vartheta^{(2)}_{ek}(pX,s) = \frac{Z_{E_1 1k}\cdot F_{ap} + Z_{E_2 1k}\cdot \Phi_{ap}}{\Delta_k}\left(\vartheta_u(s)-\lambda\frac{P(s)}{s}\right) - \frac{Z_{E_1 2k}\cdot F_{ap} + Z_{E_2 2k}\cdot \Phi_{ap}}{\Delta_k}\cdot \lambda\frac{P(s)}{s} + \lambda\frac{P(s)}{s} \tag{6.45}$$

7. The system discriminant and its derivative

All of the above transfer functions must be developed into partial fraction series for reasons given in chapter 3.

The first task is to establish and solve the discriminant equation $\Delta_k(\pm i\sigma) = 0$. Equation (6.12) for arguments $X = \pm i\sigma$, and relationships (6.19) show that the discriminant equation is

$$\boxed{\Delta_k(\pm i\sigma) = -\,i\left[(G\psi'_a-\lambda G'\psi_a)(q\sigma G'_{ap}+G_{ap}) - (GG'_a-\lambda G' G_a)(q\sigma\psi'_{ap}+\psi_{ap})\right] = 0} \tag{7.1}$$

In this condensed notation for G and ψ, the arguments are written as subscripts omitting $\pm\sigma_{kn}$, e.g. $G_k(\pm ap\sigma_{kn}) = G_{ap}$, and so on.

Note that the double sign drops out when the modified functions are used. Moreover, a relatively simple computation shows that the additional terms in $\Phi_2(-ix)$ and $\Phi_2'(-ix)$ cancel in the exceptional case $\Delta_2(-i\sigma)$. Thus, the discriminant equation (7.1) is quite universal and furnishes a series of isolated and simple roots σ_{kn}, in ascending order. In no case is $\sigma = 0$ a root.

Equation (7.1) for the three geometries:

k = 1 (plane geometry)

$$q\sigma \sin(ap-a+1)\sigma - \cos(ap-a+1)\sigma + $$
$$+ (\lambda-1)\sin\sigma \,[q\sigma \cos(p-1)a\sigma + \sin(p-1)a\sigma] = 0 \qquad (7.2)$$

k = 2 (cylindrical geometry)

$$[J_0(\sigma)Y_1(a\sigma) - \lambda J_1(\sigma)Y_0(a\sigma)][q\sigma J_1(ap\sigma) - J_0(ap\sigma)] -$$
$$- [J_0(\sigma)J_1(a\sigma) - \lambda J_1(\sigma)J_0(a\sigma)][q\sigma Y_1(ap\sigma) - Y_0(ap\sigma)] = 0 \qquad (7.3)$$

k = 3 (spherical geometry)

$$[\sin \sigma \,(\sin a\sigma + \frac{\cos a\sigma}{a\sigma}) + \lambda(\cos \sigma - \frac{\sin \sigma}{\sigma}) \cos a\sigma] \times$$
$$\times [q\sigma \,(\cos ap\sigma - \frac{\sin ap\sigma}{ap\sigma}) + \sin ap\sigma] +$$
$$+ [- \sin \sigma \,(\cos a\sigma - \frac{\sin a\sigma}{a\sigma}) + \lambda(\cos \sigma - \frac{\sin \sigma}{\sigma}) \sin a\sigma] \times$$
$$\times [q\sigma \,(\sin ap\sigma + \frac{\cos ap\sigma}{ap\sigma}) - \cos ap\sigma] = 0 \qquad (7.4)$$

The derivative of Δ_k with respect to s at $s = -\sigma_{kn}^2/\nu$ *),

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*) Eq. (5.15) shows that ν must now be taken as $\nu = R^2/a_1^2$.

i.e. $X = \pm i\sigma_{kn}$, is needed for calculating the residues as in chapter 3 and is computed as follows:

$$\frac{d\Delta_k}{ds} = \frac{d\Delta_k}{dX} \cdot \frac{\nu}{2X} \quad \text{(eq. 3.17 unchanged)} ,$$

$$= \frac{d\Delta_k}{d\sigma} \cdot \frac{\nu}{2\sigma} \quad \text{where } X = i\sigma \tag{7.5}$$

Substituting from equation (7.1) and setting $\sigma = \pm\sigma_{kn}$, we obtain

$$\left.\frac{d\Delta_k}{ds}\right|_{s=-\frac{\sigma_{kn}^2}{\nu}} = +i\frac{\nu}{2}\cdot\frac{1}{\sigma_{kn}}\Big\{[aG\psi_a'' - \lambda G''\psi_a + (1-\lambda a)G'\psi_a'][q\sigma_{kn}G_{ap}'+G_{ap}] +$$

$$+ [G\psi_a - \lambda G'\psi_a][(ap+q)G_{ap}' + apq\sigma_{kn}G_{ap}''] -$$

$$- [aGG'' - \lambda G''G_a + (1-\lambda a)G'G_a'][q\sigma_{kn}\psi_{ap}' + \psi_{ap}] -$$

$$- [GG_a' - \lambda G_a'][(ap + q)\psi_{ap}' + apq\sigma_{kn}\psi_{ap}'']\Big\} . \tag{7.6}$$

The second derivatives are eliminated by means of the respective differential equations

$$G'' = - G - \frac{k-1}{\sigma_{kn}} G' \;;\; G_a'' = - G_a - \frac{k-1}{a\sigma_{kn}} G_a' \;;\; G_{ap}'' = - G_{ap} - \frac{k-1}{ap\,\sigma_{kn}} G_{ap}', \tag{7.7}$$

and correspondingly for ψ.

$$\left.\frac{d\Delta_k}{ds}\right|_{s=-\frac{\sigma_{kn}^2}{\nu}} = +i\frac{\nu}{2}\cdot\frac{1}{\sigma_{kn}}\Big\{[(\lambda-a)G\psi_a+(1-\lambda a)G'\psi_a'-\frac{k-1}{\sigma_{kn}}(G\psi_a'-\lambda G'\psi_a)][q\sigma_{kn}G_{ap}'+G_{ap}]+$$

$$+ [G\psi_a'-\lambda G'\psi_a][(ap+q(2-k))G_{ap}'-apq\sigma_{kn}G_{ap}] -$$

$$- [(\lambda-a)GG_a+(1-\lambda a)G'G_a'-\frac{k-1}{\sigma_{kn}}(GG_a'-\lambda G'G_a)][q\sigma_{kn}\psi_{ap}'+\psi_{ap}] -$$

$$- [GG_a'-\lambda G'G_a][(ap+q(2-k))\psi_{ap}'-apq\sigma_{kn}\psi_{ap}]\Big\} . \tag{7.8}$$

As the terms with factor $\frac{k-1}{\sigma_{kn}}$ cancel because of equation (7.1), we obtain:

$$\left.\frac{d\Delta_k}{ds}\right|_{s=-\frac{\sigma^2_{kn}}{\nu}} = + i\frac{\nu}{2}\cdot\frac{1}{\sigma_{kn}}\Big\{(G\psi'_a-\lambda G'\psi_a)[(ap+q(2-k))G'_{ap} - apq\sigma_{kn}G_{ap}] -$$
$$- (GG'_a-\lambda G'G_a)[(ap+q(2-k))\psi'_{ap} - apq\sigma_{kn}\psi_{ap}] +$$
$$+ [(\lambda-a)G\psi_a + (1-\lambda a)G'\psi'_a][q\sigma_{kn}G'_{ap} + G_{ap}] -$$
$$- [(\lambda-a)GG_a + (1-\lambda a)G'G'_a][q\sigma_{kn}\psi'_{ap} + \psi_{ap}]\Big\} = \Delta'_{kn} \, . \tag{7.9}$$

This expression, which we call for brevity Δ'_{kn}, corresponds to (3.22) of Part A. It is emphasized that no sign ambiguity appears for $X = \pm i\sigma_{kn}$.

In certain cases Δ'_{kn} and Δ_{kn} have a common factor, apparently indicating multiple roots. In the particular case $q = 0$, $a = 1$, $\lambda = 1$, this common factor is the WRONSKIan $G\psi' - G'\psi$ which cannot vanish as G and ψ are independent functions. Here the WRONSKIan reduces to σ_{kn}^{1-k} in the three geometries that does not vanish in any finite domain; no multiple roots therefore occur.

8. The residues of the partial fraction series development

Equation (3.16) gives the residues of the partial fraction series developments of all transfer functions for problems (I) and (II) in the solutions (6.36) to (6.45).

We begin with the "mean" temperatures.

$$a_{kn} = \left.\frac{Z_{I1k}\cdot\bar{F}_k}{\Delta'_{kn}}\right|_{X=\pm i\sigma_{kn}} \, , \tag{8.1}$$

$$Z_{I1k}\cdot\bar{F}_k = (F_a\Phi'_a - F'_a\Phi_a)\,\frac{k}{X}\,F' \, , \tag{8.2}$$

Substituting from (6.19), we obtain

$$Z_{I1k}\cdot\bar{F}_k\Big|_{X=\pm i\sigma_{kn}} = \frac{ik}{\sigma_{kn}}\,(G_a\psi_a' - G_a'\psi_a)G'. \quad (\neq 0) \tag{8.3}$$

Note that the double sign has dropped out; this is also true for all following residues.

$$a_{kn} = \frac{ik}{\sigma_{kn}}\,\frac{(G_a\psi_a' - G_a'\psi_a)G'}{\Delta_{kn}'} . \tag{8.4}$$

Correspondingly:

$$b_{kn} = \frac{Z_{I2k}\cdot\bar{F}_k}{\Delta_{kn}'}\Big|_{X=i\sigma_{kn}}$$

$$= \frac{ik}{\sigma_{kn}}\,\frac{[G_a'(q\sigma_{kn}\psi_{ap}'+\psi_{ap}) - \psi_a'(q\sigma_{kn}G_{ap}'+G_{ap})]G'}{\Delta_{kn}'} , \tag{8.5}$$

$$c_{kn} = \frac{Z_{E_1 1k}\cdot\bar{\bar{F}}_k + Z_{E_2 1k}\cdot\bar{\bar{\Phi}}_k}{\Delta_{kn}'}\Big|_{X=i\sigma_{kn}}$$

$$= \frac{ik}{a\sigma_{kn}(p^k-1)}\,\frac{(G\psi_a'-\lambda G'\psi_a)(p^{k-1}G_{ap}'-G_a')-(GG_a'-\lambda G'G_a)(p^{k-1}\psi_{ap}'-\psi_a')}{\Delta_{kn}'} \tag{8.6}$$

$$d_{kn} = \frac{Z_{E_1 2k}\cdot\bar{\bar{F}}_k + Z_{E_2 2k}\cdot\bar{\bar{\Phi}}_k}{\Delta_{kn}'}\Big|_{X=i\sigma_{kn}}$$

$$= \frac{ik.\lambda G'}{a\sigma_{kn}(p^k-1)}\,\frac{(q\sigma_{kn}\psi_{ap}'+\psi_{ap})(p^{k-1}G_{ap}'-G_a')-(q\sigma_{kn}G_{ap}'+G_{ap})(p^{k-1}\psi_{ap}'-}{\Delta_{kn}'} \tag{8.7}$$

The residues for the central temperature are:

$$e_{kn} = \frac{Z_{I1k}}{\Delta_{kn}'}\Big|_{X=i\sigma_{kn}}$$

$$= -i \frac{G_a \psi_a' - G_a' \psi_a}{\Delta_{kn}'} , \tag{8.8}$$

$$f_{kn} = \frac{Z_{I2k}}{\Delta_{kn}'} \Bigg|_{X=i\sigma_{kn}}$$

$$= - i \frac{G_a'(q\sigma_{kn}\psi_{ap}'+\psi_{ap}) - \psi_a'(q\sigma_{kn}G_{ap}'+G_{ap})}{\Delta_{kn}'} ; \tag{8.9}$$

and for the interface temperature:

$$g_{kn} = \frac{Z_{I1k}\cdot F}{\Delta_{kn}'} \Bigg|_{X=i\sigma_{kn}}$$

$$= - i \frac{(G_a \psi_a' - G_a' \psi_a)G}{\Delta_{kn}'} , \tag{8.10}$$

$$h_{kn} = \frac{Z_{I2k}\cdot F}{\Delta_{kn}'} \Bigg|_{X=i\sigma_{kn}}$$

$$= - i \frac{[G_a'(q\sigma_{kn}\psi_{ap}'+\psi_{ap}) - \psi_a'(q\sigma_{kn}G_{ap}'+G_{ap})]G}{\Delta_{kn}'} ; \tag{8.11}$$

and finally for the boundary temperature:

$$j_{kn} = \frac{Z_{E_1 1k}\cdot F_{ap} + Z_{E_2 1k}\cdot \Phi_{ap}}{\Delta_{kn}'} \Bigg|_{X=i\sigma_{kn}}$$

$$= - i \frac{(G\psi_a' - \lambda G'\psi_a)G_{ap} - (GG_a' - \lambda G' G_a)\psi_{ap}}{\Delta_{kn}'} , \tag{8.12}$$

$$k_{kn} = \frac{Z_{E_1 2k}\cdot F_{ap} + Z_{E_2 2k}\cdot \Phi_{ap}}{\Delta_{kn}'} \Bigg|_{X=i\sigma_{kn}}$$

$$= - i \frac{\lambda G' q \sigma_{kn}(\psi'_{ap} G_{ap} - G'_{ap} \psi_{ap})}{\Delta'_{kn}} \quad . \tag{8.13}$$

9. Inverse transformation for the selected temperatures

The procedure is the same as in chapter 4.

