EUR 4678 e

COMMISSION OF THE EUROPEAN COMMUNITIES

MONTE CARLO SIMULATION OF THE ADJOINT TRANSPORT EQUATION

by

W. MATTHES

1971

Joint Nuclear Research Centre
Ispra Establishment - Italy

Nuclear Studies

LEGAL NOTICE

This report is on sale at the addresses listed on cover page 4

at the price of FF 6.65	FB 60.—	DM 4 40	Lit. 750	Fl. 4.30

When ordering, please quote the EUR number and the title which are indicated on the cover of each report.

Printed by Van Muysewinkel, Brussels
Luxembourg, August 1971

EUR 4678 e

MONTE CARLO SIMULATION OF THE ADJOINT TRANSPORT EQUATION by W. MATTHES

Commission of the European Communities
Joint Nuclear Research Centre - Ispra Establishment (Italy)
Nuclear Studies
Luxembourg, August 1971 - 44 Pages - 6 Figures - B.Fr. 60.—

Usually one starts from the adjoint transport equation to construct a Monte Carlo game for its simulation. This procedure requires a sophisticated reasoning to find out which quantity of the game is adjoint to what. Contrary to this point of view we start from the beginning with two independent games of two different kinds of particles and put the condition that the expectation value of some estimator in the two games should be equal. This leads directly to the Monte Carlo game for the adjoint flux and provides on with a large arbitrariness for the adjoint game. This arbitrariness can be used to find adjoint games with smaller variances. This method is applied to the calculation of the neutron flux in an annular air gap in the water shield of a cylindrical reactor.

EUR 4678 e

MONTE CARLO SIMULATION OF THE ADJOINT TRANSPORT EQUATION by W. MATTHES

Commission of the European Communities
Joint Nuclear Research Centre - Ispra Establishment (Italy)
Nuclear Studies
Luxembourg, August 1971 - 44 Pages - 6 Figures - B.Fr. 60.—

Usually one starts from the adjoint transport equation to construct a Monte Carlo game for its simulation. This procedure requires a sophisticated reasoning to find out which quantity of the game is adjoint to what. Contrary to this point of view we start from the beginning with two independent games of two different kinds of particles and put the condition that the expectation value of some estimator in the two games should be equal. This leads directly to the Monte Carlo game for the adjoint flux and provides on with a large arbitrariness for the adjoint game. This arbitrariness can be used to find adjoint games with smaller variances. This method is applied to the calculation of the neutron flux in an annular air gap in the water shield of a cylindrical reactor.

EUR 4678 e

MONTE CARLO SIMULATION OF THE ADJOINT TRANSPORT EQUATION by W. MATTHES

Commission of the European Communities
Joint Nuclear Research Centre - Ispra Establishment (Italy)
Nuclear Studies
Luxembourg, August 1971 - 44 Pages - 6 Figures - B.Fr. 60.—

Usually one starts from the adjoint transport equation to construct a Monte Carlo game for its simulation. This procedure requires a sophisticated reasoning to find out which quantity of the game is adjoint to what. Contrary to this point of view we start from the beginning with two independent games of two different kinds of particles and put the condition that the expectation value of some estimator in the two games should be equal. This leads directly to the Monte Carlo game for the adjoint flux and provides on with a large arbitrariness for the adjoint game. This arbitrariness can be used to find adjoint games with smaller variances. This method is applied to the calculation of the neutron flux in an annular air gap in the water shield of a cylindrical reactor.

EUR 4678 e

COMMISSION OF THE EUROPEAN COMMUNITIES

MONTE CARLO SIMULATION OF THE ADJOINT TRANSPORT EQUATION

by

W. MATTHES

1971

Joint Nuclear Research Centre
Ispra Establishment - Italy

Nuclear Studies

ABSTRACT

Usually one starts from the adjoint transport equation to construct a Monte Carlo game for its simulation. This procedure requires a sophisticated reasoning to find out which quantity of the game is adjoint to what. Contrary to this point of view we start from the beginning with two independent games of two different kinds of particles and put the condition that the expectation value of some estimator in the two games should be equal. This leads directly to the Monte Carlo game for the adjoint flux and provides on with a large arbitrariness for the adjoint game. This arbitrariness can be used to find adjoint games with smaller variances. This method is applied to the calculation of the neutron flux in an annular air gap in the water shield of a cylindrical reactor.

KEYWORDS

MONTE CARLO METHOD
TRANSPORT THEORY
STATISTICS
ADJOINT FLUX
ANNULAR SPACE
AIR
WATER
CYLINDERS
REACTORS
SHIELDING
GAMES THEORY

C O N T E N T S

Monte Carlo simulation of the adjoint neutron game *)

A) The normal game

In the following we describe a procedure to arrive at the Monte Carlo simulation of the adjoint neutron transport equation from a somewhat different point of view than usual (see for instance [1], [2] and [3]). We begin with the transport game for particles diffusing in a medium. The particles are injected into the medium by a (stationary) source and establish a (stationary) flux distribution. Using the terminology of [4] we write their transport equation in the integral form:

$$\phi(x v) = \int dx' \, \chi(x' v) \, F(x' \to x | v) \tag{1}$$

$$\chi(x v) = S(x v) + \int \phi(x v') \, C(v' \to v | x) \, dv' \tag{2}$$

The quantities introduced have the following meaning:

$$S(x v) = S_o(x v) \, W^S(x v) \tag{3}$$

$$F(x' \to x | v) = E(x' \to x | v) \, W^T(x' \to x | v) \tag{4}$$

$$C(v' \to v | x) = \sum_K \nu(K) \sigma_K(x v') \, C_K(v' \to v | x) \, W_K(v' \to v | x) \tag{5}$$

where

$S_o(x v) \, dx \, dv$ is the probability that a particle starts its history in the spatial volume-element dx around x with a velocity in the range dv around v . $S_o(x v)$ is normalized: $\int S_o(x v) \, dx \, dv = 1$.

*) Manuscript received on March 12, 1971

$W^S(x\,v)$	is the weight of the starting particle.
$E(x' \to x \mid v)$	is the probability that a particle at x' flying with velocity v in the direction $\mu_0 = v/\lvert v\rvert$ is still alive at the point x, and is given by

$$E(x' \to x \mid v) = e^{-\int_0^{\lvert x-x'\rvert} \sigma(x' + s\mu_0)\,ds} \frac{\delta(\mu_0 - \mu)}{\lvert x - x'\rvert^2}, \qquad \mu = \frac{x - x'}{\lvert x - x'\rvert} \tag{6}$$

$W^T(x' \to x \mid v)$	is a factor which multiplies the weight of the particle when passing from x' to x.
$\sigma_\kappa(x\,v)$	is the cross section at x for an incoming particle of velocity v to induce a reaction in which $\nu(\kappa)$ new particles are generated and where
$C_\kappa(v \to v' \mid x)\,dv'$	is the probability that the velocity of a particle generated in this reaction is in dv'
$W_\kappa(v \to v' \mid x)$	is a factor which multiplies the weight of the incoming particle to get the weight of this newly born particle leaving this reaction point.

The total cross section becomes

$$\sigma^{total}(x\,v) = \sum_\kappa \sigma_\kappa(x\,v) \tag{7}$$

For the later use we introduce the quantity

$$T(x \to x'|v) = E(x \to x'|v)\, \sigma^{total}(x'v) \tag{8}$$

which is the probability that the particle starting at $(x\, v)$ makes its next collision at x'.

With these definitions we have the usual interpretation in which

$\phi(x v)\, dv$	is the mean weight of all particles with velocities in the range dv crossing unit area at x perpendicular to v per unit time, and
$\chi(x v)\, dx\, dv$	is the mean weight of all particles leaving collisions in dx with velocities in dv per unit time.

The integro-differential form of the transport equation (1) and (2) can be found in the usual way (see [4]) by writing:

$$\phi(x - t\Omega, v) = \int_t^{\infty} dt'\, \chi(x - t'\Omega, v)\, F(x - t'\Omega \to x - t\Omega | v) \tag{9}$$

and forming

1) $$\frac{\partial}{\partial t} \phi(x - t\Omega, v) = -\Omega\, \mathrm{grad}_x\, \phi(x - \Omega t, v) \tag{10}$$

2) $$\frac{\partial}{\partial t} \phi(x - t\Omega, v) = -\chi(x - t\Omega, v) + \int_t^{\infty} dt'\, \chi(x - t'\Omega, v) \frac{\partial}{\partial t} F(x - t'\Omega \to x - t\Omega | v) \tag{11}$$

Under the assumption that

$$\frac{\partial}{\partial s} F(x \to x+s\Omega|v) = f(x+s\Omega, v)\, F(x \to x+s\Omega|v) \tag{12}$$

we obtain the equation for ϕ in the form:

$$\Omega \,\mathrm{grad}_x \phi(xv) + f(xv)\,\phi(xv) = X(xv) \tag{13}$$

We assume now that our problem consists in calculating the integral

$$D = \int \phi(xv)\, H(xv)\, dx\, dv \tag{14}$$

where $H(x\,v)$ is an arbitrary function.

The integral over the spatial coordinate x may extend over certain volume or over a surface. As the scoring procedures for these two cases are different we consider them separately.

a) Volume integral

Within the transport game described above the quantity D may be evaluated by noting, that it can be written in the form

$$D = \int \phi(xv)\,\sigma(xv)\, dx\, dv\, \frac{H(xv)}{\sigma(xv)} \tag{15}$$

where $\sigma(x\,v)$ is the cross section for an arbitrarily chosen reaction. The type of reaction chosen may even be different for different parts of the phase space.

