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STATISTICAL PARAMETRIC AND NON-PARAMETRIC METHODS OF DETERMINING THE RELIABILITY OF MECHANICAL COMPONENTS

by

D. BASILE and G. VOLTA

1970

Joint Nuclear Research Center
Ispra Establishment - Italy
Engineering Department
Technology

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KEYWORDS

ELEMENTARY PARTICLES
ELECTRONICS
LIFETIME
STRESSES
STATISTICS
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MECHANICAL STRUCTURES
PLANTS

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STATISTICAL PARAMETRIC AND NON-PARAMETRIC METHODS OF DETERMINING THE RELIABILITY OF MECHANICAL COMPONENTS *)

1. INTRODUCTION

1.1 Subject Matter

The theory of reliability can be divided into two main sections. The first deals with the ways of handling the available experimental material so as to discover a posteriori the statistical law of behaviour of a component. (The notion of a "component" or "system" is not to be associated with any image of a physical complex. The component is the elementary unit under consideration, for which the statistical law of behaviour is to be defined. The system is the result of the functional connexion of a number of components.)

The second section starts from the assumption of knowledge of the statistical properties of the components to deduce, by means of appropriate probabilistic models that simulate the functional relations between components, the properties of a system.

This report is a contribution to the first section. To process the experimental material, which consists of data (lifetime, breaking stresses, etc.) corresponding to events considered as random, one uses statistical methods already developed to a large extent for an immense variety of applications. The specific application of these mathematical methods to reliability problems depends on the type of component in question, the context and the purpose of the application.

The method that can and must be employed to assess the reliability of mass-produced electronic components in a design study for a data bank, for instance, is of little use to someone who wants to evaluate the reliability of mechanical components of a plant in operation so that the management can be duly adjusted at once.

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*) Manuscript received on 2 March 1970

In this report we adopted the position of someone concerned with the reliability of mechanical and electromechanical components, i.e., components for which:

- the dimensions of the available sample are always fairly small;
- the deterioration of the properties (through wear, corrosion, fatigue, etc.) with time is significant with respect to the lifetimes regarded as useful;
- the reliability analysis effected during operation, taking into account the damage that has occurred on only a fraction of a series of functioning components, can be of more immediate interest than the reliability analysis that can be obtained when the sampling procedure is completed in full.

Adopting this point of view, to which is not yet given enough consideration in the literature on reliability, we have set out the typical and suitable methods of analysis, developing for each the appropriate digital programmes.

1.2 Plan of the Report

Section 2 briefly describes the main outlines of what are called parametric methods for the statistical analysis of samples, i.e., the methods most commonly used in the case of large numbers of samples. We have dwelt more particularly on the application of these methods to cases of exponential and Weibull distributions of failure.

The range of reference works available for this matter is enormous as far as the general principles are concerned, but is far more limited when it comes to specific application to Weibull distributions. We referred chiefly to the excellent book by Lloyd and Lipow (Ref. 1).

Section 3 shows a non-parametric method which can be regarded as a direct application of a general property of the statistical variables associated with ordered events (order statistics).

This method has been insistently advocated and illustrated by L.G. Johnson (Refs. 2 and 3) of General Motors, precisely in the context of its application to mechanical components.

The method is extremely simple when suitable tabulated values are available; for small samples it is better than the parametric methods and, unlike them, enables one to take into consideration incomplete samples, such as occur in the case of a set of in-service components only a fraction of which is damaged. We describe the method and have also developed a digital programme by which the tabulated values can be obtained for samples composed of 1-50 elements and for various degrees of confidence.

Section 4 contains a critical analysis of the method of "probability papers", a method which combines the advantages of the non-parametric method with the potentialities inherent to the parametric methods. For this method we referred principally to the works by Gumbel (Ref. 4) and Weibull (Ref. 5).

Lastly, in Section 5, the various methods mentioned are applied to some real cases and the results are compared with reference to the extreme values.

1.3 Some General Concepts

1.3.1 Definition of reliability

Out of the various definitions of reliability we quote the one adopted by the IEC: "The characteristic of an item expressed by the probability that it will perform a required function under stated conditions for a stated period of time". The probability indicated, a function of time, is the complement to 1 of the probability of non-function or probability of failure.

Considerations on reliability are based on the considerations on the failure distribution, since the failure is the physically observed event.

1.3.2 Failure distribution function and failure rate function

The functions of failure distribution versus time are also indicated as life characteristics of the given component. We shall take $F(t)$ to be the failure distribution, i.e., the probability that the component will fail before time t, and $f(t)$ the corresponding density. It is also expedient to introduce a "failure rate" $v(t)$ defined as

$$v(t) = \frac{f(t)}{1 - F(t)}$$

This function is also known as the "force of mortality", "mills ratio", "intensity function" or "hazard rate". The failure rate function is useful because amongst other things, it allows of dividing the distribution functions into two main categories - the failure rate functions that increase with time, and those that decrease with time.

The fact of belonging to one or other of these categories has an immediate physical significance: an increasing f.r.f. corresponds to the existence of wear or fatigue phenomena, a decreasing f.r.f. to the running-in situation, for instance; but the subdivision also has an important formal significance: one need only know that a distribution belongs to one or the other category to be able to deduce limit statistical properties of the component concerned or of the system consisting of a number of components (Ref. 6).

1.3.3 Most commonly used continuous failure distributions

Exponential distribution

$$F(t) = 1-e^{-\lambda t} \qquad t \geq 0 \, , \; \lambda > 0$$

$$f(t) = \lambda e^{-\lambda t}$$

$$v(t) = \lambda$$

$$\text{Mean} = 1/\lambda = \tau$$

Weibull distribution

$$F(t) = 1 - e^{-\lambda(t-\theta)^{\alpha}} \qquad t \geq 0 \text{ , } \theta \geq 0 \text{ , } \lambda > 0 \text{ , } \alpha > 0$$

$$f(t) = \lambda\ \alpha(t-\theta)^{\alpha-1}\ e^{-\lambda(t-\theta)^{\alpha}}$$

$$v(t) = \lambda\ \alpha(t-\theta)^{\alpha-1}$$

$$\text{Mean} = \lambda^{1/\alpha}\Gamma(1+1/\alpha)$$

Normal distribution

$$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

$$v(t) = \frac{e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}}{\int_{t}^{\infty} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt}$$

$$\text{Mean} = \mu$$

Log-normal distribution

If $y = \ln t$ is a normal variate with mean μ and variance σ, the distribution of t is known as log-normal:

$$F(t) = \frac{1}{\sigma\ \sqrt{2\pi}} \int_{-\infty}^{\ln t} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

$$= \frac{1}{\sigma\ \sqrt{2\pi}} \int_{o}^{t} e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2} \frac{dt}{t}$$

$$f(t) = \frac{1}{t}\ \frac{1}{\sigma\ \sqrt{2\pi}}\ e^{-\frac{1}{2}\left(\frac{\ln t-\mu}{\sigma}\right)^2}$$

$$\text{Mean} = e^{\mu+\sigma^2/2}$$

The exponential distribution is characterized by a constant rate of failure function (λ). The reciprocal of λ is the mean time between two failures (MTBF).

This law interprets failure phenomena corresponding to purely random events and it also interprets phenomena of failure of complex systems, when the number of components tends to become very large, independently of the law of failure of the individual components. Furthermore it takes advantage of the fact that a system consisting of components characterized by an exponential law will likewise have an exponential failure law.

The normal and log-normal distributions are used mainly to interpret failure phenomena due to wear. They are characterized by failure rate functions that increase with time.

The Weilbull distribution, with three parameters, is more flexible than the foregoing ones. Its limit case, for $\alpha = 1$, is the exponential distribution, and it too can be used to interpret failure due to wear. Moreover it is suitable for a linear representation on log-log paper, so that it does not require special probability papers. Lastly it is an asymptotic distribution of the extreme values of a wide class of distributions (Ref. 4), for which reason it appears in particular to be inherently suited to represent the phenomena of material failure, interpreted as the failure of the weakest link in a chain.

For these reasons this distribution, proposed originally by Weibull to interpret data on tensile and fatigue failure of materials, has been increasingly used in the field of electromechanical components which we shall be considering in particular.

1.3.4 The reliability function

The reliability function R is defined as the difference between the failure distribution values corresponding to the extremes of the event (period of time intended and operating conditions encountered).

$$R = F(t_2) - F(t_1)$$

In general one assumes for the time interval (t_2, t_1), (∞, T), so that:

$$R(T) = 1 - F(T)$$

The time T is often indicated as "mission time".

On the basis of this definition R(t) is to be deduced straightaway in the cases F(t) indicated.

2. PARAMETRIC METHODS

2.1 General Scheme

The term "parametric" applied to these methods is due to the fact that, starting from the sample, they evaluate the parameters of the distribution of failure and hence of reliability, a distribution hypothesized a priori. Roughly speaking, their stages of use are as follows:

i) availability of a complete set(*) of values (sample) referring

(*) An incomplete set is one of defined dimensions but only partially defined values. Take, for instance, a fixed number of components being tested simultaneously. The set of lifetimes will be complete when the last surviving component fails; it will be incomplete for all the preceding times.

to the component's characteristic used for the reliability estimate (lifetime, breaking stress, etc.). These values obviously have to be obtained from tests or operating experience on components belonging to the same statistical population.

ii) Assumption of one or more forms of statistical distribution to which the sample is assumed to belong.

iii) Estimate of the distribution parameters, based on the sample values. The most practical and suitable procedure for this purpose is the one based on the principle of maximum likelihood.

iv) Test for goodness of fit on the various assumed distributions to see which one fits the interpretation of the sample best for a given significance level.

v) Determination of the variances of the estimated parameters and, if appropriate, of their confidence intervals.

vi) Calculation of the reliability value, by means of the distribution adopted and the estimated parameters. This reliability value will likewise be an estimated value. Hence a confidence interval will have to be established for it.

2.2 Estimate of Parameters

2.2.1 The maximum-likelihood method

This very general method is mentioned in all textbooks on statistics. Considering, for simplicity's sake, a distribution with a single parameter α, $f(t,\alpha)$, of which the mathematical form is assumed to be known, we form from the sample $(t_1, t_2, \ldots, t_n)$ the function

$$L(t_1,t_2,\ldots,t_n;\alpha) = \prod_{1}^{n}{}_i \; f(t_i,\alpha) \qquad (1)$$

known as the function of likelihood of the sample. It corresponds to the compound probability of n random independent variables, each with the same probability distribution, i.e., it corresponds to the probability of obtaining the sample under study out of all the possible samples of the same size. The method consists in determining which value of the parameter α renders it most probable that the sample under study will turn up. Thus, if we call that value α^* it must satisfy the equation

$$\left(\frac{\partial L}{\partial \alpha}\right)_{\alpha=\alpha^*} = 0 \qquad (2)$$

(or $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$ with $\mathcal{L} = \log L$)

known as the likelihood equation.

Under very general conditions, the maximum-likelihood estimate has a normal distribution when the sample dimensions tend to ∞. This asymptotic property of the maximum-likelihood estimates is most useful, because it means that the properties characteristic of a normal distribution can be attributed to those estimates. At the same time, inasmuch as it is an asymptotic property, it is the chief limitation of the method, since small samples cannot be taken into consideration (according to Ref. 1, page 172, the correct use of the normal approximation calls for sample sizes of not less than 50).

2.2.1 Determination of the variances of the estimated parameters

A distribution dependent on two parameters α, λ is considered. Let $\hat{\alpha}$ and $\hat{\lambda}$ be the values of these parameters estimated by the maximum-likelihood method from the sample values. It has been shown (Ref. 7) that by using the asymptotic property of the estimated parameters, approximated values of the $\hat{\alpha}$ and $\hat{\lambda}$ variances are obtained by constructing the matrix

$$A = \begin{Vmatrix} \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{Vmatrix} \tag{3}$$

Between A and matrix

$$B = \begin{Vmatrix} \text{Var}\,\hat{\alpha} & \text{Cov}\,(\hat{\alpha},\hat{\lambda}) \\ \text{Cov}(\hat{\alpha},\hat{\lambda}) & \text{Var}\,\hat{\lambda} \end{Vmatrix} \tag{4}$$

there is the simple relation:

$$B = - A^{-1} \tag{5}$$

It will be noted that A is a function of the real parameters α, λ; approximated values are obtained by substituting for the real, unknown values the estimated values $\hat{\alpha}, \hat{\lambda}$.
In the case where the distribution depends on a single parameter α, we obtain from the foregoing formulae:

$$\text{Var}\,\hat{\alpha} = - \left(\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right)^{-1} \tag{6}$$

2.3 Goodness of Fit

The choice of the form of distribution to which the data are assumed to belong is, a priori, arbitrary. Hence, the distributions adopted, whose parameters have been estimated on the basis of the sample, must be tested to decide which fits best with the sample. Let us briefly describe two widely used tests, namely the chi-squared test and the Kolmogoroff test. The first applies to the density of distribution, the second to the distribution. The efficiency of both methods is limited by the size of the sample. The first method is not applicable to small samples because it calls for division of the sample into classes and calculation of the frequency for each class. The second method does not have this drawback. But being based on asymptotic properties, neither is very significant when it comes to small samples.

2.3.1 The chi-squared test (Ref. 8)

The data for the sample of size n are classified in k intervals

$$t_i \pm \frac{\Delta t_i}{2} \qquad i = 1, —, k$$

and the values ν_1 are considered, corresponding to the number of sample data comprised in the i-th generic interval.

If f(t) is the density function of the assumed distribution

$$P_i = \int_{t_i - \Delta t_{i/2}}^{t_i + \Delta t_{i/2}} f(t)dt \tag{7}$$

will represent the probability that the statistical variable in question belongs to the i-th interval.

If the assumption concerning the distribution is valid, then

$$\lim_{n \to \infty} P(|\nu_i - np_i| < \epsilon) = 1 \tag{8}$$

Hence a measurement of the data's goodness of fit with the hypothesis is related to the complex of differences $(\nu_i - np_i)$.

