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EUROPEAN ATOMIC ENERGY COMMUNITY - EURATOM

A COMPARISON OF METHODS FOR COMPUTING THE EIGENVALUES AND EIGENVECTORS OF A MATRIX

by

I. GALLIGANI

1968

Joint Nuclear Research Center
Ispra Establishment - Italy

Scientific Data Processing Center - CETIS

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In order to pick out the «best» methods we collect a list of matrices with different «condition numbers» which form a representative sample of those which occur in practice. Some relationships between these «condition numbers» are discussed.

Finally the results of some computational experiments carried out on the above test-matrices are presented.

In order to pick out the «best» methods we collect a list of matrices with different «condition numbers» which form a representative sample of those which occur in practice. Some relationships between these «condition numbers» are discussed.

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SUMMARY

In this report a comparison of some methods for solving the eigenproblem of a matrix is given. An attempt has been made to establish the «efficiency», on the basis of computing time and accuracy, of each method by carrying out «experimental» calculations on «representative» problems for which exact results are known.

In order to pick out the «best» methods we collect a list of matrices with different «condition numbers» which form a representative sample of those which occur in practice. Some relationships between these «condition numbers» are discussed.

Finally the results of some computational experiments carried out on the above test-matrices are presented.

KEYWORDS

EIGENVALUES
EIGENVECTORS
MATRICES
NUMERICALS
EFFICIENCY

INTRODUCTION (*)

In this paper a comparison of some methods for solving the eigenproblem of a matrix is given. An attempt has been made to establish the "efficiency", on the basis of computing time and accuracy, of each method by carrying out "experimental" calculations on "representative" problems for which exact results are known.

In order to pick out the " best " methods we collect in Chapter I a list of matrices which form a representative sample of those which occur in practice.

Any computing problem is "ill conditioned" if values to be computed are very sensitive to small changes in the data. It is convenient to have some numbers which define the condition of a matrix with respect to the eigenproblem. These condition-numbers and some relationships between them are discussed in Chapter II.

The results of some computational experiments carried out on the above test-matrices are presented in Chapter III. The methods compared for the eigenproblem of symmetric matrices are the Jacobi, threshold Jacobi, Givens-Householder and Rutishauser schemes. The numerical experiments reported in Chapter III §1 give a more realistic picture of the accuracy of the above methods than that obtained by "a-priori error analysis", and make apparent the "efficiency" of the Givens-Householder method for determining the eigenvalues of general symmetric matrices. The Rutishauser method is efficient for determining the eigenvalues of symmetric <u>band</u> matrices.
As far as vectors are concerned the threshold Jacobi method and the Jacobi method give almost exactly orthogonal vectors. The Givens-Householder method (with inverse iteration) gives accurate eigenvectors, but eigenvectors corresponding to multiple or close eigenvalues may be far from orthogonal.

For non-Hermitian matrices the QR method and the Laguerre method are compared. The numerical experiments reported in Chapter III §2 and §3 lead to the following conclusion:

a) for finding all eigenvalues, the Laguerre method is troublesome because of the difficulty in finding "convenient" convergence-parameters;

(*) Manuscript received on May 30, 1968.

b) the convergence rate of the QR method is remarkably impressive (for the matrices dealt with in these tests the average number of iterations is less than 2.3 per eigenvalue!);

c) the Laguerre method is useful for finding some eigenvalues (especially the eigenvalues with largest modulus) and may be faster than the QR method for matrices with multiple eigenvalues when a convenient choice of the "convergence-parameters" has been made.

We have considered the iterative method of Wielandt for determining the eigenvectors of non-Hermitian matrices. The accuracy of each computed eigenvector lying in the linear m-fold subspace spanned by the true eigenvectors which correspond to an eigenvalue of multiplicity m has been tested.

CHAPTER I

A list of test-matrices for the eigenproblem

INTRODUCTION

As test-matrices we usually take matrices which form a representative sample of those which occur in practice, are general enough as to put sufficient strain on the numerical methods we have to test, and give the solution of the eigenproblem in closed form.
In §1 we give a list of test matrices with known eigenvalues.
In §2 we give a list of special test matrices. Important classes of special test matrices are the unitary matrices, the circulant matrices and the Frobenius matrices.
In §3 the tridiagonal test-matrices are considered.
When we are interested to generate test-matrices with a prescribed distribution of the eigenvalues, it is convenient to resort to matrices generated by Kronecker operations and by similarity transformations.
These test-matrices are considered in §4 and §5.

§1 A LIST OF TEST MATRICES WITH KNOWN EIGENVALUES

1.1 Symmetric test-matrices with known eigenvalues

Test matrix SM4/1

4	-2	-1	0
-2	4	0	-1
-1	0	4	-2
0	-1	-2	4

$\lambda_1 = 1$
$\lambda_2 = 3$
$\lambda_3 = 5$
$\lambda_4 = 7$

Test matrix SM4/2

0.67	0.13	0.12	0.11
0.13	0.96	0.14	0.13
0.12	0.14	0.31	0.16
0.11	0.13	0.16	0.15

$\lambda_1 = 0.0479716838$
$\lambda_2 = 0.3111488671$
$\lambda_3 = 0.6384911230$
$\lambda_4 = 1.0923883260$

Test matrix SM4/3
([1], page 269)

5	-5	5	0
-5	16	-8	7
5	-8	16	7
0	7	7	21

$\lambda_1 = \lambda_2 = 2.65728073$
$\lambda_3 = \lambda_4 = 26.34271928$

Test matrix SM4/4
([1], page 302)

1	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

$\lambda_1 = 0.03801601$
$\lambda_2 = 0.4538345$
$\lambda_3 = 2.2034461$
$\lambda_4 = 26.304703$

Test matrix SM5/1 [6]

0.81321	-0.00013	0.00014	0.00011	0.00021
-0.00013	0.93125	0.23567	0.41235	0.41632
0.00014	0.23567	0.18765	0.50632	0.30697
0.00011	0.41235	0.50632	0.27605	0.46322
0.00021	0.41632	0.30697	0.46322	0.41931

$\lambda_1 = -0.29908$
$\lambda_2 = 0.01521$
$\lambda_3 = 0.41985$
$\lambda_4 = 0.81321$
$\lambda_5 = 1.67828$

Test matrix SM5/2 [7]

$$\begin{vmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 6 & 0 & 4 & 3 \\ 3 & 0 & 7 & 6 & 5 \\ 2 & 4 & 6 & 8 & 7 \\ 1 & 3 & 5 & 7 & 9 \end{vmatrix}$$

$\lambda_1 = 22.40687532$
$\lambda_2 = 7.513724155$
$\lambda_3 = 4.848950120$
$\lambda_4 = 1.327045605$
$\lambda_5 = -1.096595181$

The eigenvectors of the test matrix SM5/2 are:

$v_1 \equiv (-0.245877938, -0.302396039, -0.453214523, -0.577177153, -0.556384583)$

$v_2 \equiv (-0.550961956, -0.709440339, 0.340179132, 0.0834109534, 0.265435677)$

$v_3 \equiv (-0.547172795, 0.312569920, -0.618112077, 0.115606593, 0.455493746)$

$v_4 \equiv (0.341013042, -0.116434620, -0.019590671, -0.682043035, 0.636071214)$

$v_5 \equiv (0.469358072, -0.542212195, -0.544452403, 0.425865662, 0.0889885036)$

Test matrix SM5/3
([1], page 255)

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix}$$

$\lambda_1 = 0.01083536$
$\lambda_2 = 0.18124190$
$\lambda_3 = 1.$
$\lambda_4 = 5.51748791$
$\lambda_5 = 92.29043483$

Test matrix SM6/1 [7]

$$\begin{vmatrix} 1 & 2 & 3 & 0 & 1 & 2 \\ 2 & 4 & 5 & -1 & 0 & 3 \\ 3 & 5 & 6 & -2 & -3 & 0 \\ 0 & -1 & -2 & 1 & 2 & 3 \\ 1 & 0 & -3 & 2 & 4 & 5 \\ 2 & 3 & 0 & 3 & 5 & 6 \end{vmatrix}$$

$\lambda_1 = 12.41133643$
$\lambda_2 = 12.41133642$
$\lambda_3 = 0.2849864395$
$\lambda_4 = 0.2849864365$
$\lambda_5 = -1.696322849$
$\lambda_6 = -1.696322851$

The eigenvectors of the test matrix SM6/1 are:

$v_1 \equiv$ (-0.221789750, -0.472911329, -0.720938140, 0.259414890, 0.357807087, 0.109956267)

$v_2 \equiv$ (0.170061798, 0.178584630, -0.138066492, 0.295915130, 0.565489671, 0.716086465)

$v_3 \equiv$ (0.669545567, -0.395331735, 0.136726362, -0.288372768, 0.463372193, -0.280810985)

$v_4 \equiv$ (0.013164189, 0.259286123, -0.199515430, -0.728887389, 0.551154856, -0.240296694)

$v_5 \equiv$ (0.503951797, 0.074032290, -0.529160563, -0.313202308, -0.521389692, 0.300995029)

$v_6 \equiv$ (0.391015688, -0.080878210, -0.418685666, -0.446284472, -0.520371701, 0.441940292)

Test matrix SM6/2
([1], page 237)

$$\begin{vmatrix} 0 & 1 & 6 & 0 & 0 & 0 \\ 1 & 0 & 2 & 7 & 0 & 0 \\ 6 & 2 & 0 & 3 & 8 & 0 \\ 0 & 7 & 3 & 0 & 4 & 9 \\ 0 & 0 & 8 & 4 & 0 & 5 \\ 0 & 0 & 0 & 9 & 5 & 0 \end{vmatrix}$$

$\lambda_1 = 16.60600885$
$\lambda_2 = 5.94293604$
$\lambda_3 = -10.06472040$
$\lambda_4 = -12.12830070$
$\lambda_5 = 2.10943466$
$\lambda_6 = -2.46535845$

Test matrix SM6/3 [4]

$$\begin{vmatrix} 253 & 121 & 66 & 11 & 11 & 0 \\ 121 & 96 & -19 & 71 & -24 & 7 \\ 66 & -19 & 137 & -117 & 73 & -14 \\ 11 & 71 & -117 & 152 & -82 & 21 \\ 11 & -24 & 73 & -82 & 57 & -14 \\ 0 & 7 & -14 & 21 & -14 & 7 \end{vmatrix}$$

$\lambda_1 = \lambda_2 = 2.533$
$\lambda_3 = \lambda_4 = 15.618$
$\lambda_5 = \lambda_6 = 332.849$

Test matrix SM8/1 [3]

611	196	-192	407	-8	-52	-49	29
196	899	113	-192	-71	-43	-8	-44
-192	113	899	196	61	49	8	52
407	-192	196	611	8	44	59	-23
-8	-71	61	8	411	-599	208	208
-52	-43	49	44	-599	411	208	208
-49	-8	8	59	208	208	99	-911
29	-44	52	-23	208	208	-911	99

$\lambda_1 = 1020.04901843$
$\lambda_2 = 1020.$
$\lambda_3 = 1019.90195436$
$\lambda_4 = \lambda_5 = 1000.$
$\lambda_6 = 0.09804864072$
$\lambda_7 = 0.0$
$\lambda_8 = -1020.04901843$

The eigenvectors of the test matrix SM8/1 are:

$v_1 \equiv (2,1,1,2, -0.004901843, -0.004901843, 0.009803686, 0.009803686)$

$v_2 \equiv (1, -2, -2, 1, 2, -2, 1, -1)$

$v_3 \equiv (2, -1, 1, -2, 10.09901951, -10.09901951, -20.19803903, 20.19803903)$

$v_4 \equiv (1, -2, -2, 1, -2, 2, -1, 1)$

$v_5 \equiv (7, 14, -14, -7, -2, -2, -1, -1)$

$v_6 \equiv (2, -1, 1, -2, -0.099019514, 0.099019514, 0.198039027, -0.198039027)$

$v_7 \equiv (1, 2, -2, -1, 14, 14, 7, 7)$

$v_8 \equiv (2, 1, 1, 2, 204.0049018, 204. 0049018, -408.0098037, -408.0098037)$

Test matrix SM8/2
([1], page 275)

1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
1	3	6	10	15	21	28	36
1	4	10	20	35	56	84	120
1	5	15	35	70	126	210	330
1	6	21	56	126	252	462	792
1	7	28	84	210	462	924	1716
1	8	36	120	330	792	1716	3432

$\lambda_1 = 2.2008514614 \; 10^{-4}$
$\lambda_2 = 6.7202144403 \; 10^{-3}$
$\lambda_3 = 8.3730245858 \; 10^{-2}$
$\lambda_4 = 5.1189155425 \; 10^{-1}$
$\lambda_5 = 1.9535387754 \; 10^{0}$
$\lambda_6 = 1.1943115534 \; 10^{1}$
$\lambda_7 = 1.4880477534 \; 10^{2}$
$\lambda_8 = 4.5436960082 \; 10^{3}$

Test matrix SM8/3
([1], page 239)

a	-b	-c	d	-e	f	g	-h
-b	a	d	-c	f	-e	-h	g
-c	d	a	-b	g	-h	-e	f
d	-c	-b	a	-h	g	f	-e
-e	f	g	-h	a	-b	-c	d
f	-e	-h	g	-b	a	d	-c
g	-h	-e	f	-c	d	a	-b
-h	g	f	-e	d	-c	-b	a

a = 11111111 e = 11108889
b = 9090909 f = 9089091
c = 10891089 g = 10888911
d = 8910891 h = 8909109

$$\lambda_k = 8 \; 10^{k-1} \qquad (k=1,2,\ldots,8)$$

Test matrix SM8/4
([1], page 244)

$$M = \begin{vmatrix} H & K \\ K & H \end{vmatrix} \qquad H = \left|\begin{array}{cc|cc} a & b & c & d \\ b & a & d & c \\ \hline c & d & a & b \\ d & c & b & a \end{array}\right| \qquad K = \left|\begin{array}{cc|cc} e & f & g & h \\ f & e & h & g \\ \hline g & h & e & f \\ h & g & f & e \end{array}\right|$$

a = 0	e = 5	$\lambda_1 = -21$	$\lambda_5 = 7$
b =-2	f = 6	$\lambda_2 = \lambda_3 = -13$	$\lambda_6 = \lambda_7 = 11$
c =-3	g = -7	$\lambda_4 = -5$	$\lambda_8 = 23$
d = 4	h = 8		

Test matrix SM8/5
([1], page 238)

$$\begin{vmatrix} 33 & -3 & 0 & -4 & 0 & 8 & 0 & -4 \\ -3 & 33 & 4 & 0 & -8 & 0 & 4 & 0 \\ 0 & 4 & 29 & 1 & -12 & -2 & -8 & -2 \\ -4 & 0 & 1 & 29 & -2 & -12 & -2 & -8 \\ 0 & -8 & -12 & -2 & 25 & 1 & -4 & -2 \\ 8 & 0 & -2 & -12 & 1 & 25 & -2 & -4 \\ 0 & 4 & -8 & -2 & -4 & -2 & 21 & 1 \\ -4 & 0 & -2 & -8 & -2 & -4 & 1 & 21 \end{vmatrix}$$

$\lambda_k = 6.k \qquad (k=1,2,...,8)$

Test matrix SM9/1 [2]

$M = (a_{ij})$ with $a_{ij} = a_{ji} = 0$ for $j \neq i,i+1$ and:

a_{ii}	$a_{i+1\ i} = a_{i\ i+1}$	λ_k
0.71507		0.83818541
0.42721	0.13952	0.75787017
0.71226	0.11389	0.74734873
0.42823	0.17385	0.43584777
0.70177	0.021681	0.42784000
0.44052	0.12899	0.42773464
0.43474	0.0035016	0.39554758
0.42862	0.0025372	0.38360934
0.42784	0.0	0.30227655

Test matrix SM21/1 [5]

$M = (a_{ij})$ with

n = 10

$a_{ii} = n+1-i \qquad (i = 1,2,...,n+1)$

$a_{ii} = i-n-1 \qquad (i = n+2,n+3,...,2n+1)$

$a_{ii+1} = a_{i+1\ i} = 1$

$a_{ij} = a_{ji} = 0$ for $j \neq i, i+1$

The eigenvalues of M to 7 decimal places are:

λ_k = 10.7461942	5.0002444
10.7461942	4.9997825
9.2106786	4.0043540
9.2106786	3.9960482
8.0389411	3.0430993
8.0389411	2.9610589
7.0039522	2.1302092
7.0039518	1.7893214
6.0002340	0.9475344
6.0002175	0.2538058
	-1.1254415

Test matrix SM21/2

$M = (a_{ij})$ with

$n = 10$

$a_{ii} = n+1-i \qquad (i = 1,2,\ldots,2n+1)$

$a_{i\,i+1} = a_{i+1\,i} = 1$

$a_{ij} = a_{ji} = 0 \quad$ for $\quad j \neq i, i+1$

The eigenvalues of M to 7 decimal places are:

λ_k = ± 10.7461942	± 4.0000002
± 9.2106786	± 3.0000000
± 8.0389411	± 2.0000000
± 7.0039520	± 1.0000000
± 6.0002257	0.0000000
± 5.0000082	

The eigenvector correct to 8 decimal places of λ_1 = 10.7461942 is:

$v_1 \equiv$ (1., .74619418, .30299994, .08590249, .01880748, .00336146, .00050815, .00006659, .00000771, .00000080, .00000007, .00000001, .0, .0, .0, .0, .0, .0, .0, .0, .0)

1.2 A list of real test matrices with known eigenvalues

Test matrix RM4/1
([1], page 228)

$$\begin{vmatrix} -2 & 1 & -3 & 2 \\ 1 & 1 & 1 & 0 \\ 4 & -2 & 5 & -3 \\ 5 & -1 & 5 & 6 \end{vmatrix} \qquad \lambda_k = k \quad (k = 1,2,3,4)$$

Test matrix RM4/2
([1], page 302)

$$\begin{vmatrix} 2 & 1 & -2 & 1 \\ -1 & 5 & -3 & 2 \\ -2 & 3 & 0 & 1 \\ -1 & 2 & -1 & 3 \end{vmatrix} \qquad \lambda_k = k \quad (k = 1,2,3,4)$$

Test matrix RM4/3
([1], page 303)

$$\begin{vmatrix} -1 & 3 & 0 & 0 \\ -1 & -1 & -2 & 0 \\ 0 & 4 & -6 & -16 \\ 0 & 0 & 10 & 18 \end{vmatrix} \qquad \begin{matrix} \lambda_1 = 2, \ \lambda_2 = 4 \\ \lambda_3, \ \lambda_4 = 2 \pm 2i \end{matrix}$$

Test matrix RM4/4
([1], page 303)

$$\begin{vmatrix} 6 & 1 & 0 & 0 \\ 9 & -3 & 1 & 0 \\ 15 & -12 & 3 & 1 \\ 23 & -23 & 3 & 4 \end{vmatrix} \qquad \begin{matrix} \lambda_1 = 7, \ \lambda_2 = 4 \\ \lambda_3 = 1, \ \lambda_4 = -5 \end{matrix}$$

Test matrix RM4/5
([1], page 274)

$$\begin{vmatrix} 17119 & 8289 & 3159 & 729 \\ -50436 & -24326 & -9216 & -2106 \\ 49554 & 23814 & 8974 & 2034 \\ -16236 & -7776 & -2916 & -656 \end{vmatrix} \qquad \lambda_k = 10^{k-1} \quad (k=1,2,3,4)$$

Test matrix RM4/6
([1], page 237)

$$\begin{vmatrix} 9 & 1 & -2 & 1 \\ 1 & 8 & -3 & -2 \\ -2 & -3 & 7 & -1 \\ 1 & -2 & -1 & 6 \end{vmatrix} \qquad \lambda_k = 3.k \qquad (k = 1,2,3,4)$$

Test matrix RM4/7
([1], page 270)

$$\begin{vmatrix} 48 & 25 & 11 & 3 \\ -154 & -82 & -37 & -10 \\ 176 & 96 & 44 & 11 \\ -64 & -34 & -14 & -1 \end{vmatrix} \qquad \begin{array}{l} \lambda_1 = \lambda_2 = \lambda_3 = 2 \\ \lambda_4 = 3 \end{array}$$

Test matrix RM4/8
([1], page 386)

$$\begin{vmatrix} -4 & 5 & -1 & -7 \\ 1 & -1 & -4 & 1 \\ -2 & 4 & 7 & -2 \\ 7 & -5 & 1 & 10 \end{vmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 3$$