The partial fraction developments of the various temperatures (6.36) to (6.45) may be written:

Problem (I):

"mean" :

$$s\bar{\theta}_{ik}^{(1)}(s) = \sum \frac{a_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} s\theta_u(s) + \sum \frac{b_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} P(s) + P(s), \tag{9.1}$$

$$s\bar{\bar{\theta}}_{ek}^{(1)}(s) = \sum \frac{c_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} s\theta_u(s) + \sum \frac{d_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} P(s) , \tag{9.2}$$

"central" :

$$s\theta_{ik}^{(1)}(0,s) = \sum \frac{e_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} s\theta_u(s) + \sum \frac{f_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} P(s) + P(s), \tag{9.3}$$

"interface":

$$s\theta_{ik}^{(1)}(X,s) = \sum \frac{g_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} s\theta_u(s) + \left(h_{k0}(s) + \sum \frac{h_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \right) P(s) + P(s), \tag{9.4}$$

"boundary" :

$$s\theta_{ek}^{(1)}(pX,s) = \sum \frac{j_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} s\theta_u(s) + \sum \frac{k_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} P(s) \quad . \tag{9.5}$$

Problem (II):

"mean" :

$$s\bar{\vartheta}_{ik}^{(2)}(s) = \sum \frac{a_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \Big(s\vartheta_u(s) - \lambda P(s)\Big) - \sum \frac{b_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \lambda P(s), \tag{9.6}$$

$$s\bar{\bar{\vartheta}}_{ek}^{(2)}(s) = \sum \frac{c_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \Big(s\vartheta_u(s) - \lambda P(s)\Big) - \sum \frac{d_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \lambda P(s) + \lambda P(s), \tag{9.7}$$

"central" :

$$s\vartheta_{ik}^{(2)}(0,s) = \sum \frac{e_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \Big(s\vartheta_u(s) - \lambda P(s)\Big) - \sum \frac{f_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \lambda P(s), \tag{9.8}$$

"interface":

$$s\vartheta_{ik}^{(2)}(X,s) = \sum \frac{g_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \Big(s\vartheta_u(s) - \lambda P(s)\Big) - \Big(h_{ko}(s) + \sum \frac{h_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}}\Big)\lambda P(s), \tag{9.9}$$

"boundary" :

$$s\vartheta_{ek}^{(2)}(pX,s) = \sum \frac{j_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \Big(s\vartheta_u(s) - \lambda P(s)\Big) - \sum \frac{k_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} \lambda P(s) + \lambda P(s) . \tag{9.10}$$

As already stated in chapter 3, the partial fraction series representation of a function is not unambiguous as, in principle, an additive integral function may occur in all the above sums. We subsequently show that this occurs only in the case of h. Whether we introduce it or not, it drops out of the subsequent computation. Only when establishing the summation formulas in Appendix III, must it be known whether the respective additional term is zero or not.

The inverse transforms of the above expressions are:

Problem (I):

$$\dot{\Theta}^{(1)}_{ik}(t) = \sum a_{kn}\, e^{-\frac{\sigma^2_{kn}}{\nu}t} \int_0^t \dot{\Theta}_u(\tau)\, e^{+\frac{\sigma^2_{kn}}{\nu}\tau}\, d\tau + \sum b_{kn}\, e^{-\frac{\sigma^2_{kn}}{\nu}t} \int_0^t \frac{\Delta W(\tau)}{(\rho c)_i}\, e^{+\frac{\sigma^2_{kn}}{\nu}\tau}\, d\tau + \frac{\Delta W(t)}{(\rho c)_i}\,, \qquad (9.11)$$

and correspondingly for the other temperatures.

Problem (II):

$$\dot{\Theta}^{(2)}_{ek}(t) = \sum c_{kn}\, e^{-\frac{\sigma^2_{kn}}{\nu}t} \int_0^t \left[\dot{\Theta}_u(\tau) - \lambda \frac{\Delta W(\tau)}{(\rho c)_i}\right] e^{+\frac{\sigma^2_{kn}}{\nu}\tau}\, d\tau - \sum d_{kn}\, e^{-\frac{\sigma^2_{kn}}{\nu}t} \int_0^t \lambda \frac{\Delta W(\tau)}{(\rho c)_i}\, e^{+\frac{\sigma^2_{kn}}{\nu}\tau}\, d\tau + \lambda \frac{\Delta W(t)}{(\rho c)_i}\,, \qquad (9.12)$$

and correspondingly for the other temperatures.

N.B. Corresponding to the s-dependent inputs in expression (9.1) to (9.10), there are only four different types of convolution integrals, two for each problem. Therefore, we need to define only four different types of "transient temperature complement functions Γ", namely:

$$e^{-\frac{\sigma^2_{kn}}{\nu}t} \int_0^t \dot{\Theta}_u(\tau)\, e^{+\frac{\sigma^2_{kn}}{\nu}\tau}\, d\tau = \Gamma^{(1)}_{akn}(t)\,, \qquad (9.13)$$

$$e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t \frac{\Delta W(\tau)}{(\rho c)_i} e^{+\frac{\sigma_{kn}^2}{\nu}\tau} d\tau = \Gamma_{bkn}^{(1)}(t) \; , \tag{9.14}$$

as in Part A, eqs. (4.4) and (4.5).

The Γ's for problem (II) are:

$$e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t \left[\dot{\Theta}_u(\tau) - \lambda \frac{\Delta W(\tau)}{(\rho c)_i}\right] e^{+\frac{\sigma_{kn}^2}{\nu}\tau} d\tau = \Gamma_{akn}^{(2)}(t) \; , \tag{9.15}$$

$$e^{-\frac{\sigma_{kn}^2}{\nu}t} \int_0^t - \lambda \frac{\Delta W(\tau)}{(\rho c)_i} e^{+\frac{\sigma_{kn}^2}{\nu}\tau} d\tau = \Gamma_{bkn}^{(2)}(t) \; , \tag{9.16}$$

They obey the differential equations

$$\boxed{\begin{aligned} \dot{\Gamma}_{akn}^{(1)}(t) + \frac{\sigma_{kn}^2}{\nu} \Gamma_{akn}^{(1)}(t) &= \dot{\Theta}_u(t) && (9.17) \\ \dot{\Gamma}_{bkn}^{(1)}(t) + \frac{\sigma_{kn}^2}{\nu} \Gamma_{bkn}^{(1)}(t) &= \frac{\Delta W(t)}{(\rho c)_i} && (9.18) \end{aligned}} \; ,$$

for Problem (I), and

$$\boxed{\begin{aligned} \dot{\Gamma}_{akn}^{(2)}(t) + \frac{\sigma_{kn}^2}{\nu} \Gamma_{akn}^{(2)}(t) &= \dot{\Theta}_u(t) - \lambda \frac{\Delta W(t)}{(\rho c)_i} && (9.19) \\ \dot{\Gamma}_{bkn}^{(2)}(t) + \frac{\sigma_{kn}^2}{\nu} \Gamma_{bkn}^{(2)}(t) &= - \lambda \frac{\Delta W(t)}{(\rho c)_i} && (9.20) \end{aligned}} \; ,$$

for Problem (II) with the same initial conditions $\Gamma_a(0) = \Gamma_b(0) = 0$.

We frequently need

$$\Gamma_{akn}^{(1)}(t) = \frac{\nu}{\sigma_{kn}^2} [\dot{\Theta}_u(t) - \dot{\Gamma}_{akn}^{(1)}(t)] \; , \tag{9.21}$$

$$\Gamma^{(1)}_{bkn}(t) = \frac{\nu}{\sigma^2_{kn}}\left[\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}^{(1)}_{bkn}(t)\right], \tag{9.22}$$

$$\Gamma^{(2)}_{akn}(t) = \frac{\nu}{\sigma^2_{kn}}\left[\dot{\Theta}_u(t) - \lambda\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}^{(2)}_{akn}(t)\right], \tag{9.23}$$

$$\Gamma^{(2)}_{bkn}(t) = \frac{\nu}{\sigma^2_{kn}}\left[-\lambda\,\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}^{(2)}_{bkn}(t)\right]. \tag{9.24}$$

With the Γ's, the solutions assume the form:

Problem (I):

$$\dot{\bar{\Theta}}^{(1)}_{ik}(t) = \sum a_{kn}\,\Gamma^{(1)}_{akn}(t) + \sum b_{kn}\,\Gamma^{(1)}_{bkn}(t) + \frac{\Delta W(t)}{(\rho c)_i}, \tag{9.25}$$

$$\dot{\bar{\bar{\Theta}}}^{(1)}_{ek}(t) = \sum c_{kn}\,\Gamma^{(1)}_{akn}(t) + \sum d_{kn}\,\Gamma^{(1)}_{bkn}(t), \tag{9.26}$$

$$\dot{\Theta}^{(1)}_{ik}(0,t) = \sum e_{kn}\,\Gamma^{(1)}_{akn}(t) + \sum f_{kn}\,\Gamma^{(1)}_{bkn}(t) + \frac{\Delta W(t)}{(\rho c)_i}, \tag{9.27}$$

$$\dot{\Theta}^{(1)}_{ik}(X,t) = \sum g_{kn}\,\Gamma^{(1)}_{akn}(t) + \sum h_{kn}\,\Gamma^{(1)}_{bkn}(t) + (h_{ko}+1)\frac{\Delta W(t)}{(\rho c)_i}, \tag{9.28}$$

$$\dot{\Theta}^{(1)}_{ek}(pX,t) = \sum j_{kn}\,\Gamma^{(1)}_{akn}(t) + \sum k_{kn}\,\Gamma^{(1)}_{bkn}(t). \tag{9.29}$$

Problem (II):

$$\dot{\bar{\Theta}}^{(2)}_{ik}(t) = \sum a_{kn}\,\Gamma^{(2)}_{akn}(t) + \sum b_{kn}\,\Gamma^{(2)}_{bkn}(t), \tag{9.30}$$

$$\dot{\bar{\bar{\Theta}}}^{(2)}_{ek}(t) = \sum c_{kn}\,\Gamma^{(2)}_{akn}(t) + \sum d_{kn}\,\Gamma^{(2)}_{bkn}(t) + \lambda\frac{\Delta W(t)}{(\rho c)_i}, \tag{9.31}$$

$$\dot{\Theta}^{(2)}_{ik}(0,t) = \sum e_{kn}\,\Gamma^{(2)}_{akn}(t) + \sum f_{kn}\,\Gamma^{(2)}_{bkn}(t), \tag{9.32}$$

$$\dot{\Theta}_{ik}^{(2)}(X,t) = \sum g_{kn}\,\Gamma_{akn}^{(2)}(t) + \sum h_{kn}\,\Gamma_{bkn}^{(2)}(t) - h_{ko}\,\lambda\,\frac{\Delta W(t)}{(\rho c)_i}\,, \tag{9.33}$$

$$\dot{\Theta}_{ek}^{(2)}(pX,t) = \sum j_{kn}\,\Gamma_{akn}^{(2)}(t) + \sum k_{kn}\,\Gamma_{bkn}^{(2)}(t) + \lambda\,\frac{\Delta W(t)}{(\rho c)_i}\,. \tag{9.34}$$

According to the procedure explained in Part A, chapter 4, we substitute from (9.21) - (9.24) into (9.25) - (9.34) and integrate once. Simplifications are then possible as soon as the numerical values of $\sum \frac{\nu}{\sigma_{kn}^2}\,a_{kn}$, $\sum \frac{\nu}{\sigma_{kn}^2}\,b_{kn}$, ..., and so on, are known.