With a choice owed to Pearson, we can establish the following magnitude as the measurement of this goodness of fit:

$$\Delta^2 = \sum_{1}^{k} {}_i \frac{(\nu_i - np_i)^2}{np_i} \qquad (9)$$

and it can be shown that Δ^2 is a random variable distributed, with $n \to \infty$, according to a χ^2 law with k-1 degrees of freedom, in the event that the parameters of the assumed distribution are known.

If, on the other hand, the parameters are estimated from the sample, the number of degrees of liberty will be lower than k-1 by as many units as there are estimated parameters.

For practical application of the test, having calculated Δ^2 and set a level of significance γ, we find from the tables a value χ^2_γ such that:

$$P(\chi^2 \geq \chi^2_\gamma) = \gamma \qquad (10)$$

The assumed distribution satisfies the test if

$$\Delta^2 < \chi^2_\gamma$$

For a valid application of the test the sample dimensions must be such that

$$np_i > 10 \qquad i = 1, \text{—}, k$$

2.3.2 Kolmogoroff test (Ref. 9)

This is a test which examines the cumulative distribution. Let F(t) be this distribution assumed to be continuous and let $S_n(t)$ be the empirical distribution of the sample of dimensions n, arranged in ascending order of values.

Furthermore let:

$$D_n = \underset{-\infty<x<\infty}{mx} |F(t) - S_n(t)| \quad (11)$$

$$Q(\lambda) = \sum_{-\infty}^{\infty}{}_k (-1)^k e^{-2k^2\lambda^2} \quad \lambda > 0 \quad (12)$$

The test is based on Kolmogoroff's theorem which states:

$$\lim_{n\to\infty} P(D_n < \frac{\lambda}{\sqrt{n}}) = Q(\lambda) \quad (13)$$

For application purposes, once D_n has been calculated and a level of significance α has been chosen, we find in the tables value λ_α for which

$$Q(\lambda_\alpha) = 1 - \alpha \quad (14)$$

The distribution in question will satisfy the test if

$$D_n < \lambda_\alpha/\sqrt{n}$$

In Appendix 1 will be found the description of the KTEST code, programmed in IBM 360/65 to effect the Kolmogoroff test on various distributions. The normal, log-normal, Weibull and exponential distributions are considered.

2.4 Reliability Estimate

When the failure distribution parameters have been estimated, we can estimate the reliability value corresponding to a time T.

$\hat{R}(T) = R(\hat{\alpha}, \hat{\lambda}, T)$

Now comes the problem of evaluating the confidence we can have in this estimate.

The general method, which is valid only for numerous samples and does not require knowledge of the distribution of the parameter estimates, i.e., of $\hat{\alpha}$, $\hat{\lambda}$, etc., makes use of R's property of being asymptotically normal (Ref. 1, page 192). Hence it is necessary to know $E(\hat{R})$ and Var $\hat{R}$, i.e., the mean value and variance of the estimate.

It has been shown (Ref. 10, page 354) that

$$E[R(\hat{\alpha},\hat{\lambda})] = R(\alpha,\lambda) + O(1/n) \qquad (15)$$

$$\begin{aligned} \mathrm{Var}[R(\hat{\alpha},\hat{\lambda})] = {} & \left(\frac{\partial R}{\partial \alpha}\right)^2_{\alpha} \mathrm{Var}\,\hat{\alpha} + \left(\frac{\partial R}{\partial \lambda}\right)^2_{\lambda} \mathrm{Var}\,\hat{\lambda} + \\ & + 2\left(\frac{\partial R}{\partial \alpha}\right)_{\alpha} \left(\frac{\partial R}{\partial \lambda}\right)_{\lambda} \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) + O(1/n^{1.5}) \end{aligned} \qquad (16)$$

Both $O(1/n^{1.5})$ and $O(1/n)$ are terms which tend towards zero as the sample dimensions increase. An estimate of $E(\hat{R})$ and $\mathrm{Var}(\hat{R})$ can be obtained by substituting $\hat{\alpha}, \hat{\lambda}$, for α, λ in (15) and (16).

This general procedure is not necessary in cases where the reliability is a function of a single parameter (see exponential distribution). In such a case a reliability confidence interval can be found directly from the parameter confidence interval. For this purpose one must know the parameter distribution or else apply the property of normal asymptotic behaviour of the estimate using the variance calculated in Section 2.2.1.

2.5 Applications

2.5.1 Exponential distribution

The failure distribution density is given by:

$$f(t,\lambda) = \lambda\, e^{-\lambda t} \qquad t \geq 0\ ,\ \lambda > 0$$

Starting from the sample $(t_1,\ldots,t_n)$ the maximum-likelihood function will be:

$$L = \lambda^n\, e^{-\lambda\, \Sigma_i t_i} \qquad (17)$$

and from the maximum-likelihood equation

$$\frac{\partial \log L}{\partial \lambda} = 0$$

we obtain

$$\frac{1}{\hat{\lambda}} = \frac{\Sigma_i t_i}{n} \tag{18}$$

i.e., the mean of the sample is the inverse of the estimate of parameter λ. An estimated value of the reliability at time T will be given by:

$$\hat{R} = R(T,\hat{\lambda}) = e^{-\hat{\lambda}T} \tag{19}$$

The calculation of the confidence interval of this estimate can be done by two different routes as already mentioned in Section 2.4. One procedure, which we might call general, entails calculation of the variance of the distribution parameter, followed by calculation of the $\hat{R}$ variance and, using the normal approximation, the $\hat{R}$ confidence interval. From (6) we obtain

$$\text{Var } \hat{\lambda} = - \left(\frac{\partial^2 \log L}{\partial \lambda^2}\right)^{-1} = \frac{\lambda^2}{n}$$

and from (15) and (16)

$$E(\hat{R}) = e^{-\lambda T}$$

$$\text{Var } \hat{R} = \left(\frac{\partial \hat{R}}{\partial \hat{\lambda}}\right)^2_{\lambda} \text{Var } \hat{\lambda} = \frac{\lambda^2 T^2}{n} e^{-2T.\lambda}$$

$$\sigma_{\hat{R}} = \frac{T.\lambda}{\sqrt{n}} e^{-\lambda T} = \frac{R \log 1/R}{\sqrt{n}} \simeq \frac{\hat{R} \log 1/\hat{R}}{\sqrt{n}}$$

The variable

$$\eta = \frac{\hat{R} - E(\hat{R})}{\sigma_{\hat{R}}} \tag{20}$$

is asymptotically a standardized normal variate. Having established a confidence level γ, we find the confidence interval for the reliability:

$$\hat{R} \pm \sigma_{\hat{R}} \cdot \eta_{(1+\gamma)/2}$$

A second procedure, valid only in the case of exponential distribution, allows one to avoid the repeated use of the asymptotic approximations employed in the previous procedure.

This second route is based on two characteristics of the exponential distribution:

- the distribution of the estimated parameter $\hat{\lambda}$ is known;
- the reliability is a monotonic function of the parameter.

It has been shown (Ref. 11, page 190) that the estimated parameter $\hat{\tau} = \frac{1}{\hat{\lambda}}$ has a gamma distribution:

$$P(\hat{\tau} \leq x) = \frac{1}{\Gamma(n)} \int_0^x (\frac{n}{\tau})^n \, t^{n-1} \, e^{-n\frac{t}{\tau}} \, dt \qquad (21)$$

by putting $\chi^2 = 2n\frac{\hat{\tau}}{\tau}$ this distribution can be reduced to a chi-squared distribution with 2n degrees of freedom. Thus (21) is equivalent to:

$$P(\chi^2 \leq y) = \frac{1}{2^n\Gamma(n)} \int_0^y t^{n-1} \, e^{-t/2} \, dt \qquad (22)$$

By using (22) we can obtain an exact evaluation of the confidence limit on $\hat{R} = R(\hat{\tau})$ at a given confidence level γ. For having fixed a value for γ, we obtain from (22):

$$P(\frac{2n\hat{\tau}}{\chi^2_{1-\gamma}} < \tau) = \gamma \qquad (23)$$

Thus $\hat{\tau}_i = \frac{2n\hat{\tau}}{\chi^2_{1-\gamma}}$ represents a lower limit of τ with confidence level γ.

As the reliability $R = e^{-T/\tau}$ is an increasing function of τ, it follows that a lower limit for the reliability at time T, with confidence level γ, will be given by:

$$R(\hat{\tau}_i) = e^{-\frac{T\,\chi^2_{1-\gamma}}{2n\,\hat{\tau}_i}} \qquad (24)$$

It is important to note that we have been able to transfer to the reliability the confidence limit calculated for $\hat{\tau}$ only inasmuch as the distribution has only one parameter. In general this is not possible where there is more than one parameter.

2.5.2 Weibull distribution

The most general form of this distribution has three parameters:

$$F(t) = 1-e^{-\lambda(t-\theta)^{\alpha}}$$

for the sake of simplicity, we shall assume $\theta = 0$; the probability density function therefore is:

$$f(t) = \alpha\lambda\, t^{\alpha-1}\, e^{-\lambda t^{\alpha}} \qquad t \geq 0\ ,\ \alpha > 0\ ,\ \lambda > 0$$

The log of the likelihood function, for a sample $(t_1, \ldots, t_n)$ is given by:

$$\mathcal{L} = n \log\alpha + n \log\lambda + (\alpha-1) \sum_1^n {}_i \log t_i - \lambda \sum_1^n {}_i\, t_i^{\alpha} \qquad (25)$$

By imposing the maximum likelihood conditions on $\mathcal{L}$:

$$\frac{\partial\mathcal{L}}{\partial\alpha} = 0 \qquad \frac{\partial\mathcal{L}}{\partial\lambda} = 0$$

we get two equations for the determination of the estimated values $\hat{\alpha}, \hat{\lambda}$, of the two parameters:

$$\hat{\lambda} = \frac{n}{\Sigma\, t_i^{\hat{\alpha}}} \tag{26}$$

$$\hat{\alpha} = \frac{n}{\hat{\lambda}\, \Sigma\, t_i^{\alpha} \log t_i - \Sigma \log t_i} \tag{27}$$

The calculation of $\hat{\alpha}$ and $\hat{\lambda}$ from these equations is done with an iteration process programmed on IBM 360/65. To obtain a reasonable initial value for $\hat{\alpha}$ the following relation is used, which expresses equality between the sample mean and the distribution mean:

$$\frac{1}{n} \Sigma\, t_i = \lambda^{-1/\alpha}\, \Gamma(1 + \frac{1}{\alpha}) \tag{28}$$

To determine the variance and covariance of the two parameters it is necessary to invert the matrix

$$\begin{Vmatrix} \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda} & \frac{\partial^2 \mathcal{L}}{\partial \lambda^2} \end{Vmatrix} \tag{29}$$

Having calculated Var $\hat{\alpha}$, Var $\hat{\lambda}$, Cov$(\hat{\alpha}\,\hat{\lambda})$ in this way, we can find the variance of the estimated reliability value. If T is the mission-time for the component for which the reliability is to be ascertained, then the estimated reliability value is

$$\hat{R} = e^{-\hat{\lambda} T^{\hat{\alpha}}}$$

Also, by reference to (15) and (16)

$$E(\hat{R}) = e^{-\lambda T^{\alpha}} \tag{30}$$

$$\mathrm{Var}\,\hat{R} = T^{2\alpha}\, e^{-2\lambda T^{\alpha}}\, (\lambda^2 \log^2 T\, \mathrm{Var}\,\hat{\alpha} + \mathrm{Var}\,\hat{\lambda} + 2\lambda \log T\, \mathrm{Cov}(\hat{\alpha},\hat{\lambda})) \qquad (31)$$

Estimated values for $E(\hat{R})$ and Var $\hat{R}$ can be obtained by replacing α and λ in the previous equations with their estimates $\hat{\alpha}$ and $\hat{\lambda}$.

Using the normal approximation we can then find a confidence interval for $\hat{R}$.

The foregoing calculations have been programmed on IBM 360/65. Appendix 2 gives a description of the VITA code employed.

3. NON-PARAMETRIC METHODS

3.1 General

Given a fairly small sample (with fewer than, say, 20 values) the mathematically laborious method described in the previous chapter yields results whose significance is not proportionate to the effort required.

The method we shall give here, however, enables the reliability corresponding to the measured values to be easily and directly evaluated, even with very small samples.

It also permits of evaluating a confidence interval, likewise in respect of the measured values.

Lastly, it allows the sample size to be taken into account in cases where the sample is incomplete: from this standpoint it offers a possibility not allowed by the method described in the previous chapter.

On the other hand, as it does not aim to evaluate the distribution but confines itself to evaluation of a series of discrete values, it does not provide indications for interpolation or extrapolation.

3.2 Statistical Properties of Ordered Samples

3.2.1 Distribution of the m-th value

Let $(t_1, --, t_m, --, t_n)$ be a sample of size n with values in order of increase. The distribution $\Phi(t)$ from which the sample was taken is unknown. The problem is to estimate the cumulative probability $\Phi(t_m)$, using for the purpose the sample's property of being ordered. If the population is sampled again, the value t'_m, arrayed in the m-th position, will in general be different from t_m and one can say that the sample order position m characterizes, by means of all the samples extractable from the population, a set of values, the t_m values, which will be distributed according to their own law of probability, whose density is:

$$\psi_n(t_m) = m\binom{n}{m}\Phi^{m-1}(t_m)\left[1 - \Phi(t_m)\right]^{n-m}\Phi'(t_m) \qquad (1)$$

This law can be determined at once by using the polynomial distribution and the sample's property of order. It is a known fact that, given three events with probabilities p_1, p_2 and p_3 at the instant of a test, the probability that in n tests the event with probability p_1 will occur n_1 times, that of probability p_2 n_2 times, and that of probability p_3 n_3 times is:

$$\frac{n!}{n_1!\,n_2!\,n_3!}\,p_1^{n_1}\,p_2^{n_2}\,p_3^{n_3}$$

If we now let the event "value of t lying between t_m and $t_m + dt_m$" correspond to p_1, then

$$p_1 = \xi(t_m)dt_m$$

where $\xi(t) = \Phi'(t)$

Similarly let the event "value of $t \leq t$" correspond to p_2, then

$$p_2 = \Phi(t_m)$$

and lastly let the event "value of $t > t_m$" correspond to p_3, so that

$$p_3 = 1 - \Phi(t_m)$$

If $n_1 = 1$, $n_2 = m - 1$, $n_3 = n - m$, the probability law obtained is actually that of the population of t_m values represented by (1). Naturally (1) and therefore the mean $\bar{t}_m$, the median $\check{t}_m$ and the modal value $\tilde{t}_m$ are unknown, in our case, because $\Phi(t)$ is unknown.