Test matrix RM4/9

$$\begin{vmatrix} 0.9205 & -0.8526 & 0.3265 & 0.3054 \\ 1.0961 & -0.6522 & 0.8152 & 0.3284 \\ 0.0677 & 0.2922 & 0.8561 & -0.1328 \\ -0.9395 & 0.8977 & -0.5330 & 0.6556 \end{vmatrix}$$

$\lambda_1 = 0.9612$ $\quad\lambda_2 = 0.8018$ $\quad\lambda_3, \lambda_4 = 0.0064 \pm 0.3981\, i$

The eigenvectors of the test matrix RM4/9 are:

$v_1 \approx (-0.1798,\ 0.3188,\ 0.9228,\ -0.1204)$
$v_2 \approx (\ 0.1890,\ 0.3465,\ 0.0309,\ 0.9113)$
$v_3, v_4 \approx (0.3722 \pm 0.5715\, i,\ 0.5133 \pm 0.3454\, i,\ -0.1410 \mp 0.2322\, i,$
$\quad - 0.2782 \mp 0.0118\, i)$

Test matrix RM4/10

$$\begin{vmatrix} 0 & 0.07 & 0.27 & -0.33 \\ 1.31 & -0.36 & 1.21 & 0.41 \\ 1.06 & 2.86 & 1.49 & -1.34 \\ -2.64 & -1.84 & -0.24 & -2.01 \end{vmatrix}$$

$$\lambda_1 = 0.03$$
$$\lambda_2 = 3.03$$
$$\lambda_3 = \lambda_4 = -1.97 \pm i$$

Test matrix RM5/1
([1], page 227)

$$\begin{vmatrix} 11 & -6 & 5 & 5 & -6 \\ 12 & -11 & 7 & 3 & -12 \\ -5 & 13 & 3 & -8 & 5 \\ 3 & -4 & 7 & 5 & -3 \\ -2 & 6 & -2 & 2 & 7 \end{vmatrix} \qquad \lambda_k = k \quad (k = 1,2,3,4,5)$$

Test matrix RM5/2
([1], page 247)

$$\begin{vmatrix} 40 & 56 & -11 & -8 & -39 \\ 17 & 10 & 8 & 5 & -17 \\ -1 & -6 & 6 & 5 & 1 \\ 13 & 30 & 8 & -19 & -13 \\ -63 & -54 & -3 & -3 & 64 \end{vmatrix} \qquad \begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -2 \\ \lambda_3 &= 6 \\ \lambda_4 &= -24 \\ \lambda_5 &= 120 \end{aligned}$$

Test matrix RM5/3
([1], page 270)

$$\begin{vmatrix} 1 & -1 & 1 & -1 & 1 \\ -4 & 2 & 0 & -2 & +4 \\ 6 & 0 & -2 & 0 & 6 \\ -4 & -2 & 0 & 2 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \qquad \begin{aligned} \lambda_1 &= \lambda_2 = \lambda_3 = +4 \\ \lambda_4 &= \lambda_5 = -4 \end{aligned}$$

Test matrix RM5/4 [9]

$$\begin{vmatrix} 15 & 11 & 6 & -9 & -15 \\ 1 & 3 & 9 & -3 & -8 \\ 7 & 6 & 6 & -3 & -11 \\ 7 & 7 & 5 & -3 & -11 \\ 17 & 12 & 5 & -10 & -16 \end{vmatrix}$$

$\lambda_1 = \lambda_2 = 1.50016 \pm 3.57064\ i$
$\lambda_3 = \lambda_4 = 1.50016 \pm 3.57064\ i$
$\lambda_5 = -1$

Test matrix RM6/1
([1], page 277)

$$\begin{vmatrix} 7 & 3 & 4 & -11 & -9 & -2 \\ -6 & 4 & -5 & 7 & 1 & 12 \\ -1 & -9 & 2 & 2 & 9 & 1 \\ -8 & 0 & -1 & 5 & 0 & 8 \\ -4 & 3 & -5 & 7 & 2 & 10 \\ 6 & 1 & 4 & -11 & -7 & -1 \end{vmatrix}$$

$\lambda_1 = 3$ $\lambda_3, \lambda_4 = 1 \pm 2\ i$
$\lambda_2 = 4$ $\lambda_5, \lambda_6 = 5 \pm 6\ i$

Test matrix RM6/2
([1], page 286)

$$\begin{vmatrix} 2 & 4 & 0 & 0 & 0 & 0 \\ -5 & +13 & -2 & 0 & 0 & 0 \\ -13 & 29 & -15 & 5 & 0 & 0 \\ -25 & 61 & -48 & 18 & 0 & 0 \\ -36 & 80 & -38 & -26 & 32 & -6 \\ -34 & 24 & 140 & -250 & 155 & -29 \end{vmatrix}$$

$\lambda_k = k\ ,\ k = 1(1)6$

Test matrix RM6/3
([1], page 372)

$$\begin{vmatrix} -1 & -6 & 6 & -14 & 21 & 10 \\ -3 & -2 & 3 & -7 & 14 & 3 \\ -4 & 16 & 0 & 6 & -12 & 4 \\ -7 & 15 & -5 & 11 & -8 & 7 \\ -1 & -5 & 3 & -7 & 15 & 1 \\ 1 & -3 & 3 & -7 & 7 & 8 \end{vmatrix}$$

$\lambda_1 = \lambda_2 = 1$
$\lambda_3 = 5$
$\lambda_4 = 6$
$\lambda_5 = \lambda_6 = 9$

Test matrix RM6/4 [8], [9]

1	1	1	1	1	1
1/2	1/3	1/4	1/5	1/6	1/7
1/3	1/4	1/5	1/6	1/7	1/8
1/4	1/5	1/6	1/7	1/8	1/9
1/5	1/6	1/7	1/8	1/9	1/10
1/6	1/7	1/8	1/9	1/10	1/11

Roots calculated in [9]	Roots calculated in [8]
2.132376	2.132376
-0.2214068	-0.2214066
$-0.3184330\ 10^{-1}$	$-0.3184328\ 10^{-1}$
$-0.8983233\ 10^{-3}$	$-0.8983258\ 10^{-3}$
$-0.1706278\ 10^{-4}$	$-0.1706200\ 10^{-4}$
$-0.1394499\ 10^{-6}$	$-0.1443702\ 10^{-6}$

Test matrix RM7/1
([1], page 275)

28	17	-16	11	9	-2	-27
-1	29	7	-6	-2	28	1
-11	-1	12	3	-8	10	11
-6	-11	12	8	-12	-5	6
-3	1	-4	3	8	4	3
14	16	-7	6	2	4	-14
-37	-18	-9	+5	7	26	38

$\lambda_k = 2^{k-1}$
(k=1(1)7)

Test matrix RM7/2
([1], page 372)

-3	11	10	4	-13	-2	6
2	3	10	9	-16	0	-2
-4	-1	1	7	4	11	4
0	8	1	2	-1	-8	0
3	-4	-2	7	7	7	-3
-7	9	6	9	-12	3	7
-6	9	0	-5	3	-2	9

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 3$
$\lambda_4, \lambda_5 = 3 \pm 4\,i$
$\lambda_6, \lambda_7 = 5 \pm 6\,i$

Test matrix RM8/1
([1], page 227)

-16	17	-11	-17	-23	29	-19	11
-72	31	-53	-59	-65	-13	-61	-31
-79	37	-56	-55	-71	-7	-67	-25
73	-31	54	59	65	13	61	31
-21	-21	-25	-19	-60	-77	-17	-47
73	-31	53	+59	66	13	61	31
43	-1	39	33	35	43	42	31
-73	31	-53	-59	-65	-13	-60	-31

$\lambda_k = k$
(k=1(1)8)

Test matrix RM8/2
([1], page 388)

-7	-9	13	4	-12	-11	-13	6
-1	5	-6	4	-1	8	1	9
3	-9	-1	7	-13	6	12	-3
17	19	-2	-6	11	8	-2	-17
9	8	2	-9	14	4	1	-9
-3	-8	-4	3	-9	9	5	3
1	6	-6	4	-1	8	2	7
-9	-7	7	8	-13	-3	-10	16

$\lambda_1, \lambda_2 = 1 \pm 2\,i$ $\lambda_3, \lambda_4 = 3 \pm 4\,i$ $\lambda_5, \lambda_6 = 5 \pm 6\,i$ $\lambda_7, \lambda_8 = 7 \pm 8\,i$

Test Matrix RM8/3
([1], page 326)

6	-11	-23	-17	-3	-19	-13	-19
11	-2	-9	-3	11	-5	1	-5
23	-9	29	21	22	-2	22	8
-17	3	-21	-19	-16	-4	-16	-14
3	11	22	16	8	22	12	18
-19	5	2	-4	-22	4	-8	-2
-13	-1	-22	-16	-12	-8	-11	-9
19	-5	8	14	18	2	9	13

$\lambda_1, \lambda_2 = 2 \pm 7\,i$ $\lambda_3, \lambda_4 = 5 \pm 3\,i$ $\lambda_5, \lambda_6 = 6 \pm 4\,i$ $\lambda_7, \lambda_8 = 1 \pm 9\,i$

Test matrix RM9/1
([1], page 371)

5	8	-9	5	-12	-4	0	4	-4
-1	-2	8	-1	5	-1	-2	3	1
3	3	16	2	3	-10	-13	0	-3
-1	9	-13	0	-2	14	13	-10	1
6	11	6	7	5	-2	-6	-5	-6
2	5	-5	-6	-2	13	5	-3	-2
-1	-3	9	2	3	-12	-6	2	1
7	1	3	-7	3	4	3	8	-7
-5	-3	-1	4	-7	-5	-2	7	6

$\lambda_k = k \quad k=1(1)9$

Test matrix RM9/2
([1], page 287)

14	1							
-91	0	1						
364	0	0	1					
-989	0	0	0	1				
1886	0	0	0	0	1			
-2509	0	0	0	0	0	1		
2236	0	0	0	0	0	0	1	
-1210	0	0	0	0	0	0	0	1
300	0	0	0	0	0	0	0	0

$\lambda_k = 1$
2
3
$1 \pm i$
$1 \pm 2i$
$2 \pm i$

Test matrix RM12/1
([1], page 384)

$$\begin{vmatrix} 24 & 3 & -20 & 10 & -1 & 5 & 24 & -11 & 16 & 40 & -22 & -29 \\ 3 & 13 & -17 & 6 & 1 & -1 & 12 & 4 & 12 & 18 & -11 & -3 \\ 4 & -5 & 7 & 2 & 12 & 5 & 9 & 4 & -7 & -7 & 1 & -4 \\ 5 & 8 & 6 & -3 & 4 & 6 & -5 & -2 & 3 & -1 & -13 & -5 \\ -23 & 40 & -18 & 26 & 13 & 15 & -3 & -17 & -26 & 18 & -17 & 23 \\ 8 & -9 & 1 & 10 & -4 & -3 & 0 & 4 & -10 & -1 & 1 & -8 \\ -19 & 10 & 13 & 2 & 1 & 0 & -4 & 2 & -2 & -13 & 9 & 19 \\ -16 & 21 & -4 & +17 & 16 & 11 & -3 & -20 & -17 & 4 & -5 & 16 \\ 3 & 5 & 9 & -7 & 4 & 6 & -5 & -2 & 7 & -4 & -8 & -3 \\ -2 & 2 & 3 & -1 & 12 & 5 & 9 & 4 & -4 & -3 & 0 & 2 \\ 7 & 6 & -17 & 6 & 1 & -1 & 12 & 4 & 12 & 18 & -8 & -7 \\ 18 & -11 & -3 & 4 & -2 & 6 & 12 & -15 & 4 & 22 & -7 & -23 \end{vmatrix}$$

$$\lambda_k = \pm(3 \pm 4\,i),\quad \pm(4 \pm 3\,i),\quad \pm 5\,i,\ \pm 5$$

1.3 A list of complex test-matrices with known eigenvalues

Test matrix CM3/1
([1], page 378)

$$\begin{vmatrix} -189-790\,i & 537-505\,i & -1626+2740\,i \\ 438-204\,i & 361+253\,i & -1788-630\,i \\ -65-144\,i & 85-115\,i & -207+581\,i \end{vmatrix}$$

$$\lambda_1 = 1+i \qquad \lambda_2 = 7-8\,i \qquad \lambda_3 = -43+51\,i$$

Test matrix CM4/1
([1], page 251)

$$\begin{vmatrix} 12 & -(1+i) & 2 & 3(1+i) \\ -(1-i) & 12 & (1-i) & -2 \\ 2 & (1+i) & 8 & -(1+i) \\ 3(1-i) & -2 & -(1-i) & 8 \end{vmatrix}$$

$$\lambda_k = 4.k \quad (k = 1,2,3,4)$$

Test matrix CM4/2
([1], page 287)

2+11 i	3-5 i		
6+ 4 i	-9+4 i	8-2 i	
14- 2 i	-30+14 i	22-9 i	-1+3 i
25- 8 i	-57+28 i	39-24 i	-2+10 i

λ_k = (1+i), (3+4 i), (4+5 i), (5+6 i)

Test matrix CM5/1 [10]

1+ 2 i	3 + 4 i	21+22 i	23+24 i	41+42 i
43+44 i	13 +14 i	15+16 i	33+34 i	35+36 i
5+ 6 i	7 + 8 i	25+26 i	27+28 i	45+46 i
47+48 i	17 +18 i	19+20 i	37+38 i	39+40 i
9+10 i	11 +12 i	29+30 i	31+32 i	49+50 i

$\lambda_1 = \lambda_2 = 0$

λ_3 = 127.387 + 132.278 i

λ_4 = -9.45999 + 7.28019 i

λ_5 = 7.07332 - 9.55839 i

Test matrix CM5/2 [11]

-0.845+0.0 i	5.2+0.103 i	.301-0.0454 i	-9.6+0.936 i	.0734+7.26 i
5.2 -0.103 i	-6.2+0.0 i	-3.39-0.407 i	+0.122+0.91 i	4.19 -3.66 i
0.301+0.0454	-3.39+0.407 i	0.019+0.0 i	0.935-0.271 i	-0.0572+2.82 i
-9.6 -0.936 i	0.122-0.91 i	.935+0.271 i	7.21+0.0 i	0.337+0.0603 i
.0734-7.26 i	4.19+3.66 i	-0.0572-2.82 i	0.337-0.0603 i	-1.23+0.0 i

λ_1 = 15.180165225

λ_2 = 5.6787293543

λ_3 = -0.83398680019

λ_4 = -5.1498456282

λ_5 = -15.921062150

Test matrix CM6/1
([1], page 376)

3	1+3 i	-6+16 i	8-14 i	-13-5 i	-2
3+i	6+2 i	-6+ 9 i	6-10 i	-10-5 i	-3-i
-1+6i	18-9 i	3+i	-3-i	-17+3 i	1-6 i
4	4-2 i	3	-3	- 8+2 i	-4
1-2i	6+4 i	-6+9 i	6-10 i	- 8-4 i	-1+2 i
1+i	-6-i	7i	2-4i	-3	-i

λ_k = 0, 1, i, (2+i), (-1-2i), (-1-2i)

Test matrix CM10/1 [12]

2+3i	3+i								
3+2i	-2-i	1+2i							
5-3i	1+2i	2+i	-1+4i						
2+6i	-2+3i	3-i	-4+2i	5+5i					
1+4i	2+2i	-3+7i	1+5i	2-3i	1+6i				
5-i	0+4i	1+5i	-8-1i	4+7i	7+i	4-2i			
5+2i	1+4i	6-5i	8+4i	4-4i	-1+5i	3+0i	-4+6i		
-4-3i	7+3i	1+6i	2-4i	3+i	1+2i	1+4i	6+3i	7-i	
5+0i	2+2i	1+3i	1+i	-4-2i	1+6i	1+2i	2+5i	0+i	3+2i
5+2i	2+6i	1-3i	7+4i	4+i	-7+0i	3-3i	5-4i	6+3i	2+5i

λ_1 = 4.16174868+3.13751356 i
λ_2 = 5.43644837-3.97142582 i
λ_3 = 2.38988759+7.26807071 i
λ_4 =-1.93520144-3.97509382 i
λ_5 =-2.44755082+0.437126175 i
λ_6 =-5.27950616-2.27596303 i
λ_7 = 1.03205812+9.29413278 i
λ_8 =-4.96687009-8.08712475 i
λ_9 = 8.81130928+1.54938266 i
λ_{10} =10.7976764 +8.62338151 i

Test matrix CM15/1

([1], page 278)

$$A = B + i\,C$$

B =

6	-12	18	-25	20	-28	40	-41	19	-23	38	-36	42	-52	41
20	-24	28	-28	21	-24	36	-26	15	-15	19	-25	31	-36	16
11	-2	-6	14	-12	9	-2	12	-11	11	-19	16	-6	-3	-8
-3	-5	-15	33	-26	25	-27	32	-22	22	-35	23	-3	11	-8
6	-32	12	12	-2	10	-20	21	-10	13	-28	4	16	10	-16
9	-20	-2	13	-24	23	-21	27	-37	41	-30	19	3	8	-17
7	-15	-2	5	-22	8	-6	0	-7	21	-4	1	16	-8	1
12	-17	3	-4	-8	3	-4	-2	-2	7	5	-4	18	-13	1
12	-17	5	-4	-6	12	-13	19	-26	23	-13	12	0	5	-17
5	-21	11	-1	4	2	-12	13	-2	7	-15	5	5	11	-16
-1	0	-12	13	-9	8	-10	15	-5	5	-15	20	-8	9	-8
4	10	-20	19	-17	14	-7	17	-16	16	-24	30	-18	4	-8
10	0	-3	3	-10	7	5	5	-16	16	-12	6	7	-26	16
8	-5	11	-18	13	-21	33	-34	12	-16	31	-29	35	-54	41
0	-1	5	-12	16	-21	29	-38	23	-21	25	-18	18	-24	19

C =

-7	-3	-2	-10	4	-1	4	-5	1	-7	18	-9	22	-10	14
-10	-13	9	-12	6	-9	16	-15	3	2	14	-19	37	-11	21
11	-34	27	-33	34	-24	29	-30	24	-35	44	-36	51	-28	17
4	-18	5	-9	23	-22	14	-21	25	-25	21	-14	27	-13	9
2	-12	-1	4	-4	9	-21	16	3	-11	13	-16	29	-19	17
15	-27	31	-37	18	10	-14	26	-23	-12	31	-25	21	-9	-6
7	-10	21	-31	19	3	2	11	-25	3	9	1	-12	15	-22
-1	-4	7	-17	14	0	4	9	-22	8	-5	15	-18	23	-22
-1	-15	18	-28	19	-3	-2	14	-10	-13	22	-12	9	7	-6
1	-8	0	-5	-2	3	-15	10	9	-13	22	-17	25	-18	17
7	-16	2	-2	8	-7	-1	-6	10	-10	14	-11	25	-16	9
1	-15	7	-5	6	4	1	-2	-4	-7	16	-16	32	-18	17
-16	2	-8	5	-11	8	-1	2	-14	19	-3	-2	22	-5	21
-14	4	-9	-3	-3	6	-3	2	-6	0	11	-2	15	-3	14
-5	4	-4	-3	0	8	-4	-1	3	-14	14	-3	5	-5	7

λ_1 = -5-3i	λ_9 = 2+8i
λ_2 = -5+2i	λ_{10} = 3+8i
λ_3 = -9+0i	λ_{11} = 6+3i
λ_4 = -9+3i	λ_{12} = 7+2i
λ_5 = -8+5i	λ_{13} = 3-7i
λ_6 = -4+7i	λ_{14} = -5-9i
λ_7 = 3+i	λ_{15} = 9-8i
λ_8 = 2+4i	

<u>CM 16/1</u> [12]

$A = (a_{kl})$ (k;l = 1,2,...,16)

$a_{kl} = a_{lk} = 0$ for $l \neq k,k+1$

For k = 1,2,...,16

a_{kk}	$a_{k\,k+1}$ ($= a_{k+1\,k}$)	Eingenvalues of A λ_k (real)	(imag.)
3+2i	4-i	+2.06853152	-2.05443045i
-1-i	2+4i	2.40341933	+2.08105512i
3-4i	3+i	2.72491267	-2.37837845i
2+3i	3-2i	2.45640400	+0.631936861i
-5+i	2-2i	2.27740066	+1.44826850i
1+2i	2+3i	0.812811959	+1.33551135i
5+2i	1+3i	-1.38565721	-1.38756051i
-2+i	-2+2i	-2.72480368	+0.657064546i
1-2i	3+3i	+1.57598142	-3.83032770i
-1-4i	-1+5i	+3.28048252	+3.27566163i
2+i	4+3i	+1.19252750	-5.44399752i
1-5i	1-6i	+3.55339888	+1.26465631i
3+i	2+i	-2.45560768	-4.69290496i
2+4i	5-i	-4.89673115	+3.62210856i
-4+3i	-3-4i	-5.65716067	+1.63200082i
1-5i	0	5.77408994	+2.83933591i

§2 SPECIAL TEST-MATRICES

The following procedure can be used for testing methods for solving the non symmetric eigenvalue problem.