As $\phi(x\,v)\,\sigma(x\,v)dxdv$ is the mean weight of all those particles entering reactions of the type chosen in dxd$v$, we obtain D if at every such reaction event we score the quantity $q(x\,v) = w \cdot H(x\,v)/\sigma(x\,v)$, where w is the weight of the particle inducing the reaction.

b) Surface integral

In this case we write the integral for D in the form

$$D = \int dF\,dv\ \phi(x v)\,(\Omega\, n)\,X(x v)\,\frac{H(x v)}{(\Omega\, n)\,X(x v)} \tag{16}$$

where

$\Omega = v/|v|$

n is the normal to the surface F at point x(on F); positive direction on the same side as v.

$X(x v)$ is the probability that a particle, when trying to cross F at x in direction v will induce some event X (e.g. the particle might be absorbed in the surface with a certain probability X).

The term $\phi(x v)(\Omega\, n)\,X(x v)$ is now the mean number of events X per sec per unit area at x induced by particles trying to cross F at x in direction v.

This means that at any event $X(x\,v)$ induced by a particle trying to cross F, x in direction v we score the quantity $q(x\,v) = w(x\,v)H(x\,v)/[(\Omega\, n)\cdot X(x\,v)]$ where $w(x\,v)$ is the weight of the incoming particle.
After playing N histories we obtain an estimated value $\tilde{D}$ for D given by

$$\tilde{D} = \frac{1}{N}\sum_{i=1}^{N}\sum_{\alpha} q_i\,(x_\alpha\, v_\alpha) \tag{17}$$

where the first sum goes over all N histories and the second sum over all reaction events α of history i, initiated by source-particle number i.

We denote by

$$\mathcal{E}_i(x_k v_k) = \sum_\alpha q_i(x_\alpha v_\alpha) \tag{18}$$

the contribution of history i to the final result where $(x_k\ v_k)$ indicates explicitly the starting point of history i.

Now we transform (1^o) to

$$\tilde{D} = \sum_{\Delta\tau_k} \frac{\Delta N_k}{N} \frac{1}{\Delta N_k} \sum_i \mathcal{E}_i(x_k v_k) \tag{19}$$

where the first sum goes over all phase space elements $\Delta\tau_k = \Delta x_k \Delta v_k$ and the second sum over all the histories i starting in $\Delta\tau_k$, and ΔN_k is the number of histories starting in $\Delta\tau_k$.

Repeating the game for many histories, we obtain in the limit for very large N for the mean value of $\tilde{D}$ the expression

$$\langle\tilde{D}\rangle = D = \int S_o(xv)\, C_1(xv)\, dx\, dv \tag{20}$$

$$C_1(xv) = \langle \mathcal{E}(xv) \rangle \text{ averaged over all histories starting in x} \tag{21}$$

and for its variance:

$$\sigma^2_{\tilde{D}} = \langle \tilde{D}^2 \rangle - \langle \tilde{D} \rangle^2 \tag{22}$$

where

$$\tilde{D}^2 = \sum_{\Delta\tau_k} \frac{\Delta N_k}{N} \frac{1}{\Delta N_k} \sum_i \mathcal{E}_i^2 (x_k v_k) \tag{23}$$

and

$$\langle \tilde{D}^2 \rangle = \int S_o(xv)\, C_2(xv)\, dx\, dv \tag{24}$$

$$C_2(xv) = \langle \mathcal{E}^2 (xv) \rangle \text{ averaged over all histories starting in } x \tag{25}$$

The equations for $C_1(xv)$ and $C_2(xv)$ are easily derived (see [5]):
Let $\tilde{\mathcal{E}}_o(xv)$ be the contribution to the final result of a history generated by a particle starting with weight unity at (xv).
The obviously:

$$\tilde{\mathcal{E}}(xv) = W_s(xv)\, \tilde{\mathcal{E}}_o(xv)$$

Assume that this particle suffers its first collision at x'. If the particle initiates a reaction of type κ which we use to calculate $\tilde{D}$, we obtain first the contribution

$$A_\kappa = W_T(x \to x'|v) \frac{H(x'v)}{\Sigma_\kappa(x'v)}$$

to $\tilde{\mathcal{E}}_o(xv)$ and second, as $\nu(\kappa)$ new particles ar created each starting with a velocity v_i' (i = 1,2.., $\nu(\kappa)$) chosen from the distribution

$C_\kappa(v \to v_i' \mid x)$, a contribution due to these newly born particles:

$$B_\kappa = W_T(x \to x'|v) \sum_{i=1}^{\nu(\kappa)} W_\kappa(v \to v_i'|x') \tilde{\mathcal{C}}_0(x', v_i') \qquad (26)$$

The probability for this event (reaction of type κ at the first collision at x') is given by:

$$E(x \to x'|v)\, \sigma_\kappa(x' v) \qquad (27)$$

If the first collision at x' is of type κ' ($\neq \kappa$, probability for this event is $E(x \to x'|v)\sigma_{\kappa'}(x'v)$) then we obtain only the contribution:

$$B_\kappa' = W_T(x \to x'|v) \sum_{i=1}^{\nu(\kappa')} W_{\kappa'}(v \to v_i'|x') \tilde{\mathcal{C}}_0(x' v_i') \qquad (28$$

Combining these results we have:

$$\tilde{\mathcal{C}}_0(x v) = \left.\begin{matrix} A_\kappa + B_\kappa \\ \\ B_{\kappa'} \end{matrix}\right\} \begin{matrix} \text{if first reaction is of type } \kappa \\ \\ \text{if first reation is of type } \kappa' \end{matrix} \qquad (2$$

To obtain the mean value $C_0(x v)$ of $\mathcal{C}_0(x v)$ we multiply each possible contribution with its probability and sum over all contributions.

This leads to

$$C_0(x v) = \int F(x \to x'|v) \left\{ H(x'v) + \int C(v \to v'|x') C_0(x'v') dv' \right\} dx' \qquad (3$$

As this equation is adjoint to equation (1) we can put

$$C_0(x v) = \phi^\dagger(x v) \qquad (3.$$

and have:

$$\phi^{+}(x v) = \int F(x \to x'|v)\ \chi^{+}(x'v)\, dx' \tag{32}$$

$$\chi^{+}(x v) = H(x v) + \int C(v \to v'|x)\ \phi^{+}(x v')\, dv' \tag{33}$$

In the same way one can derive the equation for $C_2(x v)$ by first squaring the contributions and then averaging. This equation takes a complicated form similar to the adjoint equation for $C_o(x v)$ above, with modified source and modified kernels. Only for $\nu(K) = 1$ and all weight factors unity it reduces to a simple form which is given for instance in [5].

When we calculate D with the Monte Carlo method by giving an estimated value $\tilde{D}$ we can therefore with (16) simultaneously calculate an estimation for its variance by evaluating $\tilde{D}^2$. Generally we want to get $\tilde{D}$ with high accuracy, that means, with a small variance. But for a given game the statistical behaviour of $\tilde{D}$ and $\tilde{D}^2$ is well determined. A change in the value of the variance can therefore only be obtained by playing another game, changing some or all of the characteristic features of the given game. This change has to be done in such a way, that the new game leads to the same $\langle\tilde{D}\rangle$ but, hopefully, to smaller variance. Such a procedure might indeed lead to a smaller variance but unfortunately it might also lead to an increase in the computing time per history. So the only condition of a small variance is not reasonable. What we need is a criterion which leads to a minimal variance of the result under the condition of fixed cost (computing time). If therefore $T_i(x_* v_*)$ is the random variable giving the computing time for history i starting in $(x_* v_*)$ we want to find a changed game which makes $\sigma^2_{\tilde{D}}$ smaller, keeping $\langle\tilde{T}\rangle$ constant.

We do not try to solve this general problem here. In the next chapter we rather consider the different possibilities for constructing a changed game which leads to the same expectation value D for $\tilde{D}$.

B) The changed game

We consider the Monte Carlo simulation of the diffusion of two types of particles a and b. The equations for the relevant mean values of the two diffusion processes are for the a-game:

$$\phi_a(xv) = \int dx' \, X_a(x'v) \, F_a(x' \to x \mid v) \tag{34}$$

$$X_a(xv) = S_a(xv) + \int \phi_a(xv') \; C_a(v' \to v \mid x) \, dv' \tag{35}$$

and for the b-game:

$$\phi_b(xv) = \int dx' \; X_b(x'v) \; F_b(x' \to x \mid v) \tag{36}$$

$$X_b(xv) = S_b(xv) + \int dv' \, \phi_b(xv') \; C_b(v' \to v \mid x) \tag{37}$$

We are interested in the quantity

$$D = \int \phi_a(xv) \; H_a(xv) \; dx \, dv \tag{38}$$

It might turn out that the variance of the quantity $\tilde{D}$ obtained for a fixed computing time by playing the a-game is not tolerable. We therefore try to estimate D by playing another game, the b-game. The question is:

How can we choose the characteristics $\{S_b, F_b, C_b, H_b\}$ of the b-game to obtain the equality

$$\int \phi_a(xv)\, H_a(xv)\, dx\, dv = \int \phi_b(xv)\, H_b(xv)\, dx\, dv \tag{39}$$

for the expectation values.

The calculation of the quantity D can then be done by simulating the b-game, utilizing $\phi_b(xv)$ and $H_b(xv)$.

Out of the different possibilities (see [4]) for choosing an allowed b-game we take the following:

$$S_b(xv) = H_a(x, -v) \tag{40}$$

$$H_b(xv) = S_a(x, -v) \tag{41}$$

$$F_b(x' \to x \mid v) = F_a(x \to x' \mid -v) \tag{42}$$

$$C_b(v' \to v \mid x) = C_a(-v \to -v' \mid x) \tag{43}$$

If we put:

$$\begin{aligned} \phi_a^{\dagger}(xv) &= \phi_b(x, -v) \\ \chi_a^{\dagger}(xv) &= \chi_b(x, -v) \end{aligned} \tag{44}$$

and insert the relations (40) - (43) in (36) and (37) we obtain the adjoint equations to (34), (35).