We again compute the limits of the transfer functions in question for $s \to 0$, or $X \to 0$. To avoid delaying the analysis we simply state the results derived in Appendix II. We consider all the transfer functions in (6.36) to (6.40), as the transfer functions of problems (I) and (II) are identical.

$$\lim_{X\to 0} \frac{Z_{I1k}\cdot\overline{F}_k}{\Delta_k} = +1\,, \tag{9.35}$$

$$\lim_{X\to 0} \frac{Z_{I2k}\cdot\overline{F}_k}{\Delta_k} = -1\,, \tag{9.36}$$

$$\lim_{X\to 0} \frac{Z_{E_1 1k}\cdot\overline{\overline{F}}_k + Z_{E_2 1k}\cdot\overline{\overline{\Phi}}_k}{\Delta_k} = +1\,, \tag{9.37}$$

$$\lim_{X\to 0} \frac{Z_{E_1 2k}\cdot\overline{\overline{F}}_k + Z_{E_2 2k}\cdot\overline{\overline{\Phi}}_k}{\Delta_k} = 0\,, \tag{9.38}$$

$$\lim_{X\to 0} \frac{Z_{I1k}}{\Delta_k} = +1\,, \tag{9.39}$$

$$\lim_{X\to 0} \frac{Z_{I2k}}{\Delta_k} = -1\,, \tag{9.40}$$

$$\lim_{X\to 0} \frac{Z_{I1k}\cdot F}{\Delta_k} = +1\,, \tag{9.41}$$

$$\lim_{X\to 0} \frac{Z_{I2k}\cdot F}{\Delta_k} = -1 \quad , \tag{9.42}$$

$$\lim_{X\to 0} \frac{Z_{E_1 1k}\cdot F_{ap} + Z_{E_2 1k}\cdot \Phi_{ap}}{\Delta_k} = +1 \quad , \tag{9.43}$$

$$\lim_{X\to 0} \frac{Z_{E_1 2k}\cdot F_{ap} + Z_{E_2 2k}\cdot \Phi_{ap}}{\Delta_k} = 0 \quad , \tag{9.44}$$

But these values must be identical with the respective partial fraction's sum for s = 0, provided that there is no additional integral function.

Hence,

$$\lim_{s\to 0} \sum \frac{a_{kn}}{s+\frac{\sigma_{kn}^2}{\nu}} = \nu \sum \frac{a_{kn}}{\sigma_{kn}^2} = +1 \quad , \tag{9.45}$$

$$\nu \sum \frac{b_{kn}}{\sigma_{kn}^2} = -1 \quad , \tag{9.46}$$

$$\nu \sum \frac{c_{kn}}{\sigma_{kn}^2} = +1 \quad , \tag{9.47}$$

$$\nu \sum \frac{d_{kn}}{\sigma_{kn}^2} = 0 \quad , \tag{9.48}$$

$$\nu \sum \frac{e_{kn}}{\sigma_{kn}^2} = +1 \quad , \tag{9.49}$$

$$\nu \sum \frac{f_{kn}}{\sigma_{kn}^2} = -1 \quad , \tag{9.50}$$

$$\nu \sum \frac{g_{kn}}{\sigma_{kn}^2} = +1 \quad , \tag{9.51}$$

$$h_{ko} + \nu \sum \frac{h_{kn}}{\sigma_{kn}^2} = -1 \ , \tag{9.52}$$

$$\nu \sum \frac{j_{kn}}{\sigma_{kn}^2} = +1 \ , \tag{9.53}$$

$$\nu \sum \frac{k_{kn}}{\sigma_{kn}^2} = 0 \ . \tag{9.54}$$

The only special case, i.e. the one with h_{ko}, is explained in the example B of Appendix III.

Substituting (9.21) and (9.22) into (9.25) yields

$$\dot{\bar{\Theta}}_{ik}^{(1)}(t) = \nu \sum \frac{a_{kn}}{\sigma_{kn}^2} [\dot{\Theta}_u(t) - \dot{\Gamma}_{akn}^{(1)}(t)] + + \nu \sum \frac{b_{kn}}{\sigma_{kn}^2} \left[\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}_{bkn}^{(1)}(t)\right] + \frac{\Delta W(t)}{(\rho c)_i} \ . \tag{9.55}$$

This equation corresponds to (4.11). The terms independent of the index n are taken outside the summation, and the sum formulas (9.45) and (9.46) are applied.

$$\dot{\bar{\Theta}}_{ik}^{(1)}(t) = \dot{\Theta}_u(t) - \nu \sum \frac{1}{\sigma_{kn}^2} \left[a_{kn} \ \dot{\Gamma}_{akn}^{(1)}(t) + b_{kn} \ \dot{\Gamma}_{bkn}^{(1)}(t)\right] . \tag{9.56}$$

We now show what would happen if there were additional constant terms a_{ko} and b_{ko}. The last term of (9.25) would be $a_{ko}\dot{\Theta}_u(t)$ + $(b_{ko}+1) \frac{\Delta W(t)}{(\rho c)_i}$ instead of simply $\frac{\Delta W(t)}{(\rho c)_i}$.

Substitution of (9.21) and (9.22) into this new (9.25) yields

$$\dot{\bar{\Theta}}_{ik}^{(1)}(t) = \nu \sum \frac{a_{kn}}{\sigma_{kn}^2} \left[\dot{\Theta}_u(t) - \dot{\Gamma}_{akn}^{(1)}(t)\right] + + \nu \sum \frac{b_{kn}}{\sigma_{kn}^2} \left[\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}_{bkn}^{(1)}(t)\right] + a_{ko}\dot{\Theta}_u(t) + (b_{ko}+1) \frac{\Delta W(t)}{(\rho c)_i} \ . \tag{9.57}$$

But by considering modified sum formulas (9.45): $\nu\sum \frac{a_{kn}}{\sigma_{kn}^2} = + 1 - a_{ko}$, and (9.46): $\nu\sum \frac{b_{kn}}{\sigma_{kn}^2} = - 1 - b_{ko}$, eq. (9.57) becomes

$$\dot{\bar{\Theta}}_{ik}^{(1)}(t) = \dot{\Theta}_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[a_{kn}\,\dot{\Gamma}_{akn}^{(1)}(t) + b_{kn}\,\dot{\Gamma}_{bkn}^{(1)}(t)\right]. \quad (9.58)$$

This interesting result shows that (9.56) applies even if the coefficients a_{ko} and b_{ko} are not zero, and their possible existence is irrelevant to the subsequent treatment. Applying (9.45) and (9.46) on particular examples (see Appendix III) shows whether a_{ko} and b_{ko} are zero or not. All the other coefficients behave in the same way and do not enter the subsequent treatment. The corresponding equation (4.17) in Part A is similarly valid when $a_{ko} \neq 0$, and the subsequent analysis is not changed.

It can easily be shown that the other temperatures have the same form as $\dot{\bar{\Theta}}_{ik}^{(1)}(t)$ in (9.56). On integrating these equations, the following list is obtained:

$$\bar{\Theta}_{ik}^{(1)}(t) = \Theta_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[a_{kn}\,\Gamma_{akn}^{(1)}(t) + b_{kn}\,\Gamma_{bkn}^{(1)}(t)\right], \quad (9.59)$$

$$\bar{\bar{\Theta}}_{ek}^{(1)}(t) = \Theta_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[c_{kn}\,\Gamma_{akn}^{(1)}(t) + d_{kn}\,\Gamma_{bkn}^{(1)}(t)\right], \quad (9.60)$$

$$\Theta_{ik}^{(1)}(0,t) = \Theta_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[e_{kn}\,\Gamma_{akn}^{(1)}(t) + f_{kn}\,\Gamma_{bkn}^{(1)}(t)\right], \quad (9.61)$$

$$\Theta_{ik}^{(1)}(X,t) = \Theta_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[g_{kn}\,\Gamma_{akn}^{(1)}(t) + h_{kn}\,\Gamma_{bkn}^{(1)}(t)\right], \quad (9.62)$$

$$\Theta_{ek}^{(1)}(pX,t) = \Theta_u(t) - \nu\sum \frac{1}{\sigma_{kn}^2}\left[j_{kn}\,\Gamma_{akn}^{(1)}(t) + k_{kn}\,\Gamma_{bkn}^{(1)}(t)\right]. \quad (9.63)$$

For problem II, we simply change the superscript $^{(1)}$ to $^{(2)}$.

Going back to equations (9.59) etc., we substitute once more the Γ's from (9.21) to (9.24):

$$\bar{\Theta}^{(1)}_{ik}(t) = \Theta_u(t)-\nu^2\sum \frac{1}{\sigma^*_{kn}}\left[a_{kn}\left(\dot{\Theta}_u(t)-\dot{\Gamma}^{(1)}_{akn}(t)\right)+b_{kn}\left(\frac{\Delta W(t)}{(\rho c)_i}-\dot{\Gamma}^{(1)}_{bkn}(t)\right)\right], \tag{9.64}$$

and similarly for the other four temperatures.

$$\bar{\Theta}^{(2)}_{ik}(t) = \Theta_u(t)-\nu^2\sum \frac{1}{\sigma^*_{kn}}\left[a_{kn}\left(\dot{\Theta}_u(t)-\lambda\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}^{(2)}_{akn}(t)\right) + \right.$$

$$\left. + b_{kn}\left(- \lambda\frac{\Delta W(t)}{(\rho c)_i} - \dot{\Gamma}^{(2)}_{bkn}(t)\right)\right], \tag{9.65}$$

and similarly for the other four temperatures.

N.B. The various temperatures and the power input are all measured from the initial steady state values.

The stationary parts of the above solutions obtained by omitting the dotted quantities and taking constant W_o must therefore be compared with the direct solutions of the stationary problem. This gives sum formulas for the $\frac{a_{kn}}{\sigma^*_{kn}}$ etc., as in Part A.

10. The stationary solutions

The stationary solutions of both problems (I) and (II) can be evaluated easily. In this chapter only, we denote stationary temperatures by Θ, without an extra index. In most cases, a uniform notation for all three geometries is not convenient because of the logarithm for k = 2 that replaces the powers for k = 1, 3.