Turning a positive definite symmetric matrix H around its horizontal middle axis produces a (usually) non symmetric matrix $\hat{I}.H$ which is similar to a symmetric matrix M. Indeed it is:

$$\hat{I} = \begin{vmatrix} 0 & 0 & 0 \dots 0 & 1 \\ 0 & 0 & 0 \dots 1 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & 0 \dots 0 & 0 \\ 1 & 0 & 0 \dots 0 & 0 \end{vmatrix}$$

If RR^T is the Choleski decomposition of H ($H = RR^T$ where R is a lower triangular matrix) the matrix $M = R^T(\hat{I}H)(R^T)^{-1}$ is similar to the non symmetric matrix $\hat{I}H$. The matrix $M = R^T\hat{I}RR^T(R^T)^{-1} = R^T\hat{I}R$ is a symmetric matrix with zero elements below the secondary diagonal. The eigenvalues of the non symmetric matrix $\hat{I}H$ are the eigenvalues of the symmetric matrix $M = R^T\hat{I}R$, where $H = RR^T$, which can be determined by standard methods for symmetric matrices.

Example (Rutishauser):

H = Pascal matrix of order 5:

$$H = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix} \qquad R = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{vmatrix}$$

$$\hat{I}H = \begin{vmatrix} 1 & 5 & 15 & 35 & 70 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

$$M = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 10 & 10 & 5 & 1 \\ 10 & 10 & 5 & 1 & 0 \\ 10 & 5 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

The eigenvalues of $\hat{I}H$ are the eigenvalues of M. The eigenvalues of M are determined by using standard methods for symmetric matrices.

2. <u>The test matrices $A^{(n)}$ $\bar{A}^{(n)}$</u> [12], [13]

The Hessenberg matrices $A^{(n)}$ and $\bar{A}^{(n)}$ have the same eigenvalues. $\bar{A}^{(n)} = \hat{I}\, A^{(n)}\, \hat{I}$ where $\hat{I}$ has been defined in 1).
$\det A^{(n)} = \det \bar{A}^{(n)} = 1$

$$A^{(n)} = \begin{vmatrix} n & (n-1) & (n-2) & \dots & 3 & 2 & 1 \\ (n-1) & (n-1) & (n-2) & \dots & 3 & 2 & 1 \\ 0 & (n-2) & (n-2) & \dots & 3 & 2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{vmatrix}$$

$$\bar{A}^{(n)} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 2 & 0 & \dots & 0 & 0 \\ 1 & 2 & 3 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & 4 & \dots & (n-1) & (n-1) \\ 1 & 2 & 3 & 4 & \dots & (n-1) & n \end{vmatrix}$$

For n = 12 the eigenv...ues of $A^{(n)}$ (and $\bar{A}^{(n)}$) are:

λ_1 = .0310280606	λ_7 = 1.5539887091
λ_2 = .0495074292	λ_8 = 3.5118559486
λ_3 = .0812276592	λ_9 = 6.9615330856
λ_4 = .1436465198	λ_{10} = 12.3110774009
λ_5 = .2847497206	λ_{11} = 20.1989886459
λ_6 = .6435053190	λ_{12} = 32.2288915016

The largest eigenvalues of $A^{(n)}$ are very well conditioned and the smallest very ill-conditioned.

3. <u>The test matrix $T^{(n)}$</u> [14]

(Algorithm 52):

The elements of the test matrix $T^{(n)}$ of order n are defined by:

$$T^{(n)} = (t_{ij})$$
$$t_{nn} = -1/c$$
$$t_{in} = t_{ni} = i/c \qquad (i = 1,2,\ldots,n-1)$$
$$t_{ii} = (c-i^2)/c$$
$$t_{ij} = t_{ji} = -(i.j)/c \quad (i = 2,3,\ldots,n-1;\ j=1,2,\ldots i-1)$$

where: $c = n(n-1)(2n-5)/6$

(The n-th row and the n-th column of the inverse matrix $(T^{(n)})^{-1}$ are the set: 1,2,...,n. The matrix formed by deleting the n-th row and the n-th column of $(T^{(n)})^{-1}$ is the identity matrix of order n-1.

The determinant of $T^{(n)}$ is: det $T^{(n)} = t_{nn}$). All but two of the eigenvalues of $T^{(n)}$ are unity while the two remaining are given by the expressions 6/(p(n-1)) and p/(n.(5-2n)) where

$$p = 3 + \sqrt{\frac{3(n-1)(4n-3)}{n+1}}$$

For example:

n	eigenvalues differing from unity	
10	.043532383	-.083532383
20	.016366903	-.024938332

4. The Hilbert matrix $H^{(n)}$

Elements of the Hilbert matrix $H_n^{(n)}$ of order n are defined by:

$$H^{(n)} = (h_{ij})$$

where $h_{ij} = 1/(i+j-1) \qquad i=1,\dots,n;\ j = 1,\dots,n.$

(The inverse matrix of the Hilbert matrix is given by:

$$a_{11} = n^2$$

$$a_{ij} = \frac{(-1)^{i+j}(n+i-1)!\,(n+j-1)!}{(i+j-1)\left[(i-1)!\,(j-1)!\right]^2(n-i)!\,(n-j)!}$$

The determinant of the Hilbert matrix is given by:

$$\det H^{(n)} = \frac{(1!2!\ \dots\ (n-1)!)^4}{1!2!\ \dots\ (2n-1)!}\quad)$$

The eigenvalues and eigenvectors for Hilbert matrices of order 3 through 10 are computed in [15].

For example the computed eigenvalues of $H^{(4)}$ are:

$$\lambda_1 = 1.500214280059243\ \ 10^{-0}$$
$$\lambda_2 = 1.691412202214500\ \ 10^{-1}$$
$$\lambda_3 = 6.738273605760748\ \ 10^{-3}$$
$$\lambda_4 = 9.670230402258689\ \ 10^{-5}$$

5) The Eberlein's test matrix $E_s^{(n)}$ [16]

$$E_s^{(n)} \equiv (e_{ij})$$

with:

$$e_{ii} = -\left[(2i+1)n+is-2i^2\right]$$
$$e_{ii+1} = (i+1)(n+s-i)$$
$$e_{ii-1} = i(n-i+1) \qquad\qquad i,j = 0,1,2,\dots,n.$$
$$e_{ij} = 0 \quad |i-j| > 1$$

s is an arbitrary parameter and (n+1) is the order of the matrix.

The eigenvalues of $E_s^{(n)}$ are

$$\lambda_j = -j(s+j+1) \qquad j = 0,1,2,\ldots,n$$

In [16] the corresponding components of the left eigenvectors and of the right eigenvectors are given.
When $s = -2.-3,\ldots,-2n$, the matrix $E_s^{(n)}$ is defective with two or more **pairs** of eigenvectors **coalescing.** In the range $-2n < s < -2$ at last **a pair of** eigenvectors is nearly parallel, and the positive eigenvalues of $E_s^{(n)}$ are ill-conditioned (especially for $s < -(n+1)$).

6. Brenner's test matrix $B_{\alpha,\beta}^{(n)}$ [17]

Let Q be the nxn matrix whose entries are all 1's. (The matrix Q has rank 1). The eigenvalues of

$$B_{\alpha,\beta}^{(n)} = \alpha I + \beta Q$$

are: α(n-1 times) and $\alpha+\beta n$.
The eigenvalues of the matrix

$$(B_{\alpha,\beta}^{(n)})^{-1} = \{I - \frac{\beta}{\alpha+\beta n} Q\} \cdot \frac{1}{\alpha}$$

are: $1/\alpha$ (n-1 times) and $1/(\alpha+\beta n)$.

7. Test matrix $R^{(n)}$ [18]

The test matrix $R^{(n)}$

$$\begin{vmatrix} 1 & & & & & & \\ 1 & -1 & & & & & \\ 1 & -2 & 1 & & & & \\ 1 & -3 & 3 & -1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ 1 & -5 & 10 & -10 & 5 & -1 & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{vmatrix}$$

has $\left[\frac{n}{2}\right]$ eigenvalues equal to -1 and $(n-\left[\frac{n}{2}\right])$ eigenvalues equal to +1.

The symmetric positive definite test-matrix $P^{(n)} = R^{(n)}.(R^{(n)})^T$ with elements

$$p_{ij} = \binom{i+j-2}{i-1} \qquad i,j = 1,2,\dots,n$$

and the matrix $(P^{(n)})^{-1} = (R^{(n)})^T R^{(n)}$ have the same eigenvalues ($R^{(n)}.R^{(n)} = I$).

8. Test matrix $L^{(p)}$ [18]

Let $n = p-1$, where p is an odd prime. The elements l_{ij} of the Lehmer's matrix $L^{(p)}$ are:

$$l_{ij} = \left(\frac{i+j}{p}\right)$$

where $\left(\frac{i+j}{p}\right)$ is the Legendre-Jacobi quadratic reciprocity symbol, i.e.:

$$\left(\frac{i+j}{p}\right) = \begin{cases} 0 & \text{if } p \text{ divides } i+j \\ 1 & \text{if } i+j \text{ is congruent to a square, modulo } p \\ -1 & \text{otherwise.} \end{cases}$$

Example: p=5, n=4.

$$L^{(5)} = \begin{vmatrix} -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{vmatrix}$$

The eigenvalues of $L^{(p)}$ are 1, -1, $+\sqrt{p}$ ($\frac{n-2}{2}$ times), $-\sqrt{p}$ ($\frac{n-2}{2}$ times).
The inverse matrix of $L^{(p)}$ is the matrix

$$(L^{(p)})^{-1} = \left\{\frac{1}{p}\left[\left(\frac{i+j}{p}\right) - \left(\frac{i}{p}\right) - \left(\frac{j}{p}\right)\right]\right\} \qquad (i,j = 1,2,\dots,n)$$

Example:

$$(L^{(5)})^{-1} = \frac{1}{5} \begin{vmatrix} -3 & -1 & 1 & -2 \\ -1 & 3 & 2 & 1 \\ 1 & 2 & +3 & -1 \\ -2 & 1 & -1 & -3 \end{vmatrix}$$

The matrix $(L^{(p)})^2$ is positive definite with eigenvalues 1 (with multiplicity two) and p (with multiplicity (n-2)).

$$(L^{(p)})^2 = \left\{ \left[p\,\delta_{ij} - 1 - \left(\frac{ij}{p} \right) \right] \right\} \qquad (i,j = 1,2,\dots,n)$$

9. Frobenius' test matrices $F^{(n)}$

The eigenvalues of the nxn matrix

$$F^{(n)} = \begin{vmatrix} 0 & 0 & \dots & 0 & p_1 \\ 1 & 0 & \dots & 0 & p_2 \\ 0 & 1 & \dots & 0 & p_3 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & p_n \end{vmatrix}$$

are the zeros of the polynomial:

$$\lambda^n - p_n \lambda^{n-1} - p_{n-1} \lambda^{n-2} - \dots - p_1 = 0$$

If $F^{(n)}$ is a non-derogatory matrix (i.e. there is only one Jordan matrix associated with each distinct eigenvalue λ_k of $F^{(n)}$ and therefore only one eigenvector associated with each distinct λ_k) the eigenvector v_k corresponding to the eigenvalue λ_k of $F^{(n)}$ is given by:

$$v_k = (\lambda_k^{n-1}, \lambda_k^{n-2}, \dots, \lambda_k, 1)$$

Frobenius' test matrix $F_1^{(n)}$ [19] with

$$p_k = (-1)^{n-k} \binom{n+k-1}{n-k+1} \qquad k = 1,2,\dots,n$$

has the eigenvalues

$$\lambda_n(F_1^{(n)}) = 2(1-\cos \frac{(2k-1)\pi}{2n+1})$$

Since there exists in literature an extensive list of polynomials with known zeros, the Frobenius matrices are a wide class of test-matrices.

10. Circulant test matrices $C^{(n)}$

A circulant matrix is one of the form:

$$C^{(n)} = \begin{vmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_n & c_1 & c_2 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ c_2 & c_3 & c_4 & \dots & c_1 \end{vmatrix}$$

and is specified therefore by its first row:

$$c_n = \{c_1\ c_2\ \dots\ c_n\}$$

The eigenvalues of $C^{(n)}$ are the numbers

$$\lambda_k = c_1+c_2z_k+c_3z_k^2+\ \dots\ +c_nz_k^{n-1}$$

where $z_k = \cos\left(\frac{2\pi k}{n}\right) + \sqrt{-1}\,\sin\left(\frac{2\pi k}{n}\right) \quad (k = 1,2,\dots, n).$

The eigenvector of $C^{(n)}$ corresponding to λ_k is:

$$v_k = \frac{1}{\sqrt{n}}\,(z_k^{n-1}, z_k^{n-2},\dots, z_k, 1).$$

The Brenner's test matrix is a particular circulant matrix of the form

$$C^{(n)} = \{\alpha+\beta\ \ \beta\ \ \beta\ \dots\ \beta\}$$

Test matrix $C^{(8)}$ ([1], page 256)

$$C^{(8)} = \{1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\}$$

$\lambda_1 = \lambda_2 \quad -6.82842712$

$\lambda_3 = \lambda_4 = \lambda_5 = 0$

$\lambda_6 = \lambda_7 = -1.17157288$

$\lambda_8 = 24$

Test matrix $C^{(16)}$ ([1], page 240)

$$C^{(16)} = \{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0$
$\lambda_8 = \lambda_9 = -26.27414237$
$\lambda_{10} = \lambda_{11} = -\ 3.23982881$
$\lambda_{12} = \lambda_{13} = -\ 1.03956613$
$\lambda_{14} = \lambda_{15} = -\ 1.44646269$
$\lambda_{16} = 80$

Remark

Circulant matrices are related to the numerical solution of elliptic and parabolic differential equations with periodic conditions.

Remark

The eigenvalues and the eigenvectors of block-circulant matrices of order p.m can be found by computing the eigenvalues and the eigenvectors of p sub-matrices of order m.
(J. Ponstein: Splitting certain eigenvalue eigenvector problems, Numer. Math. 8, 412 (1966))

11. Unitary test matrices $U^{(n)}$

The complex matrices U which satisfy the condition

$$U \cdot U^* = U^* \cdot U = I \qquad (* \text{ denotes conjugate transpose})$$

are called unitary.
Any eigenvalue of U has absolute value 1.

Test matrix $U_1^{(n)}$

Let $r_k = e^{\frac{2\pi k}{n} i}$ $(k = 1,2,\ldots,n-1)$

The matrix

$$U_1^{(n)} = \frac{1}{\sqrt{n}} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & r_1 & \cdots & r_1^{n-1} \\ 1 & r_2 & \cdots & r_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & r_{n-1} & \cdots & r_{n-1}^{n-1} \end{vmatrix}$$

is unitary.

Test matrix $U_2^{(n)}$

The nxn real matrix $U_2^{(n)}$ with elements

$$u_{ij} = \sqrt{\frac{2}{n+1}}\ \sin\ \frac{ij\pi}{n+1}$$

has $\left[\frac{n}{2}\right]$ eigenvalues equal to -1 and $(n - \left[\frac{n}{2}\right])$ eigenvalues equal to +1.

12. Test matrix $D^{(n)}$ ([20], page 74)

The skew-symmetric matrix

$$D^{(n)} = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & 0 & 1 & \dots & 1 \\ -1 & -1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & 0 \end{vmatrix} \qquad (\text{order of } D^{(n)} = n)$$

has the following eigenvalues:

$$\lambda_k(D^{(n)}) = -i\ \text{cotg}\ ((2k-1)\ \frac{\pi}{2n}) \qquad (k=1,2,\dots n).$$

§3 TRIDIAGONAL TEST-MATRICES

Three orthogonal polynomials with consecutive indices are related by a recursion relation of the form:

$$P_n(x) = (x-\beta_n)P_{n-1}(x) - \gamma_n\ P_{n-2}(x) \qquad n = 2,3\dots$$

$$P_o(x) = 1$$

$$P_1(x) = x-\beta_1$$

The polynomial $P_n(x)$ can be expressed as the determinant

$$P_n(x) = (-1)^n\ \det(J^{(n)}-xI)$$

where:

$$J^{(n)} = \begin{vmatrix} \beta_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ \gamma_2 & \beta_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & \gamma_3 & \beta_3 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & & \gamma_n & \beta_n \end{vmatrix}$$

Thus the eigenvalues of the tridiagonal matrix $J^{(n)}$ are the zeros of the orthogonal polynomial $P_n(x)$. There are extensive tables of the zeros of the orthogonal polynomials [21] so that we can consider $J^{(n)}$ and anypolynomial $P(J^{(n)})$ as a test matrix. The inverse $(J^{(n)})^{-1}$ of some tridiagonal matrices $J^{(n)}$ are easy to construct; thus also these matrices have been used for testing purposes.

Test matrix $J^{(n)}_{\alpha,\beta}$ [22]

$$J^{(n)}_{\alpha,\beta} = \begin{vmatrix} 2+\alpha & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 2+\beta \end{vmatrix}$$

For $|\alpha| < 1$ and $|\beta| < 1$, the eigenvalues of $J^{(n)}_{\alpha,\beta}$ are:

$$\lambda_k(J^{(n)}_{\alpha\,\beta}) = 2(1-\cos\theta_k) \qquad (k = 1,2,\dots,n)$$

where the θ_k are the n distinct roots of equation

$$\sin(n+1)\theta + (\alpha+\beta)\sin n\theta + \alpha\beta\sin(n-1)\theta = 0$$

in the range $0 < \theta < \pi$.

In particular ($k = 1,2,\dots,n$):

$$\lambda_k(J^{(n)}_{0,1}) = \lambda_k(J^{(n)}_{1,0}) = 2(1-\cos\frac{2k\pi}{2n+1})$$

$$\lambda_k(J^{(n)}_{0,-1}) = \lambda_k(J^{(n)}_{-1,0}) = 2(1-\cos\frac{(2k-1)\pi}{2n+1})$$

$$\lambda_k(J^{(n)}_{1,1}) = 2(1-\cos\frac{k\pi}{n})$$

$$\lambda_k(J^{(n)}_{-1,-1}) = 2(1-\cos\frac{(k-1)\pi}{n})$$

$$\lambda_k(J_{0,0}^{(n)}) = 2(1-\cos\frac{k\pi}{n+1}) = 4\sin^2(\frac{k\pi}{2(n+1)})$$

The eigenvectors of $J_{0,0}^{(n)}$ are $v_{ik} = \sqrt{\frac{2}{n+1}}\sin(\frac{ki\pi}{n+1})$ $(i,k=1,2,\ldots,n)$. The determinant of $J_{0,0}^{(n)}$ is equal to n+1. The test matrices $(J_{0,0}^{(n)})$ and $(J_{0,0}^{(n)})^3$ have the form:

$$(J_{0,0}^{(n)})^2 = \left|\begin{array}{cccccccc} 5 & -4 & 1 & & & & & \\ -4 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & & 1 & -4 & 6 & -4 & 1 \\ & & & & 1 & -4 & \ddots\ 6 & -4 \\ & & & & & 1 & -4 & 5 \end{array}\right|$$

$$(J_{0,0}^{(n)})^3 = \left|\begin{array}{cccccccccc} +14 & -14 & +6 & -1 & & & & & & \\ -14 & +20 & -15 & +6 & -1 & & & & & \\ +6 & -15 & +20 & -15 & +6 & -1 & & & & \\ -1 & +6 & -15 & +20 & -15 & +6 & -1 & & & \\ & -1 & +6 & -15 & +20 & -15 & +6 & -1 & & \\ & & & & \ddots & & \ddots & & & \\ & & & -1 & +6 & -15 & +20 & -15 & +6 & -1 \\ & & & & -1 & +6 & -15 & +20 & -15 & +6 \\ & & & & & -1 & +6 & -15 & +20 & -14 \\ & & & & & & -1 & +6 & -14 & +14 \end{array}\right|$$

The eigenvalues of $(J_{0,0}^{(n)})^2$ and $(J_{0,0}^{(n)})^3$ are $\lambda_k^2(J_{0,0}^{(n)})$ and $\lambda_k^3(J_{0,0}^{(n)})$, respectively, and their eigenvectors are the same eigenvectors as $J_{0,0}^{(n)}$.