This can also be seen by using the integro-differential equations for the two games (see (4)):

$$\Omega \operatorname{grad} \phi_a + f_a \phi_a = \chi_a \tag{45}$$

$$\Omega \operatorname{grad} \phi_b + f_b \phi_b = \chi_b \tag{46}$$

Using the relations (42) - (43) we obtain for the b-equation:

$$\Omega \operatorname{grad} \phi_b + f_a(x, -v)\phi_b = S_b(xv) + \int C_a(v \to v')\phi_b(xv')\,dv' \qquad (47)$$

Replacing v by $-v$ we obtain

$$-\Omega \operatorname{grad} \phi_a^+(xv) + f_a(xv)\,\phi_a^+(xv) = H_a(xv) + \int C_a(v \to v')\,\phi_a^+(xv')\,dv' \qquad (48)$$

and this equation is adjoint to (45).

The b-game may therefore be considered as "adjoint" to the a-game.

In the same way we can say, that the a-game is adjoint to the b-game, as due to the equations (40)-(43) the property of being "adjoint to each other" is a reciprocal one. This reciprocity has also the following consequence:

If we identify the a-particle with neutrons and if we play for the neutrons the usual transport game, we know that introducing a weight function which is proportional to an approximation of the "adjoint flux" leads to a smalle variance. Similarly we can now play the b-game introducing a weight functio (for the b-particles) which is proportional to an approximation of the actual neutron flux.

With the help of equations (4) and (5) we can write (42) and (43) in a more explicit form. For (42) we use the fact that in general

$$E(x \to x' \mid v) = E(x' \to x \mid -v) \qquad (49)$$

and for (43) that the scattering kernels depend usually on the inner product of v and v' and the cross sections only on the absolute value of the velocity. Then we obtain for (42):

$$\sigma_a^{total}(x\,v)\, T_b(x' \to x|v)\, W_b^T(x' \to x|v) = \sigma_b^{total}(x\,v)\, T_a(x' \to x|v)\, W_a^T(x \to x'|-v) \quad (50)$$

and for (43):

$$\sum \nu^b(\kappa)\, \sigma_\kappa^b(x\,v')\; C_\kappa^b(v' \to v/x)\, W_\kappa^b(v' \to v/x) =$$
$$\sum \nu^a(\kappa)\, \sigma_\kappa^a(x\,v)\; C_\kappa^a(v \to v'/x)\, W_\kappa^a(v \to v'/x) \quad (51)$$

The characteristics of the b-game have to be chosen such that these equations are satisfied.

C) <u>Choice of the adjoint game</u>

As an example we choose the b-game in the following way

a) <u>Source routine</u>

$$S_b(x\,v) = H_a(x, -v) \quad (52)$$

where $S_b(x\,v)$ has somehow to be separated into a normalized part $S_b^o(x\,v)$ and a weight function $W_b^S(x\,v)$.

b) <u>Transport routine</u>

$$T_b(x' \to x|v) = T_a(x' \to x|v) \quad (53)$$

$$W_b^T(x' \to x | v) = W_a^T(x \to x' | -v) \frac{\sigma_b^{total}(xv)}{\sigma_a^{total}(xv)} \tag{54}$$

Equation (53) implies that the geometric transport of the b-particles from the chosen starting point $(x' v)$ to the next collision point (xv) can be made using the transport kernel of the a-game. The weight of the b-particle when flying from x' to x has then to be multiplied by $W_b(x' \to x, v)$.

c) Collision Routine

$$\nu^b(\kappa) = \nu^a(\kappa) \tag{55}$$

$$C_\kappa^b(v' \to v | x) = \frac{\sigma_\kappa^a(xv)}{\rho_\kappa^a(xv)} C_\kappa^a(v \to v' | x) \quad \begin{Bmatrix}\text{normalized}\\ \text{collision}\\ \text{kernel}\end{Bmatrix} \tag{56}$$

$$W_\kappa^b(v' \to v | x) = \frac{\rho_\kappa^a(xv')}{\sigma_b^\kappa(xv')} W_\kappa^a(v \to v' | x) \tag{57}$$

where

$$\rho_\kappa^a(xv') = \int \sigma_\kappa^a(xv) C_\kappa^a(v \to v'' | x) \, dv \tag{58}$$

d) Cross Sections:

$$\sigma_\kappa^b(xv) = \rho_\kappa^a(xv) \tag{59}$$

This gives the probabilities for the different types of reaction events κ :

$$p_\kappa(xv) = \frac{\sigma_\kappa^b(xv)}{\sum_\kappa \sigma_\kappa^b(xv)} = \frac{\rho_\kappa^q(xv)}{\sum_\kappa \rho_\kappa^q(xv)} \tag{60}$$

e) Scoring routine

$$H_b(xv) = S_a(x,-v) \tag{61}$$

This b-game can now be used to calculate quantities defined for the a-game. In the application described below we identify the a-particles with neutrons. The weight factors in the original neutron game are allunity and we obtain for the weight factors for the "adjoint" game from (54) and (57):

$$W_b^T(x' \Rightarrow x|v) = \sigma_b^{total}(xv) / \sigma_a^{total}(xv) \tag{62}$$

$$W_\kappa^b(v' \Rightarrow v|x) = 1. \tag{63}$$

By a simple procedure we can also avoid the geometric weight factor W_b^T in the b-game. Instead of letting one b-particle enter the collision with weight W_b^T, we let

$$\nu = [W_b^T] \quad \text{particles (if } r < d)$$

or

$$\nu = [W_b^T]+1 \quad \text{particles (else)}$$

enter the collision point, where r is a random number, equally distributed between 0 and 1 and

$$d = W_b^T - [W_b^T]$$

This procedure might lead to the end of a history if $[W_b^T] = 0$. Obviously, if $\nu > 1$, then each of these particles has to be followed separately from this collision point on.

D) Evaluation of the C^b distribution

For the actual evaluation of $S_\kappa^a(x\,v)$ we recall that the scattering kernels C_κ^a can usually be written in the form (we express now the kernels in terms of energy E and direction $\Omega = v/|v|$):

$$C_\kappa^a(E,\Omega \rightarrow E',\Omega') = C_\kappa^a(E \rightarrow E') \frac{1}{2\pi} \delta(\mu - f_\kappa(E,E')) \qquad (64)$$

Here

$$\mu = (\Omega\,\Omega')$$

$C_\kappa^a(E \rightarrow E')dE'$ is the (normalized) probability, that the energy of an a-particle leaving a reaction of type κ (induced by an a-particle of energy E) has an energy in the interval dE' around E', and

$f_\kappa(E, E')$ is the cosine of the scattering angle (for instance in the Laboratory system) compatible with the pair (E,E') due to the impulse- and energy-conservation laws.

The factor $\frac{1}{2\pi}$ expresses azimuthal equidistribution of the direction around the incoming direction Ω . Inserting (64) in (58) (with dv replaced by $dE\, d\Omega$) and integrating over $d\Omega = d\mu\, d\varphi$ leads to (we skip now the index κ,α and the variable x):

$$\rho(E') = \int_{EL}^{ER} dE\ \sigma(E')\, C(E \rightarrow E') \tag{65}$$

The integration ranges over an energy interval $[EL,ER]$ over which we want C_κ^b to be normalized.

Using the relation

$$C(E \rightarrow E')\, dE' = P(\mu_s / E)\, d\mu_s \tag{66}$$

where $P(\mu_s/E)\, d\mu_s$ is the probability that the scattering angle μ_s lies in the range $d\mu_s$ around μ_s (s = c for center of mass system, s = L for laboratory system), we obtain

$$\rho(E') = \int_{EL}^{ER} \sigma(E)\, P(\mu_s(E \rightarrow E')/E) \left| \frac{d\mu_s}{dE} \right| dE \tag{67}$$

Transforming finall to the lethargy scale

$$E = E_0\, 10^{-u}$$

we arrive at

$$\rho(u') = \int_{uL}^{uR} \sigma(u)\, P(\mu_s(u \rightarrow u')/u) \left| \frac{d\mu_s}{du} \right| \frac{E}{E'}\, du \tag{68}$$

Using the form (64) also for C^b, integrating (56) over Ω and transforming to lethargy gives:

$$C^b(u' \to u) = \frac{\sigma^q(u)}{\rho^q(u)} P(\mu(u \to u')/u) \left|\frac{d\mu}{du}\right| \frac{E}{E'} \tag{69}$$

This scheme suggests that for the b-game we pick first the lethargy u of the b particle after the collision out of the distribution $C^b(u' \to u)$ and calculate afterwards the corresponding scattering angle due to $\mu = f(u, u')$. Note that tables usually do not contain the distribution $P(\mu/u)$ but the combination:

$$\overset{\text{(tabulated)}}{\sigma(\mu(u \to u')/u)} = \frac{1}{2\pi} \sigma(u)\, P(\mu(u \to u')/u) \tag{70}$$

In the example described below we deal only with the reaction types of elastic scattering and inelastic scattering on individual levels.

We have especially for the case of

a) <u>elastic scattering:</u>

$$\begin{aligned} \mu_c(u \to u') &= A_0 + A_1\, 10^{u-u'} \\ A_0 &= -(A^2+1)/(2A) \\ A_1 &= (A+1)^2/(2A) \end{aligned} \tag{71}$$

where A is the atomic weight.