FOURIER's equation is: Problem (I):

$$\frac{d^2\Theta^{(1)}_{ik}}{dr^2} + \frac{k-1}{r}\,\frac{d\Theta^{(1)}_{ik}}{dr} + \frac{W}{\lambda_i} = 0\ . \tag{10.1}$$

$$\frac{d^2\Theta^{(1)}_{ek}}{dr^2} + \frac{k-1}{r}\,\frac{d\Theta^{(1)}_{ek}}{dr} = 0\ . \tag{10.2}$$

Problem (II):

$$\frac{d^2\Theta_{ik}^{(2)}}{dr^2} + \frac{k-1}{r}\frac{d\Theta_{ik}^{(2)}}{dr} = 0 , \qquad (10.3)$$

$$\frac{d^2\Theta_{ek}^{(2)}}{dr^2} + \frac{k-1}{r}\frac{d\Theta_{ek}^{(2)}}{dr} + \frac{W}{\lambda_e} = 0 , \qquad (10.4)$$

with common boundary conditions (5.6) to (5.9); (5.9) may also be written as

$$- qR\frac{d}{dr}\Theta_{ek}\Big|_{r=pR} = \Theta_{ek}(pR) - \Theta_u . \qquad (10.5)$$

The general solutions are

Problem (I) :

$$\left.\begin{aligned} \Theta_{ik}^{(1)}(r) &= \Theta_u + \frac{1}{\lambda}\frac{W(R^2-r^2)}{2k\lambda_e} + \frac{WR^2}{k\lambda_e}\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right], && (10.6)\\ \Theta_{ek}^{(1)}(r) &= \Theta_u - \frac{WR^k}{k\lambda_e}\frac{r^{2-k}}{2-k} + \frac{WR^2}{k\lambda_e}\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} , && (10.7)\end{aligned}\right\} k = 1,3$$

and

$$\left.\begin{aligned} \Theta_{i2}^{(1)}(r) &= \Theta_u + \frac{1}{\lambda}\frac{W(R^2-r^2)}{4\lambda_e} + \frac{WR^2}{2\lambda_e}\left(\frac{q}{p} + \ln p\right) , && (10.8)\\ \Theta_{e2}^{(1)}(r) &= \Theta_u - \frac{WR^2}{2\lambda_e}\ln r + \frac{WR^2}{2\lambda_e}\left(\frac{q}{p} + \ln pR\right) . && (10.9)\end{aligned}\right\} k = 2$$

Problem (II):

$$\left.\begin{aligned} \Theta_{ik}^{(2)}(r) &= \Theta_u + \frac{WR^2}{k\lambda_e}\left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} + \frac{k}{2(2-k)}\right](= \text{const.}), && (10.10)\\ \Theta_{ek}^{(2)}(r) &= \Theta_u - \frac{Wr^2}{2k\lambda_e} + \frac{WR^k}{k\lambda_e}\cdot\frac{r^{2-k}}{2-k} + \frac{WR^2}{k\lambda_e}\left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k}\right], && (10.11)\end{aligned}\right\} k=1,3$$

and

$$\left.\begin{aligned}\Theta_{i2}^{(2)}(r) &= \Theta_u + \frac{WR^2}{2\lambda_e}\left[pq + \frac{p^2}{2} - \frac{q}{p} - \ln p - \frac{1}{2}\right] (=\text{const.}), & (10.12)\\ \Theta_{e2}^{(2)}(r) &= \Theta_u - \frac{Wr^2}{4\lambda_e} + \frac{WR^2}{2\lambda_e}\ln r + \frac{WR^2}{2\lambda_e}\left[pq + \frac{p^2}{2} - \frac{q}{p} - \ln pR\right] & (10.13)\end{aligned}\right\} k=2$$

By applying the averaging rules (6.20) and (6.22), left hand sides only, or by specifying to the selected fixed coordinates $r=0,R,pR$, respectively, one obtains the following set:

Problem (I):

$$\left.\begin{aligned}\bar{\Theta}_{ik}^{(1)} &= \Theta_u + \frac{WR^2}{k\lambda_e}\left[\frac{1}{\lambda(k+2)} + \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right], & (10.14)\\ \bar{\bar{\Theta}}_{ek}^{(1)} &= \Theta_u + \frac{WR^2}{k\lambda_e}\left[-\frac{k}{2(2-k)}\frac{p^2-1}{p^k-1} + \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k}\right], & (10.15)\\ \Theta_{ik}^{(1)}(0) &= \Theta_u + \frac{WR^2}{k\lambda_e}\left[\frac{1}{2\lambda} + \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right], & (10.16)\\ \Theta_{ik}^{(1)}(R) &= \Theta_u + \frac{WR^2}{k\lambda_e}\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right], & (10.17)\\ \Theta_{ek}^{(1)}(pR) &= \Theta_u + \frac{WR^2}{k\lambda_e}\cdot qp^{1-k}, & (10.18)\end{aligned}\right\} k=1,3$$

$$\left.\begin{aligned}\bar{\Theta}_{i2}^{(1)} &= \Theta_u + \frac{WR^2}{2\lambda_e}\left[\frac{1}{4\lambda} + \frac{q}{p} + \ln p\right], & (10.19)\\ \bar{\Theta}_{e2}^{(1)} &= \Theta_u + \frac{WR^2}{2\lambda_e}\left[\frac{q}{p} + \frac{1}{2} - \frac{\ln p}{p^2-1}\right], & (10.20)\\ \Theta_{i2}^{(1)}(0) &= \Theta_u + \frac{WR^2}{2\lambda_e}\left[\frac{1}{2\lambda} + \frac{q}{p} + \ln p\right], & (10.21)\\ \Theta_{i2}^{(1)}(R) &= \Theta_u + \frac{WR^2}{2\lambda_e}\left[\frac{q}{p} + \ln p\right]; & (10.22)\\ \Theta_{i2}^{(1)}(pR) &= \Theta_u + \frac{WR^2}{2\lambda_e}\cdot\frac{q}{p}. & (10.23)\end{aligned}\right\} k=2$$

Problem (II):

$$\left.\begin{aligned}
\bar{\Theta}^{(2)}_{ik} &= \Theta^{(2)}_{ik}(0) = \Theta^{(2)}_{ik}(R) = \Theta_u + \frac{WR^2}{k\lambda_e} \times \\
&\quad \times \left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k} + \frac{k}{2(2-k)} \right], && (10.24) \\
\bar{\bar{\Theta}}^{(2)}_{ek} &= \Theta_u + \frac{WR^2}{k\lambda_e} \left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k} - \frac{k}{2(k+2)} \frac{p^{k+2}-1}{p^k-1} + \right. \\
&\quad \left. + \frac{k}{2(2-k)} \frac{p^2-1}{p^k-1} \right], && (10.25) \\
\Theta^{(2)}_{ek}(pR) &= \Theta_u + \frac{WR^2}{k\lambda_e} q(p - p^{1-k}) , && (10.26)
\end{aligned}\right\} \quad k = 1,3$$

$$\left.\begin{aligned}
\bar{\Theta}^{(2)}_{i2} &= \Theta^{(2)}_{i2}(0) = \Theta^{(2)}_{i2}(R) = \Theta_u + \frac{WR^2}{2\lambda_e} \times \\
&\quad \times \left[pq + \frac{p^2}{2} - \frac{q}{p} - \ln p - \frac{1}{2} \right], && (10.27) \\
\bar{\bar{\Theta}}^{(2)}_{e2} &= \Theta_u + \frac{WR^2}{2\lambda_e} \left[pq + \frac{p^2}{2} - \frac{q}{p} - \frac{1}{4}(p^2+1) - \frac{1}{2} + \frac{\ln p}{p^2-1} \right], && (10.28) \\
\Theta^{(2)}_{e2}(pR) &= \Theta_u + \frac{WR^2}{2\lambda_e} \left[pq - \frac{q}{p} \right] . && (10.29)
\end{aligned}\right\} \quad k=2$$

11. The final form of the solutions

Comparing the stationary parts of the sets (9.64) and (9.65) with the corresponding solutions of chapter 10 shows that

$$\left.\begin{aligned}
\Theta^{(1)}_{ik,stat} &= \Theta_u - \nu^2 \frac{W}{(\rho c)_i} \sum \frac{b_{kn}}{\sigma^4_{kn}} \\
&= \Theta_u + \frac{WR^2}{k\lambda_e} \left[\frac{1}{\lambda(k+2)} + \left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k} - \frac{1}{2-k} \right],
\end{aligned}\right\} k=1,3 \qquad (11.1)$$

$$\bar{\Theta}^{(1)}_{i2,stat} = \Theta_u - \nu^2 \frac{W}{(\rho c)_i} \sum \frac{b_{2n}}{\sigma^4_{2n}} = \Theta_u + \frac{WR^2}{2\lambda_e} \left[\frac{1}{4\lambda} + \frac{q}{p} + \ln p \right] . \quad k=2 \qquad (11.2)$$

By setting $\nu^2 \frac{W}{(\rho c)_i} = \nu \frac{WR^2}{\lambda\lambda_e}$, we obtain

$$\nu \sum \frac{b_{kn}}{\sigma_{kn}^4} = - \frac{1}{k(k+2)} - \frac{\lambda}{k}\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right], \qquad k=1,3 \quad (11.3)$$

$$\nu \sum \frac{b_{2n}}{\sigma_{2n}^4} = - \frac{1}{8} - \frac{\lambda}{2}\left(\frac{q}{p} + \ln p\right) . \qquad k=2 \quad (11.4)$$

Similarly from $\bar{\bar{\Theta}}^{(1)}_{ek,stat}$:

$$\nu \sum \frac{d_{kn}}{\sigma_{kn}^4} = - \frac{\lambda}{k}\left[- \frac{k}{2(2-k)} \frac{p^2-1}{p^k-1} + \left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k}\right], \qquad k=1,3 \quad (11.5)$$

$$\nu \sum \frac{d_{2n}}{\sigma_{2n}^4} = - \frac{\lambda}{2}\left(\frac{q}{p} + \frac{1}{2} - \frac{\ln p}{p^2-1}\right) ; \qquad k=2 \quad (11.6)$$

from $\Theta^{(1)}_{ik,stat}(0)$:

$$\nu \sum \frac{f_{kn}}{\sigma_{kn}^4} = - \frac{1}{2k} - \frac{\lambda}{k}\left[\left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k} - \frac{1}{2-k}\right], \qquad k=1,3 \quad (11.7)$$

$$\nu \sum \frac{f_{2n}}{\sigma_{2n}^4} = - \frac{1}{4} - \frac{\lambda}{2}\left(\frac{q}{p} + \ln p\right) , \qquad k=2 \quad (11.8)$$

from $\Theta^{(1)}_{ik,stat}(R)$:

$$\nu \sum \frac{h_{kn}}{\sigma_{kn}^4} = - \frac{\lambda}{k}\left[\left(\frac{q}{p} + \frac{1}{2-k}\right) p^{2-k} - \frac{1}{2-k}\right], \qquad k=1,3 \quad (11.9)$$

$$\nu \sum \frac{h_{2n}}{\sigma_{2n}^4} = - \frac{\lambda}{2}\left(\frac{q}{p} + \ln p\right) , \qquad k=2 \quad (11.10)$$

and from $\Theta^{(1)}_{ek,stat}(pR)$:

$$\nu \sum \frac{k_{kn}}{\sigma_{kn}^4} = - \frac{\lambda}{k}\, qp^{1-k} . \qquad k=1,2,3 \quad (11.11)$$

The corresponding summation formulas from problem (II) are:

$$\nu\sum \frac{a_{kn}+b_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} + \frac{k}{2(2-k)}\right], \quad k=1,3 \quad (11.12)$$

$$\nu\sum \frac{a_{2n}+b_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[\frac{p}{q} + \frac{p^2}{2} - \frac{q}{p} - \ln p - \frac{1}{2}\right]. \qquad k=2 \quad (11.13)$$

Now, by subtracting (11.3) or (11.4) respectively:

$$\nu\sum \frac{a_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left\{pq + \frac{p^2}{2} + (\lambda-1)\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right]\right\} - \frac{1}{2(k+2)}, \quad k=1,3 \quad (11.14)$$

$$\nu\sum \frac{a_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left\{pq + \frac{p^2}{2} + (\lambda-1)\left(\frac{q}{p} + \ln p\right)\right\} - \frac{1}{8}. \qquad k=2 \quad (11.15)$$

This linking between problems (I) and (II) justifies the simultaneous treatment of the two problems.