<u>Test matrix $J_{a,b,c}^{(n)}$</u>

$$J_{a,b,c}^{(n)} = \left|\begin{array}{ccccccc} a & b & 0 & 0 & \ldots & 0 & 0 \\ c & a & b & 0 & \ldots & 0 & 0 \\ 0 & c & a & b & \ldots & 0 & 0 \\ & & & & & & \\ 0 & 0 & 0 & 0 & \ldots & a & b \\ 0 & 0 & 0 & 0 & \ldots & c & a \end{array}\right| \qquad (\text{order of } J_{a,b,c}^{(n)} = n)$$

The eigenvalues of $J^{(n)}_{a,b,c}$ are (bc > 0) :

$$\lambda_k(J^{(n)}_{a,b,c}) = a-2\sqrt{b.c}\,\cos\left(\frac{k\pi}{n+1}\right) \qquad (k=1,2,\dots,n)$$

Test matrix $\bar{J}^{(n)}_{p,q,r}$ (Rutherford, Todd)

$$\bar{J}^{(n)}_{p,q,r} = \begin{vmatrix} p-r & 2q & r & & & & \\ 2q & p & 2q & r & & & \\ r & 2q & p & 2q & r & & \\ & & \ddots & \ddots & \ddots & & \\ & r & 2q & p & 2q & r & \\ & & r & 2q & p & 2q & \\ & & & r & 2q & p-r & \end{vmatrix} \qquad (\text{order of } \bar{J}^{(n)}_{p,q,r} = n)$$

The eigenvalues of $\bar{J}^{(n)}_{p,q,r}$ are: (k = 1,2,...,n)

$$\lambda_k(\bar{J}^{(n)}_{p,q,r}) = (p-2r) - \frac{1}{r}\left(q^2 - (q-2r\cos k\theta)^2\right) \qquad \theta = \frac{\pi}{n+1}$$

The result has been obtained by relating $\bar{J}^{(n)}_{p,q,r}$ to $(J^{(n)}_{a,b,c})^2$.

The elements a_{ij} of the inverse matrix of $J^{(n)}_{2,-1,-1}$ are given by:

$$a_{ij} = \begin{cases} \dfrac{i(n-i+1)}{n+1} & \text{for } i=j \\ a_{ij-1} - i/(n+1) & \text{for } j>i \\ a_{ji} = a_{ij} & \text{for } j<i \end{cases}$$

and n is the order of $J^{(n)}_{2,-1,-1}$.

The elements a_{ij} of the inverse matrix of $J^{(n)}_{2,1,-1}$ are given by:

$$a_{ij} = \min(i,j) - \frac{1}{2}$$

Test matrix $K^{(n)}$

The inverse matrix of $J^{(n)}_{-1,0}$ has the form:

$$K^{(n)} = \begin{vmatrix} n & n-1 & n-2 & \dots & 2 & 1 \\ n-1 & n-1 & n-2 & \dots & 2 & 1 \\ n-2 & n-2 & n-2 & \dots & 2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{vmatrix}$$

The eigenvalues of $K^{(n)}$ are:

$$\mu_k = \lambda_k(K^{(n)}) = \left(2-2\cos\frac{2k-1}{2n+1}\pi\right)^{-1} \qquad (k=1,2,\dots,n)$$

In [13] are given the values of $\lambda_k(K^{(n)})$ for n = 12

μ_1 = 0.25398978	μ_7 = 0.61529474
μ_2 = 0.26648096	μ_8 = 0.87074533
μ_3 = 0.28918975	μ_9 = 1.3790212
μ_4 = 0.32555754	μ_{10} = 2.6180340
μ_5 = 0.38196601	μ_{11} = 7.1201222
μ_6 = 0.47045960	μ_{12} =63.409139

Test matrix $J1^{(n)}_{\alpha,\beta}$

The tridiagonal $J1^{(n)}_{\alpha,\beta}$ is related to the Jacobi polynomials.

$$J1^{(n)}_{\alpha,\beta} = \left\{ \frac{2(K+\alpha-1)(K+\beta-1)}{(2K+\alpha+\beta-1)(2K+\alpha+\beta-2)} \; ; \; -\frac{\alpha^2-\beta^2}{(2K+\alpha+\beta)(2K+\alpha+\beta-2)} \; ; \; \frac{2K(K+\alpha+\beta)}{(2K+\alpha+\beta)(2K+\alpha+\beta-1)} \right\}$$

(K = 1,2,...,n)

The eigenvalues of $J1^{(n)}_{\alpha,\beta}$ are the zeros of the Jacobi polynomials of order n. ([21], page 164 for $\alpha=\beta=1$; page 167 for $\alpha=\beta=3/2$; page 174 for $\alpha=0$ $\beta=1,2,3,4$)

Text matrix $J_1^{(n)}$

The test matrix $J_1^{(n)}$ is a particular case of the matrix $J1_{\alpha,\beta}^{(n)}$ for $\alpha=\beta=0$. ($J_1^{(n)} \equiv J1_{0,0}^{(n)}$).

$$J_1^{(n)} = \begin{vmatrix} 0 & 1 & & & & \\ 1/3 & 0 & 2/3 & & & \\ & 2/5 & 0 & 3/5 & & \\ & & 3/7 & 0 & 4/7 & \\ & & & \ddots & \ddots & \ddots \\ & & & & (\frac{n-1}{2n-1}) & 0 \end{vmatrix} \qquad \text{(order of } J_1^{(n)} = n)$$

The eigenvalues of $J_1^{(n)}$ are the zeros of the Legendre polynomials of order n ([21], page 100)

Test matrix $J_2^{(n)}$

The test matrix $J_2^{(n)}$ is a particular case of the matrix $J1_{\alpha,\beta}^{(n)}$ for $\alpha=\beta= -1/2$ ($J_2^{(n)} \equiv J1_{-1/2,-1/2}^{(n)}$)

$$J_2^{(n)} = \frac{1}{2} \begin{vmatrix} 0 & 2 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & \ddots & \ddots \\ & & & 1 & 0 \end{vmatrix} \qquad \text{(order of } J_2^{(n)} = n)$$

The eigenvalues of $J_2^{(n)}$ are the zeros of the Thebychev polynomials of first kind:

$$\lambda_k(J_2^{(n)}) = \cos(K-1/2)\frac{\pi}{n} \quad (K = 1,2,\dots,n)$$

([21], page 158)

Test matrix $J_3^{(n)}$

The test matrix $J_3^{(n)}$ is a particular case of the matrix $J1_{\alpha,\beta}^{(n)}$ for $\alpha=\beta=1/2$.

$$J_3^{(n)} = \frac{1}{2} \begin{vmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 \ddots & 1 \\ & & & 1 & \ddots 0 \end{vmatrix} \qquad (\text{order of } J_3^{(n)} = n)$$

The eigenvalues of $J_3^{(n)}$ are the zeros of the Thebychev polynomials of second kind:

$$\lambda_k(J_3^{(n)}) = \cos \frac{k\pi}{n+1} \qquad (K = 1,2,\dots,n)$$

([24], page 161)

Test matrix $J2_{\alpha}^{(n)}$

The test matrix $J2_{\alpha}^{(n)}$ is related to the gemeralized Laguerre polynomials.

$$J2_{\alpha}^{(n)} = \left\{ -(K+\alpha-1); (2K+\alpha-1); -K \right\}$$

$(K = 1,2,\dots,n)$

The eigenvalues of $J2_{\alpha}^{(n)}$ are the zeros of the generalized Laguerre polynomials.

Test matrix $J_4^{(n)}$

The test matrix $J_4^{(n)}$ is a particular case of the matrix $J2_{\alpha}^{(n)}$ for $\alpha=0$.

$$J_4^{(n)} = \begin{vmatrix} 1 & -1 & & & & \\ -1 & 3 & -2 & & & \\ & -2 & 5 \ddots & -3 & & \\ & & \ddots & -(n-2) & (2n-3) & -(n-1) \\ & & & & -(n-1) & (2n-1) \end{vmatrix}$$

The eigenvalues of $J_4^{(n)}$ are the zeros of the Laguerre polynomials of order n ([21], page 254).

Test matrix $J_5^{(n)}$

$$J_5^{(n)} = \begin{vmatrix} 0 & 1/2 & & & & \\ 1 & 0 & 1/2 & & & \\ & 2 & 0 & 1/2 & & \\ & & & \ddots & & \\ & & & (n-2) & 0 & 1/2 \\ & & & & (n-1) & 0 \end{vmatrix}$$

The eigenvalues of $J_5^{(n)}$ are the zeros of the Hermite polynomials of order n ([21], page 218).

Test matrix $J3_\alpha^{(n)}$

The tridiagonal matrix $J3_\alpha^{(n)}$ is related to the Lamé polynomials.

$$J3_\alpha^{(n)} \equiv (a_{ij}) \quad i,j = 1,2,\ldots,n$$

$$\begin{cases} a_{ii} = 4(i-1)^2(1+\alpha) \\ a_{ii+1} = 2i(2i-1) \\ a_{i+1i} = -(2n-2i+4)(2n+2i-3)\alpha \\ a_{ij} = 0 & |i-j| > 1 \end{cases}$$

Test matrix J_6 [23]

The test matrix J_6 is a particular case of the matrix $J3_\alpha^{(n)}$ for n=13 and $\alpha = 0.9$. The non zero elements and the eigenvalues of J_6 are:

$a_{i+1\ i}$	a_{ii}	$a_{i\ i+1}$	λ_i
	0	2	22.7677122
-540	7.6	12	110.037603
-534.6	30.4	30	189.702991
-522	68.4	56	261.758027
-502.2	121.6	90	326.192938
-475.2	190	132	382.990354
-441	273.6	182	432.116798
-399.6	372,4	240	473.500040
-351	486.4	306	506.948122
-295.2	615.6	380	531.252512
-232.2	760	462	545.029856
-162	919.6	552	565.168267
-84.6	1094.4		592.534780

§4. TEST-MATRICES GENERATED BY KRONECKER OPERATIONS

Let $A = (a_{ij})$ and $B = (b_{kl})$ denote NxN and MxM matrices, respectively. The Kronecker product (tensor product, direct product) of A and B, denoted by $A \otimes B$, can be written as N.MxN.M matrix in block partition form:

$$A \otimes B = \begin{vmatrix} a_{11}B & a_{12}B & \dots & a_{1N}B \\ a_{21}B & a_{22}B & \dots & a_{2N}B \\ \dots & \dots & \dots & \dots \\ a_{N1}B & a_{N2}B & \dots & a_{NN}B \end{vmatrix}$$

If V and W are eigenvectors of A and B with eigenvalues λ and μ, respectively, then $V \otimes W$ is an eigenvector of $A \otimes B$ with eigenvalue $\lambda . \mu$.

The Kronecker sum of A and B, denoted by $A \oplus B$, can be written as a (N+M)x(N+M) matrix in block partition form:

$$A \oplus B = \begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}$$

The matrix $A \oplus B$ has the eigenvalues of A and of B.
Thus the matrix $A_1 \oplus A_2 \oplus \dots \oplus A_m$ has the eigenvalues of A_1 and of $A_2, \dots,$

and of A_m.
The matrix $A_1 \oplus A_2 \oplus \dots \oplus A_m$ and the matrix

$$\begin{vmatrix} A_1 & A_{12} & A_{13} & \dots & A_{1m} \\ 0 & A_2 & A_{23} & \dots & A_{2m} \\ 0 & 0 & A_3 & \dots & A_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & A_m \end{vmatrix}$$

have the same eigenvalues.
The N.M eigenvalues of the matrix $A \otimes I_M + I_N \otimes B$ are $\lambda_i + \mu_j$ $(i=1,2,\dots,N; j=1,2,\dots,M)$.

<u>Test matrix TP_1</u> [24]

$$TP_1 = \begin{vmatrix} A & 2A \\ 4A & 3A \end{vmatrix} \qquad A = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 10^{-5} & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\lambda_k(TP_1) = 0.5\, e^{\frac{2\pi k}{5} i}$$

$$\lambda_{k+5}(TP_1) = 0.1\, e^{\frac{2\pi k}{5} i} \qquad k = 1,2,3,4,5$$

<u>Test matrix TP_2</u> [8]

$$TP_2 = \begin{vmatrix} A & 2A \\ 4A & 3A \end{vmatrix} \qquad A = \begin{vmatrix} 5B & -B \\ 5B & B \end{vmatrix}$$

$$B = \begin{vmatrix} -2 & 2 & 2 & 2 \\ -3 & 3 & 2 & 2 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 5 \end{vmatrix}$$

$$\lambda_k(TP_2) = 15 \pm 5i,\ -3 \pm i,\ 45 \pm 15i,\ 60 \pm 20i,\ 30 \pm 10i,\ -6 \pm 2i,\ -9 \pm 3i, -12 \pm 4i.$$

Test matrix TP_3 [8]

$$TP_3 = \begin{vmatrix} 8A & 4A \\ -5A & -A \end{vmatrix} \qquad A = \begin{vmatrix} B & 2B \\ 4B & 3B \end{vmatrix}$$

$$B = \begin{vmatrix} 4C & 3C \\ -2C & -C \end{vmatrix} \qquad C = \begin{vmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$\alpha = 1 \pm \sqrt{-3}$$

$$\lambda_k(TP_3) = 3, 6, -15, -30, \pm 1.5\alpha,\ \pm 3\alpha,\ \pm 7.5\alpha,\ \pm 15\alpha,$$
$$4, 8, -20, -40, \pm\ 2\alpha,\ \pm 10\alpha,\ \pm 20\alpha$$

Test matrix TP_4 [25]

$$TP_4 = \begin{vmatrix} A & 2A \\ 4A & 3A \end{vmatrix} \qquad A = \begin{vmatrix} 3B & 3B \\ 5B & B \end{vmatrix}$$

$$B = \begin{vmatrix} 6C & -C & C & 0 \\ 8C & 0 & C & 2C \\ -2C & 0 & C & 2C \\ 5C & -C & -C & C \end{vmatrix} \qquad C = \begin{vmatrix} -2 & 2 & 2 & 2 \\ -3 & 3 & 2 & 2 \\ -2 & 0 & 4 & 2 \\ -1 & 0 & 0 & 5 \end{vmatrix}$$

$$\alpha = 3 \pm \sqrt{-1}\ , \qquad \beta = 1 \pm 2\sqrt{-1}\ , \qquad \gamma = \alpha, \beta$$

$$\lambda_k(TP_4) = 120\gamma,\ -40\gamma,\ -24\gamma,\ 8\gamma,\ 90\gamma,\ -30\gamma,\ -18\gamma,\ 6\gamma,\ 60\gamma,\ -20\gamma,$$
$$-12\gamma,\ 4\gamma,\ 30\gamma,\ -10\gamma,\ -6\gamma,\ 2\gamma$$

§5 TEST-MATRICES GENERATED BY SIMILARITY TRANSFORMATIONS

We summarize the method of J.M. Ortega [26] for obtaining test matrices with a prescribed distribution of the eigenvalues. The matrices generated also have known eigenvectors. The eigenvalues of the matrix R are known, then the test matrix

$$S = R + uv^{*}R - \alpha Ruv^{*} - \alpha(v^{*}Ru)uv^{*}$$

where u and v are vectors (* denotes conjugate transpose)

$$\alpha = \frac{1}{1+v^{*}u}$$

has the same eigenvalues as R. The test matrix S is generated with $O(n^2)$ operations.

For testing accuracy of routines, the matrix S must be generated exactly. Some special choices of u,v,R facilitate the computation of the test matrix S.

Symmetric test matrices

Let $\sum_{i=1}^{n} v_i^2 = 1$, $u = -2v$ and $R = \text{diag}(d_1 d_2 \ldots d_n)$.

Then

$$S = D-2vv^{T}D-2Dvv^{T} + 4(v^{T}Dv)vv^{T}$$

is an nxn symmetric matrix with eigenvalues $d_1, d_2, \ldots d_n$ and eigenvectors which are the columns of $I-2vv^{T}$.

In particular, if $v^{T} = (n^{-1/2} n^{-1/2} \ldots n^{-1/2})$ then

$$S = \frac{1}{n^2}\left[n^2 d_i \delta_{ij} - 2n(d_i+d_j) + 4\left(\sum_{k=1}^{n} d_k\right)\right]$$

(δ_{ij} is the Kronecker symbol).

Real and complex test matrices

Let $R = \text{diag}(d_1 d_2 \ldots d_n)$, $n = 2m$, $u^{T} = (1\ 1 \ldots 1)$ and $v^{T} = (1\ 1 \ldots 1\ {-1} \ldots {-1})$, then the elements s_{ij} of the test matrix S are:

$$s_{ij} = \begin{cases} d_i\delta_{ij} - (d_i - d_j + \sigma) & 1 \leqslant j \leqslant m \\ d_i\delta_{ij} + (d_i - d_j + \sigma) & m+1 \leqslant j \leqslant n \end{cases}$$

with

$$\sigma = \sum_{k=1} d_k - \sum_{k=m+1}^{n} d_k$$

The matrix S is non symmetric with eigenvalues $d_1\ d_2 \ldots d_n$ and eigenvectors which are parallel to the columns of $I+uv^T$.

If $d_i (1 \leqslant i \leqslant n)$ is a real number the matrix S is a real test matrix (with real eigenvalues). If $d_i (1 \leqslant i \leqslant n)$ is a complex number the matrix S is a complex test matrix.

CHAPTER II

THE CONDITION NUMBERS OF THE ALGEBRAIC EIGENPROBLEM

INTRODUCTION

Any computing problem is <u>ill-conditioned</u> if the values to be computed are very sensitive to small changes in the data. A matrix may have some eigenvalues which are very sensitive to perturbations in its elements while others are insensitive. Similarly some of the eigenvectors may be ill-conditioned while others are well-conditioned. Besides an eigenvector may be ill-conditioned when the corresponding eigenvalue is not.

It is convenient to have some number which defines the condition of a matrix with respect to the eigenproblem and to call such a number the "<u>spectral condition number</u>"

It is evident that such a single number would have severe limitations. Indeed if any one of the eigenvalues were very sensitive, then the "spectral condition number" would have to be large, even if some other eigenvalues were very insensitive.

A compromise is provided by introducing numbers which govern the sensitivity of the individual eigenvalues and which are called the "<u>condition numbers of the matrix</u>" with respect to the eigenvalue problem.

Some relationships between these condition numbers are given. Besides these numbers are related by inequalities to the "departure from normality", the "discriminant" of the eigenvalues and the Gram-determinant of the eigenvectors of the original matrix.
Finally the ill-condition of the eigenvectors of a matrix is discussed. The main result is that an eigenvector of a symmetric matrix (which is well conditioned with respect to the eigenvalue problem) is poorly conditioned if its eigenvalue is close to the remaining eigenvalues.
When an approximate eigensystem of a matrix has been computed, it is useful to have some procedure which will give a-posteriori bounds for its errors. In §6 we summarize some results.
In this report we use the following notations.
The norms of a vector x ($x^T \equiv (x_1 x_2 \ldots x_n)$) are defined by

$$|x|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p} \qquad (p = 1,2,\infty)$$

where $|x|_\infty$ is interpreted as $\max\limits_i |x_i|$.
The norm $|x|_2$ is the Euclidean lenght of the vector x.
The matrix norm subordinate to $|x|_p$ is denoted by $|A|_p$. ($A = (a_{ij})_{i,j=1}^n$).

$$|A|_1 = \max_j \sum_{i=1}^n |a_{ij}|$$

$$|A|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$$

$$|A|_2 = \max_{x \neq 0} \frac{|Ax|_2}{|x|_2} = \sqrt{\rho(A^* A)}$$

where $\rho(B)$ is the spectral radius of the nxn matrix B with eigenvalues $\lambda_1 \lambda_2 \ldots \lambda_n$. ($\rho(B) = \max\limits_i |\lambda_i|$).
The matrix A^* is defined by $A^* = (\bar{A})^T$ where $\bar{A}$ denotes the matrix whose elements are the complex conjugate of those of A and B^T denotes the transpose matrix of B. There is a second important norm which is compatible with the vector norm $|x|_2$. This is the so-called Euclidean or Schur or Frobenius norm and it is denoted by $|A|_E$. It is defined by

$$|A|_E = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\right)^{1/2}$$

§1 SPECTRAL CONDITION NUMBER

We consider the "spectral condition number" of a nxn matrix A with respect to its eigenproblem when A has linear elementary divisors.
In this case there is a non-singular matrix H such that

$$H^{-1}AH = \mathrm{diag}(\lambda_i)$$

having its columns parallel to a complete set of right-eigenvectors of A and such that H^{-1} has its rows parallel **to** a complete set of left-eigenvectors **of** A. λ_i is the ith eigenvalue of A. Normalized right and left eigenvectors corresponding to λ_i are given by:

$$x_i = \frac{He_i}{|He_i|_2} \qquad\qquad y_i = \frac{(H^{-1})^T e_i}{|(H^{-1})^T e_i|_2}$$

where e_i is the ith column of the identity matrix I.
The number

$$\boxed{k(H) = |H^{-1}|_2 |H|_2}$$

defines the "<u>spectral condition number</u>" of A.
The overall sensitivity of the eigenvalues of A is dependent on the magnitude of k(H) since the following theorem holds ([1], page 87 ; [2])
<u>Theorem</u> (Bauer-Fike)
Let the matrix A be of order n and have n linearly independent eigenvectors with eigenvalues λ_i $(1 \leqslant i \leqslant n)$. For any fixed matrix B, define the perturbed matrix $A(\varepsilon) = A+\varepsilon B$. Then each eigenvalue $\lambda(\varepsilon)$ of $A(\varepsilon)$ satisfies

$$\min_{1 \leqslant i \leqslant n} |\lambda(\varepsilon)-\lambda_i| \leqslant |\varepsilon| \; |B|_p |H^{-1}|_p |H|_p$$

for any p-norm, with $p = 1,2,\infty$.