Therefore:

$$\left|\frac{d\mu_c}{du'}\right| \frac{E}{E'} = A_1 \ln(10) = C \tag{72}$$

and finally:

$$\rho(w) = C \int_{xL}^{xR} dx\, \sigma(x)\, P(\mu_c(x \to w)/x) \tag{73}$$

$$C^b(w \to x) = \frac{\sigma(x)}{\rho(w)} P(\mu_c(x \to w)/x) \tag{74}$$

$$xL = MAX(W_0, w - \varepsilon) \tag{75}$$

$$XR = W; \qquad \varepsilon = \log\left(\frac{A+1}{A-1}\right)^2 \tag{76}$$

If the lethargy X after the collision of a b-particle is determined out of the distribution $C^b(w \to x)$, we find the cosine of the scattering angle in the Laboratory system with the help of the transformations:

$$\mu_c = A_0 + A_1\, 10^{x-w} \tag{77}$$

$$\mu_L = (1 + A\mu_c) / (1 + A^2 + 2A\mu_c)^{\frac{1}{2}} \tag{78}$$

b) inelastic scattering on level γ

Assuming isotropic scattering in the CM-system

$$P(\mu_c/u) = \frac{1}{2} \qquad (79)$$

we have (see Appendix A):

$$\mu_c(u \to u') = \frac{(A+1)^2 F_1 - A^2 F_2 - 1}{2A\sqrt{F_2}} \qquad (80)$$

$$F_1 = 10^{u-u'}$$

$$F_2 = 1 - 10^{u-u_\gamma}$$

where u_γ is the lethargy value of $\frac{A+1}{A} \cdot \gamma$ and γ is the niveau energy of the level.

Then

$$\left| \frac{d\mu_c}{du'} \right| \frac{E}{E'} = \frac{C}{\sqrt{F_2}} \qquad (81)$$

and finally

$$p(w) = \frac{c}{2} \int_{xL}^{xR} dx \, \sigma(x) \qquad (82)$$

$$c^{b}(w \to x) = \frac{c}{2} \frac{\sigma(x)}{p(w)} \qquad (83)$$

where we made the substitution

$$\mathfrak{S}(x) = \frac{\sigma(x)}{\sqrt{F_2}} \tag{84}$$

As for the integration limits we observe that for inelastic scattering a lethargy value w after the collision can be reached from all lethargy values x (before the collision) lying in the range (see Appendix A):

$$x_-(w) \leq x \leq x_+(w) \tag{85}$$

where

$$X_\pm(w) = u_y + \log\left(1 - \frac{z_\mp^2(w)}{A^2}\right) \tag{86}$$

$$z_\mp(w) = \frac{\left| R \mp \sqrt{R\left(1-\frac{1}{A^2}\right) + \frac{1}{A^2}} \right|}{R + \frac{1}{A^2}} \tag{87}$$

$$R = 10^{Q}$$

$$Q = W - u_y - 2\log(1+A)$$

The integration boundaries are then

$$XL(w) = MAX\left(w_0, X_1, X_-(w)\right) \tag{88}$$

$$XR(w) = MIN\left(X_2, X_+(w)\right) \tag{89}$$

where X_1 and X_2 are such that $\sigma(x) \neq 0$ within the interval $[X_1, X_2]$

Clearly, if

$$XR(w) \leq XL(w) \qquad (90)$$

then

$$\rho(w) = 0.$$

An example for the functions XL(w) and XR(w) is given in fig.(1) .

After having determined X we find the cosine of the scattering angle in the Lab. System μ_L out of

$$\mu_c = \frac{(1+A)^2 F_1 - A^2 F_2 - 1}{2 A \sqrt{F_2}} \qquad (91)$$

$$\mu_L = \frac{1 + A \mu_c \sqrt{F_2}}{(1 + A^2 F_2 + 2 A \mu_c \sqrt{F_2})^{\frac{1}{2}}} \qquad (92)$$

E) <u>Example</u>

As an application for constructing the b-game we want to calculate the neutron flux as a function of height z in an annular air gap within a cylindrical water shield surrounding concentrically a cylindrical reactor of finite height as sketched in fig.(2) .

The average flux in the energy interval Δ E around E at height z, averaged over the annular cylindrical volume element Δv of height Δz and radial width ΔR is given by:

$$\bar{\Phi}(z, E) = \frac{1}{\Delta\tau} \int_{\Delta\tau} \phi(x, E, \Omega)\, dx\, dE\, d\Omega$$

$$\Delta\tau = \Delta V \cdot \Delta E \cdot 4\pi \tag{93}$$

This expression has the form (7) with

$$H(xv) = \begin{cases} \frac{1}{\Delta\tau} & \text{within} \quad \Delta\tau \\ 0 & \text{else} \end{cases} \tag{94}$$

This means that if we want to calculate $\bar{\Phi}$ by using the b-game

$$\bar{\Phi} = \int \phi_b(x E \Omega)\, H_b(x E \Omega)\, dx\, dE\, d\Omega \tag{95}$$

we have for the source of the b-particles (see (52)):

$$S_b(x E \Omega) = H_a(x, E, -\Omega) = \frac{1}{\Delta\tau} \tag{96}$$

which is already normalized over $\Delta\tau$.

As the direct neutron game (a-game) is played without weights (all weight factors unity) we start the b-particles:

1) equally distributed over the spatial volume Δ v

2) equally distributed over the energy interval Δ E

3) isotropically, and

4) with unit weight

This source establishes a flux distribution $\phi_b(x,E,\Omega)$ of b-particles which finally leads to $\bar{\phi}$ due to (95) using for H_b the relation (41)

$$H_b(x,E,\Omega) = S_a(x,E,-\Omega) \tag{97}$$

The original neutron source distribution S_a is defined over the reactor-surface and taken to be

$$S_a(x,E,\Omega) = S_0 \frac{f(E)}{2\pi} y(\Omega) \cos\left(\pi\left(\frac{z}{H} - \frac{1}{2}\right)\right) \tag{98}$$

where

S_0: Source intensity (Source intensity (Total number of neutrons emitted from F per sec)

f(E): normalized fission spectrum

$$y(\Omega) = \begin{cases} 1 & \text{if } \Omega \text{ points into the shield} \\ 0 & \text{else} \end{cases}$$

As we consider the reactor surface F to represent the neutron source, any neutron crossing F from outside has to be considered as absorbed by F. Now the geometric transport in the game for the b-particles is performed with the transport kernel T_a for the neutrons. Therefore also the b-particles are to consider as being totally absorbed when crossing F from outside. In this case we take the event X (see (9)) to be the absorption and have $X \equiv 1$.

To obtain $\bar{\phi}$ we have to score for each b-particle which crosses F at x in direction Ω from outside

$$\bar{\phi} = \bar{\phi} + \frac{f(E)}{(\Omega n)} \cos\left(\pi\left(\frac{z}{H} - \frac{1}{2}\right)\right) \quad (n \text{ pointing inside}) \tag{99}$$

and the final result after N histories becomes

$$\bar{\phi} = \frac{1}{N} \bar{\phi}_s \cdot \frac{S_0}{2\pi} \cdot \frac{\lambda}{R} \tag{100}$$

F) Computer programme for the evaluation of the adjoint cross sections and scattering distributions

a) Elastic Scattering

We have to calculate the functions $\rho(w)$ and $C^b(w \rightarrow x)$ as given in the expressions (73) and (74).

As input data we use a discrete set of values for the elastic cross section $\sigma(x)$ at the lethargy values X(I) (I = 1,2...MX):

Lethargy X(I)	Cross Section σ(I)
x(1)	σ(1)
x(2)	σ(2)
⋮	⋮
x(Mx)	σ(Mx)

where the lethargy intervall $[x(1), x(Mx)]$ covers the lethargy range $[w_o, w_1]$ relevant for the problem.

The tabulated differential elastic scattering distributions are given for a set of lethargy values y(I) (I = 1,2...My) and have the form:

μ_c	$\sigma^{(T)}(\mu_c \mid y(I))$
1.0	xxx
0.9	xxx
0.8	xxx
⋮	⋮
-0.9	.
-1.0	xxx

where $\overline{\phi_S}$ is the stored quantity due to (99) for all b-particles starting in the volume element ΔV considered, and R is a constant which normalizes the spatial and spectral source distribution to one (e.g. $R = F \cdot \int_{-H/2}^{H/2} dz \cdot \cos\left(\frac{z}{2}\frac{\pi}{H}\right)$ for a spatial cosine source distribution and a normalized fission spectrum).

Figs. 6a,b show the result of the calculations done for different source distributions (with S_o = 1 neutron/F·sec) (flat or cosine over the reactor surface F, and flat or fission spectrum over the energy). The height H (= 60.0 cm) of the assembly was divided in 10 intervals and 3000 b-particles were started in each interval v_1 v_2 ... v_{10} (see Fig. 2). Finally the values for v_1 and v_{10}, v_2 and v_9 etc. were added to obtain a smoother curve. The calculation time was about 1 hour for 30.000 histories. The values $\overline{\phi}(z)$ plotted in Figs. 6a and 6b are the scored quantities due to (99). To obtain the actual numerical values of the flux one has to apply (100) (N = 6000).

where again the interval $[y(1),\ y(My)]$ covers the range $[w_o, w_1]$

We put

$$\mu_c = -1 + \frac{K-1}{10}$$

and form the tables:

$$Q(I,K) = \frac{\sigma^{(T)}(\mu_c / Y(I))}{\sigma(Y(I))} \cdot 2\pi \qquad (101)$$

K = 1,2..... 21

I = 1,2..... My

As <u>output</u> data we tabulate the function ρ (w) at a discrete set of MW equally spaced (by DW) lethargy values $W(I) = W_o + (I-1) \cdot DW$ (I =1,2...MW), and the distributions $C^b(w \rightarrow x)$ will be presented at a discrete set of MV equally spaced (by DV) lethargy values $V(I) = W_o + (I-1) \cdot DV$ (I = 1,2...MV) by a table of numbers C(I,K) (I = 1,2...MV, K = 1,2 ....NV) which are solutions of the equation:

$$\int_{XL}^{C(I,K)} \sigma(x)\, P(\mu_c(x \rightarrow V(I)))\,dx = \frac{K-1}{NV-1}\, \rho(V(I)) \qquad (102)$$

Note that $C(1,K) = W_o$ for all K and $\{C(I,1) = XL,\ C(I,NV) = V(I)\}$ for all I. The quantities $\{MW,\ DW,\ MV,\ DV\}$ are such that W(MW) = V(MV).