3.2.2 Probability distribution for m-th value

By performing in (1) the variate transformation

$$\Phi_m = \Phi(t_m) \tag{2}$$

we obtain

$$\chi_n(\Phi_m) = m\binom{n}{m}\Phi_m^{m-1}(1-\Phi_m)^{n-m} \tag{3}$$

in which

$$0 \leq \Phi_m \leq 1$$

Thus $\chi_n(\Phi_m)$ represents the probability density for the distribution of the cumulative probability values appropriate to the values of t_m.

The chief interest of (3) lies in the fact that the Φ_m distribution does not depend on the unknown $\Phi(t)$ distribution. It will be recognized that $\chi_n(\Phi_m)$ is a beta distribution.

In general the probability density for the beta distribution is:

$$\xi(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\ \Gamma(\beta+1)} \quad x^{\alpha}(1-x)^{\beta}$$

for $0 \geq x \geq 1$ with entire $\alpha, \beta > -1$.
With $\alpha = m - 1$ and $\beta = n - m$ one obtains expression (3).

3.2.3 Estimate of the $\Phi(t_m)$ probability - Median ranks

If $\eta(p)$ is taken to represent the cumulative distribution of Φ_m, then

$$\eta(p) = \int_o^p \chi_n(\Phi_m)d\Phi_m$$

It is readily apparent that, integrating item by item successively, we shall obtain

$$\eta(p) = \sum_m^n {}_i\binom{n}{i}p^i(1-p)^{n-i} \qquad (4)$$

or

$$\eta(p) = 1 - \sum_o^{m-1} {}_i \binom{n}{i} p^i(1-p)^{n-i} \qquad (5)$$

Relation (5) enables us to solve the problem stated at the outset, namely, that of obtaining an estimate of the probability $\Phi(t_m)$ and assigning a confidence level for that estimate.

For in (5), p is a value of Φ_m such that the probability of a value $\Phi_m \leq p$ is $\eta(p)$; hence it can be said that p is the estimate of $\Phi(t_m)$ with confidence level $\eta(p)$.

In other terms this means that if we assign the cumulative probability p to the sample observation t_m, there will be 100 $\eta(p)$ samples, out of 100 extractable from the population, in which the value $\Phi(t_m)$ will be lower than p. It is perhaps needless to remark that the use of (5) to estimate $\Phi(t_m)$ does not entail knowledge of the value t_m; it is merely assumed that t_m is the largest of the m values observed, i.e., that the sample is ordered in increasing values.

Hence (5) lends itself to the construction of tables for p, each one characterized by a value of $\eta(p)$. These are double-entry tables in which, for every n, the p values are given in line with m = 1, 2, __, n. It can be shown that with $\eta(p)$, m, n fixed, there is a single solution of (5) lying between 0 and 1.

Appendix 3 gives the text of the RANKS programme processed on IBM 360/65 for the solution of (5), and also, for $\eta(p) = .05$, .5, .95, the tables of the p values obtained, for sample sizes up to 20. The p values obtained with $\eta(p) = .5$ are known as "median ranks" and are particularly recommended by Johnson (Refs. 2 and 3), who was the first to use them. An interesting aspect of (5) is that confidence belts can be constructed. For this purpose the tables for $\eta(p) = .05$ and .95 are provided. Their use is immediate: they permit of stating that the unknown real probability $\Phi(t_m)$ lies, with 90% probability, in the interval bounded by ${}_mP_{.95}$ and ${}_mP_{.05}$.

3.2.4 Mean ranks and modal value

Other interesting aspects of the distribution of the cumulative probabilities $\eta(p)$ can be found by calculating, in addition to the median already noticed, the mean value and the modal value; as regards the mean value we have:

$$\bar{\Phi}_m = \int_o^1 \chi_n(\Phi_m)\Phi_m d\Phi_m \qquad (6)$$

Noting that:

$$\int_o^1 \Phi_m^m(1-\Phi_m)^{n-m}d\Phi_m = \frac{\Gamma(m+1)\ \Gamma(n-m+1)}{\Gamma(n+2)}$$

we have

$$\bar{\Phi}_m = m\binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{m}{n+1} \qquad (7)$$

The modal value is obtained from (3) as the solution of $\chi_n' = 0$:

$$\tilde{\Phi}_m = \frac{m-1}{n-1} \qquad (8)$$

The advantage of these estimates as against the median ranks is their very simple form which permits of immediate calculation for all values of m and any size of sample.
Furthermore the confidence level which, by means of (5), can be associated with each estimate is not constant as for the median ranks, but varies with m or with the sample size.

4. METHOD OF PROBABILITY PAPERS

4.1 General

This method has the same objectives as the method described in Section 2, i.e., it aims at deriving a distribution from the sample.

As in the parametric method, the first step in this method is to choose a form of distribution, and then to select a "paper" in which that form of distribution is linear.

Having chosen the paper and therefore the linearization of the function, we now have to represent the sample values.

Next we trace, by means of a suitable regression, the straight line which best interpolates these points, and in this way we obtain the parameters of the desired basic distribution.

4.2 Linearization

Let $\Phi(t, \alpha, \beta)$ be the cumulative probability of a statistical variable t and let α, β be the distribution parameters. If there is a linear transformation

$$y = \alpha(t - \beta) \qquad (1)$$

such that the distribution

$$F(y) = \Phi(\beta+y/\alpha, \alpha, \beta) \qquad (2)$$

is independent of the parameters α, β, it is possible to construct a probability paper for the distribution Φ. On this $\Phi(t, \alpha, \beta)$ will then be represented by the straight line (1). F(y) is called the "standard form" of the distribution and is usually tabulated. If there are three parameters α, β, γ, there is more than one linearization possible. For instance in the case of a complete Weibull distribution

$$W(t) = 1 - e^{-\left[(t-\beta)\alpha\right]^{\gamma}} \qquad (3)$$

it is possible, for each fixed value of γ, to effect the linearization (1) and hence to refer to a standard form relating to the fixed γ value. Generally, however, the Weibull distribution is used in the incomplete form obtained with $\beta = 0$. Obviously linearization of type (1) is then out of the question. In that case we effect a logarithmic transformation which leads to

$$\frac{1}{\gamma} \log \ln \frac{1}{1-W(t)} = \log t + \log \alpha \qquad (4)$$

which on log-log paper with coordinates t, $\ln \frac{1}{1-W(t)}$ is a linearization of (3). In this case one can no longer speak of a standard form for the distribution.

4.3 The Plotting Position

As already remarked, the crucial problem in using probability papers lies in the choice of the probability value to assign to the generic value of the sample. It will be seen that the manner of choice can be exacting, taking into account the type of distribution that the data have to fit, or approximate (although fulfilling certain criteria), disregarding that distribution.

4.3.1 Distribution-dependent plotting

We have already seen in Section 3 that in an ordered sample of size n the m-th position designates, through all the possible ordered samples extractable from the population, a new distribution, that of the m-th value, whose density function is:

$$\psi_n(t_m) = m\binom{n}{m}\Phi^{m-1}(t_m)\left[1 - \Phi(t_m)\right]^{n-m}\Phi'(t_m) \qquad (5)$$

the transformation (1) will provide a value corresponding to each t_m, namely

$$y_m = \alpha(t_m-\beta) \qquad (6)$$

belonging to the distribution of the m-th reduced value. Applying the mean operator to (6), we obtain:

$$E(t_m) = \beta + \frac{1}{\alpha} E(y_m) \qquad (7)$$

The plotting position proposed by Weibull (Ref. 5, p. 198) is

$$P_m = F(E(y_m)) \qquad (8)$$

F being distribution (2), i.e., the standard form of the hypothesized distribution.

Consequently, with the m-th observation of the ordered sample we must associate a cumulative probability given by the value of the standard distribution at the mean of the reduced variables relating to the m-th position; the least squares will therefore be effected on the points t_m, $E(y_m)$. The mean $E(y_m)$ must be calculated from the distribution of y_m which is determined by F(y) and is thus independent of the unknown parameters. The distribution of y_m is found by operating in (5) the change of variables given by (1):

$$\theta_n(y_m) = m\binom{n}{m}F^{m-1}(y_m)\left[1 - F(y_m)\right]^{n-m}F'(y_m) \qquad (9)$$

Hence

$$E(y_m) = \int_{-\infty}^{\infty} y_m \, \theta_n(y_m)dy_m \qquad (10)$$

or else, writing

$$F(y) = u \; , \quad y = G(u) \qquad (11)$$

$$E(y_m) = \int_0^1 n\binom{n}{m} G(u) \, u^{m-1}(1-u)^{n-m}du \qquad (12)$$

It will be seen from (12) that $E(y_m)$ depends only on m, n and on the standard form of the assumed distribution. If the plotting position (8) is used and the fitting is done with the least-squares method (minimizing the deviations Δt_i) the estimates $\hat{\alpha}$, $\hat{\beta}$, are not affected by systematic errors (Ref. 5, p. 198). It must be pointed out that position (8) can only be used when the distribution can be brought to a standard form by means of (1). This is not the case, for instance, with the usual Weibull distribution, with $\beta = 0$ (see (3)).

4.3.2 Distribution-independent plotting

Whilst the plotting position (8) recommended by Weibull is the strictest because it does not introduce systematic errors in the parameter estimates, it has the drawback of depending on the preselected form of distribution and hence of requiring the use of tables of the values $E(y_m)$.

Where these tables are not available and one can make do with a certain degree of approximation in the estimate, it is possible to use other plotting positions which are independent of the distribution and have very simple forms.

For example, if the sample is in order of increasing values we can, by convention, assign the cumulative probability m/n to the ordered value t_m. If, on the other hand, the sample is in order of decreasing values, by the same convention we shall assign to the value t_m (which is the (n - m + 1) away from the highest value) the probability

$$1 - \frac{n - m + 1}{n} = \frac{m - 1}{n} \qquad (13)$$

Hence it is clear that the choice of a distribution-independent plotting position involves a certain arbitrariness and at the same time an ambiguity which can be resolved only where a criterion is specified for the most rational choice of position.

The problem has been tackled by Gumbel (Ref. 4, p. 29) who set some criteria for the purpose. They can be summed up as follows:

a) <u>the plotting position must be such that all the sample observations can be represented on the probability paper.</u>

This criterion is not met by the positions m/n and (m - 1)/n, since a probability 1 corresponds to tn in the first and a probability 0 to t_1 in the second. Furthermore, as the probability papers are constructed for unlimited variables, they do not contain the probability values 0 and 1.

An attempt to overcome this difficulty has been made by introducing the position

$$\frac{m - 1/2}{n} \qquad (14)$$

the arithmetic mean of the two previous positions (mid-ranks). But this position, too, is not very satisfactory if tested with the following criterion:

b) <u>the return period of a value equal to or greater than the largest observation</u> (i.e., the number of trials needed on average to obtain a value greater than or equal to the largest observation) <u>and the return period of a value smaller than the smallest observation</u> (i.e., the number of trials needed on average to obtain a number smaller than the smallest observation) must tend to n, the number of observations. The return period is defined as the mean of the geometric distribution, relative to an event with probability p. Given an event with probability p at each test, the probability that it will occur for the first time at the v-th test will be

$$w(v) = pq^{v-1} \qquad\qquad q = 1 - p$$

The mean value of v is $\bar{v} = 1/p$ and represents the return period of the event with probability p.

Hence the return period of a value greater than or equal to the m-th value of an ordered sample is:

$$T_s(t_m) = \frac{1}{1-\Phi(t_m)} \tag{15}$$

So the return period of t_n, using position (14), is:

$$T_s(t_n) = \frac{1}{1 - \frac{n-1/2}{n}} = 2n \tag{16}$$

which corresponds to an admission that an event t_n, which has occurred once in n trials, occurs on average once in every 2n trials. Similarly, considering the return period of a value smaller than t_1, we have:

$$T_i(t_1) = \frac{1}{\Phi(t_1)} = 2n \tag{17}$$

position (14) consequently gives an over-optimistic result precisely at the extreme values which, in many circumstances and in failure phenomena in particular, are the most significant ones. Moreover, the position m/n and (m - 1)/n are not satisfactory from the standpoint of the return period; the return period of a value greater than or equal to t_m, for the position m/n, is:

$$T_s(t_m) = \frac{n}{n - m} \tag{18}$$

and is no longer defined for t_n, while the return period of a value smaller than t_m, for the position (m - 1)/n, is:

$$T_i(t_m) = \frac{n}{m - 1} \tag{19}$$

and is no longer defined for t_1.

It is interesting now to consider, from the standpoint of the plotting position, the magnitudes discussed in Section 3 and defined on the basis of the distribution $\lambda_n(\Phi_m)$ of the probabilities appropriate to the m-th value of an ordered sample.
The modal value

$$\tilde{\Phi}_m = \frac{m-1}{n-1} \tag{20}$$

is not acceptable since it does not satisfy either the first or the second of the preceding criteria.

The median value $\check{\Phi}_m$ defined by

$$\sum_{0}^{m-1}{}_i \binom{n}{i} \check{\Phi}_m^i (1 - \check{\Phi}_m)^{n-i} = 1/2 \tag{21}$$

satisfies the first criterion but not the second. The return period for t_n has the value

$$T_s(t_n) = \frac{1}{1-\Phi_n} \tag{22}$$

But from (21) we find $\Phi_n = 2^{-1/n}$ and therefore, for high values of n,

$$T_s(t_n) \simeq 1.44\ n \tag{23}$$

Also it can happen that $T_i(t_1) = T_s(t_n)$, so that the use of the median ranks as plotting position attributes to the extreme values a return period which exceeds n by 44% and therefore does not satisfy the second criterion.