Thus the eigenvalue problem of the matrix A is "ill-conditioned" if k(H) is "large" (with respect to 1).
When A is an hermitian matrix ($A = A^*$) the "spectral condition number" k(H) = 1; thus the eigenvalue problem of hermitian matrix is well-conditioned.

More precisely the following theorem holds:
If $A(\varepsilon) = A+\varepsilon B$, where A and B and $A(\varepsilon)$ are hermitian matrices having the eigenvalues λ_i, μ_i and $\lambda_i(\varepsilon)$ arranged in non-increasing order, then

$$|\lambda_i(\varepsilon)-\lambda_i| \leq |\varepsilon| \ |B|_2 \quad (\leq |\varepsilon| \ (\sum_{i=1}^{n} \mu_i^2)^{1/2})$$

An useful bound for the matrix $(\Lambda(\varepsilon)-\Lambda)$, where $\Lambda(\varepsilon) = \mathrm{diag}(\lambda_i(\varepsilon))$ and $\Lambda = \mathrm{diag}(\lambda_i)$, when A and B are hermitian matrices is given by the following theorem ([1],page 104)
Theorem (Wielandt-Hoffman)
If $A(\varepsilon) = A+\varepsilon B$, where A, B and $A(\varepsilon)$ are hermitian matrices having the eigenvalues λ_i, μ_i and $\lambda_i(\varepsilon)$ arranged in non-increasing order, than

$$|\Lambda(\varepsilon) - \Lambda|_E \leqslant |\varepsilon|\cdot|B|_E \ (=|\varepsilon| \ (\sum_{i=1}^{n} \mu_i^2)^{1/2}) \quad .$$

A consequence of the Courant-Fischer characterisation of the eigenvalues of Hermitian matrices gives the following theorem:
If $C = A+B$ where A,B and C are nxn hermitian matrices having the eigenvalues α_i, β_i and γ_i arranged in non increasing order, then

$$\alpha_i+\beta_n \leqslant \gamma_i \leqslant \alpha_i + \beta_1$$

and

$$\sum_i (\gamma_i-\alpha_i)^2 \leqslant \sum_i \beta_i^2 = |B|_E^2$$

§2 THE CONDITION NUMBERS OF THE MATRIX (WITH RESPECT TO THE EIGENVALUE PROBLEM)

We introduce a condition number which serves as a measure of the effect of the perturbation of on each eigenvalue of A.
The matrix A has linear elementary divisors; thus $H^{-1}AH = \mathrm{diag}(\lambda_i)$, where the columns of H are parallel to a complete set of the right-eigenvectors of A and the rows of H^{-1} are parallel to a complete set of left-eigenvectors of A.

$$\boxed{q_i = \frac{1}{y_i^T x_i}} \qquad i = 1,2,\dots n$$

define the "condition numbers of A" with respect to the eigenvalues λ_i.
It is:

$$q_i = \frac{|(H^{-1})^T e_i|_2 \ |He_i|_2}{((H^{-1})^T e_i)^T (He_i)} = |(H^{-1})^T e_i|_2 \ |He_i|_2$$

We may take the ith column of H to be x_i and the ith row of H^{-1} to be $q_i y_i^T$.
With H in this form we have:

$$H^{-1}(A+\varepsilon B)H = \mathrm{diag}(\lambda_i) + \varepsilon(q_i(y_i^T Bx_j))_{ij=1}^n$$

$$= \mathrm{diag}(\lambda_i) + \varepsilon(q_i \beta_{ij})_{ij=1}^n$$

where $\beta_{ij} = y_i^T Bx_j$.
An application of the Gerschgorin's theorem shows that the eigenvalues of $(A+\varepsilon B)$ lie in circular discs with centres $(\lambda_i+\varepsilon\ \beta_{ii} q_i)$ and radii $\sum_{j\neq i} |\varepsilon\ (q_i\ \beta_{ij})|$. If $|b_{ij}| < 1$, since $|B|_2 \leqslant |B|_E \leqslant n$, we have

$$|\beta_{ij}| \leqslant |y_i^T|_2 \ |Bx_j|_2 \leqslant |B|_2 \ |y_i^T|_2 \ |x_j|_2 \leqslant n .$$

Thus the ith disc is of radius less than $n(n-1)|\varepsilon\ \ q_i|$.
If λ_1 is a simple eigenvalue of A, for sufficiently small ε he first disc is isolated and therefore contains precisely one eigenvalue.
If λ_1 is a multiple eigenvalue of A with multiplicity m, there are m discs with centres $\lambda_1+\varepsilon\ q_i\beta_{ii}$ (i=1,2,...,m) whose corresponding radii are all of order ε. For sufficiently small ε, this group of m discs will be isolated from the other discs and in their union there are m eigenvalues of $A+\varepsilon B$.
When $|q_i|$ is "large" (with respect to 1), the eigenvalue problem for finding λ_i of the matrix A is <u>ill-conditioned</u>.
When A is an hermitian matrix, we have $|q_i| = 1$ for i = 1,2,...,n.

§3 PROPERTIES OF CONDITION NUMBERS

Some relationships between the numbers k(H) and q_i are ([1], page 88):

$$1 \leqslant |q_i| \leqslant k(H)$$

$$1 \leqslant k(H) \leqslant \sum_{j=1}^{n} |q_j|$$

It is ([1], page 56):

$$K(H) = \frac{\sigma_{max}(H)}{\sigma_{min}(H)}$$

where $\sigma_{min}(H)$ and $\sigma_{max}(H)$ are the least and the greatest of the "singular values" $\sigma_i(H) = \sqrt{\lambda_i(H^*H)}$ and $\lambda_i(H^*H)$ are the eigenvalues of the matrix H^*H. The inequality of Kantorovich ([3], page 81) gives the result

$$|q_i| = \left| \frac{||He_i||_2 \cdot ||(H^{-1})^T e_i||_2}{((H^{-1})^T e_i)^T \cdot (He_i)} \right| = \frac{|e_i^T H^{-1}|_2 \cdot |He_i|_2}{|e_i^T \cdot e_i|}$$

$$= \frac{|e_i^T H^{-1}|_2 \cdot |He_i|_2}{|e_i^T|_2 \cdot |e_i|_2} \leqslant \frac{\sigma^2_{max}(H) + \sigma^2_{min}(H)}{2\sigma_{max}(H) \cdot \sigma_{min}(H)}$$

Thus:

$$|q_i| \leqslant \frac{1}{2} \, (k(H) + (k(H))^{-1})$$

The condition numbers k(H) and $|q_i|$ are related by inequalities to the "departure from normality" of A, the "discriminant" of the eigenvalues of A and the Gram-determinant of the eigenvectors of A. [4], [5]. If all eigenvalues of A are simple then

$$\sum_{j=1}^{n} |q_j| \leqslant n\left[\frac{1}{2} \, (k(H) + (k(H)^{-1})\right]$$

If λ_i is a simple eigenvalue of A, then

$$|q_i| = |adj(\lambda_i I - A)|_E \cdot \Pi_i |\lambda_i - \lambda_k|^{-1}$$

where Π_i denotes the product over all $k \neq i$ in $1 \leqslant k \leqslant n$ and adj C denotes the matrix $\{\tilde{c}_{ij}\}$ whose element $\tilde{c}_{ij}$ is the cofactor of the element in the ith row and jth column of the matrix C. (adjC = (det C).C^{-1}).

The "<u>departure from normality</u>" of A is defined by:

$$D = \left\{ |A|^2_E - \sum_{i=1}^{n} |\lambda_i|^2 \right\}^{1/2}$$

A bound for D is given in [6].

$$D^2 \leqslant \sqrt{\frac{n^3-n}{12}} \; |A^*A - AA^*|_E$$

Clearly D = 0 when A is a normal matrix.

If λ_i is a single eigenvalue of A then

$$|q_i| \leq \left\{1 + \frac{D^2}{(n-1)\delta_i^2}\right\}^{\frac{n-1}{2}}$$

where $\delta_i = \min |\lambda_i - \lambda_k|$ over all $k \neq i$ in $1 \leq k \leq n$.

An immediate inequality is:

$$|q_i| \leq \exp\left(\frac{D^2}{2\delta_i^2}\right)$$

If all eigenvalues of A are simple, then the discriminant $\Delta = \prod_{i \neq j} (\lambda_i - \lambda_j)^2$ of the characteristic polynomial $\det(\lambda I - A)$ is different from zero. We have the following theorem:

If $\Delta \neq 0$ then:

$$\frac{k(H)+(k(H))^{-1}}{2} \leq \frac{1}{\sqrt{\Delta}}\left\{\frac{2}{n-1}\left(|A|_E^2 - \frac{1}{n}|\text{trace } A|^2\right)\right\}^{\frac{n(n-1)}{4}}$$

When A has linear elementary divisors an understimate of k(H) is:

$$(k(H))^4 \geq 1 + \frac{1}{2}\left\{\frac{|A^*A - AA^*|_E}{|A^2|_E}\right\}^2$$

If all the eigenvalues of A are simple then:

$$\left(\frac{1}{\det(H^*H)}\right)^{\frac{1}{2-2n}} \leq \frac{k(H)+(k(H))^{-1}}{2} \leq \left(\frac{1}{\det(H^*H)}\right)^{1/2}$$

If A is a real matrix

$$\left(\frac{1}{D_1}\right)^{\frac{1}{n-1}} \leq (K(H))^2 \leq \frac{1+\sqrt{1-D_1}}{1-\sqrt{1-D_1}}$$

$$\left(\frac{1}{D_2}\right)^{\frac{1}{n-1}} \leq (K(H))^2 \leq \frac{1+\sqrt{1-D_2}}{1-\sqrt{1-D_2}}$$

$$\left(\frac{1}{D_1 D_2}\right)^{\frac{1}{2n}} \leq \frac{k(H)+(k(H))^{-1}}{2} \leq \frac{n}{2}\left(\frac{1}{D_1 D_2}\right)^{\frac{1}{2n}} + \left(1 - \frac{n}{2}\right)$$

where

$$D_1 = \frac{n^n \cdot \det(H^T H)}{[\mathrm{trace}(H^T H)]^n}$$

$$D_2 = \frac{n^n . [\det(H^T H)]^{n-1}}{[\det(H^T H).\mathrm{trace}((H^T H)^{-1})]^n} \cdot \left[\frac{\mathrm{trace}(H^T H)}{\det(H^T H)}\right]^n$$

The matrices A and RAR^*, where R is an unitary matrix, have the same condition numbers:

$$k(H) = k(RH)$$

$$q_i = q_i' \qquad (i = 1,2,\ldots,n)$$

where $q_i = y_i^T x_i$ and $q_i' = (\bar{R}y_i)^T(Rx_i)$

Hence the sensitivities of the eigenvalues are invariant under unitary transformations. However, in general, it is possible that the problem of finding λ_i, an eigenvalue of A, is <u>ill-conditioned</u>, although the problem of finding the same λ_i as an eigenvalue of $B = P^{-1}AP$ is <u>well-conditioned</u>. This fact is illustrated by the following example([7], page 146). Let be given

$$A = \begin{vmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \lambda_n \end{vmatrix} \qquad \lambda_i \neq \lambda_j \quad (i,j=1,2,\ldots n)$$

with the modal matrix

$$X = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & \delta & 0 & \ldots & 0 \\ 0 & 0 & \delta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \delta \end{vmatrix}$$

Then

$$X^{-1} = \begin{vmatrix} 1 & -1/\delta & -1/\delta & \ldots & -1/\delta \\ 0 & 1/\delta & 0 & \ldots & 0 \\ 0 & 0 & 1/\delta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & 1/\delta \end{vmatrix}$$

The "condition number" of A with respect to λ_1 is

$$|q_1| = \sqrt{1 + \frac{(n-1)}{\delta^2}}$$

which is a large number for small δ. Therefore, for small δ, the eigenvalue problem for finding λ_1 is ill-conditioned.
But the eigenvalue problem for finding the eigenvalue λ_i of the matrix $B = P^{-1}AP$ is well conditioned if $P = X$.
Indeed the right eigenvector and the left eigenvector corresponding to λ_1 of

$$B=P^{-1}AP \equiv \begin{vmatrix} \lambda_1 & \lambda_1-\lambda_2 & \lambda_1-\lambda_3 & \cdots & \lambda_1-\lambda_n \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{vmatrix}$$

are, respectively, $(1,0,0,\ldots,0)$ and $(1,1,1,\ldots,1)$. Thus the "condition number" of B with respect to λ_1 is $q_1' = \sqrt{n}$ which does not depend by δ.

§4 THE CONDITION NUMBERS OF PARTICULAR MATRICES

a) The eigenvalues corresponding to non-linear elementary divisors must be regarded, in general, as ill-conditioned. The following example, due to G.E. Forsythe, serves to illustrate this case. The nxn Forsythe matrix

$$A(\varepsilon) = \begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \\ \varepsilon & & 1 & 1 \end{vmatrix}$$

has characteristic polynomial $(\lambda-1)^n \pm \varepsilon = 0$. If $\varepsilon = 0$, all eigenvalues are unity while if $\varepsilon \neq 0$ the eigenvalues differ from unity by $|\varepsilon|^{1/n}$. Thus if $n = 10$ and $\varepsilon = 10^{-10}$, then $|\varepsilon|^{1/n} = 10^{-1}$ and a change of one element of the matrix has produced a change in the eigenvalues 10^9 times as large.

b) Even if the eigenvalues are distinct and well separated, they may be ill-conditioned. ([1], page 90) Consider the 20x20 matrix A defined by

$$A = \begin{vmatrix} 20 & 20 & & & & \\ & 19 & 20 & & & \\ & & 18 & 20 & & \\ & & & \ddots & \ddots & \\ & & & & 2 & 20 \\ & & & & & 1 \end{vmatrix}$$

The eigenvalues of A are $\lambda_k = k$ ($k = 1,2,\ldots,20$) and the right eigenvector x_k of A corresponding to $\lambda_k = k$ has the components:

$$\left[1;\ \frac{20-k}{(-20)};\ \frac{(20-k)(19-k)}{(-20)^2};\ \ldots;\ \frac{(20-k)!}{(-20)^{20-k}};\ 0;\ \ldots;\ 0\right]$$

the left eigenvector y_k of A has components $\left[0;0;\ldots;0;\ \frac{(k-1)!}{20^{k-1}};\ldots;\ \frac{(k-1)(k-2)}{20^2};\ \frac{(k-1)}{20};\ 1\right]$.

We have

$$|q_k| = \frac{1}{|y_k^T x_k|} = \frac{20^{19}}{(20-k)!(k-1)!} \qquad (k=1,2,\ldots,20)$$

which is a large number for all values of k.

c) The matrices of class $A^{(n)}$ considered in chapter 1 §2, 2) have eigenvalues of widely varying condition.
The largest eigenvalues of the matrix $A^{(n)}$ are well-conditioned and the smallest very ill-conditioned.
The following consideration makes evident that same of the eigenvalues of $A^{(n)}$ must be very sensitive to small changes in same matrix elements. It will readily be verified that the $\det A^{(n)} = 1$ for all n. If the (1,n) element of $A^{(n)}$ is changed to $(1+\varepsilon)$ the determinant becomes $1 \pm \varepsilon(n-1)!$. If $\varepsilon = 10^{-7}$ and n=18, the determinant is changed from 1 to $(1-17!10^{-7})$, that is, approximately $-3.55\ 10^7$. Now the determinant of a matrix is equal to the product of its eigenvalues. Therefore at least one eigenvalue of the perturbed matrix must be very different from that of the original. It can be shown that for n=12 the smaller eigenvalues of the perturbed matrix are changed beyond recognition while the larger ones are scarcely affected by the perturbation. With increasing values of n the smallest eigenvalues become progressively worse conditioned.

§5 ILL-CONDITIONING OF THE EIGENVECTOR OF A MATRIX

The eigenvalues of a matrix can be insensitive to small changes in the matrix; the same cannot true of the eigenvectors. In fact, the eigenvalues of a (real) symmetric matrix are well-conditioned while the eigenvectors need not even be continuous functions of the matrix elements. The following example is due to J.W. Givens. Let:

$$A = \begin{vmatrix} 1+\varepsilon \cos \frac{2}{\varepsilon} & -\varepsilon \sin \frac{2}{\varepsilon} \\ -\varepsilon \sin \frac{2}{\varepsilon} & 1-\varepsilon \cos \frac{2}{\varepsilon} \end{vmatrix}$$

then A has eigenvalues $1 \pm \varepsilon$ and eigenvectors $(\sin \frac{1}{\varepsilon}, \cos \frac{1}{\varepsilon})$, $(\sin \frac{1}{\varepsilon}, -\cos \frac{1}{\varepsilon})$ so that as $\varepsilon \to 0$, the eigenvectors do not tend to a limit. Thus arbitrarily small changes in the coefficients of A can change the eigenvectors completely. The sensitivity of an eigenvector of a (real) symmetric matrix is connected with the separation of its eigenvalue from the remaining eigenvalues. Indeed the following theorem holds ([8], page 101).

Theorem (J. Ortega)

Let A and A+E be symmetric with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ and corresponding normalized eigenvectors $u_1\ u_2\ \dots\ u_n$ and $v_1\ v_2\ \dots\ v_n$. Then if

$$|\lambda_k - \lambda_i| \geq \alpha > |E|_2 \qquad i \neq k$$

we have

$$|u_k - v_k|_2 \leq \gamma \sqrt{1+\gamma^2}$$

where

$$\gamma = \frac{|E|_2}{\alpha - |E|_2}$$

Thus if an eigenvalue is well separated, the above theorem shows that the corresponding eigenvector is well conditioned. For example, let $|E|_2 \leq n\ 10^{-8}$ and $\min_{i \neq k} |\lambda_k - \lambda_i| = 10^{-3} = \alpha$. Then $\gamma \approx n\ 10^{-5}$ is a bound on the eigenvector error.

If λ_k is a multiple eigenvalue, then the above theorem can be strengthened to show that the eigenspace of λ_k is well conditioned provided again that λ_k is well separated from its neighbors.