The computer programme for the calculation of the quantities ρ (W(I) and C(I,K) proceeds as follows:

a) For a given lethargy value W prepare the tables $\{A(I), F(I)\ I = 1,2...MA\}$, where

$$F(I) = \sigma(A(I))\, P(\mu_c(A(I) \rightarrow W)) \qquad (103)$$

in the range (see fig.3a):

$$XL = A(1) \leq A(I) \leq A(MA) = XR \qquad (104)$$

with

$$A(I) = X(K) \quad \text{if} \quad XL \leq X(K) \leq XR \qquad (105)$$

b) Integrate the function F(I) (using trapezoidal rule) from A(1) to A(MA) and form the table (see fig.(3b)):

$$H(I) = \int_{A(1)}^{A(I)} F(x)\,dx \qquad (I = 2,3\ldots MA) \qquad (106)$$

c) Divide H(MA) in (NV-1) equal parts and determine the C(I,K) values (see fig.(3c)).

To find the lethargy X after an elastic collision, when the incoming particle has lethargy W, we have in general to interpolate between two lethargy-distribution curves. Assume that: $V(I) < W < V(I+1)$. Then we have (see fig.(4)):

$$X = C(I) + B\left\{ C(I+1) - C(I) \right\} \qquad (107)$$

where

$C(I) = C(I,K) + DK \left\{ C(I,K+1) - C(I,K) \right\}$

$B \quad = (W - V(I))/DV$

$H \quad = r(NV-1) + 1$

$K \quad = [H]$, $DK = H - K$

r = Random number equally distributed between 0 and 1.

b) <u>Inelastic scattering on level</u> γ

We have to calculate the functions $\rho(W)$ and $C^b(W \rightarrow x)$ as given in the expressions (82) and (83).

As <u>input</u> data we use a discrete set of values for the inelastic scattering cross section $\sigma(x)$ at the lethargy values X(I) (I = 1,2... Mx) and form the table

$$\sigma(I) = \frac{\sigma(x(I))}{\sqrt{1 - 10^{x(I) - u_{\gamma}}}} \qquad (108)$$

As <u>output</u> data the function $\rho(W)$ will be tabulated at a discrete set of MW equally spaced (by DW) lethargy values $W(I) = W_o + (I-1)\,DW$ $(I = 1,2..MW)$, and the distributions $C^b(W \to x)$ will be presented at a discrete set of MV equally spaced (by DV) lethargy values $V(I) = W_o + (I-1) \cdot DV$ $(I = 1,2...MV)$ by a table of numbers $C(I,K)$ $(I = 1,2...MV,\ K = 1,2...NY)$ which are solutions of the equation:

$$\int_{XL(V(I))}^{C(I,K)} \sigma(x)\,dx = \frac{K-1}{NY-1}\,\rho(V(I)). \qquad (109)$$

The quantities $\{MW, DW, MV, DV\}$ are such that $W(MW) = V(MV)$.

For lethargy values W for which $\rho(W) = 0$, an inelastic collision will never occur and the corresponding table C(W,K) will never be used. Their content is therefore irrelevant and will be put equal to zero.

The computer programme for the calculation of the quantities $\rho(W(I))$ and C(I,K) proceeds in the same way as in the elastic case with the function $F(I) = \sigma(I)$.

To find the lethargy X after an inelastic collision, when the incoming particle has lethargy W, we have in general to interpolate between two lethargy-distribution curves.

Assume that $V(I) < W < V(I+1)$ and that

a) the values C(I,K) and C(I+1,K) are all different from zero.

   Then we can use the same formulae for X as for the elastic scattering see (107).

b) $C(I,K) = 0$ and $C(I+1, K) \neq 0$.
   We can use the same formulae for X as in case a) if we put

$$B = (W - WM) \,/\, (V(I+1) - WM) \qquad (110)$$

c) C(I,K) $\neq$ 0 and C(I+1, K) = 0

We can use the same formulae for X as in case a) if we put

$$B = (W - V(I)) \,/\, (WP - V(I)) \qquad (111)$$

$$C(I+1) = C(I, NV)$$

Here WM(WP) is the lethargy value at the left (right) of which ρ(W) is identical to zero (see fig.(5)).

Literature

1)B. Eriksson et al., Monte Carlo Integration of the Adjoint Neutron Transport Equation, Nuclear Science and Engineering, 37, 410-422 (1969).

2)Leo B. Levitt et al., A New Non-Multigroup adjoint Monte Carlo Technique, Nuclear Science and Engineering, 37, 278-287 (1969).

3)The MORSE-Code, Contributed by Neutron Physics Division, Oak Ridge Nationa Laboratories (1969).

4)G. Goertzel and M.H. Kalos, Monte Carlo Methos in Transport Problems, Progress in Nuclear Energy, D.J. Hughes, Ed. Series I, Z. 315-369, Pergamon Press, NY (1958).

5)R. Coveyou et al., Adjoint and Importance in Monte Carlo Application, Nuclear Science and Engineering, 27, 210-234 (1967).

Figure Captions

Fig. 1: Inelastic scattering of an adjoint particle on the 6.065 MeV-Level of Oxygen (A = 16.0). If the lethargy of the (adjoint) particle before the collision is W, then the lethargy X after the collisions between XL and XR ($XL \leq X \leq XR$).

Fig. 2: Geometrical arrangement for the example. The surface of the inner cylinder with radius R_1 represents the neutron source. The calculation gives the neutron flux averaged over the annular rings of equal volume v_1, v_2 ....v_N. Regions (2) and (4) are filled with water, Region (3) is empty.

Fig. 3: The graphs in Fig. 3a, 3b and 3c show the individual steps in the calculation of the C(I,K) table as described in the text.

Fig. 4: Double interpolation between two distribution curves given for the lethargy values v(I) and v(I+1). W is the lethargy of the (adjoint) particle before the collision (elastic or inelastic) and X the lethargy after the collision. For the explicit formulae see the text.

Fig. 5: General behaviour of ρ (w) for inelastic scattering. ρ(w) is different from zero only within the intervall [WM, WP] .

Fig. 6 Neutron flux averaged over the annular rings of equal volume v_1, v_2,...v_N as a function of z for the special geometrical situation

R_1 = 10 cm, R_2 = 25 cm, R_3 = 26 cm, R_4 = 30 cm, H = 60 cm.

$\bar{\Phi}$ (z) is the neutron flux averaged over the energy interval [10 eV - 20 eV] , over the volume element and all directions for different source distributions.

FL-FL: source distribution flat over the cylinder surface F and over the energy range 10 Mev - lev

FL-FI: source distribution flat over the cylinder surface F and fission spectrum over the energy

CO-FL: source distribution "cosine" over the cylinder surface F and flat over the energy

CO-FI: source distribution "cosine" over the cylinder surface F and fission spectrum over the energy.

Appendix A:

Inelastic Scattering on level γ

We introduce the following quantities:

$\{\vec{v}_o, \vec{V}_o\}$ velocities of neutron and nucleus before the collision in the laboratory system. We assume $\vec{V}_o = 0$ and put $E_o = \frac{1}{2} m v_o^2$.

$\{\vec{v}_1, \vec{V}_1\}$ velocities of neutron and nucleus after the collision in the laboratory system. We put $E_1 = \frac{1}{2} m v_1^2$.

This means:

Velocity of the center of mass

$$\vec{V}_c = \vec{v}_o \left(\frac{1}{1+A}\right) \qquad A = \frac{M}{m} \tag{1}$$

Energy available for the reaction in the CM system:

$$D = \frac{1}{2} m v_o^2 - \frac{1}{2}(m + M) V_c^2 = E_o \frac{A}{1+A} \tag{2}$$

This shows that the minimum energy ε of the neutron in the laboratory system necessary to exite the level γ is given by:

$$\gamma = \varepsilon \frac{A}{1+A} \tag{3}$$

or:

$$\varepsilon = \gamma \frac{1+A}{A} \tag{4}$$

$\left\{\vec{v}_0', \vec{V}_0'\right\}$ velocities of neutron and nucleus before the collision in the CM system.

$\left\{\vec{v}_1', \vec{V}_1'\right\}$ velocities of neutron and nucleus after the collision in the CM system.

Momentum conservation in the CM system:

$$m\vec{v}_0' + M\vec{V}_0' - m\vec{v}_1' + M\vec{V}_1' = 0 \tag{5}$$

This gives

$$\vec{V}_1' = -\frac{m}{M}\vec{v}_1' \tag{6}$$

Energy conservation in the CM system:

$$\frac{1}{2} m v_1'^2 + \frac{1}{2} M V_1'^2 = D - \gamma \tag{7}$$

This gives, with (2) and (4):

$$v_1'^2 = E_0 \frac{2}{m}\left(\frac{A}{1+A}\right)^2 \left(1 - \frac{\varepsilon}{E_0}\right) \tag{8}$$

Now we use the relation:

$$\vec{v}_1 = \vec{V}_c + \vec{v}_1' \tag{9}$$

and obtain, with (1) and (8)

$$E_1 = \frac{1}{2} m v_1^2 = \frac{E_0}{(1+A)^2}\left\{1 + A^2\left(1 - \frac{\varepsilon}{E_0}\right) + 2\mu_c A \sqrt{1 - \frac{\varepsilon}{E_0}}\right\} \tag{10}$$

This gives us finally:

$$\mu_c(E_0 \to E_1) = \frac{\frac{E_1}{E_0}(1+A)^2 - 1 - A^2\left(1 - \frac{\varepsilon}{E_0}\right)}{2A\sqrt{1-\frac{\varepsilon}{E_0}}} \qquad (11)$$