Similarly from $\overset{=}{\Theta}{}^{(2)}_{ek,stat}$:

$$\nu\sum \frac{c_{kn}+d_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left[pq+\frac{p^2}{2} - \left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k} - \frac{k}{2(k+2)}\,\frac{p^{k+2}-1}{p^k-1} + \frac{k}{2(2-k)}\,\frac{p^2-1}{p^k-1}\right], \quad k=1,3 \quad (11.16)$$

$$\nu\sum \frac{c_{2n}+d_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[pq + \frac{p^2}{2} - \frac{q}{p} - \frac{1}{4}(p^2+1) - \frac{1}{2} + \frac{\ln p}{p^2-1}\right], \quad k=2 \quad (11.17)$$

$$\nu\sum \frac{c_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left\{pq + \frac{p^2}{2} - \frac{k}{2(k+2)}\,\frac{p^{k+2}-1}{p^k-1} + (\lambda-1)\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{k}{2(2-k)}\,\frac{p^2-1}{p^k-1}\right]\right\}, \quad k=1,3 \quad (11.18)$$

$$\nu\sum \frac{c_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left\{pq + \frac{p^2}{2} - \frac{1}{4}(p^2+1) + (\lambda-1)\left[\frac{q}{p} + \frac{1}{2} - \frac{\ln p}{p^2-1}\right]\right\}; \quad k=2 \quad (11.19)$$

from $\Theta^{(2)}_{ik,stat}(0)$:

$$\nu\sum \frac{e_{kn}+f_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} + \frac{k}{2(2-k)}\right], \quad k=1,3 \qquad (11.20)$$

$$\nu\sum \frac{e_{2n}+f_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[pq + \frac{p^2}{2} - \frac{q}{p} - \ln p - \frac{1}{2}\right], \quad k=2 \qquad (11.21)$$

$$\nu\sum \frac{e_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left\{pq + \frac{p^2}{2} + (\lambda-1)\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right]\right\}, \quad k=1,3 \qquad (11.22)$$

$$\nu\sum \frac{e_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[pq + \frac{p^2}{2} + (\lambda-1)\left(\frac{q}{p} + \ln p\right)\right]; \quad k=2 \qquad (11.23)$$

from $\Theta^{(2)}_{ik,stat}(R)$:

$$\nu\sum \frac{g_{kn}+h_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left[pq + \frac{p^2}{2} - \left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} + \frac{k}{2(2-k)}\right], \quad k=1,3 \qquad (11.24)$$

$$\nu\sum \frac{g_{2n}+h_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[pq + \frac{p^2}{2} - \frac{q}{p} - \ln p - \frac{1}{2}\right], \quad k=2 \qquad (11.25)$$

$$\nu\sum \frac{g_{kn}}{\sigma_{kn}^4} = \frac{1}{k}\left\{pq + \frac{p^2}{2} - \frac{1}{2} + (\lambda-1)\left[\left(\frac{q}{p} + \frac{1}{2-k}\right)p^{2-k} - \frac{1}{2-k}\right]\right\}, \quad k=1,3 \qquad (11.26)$$

$$\nu\sum \frac{g_{2n}}{\sigma_{2n}^4} = \frac{1}{2}\left[pq + \frac{p^2}{2} - \frac{1}{2} + (\lambda-1)\left(\frac{q}{p} + \ln p\right)\right], \quad k=2 \qquad (11.27)$$

and from $\Theta^{(2)}_{ek,stat}(pR)$:

$$\left.\begin{aligned} \nu\sum \frac{j_{kn}+k_{kn}}{\sigma_{kn}^4} &= \frac{q}{k}\left(p - p^{1-k}\right), && (11.28) \\ \nu\sum \frac{j_{kn}}{\sigma_{kn}^4} &= \frac{q}{k}\left[p + (\lambda-1)\, p^{1-k}\right]. && (11.29) \end{aligned}\right\} k=1,2,3$$

So we are able to compile the list of final solutions:

<u>Problem (I):</u>

for k=1,3:

$$\overline{\Theta}^{(1)}_{ik}(t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\cdot\left[\frac{1}{\lambda(k+2)}+\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}-\frac{1}{2-k}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{kn}}\left[a_{kn}\left(\dot{\Gamma}^{(1)}_{akn}(t)-\dot{\Theta}_u(t)\right)+b_{kn}\dot{\Gamma}^{(1)}_{bkn}(t)\right], \qquad (11.30)$$

$$\overline{\overline{\Theta}}^{(1)}_{ek}(t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[-\frac{k}{2(2-k)}\frac{p^2-1}{p^k-1}+\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{kn}}\left[c_{kn}\left(\dot{\Gamma}^{(1)}_{akn}(t)-\dot{\Theta}_u(t)\right)+d_{kn}\dot{\Gamma}^{(1)}_{bkn}(t)\right], \qquad (11.31)$$

$$\Theta^{(1)}_{ik}(0,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[\frac{1}{2\lambda}+\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}-\frac{1}{2-k}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{kn}}\left[e_{kn}\left(\dot{\Gamma}^{(1)}_{akn}(t)-\dot{\Theta}_u(t)\right)+f_{kn}\dot{\Gamma}^{(1)}_{bkn}(t)\right], \qquad (11.32)$$

$$\Theta^{(1)}_{ik}(R,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}-\frac{1}{2-k}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{kn}}\left[g_{kn}\left(\dot{\Gamma}^{(1)}_{akn}(t)-\dot{\Theta}_u(t)\right)+h_{kn}\dot{\Gamma}^{(1)}_{bkn}(t)\right], \qquad (11.33)$$

$$\Theta^{(1)}_{ek}(pR,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\cdot qp^{1-k}+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{kn}}\left[j_{kn}\left(\dot{\Gamma}^{(1)}_{akn}(t)-\dot{\Theta}_u(t)\right)+k_{kn}\dot{\Gamma}^{(1)}_{bkn}(t)\right], \quad k=1,2,3 \qquad (11.34)$$

and for k=2:

$$\overline{\Theta}^{(1)}_{i2}(t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[\frac{1}{4\lambda}+\frac{q}{p}+\ln p\right]+$$

$$+\nu^2\sum\frac{1}{\sigma^4_{2n}}\left[a_{2n}\left(\dot{\Gamma}^{(1)}_{a2n}(t)-\dot{\Theta}_u(t)\right)+b_{2n}\dot{\Gamma}^{(1)}_{b2n}(t)\right], \qquad (11.35)$$

$$\overset{=(1)}{\Theta}_{e2}(t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[\frac{q}{p}+\frac{1}{2}-\frac{\ln p}{p^2-1}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[c_{2n}\left(\overset{\bullet(1)}{\Gamma}_{a2n}(t)-\dot{\Theta}_u(t)\right)+d_{2n}\overset{\bullet(1)}{\Gamma}_{b2n}(t)\right], \qquad (11.36)$$

$$\overset{(1)}{\Theta}_{i2}(0,t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[\frac{1}{2\lambda}+\frac{q}{p}+\ln p\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[e_{2n}\left(\overset{\bullet(1)}{\Gamma}_{a2n}(t)-\dot{\Theta}_u(t)\right)+f_{2n}\overset{\bullet(1)}{\Gamma}_{b2n}(t)\right], \qquad (11.37)$$

$$\overset{(1)}{\Theta}_{i2}(R,t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[\frac{q}{p}+\ln p\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[g_{2n}\left(\overset{\bullet(1)}{\Gamma}_{a2n}(t)-\dot{\Theta}_u(t)\right)+h_{2n}\overset{\bullet(1)}{\Gamma}_{b2n}(t)\right]. \qquad (11.38)$$

<u>Problem (II):</u>

for k=1,3:

$$\overset{-(2)}{\Theta}_{ik}(t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[pq+\frac{p^2}{2}-\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}+\frac{k}{2(2-k)}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{kn}^4}\left[a_{kn}\left(\overset{\bullet(2)}{\Gamma}_{akn}(t)-\dot{\Theta}_u(t)\right)+b_{kn}\overset{\bullet(2)}{\Gamma}_{bkn}(t)\right], \qquad (11.39)$$

$$\overset{=(2)}{\Theta}_{ek}(t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[pq+\frac{p^2}{2}-\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}-\frac{k}{2(k+2)}\,\frac{p^{k+2}-1}{p^k-1}+\frac{k}{2(2-k)}\,\frac{p^2-1}{p^k-1}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{kn}^4}\left[c_{kn}\left(\overset{(2)}{\Gamma}_{akn}(t)-\dot{\Theta}_u(t)\right)+d_{kn}\overset{\bullet(2)}{\Gamma}_{bkn}(t)\right], \qquad (11.40)$$

$$\overset{(2)}{\Theta}_{ik}(0,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[pq+\frac{p^2}{2}-\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}+\frac{k}{2(2-k)}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{kn}}\left[e_{kn}\left(\overset{\bullet(2)}{\Gamma}_{akn}(t)-\dot{\Theta}_u(t)\right)+f_{kn}\overset{\bullet(2)}{\Gamma}_{bkn}(t)\right], \qquad (11.41)$$

$$\Theta_{ik}^{(2)}(R,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\left[pq+\frac{p^2}{2}-\left(\frac{q}{p}+\frac{1}{2-k}\right)p^{2-k}+\frac{k}{2(2-k)}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{kn}}\left[g_{kn}\left(\dot{\Gamma}_{akn}^{(2)}(t)-\dot{\Theta}_u(t)\right)+h_{kn}\dot{\Gamma}_{bkn}^{(2)}(t)\right], \quad (11.42)$$

$$\Theta_{ek}^{(2)}(pR,t)=\Theta_u(t)+\frac{W(t)R^2}{k\lambda_e}\cdot q(p-p^{1-k})+\nu^2\sum\frac{1}{\sigma_{kn}^4}\left[j_{kn}\left(\dot{\Gamma}_{akn}^{(2)}(t)-\dot{\Theta}_u(t)\right)+\right.$$

$$\left.+k_{kn}\dot{\Gamma}_{bkn}^{(2)}(t)\right], \quad k=1,2,3 \quad (11.43)$$

and for k=2:

$$\bar{\Theta}_{i2}^{(2)}(t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[pq+\frac{p^2}{2}-\frac{q}{p}-\ln p-\frac{1}{2}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[a_{2n}\left(\dot{\Gamma}_{a2n}^{(2)}(t)-\Theta_u(t)\right)+b_{2n}\dot{\Gamma}_{b2n}^{(2)}(t)\right], \quad (11.44)$$

$$\bar{\bar{\Theta}}_{e2}^{(2)}(t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[pq+\frac{p^2}{2}-\frac{q}{p}-\frac{1}{4}(p^2+1)-\frac{1}{2}+\frac{\ln p}{p^2-1}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[c_{2n}\left(\dot{\Gamma}_{a2n}^{(2)}(t)-\dot{\Theta}_u(t)\right)+d_{2n}\dot{\Gamma}_{b2n}^{(2)}(t)\right], \quad (11.45)$$

$$\Theta_{i2}^{(2)}(0,t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[pq+\frac{p^2}{2}-\frac{q}{p}-\ln p-\frac{1}{2}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[e_{2n}\left(\dot{\Gamma}_{a2n}^{(2)}(t)-\dot{\Theta}_u(t)\right)+f_{2n}\dot{\Gamma}_{b2n}^{(2)}(t)\right],$$

(11.46)

$$\Theta_{i2}^{(2)}(R,t)=\Theta_u(t)+\frac{W(t)R^2}{2\lambda_e}\left[pq+\frac{p^2}{2}-\frac{q}{p}-\ln p-\frac{1}{2}\right]+$$

$$+\nu^2\sum\frac{1}{\sigma_{2n}^4}\left[g_{2n}\left(\dot{\Gamma}_{a2n}^{(2)}(t)-\dot{\Theta}_u(t)\right)+h_{2n}\dot{\Gamma}_{b2n}^{(2)}(t)\right].$$

(11.47)

In all expressions, the very uniform last (sum-) term is the additional transient complement to the stationary solution.

The final form of the solutions for the one-layer problem (Part A) (4.25), (4.32), (4.33), is, when expressed in the same uniform notation:

$$\bar{\Theta}_k(t) = \Theta_u(t) + \frac{W(t)R^2}{k\lambda}\left(\frac{1}{k+2} + q\right) + \nu^2 \sum \frac{1}{\sigma_{kn}^4}\left[a_{kn}\left(\dot{\Gamma}_{akn}(t) - \dot{\Theta}_u(t)\right) - a_{kn}\dot{\Gamma}_{bkn}(t)\right], \tag{11.48}$$

$$\Theta_k(0,t) = \Theta_u(t) + \frac{W(t)R^2}{k\lambda}\left(\frac{1}{2} + q\right) + \nu^2 \sum \frac{1}{\sigma_{kn}^4}\left[e_{kn}\left(\dot{\Gamma}_{akn}(t) - \dot{\Theta}_u(t)\right) - e_{kn}\dot{\Gamma}_{bkn}(t)\right], \tag{11.49}$$

$$\Theta_k(R,t) = \Theta_u(t) + \frac{W(t)R^2}{k\lambda}\cdot q + \nu^2 \sum \frac{1}{\sigma_{kn}^4}\left[j_{kn}\left(\dot{\Gamma}_{akn}(t) - \dot{\Theta}_u(t)\right) - j_{kn}\dot{\Gamma}_{bkn}(t)\right], \tag{11.50}$$

using the appropriate quantities a_{kn}, e_{kn}, j_{kn}, σ_{kn}, λ, and ν for Part A.