Now we study the perturbations in the eigenvectors of a matrix A which has distinct eigenvalues. We have therefore

$$H^{-1}A\,H = \text{diag}(\lambda_i)$$

and the columns of H form a complete set of eigenvectors $x_1\ x_2 \dots x_n$ of A. We denote the "corresponding eigenvectors of A+εB by $x_1(\varepsilon)\ x_2(\varepsilon)\dots x_n(\varepsilon)$, and the eigenvectors of $H^{-1}(A+\varepsilon B)H$ by $z_1(\varepsilon)\ z_2(\varepsilon)\dots z_n(\varepsilon)$ so that $x_i(\varepsilon)=H\ z_i(\varepsilon)$. We now consider a particular vector $z_j(\varepsilon)$. It is clear that the jth component of $z_j(0)$ is unity, and all the rest zero, since $x_j(0) = H\ z_j(0)$ and $x_j(0)$ is the jth column of H. We assume that for sufficiently small ε the ith component of $z_j(\varepsilon)$ is the largest, and we normalize so that this component is unity. Then the equation

$$\lambda_j(\varepsilon)\ z_j(\varepsilon) = H^{-1}(A+\varepsilon B)H\ z_j(\varepsilon) =$$

$$= (\text{diag}(\lambda_i) + \varepsilon(q_i\ \beta_{ij})_{ij=1}^{n})z_j(\varepsilon),$$

where β_{ij} has been defined in §3, gives for the kth component of $z_j(\varepsilon)$, with $k \neq j$, the result

$$\lambda_i(\varepsilon)z_{kj}(\varepsilon) = \lambda_k\ z_{kj}(\varepsilon) + \varepsilon q_k \sum_{h=1}^{n} (\ \beta_{kh}\ z_{hj}(\varepsilon))$$

Thus:

$$|\lambda_j(\varepsilon)-\lambda_k|\cdot|z_{kj}(\varepsilon)| \leqslant \varepsilon|q_k| \left\{\sum_{n=1}^{n} |\beta_{kh}|\right\}$$

Finally:

$$|z_{kj}(\varepsilon)| \leqslant \frac{\varepsilon|q_k| \sum_{n=1}^{n} |y_k^T\ Bx_n|}{|\lambda_j(\varepsilon)-\lambda_k|}$$

where

$$|q_k| = \frac{1}{|y_k^T\ x_k|}$$

§6 A POSTERIORI ERROR BOUNDS FOR THE EIGENVALUES AND THE EIGENVECTORS

When an approximate eigensystem of a matrix has been computed, it is useful to have some procedure which will give bounds for its errors. Here we summarize some results contained in ([7], page 140) and [9].

Hermitian matrices . Let λ and x be an approximate eigenvalue and the corresponding eigenvector with $|x|_2 = 1$ of the hermitian matrix A and let $\eta \equiv Ax-\lambda x$. Then there is an eigenvalue λ_i of A so that

$$|\lambda_i-\lambda| \leqslant |\eta|_2 = \varepsilon$$

If we compute the Rayleigh quotient

$$\mu = \frac{x^* Ax}{x^* x} = x^* Ax \text{ and if } |\lambda_j-\mu| \geqslant \alpha, \quad j \neq i,$$

then

$$|\mu-\lambda_i| \leqslant \frac{\varepsilon^2}{\alpha} \left(1 - \frac{\varepsilon^2}{\alpha^2}\right)^{-1}$$

so that if α is large compared to ε then μ is a better approximation than λ to λ_i. In this case the computation of μ may be considered a correction procedure. Besides, if u_i is a normalized eigenvector of λ_i, then

$$|x-u_i|_2 \leqslant \frac{\varepsilon^2}{\alpha^2} \sqrt{1 + \frac{\varepsilon^2}{\alpha^2}}$$

Non-hermitian matrices . For non-hermitian matrices it is not possible to obtain a-posteriori estimates for the error without being given some information about all of the eigenvectors of the matrix. An useful estimate for the eigenvalues of a general matrix is given by the following theorems.

Theorem (Franklin)

Let A be a matrix of order n, and have a set of n linearly independent eigenvectors $\{u_i\}$, and eigenvalues $\{\lambda_i\}$. Let λ and x, be an approximate eigenvalue and the corresponding eigenvector with $|x|_2 = 1$ of the matrix A. If for some $\varepsilon > 0$,

$$|Ax-\lambda x|_2 \leqslant \varepsilon |Ax|_2$$

then:

$$\min_{\lambda_j \neq 0} |1 - \frac{\lambda}{\lambda_j}| \leq \varepsilon\ |U|_2 \cdot\ |U^{-1}|_2$$

where U is the modal matrix which contains in the ith column the vector u_i.

The following theorem is a generalization of that given for hermitian matrices.

Theorem

Let A be a matrix of order n, and have a set of n linearly independent eigenvectors $\{u_i\}$ and eigenvalues $\{\lambda_i\}$. Let λ and x be an approximate eigenvalue and the corresponding eigenvector with $|x|_2 = 1$ of the matrix A. If for some $\varepsilon > 0$,

$$|Ax-\lambda x|_2 \leq \varepsilon |x|_2$$

then:

$$\min_i |\lambda_i-\lambda| \leq \varepsilon |U^{-1}|_2 \cdot |U|_2$$

where U is the modal matrix which contains in the ith column the vector u_i.

CHAPTER III

NUMERICAL EXPERIMENTS

INTRODUCTION

In this report we need the following quantities.

$\bar{\lambda}_i$ is the "true" ith eigenvalue of the given test matrix A.

λ_i is the ith computed eigenvalue of A.

X is the computed modal matrix.

The ith column of X contains the computed normalized eigenvector x_i corresponding to λ_i of A.

$\max_i |Ax_i-\lambda_i x_i|_E$ is the "maximum radius of indeterminacy" of the eigenvalues of A.

Let $\tilde{x}$ be a normalized eigenvector corresponding to the "true" eigenvalue $\bar{\lambda}$ of A.

Let x be a computed normalized eigenvector corresponding to the computed eigenvalue λ of A. Then $|x-\tilde{x}|$ is the "absolute error" of the vector x.

§1 SYMMETRIC MATRICES

1.1 Methods tested

For the solution of the eigenproblem of a real symmetric matrix the following methods are taken into account:

1) The Jacobi method,
2) The threshold Jacobi method,
3) The Givens-Householder method,
4) The Rutishauser method.

Four routines are selected from a scientific library which represent these different methods.

1) The Jacobi method: Routine HDIAG (Share program, SDA 705). This is the original version of the Jacobi method in which plane rotations are used to annihilate all off-diagonal elements of the matrix using the maximum off diagonal element as a pivot at each stage. The eigenvectors are obtained by computing the product of the plane rotations. ([1], page 266), [2].

2) The threshold Jacobi method : Routine EIGEN (System/360 Scientific Subroutine Package). This is a variation of the Jacobi method, in which plane rotations are used to annihilate, in a regular sequence, only those off-diagonal elements of the matrix which are greater than some preset value (threshold). When all elements are less than this preset value in absolute value, the threshold is lowered and the process continues until some final "tolerance" τ is satisfied. ([3], chapter 7).

3) The Givens-Householder method: Routine BIGM (Share program SDA 3202). With this method the eigenproblem is solved in three steps.
   a) A symmetric tridiagonal matrix similar to the original matrix is obtained by an orthogonal transformation which does not depend on

plane rotations(*) (Householder's reduction).

b) The eigenvalues of the original matrix are computed by the use of Sturm's sequence derived from the tridiagonal matrix (Givens' procedure).

c) The Wielandt inverse power method is used to calculate the eigenvectors of the tridiagonal matrix. Then the orthogonal transformations are applied in reverse order to obtain the eigenvectors of the original matrix ([1], page 290), ([4], chapter 4).

4) The Rutishauser method : routine LRCH 5 . This routine computes only the eigenvalues of a band-symmetric matrix with the LR transformation method. The method bases essentially on the fact, that by starting with the given matrix $A = A_o$, the decomposition of A_s into the product $L_s R_s$ and the reconbination of L_s and R_s by forming their product $A_{s+1} = R_s . L_s$ generate an infinite sequence of similar matrices $A_1, A_2, \ldots$ which under certain conditions converge to a diagonal matrix.

1.2 Description of the tests

A preliminary test of the above methods 1), 2), 3) is made with some test-matrices collected in chapter 1 §1 and §2.
In Tables 1 and 2 we give the results of this test. The eigenvalues and the eigenvectors are calculated in "single precision" on IBM 360/65 (floating point arithmetic). The input test matrices are given or constructed in "double precision".

In almost all these test matrices we have observed:

a) the "maximum radius of indeterminacy" is related to the eigenvalue of greatest modulus;

b) the "maximum relative error" is related to the eigenvalue of smallest modulus.

(*) If we apply plane rotations we have the Givens' reduction. Since Householder's reduction is about twice as fast as Givens' reduction and equally accurate, we consider only the Householder method.

In fig. 1 the behaviour of the "computation time" on IBM 360/65 vs. "order" of test matrix is given. In Table 2 (last column) the "computation time" for the Givens-Householder method (BIGM routine) is subdivided into "time for the reduction of the original matrix to tridiagonal form", "time for the eigenvalue calculation" and "time for the eigenvector calculation".

In order to check the performance of Jacobi method and the threshold Jacobi method with respect to the Givens-Householder method when applied to perturbated diagonal matrices, the following M_n matrices of order n are tested:

1) $M_{20} \equiv (m_{ij})$ with $m_{ii} = 2^i$

$$m_{ij} = 10^{-6} \qquad (i \neq j)$$

2) $M_{75} \equiv (m_{ij})$ with $m_{ii} = i$

$$m_{ij} = 10^{-6} \qquad (i \neq j)$$

The calculations are performed in "single precision" by the Jacobi method (HDIAG routine) and the threshold Jacobi method (EIGEN routine) and the Givens-Householder method (BIGM routine) on the IBM 360/65. The three methods calculate the eigenvalues with full machine accuracy.

For the matrix M_{20}, the routine HDIAG calculates 15 eigenvectors with "absolute error" less than 10^{-7} and 5 eigenvectors with "absolute error" less than 10^{-5}. The computation time is 1.5 sec (The total number of plane rotations in 190).

The routine EIGEN calculates 13 eigenvectors with "absolute error" less than 10^{-7} and 7 eigenvectors with "absolute error" less than 10^{-5}. The computation time is 1.5 sec.

The routine BIGM calculates 11 eigenvectors with "absolute error" less than 10^{-7} and 9 eigenvectors with "absolute error" less than 10^{-5}. The computation time

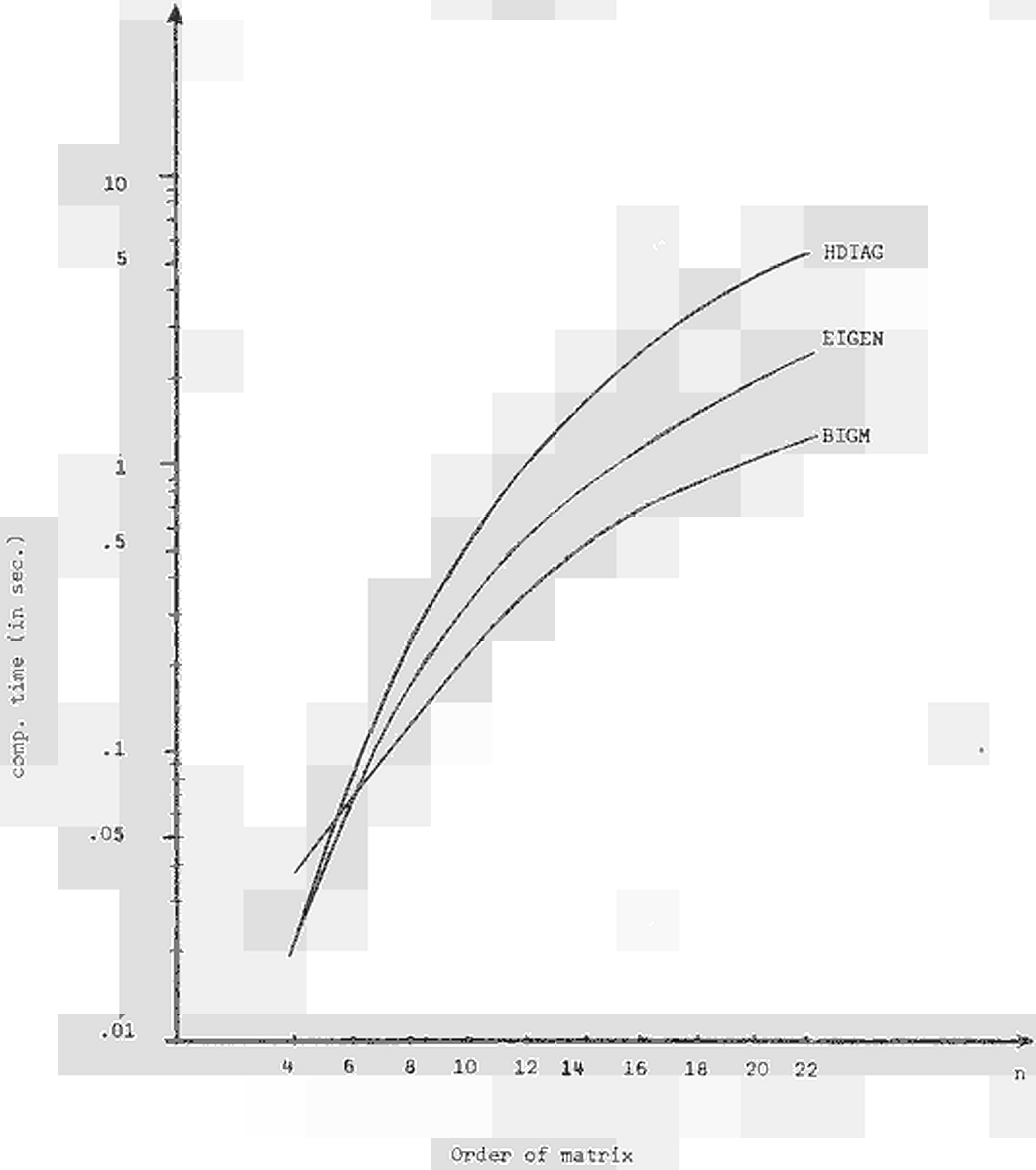
TABLE 1

n: order of the test matrix	Euclidean norm of the test matrix	$\left\{\left(\sum_{i=1}^{n} \lvert\lambda_i-\tilde{\lambda}_i\rvert^2\right)\Big/\left(\sum_{i=1}^{n}\lvert\lambda_i\rvert^2\right)\right\}^{1/2}$			$\max_i \{\lvert Ax_i-\lambda_i x_i\rvert_E\}$			$\frac{1}{\sqrt{n}}\left(\lvert I-X^TX\rvert_E\right)$		
		HDIAG	EIGEN	BIGM	HDIAG	EIGEN	BIGM	HDIAG	EIGEN	BIGM
4	9.1651	5.30(-6)	5.12(-7)	1.62(-6)	3.81(-6)	3.06(-6)	2.99(-5)	7.15(-6)	8.43(-8)	3.60(-6)
4	1.3039	4.95(-6)	6.83(-6)	4.02(-6)	4.00(-6)	9.21(-6)	6.27(-6)	5.94(-6)	1.35(-6)	1.55(-6)
4	37.4433	1.09(-6)	1.16(-6)	3.22(-6)	3.89(-5)	3.79(-5)	1.35(-4)	1.79(-6)	5.43(-8)	---
4	26.4008	3.47(-7)	6.38(-6)	1.21(-6)	9.15(-5)	1.69(-4)	6.76(-5)	4.75(-6)	1.26(-6)	2.98(-6)
5	1.9349	3.73(-6)	4.62(-6)	2.06(-6)	5.89(-6)	8.55(-6)	8.58(-6)	4.61(-6)	1.23(-6)	3.47(-6)
5	24.1868	4.44(-6)	7.06(-6)	1.63(-6)	7.78(-5)	1.68(-4)	6.50(-5)	7.60(-6)	2.41(-6)	2.55(-6)
5	92.4608	1.01(-5)	3.47(-6)	1.02(-6)	9.35(-4)	3.25(-4)	1.52(-4)	8.65(-6)	2.69(-6)	5.89(-6)
6	17.7200	7.20(-6)	2.44(-6)	3.52(-6)	9-09(-5)	3.16(-5)	6.34(-5)	7.07(-6)	2.20(-6)	---
6	23.8747	8.24(-6)	9.59(-6)	6.95(-7)	1.35(-4)	2.20(-4)	8.44(-5)	8.90(-6)	4.47(-7)	4.32(-6)
6	471.2520	3.32(-6)	5.30(-6)	5.60(-6)	7.95(-4)	1.67(-3)	2.93(-3)	7.42(-6)	1.71(-6)	---
8	2482.26	6.14(-6)	3.85(-6)	3.77(-6)	8.03(-3)	4.75(-3)	6.41(-3)	5.82(-6)	2.92(-6)	---
8	4546.15	3.32(-6)	1.39(-5)	1.05(-6)	1.59(-2)	6.33(-2)	8.79(-3)	1.24(-5)	1.23(-6)	5.75(-5)
9	8.4853	2.55(-6)	1.72(-6)	3.25(-6)	2.44(-5)	1.32(-5)	2.18(-5)	7.16(-6)	1.30(-6)	---
11	23.8328	2.13(-5)	7.46(-6)	4.77(-6)	2.85(-4)	8.19(-5)	3.64(-5)	1.84(-5)	1.08(-5)	---
12	63.8905	2.15(-5)	9.55(-6)	2.98(-6)	1.38(-3)	6.02(-4)	5.24(-5)	1.83(-5)	1.25(-5)	4.35(-6)
21	28.4605	1.36(-5)	8.58(-6)	4.21(-7)	1.18(-4)	6.93(-5)	7.11(-6)	1.56(-5)	1.12(-5)	---

TABLE 2

Test matrix	True eigenvalues	Routine	Number of eigenvalues with c exact figures c = 0	1	2	3	4	5	6	7	8	Computation time in seconds per IBM 360/65 (single precision)
SM8/3 order = 8	$\lambda_k = 8.10^{k-1}$ (k = 1,2,...,8)	BIGM	2	1	1	1	1	2				.02 + .08 + .06 = .16
		EIGEN	2		1	1	1	3				.18
		HDIAG	1	1	1	1	2	2				.30 (iter. n = 91)
SM8/4 order = 8	λ_k = 23,11,11,7 -5,-13-13-13-21	BIGM							6	1	1	.02 + .06 = .14
		EIGEN						4	4			.20
		HDIAG						3	5			.24 (iter. n = 69)
SM8/5 order = 8	$\lambda_k = 6.k$ (k = 1,2,...,8)	BIGM						4	4			.02 + .08 + .04 = .14
		EIGEN						7	1			.18
		HDIAG						3	5			.12 (iter. n = 36)
$C^{(8)}$ order = 8	λ_k = 24, -6.82842712 (2 times) -1.17157288 (2 times) 0. (3 times)	BIGM					2	3	2	1		.02 + .10 + .06 = .18
		EIGEN						5	3			.18
		HDIAG					3	3	2			.26 (iter. n = 76)
$C^{(16)}$ order = 16	λ_k = 80, 0 (7 times) -26.27414237 (2 times) - 3.23982881 (2 times) - 1.44646269 (2 times) - 1.03956613 (2 times)	BIGM				2	4	6	3	1		.12 + .30 + .30 = .72
		EIGEN					10	5	1			1.26
		HDIAG				2	5	7	2			2.55 (iter. n = 388)
$U^{(49)}$	$\lambda_k = -1$, ($\frac{p}{2}$ times) $= +1$, (n- $\frac{n}{2}$) times	BIGM					31	16	2			2.55 + .38 + 5.04 = 7.97
		EIGEN					38	11				43.81
		HDIAG						49				133.39 (iter. n = 7018)

Fig. 1


10
5
1
.5
.1
.05
.01
comp. time (in sec.)
HDIAG
EIGEN
BIGM
4
6
8
10
12
14
16
18
20
22
n
Order of matrix

is 1.26 sec.

For the matrix M_{70}, the routines HDIAG and EIGEN calculate all the eigenvectors with "absolute error" less than 10^{-6}. The computation time of HDIAG is 82.3 sec (The total number of plane rotations is 2775). The computation time of EIGEN is 78.5 sec.

The routine BIGM calculates 25 eigenvectors with "absolute error" less than 10^{-5}, and 45 eigenvectors with "absolute error" less than 10^{-3}. The computation time of BIGM is 30.5 sec.

Three test matrices are constructed by using a tensor product of lower order matrices whose eigenvalues are known. The order of these matrices is n = 24, n = 48 and n = 96. If the calculation of the eigenproblem of these test matrices is performed in "double precision" on IBM 360/65, the routines BIGM and EIGEN give all the eigenvalues with at least 8 exact figures. The computation time of BIGM is 3.4 sec. (n = 24), 16.2 sec. (n = 48) and 86.7 sec. (n = 96), respectively. The computation time of EIGEN is 7.6 sec. (n = 24), 62,2 sec. (n = 48) and 456.2 sec. (n = 96) respectively.