As μ_c can only range from (-1) to (+1) we obtain the range for the energy E_1 after the collision:

$$\underbrace{E_0 \frac{\left[A\sqrt{1-\frac{\varepsilon}{E_0}} - 1\right]^2}{(1+A)^2}}_{\mu_c = -1} \leq E_1 \leq \underbrace{E_0 \frac{\left[A\sqrt{1-\frac{\varepsilon}{E_0}} + 1\right]^2}{(1+A)^2}}_{\mu_c = +1} \qquad (12$$

We introduce

$$z = A\sqrt{1 - \varepsilon/E_0} \qquad E_0 = \varepsilon / \left(1 - \frac{z^2}{A^2}\right) \qquad (13)$$

$$\xi = \frac{\varepsilon}{E_1} \frac{1}{(1+A)^2} \qquad (14)$$

and obtain from (12):

$$\xi_+(z) \leq \xi \leq \xi_-(z) \qquad (15)$$

with

$$\xi_\pm(z) = \frac{1 - \frac{z^2}{A^2}}{(z \pm 1)^2} \quad ; \quad \xi_+(z) = \xi_-(-z); \qquad (16)$$

For a fixed ξ we obtain from this unequality:

$$Z_-(\xi) \leq Z \leq Z_+(\xi) \tag{17}$$

with

$$Z_\pm(\xi) = \frac{\left| \xi \pm \sqrt{\xi\left(1-\frac{1}{A^2}\right)+\frac{1}{A^2}} \right|}{\xi + \frac{1}{A^2}} \tag{18}$$

Expressing Z back in terms of E_o, transforming to the lethargy scale

$$\frac{\varepsilon}{E_o} = 10^{x - x_\gamma} \tag{19}$$

and taking the logarithm of (18) finally leads to

$$X_-(w) \leq X \leq X_+(w) \tag{20}$$

where

$$X_\pm(w) = \log\left(1 - \frac{Z_\mp^2(w)}{A^2}\right) + x_\gamma \tag{21}$$

ξ = $10^{w-x_\gamma} - 2 \cdot \log(1+A)$.

x = Lethargy value before the collision

w = Lethargy value after the collision

x_γ = Lethargy value of the level (ε).

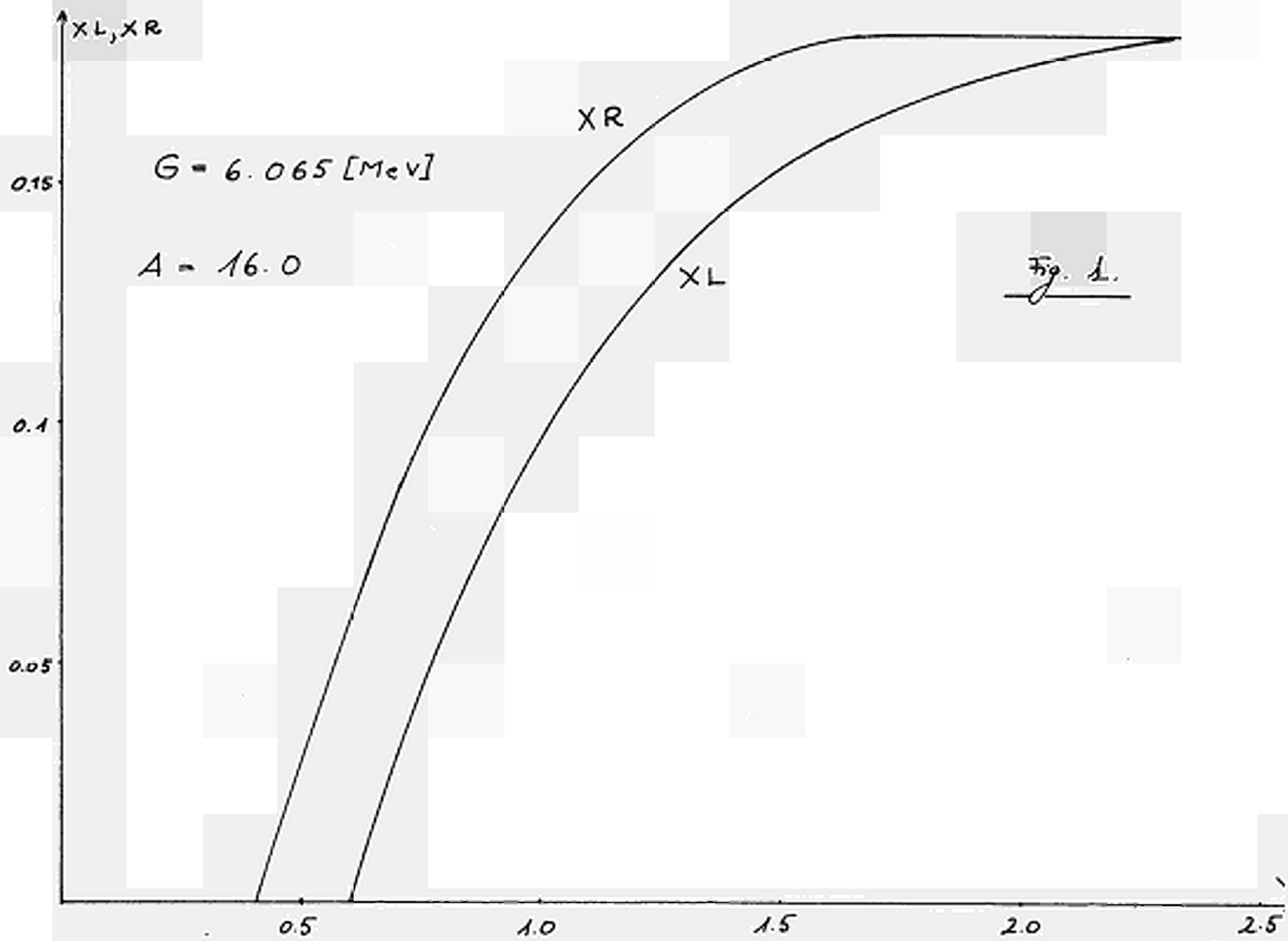

XL, XR
G = 6.065 [MeV]
A = 16.0
XR
XL
0.15
0.1
0.05
0.5
1.0
1.5
2.0
2.5

Fig. 1.