Lastly, the mean of $\chi_n(\Phi_m)$

$$\bar{\Phi}_m = \frac{m}{n+1} \tag{24}$$

satisfies both criteria, at any rate for high values of n, since the return period for the extremes has the value n + 1.

This plotting position (mean ranks) appears to Gumbel (Ref. 4) to be the recommendable.

4.4 Least Squares Method

Let us briefly review the formulae expressing the distribution parameter estimates obtained by the least squares method. The values of the estimates are naturally different according to whether we minimize the deviations on the observed variable or on the reduced variable. It should be remarked that if the mean of the reduced variables y_m is used as plotting position, estimates free of systematic error will be obtained only by minimizing the deviations of the observed variable (Ref. 5, p. 198). With reference to (1) we then have:

A) $$\sum_1^n {}_m (t-t_m)^2 = \min$$

$$\frac{1}{\hat{\alpha}_A} = \frac{\overline{ty} - \bar{t}\,\bar{y}}{\sigma_n^2} \qquad \hat{\beta}_A = \bar{t} - \bar{y}/\hat{\alpha}_A \tag{25}$$

where

$$\bar{t} = \frac{1}{n}\sum_{1}^{n}{}_m\, t_m \,,\quad \bar{y} = \frac{1}{n}\sum_{1}^{n}{}_m\, y_m \,,\quad \overline{ty} = \frac{1}{n}\sum_{1}^{n}{}_m\, t_m y_m$$

$$\overline{y^2} = \frac{1}{n}\sum_{1}^{n}{}_m\, y_m^2 \qquad \sigma_n^2 = \overline{y^2} - \bar{y}^2$$

Note that, the plotting position and distribution having been chosen, σ_n is a function of the sample size only.

B)

$$\sum_{1}^{n}{}_m\, (y-y_m)^2 = \min$$

$$\frac{1}{\hat{\alpha}_B} = \frac{s_t^2}{\overline{ty} - \bar{t}\bar{y}} \qquad \hat{\beta}_B = \bar{t} - \bar{y}/\hat{\alpha}_B \tag{26}$$

where

$$s_t^2 = (\overline{t^2} - \bar{t}^2)\,\frac{n}{n-1}$$

C) A third parameter estimate consists in minimizing the deviation of the points in parallel to a straight line determined by the condition ty = 0. The gradient of this line is equal and opposite to (1). In this case:

$$\frac{1}{\hat{\alpha}_C} = \left(\frac{1}{\hat{\alpha}_A\, \hat{\alpha}_B}\right)^{0.5} \tag{27}$$

$$\hat{\beta}_C = \bar{t} - \sqrt{(\bar{t} - \hat{\beta}_A)(\bar{t} - \hat{\beta}_B)}$$

If the observations are highly concentrated around (1), i.e., if the degree of correlation is high, the difference between the estimates obtained in the first two systems are small and the parameters estimated with the third system are roughly the arithmetic mean of the parameters estimated with the first two.

4.5 <u>Building of Control Band</u>

Having solved the fitting problem, i.e., determined the estimate values $\hat{\alpha}$, $\hat{\beta}$, our next task is to construct a control band on either side of the straight line $y = \hat{\alpha}(t - \hat{\beta})$, i.e., to delimit a zone within which, with a pre-established confidence level, we shall find the m-th observation of an ordered sample extractable from the population.

For this purpose the distribution of the m-th value of the sample, expressed by (5), must be taken into consideration. Naturally $\Psi_n(t_m)$ is unknown, because $\Phi(t_m)$ is unknown; on the other hand, $\Theta_n(y_m)$ is known, since it is expressed by (9) as a function of the standard form $F(y_m)$. Furthermore, the two distributions are formally equal and hence the properties of the one that leave the parameters out of account are also properties of the other. In particular it has been shown (Ref. 4, p. 48) that the asymptotic form of (5), for central values of m, is normal with a mean value $\bar{t}_m$ obtainable from:

$$\Phi(\bar{t}_m) = \frac{m}{n+1} \tag{28}$$

and variance

$$\sigma^2(t_m) = \frac{\Phi(\bar{t}_m)(1-\Phi(\bar{t}_m))}{n\ \Phi'^2(\bar{t}_m)} \tag{29}$$

If $\sigma^2(t_m)$ were known, then the control band problem would be solved, at least under the conditions for the validity of the asymptotic form. But $\sigma^2(t_m)$ is not known because it depends on $\Phi'(\bar{t}_m)$. It is therefore necessary to use the preceding observation which also has an asymptotically normal distribution $\Theta_n(y_m)$ with variance

$$\sigma^2(y_m) = \frac{F(\bar{y}_m)(1-F(\bar{y}_m))}{n\, F'^2(\bar{y}_m)} \qquad (30)$$

which is independent of the α, β parameters and can only be calculated on the basis of the adopted distribution.

The standard error for the reduced variable y_m is therefore a pure number

$$\sqrt{n}\, \sigma(y_m) = \frac{\sqrt{F(1-F)}}{F'^2} \qquad (31)$$

which can be determined, for each m, from the knowledge of $F(\bar{y}_m) = m/(n + 1)$ and also of $F'(\bar{y}_m)$ which can be found in the standard form tables beside $F(\bar{y}_m)$.

The standard error on t_m is then obtained from (1) and (31):

$$\sigma(t_m) = \frac{\left[\sqrt{n}\, \sigma(y_m)\right]}{\alpha \sqrt{n}} \qquad (32)$$

and if α has been estimated, (32) can be used to construct the control curves. These will be obtained by connecting the points

$$\hat{t}_m \pm k\sigma(\hat{t}_m) \qquad (33)$$

$\hat{t}_m$ being a point on the estimated straight line and k a coefficient dependent on the degree of confidence attributed to the control band. For example, k = 1.96 expresses the probability 0.95 that for any m - within the limits of the hypothesis on the central values - the observation t_m of the generic sample lies within the interval $\hat{t}_m \pm 1.96\, \sigma(\hat{t}_m)$. On these bases it is not possible to calculate the control band at the extreme values. As a rule one assumes that the foregoing considerations are valid in the probability interval 0.15 - 0.85. Outside that interval the asymptotic distribution of t_m ceases to be normal (Ref. 4, p. 49).

Another way of constructing control bands, which has the advantage of being independent of the standard form of the distribution adopted and of being valid even at the extreme sample values, is the method which is mentioned at the end of Section 3 and is based on knowledge of the tabulated values of p for a certain confidence level $\eta(p)$.

5. APPLICATIONS

By way of example, three sets of data concerning times and breaking stresses of mechanical components are processed below by the methods we have described. One of the samples examined is incomplete, i.e., this is a case of failure times drawn from a sample which includes components still in operation; the other two samples are complete. The available data are processed with the KTEST code (Appendix 1) to establish which distribution interprets them best. Owing to the smallness of the samples, the Kolmogoroff test is ineffective in two cases because the level of significance reaches the max. value 1 in three out of the four distributions tried. For the third set, however, (stresses to failure) the test gives as the limit level of significance the values 97.6% for the Weibull distribution, 98.5% for the log-normal, and 97.5% for the normal and the exponential.

For greater simplicity and for the purposes of example, we shall assume, however, that the sets of data can be interpreted by a Weibull distribution, linearized as in expression (4), Section 4.

The scales of (1 - R), R being the reliability, and of Y = log ln 1/R are entered on the two horizontal axes of the relevant probability paper, whilst the observed variable and its logarithm are entered on the vertical axes.

5.1 Intermetallic Weld Failure Stresses

The following results were obtained from a series of shearing strength tests:

σ_i (kg/mm^2)

6.73 , 6.74 , 10.1 , 10.5 , 10.7 , 12.6 , 13.3 , 13.8
14.7 , 14.75 , 15. , 15.5 , 16.3 , 16.7 , 17.1 , 17.2
17.24 , 17.3 , 17.5 , 18.1 , 18.24 , 20.2 , 20.3 , 21.2
21.9 , 22.6 , 23.1 , 24.5

Assuming a Weibull distribution and using the non-parametric method of mean ranks on probability paper, we obtain:

$$R(\sigma) = e^{-\left(\frac{\sigma}{19}\right)^3}$$

The maximum-likelihood method, however, gives:

$$R(\sigma) = e^{-\left(\frac{\sigma}{17.95}\right)^{4.13}}$$

Fig. 1 shows the two corresponding straight lines. The following table compares the strength values obtained with the two methods for given values of reliability R.

	$\sigma_{.95}$	$\sigma_{.99}$	$\sigma_{.9999}$
max. likelihood	8.7	5.88	1.93
mean ranks	7.2	4.2	0.89

If, however, we set a working strength of 6 kg/mm^2, the max. likelihood method gives a reliability value of .989 and a lower limit, with confidence level of 98%, given by

$$.989 - \sigma_R \, \zeta_{.95} = .975$$

where σ_R^2 is the reliability variance calculated from (31) in Section 2 ($\sigma_R^2 = 6.91 . 10^{-5}$) and $\zeta_{.95}$ is the reduced normal variable at the 95% level.

Similarly, the non-parametric method gives a reliability value of .969 and a lower limit of .92 with confidence level 95%. The lower limit value is obtained by extrapolating the straight line which interpolates the 95% ranks calculated with the RANKS code.

5.2 Mechanical Seals on Pumps

A series of endurance tests yielded the following results:

t_i (hours)

750, 900, 1018, 1200, 1250, 1500, 1500,

Assuming a Weibull distribution, we obtain the following expressions for the reliability:

a. with the max. likelihood method

$$R(t) = \exp\left(-\left(\frac{t}{1270}\right)^{5.09}\right)$$

b. with mean ranks plotting position on probability paper

$$R(t) = \exp\left(-\left(\frac{t}{1320}\right)^{3.5}\right)$$

c. with median ranks plotting position on probability paper

$$R(t) = \exp\left(-\left(\frac{t}{1300}\right)^{3.9}\right)$$

The corresponding straight lines are shown in Figs. 2 and 3. The table below gives the time-to-failure values with the three methods for set reliability values.

	max. likelihood	mean ranks	median ranks
$t_{.95}$	700	560	596
$t_{.99}$	515	355	400

On the other hand with a set mission time of 600 hours, the maximum-likelihood method gives a reliability value of .979 and the lower limit, with confidence level 95% is:

$.979 - \sigma_R \, 1.65 = .932$

where $\sigma_R^2 = 8.02 . 10^{-4}$ is the variance of R(600) calculated from (31), Section 2.

For T = 600 h the non-parametric methods give .954 (median ranks) and 0.937 (mean ranks). The lower limit, with confidence level 95%, calculated by extrapolating the 0.95 ranks, is .82.

5.3 Electromagnetic Valves

A set of twelve electromagnetic valves was reduced to six components in working order after a service of 3250 hours. The failure times of the eliminated components were:

t_i (hours)

1200, 1450, 2100, 2600, 3000, 3250

This sample cannot be treated by the maximum-likelihood method because the information contained in the fact that six valves are still working would be lost. With the non-parametric methods, however, this information can be taken into account and the reliability estimate is naturally different from what it would be if a sample of six were considered.

Again assuming a Weibull distribution, the use of probability papers gives a reliability estimate in accordance with the following expressions:

a. plotting with mean ranks

$$R(t) = \exp\left(- \left(\frac{t}{4300}\right)^{1.854}\right)$$

b. plotting with median ranks

$$R(t) = \exp\left(- \left(\frac{t}{4050}\right)^{2.23}\right)$$

represented in Figs. 4 and 5.

For set reliability values, the two estimates give the following values:

	mean ranks	median ranks
$t_{.95}$	900	1050
$t_{.99}$	357	510

5.4 Comments on the Results Obtained with the Various Methods

It emerges very clearly from the results that for the lower extreme values of the distribution

- the maximum-likelihood method gives more optimistic results than the median ranks method, which in turn gives more optimistic results than the mean ranks method;
- the difference between the results obtained with the various methods increases directly with the reliability sought and inversely with the Weibull distribution parameter α;
- these differences in results are not tied to the sample size but rather depend on the statistical behaviour of the extreme values, which is evaluated differently with each method.

When, as in the case of mechanical components, the chief concern is to evaluate the extreme values, the use of the parametric method and probability papers offers advantages of greater simplicity than the classical maximum-likelihood method and also permits a far more realistic evaluation.

It amounts to obtaining a reliability estimate with a confidence level $A \geq 0.5$.

For, referring ot (5), Section 3, and considering the first value of an ordered sample, we have:

$$\eta(p) = 1 - (1 - p)^n$$

or, calling the confidence level A and the reliability R:

$$A = 1 - R^n$$

This relation is graphed in Fig. 6 in respect of various values of n. The median rank relating to the lower end of the sample is obtained, for each n, from the intersection of the corresponding curve with the horizontal A = 0.5. If, however, the points corresponding to the reliability estimated with the mean rank are plotted on the curves, it will be seen that these estimates are equivalent to those obtained from (5), Section 3, for $A > 0.5$. If the maximum-likelihood estimates were plotted instead, one would find values for A smaller than 0.5. Another interesting consideration is that, in Fig. 6, the curves grow denser as n increases. This means that, given a certain value for the confidence level A, the reliability gain in the extreme sample value is progressively slighter as n increases; hence one could evaluate a maximum sample size such that trials on bigger samples would not introduce significant improvements in the reliability values.

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APPENDIX 1 - Description of KTEST code

With reference to Section 2.3.2, the KTEST code, written in FORTRAN H for IBM 360/65, performs the Kolmogoroff test on the Weibull, normal, log-normal and exponential distributions. The estimated distributions are determined from the sample data by the method of probability papers, i.e., the fitting to the data is done with the linearized form of the distribution (Section 4), after each sample value has been assigned its appropriate probability according to the non-parametric method selected.

The code performs the following operations:

a) It defines the regression variables for each of the distributions studied and calculates their values to correspond with the sample data.

b) It performs the fitting by the least-squares method (minimizing the deviations of the measured variable) and then determines an estimate of the parameters of each distribution.

c) It calculates the cumulative probabilities appropriate to the sample values, using the estimated distribution.

d) It performs the Kolmogoroff test, comparing the calculated probabilities and those assigned to the sample values by the non-parametric method adopted.