The eigenvalues and the eigenvectors of the above three test matrices are calculated also in "single precision" on IBM 360/65 with the routines HDIAG, EIGEN and BIGM. The results are summarised in Table 3.

In the following tables we give the number of the eigenvectors, corresponding to single eigenvalues, whose "absolute error" is less than 10^{-d}.

Seven test matrices are constructed by using the technique of J.M. Ortega described in chapter 1, §5, in which the eigenvalues are chosen single, multiple, close with some typical distribution and also are given by random numbers. We summarize in Table 4 the results obtained by solving the eigenproblem of the above test matrices, with the routines HDIAG, EIGEN and BIGM in "single precision" (s.p.) and in "double precision" (d.p.) on IBM 360/65, and with the routines HDIAG and BIGM in "single precision" on IBM 7090.

The "tolerance" τ of the threshold Jacobi method is defined as

$$\tau = \frac{\eta \left\{ \sum_{i \leq j} 2 a_{ij}^2 \right\}^{1/2}}{n}$$ where $\{a_{ij}\}$ is the input matrix of order n.

The results of Table 4 are obtained by taking $\eta = 10^{-6}$ for "single precision" calculations and $\eta = 10^{-12}$ for "double precision" calculations.
No improvement in the accuracy of the results was obtained by taking $\eta = 10^{-15}$ instead of $\eta = 10^{-12}$.

A comparison between the Givens-Householder method and the Rutishauser method is made on band symmetric matrices (whose bandwidth is small compared to the order of the matrix). The calculations are performed in "single precision" and in "double precision" on IBM 360/65. The "tolerance" in the routine LRCH is $\varepsilon = 10^{-6}$ ("single precision" calculations) and $\varepsilon = 10^{-14}$ ("double precision" calculations).

In Table 5 we give:

a) the number of eigenvalues with c (c = 0,1,2,...,7) exact figures obtained by the routines BIGM and LRCH when the calculations are performed in "single precision";
b) the machine time (in seconds) for computing the eigenvalues in "single precision" (s.p.) and in "double precision" (d.p.).

The eigenvalues are computed in decreasing order, beginning with the highest eigenvalue. When the calculations are performed in "double precision", the routines BIGM and LRCH show high accuracy. The maximum error in any eigenvalue is a few units in the last place of the larger eigenvalues.

The "computing time" of LRCH varies considerably with the order in which the eigenvalues of the test matrix are calculated. For example, the machine time for computing in increasing order the eigenvalues of $(J_{00}^{(n)})^3$ in "double precision" is .38, 1.22, 2.49, 4.11, 6.33 seconds, for n = 10, 20, 30, 40, 50, respectively.

1.3 Discussion of the test results

The results of the above test matrices may be summarized in the following way.

TABLE 3

Test matrix	Routine	Number of eigenvalues with c exact figures							Number of eigenvectors with "abs. err" ≤ 10^{-d}							Computation time (in sec.)
		c = 1	2	3	4	5	6	7	d = 1	2	3	4	5	6	7	
1: Order of the matrix = 24	HDIAG				14	10				2	16	6				9.4
Range of the eigenvalues [-317, +317]	EIGEN				16	8				3	14	7				5.2
Number of simple eigenvectors = 24	BIGM				12	10				5	12	7				1.8 (.3+.7+.8)
2: Order of the matrix = 48	HDIAG		18	30						6	28	14				76.1
Range of the eigenvalues [-634, +634]	EIGEN			48							26	22				41.5
Number of simple eigenvectors = 48	BIGM			18	28	2				4	30	14				9.7 (2.4+2.5+4.8)
3: Order of the matrix = 96	HDIAG		32	64						12	48	36				604.3
Range of the eigenvalues [-3170, +3170]	EIGEN		40	54	2					18	56	22				387.6
Number of simple eigenvectors = 96	BIGM		2	70	24					12	54	30				58.1 (17.7+9.1+31.3)

TABLE 4

Test matrix	Routine	Number of eigenvalues with c exact figures: c = 0	1	2	3	4	5	6	7	8	+	Number of eigenvectors with "abs. err." $< 10^{-d}$: d = 0	1	2	3	4	5	6	7	8	+	Computation time (in sec.)	Observations
1:																							
Order of matrix = 10	HDIAG (IBM-7090)								4	6		2				3	3					15.0	Two eigen-
Range of eigenvalues	BIGM (IBM-7090)								3	7		2			1	2	3					0.7	values are equal
[-0.0001, +0.05]	HDIAG (s.p.)							4	6						2	4	2					0.6	in the first six
Number of simple	EIGEN (s.p.)							3	7						3	3	1	1				0.3	figures
eigenvectors = 10	BIGM (s.p.)							1	9			2			3	3						0.3	
																						(.05+.15+.10)	
	EIGEN (d.p.)									5	5			2	1	4	1	1	1			0.5	
	BIGM (d.p.)										10			2	2				6			0.6	
2:																							
Order of matrix = 30	HDIAG (IBM-7090)						2	13	15								3	6				45.0	Eight eigen-
Range of eigenvalues	BIGM (IBM-7090)						3	9	18								5	4				10.0	values are equal
[-3.550001, 4.2]	HDIAG (s.p.)					7	19	4							1	3	5					19.0	in the first six
Number of simple	EIGEN (s.p.)					2	22	6								2	7					7.4	figures
eigenvectors = 17	BIGM (s.p.)						22	8						4	3	2						2.8	
																						(.6+.8+1.4)	
	EIGEN (d.p.)									7	23								1	8	8	11.0	
	BIGM (d.p.)										30								1	7	9	4.8	
																						(.8+2.4+1.6)	
3:																							
Order of matrix = 40	HDIAG (IBM-7090)					1	2	25	12								4	4				90.0	Ten eigen-
Range of eigenvalues	BIGM (IBM-7090)					3	3	25	9					1	1		6					16.6	values differ in
[-0.0001, +6.000001]	HDIAG (s.p.)					14	22	4								2	5	1				45.2	the 7th figure
Number of simple	EIGEN (s.p.)					14	19	7								1	7					16.8	
eigenvectors = 18	BIGM (s.p.)					7	27	6						1	3	4						5.6	
																						(1.4+1.2+3.0)	
	EIGEN (d.p.)									3	37								1	17		26.3	
	BIGM (d.p.)										40									18		9.1	
																						(1.6+4.1+3.4)	

TABLE 4 (continued)

Test matrix	Routine	Number of eigenvalues with c exact figures										Number of eigenvectors with "abs. err." 10^{-d}										Computation time (in seconds)	Observations
		c = 0	1	2	3	4	5	6	7	8	+	d = 0	1	2	3	4	5	6	7	8	+		
4: Order of matrix = 60 Range of eigenvalues -43.7, +24.0 Number of simple eigenvectors = 56	HDIAG (IBM 7090)				2	9	3	44	2				1	3	4	11	17	12				280.0	Eight eigen-values differ in the 7th figure
	BIGM (IBM 7090)				7	4	5	30	14				1	2	5	12	15	13				49.0	
	HDIAG (s.p.)				5	17	37	1				1	4	7	6	23	7					158.0	
	EIGEN (s.p.)				40	13	7					1	2	2	10	29	4					63.7	
	BIGM (s.p.)				8	47	5					3	5	8	10	8	14					16.7 (4.5+3.6+8.6)	
	EIGEN (d.p.)									2	58									8	48	95.2	
	BIGM (d.p.)										60									5	51	26.7 (5.2+11.6+9.9)	
5: Order of matrix = 72 Range of eigenvalues 0.034212, 0.95544 Number of simple eigenvectors = 24	HDIAG (IBM 7090)						8	41	18	5						2	6	14	2			350.0	
	BIGM (IBM 7090)						14	37	21							4	12	8				76.0	
	EIGEN (d.p.)										72										24	191.8	
	BIGM (d.p.)										72										24	37.1 (8.9+11.8+16.4)	
6: Order of matrix = 86 Range of eigenvalues 0.00943, 494.33348 Number of simple eigenvectors = 50	HDIAG (IBM 7090)					10	34	26	16						4	15	13	9	3			730.0	Six eigen-values differ in the 7th figure
	BIGM (IBM 7090)			3	14	21	28	13	7						7	21	11	5				118.0	
	EIGEN (d.p.)									1	85									4	46	297.9	
	BIGM (d.p.)										86										50	59.9 (14.9+18.5+26.5)	
7: Order of matrix = 94 Range of eigenvalues 0.03421, 477.27470 Number of simple eigenvectors = 58	HDIAG (IBM 7090)					21	22	31	20						9	11	25	13				960.0	
	BIGM (IBM 7090)				8	27	35	24						7	12	8	15	16				151.0	
	EIGEN (d.p.)									2	92									6	52	444.3	
	BIGM (d.p.)										94										58	81.3 (19.3+28.1+33.9)	

certain conditions converge to an upper triangular matrix. The "shifts" K_s are chosen so that $K_s \to \lambda_n$ as rapidly as possible, and all A_s remain real also when λ_n is a complex eigenvalue of A.

The Laguerre method: routine EIG5 (Share program SDA 3098). The method consists of two parts. Firstly the given matrix is reduced to Hessemberg form A by elementary similarity transformations. The second stage is the iterative search for the eigenvalues of A with the Laguerre method. Let $P(z) = \det(A-zI)$ be the polynomial in z with roots equal to the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the matrix A. Given an approximation to one of the roots, the Laguerre, method uses P(z), P'(z) and P"(z) to obtain better approximation. The polynomials P(z),P'(z) and P"(z) are evaluated with the Hyman method [8] The numerical criterion used by the routine EIG5 for a given number to be an acceptable approximation to a zero of a polynomial is defined in the following way. Let z be the current iterate, Δz the computed increment and L the modulus of the largest eigenvalue yet found.

$$|P(z)| < \eta_1 \ |z| \ |P'(z)|$$

$$|\Delta z| < \eta_2 \max \{|z \ |, 10^{-3}L\} \quad \text{(cubic convergence)}$$

$$|\Delta \ | < \eta_3 \max \{|z \ |, 10^{-2}L\} \quad \text{(linear convergence)}$$

$$(|z| = |Re(z)| + |Im(z)|.$$

2.2 Description of the tests

A preliminary test of these routines is made with some matrices with known eigenvalues collected in chapter I §1) and §2. In Table 6 we give the computed eigenvalues of the Eberlein's matrix when the calculations are performed on IBM 7090 and on IBM 360/65 ("double precision" arithmetic).

The eigenvalues of several real matrices constructed by Kronecker operations and by Ortega's similarity transformations are calculated with the above methods. The eigenvalues of these matrices are chosen real and conjugate complex, single and multiple with small and large distances from each other. Some eigenvalue satisfy typical distributions, others are given by random numbers.

TABLE 6

Eberlein's test matrices

True eigenvalues		Eigenvalues calculated by QREI on IBM 7090	Eigenvalues calculated by QREI on IBM 360/65 (d.p.)	Eigenvalues calculated by EIG5 on IBM 7090	Eigenvalues calculated by EIG5 on IBM 360/65 (d.p.)
(N=6					
s=-6.5)	-3.	-3.	-3.	-3.	-3.0
	0.	.0005	0.	0.00007	0.
	2.5	2.496	2.5	2.499	2.5
	4.5	4.514	4.5	4.501	4.5
	6.	5.972	6.0	5.994	6.0
	7	7.032	7.0	6.987	7.0
	7.5	7.485	7.5	7.5	7.5
(N=10					
s=-14)	0.	-0.0001	-0.0	0.	0.0
	12.	11.998	12.0	12.	12.0
	22.	22.35	22.0	21.840+0.004 i	22.0
	30.,30.	25.541	29.999998±0.002309 i	26.753+0.729 i	30.000457
		30.297±5.659i		30.962+4.237 i	35.999987+0.007808i
	36.,36.	37.828±7.491i	35.999982±0.008972 i	34.136+6.674 i	35.999989-0.007415i
				37.846+6.397 i	39.992283
	40.,40.	43.957±5.399i	39.999969±0.015006 i	41.728+5.817 i	40.001723
				44.226+3.758 i	40.008315
	42.,42.	46.702	42.000051±0.008174 i	45.474+0.012 i	41.999046
				45.740+0.214 i	42.002293

The results of these experiments are summarized in the Tables 7 to 10.
The appropriate choice of the "starting value" for the search for the eigenvalues is of the utmost importance in the success of the Laguerre method. For example, if we start the search for the eigenvalues from the origin, we are unable to find all the eigenvalues of some test-matrices. All these eigenvalues are determined if the starting value has modulus greater than the eigenvalue of greatest modulus.
In Table 7 we give the results obtained by changing the "starting value" z_o of the routine EIG5 on a 30-order well conditioned matrix with real eigenvalues.
The first column contains the "true" eigenvalues, the second column the eigenvalues obtained by the routine QREI on IBM 360/65 ("single precision" arithmetic) and the other columns the eigenvalues obtained by the routine EIG5 on IBM 360/65 ("double precision" arithmetic) with the "starting value" produced by the routine and described in [8], §12) and with the "starting value" equal to (1000.,0.).
The routine QREI solves exactly this eigenproblem when the calculations are performed on IBM 360/65 in "double precision".
However, if no good initial guess at the eigenvalue of greatest modulus can be made, we recommend the usage of the "starting value" produced by the routine EIG5.
In Table 8 we give the results obtained by changing the convergence parameters on a 30-order well-conditioned matrix with real eigenvalues. The first column contains the "true" eigenvalues, the second column the eigenvalues obtained by the routine QREI and the other columns the eigenvalues obtained by the routine EIG5 with different values of η_2.
The calculations are performed on IBM 7090 with $\eta_1 = 10^{-7}$ and $\eta_2 = 10^{-6}$.

In Table 9 we give for each test matrix of order n:

1) the number of eigenvalues with c (c=0,1,...) correct figures (*), obtained by the routine QREI and EIG5, respectively, on IBM 7090;
2) the computing time on IBM 7090 (in seconds) for determining the eigenvalues.

(*) Useful measures of the "accuracy" of the computed eigenvalues λ_i with respect to the "true" eigenvalues $\tilde{\lambda}_i$ of a matrix of order n are the "spectral variation" of $\Lambda \equiv \mathrm{diag}\,\{\lambda_i\}$ with respect to $\tilde{\Lambda} \equiv \mathrm{diag}\{\tilde{\lambda}\}$ and the "eigenvalue variation" of $\tilde{\Lambda}$ with respect to Λ [9].

TABLE 7

(The columns headed I indicate the number of iterations required)

λ_i	μ_i (s.p.)	ν_i (d.p.) z_o given by EIG5	I	ν_i (d.p.) $z_o = (10^3, 0)$	I
439.15247	439.10474	439.15247	11	439.15247	6
418.07643	418.02002	418.07643	3	418.07643	3
383.93657	383.89209	(1988.95300)	1	383.93657	4
370.55685	370.51953	370.55685	5	370.55685	3
325.89538	325.86548	325.89538	6	325.89539	6
325.89538	325.86279	325.89538	1	325.89537	1
308.14268	308.10181	(-1784.14900)	1	308.14268	5
277.31125	277.28101	(1663.21620)	1	277.31125	6
277.31125	277.27856	(1663.21680)	1	277.31125	1
173.40178	173.39308	$(28.99149 \pm 2036.85180i)$	1	173.40317	6
173.40178	173.39282		1	173.40086	1
157.12406	157.11682	$(-844.88169 \pm 1627.55150i)$	1	157.12406	6
157.12406	157.11331		1	157.12406	1
104.35584	104.35118	104.35584	12	104.35585	6
104.35584	104.35001	104.35584	1	104.35583	1
87.68912	87.68767	87.68951	6	87.68951	2
24.21237	24.21165	24.21237	1	24.21237	6
24.21237	24.21121	24.21237	6	24.21237	1
0.958204	0.961676	.958204	2	0.958204	4
0.852725	0.852761	.8521725	2	0.852755	4
0.846036	0.835038	.846036	5	0.846036	2
0.762287	0.761726	.762227	4	0.762227	3
0.647369	0.655146	.647369	4	0.647369	3
0.433477	0.442499	.433477	3	0.433477	3
0.373408	0.351100	.373408	3	0.373408	3
0.307700	0.315524	.307700	2	0.307700	3
0.271217	0.281016	.271217	3	0.211217	2
0.219962	0.221796	.219962	3	0.219962	2
0.187560	0.190150	.187560	2	0.187560	2
0.179734	0.177824	.179734	15	0.179734	4

TABLE 8

(The columns headed I indicate the number of iterations required)

λ_i	μ_i	$\nu_i(\varepsilon=10^{-3.5})$	I	$\nu_i(\varepsilon=10^{-4.5})$	I	$\nu_i(\varepsilon=10^{-5.5})$	I
-3.550001	-3.5500001	-3.5504094	8	-3.5500143	10	-3.5500062	12
-3.55	-3.5499943	-3.5497987	9	-3.5499942	1	-3.5499976	1
-3.55	-3.5499848	-3.5496697	1	-3.5497984	5	-3.5497995	6
-3.000001	-2.9999958	-2.9999958	6	-2.9999957	6	-2.9999971	6
-3.	-2.9999673	-2.9999024	1	-2.9999022	1	-2.9999061	2
-2.500001	-2.4999983	-2.4999949	6	-2.4999948	6	-2.4999957	9
-2.5	-2.4999807	-2.4999181	1	-2.4999174	1	-2.4999176	1
-2.	-1.9999998	-2.0001050	10	-2.0000060	12	-2.0000003	1
-2.	-1.9999998	-2.0000000		-2.0000000		-2.0000000	
-2.	-1.9999998	-1.9999535	1	-1.9999923	4	-1.9999992	14
-2.	-1.9999998	-1.9998766	6	-1.9999918	1	-1.9999923	9
-2.	-1.9999808	$-1.99996-.20710^{-5}i$	1	-1.9998953	6	-1.9998776	1
0.	-0.0000047464	$-1.99996+.31110^{-5}i$	4	-0.0193479	16	-0.000014764	16
0.	0.	0.		0.		0.	
0.00001	.000013266	$.49994-.24310^{-4}i$	1	.00020926	15	.00020952	17
.5	.49998821	$.49997+.22310^{-4}i$	2	.49987642	1	.49987444	5
.500001	.49999973	.49998862	6	.49999718	9	.49999716	10
.500001	.50000187	.50000191		.50000191		.50000191	
.8	.79999141	.79996708	6	.79996768	6	.79996716	4
1.1	1.0999934	$.249998+.337910^{-5}i$	2	$2.49998+.337910^{-5}i$	2	1.0999565	4
1.5	1.4999942	1.4998724	5	1.4998733	5	1.4998740	4
2.	1.9999946	1.9999733	12	1.9999725	12	1.9999725	4
2.5	2.4999880	2.4999772	1	2.4999772	1	2.4998935	1
2.5	2.4999934	2.4999893	6	2.4999893	6	2.4999999	
2.500001	2.4999997	2.4999999		2.4999999		2.5000011	8
3.	2.9999927	2.9998665	3	2.9998671	4	2.9998671	4
3.5	3.4999933	3.5000008	3	3.5000001	3	3.5000001	4
4.	3.9999846	3.9999298	2	3.9999308	3	3.9999308	3
4.1	4.0999845	4.0999711	3	4.0999701	3	4.0999701	3
4.2	4.1999842	4.1999017	10	4.1999023	7	4.1999023	7
1.55001		1.4497436	116	2.9306612	146	1.5499131	156

TABLE 9 (Routine QREI)

Number of correct figures																				
8	1	2	2	6		4	6	8		4		1							12	
7	8	4	5	13		4	6	17	7	19	16	3	1	4		2	9	24	71	72
6	1	1	6	2	3	6	7	14	17	21	12	7	1	13	2	6	13	24	13	20
5		3	3	3	14	6	8	1	24	4	30	15	8	10	21	10	24	28		4
4			4		7	9	11				6	24	26	7	27	22	9	30		
3						1	2					4	13	8	7	18	12			
2												6	11	18	15	20	14			
1																	3			
																	2			
order of the matrix	10	10	20	24	24	30	40	40	48	48	54	60	60	60	72	78	86	96	96	96
Computing time	1.2	1.2	4.	7.	9.	15.0	20.	18.	24.	22.	45.	45.	55.	60.	100.	95.	120.	210.	150.	175.