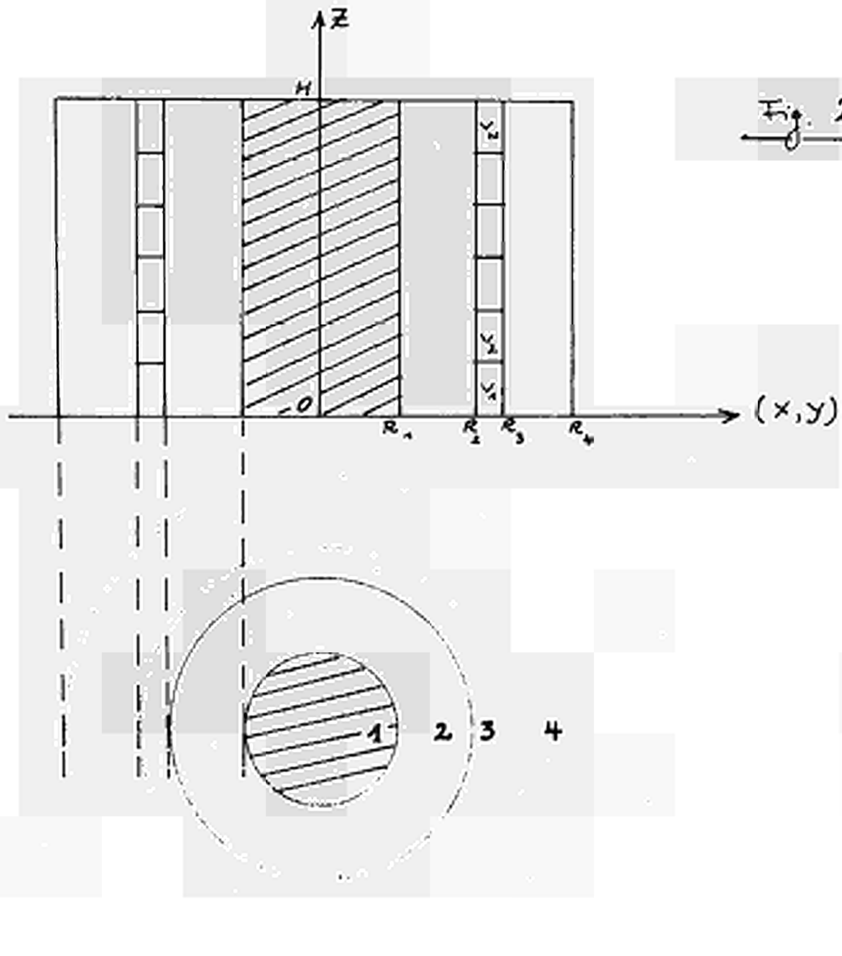

Z
H
0
R_1
R_2
R_3
R_4
(x, y)
1
2
3
4

Fig. 2

Fig. 3a

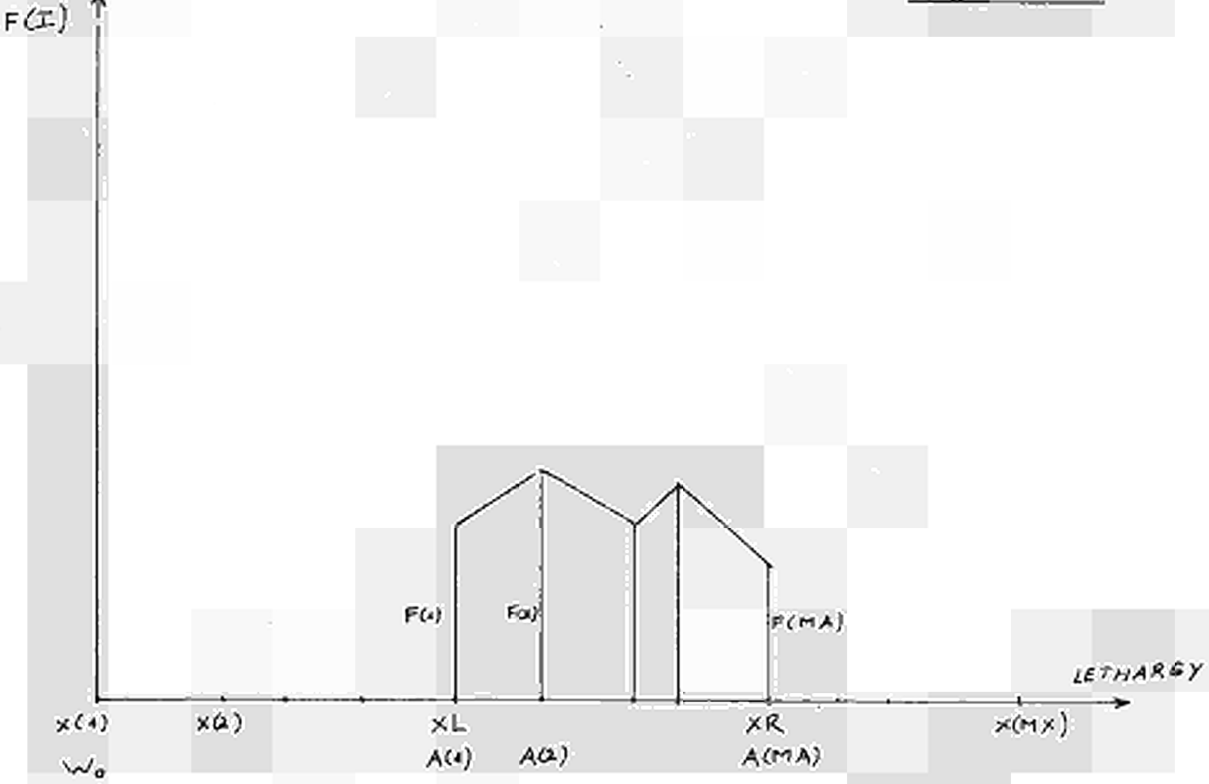


Fig. 3b

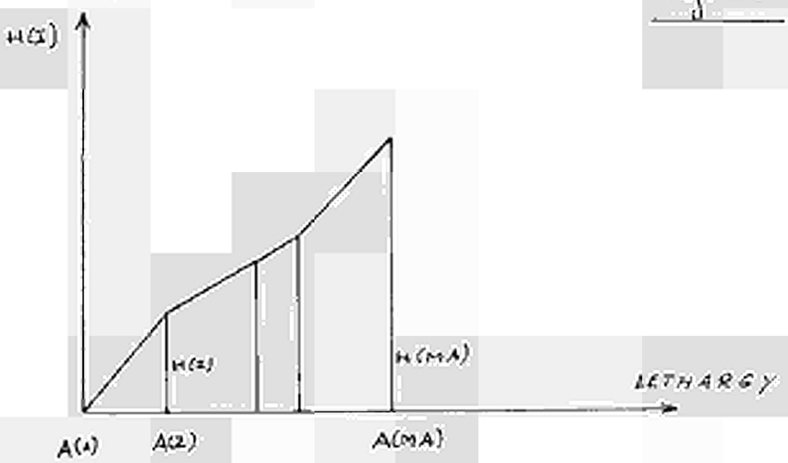


Fig. 3c

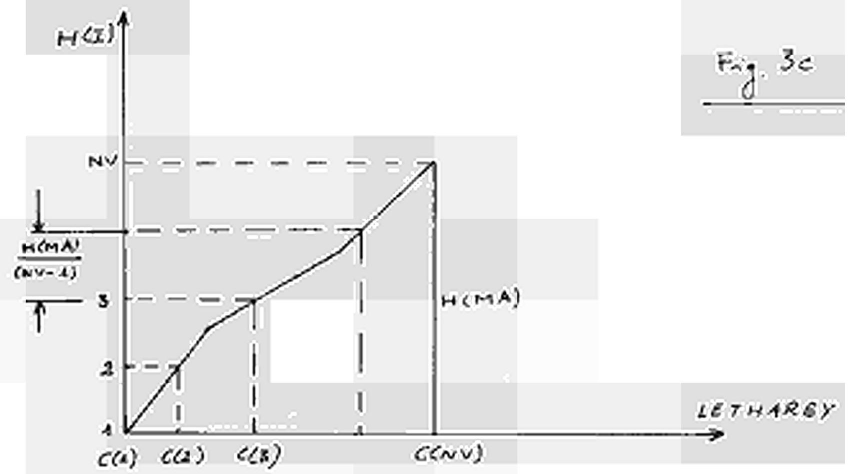


Fig. 4

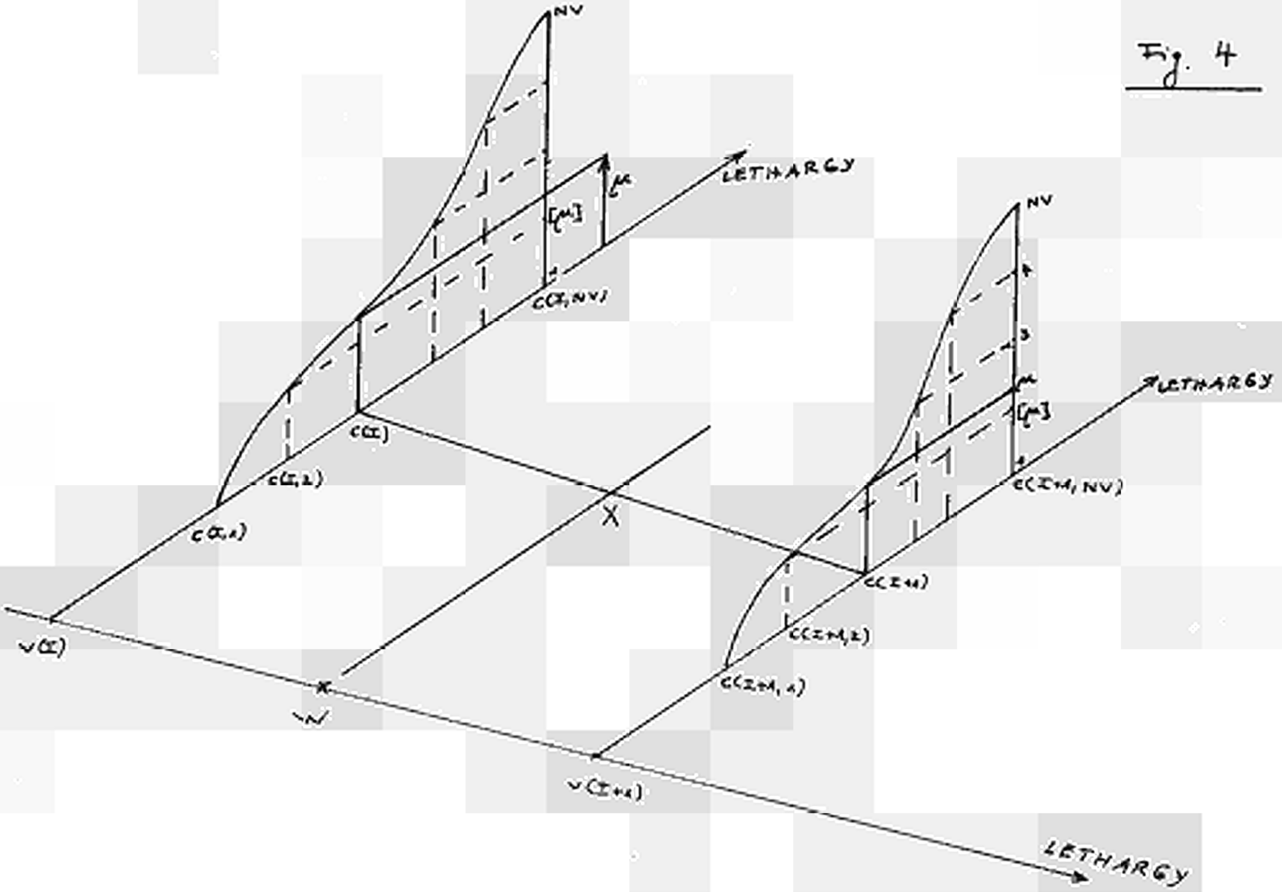
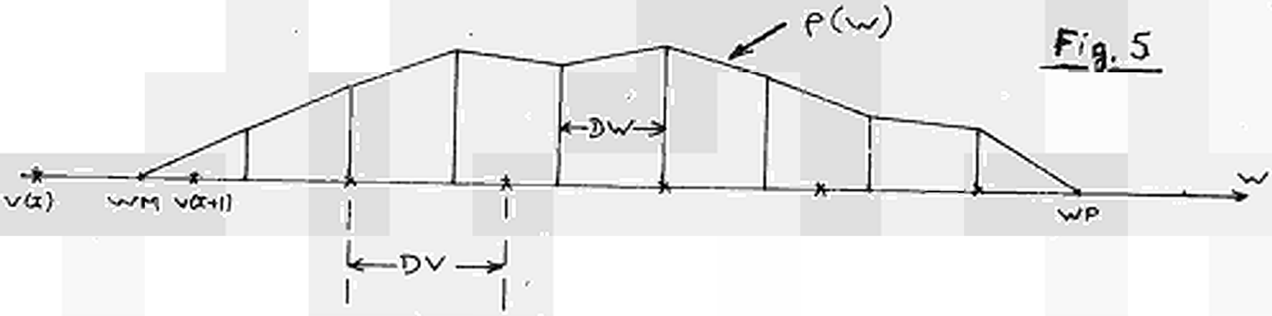
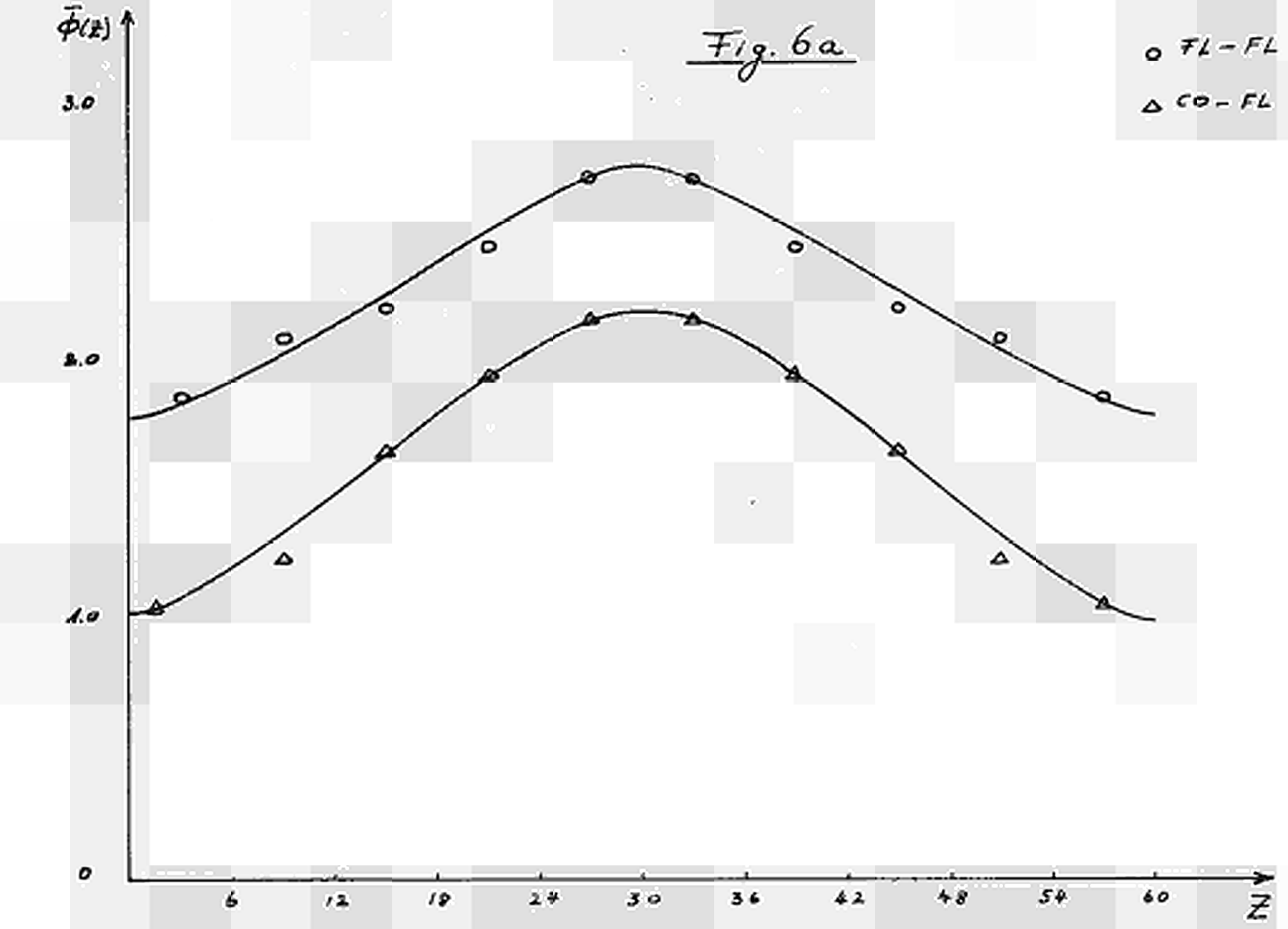
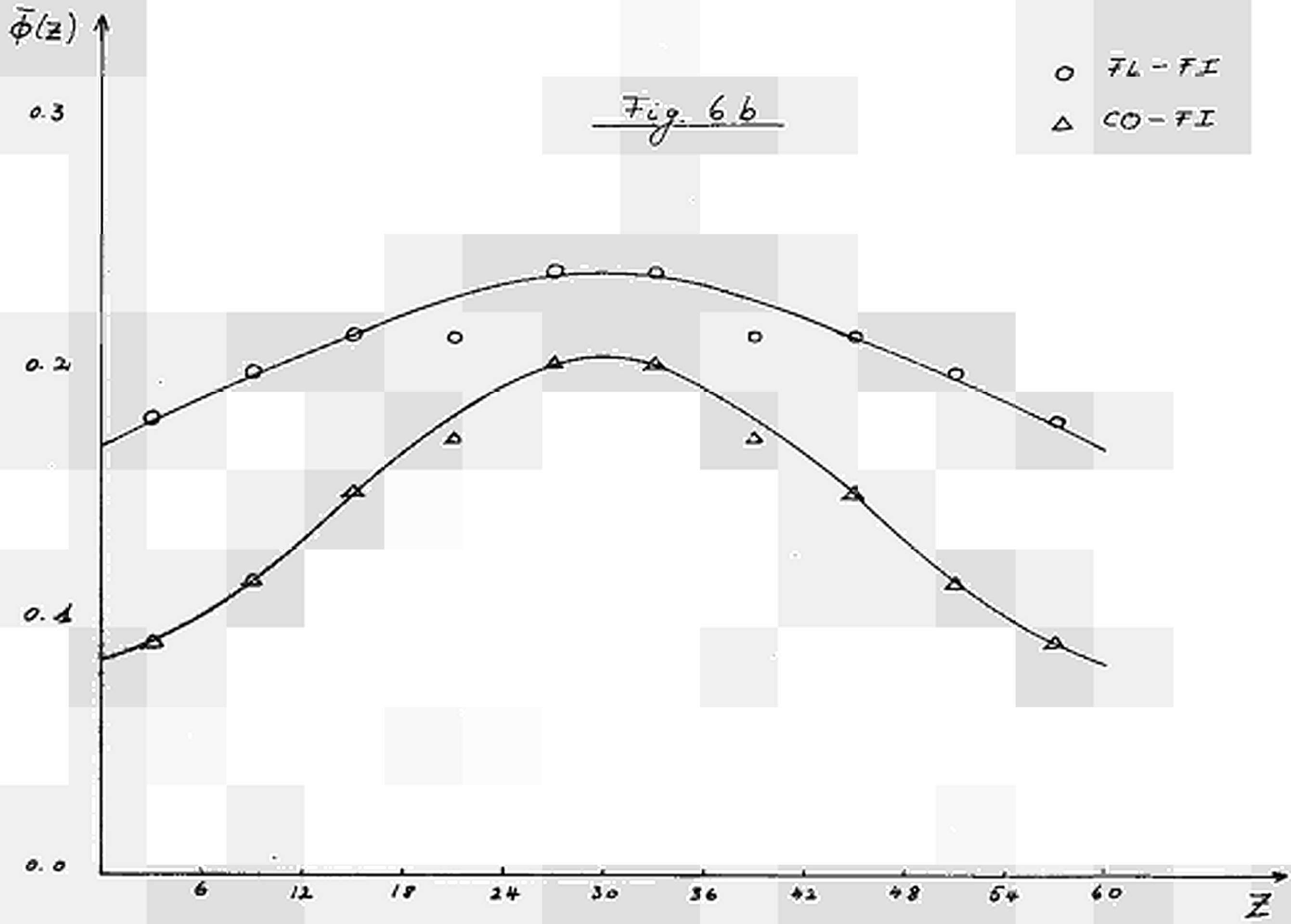

NV
LETHARGY
μ
[μ]
C(I,NV)
C(I)
C(I,2)
C(I,1)
V(I)
X
W
V(I+1)
NV
LETHARGY
μ
[μ]
C(I+1,NV)
C(I+1)
C(I+1,2)
C(I+1,1)
LETHARGY

Fig. 5


P(W)
DW
V(I)
WM
V(I+1)
DV
WP
W



$\bar{\Phi}(z)$
Fig. 6a
○ FL – FL
△ CO – FL
3.0
2.0
1.0
0
6
12
18
24
30
36
42
48
54
60
Z



$\bar{\Phi}(z)$
Fig. 6b
○ FL – FI
△ CO – FI
0.3
0.2
0.1
0.0
6
12
18
24
30
36
42
48
54
60
Z

To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

Alfred Nobel

SALES OFFICES

All reports published by the Commission of the European Communities are on sale at the offices listed below, at the prices given on the back of the front cover. When ordering, specify clearly the EUR number and the title of the report which are shown on the front cover.