The regression variables x_i, y_i are defined as follows, t_i being the i-th value of the ordered sample and P_i the probability value attributed to t_i:

1. Weibull distribution

$$x_i = \log t_i \qquad\qquad y_i = \log \ln \frac{1}{1 - P_i}$$

with reference to (4), Section 4.2.

2. Normal distribution

$x_i = t_i$ y_i obtained by solving the equation

$$\frac{1}{\sqrt{\pi}} \int_0^{y_i} e^{-\eta^2} d\eta + 0.5 - P_i = 0$$

3. Log-normal distribution

$x_i = \log t_i$ y_i as for the normal distribution

4. Exponential distribution

$x_i = t_i$ $y_i = \log \frac{1}{1 - P_i}$

The regression in every case is of the form $y = \alpha(x - \beta)$. The coefficients α, β , tied to the distribution parameters, are determined by the least-squares method.

In applying the test the boundary level of significance is determined for each distribution, i.e., the level that constitutes the upper limit of the probability with which the distribution hypothesis can be accepted. Hence the data are interpreted best from the distribution that has the highest boundary level of significance. Referring to Section 2.3.2, this level is given by

$$\alpha = 1 - Q(\lambda)$$

where $\lambda = D_n \sqrt{n}$ and $Q(\lambda)$ is the function (12).

<u>Subprogrammes employed</u>:

1. FUNCTION ZERO(A1, B1, P, PREC)

Calculates by the bisection method the regression variable y in the case of normal or log-normal distribution. A1, B1 are the limits of the interval containing the desired radix

(A1 = -3, B1 = 3), P is the value of the non-parametric estimate, PREC is the precision with which y is obtained.

2. FUNCTION DMAX(A,K)

Calculates the largest of the values (all positive) of a matrix A of K dimensions. It is used to determine D_n, the maximum difference between the calculated probability and that attributed to the generic value of the sample.

3. FUNCTION Q(Y)

Calculates the value of the asymptotic distribution of $D_n \sqrt{n}$, for a given Y, with reference to (12) of Section 2.3.2.
This subprogramme is supplied by the IBM library under the name of SUBROUTING SMIRN. Reference should be made to the library for a description of the method employed.

Input data

N	Sample size (max 40)
ND	Number of distributions examined
VITA(J)	Matrix of sample values
PR(J)	Matrix of probability values attributed to the sample values

```
                C
                C
                C      QUESTO PROGRAMMA ESEGUE IL TEST DI KOLMOGOROV SU DIVERSE DISTRIB.
                C      STATISTICHE CHE VENGONO STIMATE, A PARTIRE DAI DATI DEL CAMPIONE,
                C      ESEGUENDO IL FITTING CON LE FORME LINEARIZZATE- VIENE COSI
                C      DETERMINATA LA DISTRIBUZIONE CHE MEGLIO SI ADATTA AI VALORI DEL
                C      CAMPIONE-
                C
ISN 0002               DIMENSION VITA(40),PR(40),X(40),Y(40),VRD(40),D(40),CLL(10),PRC(40
                      1)
ISN 0003               READ (5,1) N,ND
ISN 0004             1 FORMAT (2I3)
ISN 0005               READ (5,2) (VITA(J),J=1,N)
ISN 0006             2 FORMAT (8E10.5)
ISN 0007               READ (5,3) (PR(J),J=1,N)
ISN 0008             3 FORMAT (8E10.5)
ISN 0009               WRITE (6,4)
ISN 0010             4 FORMAT (//30X'FITTING A DISTRIBUZIONE DI WEIBULL')
ISN 0011               WRITE (6,5)
ISN 0012             5 FORMAT (///10X'J',7X'VITA(J)',8X'PR(J)',11X'X(J)',11X'Y(J)',9X'PRC
                      1(J)',11X'D(J)'//)
                C
                C      DEFINIZIONE VARIAB. REGR. PER DISTRIB. DI WEIBULL
                C
ISN 0013               I=0
ISN 0014            10 DO 100 J=1,N
ISN 0015               X(J)=ALOG10(VITA(J))
ISN 0016           100 Y(J)=ALOG10(ALOG(1./(1.-PR(J))))
ISN 0017               GO TO 2000
                C
                C      DEFINIZ. VARIAB. REGR. PER DISTRIB. LOG-NORMALE
                C      X(J) COME IN DO 100
                C
ISN 0018            20 DO 200 J=1,N
ISN 0019           200 Y(J)=ZERO(-3.,3.,PR(J),1.E-4)
ISN 0020               WRITE (6,11)
ISN 0021            11 FORMAT (//30X'FITTING A DISTRIBUZ. LOG. NORMALE')
ISN 0022               WRITE (6,5)
ISN 0023               GO TO 2000
                C
                C      DEFINIZ. VARIAB. REGR. PER DISTRIB. NORMALE
                C      Y(J) COME IN DO 200
                C
ISN 0024            30 DO 300 J=1,N
ISN 0025           300 X(J)=VITA(J)
ISN 0026               WRITE (6,12)
ISN 0027            12 FORMAT (//30X'FITTING A DISTRIBUZ. NORMALE')
ISN 0028               WRITE (6,5)
```

```
      GO TO 2000
C
C
C     DEFINIZ. VARIAB. REGR. PER DISTRIB. ESPONENZIALE
C     X(J) COME IN DO 300
C
C
   40 DO 400 J=1,N
  400 Y(J)=ALOG(1./(1.-PR(J)))
      WRITE (6,13)
   13 FORMAT (//30X'FITTING A DISTRIBUZ. ESPONENZIALE')
      WRITE (6,5)
      GO TO 2000
C
C
C     CALCOLO COEFF. REGRESSIONE LINEARE Y=ALFA*(X-BETA)
C     DEVIAZ. QUADR. MIN IN DIREZ. X
C
 2000 SXY=0.
      SX=0.
      SY=0.
      SY2=0.
      DO 2001 J=1,N
      SXY=SXY+X(J)*Y(J)
      SX=SX+X(J)
      SY=SY+Y(J)
 2001 SY2=SY2+Y(J)**2
      Z=N
 2002 ALFA=(Z*SY2-SY**2)/(Z*SXY-SX*SY)
 2003 BET=SX-SY/ALFA
      BETA=BET/Z
      IF(I)210,210,209
  209 CONTINUE
      GO TO (220,230,240),I
C
C
C     CALCOLO PROB. CON DISTRIB. DI WEIBULL STIMATA
C
  210 B=10.**BETA
      DO 215 J=1,N
      PRC(J)=1.-1./EXP((VITA(J)/B)**ALFA)
  215 D(J)=ABS(PR(J)-PRC(J))
      WRITE (6,6) (J,VITA(J),PR(J),X(J),Y(J),PRC(J),D(J),J=1
    6 FORMAT (8X,I3,6E15.5)
      DMX=DMAX(D,N)
      WRITE (6,14) DMX
   14 FORMAT (/8X,'DMX='F15.5)
      FN=FLOAT(N)
      TETA=DMX*SQRT(FN)
```

```
ISN 0063          CLL(1)=1.-Q(TETA)
ISN 0064          WRITE (6,7) ALFA,BETA
ISN 0065        7 FORMAT (///3X'I COEFF. DELLA REGRESSIONE SONO ALFA='E15.5,3X'BETA=
                 1'E15.5)
ISN 0066          WRITE (6,8) CLL(1)
ISN 0067        8 FORMAT (//8X'IL LIVELLO DI CONF. LIMITE SECONDO IL TEST DI KOLMOGO
                 1ROV E''='F15.5)
ISN 0068          I=I+1
ISN 0069          GO TO 20
            C
            C
            C     CALCOLO PROB. CON DISTRIB. LOG-NORMALE STIMATA
            C
ISN 0070      220 DO 225 J=1,N
ISN 0071          VRD(J)=ALFA*(ALOG10(VITA(J))-BETA)
ISN 0072      221 PRC(J)=0.5+0.5*ERF(VRD(J))
ISN 0073      225 D(J)=ABS(PR(J)-PRC(J))
ISN 0074          WRITE (6,6) (J,VITA(J),PR(J),X(J),Y(J),PRC(J),D(J),J=1,N)
ISN 0075          DMX=DMAX(D,N)
ISN 0076          WRITE (6,14) DMX
ISN 0077          FN=FLOAT(N)
ISN 0078          TETA=DMX*SQRT(FN)
ISN 0079          CLL(2)=1.-Q(TETA)
ISN 0080          WRITE (6,7) ALFA,BETA
ISN 0081          WRITE (6,8) CLL(2)
ISN 0082          I=I+1
ISN 0083          GO TO 30
            C
            C
            C     CALCOLO PROB. CON DISTRIB. NORMALE STIMATA
            C
ISN 0084      230 DO 235 J=1,N
ISN 0085          VRD(J)=ALFA*(VITA(J)-BETA)
ISN 0086      231 PRC(J)=0.5+0.5*ERF(VRD(J))
ISN 0087      235 D(J)=ABS(PR(J)-PRC(J))
ISN 0088          WRITE (6,6) (J,VITA(J),PR(J),X(J),Y(J),PRC(J),D(J),J=1,N)
ISN 0089          DMX=DMAX(D,N)
ISN 0090          WRITE (6,14) DMX
ISN 0091          FN=FLOAT(N)
ISN 0092          TETA=DMX*SQRT(FN)
ISN 0093          CLL(3)=1.-Q(TETA)
ISN 0094          WRITE (6,7) ALFA,BETA
ISN 0095          WRITE (6,8) CLL(3)
ISN 0096          I=I+1
ISN 0097          GO TO 40
            C
            C
            C     CALCOLO PROB. CON DISTRIB. EXP. STIMATA
```

```
      C
ISN 0098     240 DO 245 J=1,N
ISN 0099         VRO(J)=ALFA*(VITA(J)-BETA)
ISN 0100         PRC(J)=1.-1./EXP(VRO(J))
ISN 0101     245 D(J)=ABS(PR(J)-PRC(J))
ISN 0102         WRITE (6,6) (J,VITA(J),PR(J),X(J),Y(J),PRC(J),D(J),J=1,N)
ISN 0103         DMX=DMAX(D,N)
ISN 0104         WRITE (6,14) DMX
ISN 0105         FN=FLOAT(N)
ISN 0106         TETA=DMX*SQRT(FN)
ISN 0107         CLL(4)=1.-Q(TETA)
ISN 0108         WRITE (6,7) ALFA,BETA
ISN 0109         WRITE (6,3) CLL(4)
ISN 0110         CLMX=DMAX(CLL,NO)
ISN 0111         WRITE (6,9) CLMX
ISN 0112       9 FORMAT (///8X'LA DISTRIB. CHE MEGLIO INTERPRETA IL CAMPIONE E''QUE
                1LLA CON CLL='E15.5)
ISN 0113         STOP
ISN 0114         END
```

```
                C
                C
ISN 0002              FUNCTION DMAX(A,K)
                C
ISN 0003              DIMENSION A(1)
ISN 0004              X=A(1)
ISN 0005              DO 3 I=2,K
ISN 0006              IF(A(I).LE.X) GO TO 3
ISN 0008              X=A(I)
ISN 0009            3 CONTINUE
ISN 0010              DMAX=X
ISN 0011              RETURN
ISN 0012              END
```

```
                C
                C
ISN 0002              FUNCTION ZERO(A1,B1,P,PREC)
                C
ISN 0003              A=A1
ISN 0004              B=B1
ISN 0005            1 IF(ABS(A-B)-PREC)2,2,3
ISN 0006            3 C=0.5*(A+B)
ISN 0007              PP=0.5-P
ISN 0008            4 W=0.5*ERF(A)+PP
ISN 0009            5 U=0.5*ERF(C)+PP
ISN 0010              IF(U*W)6,7,8
ISN 0011            6 B=C
ISN 0012              GO TO 1
ISN 0013            8 A=C
ISN 0014              GO TO 1
ISN 0015            7 ZERO=C
ISN 0016              GO TO 9
ISN 0017            2 ZERO=A
ISN 0018            9 RETURN
ISN 0019              END
```

```
                C
                C
ISN 0002              FUNCTION Q(Y)
                C
ISN 0003              IF(Y-.27)1,1,2
ISN 0004            1 Q=0.0
ISN 0005              GO TO 9
ISN 0006            2 IF(Y-1.0)3,6,6
ISN 0007            3 Q1=EXP(-1.233701/Y**2)
ISN 0008              Q2=Q1*Q1
ISN 0009              Q4=Q2*Q2
ISN 0010              Q8=Q4*Q4
ISN 0011              IF(Q8-1.0E-25)4,5,5
ISN 0012            4 Q8=0.0
ISN 0013            5 Q=(2.506628/Y)*Q1*(1.0+Q8*(1.0+Q8*Q8))
ISN 0014              GO TO 9
ISN 0015            6 IF(Y-3.1)8,7,7
ISN 0016            7 Y=1.0
ISN 0017              GO TO 9
ISN 0018            8 Q1=EXP(-2.0*Y*Y)
ISN 0019              Q2=Q1*Q1
ISN 0020              Q4=Q2*Q2
ISN 0021              Q8=Q4*Q4
ISN 0022              Q=1.0-2.0*(Q1-Q4+Q8*(Q1-Q8))
ISN 0023            9 RETURN
ISN 0024              END
```

APPENDIX 2 - Description of VITA code

With reference to Section 2.5.2 of the text, the VITA code programmed in FORTRAN H for IBM 360/65 performs the following operations:

a) It estimates the shape and scale parameters, α, λ of the incomplete Weibull distribution.

b) It calculates the estimate variances $\hat{\alpha}$, $\hat{\lambda}$, inverting the matrix (29) and making use of (5).

c) It calculates the reliability value at a given mission time.

d) It calculates the reliability variance, making use of relation (31).

e) It tabulates the estimated Weibull distribution and its probability density.