Appendix I

BESSEL functions with negative and imaginary argument

Some properties of BESSEL functions of order n = 0 and n = 1 are compiled which are useful for the main analysis.

The usual series-definitions of these BESSEL functions are [2]:

$$J_0(x) \equiv 1 + \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2} ,$$

$$I_0(x) \equiv 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2} ,$$

$$J_1(x) \equiv \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2^{2n-1}(n-1)!n!} ,$$

$$I_1(x) \equiv \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2^{2n-1}(n-1)!n!} ,$$

$$\frac{\pi}{2} Y_0(x) \equiv (\gamma + \ln \frac{x}{2}) J_0(x) - \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2} (1 + \frac{1}{2} + \ldots + \frac{1}{n}) ,$$

$$K_0(x) \equiv -(\gamma + \ln \frac{x}{2}) I_0(x) + \sum_{n=1}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2} (1 + \frac{1}{2} + \ldots + \frac{1}{n}) ,$$

$$\frac{\pi}{2} Y_1(x) \equiv - \frac{1}{x} + (\gamma + \ln \frac{x}{2}) J_1(x) - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2^{2n-1}(n-1)!n!} \times$$

$$\times (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \frac{1}{2n}) ,$$

$$K_1(x) = \frac{1}{x} + (\gamma + \ln\frac{x}{2})\, I_1(x) - \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2^{2n-1}(n-1)!\,n!} \times$$

$$\times \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \frac{1}{2n}\right) \quad .$$

γ is EULER's constant .5772...

We now rotate the arguments of all these functions through angles $\pm\,\pi$ in the complex domain. For the functions with logarithmic singularities at the origin, we make a cut along on the negative real axis so that they are many-valued there. We define the principal values to correspond with the positive (counter-clockwise) rotation π from +x, e.g. $Y_0(-x) \equiv Y_0(e^{+i\pi}\,x)$, and not $Y_0(e^{-i\pi}\,x)$.

The following table can then be deduced from the above equations:

$$J_0(+ix) = +\, I_0(x) \qquad J_1(+ix) = +\, iI_1(x)$$
$$J_0(-x) = +\, J_0(x) \qquad J_1(-x) = -\, J_1(x)$$
$$J_0(-ix) = +\, I_0(x) \qquad J_1(-ix) = -\, iI_1(x)$$

$$I_0(+ix) = +\, J_0(x) \qquad I_1(+ix) = +\, iJ_1(x)$$
$$I_0(-x) = +\, I_0(x) \qquad I_1(-x) = -\, I_1(x)$$
$$I_0(-ix) = +\, J_0(x) \qquad I_1(-ix) = -\, iJ_1(x)$$

$$\frac{\pi}{2}\, Y_0(+ix) = -\, K_0(x) + i\frac{\pi}{2}I_0(x) \qquad \frac{\pi}{2}\, Y_1(+ix) = -\, \frac{\pi}{2}\, I_1(x) + iK_1(x)$$
$$\frac{\pi}{2}\, Y_0(-x) = +\, \frac{\pi}{2}\, y_0(x) + i\pi J_0(x) \qquad \frac{\pi}{2}\, Y_1(-x) = -\, \frac{\pi}{2}\, Y_1(x) - i\pi J_1(x)$$
$$\frac{\pi}{2}\, Y_0(-ix) = -\, K_0(x) - i\frac{\pi}{2}I_0(x) \qquad \frac{\pi}{2}\, Y_1(-ix) = -\, \frac{\pi}{2}\, I_1(x) - iK_1(x)$$

$$K_0(+ix) = -\, \frac{\pi}{2}\, Y_0(x) - i\frac{\pi}{2}J_0(x) \qquad K_1(+ix) = -\, \frac{\pi}{2}\, J_1(x) + i\,\frac{\pi}{2}Y_1(x)$$
$$K_0(-x) = +\, K_0(x) - i\pi I_0(x) \qquad K_1(-x) = -\, K_1(x) - i\pi I_1(x)$$
$$K_0(-ix) = -\, \frac{\pi}{2}\, Y_0(x) + i\,\frac{\pi}{2}J_0(x) \qquad K_1(-ix) = -\, \frac{\pi}{2}\, J_1(x) - i\,\frac{\pi}{2}Y_1(x)$$

The difference between consecutive sheets at the cut (i.e. for argument - x), is $\pm 2\pi i J_n(x)$ for the functions $\frac{\pi}{2} Y_n(x)$, and $\pm 2\pi i I_n(x)$ for the functions $K_n(x)$ $(n = 0,1)$.

The results indicate that

$$\frac{\pi}{2} Y_n(-ix) \text{ is conjugate complex to } \frac{\pi}{2} Y_n(+ix)$$

and $K_n(-ix)$ is conjugate complex to $K_n(+ix)$.

$\frac{\pi}{2} Y_n(-x)$ and $K_n(-x)$, both complex-valued, are self-conjugate since the imaginary parts at the two sheets differ only in sign.

Appendix II

Computation of some limits involving the "fundamental functions"

In general, the limits for $X \to 0$ of the transfer functions needed in Chapter 9 lead to indefinite forms. Applying L'HOPITAL's rule is not convenient for their evaluation because of the very complex formulas due to repeated differentiation.

It is far better to consider the first terms of the TAYLOR expansions of the fundamental functions about $X = 0$, if there is no singularity, or, in addition, terms in $\frac{1}{X}$, $\frac{1}{X^2}$, ..., and ln X, defining the type of singularity in the origin.

According to the definitions (6.17) of the functions F_k and Φ_k, and of their derivatives (6.18), one finds the following behaviour near $X = +0$ (for the BESSEL functions cf. Appendix I):

k	$F_k(+0)$	$F_k'(+0)$	$\Phi_k(+0)$	$\Phi_k'(+0)$
1	1	$0\ (\sim +X)$	$0\ (\sim -iX)$	$-i$
2	1	$0\ (\sim +\frac{X}{2})$	$-\infty\ (\sim +\ln X)$	$+\infty\ (\sim +\frac{1}{X})$
3	1	$0\ (\sim +\frac{X}{3})$	$-i\infty\ (\sim -\frac{i}{X})$	$+i\infty(\sim +\frac{i}{X^2})$
k	1	$0\ (\sim +\frac{X}{k})$	$(\sim -i^k\,\frac{X^{2-k}}{2-k})$	$(\sim -i^kX^{1-k})$

The uniform notation for general k has been achieved by a suitable choice of the coefficients of ψ_k in (6.16), or Φ_k in (6.17), respectively. We define $X^{2-k}/(2-k)$ to mean ln X, when $k = 2$.

Substituting the above expressions into (6.12) for the system determinant Δ_k shows that

$$\lim_{X\to 0} \Delta_k(X) \sim \left[1.(-i^k)(aX)^{1-k} - \lambda\,\frac{X}{k}\,(-i^k)\,\frac{X^{2-k}}{2-k}\right]\left[qX\,\frac{apX}{k} + 1\right] -$$

$$- \left[1.\frac{aX}{k} - \lambda\,\frac{X}{k}.1\right]\left[qX(-i^k)(apX)^{1-k} + (-i^k)\,\frac{(apX)^{2-k}}{2-k}\right]$$

$$\sim -\,i^k(aX)^{1-k}\ ,$$

as the lowest power of X dominates; i.e. X^{1-k} in the first term.

Similarly, one finds

$$\lim_{X\to 0} \overline{F}_k(X) = \lim_{X\to 0} \frac{k}{X} F'_k(X) = +1,$$

$$\lim_{X\to 0} \overline{\overline{F}}_k(X) = \lim_{X\to 0} \frac{k}{aX} \frac{p^{k-1}F'_k(apX) - F'_k(aX)}{p^k-1} = +1,$$

$$\begin{aligned}\lim_{X\to 0} \overline{\overline{\Phi}}_k(X) = \lim_{X\to 0} \frac{k}{aX} \frac{p^{k-1}\Phi'_k(apX) - \Phi'_k(aX)}{p^k-1} &= 0 \left(\sim -\tfrac{i}{2}(p+1)aX\right) && k=1\\ &= -\infty(\sim \ln aX) && k=2\\ &= -i\infty\left(\sim \tfrac{3}{2}\,\frac{p+1}{p^2+p+1}\cdot\frac{1}{aX}\right) && k=3\end{aligned}$$

Nevertheless, equations (9.37) and (9.38) still remain valid also for k=2 and k=3.

The limits of the six transfer function numerators in (6.6) to (6.11) are:

$$\lim_{X\to 0} Z_{I1k}(X) = \lim_{X\to 0} \left\{F(aX)\,\Phi'(aX) - F'(aX)\,\Phi(aX)\right\} \sim -\,i^k(aX)^{1-k}$$

$$\lim_{X\to 0} Z_{I2k}(X) = \lim_{X\to 0} \Big\{F'(aX)[qX\Phi'(apX) + \Phi(apX)] - \Phi'(aX)[qXF'(apX) + F(apX)]\Big\} \sim +\,i^k(aX)^{1-k}$$

$$\lim_{X\to 0} Z_{E_11k}(X) = \lim_{X\to 0} \left\{F(X)\Phi'(aX) - \lambda F'(X)\Phi(aX)\right\} \sim -\,i^k(aX)^{1-k}$$

$$\lim_{X\to 0} Z_{E_21k}(X) = \lim_{X\to 0} \left\{-\,F(X)F'(aX) + \lambda F'(X)F(aX)\right\} = 0$$

$$\lim_{X\to 0} Z_{E_12k}(X) = \lim_{X\to 0} \left\{\lambda F'(X)[qX\Phi(apX) + \Phi(apX)]\right\}$$

$$\sim -\,i^k \frac{\lambda}{k}\left(\frac{q}{ap} + \frac{1}{2-k}\right)(ap)^{2-k}X^{3-k}$$

$$= \begin{cases} 0 & k=1,2 \\ + i \frac{\lambda}{3} \left[\frac{q}{(ap)^2} - \frac{1}{ap} \right] = \text{const.} & k=3 \end{cases}$$

$$\lim_{X \to 0} Z_{E_2 2k}(X) = \lim_{X \to 0} \left\{ - \lambda F'(X) [qXF'(apX) + F(apX)] \right\} = 0$$

By combining all these results, the limits (9.35) to (9.44) are obtained.

Appendix III

Examples and summations of number-theoretical interest

This appendix illustrates not only the procedure followed in the main part but also provides some results of pure mathematical interest.

In the course of the development, one meets systematically summations of the types

$$\nu\sum_{n=1}^{\infty}\frac{\varepsilon_{kn}}{\sigma_{kn}^{2}} = \dots\ , \text{ and } \nu\sum_{n=1}^{\infty}\frac{\varepsilon_{kn}}{\sigma_{kn}^{4}} = \dots\ (\varepsilon_{kn} = a_{kn},\ b_{kn},\ \dots)\ .$$

As the ε_{kn} are computable coefficients, these formulas can be checked in some cases by specifying the parameters k, p, q, a, and λ.

We consider two cases, one each from Part A and B.

Example A:

We consider all three geometries k = 1,2,3. Let, for simplicity, R = a, so that $\nu = 1$.

First, we treat the case q = 0 (boundary condition of the first kind, i.e. given boundary temperature). The discriminant equation (3.7) and solutions are

$$\left.\begin{array}{lll} k = 1 & \cos\sigma_1 = 0 & \sigma_{1n} = (2n-1)\frac{\pi}{2} \\ k = 2 & J_0(\sigma_2) = 0 & \sigma_{2n} = j_{on}{}^{*)} \\ k = 3 & \frac{\sin\sigma_3}{\sigma_3} = 0 & \sigma_{3n} = n\pi \end{array}\right\}\quad n = 1,2,3,\dots$$

---

*) j_{on}, the zeros of $J_o(\sigma)$, and $J_1(j_{on})$ are well-known; see e.g. [2]. These zeros must not be confused with the residues j_{kn} for the boundary temperature.