OFFICE FOR OFFICIAL PUBLICATIONS
OF THE EUROPEAN COMMUNITIES

P.O. Box 1003 - Luxembourg 1
(Compte chèque postal N° 191-90)

BELGIQUE — BELGIË
MONITEUR BELGE
Rue de Louvain, 40-42 - B-1000 Bruxelles
BELGISCH STAATSBLAD
Leuvenseweg 40-42 - B-1000 Brussel

DEUTSCHLAND
VERLAG BUNDESANZEIGER
Postfach 108 006 - D-5 Köln 1

FRANCE
SERVICE DE VENTE EN FRANCE
DES PUBLICATIONS DES
COMMUNAUTÉS EUROPÉENNES
rue Desaix, 26 - F-75 Paris 15e

ITALIA
LIBRERIA DELLO STATO
Piazza G. Verdi, 10 - I-00198 Roma

LUXEMBOURG
OFFICE DES
PUBLICATIONS OFFICIELLES DES
COMMUNAUTÉS EUROPÉENNES
Case Postale 1003 - Luxembourg 1

NEDERLAND
STAATSDRUKKERIJ
en UITGEVERIJBEDRIJF
Christoffel Plantijnstraat - Den Haag

UNITED KINGDOM
H. M. STATIONERY OFFICE
P.O. Box 569 - London S.E.1

Commission of the
European Communities
D.G. XIII - C.I.D.
29, rue Aldringen
Luxembourg

CDNA04678ENC