As regards a) above, we may note that the resolving equation is as follows:

$$\hat{\alpha} = \frac{n \Sigma_i t_i^{\hat{\alpha}}}{n \Sigma_i t_i^{\hat{\alpha}} \log t_i - \Sigma_i t_i^{\hat{\alpha}} \Sigma_i \log t_i} \qquad (A2.1)$$

obtained by substituting (26) in (27).
The value for $\hat{\alpha}$ that starts the iteration is obtained from (28), equivalent to

$$\alpha = \frac{\log \frac{\Sigma_i t_i^{\alpha}}{n}}{\log \frac{m}{\Gamma(1+1/\alpha)}} \qquad (A2.2)$$

m being the sample mean.

Both (A2.1) and (A2.2) are equations of the type $\zeta = f(\zeta)$ which can be solved by an iteration process represented by the formula (Ref. 1, p. 184)

$$\zeta_o^1 = \zeta_o + \frac{(\zeta_1 - \zeta_o)^2}{2\zeta_1 - \zeta_o - \zeta_2} \qquad \text{(A2.3)}$$

where ζ_o is a trial initial value, $\zeta_1 = f(\zeta_o)$, $\zeta_2 = f(\zeta_1)$ and ζ_o^1 the initial value for the second iteration. The process usually converges very fast, if $f'(\bar{\zeta}) \neq 1$ where $\bar{\zeta}$ is the desired radix, because the error for each successive iteration is an infinitesimal of higher order than the one in the preceding iteration.

<u>Subprogrammes employed</u>:

1. FUNCTION TETA(A) calculates $\sum_i t_i^{\alpha}$
2. FUNCTION STAR1(A) calculated the function (A2.2)
3. FUNCTION STAR2(A) calculates the function (A2.1)
4. FUNCTION ZERO(Y, STAR, PREC) carries out the iteration according to (A2.3) on the generic function STAR, where Y is the trial initial value. It stops the process when the result of the difference in the values of two successive interations is smaller than PREC, i.e., than the set precision value.

<u>Input data</u>:

N	Sample size
A1	Initial value for the iteration on (A2.2)
PREC	Precision of iteration process
NOIT	Limit number of iterations permitted
T	Mission time
TB	Control indicator to effect (TB $\neq$ 0) or not effect (TB = 0) tabulation of the distribution values
VITA(J)	Is the whole of the sample values, for J = 1,...,N

```
              C      PROGRAMMA PRINCIPALE
              C
ISN 0002             DIMENSION TM(100),P(100),PP(100),S(100),Z(100)
ISN 0003             COMMON/PIPPO/VITA(100)//N,J,B,NOIT
              C
ISN 0004             READ(5,1)N,A1,PREC,NOIT,T,TB
ISN 0005           1 FORMAT (I2,2E9.2,I4,E9.2,I2)
ISN 0006             READ (5,2) (VITA(J),J=1,N)
ISN 0007           2 FORMAT (6E10.3)
ISN 0008             WRITE (6,100)
ISN 0009         100 FORMAT (30X'DATI DI PARTENZA'///)
ISN 0010             WRITE (6,101)
ISN 0011         101 FORMAT (10X'TEMPI DI VITA'//)
ISN 0012             WRITE (6,102) (J,VITA(J),J=1,N)
ISN 0013         102 FORMAT (10X,I5,E13.4)
ISN 0014             WRITE (6,103) A1,PREC
ISN 0015         103 FORMAT (//10X'A1='E9.2,3X,'PREC='E9.2///)
              C
              C      VALORE DI PRIMO TENTATIVO
              C
ISN 0016             N=N
ISN 0017             J=J
ISN 0018             EXTERNAL STAR1
ISN 0019             CO=ZERO(A1,STAR1,PREC)
ISN 0020             IF(CO) 10,10,15
ISN 0021          15 CONTINUE
ISN 0022             WRITE (6,104)
ISN 0023         104 FORMAT (35X'RISULTATI'//)
ISN 0024             WRITE (6,105) CO,B
ISN 0025         105 FORMAT (10X'CO='F12.4,3X,'NO ITERAZIONI PER CO='F4.0/)
ISN 0026             WRITE (6,106)
ISN 0027         106 FORMAT (15X'PARAMETRI WEIBULL'/)
              C
              C      WEIBULL SHAPE PARAMETER ALFA
              C
ISN 0028             EXTERNAL STAR2
ISN 0029             ALFA=ZERO(CO,STAR2,PREC)
              C
              C      WEIBULL SCALE PARAMETER AMDA
              C
ISN 0030             AN=N
ISN 0031             TI=TETA(ALFA)
ISN 0032             AMDA=AN/TI
ISN 0033             WRITE (6,107) ALFA,B,AMDA
ISN 0034         107 FORMAT (5X'ALFA='E12.3,2X'NO ITERAZ. PER ALFA='F3.0,2X'AMDA='E12.3
                    1)
              C
              C      CALCOLO MATRICE COVARIANZA
              C
```

```
ISN 0035          SL1=0
ISN 0036          SL2=0
ISN 0037          DO 30 J=1,N
ISN 0038          S(J)=VITA(J)**ALFA*(ALOG(VITA(J)))**2
ISN 0039          SL1=SL1+S(J)
ISN 0040          Z(J)=VITA(J)**ALFA*ALOG(VITA(J))
ISN 0041       30 SL2=SL2+Z(J)
ISN 0042       31 F1=-AN/ALFA**2-AMDA*SL1
ISN 0043       32 F2=-AN/AMDA**2
ISN 0044       33 F12=-SL2
ISN 0045       34 DET=F1*F2-F12**2
ISN 0046          IF(DET)35,39,35
ISN 0047       35 VAR1=-F2/DET
ISN 0048       36 VAR2=-F1/DET
ISN 0049       37 CVR12=F12/DET
ISN 0050          WRITE (6,38) VAR1,VAR2,CVR12
ISN 0051       38 FORMAT (//5X'VAR.ALFA='E12.3,2X'VAR.AMDA='E12.3,2X'CVR.ALFA-AMDA='
                 1E12.3//)
ISN 0052          GO TO 41
ISN 0053       39 WRITE (6,40)
ISN 0054       40 FORMAT (//5X'DETERMINANTE=0')
             C
             C    CALCOLO RELIABILITY E SUA VARIANZA
             C
ISN 0055       41 R=1./EXP(AMDA*T**ALFA)
ISN 0056       42 VR1=T**(2.*ALFA)/EXP(2.*AMDA*T**ALFA)
ISN 0057       43 VR2=AMDA**2*ALOG(T)**2*VAR1+VAR2+2.*AMDA*ALOG(T)*CVR12
ISN 0058       44 VARR=VR1*VR2
ISN 0059          WRITE (6,45) T,R,VARR
ISN 0060       45 FORMAT (//5X'PER T='F9.3,2X'LA RELIABILITY E='F9.5,2X'LA SUA VARI
                 1ANZA E='E12.4//)
             C
             C    TABULAZIONE WEIBULL
             C
ISN 0061          IF(TB)46,11,46
ISN 0062       46 TMX=2.*VITA(N)
ISN 0063          TMIN=0.1*VITA(1)
ISN 0064          TM(1)=TMIN
ISN 0065          DO 50 K=1,99
ISN 0066          TM(K+1)=TM(K)+(TMAX-TMIN)/99.
ISN 0067          P(K)=1.-1./EXP(AMDA*TM(K)**ALFA)
ISN 0068       50 PP(K)=AMDA*ALFA*(TM(K)**(ALFA-1.))*(1-P(K))
ISN 0069          WRITE (6,108)
ISN 0070      108 FORMAT (//30X'TABULAZIONE WEIBULL'//)
ISN 0071          WRITE (6,109) (TM(K),P(K),PP(K),K=1,100)
ISN 0072      109 FORMAT (5X,E16.3,E23.4,E23.4)
ISN 0073          GO TO 11
ISN 0074       10 WRITE (6,9)
ISN 0075        9 FORMAT (3X'PRIMA ITERAZ. NON CONV.')
ISN 0076       11 STOP
```

```
      FUNCTION TETA(A)
      DIMENSION T(100)
      COMMON/PIPPO/VITA(100)//N,J
      TETA=0
      DO 10 J=1,N
      T(J)=VITA(J)**A
   10 TETA=TETA+T(J)
      RETURN
      END
```

```
      FUNCTION STAR1(A)
      COMMON/PIPPO/VITA(100)//N,J
      TI=TETA(A)
      AN=N
      TETA1=ALOG(TI/AN)
      SM=0
      DO 10 J=1,N
   10 SM=SM+VITA(J)
      EDIA=SM/AN
      X=1.+1./A
      TETA2=ALOG(EDIA/GAMMA(X))
      STAR1=TETA1/TETA2
      RETURN
      END
```

```
      FUNCTION STAR2(A)
      COMMON/PIPPO/VITA(100)//N,J
      DIMENSION S(100),Z(100)
      TI=TETA(A)
      AN=N
      TETA3=AN*TI
      SL=0
      DO 10 J=1,N
      S(J)=VITA(J)**A*ALOG(VITA(J))
   10 SL=SL+S(J)
      TETA4=AN*SL
      SZ=0
      DO 20 J=1,N
      Z(J)=ALOG(VITA(J))
   20 SZ=SZ+Z(J)
      TETA5=SZ*TI
      TETA6=TETA4-TETA5
      STAR2=TETA3/TETA6
      RETURN
      END
```

```
      FUNCTION ZERO(Y,STAR,PREC)
      COMMON N,J,B,NOIT
      B=0
    5 C1=Y
    6 X1=STAR(C1)
      IF(ABS(X1-C1)-PREC)10,10,20
   20 X2=STAR(X1)
   21 C2=C1+(X1-C1)**2/(2.*X1-C1-X2)
      C1=C2
      B=B+1
      IF(B-NOIT)6,12,12
   12 ZERO=0
   10 ZERO=X1
      RETURN
      END
```

APPENDIX 3 - Description of RANKS code[1]

With reference to Section 3.2.3, the RANKS code, written in FORTRAN H for IBM 360/65, solves the following equation in p:

$$1-\eta(p) - \sum_{o}^{m-1}{}_{i} \binom{n}{i} p^{i}(1-p)^{n-i} = 0 \qquad (A3.1)$$

for given values of $\eta(p)$, n, m, and all values of i between 0 and m-1. The bisection method is used for the solution.

Subprogrammes employed:

1. FUNCTION PIPPO(Z) calculates the function (A3.1)
2. FUNCTION ZERO(A1, B1, Y, PREC) applies the bisection method to the Y function to find the Y radix in the interval A1, B1 with precision PREC.

Input data:

AI(K) values of confidence level $\eta(p)$ for K = 1,2,3

Remarks:

- the sample dimensions must be ≤ 40
- the precision of the bisection method is 5.10^{-5}
- A1 = 0, B1 = 1

---

(1) The original version of the code, in F2V3 on IBM 7090, is given in Ref. 12.

```
C       QUESTO PROGRAMMA CALCOLA UNA STIMA
C       NON PARAMETRICA DELLA PROBABILITA' CUMULATIVA
C       COMPETENTE AL VALORE J-ESIMO DI UN
C       CAMPIONE ORDINATO DI DIMENSIONI N -
C       ESSA E' OTTENUTA RISOLVENDO RISPETTO
C       A Z L'EQUAZIONE
C
C
C       (I-A)=SOMMA((Z**I)*(1-Z)**(N-I))*N /I!(N-I)!
C
C
C       PER I VARIABILE TRA O E(J-1)
C       A RAPPRESENTA IL LIVELLO DI CONFIDENZA
C       DELLA STIMA-LA RISOLUZIONE E' EFFETTUATA
C       CON IL METODO DI BISEZIONE
C       PROGRAMMA PRINCIPALE
C
        DIMENSION RANK(40,40),AI(3)
        COMMON N,J,A
        READ (5,1) AI
      1 FORMAT (3E10.4)
        DO 10 K=1,3
        A=AI(K)
        DO 10 J=1,40
        DO 10 J=1,N
        N=N
        J=J
        EXTERNAL PIPPO
        RANK(N,J)=ZERO(0.,1.,PIPPO,5.E-5)
        WRITE (6,2) J,N,A,RANK(N,J)
      2 FORMAT (1H,3X,'J='I2,5X,'N='I2,5X,'A='E10.4,5X,'RANK='E11.5)
     10 CONTINUE
        STOP
        END


        FUNCTION PIPPO(Z)
C
C
        DIMENSION T(50)
        COMMON N,J,A
C
        IF(J-1)98,99,98
     98 T(1)=1.-A
        T(2)=-(1.-Z)**N
        B=Z/(1.-Z)
        DO 100 K=2,J
    100 T(K+1)=T(K)*B*FLOAT(N-K+2)/FLOAT(K-1)
        S=0.
        JO=J+1
        DO 110 L=1,JO
    110 S=S+T(L)
        PIPPO=S
        GO TO 120
     99 PIPPO=1.-A-(1.-Z)**N
    120 CONTINUE
        RETURN
        END


        FUNCTION ZERO(A1,B1,Y,PREC)
C
C
        A=A1
        B=B1
     21 IF(ABS(A-B)-PREC)20,20,22
     22 C=0.5*(A+B)
        IF(Y(A)*Y(C))23,24,25
     23 B=C
        GO TO 21
     25 A=C
        GO TO 21
     24 ZERO=C
        GO TO 26
     20 ZERO=A
     26 RETURN
        END
```