The derivative (3.22) of Δ_k is

$$\Delta'_{kn} = \begin{cases} \dfrac{(-1)^{n-1}}{2\sigma_{1n}} & k = 1 \\[2ex] \dfrac{J_1(j_{on})}{2\sigma_{2n}} & k = 2 \\[2ex] \dfrac{(-1)^{n-1}}{2\sigma_{3n}^2} & k = 3 \end{cases}$$

Hence, from (3.23) to (3.25):

k	a	e	j
1	$a_{1n} = +2$	$e_{1n} = (-1)^{n-1}.2\sigma_{1n}$	$j_{1n} = 0$
2	$a_{2n} = +4$	$e_{2n} = \dfrac{2\sigma_{2n}}{J_1(j_{on})}$	$j_{2n} = 0$
3	$a_{3n} = +6$	$e_{3n} = (-1)^{n-1}.2\sigma_{3n}^2$	$j_{3n} = 0$

We now apply the summations (4.14) to (4.16) from n = 1 to infinity:

$$\sum \frac{a_{1n}}{\sigma_{1n}^2} = 2.\frac{4}{\pi^2}\sum \frac{1}{(2n-1)^2} = 1 \qquad \rightarrow \sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \quad *)$$

$$\sum \frac{a_{2n}}{\sigma_{2n}^2} = 4.\sum \frac{1}{j_{on}^2} = 1 \qquad \rightarrow \sum \frac{1}{j_{on}^2} = \frac{1}{4} \quad \text{[3], p.502}$$

$$\sum \frac{a_{3n}}{\sigma_{3n}^2} = 6.\frac{1}{\pi^2}\sum \frac{1}{n^2} = 1 \qquad \rightarrow \sum \frac{1}{n^2} = \frac{\pi^2}{6} \quad *)$$

$$\sum \frac{e_{1n}}{\sigma_{1n}^2} = 2.\frac{2}{\pi}\sum \frac{(-1)^{n-1}}{2n-1} = 1 \qquad \rightarrow \sum \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \quad *)$$

$$\sum \frac{e_{2n}}{\sigma_{2n}^2} = 2.\sum \frac{1}{j_{on}J_1(j_{on})} = 1 \qquad \rightarrow \sum \frac{1}{j_{on}J_1(j_{on})} = \frac{1}{2} \quad \text{[4],p.199}$$

$$\sum \frac{e_{3n}}{\sigma_{3n}^2} = 2.\sum (-1)^{n-1} = 1 \qquad \ldots$$

$$\sum \frac{j_{kn}}{\sigma_{kn}^2} = \ldots = 1$$ (this paradox is discussed below in case q = 1).

At the right hand side, we denote commonly known summations with an asterisk, which therefore confirm our computations. For less known summations, references are given. The summation for $\frac{e_{3n}}{\sigma_{3n}^2}$ shows a limiting case which is no longer convergent.
This indicates that the rate of convergence of the main solutions (11.39) to (11.50) can vary considerably with the system parameters.

Similarly, for the sum formulas (4.24), (4.30), (4.31):

$$\sum \frac{a_{1n}}{\sigma_{1n}^4} = 2.\frac{16}{\pi^4}\sum \frac{1}{(2n-1)^4} = \frac{1}{3} \qquad \rightarrow \sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} \qquad *)$$

$$\sum \frac{a_{2n}}{\sigma_{2n}^4} = 4.\sum \frac{1}{j_{on}^4} = \frac{1}{8} \qquad \rightarrow \sum \frac{1}{j_{on}^4} = \frac{1}{32} \qquad [3],\text{p.}502$$

$$\sum \frac{a_{3n}}{\sigma_{3n}^4} = 6.\frac{1}{\pi^4}\sum \frac{1}{n^4} = \frac{1}{15} \qquad \rightarrow \sum \frac{1}{n^4} = \frac{\pi^4}{90} \qquad *)$$

$$\sum \frac{e_{1n}}{\sigma_{1n}^4} = 2.\frac{8}{\pi^3}\sum \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{1}{2} \qquad \rightarrow \sum \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32} \qquad *)$$

$$\sum \frac{e_{2n}}{\sigma_{2n}^4} = 2.\sum \frac{1}{j_{on}^3 J_1(j_{on})} = \frac{1}{4} \qquad \rightarrow \sum \frac{1}{j_{on}^3 J_1(j_{on})} = \frac{1}{8} \qquad [4],\text{p.}201$$

$$\sum \frac{e_{3n}}{\sigma_{3n}^4} = 2.\frac{1}{\pi^2}\sum \frac{(-1)^{n-1}}{n^2} = \frac{1}{6} \qquad \rightarrow \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} \qquad *)$$

$$\sum \frac{j_{kn}}{\sigma_{kn}^4} = \dots = 0.$$

Now, we pass to case q = 1. The discriminant equation (3.7) is:

$k = 1 \qquad \sigma_1 \tan \sigma_1 = 1$

$k = 2 \qquad \sigma_2 J_1(\sigma_2) = J_0(\sigma_2)$

$k = 3 \qquad \cos \sigma_3 = 0$ with the solution $\sigma_{3n} = (2n-1)\frac{\pi}{2}$.

From formulas (3.23) to (3.25):

k	a	e	j
1	$a_{1n} = \frac{2}{2+\sigma_{1n}^2}$	$e_{1n} = \frac{2\sigma_{1n}}{\sin\sigma_{1n}} \cdot \frac{1}{2+\sigma_{1n}^2}$	$j_{1n} = \frac{2\sigma_{1n}^2}{2+\sigma_{1n}^2}$
2	$a_{2n} = \frac{4}{1+\sigma_{2n}^2}$	$e_{2n} = \frac{2\sigma_{2n}}{J_1(\sigma_{2n})} \cdot \frac{1}{1+\sigma_{2n}^2}$	$j_{2n} = \frac{2\sigma_{2n}^2}{1+\sigma_{2n}^2}$
3	$a_{3n} = \frac{6}{\sigma_{3n}^2}$	$e_{3n} = (-1)^{n-1} \cdot 2\sigma_{3n}$	$j_{3n} = +2$

Unfortunately, there are no simple formulas for the σ_{1n} and σ_{2n} though they have been tabulated ([4], Appendix IV) so that the expressions for k = 1,2 cannot be further simplified.

Applying the summations (4.14) to (4.16) shows that

$$\sum \frac{a_{1n}}{\sigma_{1n}^2} = 2\sum \frac{1}{2\sigma_{1n}^2+\sigma_{1n}^4} = 1 \,, \qquad \text{[4],pp.121, 122}$$

$$\sum \frac{a_{2n}}{\sigma_{2n}^2} = 4\sum \frac{1}{\sigma_{2n}^2+\sigma_{2n}^4} = 1 \,, \qquad \text{[4],p. 202}$$

$$\sum \frac{a_{3n}}{\sigma_{3n}^4} = 6.\frac{16}{\pi^4}\sum \frac{1}{(2n-1)^4} = 1 \,, \qquad \rightarrow \sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{96} \qquad *)$$

$$\sum \frac{e_{1n}}{\sigma_{1n}^2} = 2\sum \frac{1}{\cos\sigma_{1n}.(2+\sigma_{1n}^2)} = 1 \,, \qquad \text{[4],p.122}$$

$$\sum \frac{e_{2n}}{\sigma_{2n}^2} = 2\sum \frac{1}{J_0(\sigma_{2n}).(1+\sigma_{2n}^2)} = 1 \,, \qquad \text{[4],p.202}$$

$$\sum \frac{e_{3n}}{\sigma_{3n}^2} = 2.\frac{2}{\pi}\sum \frac{(-1)^{n-1}}{2n-1} = 1 \,, \qquad \rightarrow \sum \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4} \qquad *)$$

$$\sum \frac{j_{1n}}{\sigma_{1n}^2} = 2\sum \frac{1}{2+\sigma_{1n}^2} = 1 \,, \qquad \text{[4],p.122}$$

$$\sum \frac{j_{2n}}{\sigma_{2n}^2} = 2\sum \frac{1}{1+\sigma_{2n}^2} = 1 \,, \qquad \text{[4],p.202}$$

$$\sum \frac{j_{3n}}{\sigma_{3n}^2} = 2.\frac{4}{\pi^2}\sum \frac{1}{(2n-1)^2} = 1 \ , \qquad \rightarrow \sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \qquad *)$$

The series for k = 1 and k = 2 converge particularly rapidly because of the high powers of σ_{1n} and σ_{2n} respectively. In principle, the results for the σ_{1n} and σ_{2n} would be as interesting as for the σ_{3n} if they were in common use as e.g. the j_{on}.

N.B. For q = 1, the j_{kn} are finite numbers so that the summations $\sum \frac{j_{kn}}{\sigma_{kn}^2}$ converge to unity without difficulty.

This is also true for q < 1, as long as q ≠ 0; only the convergence of the series deteriorates as $q \rightarrow 0$. In the actual case q = 0, we have $j_{kn} = 0$ (cf. 3.25), and the series contains infinitely many vanishing terms. If however we define this summation by letting $q \rightarrow 0$ instead of putting q = 0, the summation remains unity, and thi explains the paradox found for this summation in the previous case q = 0.

Example B

We take the following special case in Part B:

k = 1 plane geometry,

p = 2 i.e. $R_e = 2R_i$; equal layer thicknesses,

q = 0 i.e. $\alpha = \infty$; boundary condition of the first kind, i.e. boundary temperature $\Theta_e(pR) = \Theta_u$,

$\left.\begin{array}{l} a = 1 \\ \lambda = 1 \end{array}\right\}$ eaual material properties for both layers,

$\nu = 1$ time constant equal unity.

Then the discriminant equation (7.1) reduces to:

$$G_1(ap\sigma) = \cos 2\sigma = 0$$

with the solutions

$$\sigma_n = \pm (2n-1)\frac{\pi}{4} \qquad n = 1,2,\ldots$$

As emphasized in the text, when $+\sigma_n$ is a root, so is $-\sigma_n$. But the values of the summation are independent of the sign of σ_n.

We compile the required fundamental functions:

$$G_1(\sigma_n) = G_1(a\sigma_n) = +\cos(2n-1)\frac{\pi}{4} \quad = \tfrac{1}{2}\sqrt{2}$$

n=	1	2	3	4	5	6	*)
.	+1	-1	-1	+1	+1	-1	-++-.

$$G_1(ap\sigma_n) = +\cos(2n-1)\frac{\pi}{2} = 0$$

$$G_1'(\sigma_n) = G_1'(a\sigma_n) = -\sin(2n-1)\frac{\pi}{4} \quad = \tfrac{1}{2}\sqrt{2}$$

n=	1	2	3	4	5	6	
.	-	-	+	+	-	-	++--.

$$G_1'(ap\sigma_n) = -\sin(2n-1)\frac{\pi}{2} = (-1)^n$$

$$G_1'(ap\sigma_n) = -\sin(2n-1)\frac{\pi}{2} = (-1)^n$$

$$\psi_1(\sigma_n) = \psi_1(a\sigma_n) = +\sin(2n-1)\frac{\pi}{4} \quad = \tfrac{1}{2}\sqrt{2}$$

n=	1	2	3	4	5	6	
.	+	+	-	-	+	+	--++.

$$\psi_1(ap\sigma_n) = +\sin(2n-1)\frac{\pi}{2} = (-1)^{n-1}$$

$$\psi_1'(\sigma_n) = \psi_1'(a\sigma_n) = +\cos(2n-1)\frac{\pi}{4} \quad = \tfrac{1}{2}\sqrt{2}$$

n=	1	2	3	4	5	6	
.	+	-	-	+	+	-	-++-.

$$\psi_1'(ap\sigma_n) = +\cos(2n-1)\frac{\pi}{2} = 0\ .$$

$$\Delta_n' = +\frac{i}{2\sigma_n}\cdot 2G_1'(ap\sigma_n) = +\frac{i}{\sigma_n}(-1)^n \qquad \text{(cf. 7, 10).}$$

$$a_n = (-1)^n G_1'(\sigma_n) = \qquad \tfrac{1}{2}\sqrt{2}$$

n=	1	2	3	4	5	6	
.	+	-	-	+	+	-	-++- ...