TABLE 1 - 0.05 RANKS

Sample size = N

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	.0500	.0253	.0169	.0127	.0102	.0035	.0073	.0064	.0057	.0051	.0045	.0042	.0039	.0036	.0034	.0032	.0030	.0025	.0027	.0025
2		.2236	.1353	.0976	.0764	.0628	.0534	.0464	.0410	.0367	.0333	.0305	.0280	.0260	.0242	.0227	.0213	.0201	.0190	.0180
3			.3684	.2486	.1892	.1531	.1288	.1111	.0977	.0872	.0788	.0719	.0660	.0611	.0563	.0531	.0499	.0470	.0445	.0421
4				.4729	.3426	.2713	.2253	.1929	.1687	.1500	.1351	.1228	.1126	.1040	.0967	.0902	.0846	.0797	.0753	.0713
5					.5493	.4182	.3412	.2892	.2513	.2224	.1995	.1810	.1656	.1527	.1417	.1321	.1237	.1164	.1099	.1041
6						.6069	.4793	.4003	.3449	.3035	.2712	.2453	.2239	.2061	.1909	.1773	.1664	.1563	.1475	.1395
7							.6518	.5293	.4503	.3934	.3458	.3152	.2870	.2635	.2437	.2267	.2119	.1989	.1875	.1773
8								.6877	.5709	.4931	.4356	.3908	.3548	.3250	.3000	.2786	.2601	.2440	.2297	.2170
9									.7169	.6058	.5299	.4727	.4274	.3904	.3596	.3334	.3108	.2912	.2739	.2586
10										.7411	.6356	.5619	.5053	.4600	.4225	.3910	.3640	.3406	.3201	.3019
11											.7616	.6513	.5899	.5343	.4392	.4516	.4197	.3922	.3681	.3469
12												.7791	.6837	.6146	.5602	.5156	.4781	.4460	.4181	.3936
13													.7942	.7032	.6365	.5834	.5394	.5022	.4700	.4420
14														.8073	.7206	.6562	.6043	.5611	.5242	.4522
15															.8159	.7360	.6738	.6233	.5809	.5444
16																.8292	.7498	.6897	.6406	.5990
17																	.8384	.7623	.7042	.6563
18																		.8467	.7736	.7174
19																			.8541	.7839
20																				.8609

TABLE 2 - 0.5 RANKS

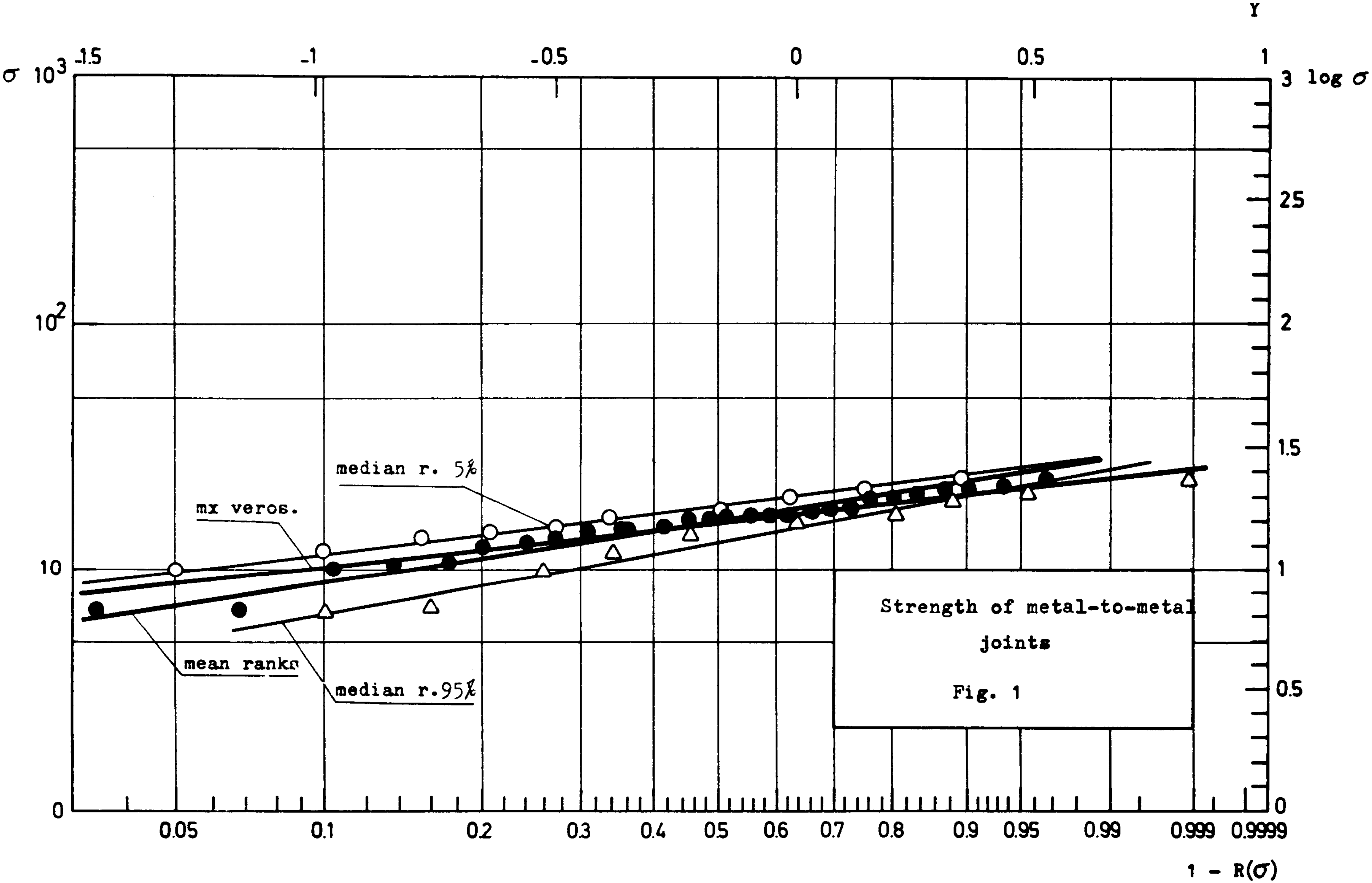
Sample size = N

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	.5000	.2929	.2063	.1591	.1294	.1091	.0943	.0830	.0741	.0670	.0611	.0561	.0519	.0483	.0451	.0424	.0399	.0377	.0358	.0341
2		.7071	.5000	.3857	.3138	.2644	.2285	.2011	.1796	.1623	.1479	.1360	.1258	.1170	.1094	.1027	.0968	.0915	.0868	.0825
3			.7937	.6143	.5000	.4214	.3641	.3205	.2862	.2585	.2358	.2167	.2004	.1865	.1043	.1636	.1542	.1458	.1382	.1315
4				.8409	.6862	.5786	.5000	.4402	.3931	.3551	.3238	.2975	.2753	.2561	.2394	.2247	.2118	.2002	.1899	.1805
5					.8705	.7355	.6359	.5598	.5000	.4517	.4119	.3785	.3502	.3257	.3045	.2859	.2694	.2547	.2415	.2296
6						.8909	.7715	.6795	.6069	.5483	.5000	.4595	.4250	.3954	.3697	.3470	.3270	.3092	.2932	.2788
7							.9057	.7989	.7137	.6449	.5881	.5405	.5000	.4651	.4348	.4082	.3847	.3637	.3449	.3279
8								.9170	.8204	.7414	.6762	.6215	.5749	.5349	.5000	.4694	.4423	.4182	.3966	.3771
9									.9259	.8377	.7642	.7024	.6498	.6046	.5652	.5306	.5000	.4727	.4483	.4262
10										.9330	.8520	.7833	.7247	.6742	.6303	.5918	.5576	.5273	.5000	.4754
11											.9389	.8640	.7995	.7439	.6955	.6529	.6153	.5818	.5517	.5246
12												.9438	.8742	.8135	.7606	.7141	.6729	.6363	.6034	.5737
13													.9481	.8830	.8257	.7752	.7306	.6908	.6551	.6229
14														.9517	.8906	.8363	.7882	.7453	.7068	.6720
15															.9548	.8973	.8458	.7997	.7585	.7212
16																.9576	.9032	.8542	.8101	.7703
17																	.9600	.9085	.8617	.8194
18																		.9622	.9132	.8685
19																			.9642	.9175
20																				.9659

TABLE 3 - 0.95 RANKS

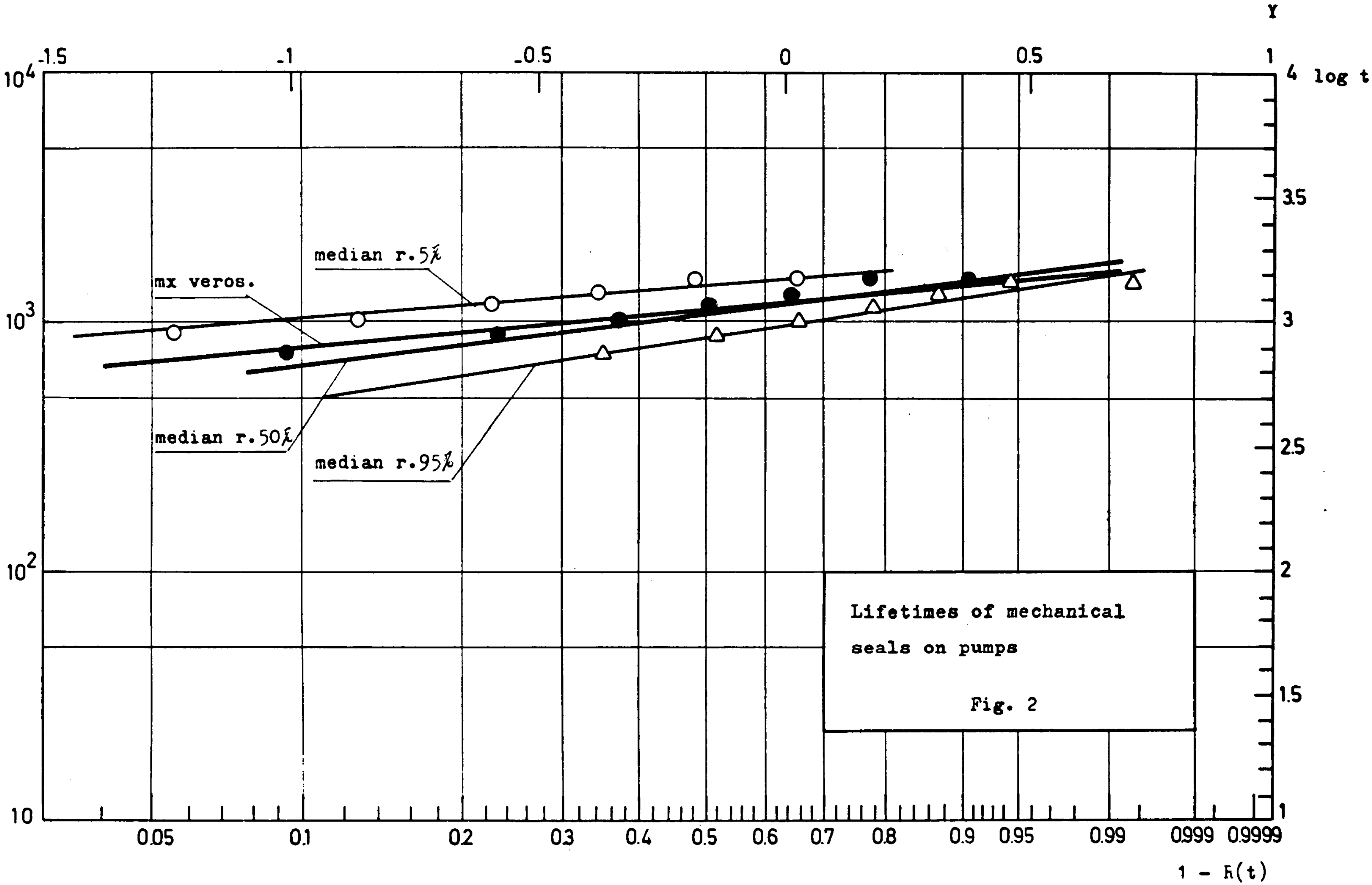
Sample size = N

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	.9500	.7764	.6316	.5271	.4507	.3930	.3481	.3123	.2831	.2589	.2384	.2209	.2058	.1926	.1810	.1707	.1616	.1533	.1458	.1391
2		.9747	.8646	.7514	.6574	.5818	.5207	.4707	.4291	.3941	.3643	.3387	.3163	.2967	.2794	.2639	.2501	.2375	.2263	.2161
3			.9830	.9024	.8107	.7286	.6587	.5997	.5496	.5069	.4701	.4381	.4101	.3854	.3634	.3438	.3262	.3102	.2958	.2826
4				.9872	.9236	.8468	.7747	.7108	.6550	.6066	.5644	.5273	.4946	.4656	.4398	.4166	.3956	.3767	.3594	.3437
5					.9898	.9371	.8712	.8071	.7486	.6964	.6502	.6091	.5725	.5400	.5107	.4844	.4605	.4389	.4191	.4010
6						.9915	.9466	.8889	.8312	.7776	.7287	.6848	.6452	.6096	.5774	.5483	.5219	.4978	.4758	.4556
7							.9927	.9536	.9023	.8499	.8004	.7547	.7129	.6750	.6404	.6090	.5803	.5540	.5299	.5078
8								.9936	.9590	.9127	.8649	.8190	.7760	.7364	.7000	.6666	.6360	.6075	.5819	.5580
9									.9943	.9632	.9212	.8771	.8363	.7939	.7563	.7214	.6891	.6594	.6319	.6064
10										.9949	.9667	.9281	.8873	.8473	.8091	.7733	.7399	.7088	.6799	.6530
11											.9953	.9695	.9339	.8959	.8583	.8222	.7881	.7560	.7260	.6990
12												.9957	.9719	.9389	.9033	.8679	.8336	.8010	.7703	.7413
13													.9960	.9740	.9431	.9097	.8762	.8436	.8125	.7829
14														.9963	.9758	.9468	.9153	.8835	.8525	.8227
15															.9966	.9773	.9501	.9203	.8901	.8604
16																.9962	.9787	.9530	.9247	.8959
17																	.9948	.9799	.9555	.9286
18																		.9930	.9810	.9578
19																			.9909	.9819
20																				.9886



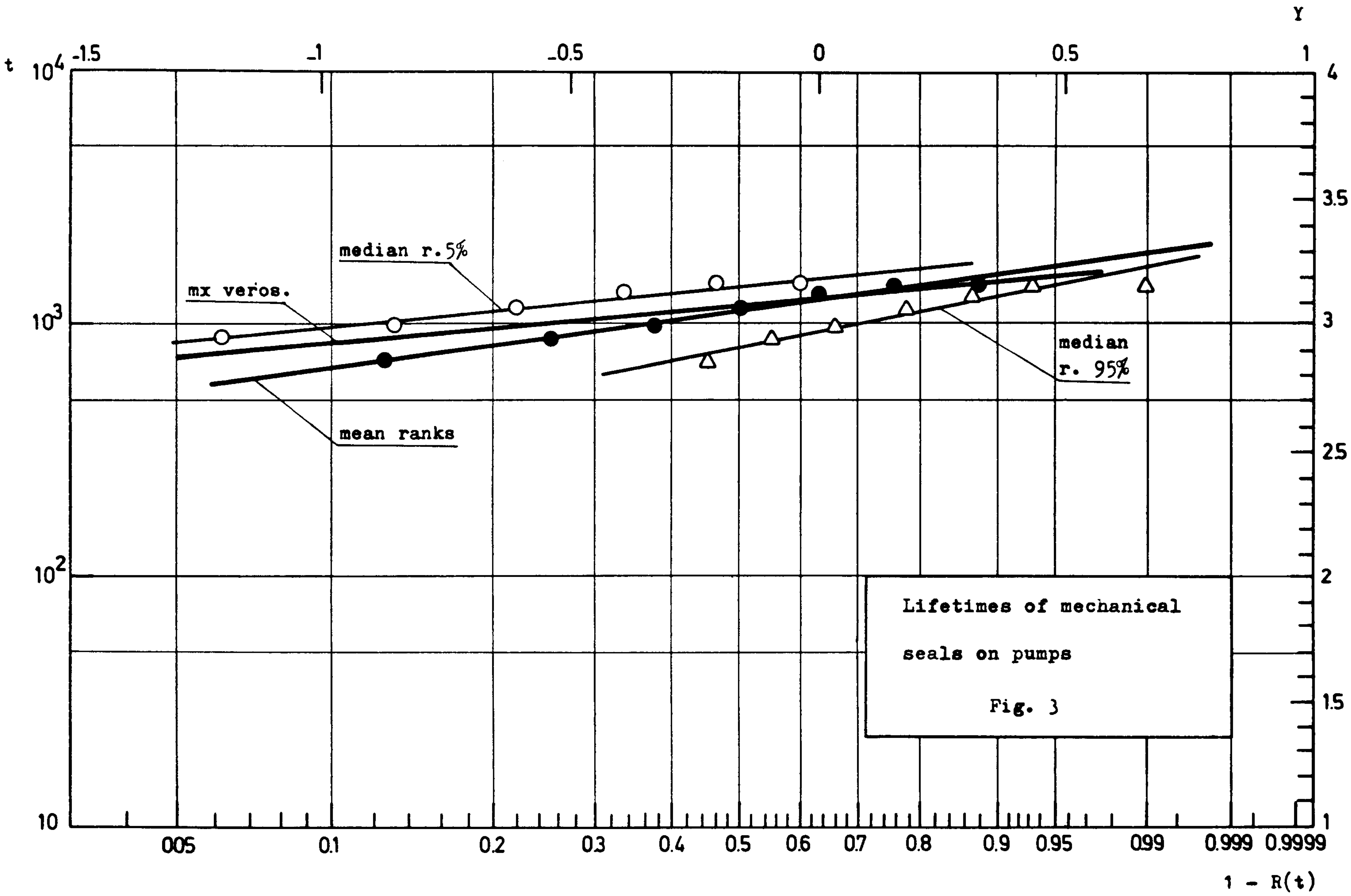
Strength of metal-to-metal joints

Fig. 1



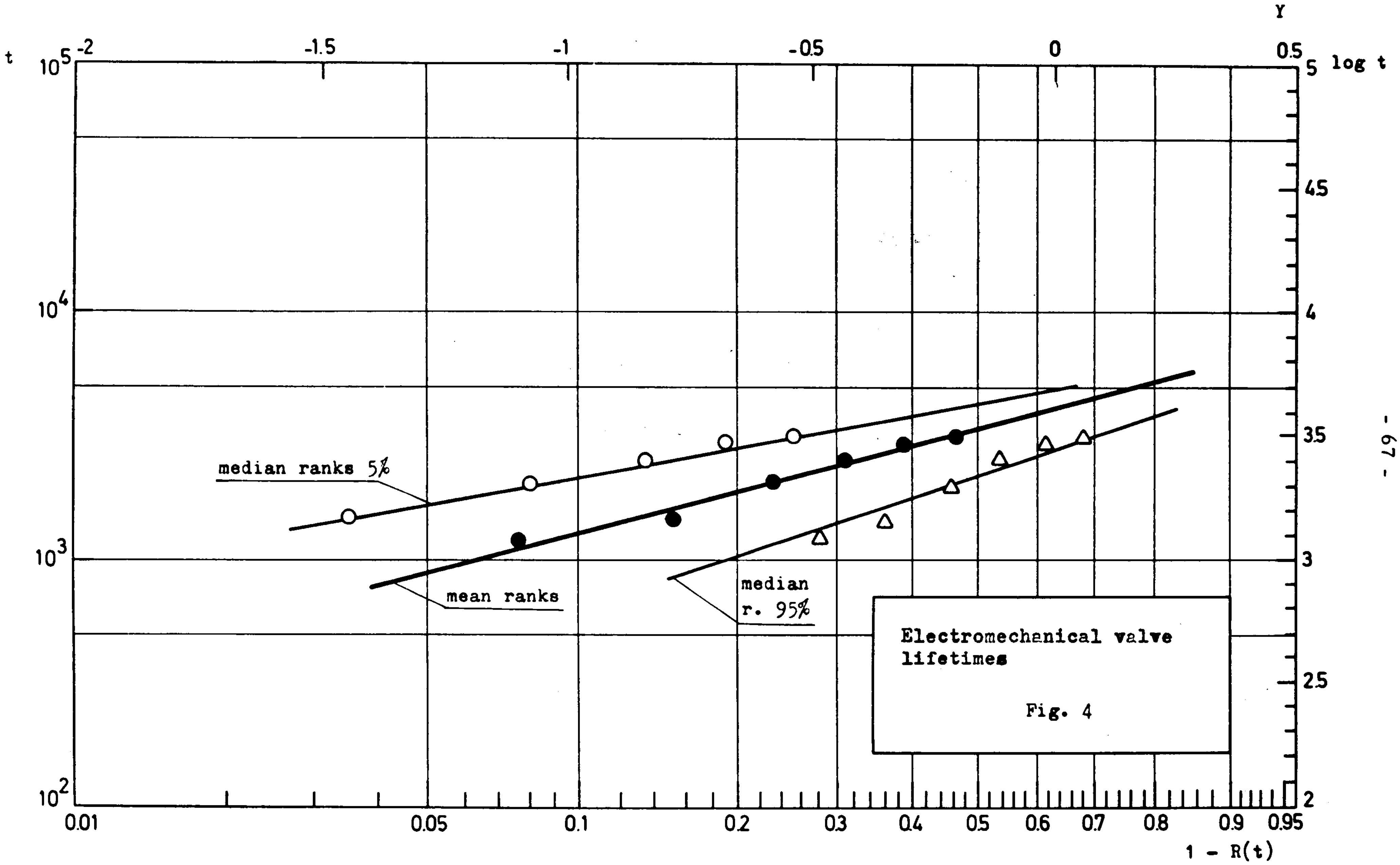
Lifetimes of mechanical seals on pumps

Fig. 2



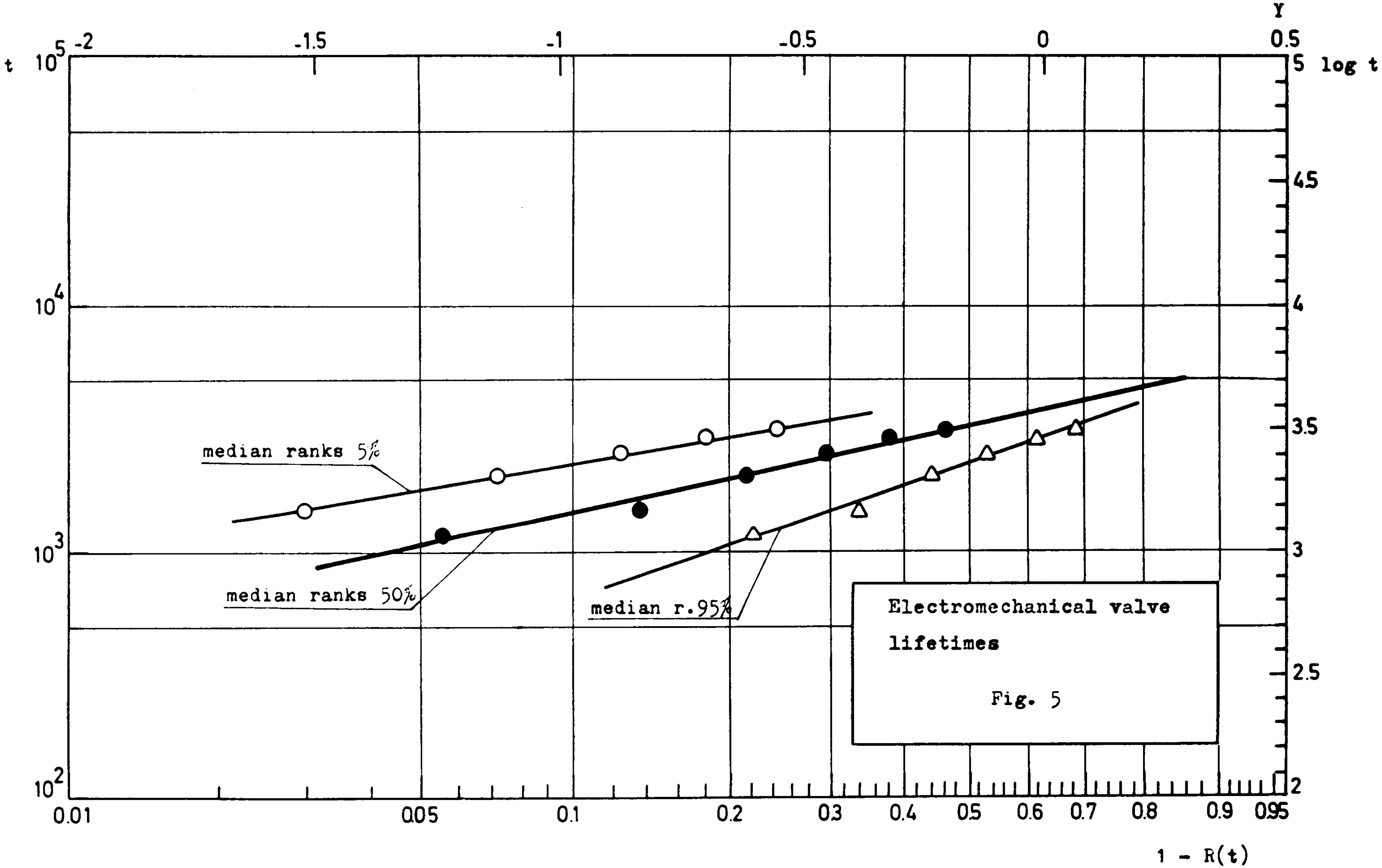
Lifetimes of mechanical seals on pumps

Fig. 3



Electromechanical valve lifetimes

Fig. 4



Electromechanical valve lifetimes

Fig. 5

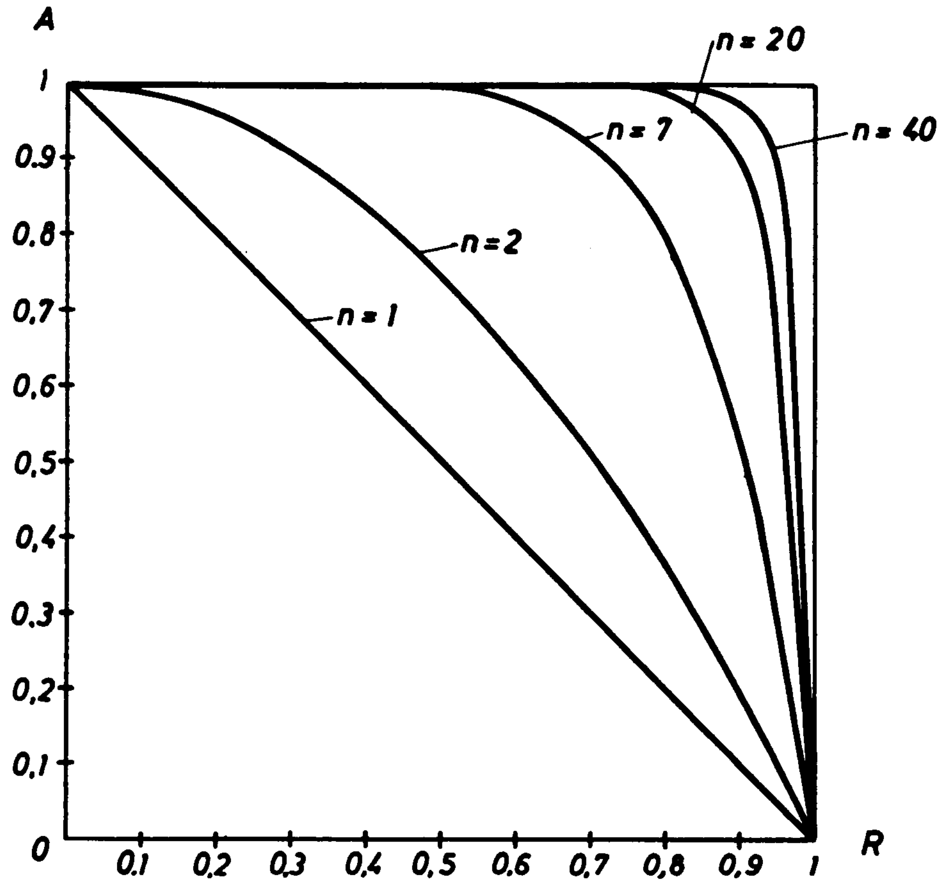


FIG. 6 Variations of confidence level A vs reliability R for various sample sizes

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Alfred Nobel

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