(cf. 8.4)

---

*) The sign sequence +--++--... may be represented by

$$\frac{i^n}{2}\left\{[1+(-1)^n]-i[1-(-1)^n]\right\} \qquad n=1,2,\ldots\ ,$$

and the sequence ++--++--... by

$$\frac{i^{n-1}}{2}\left\{[1+(-1)^{n-1}]-i[1-(-1)^{n-1}]\right\}\ .\ n=1,2,\ldots\ .$$

For practical purposes, however, these are too clumsy.

$$b_n = (-1)^n G_1'(a\sigma_n)\psi_1(ap\sigma_n)G_1'(\sigma_n) = -\tfrac{1}{2} \qquad \text{(cf.8.5)} ,$$

$$c_n = (-1)^n[G_1'(ap\sigma_n)-G_1'(a\sigma_n)] = +1 - a_n \qquad \text{(cf.8.6)} ,$$

$$d_n = (-1)^n G_1'(\sigma_n)\psi_1(ap\sigma_n)[G_1'(ap\sigma_n)-G_1'(a\sigma_n)] = +\tfrac{1}{2} - a_n \quad \text{(cf.8.7)} ,$$

$$e_n = (-1)^{n-1}.\sigma_n \qquad \text{(cf.8.8)} ,$$

$$f_n = (-1)^{n-1}.\sigma_n G_1'(a\sigma_n)\psi_1(ap\sigma_n) \qquad = \sigma_n\tfrac{1}{2}\sqrt{2}\ . \begin{array}{c|c|c|c|c|c|c|l} n= & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & - & - & + & + & - & - & ++ \end{array}$$

$$\text{(cf.8.9)} ,$$

$$g_n = (-1)^{n-1}.\sigma_n G(\sigma_n) \qquad = \sigma_n\tfrac{1}{2}\sqrt{2}\ . \begin{array}{c|c|c|c|c|c|c|l} n= & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline & + & + & - & - & + & + & -- \end{array}$$

$$\text{(cf.8.10)} ,$$

$$= -f_n$$

$$h_n = (-1)^{n-1}.\sigma_n G_1'(a\sigma_n)\psi_1(ap\sigma_n)G_1(\sigma_n) = (-1)^{n-1}\ \sigma_n.(-\tfrac{1}{2}) \quad \text{(cf.8.11)},$$

$$j_n = (-1)^{n-1}.\sigma_n G_1(ap\sigma_n) = 0 \quad \text{(N.B. Not true in general for } q=0) \quad \text{(cf.8.12)},$$

$$k_n = (-1)^{n-1}.\sigma_n.q = 0 \quad \text{(N.B. Always true if } q=0) \quad \text{(cf.8.13)}.$$

Now, by substituting $(2n-1)\frac{\pi}{4}$ for σ_n, all summations may be checked.

By (9.45) :

$$\sum \frac{a_n}{\sigma_n^2} = \tfrac{1}{2}\sqrt{2}.\frac{16}{\pi^2}\left[\sum_{\substack{n=1,\\4,5,\\8,9,\\ \dots}} \frac{1}{(2n-1)^2} - \sum_{\substack{n=\\2,3,\\6,7,\\ \dots}} \frac{1}{(2n-1)^2} \right] = +1 .$$

This rather strange formula can be found in [5],p.360, by properly choosing the parameters. Of course, the analytical representation of the sign sequence,as given in the preceding footnote, may be inserted, if desired.

(9.46) shows that:

$$\sum \frac{b_n}{\sigma_n^2} = - \frac{1}{2} \cdot \frac{16}{\pi^2} \sum \frac{1}{(2n-1)^2} = - 1 .$$

This yields the known summation $\sum \frac{1}{(2n-1)^2} = \frac{\pi}{8}$, which provides a check.

Now, from (9.47):

$$\sum \frac{c_n}{\sigma_n^2} = \sum \frac{1}{\sigma_n^2} - \sum \frac{a_n}{\sigma_n^2} = \frac{16}{\pi^2} \sum \frac{1}{(2n-1)^2} - 1 = 2 - 1 = + 1 ,$$

and from (9.48):

$$\sum \frac{d_n}{\sigma_n^2} = + \frac{1}{2} \sum \frac{1}{\sigma_n^2} - \sum \frac{a_n}{\sigma_n^2} = + 1 - 1 = 0 .$$

Further (cf. 9.49):

$$\sum \frac{e_n}{\sigma_n^2} = \frac{4}{\pi} \sum \frac{(-1)^{n-1}}{2n-1} = + 1 .$$

This leads to $\sum \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$, a known summation, too.

From (9.50):

$$\sum \frac{f_n}{\sigma_n^2} = \frac{1}{2} \sqrt{2} \cdot \frac{4}{\pi} \left[\sum_{\substack{n= \\ 3,4, \\ 7,8, \\ \dots}} \frac{1}{2n-1} - \sum_{\substack{n= \\ 1,2, \\ 5,6, \\ \dots}} \frac{1}{2n-1} \right] = - 1 .$$

Also this interesting formula proves to be correct numerically.

From (9.51):

$$\sum \frac{\varepsilon_n}{\sigma_n^2} = - \sum \frac{f_n}{\sigma_n^2} = + 1 .$$

Finally, we have the interesting case (cf. 9.52):

$$\sum \frac{h_n}{\sigma_n^2} = -\frac{1}{2}\cdot\frac{4}{\pi}\sum \frac{(-1)^{n-1}}{2n-1} = -\frac{1}{2} \;,$$

so that $h_o = -1 - \sum \frac{h_n}{\sigma_n^2} = -\frac{1}{2}$.

This explains the introduction of h_o in Chapter 9. As in the example q=0 in Part A, the j_n summation is an infinite sum of (in the limit) vanishing terms. The k_n summation is trivial.

Now, we investigate the $\frac{1}{\sigma_n^4}$ summations.
From (11.3):

$$\sum \frac{b_n}{\sigma_n^4} = -\frac{1}{2}\cdot\frac{256}{\pi^4}\sum \frac{1}{(2n-1)^4} = -\frac{1}{3} - (2-1) = -\frac{4}{3} \quad .$$

Consequently, $\sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$, a known summation.
From (11.14):

$$\sum \frac{a_n}{\sigma_n^4} = \frac{1}{2}\sqrt{2}\cdot\frac{256}{\pi^4}\Bigg[\sum_{\substack{n=1,\\4,5,\\8,9,\\\dots}} \frac{1}{(2n-1)^4} - \sum_{\substack{n=\\2,3,\\6,7,\\\dots}} \frac{1}{(2n-1)^4}\Bigg] = 2 - \frac{1}{6} = +\frac{11}{6} \quad .$$

[5],p.364

This formula checks numerically.

$$\sum \frac{d_n}{\sigma_n^4} = +\frac{1}{2}\sum \frac{1}{\sigma_n^4} - \sum \frac{a_n}{\sigma_n^4} = \frac{1}{2}\cdot\frac{256}{\pi^4}\sum \frac{1}{(2n-1)^4} - \frac{11}{6} = \frac{4}{3} - \frac{11}{6} = -\frac{1}{2}$$

which checks with (11.5),

$$\sum \frac{c_n}{\sigma_n^4} = \sum \frac{1}{\sigma_n^4} - \sum \frac{a_n}{\sigma_n^4} = \frac{8}{3} - \frac{11}{6} = +\frac{5}{6} \;.$$

which checks with (11.18),

$$\sum \frac{f_n}{\sigma_n^4} = \frac{1}{2} \sqrt{2} . \frac{64}{\pi^3} \left[\sum_{\substack{n= \\ 3,4, \\ 7,8, \\ \dots}} \frac{1}{(2n-1)^3} - \sum_{\substack{n= \\ 1,2, \\ 5,6, \\ \dots}} \frac{1}{(2n-1)^3} \right] = - \frac{3}{2} \; . \qquad [5], p.364$$

This summation has also been checked numerically.

From (11.22):

$$\sum \frac{e_n}{\sigma_n^4} = \frac{64}{\pi^3} \sum \frac{(-1)^{n-1}}{(2n-1)^3} = + 2, \text{ leading to } \sum \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32} \; ,$$

which is known.

$$\sum \frac{h_n}{\sigma_n^4} = - \frac{1}{2} \sum \frac{e_n}{\sigma_n^4} = - 1 \; ,$$

which checks with (11.9) ,

$$\sum \frac{g_n}{\sigma_n^4} = - \sum \frac{f_n}{\sigma_n^4} = + \frac{3}{2} \; ,$$

which checks with (11.26).

Formulas (11.11) for k_n and (11.29) for j_n are trivial in case $q = 0$.

LIST OF SYMBOLS

As the working equations are all in a dimensionless form, any consistent set of physical units can be used.

a^2	thermal diffusivity, Part A
a^2	ratio of thermal diffusivities ($= a_i^2/a_e^2$), Part B
$a_{kn},\ldots,k_{kn}$	coefficients of the various partial fraction series
A_{k1}, A_{k2}	s-dependent coefficients of the general solution, Part A
F,Φ	pairs of fundamental solutions of reduced eq.(1.8)
F_{ap}	(and similar indexed functions), abbreviation for $F_k(apX)$, see Chapter 6
G,ψ	pairs of fundamental solutions of the "modified" equation (6.14)
I_k, E_{k_1},E_{k_2}	s-dependent coefficients of the general solution, Part B
k	geometric index (1 = plane, 2 = cylindrical, 3 = spherical)
$N^{(\varepsilon)}$	numerator of any transfer function
p	ratio of "radii" ($= R_e/R_i$)
P(s)	$W(s)/\rho c$
q	reciprocal NUSSELT number ($= \lambda/\alpha R$ Part A; $= \lambda_e/\alpha R_i$ Part B)
r	space variable
R	radius (or layer half-thickness)
s	complex LAPLACE variable
t	time
W(t)	heat source per unit time and volume (given)
W_o	steady heat source (initial value)
$\Delta\tilde{W}(s)$	LAPLACE-transformed heat source (above steady state level)
x	transformed space variable ($= \frac{\sqrt{s}}{a}r$ Part A; $= \frac{\sqrt{s}}{a_i}r$ Part B)
X	non-dimensional radius
Z	numerators of specified transfer functions
α	heat transfer coefficient
Δ	the system discriminant

ε_{ko}	typical additive integral function for a partial fraction series
ε_{kn}	typical development coefficients in a partial fraction series
Γ_a	transient complement function with input $\dot{\Theta}_u$
Γ_b	transient complement function with input $\Delta W/\rho c$ (for Part B, problem II, see 9.19 and 9.20)
λ	thermal conductivity, Part A
λ	ratio of thermal conductivities ($= \lambda_i/\lambda_e$), Part B
ν	time constant ($= R_i^2/a_i^2$)
ρc	specific heat per unit volume
σ_{kn}	the zeros of Δ (eigenvalues of the problem)
ϑ	LAPLACE-transformed temperature
Θ	temperature
$\Theta_u(t)$	ambient temperature (given)

Indexes:

e	external layer
i	internal layer
n	summation index ($n = 1,\ldots,\infty$, where not otherwise specified)
$(_1)$	"problem I", Part B
$(_2)$	"problem II", Part B
$\overline{}$	averaged over internal layer
$=$	averaged over external layer

REFERENCES

[1] R. PALINSKI, "Lösung der instationären Wärmeleitungsgleichung mit zeitabhängiger Umgebungstemperatur und zeitlich veränderlicher und räumlich ungleichmässig verteilter Wärmequelle", Nukleonik 6, Heft 6, pp. 289-303 (1964)

R. PALINSKI, "Lösung der instationären Wärmeleitungsprobleme für hohlzylindrische Körper mit zeitabhängiger Umgebungstemperatur und Flächenbelastung". Nukleonik 7, Heft 2, pp. 93-106 (1965)

R. PALINSKI, "A Mathematical Method for the Dynamic Behaviour of a Power Reactor with Particular Regard to Non-Stationary Heat Transfer", Nukleonik 12, Heft 3, pp. 134-142 (1969)

[2] British Association for the Advancement of Science, Mathematical Tables, Vol. VI, (BESSEL functions I, orders zero and unity), Cambridge University Press, 1958

[3] G.N. WATSON, "A Treatise on the Theory of BESSEL Functions", Cambridge University Press, 2nd ed. 1958

[4] H.S.CARSLAW, and J.C. JAEGER, "Conduction of Heat in Solids", Clarendon Press, Oxford, 2nd ed. 1962

[5] T.J. BROMWICH, "An Introduction to the Theory of Infinite Series", MacMillan, London, 2nd ed.

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