TABLE 9 (Routine EIG5)

Number of correct figures																				
8			2	4		3	2	5		4	2									
7	9	3	8	14		4	1	19	5	16	15	5	1	5		2		24		
6	1	5	3	4		6	13	15	22	24	4	2	9	12	5	5	2	20	80	72
5			2	2	15	9	11	1	21	4	24	9	14	12	24	13	21	30	16	18
4		2			9	6	15				7	20	3	5	32	9	20	22		6
3			1			2	5					17	14	8	1	1	15			
2												7	1	5		10	16			
1			4										1	5		26	4			
													17	8	10	12	8			
order of matrix	10	10	20	24	24	30	40	40	48	48	54	60	60	60	72	78	86	96	96	96
Computing time	1.5	2.	10.	7.	8.	30.	50.	45.	60.	55.	120.	120.	330.	350.	320.	660.	450.	360.	360.	310.

For example in Table 9 the "point" 9 with abscissa 30 and ordinate 4 indicates that a test matrix of order 30 has 9 eigenvalues with 4 correct figures.

In all these experiments we have observed that for well-conditioned matrices the average number of iterations of the QR method is less than 2.3 per eigenvalue. For the Eberlein's ill-conditioned matrix this number is 3.0. For well-conditioned matrices the average number of iterations of the Laguerre method is 4.5 per eigenvalue. The agreement of the sum of the eigenvalues computed by the routine EIG5 with the trace of the original matrix constitutes a quite good check on their accuracy.

In Table 10 we give the trace of the original matrix and the sum of the eigenvalues of the test matrices of Table 9 computed by the routine EIG5 on IBM 7090.

When the matrices A have a "large" P-condition number ($P = |\lambda_{max}(A)|/|\lambda_{min}(A)|$, the QR method and the Laguerre method are unable to give all the eigenvalues. Sometimes the Laguerre method gives all the eigenvalues by replacing the "convergence test" $|\Delta z| < \eta_2 \max\{|z|, 10^{-3}L\}$ with $|\Delta z| < \eta_2 |z|$.

2.3 Discussion of the test results

The results of the above tests may be summarized in the following way.

a) For finding all eigenvalues, the Laguerre method is trouble-some because of the difficulty in finding "convenient" convergence parameters.

b) The convergence rate of the QR method is remarkably impressive. This method is very "efficient" with respect to accuracy, and computing time for determining the eigenvalues of real matrices.

c) The Laguerre method is useful for finding some eigenvalues (especially those with largest modulus) and may be faster than the QR method for well-conditioned matrices with multiple eingenvalues when a convenient choice of the "convergence parameters" has been made.

§3. COMPLEX MATRICES

3.1 Methods tested

For the solution of the eigenproblem of a complex matrix t e following

TABLE 10

Order of matrix	Trace of matrix	Sum of computed eigenvalues	Average number of iterations per eigenvalue
10	.12001110	.12001117	3.1
10	38.000002	38.000119	3.8
20	4118.	4117.9619	6.4
24	4280.	4280.0005	2.7
24	10.630371	10.630367	2.9
30	1.5499329	1.5499131	5.2
40	39.549808	39.548140	5.3
40	63280.0070	63279.945	5.8
48	1872.0000	1872.0000	2.4
48	144.	143.99995	2.5
54	26.619996	26.667338	4.5
60	-194.88535	-194.87905	3.4
60	7758.5782	7758.5804	8.5
60	7767.3952	7850.5532	7.6
72	34.284957	34.698833	4.8
78	7776.2060	7734.9364	7.1
86	11441.941	11442.034	4.6
96	0.	-0.00582886	3.2
96	0.	-0.00170898	2.3
96	0.	0.01058006	2.7

methods are taken into account:

1) The QR method,
2) The Laguerre method.

Two routines are selected from a scientific library which represent these different methods.

The QR-method: Routine AMAT. (Share program SDA 3441) The method consists of two parts. Firstly the given matrix is reduced to Hessemberg from A by elementary similarity transformations. The second stage bases essentially on the fact that, by starting with the matrix $A = A_o$, the decomposition of A_s into the product $Q_s R_s$ (Q_s unitary matrix and R_s upper triangular matrix) and the recombination of Q_s and R_s by forming their product $A_{s+1} = (a_{ij}^{(s+1)}) = R_s Q_s$ generate an infinite sequence of similar matrices A_1, A_2, ... which under certain conditions converge to an upper-triangular matrix. ([1], p.515).

This process makes the element $a_{n\,n-1}^{(s+1)}$ of the upper Hessemberg matrix A_{s+1} converge to zero and therefore $a_{n\,n}^{(s+1)}$ converges to an eigenvalue of A. Whe convergence (i.e. $a_{n\,n-1}^{(s+1)}$ negligible) is met, the Hessemberg matrix A_{s+1} is deflated and the process proceeds with its leading principal submatrix of order one less. If $a_{n-1\,n-2}^{(s+1)}$ becomes negligible the eigenvalues of the lower right hand matrix of order two are calculated and the process proceeds with the leading principal submatrix of order two less.

The convergence-test is:

$$|a_{n\,n-1}^{(s+1)}| \leq \varepsilon_1 \, |a_{n\,n}^{(s+1)}| \qquad (\text{for } |a_{n\,n}| \neq 0)$$

or

$$|a_{n\,n-1}^{(s+1)}| < \varepsilon_2 \, (\sqrt{n} \quad \|A\|_\infty)$$

and

$$|a_{n-1\,n-2}^{(s+1)}| < \varepsilon_2 \, (\sqrt{n} \quad \|A\|_\infty)$$

$$(|z| = |Re(z)| + |Im(z)|)$$

The Laguerre Method: Routine EIG4. (Share program SDA 3099) The method consists of two parts. Firstly the given matrix is reduced to Hessemberg form A by elementary similarity transformations. The second stage is the iterative search for the eigenvalues of A with the Laguerre method. Let $P(z) = \det(A-zI)$ be the complex polynomial in z with roots equal to the eigenvalues $\lambda_1, \lambda_2 \dots \lambda_n$ of the matrix A. Given an approximation to one of the roots, say λ_n, the Laguerre method uses $P(z)$, $P'(z)$ and $P''(z)$, to obtain a better approximation. The polynomials $P(z)$, $P'(z)$ and $P''(z)$ are evaluated with the Hyman method [8].

The numerical criterion used by the routine for a given number to be an acceptable approximation to a zero of a polynomial is defined in the following way.
Let z be the current iterate, Δz the computed increment and L the modulus of the largest eigenvalue yet found.

$$|P(z)| < \eta_1 \; |z| \quad |P'(z)|$$

$$|\Delta z| < \eta_2 \max\left(|z|, 10^{-3} L\right) \quad \text{(cubic convergence)}$$

$$|\Delta z| < \eta_3 \max\left(|z|, 10^{-2} L\right) \quad \text{(linear convergence)}$$

$$(|z| = |Re(z)| + |Im(z)|$$

3.2 Description of the tests

Study 1: (see Table 11)

A preliminary test of the above methods is made with some test-matrices, collected in chapter I §1). In Table 11 we give the results of this test. The eigenvalues are calculated in "single precision (s.p.)" and in "double precision (d.p.)" on IBM 360/65. (The eigenvalues of the test matrix CM 15/1 are calculated also on IBM 7090.)

The "convergence-parameters" are:

(s.p.) $\varepsilon_1 = 10^{-7}$, $\varepsilon_2 = 10^{-10}$

$\eta_1 = 10^{-7}$, $\eta_2 = 10^{-3}$, $\eta_3 = 10^{-5}$

(d.p.) $\varepsilon_1 = 10^{-15}$, $\varepsilon_2 = 10^{-15}$

$\eta_1 = 10^{-10}$, $\eta_2 = 10^{-5}$, $\eta_3 = 10^{-7}$

When the calculations are performed in "double precision" the eigenvalues of the matrices CM 3/1, CM 4/1, CM 4/2, CM 4/1, CM 5/2, CM 15/1 are calculated with at least 12 exact figures. Two eigenvalues of CM 6/1 are calculated with 6 exact figures, the other eigenvalues have at least 12 exact figures. The routines AMAT and EIG4 calculate in double precision the "true eigenvalues" of CM 10/1.

Some test matrices are constructed by using the technique of J.M. Ortega described in chapter I, §5, in which the eigenvalues are chosen conjugate complex and complex, single and multiple with small and large distances from each other.

In Table 12 to 14 we give for each test matrix:

1) the number of eigenvalues with c (c = 0,1,...) correct figures, obtained with the routines AMAT and EIG4, respectively, (single precision) and:
2) the computing time on IBM 360/65 (in seconds) for determining the eigenvalues in "single precision" (s.p.) and in "double precision" (d.p.). (For example in Table 12 the "point" 11 with abscissa 16 and ordinate 3 indicates that a "test matrix" of order 16 has 11 eigenvalues with 3 correct figures)

a) Complex matrices with generally *) distinct complex eigenvalues

Study 2: (see Table 12)
The real part and the imaginary part of the eigenvalues of these matrices are uniformly distributed in the intervals (-1,1), (-100,100), (-1000,1000) and (-1,1000), or are integer numbers.
The eigenvalues of these test matrices are determined with the routines AMAT and EIG4 in "single precision" (s.p.) and in "double precision" (d.p.) on IBM 360/65.

When the calculations are performed in "double precision" the routines AMAT and EIG4 determine "generally" the eigenvalues with at least 10

----

*) A matrix has generally distinct eigenvalues when only few (less than 10%) eigenvalues of the matrix are multiple. The other eigenvalues are well separated.

TABLE 11

Test Matrix CM 3/1

True eigenvalues	Eigenvalues calculated by AMAT (s.p.)	Eigenvalues calculated by EIG4 (s.p.)
(1, 1)	(.9999, 1.0015)	(1.0019, 1.0036)
(7,-8)	(7.0011, -8.0009)	(7.0000, -8.0007)
(-43,51)	(-43,0011, 50.9993)	(-43.0035, 50.9969)
Computation time (s.p.)	.02 sec	.16 sec
on IBM 360/65 (d.p.)	.04 sec	.08 sec

Test Matrix CM 4/1

True eigenvalues	Eigenvalues calculated by AMAT (s.p.)	Eigenvalues calculated by EIG4 (s.p.)
(4,0)	(3.9999971, 0)	(4. , 0)
(8,0)	(7.9999990, 0)	(8.0000029, 0)
(12,0)	(12. , 0)	(12. , 0)
(16,0)	(15.999989 , 0)	(16. , 0)
Computation time (s.p.)	.04 sec	.16 sec
on IBM 360/65 (d.p.)	.04 sec	.08 sec

Test Matrix CM 4/2

True eigenvalues	Eigenvalues calculated by AMAT (s.p.)	Eigenvalues calculated by EIG4 (s.p.)
(1,1)	(.9999, 1.0000)	(1.0000, .9999)
(3,4)	(3.0005, 3.9998)	(3.0001,3.9995)
(4,5)	(3.9994, 4.9999)	(3.9994,4.9999)
(5,6)	(5.0000, 6.0001)	(5.0000,6.0001)
Computation time (s.p.)	.08 sec	.22 sec
on IBM 360/65 (d.p.)	.12 sec	.14 sec

TABLE 11 (continued)

Test Matrix CM 5/1

True eigenvalues	Eigenvalues calculated by AMAT (s.p.)	Eigenvalues calculated by EIG4 (s.p.)
(0,0)	(.00003, .00004)	(.00005, .00006)
(0,0)	(-.00001, -.00002)	(.00007, .00006)
(127.387,132.278)	(127.3866 ,132.278)	(127.3867 ,132.2782)
(-9.45999,7.28019)	(-9.46005, 7.28017)	(-9.46003, 7.28024)
(7.07332,-9.55839)	(7.07326, -9.55823)	(7.07332, -9.55843)
Computation Time (s.p.)	.04 sec	.24 sec
on IBM 360/65 (d.p.)	.14 sec	.16 sec

Test Matrix CM 5/2

True eigenvalues	Eigenvalues calculated by AMAT (s.p.)	Eigenvalues calculated by EIG4 (s.p.)
(15.180165225,0)	(15.179976,6.9 10^{-7})	(15.180162, 1.3 10^{-6})
(5.6787293543,0)	(5.678727,5.8 10^{-7})	(5.678731,-1.0 10^{-6})
(-0.83398680019,0)	(-.8339851,5.2 10^{-6})	(-.833982,-2.6 10^{-6})
(-5.1498456282,0)	(-5.149843,1.2 10^{-6})	(-5.149846,-3.3 10^{-6})
(-15.921062150,0)	(-15.921054,-1.3 10^{-6})	(-15.921056,-1.7 10^{-6})
Computation time (s.p.)	.12 sec	.16 sec
on IBM 360/65	.16 sec	.22 sec

Test matrix CM 6/1

True eigenvalues	Eigenvalues computed by AMAT (s.p.)	Eigenvalues computed by EIG4 (s.p.)
(0,0)	(.0021, -.0015)	(-.00007, -.00009)
(1,0)	(1.0000, .0008)	(1.0000 , -.00009)
(0,1)	(-.0013, 1.0006)	(-.00007, 1.0004)
(2,1)	(1.9998, 1.0001)	(2.0000 , 1.0001)
(-1.-2)	(-1.0312,-1.9693)	(-1.0242 ,-1.9921)
(-1,-2)	(- .9699,-2.0308)	(-.9797 ,-1.9960)
Computation time (s.p.)	.20 sec	.64 sec
on IBM 360/65 (d.p.)	.26 sec	.52 sec

TABLE 11 (continued)

Test Matrix CM 10/1

True eigenvalues	Eigenvalues computed by AMAT (s.p.)	Eigenvalues computed by EIG4 (s.p.)
(4.16174868, 3.13751356)	(1.161742. 3.137547)	(4.161748, 3.137513)
(5.43644837,-3.97142582)	(5.436461,-3.971467)	(5.436452,-3.971433)
(2.38988759, 7.26807071)	(2.389897, 7.268000)	(2.389887, 7.268074)
(-1.93520144,-3.97509382)	(-1.935202,-3.975090)	(-1.935202,-3.975095)
(-2.44755082, 0.43712617)	(-2.447538, 0.437127)	(-2.447554, 0.437128)
(-5.27950616,-2.27596303)	(-5.279502,-2.275943)	(-5.279517,-2.275968)
(1.03205812, 9.29413278)	(1.032139, 9.294246)	(1.032053, 9.294141)
(-4.96687009,-8.08712475)	(-4.966941,-8.087037)	(-4.966870,-8.087138)
(8.81130928, 1.54938266)	(8.811319, 1.549407)	(8.811318, 1.549384)
(10.7976764 , 8.62338151)	(10.797742, 8.623438)	(10.797698, 8.623397)
Computation time (s.p.)	.60 sec	.86 sec
on IBM 360/65 (d.p.)	1.00 sec	1.20 sec

TABLE 11 (continued)

Test Matrix CM 15/1

True eigenvalues	Eigenvalues calculated by AMAT		Eigenvalues calculated by EIG4	
	on IBM 7090	on IBM 360/65 (s.p.)	on IBM 7090	on IBM 360/65 (s.p.)
(-5,+2)	(-4.99996,+1.99973)	(-5.00414,+2.00604)	(-5.00004, 1.99985)	(-4.99879, 1.98125)
(-9, 0)	(-9.00008,+0.00040)	(-8.99142, 0.011035)	(-8.99975,-0.00019)	(-8.98847, 0.01272)
(-9,+3)	(-8.99970,+2.99976)	(-8.98056,+2.98651)	(-9.00047,+2.99983)	(-9.00899,+2.97734)
(-8,+5)	(-8.00020,+5.00028)	(-8.01275,+4.99653)	(-7.99952,+5.00013)	(-8.00281,+5.02026)
(-5,-3)	(-5.00018,-3.00008)	(-5.00546,-2.99760)	(-4.99982,-2.99987)	(-5.00923,-2.98682)
(-4,+7)	(-3.99959,+6.99991)	(-3.99714,+6.99055)	(-4.00015,+6.99979)	(-3.98898,+6.99547)
(+2,+4)	(+1.99992,+4.00013)	(+2.01528,+3.99678)	(+1.99958,+3.99995)	(+1.99738,+4.01302)
(+3,+1)	(+3.00029,+1.00048)	(+3.00521,+1.02415)	(+3.00085,+1.00004)	(+2.97291,+1.01870)
(+2,+8)	(+2.00008,+7.99952)	(+1.98758,+7.98532)	(+2.00006,+8.00002)	(+2.00414,+7.98699)
(+3,+8)	(+2.99990,+8.00009)	(+3.00207,+8.00775)	(+2.99996,+7.99997)	(+2.99533,+8.00194)
(+6,+3)	(+5.99952,+2.99921)	(+5.98345,+2.96992)	(+5.99918,+2.99987)	(+6.03148,+3.00327)
(+7,+2)	(+6.99994,+2.00051)	(+6.99618,+2.02126)	(+7.00037,+2.00038)	(+6.98248,+1.98051)
(+3,-7)	(+2.99998,-6.99999)	(+3.00003,-6.99901)	(+2.99998,-7.00001)	(+3.00203,-7.00200)
(-5,-9)	(-4.99998,-8.99996)	(-4.99850,-9.00014)	(-5.00002,-8.99999)	(-4.99947,-9.00151)
(+9,-8)	(+8.99998,-7.99998)	(+9.00023,-7.99982)	(+9.00000,-8.00000)	(+9.00032,-7.99986)
Computation time				
on IBM 7090	8.1 sec		11.6 sec	
on IBM 360/65 (s.p.)		1.9 sec		2.6 sec
(d.p.)		2.7 sec		3.4 sec

TABLE 12 (Routine AMAT)

Every point of the abscissa characterizes one test-matrix

Number of correct figures \ order of the matrix	6	6	10	10	10	12	16	20	20	24	28	30	36	38	40
7		1													
6	2	1			1										
5	3	3	6	2	9	7				1	1				
4	1	1	3	4		5	4	12		19	8		1		3
3			1	1			11	8	18	4	19	27	28	26	22
2				2			1		2			3	7	12	15
1				1											
0															
Computing time (s.p.)	.1	.1	.4	.4	.6	.9	1.7	3.1	3.3	5.4	8.8	9.7	17.0	20.8	23.9
Computing time (d.p.)	.2	.2	.6	.6	.7	.9	2.3	3.7	4.2	7.6	10.6	14.0			

TABLE 12 -(Routine EIG4)

Every point of the abscissa characterizes one test-matrix

Number of correct figures															
7						2									
6	4				3	1									
5	2	6		2	7	6				2					
4			10	4		3	3	4	1	16	7	1		1	2
3				1			11	16	16	6	21	26	27	24	24
2				2			2		3			3	9	13	14
1				1											
0															
order of the matrix	6	6	10	10	10	12	16	20	20	24	28	30	36	38	40
Computing time (s.p.)	.3	.4	.9	1.0	.8	1.0	2.9	5.2	5.6	9.4	13.1	16.1	27.5	32.2	37.5
Computing time (d.p.)	.3	.4	1.3	1.3	1.1	1.4	3.9	6.8	7.1	11.6	14.4	23.0			

exact figures. The "convergence-parameters" are the same as in Study 1. In all these experiments there is good agreement of the trace of the original test matrix with the sum of the eigenvalues computed (in "single precision") by the routine EIG4.

b) Complex matrices with generally distinct complex eigenvalues of equal modulus

Study 3: (see Table 13)

The real part and the imaginary part of the eigenvalues of the test matrices considered satisfy some typical distributions (i.e. linear distributions, geometric distributions).

The eigenvalues of these matrices are determined with the routines AMAT and EIG4 in "single precision" (s.p.) and "double precision" (d.p.) on IBM 360/65.

When the calculations are performed in "double precision", the routines AMAT and EIG4 determine "generally" the eigenvalues with at least 10 exact figures when the order of the matrices is less than 25 and with at least 7 exact figures in the other tests. The "convergence parameters" are the same as in Study 1.

When the order of the matrices is less than 25, there is good agreement of the trace of the original matrix with the sum of the eigenvalues computed (in "single precision") by the routine EIG4.

The test matrices labelled with * in Table 13 have spectral radii of order 10^4. The other matrices have spectral radii less than 5 10^2.

c) Complex matrices with multiple and close complex eigenvalues

Study 4: (see Table 14)

The real part and the imaginary part of the eigenvalues of the test-matrices generated with the Ortega's algorithm satisfy some typical distributions.

The test-matrices labelled with * in Table 14 have the form:

$$T^{(n)} + i\ (T^{(n)})^{-1}$$

where the real matrix $T^{(n)}$ is defined by algorithm No. 52 (Collected Algorithms from C.A.C.M.). These test-matrices have the eigenvalue $(1 + \sqrt{-1})$ with multiplicity n-2.

TABLE 13 (Routine AMAT)

Every point of the abscissa characterizes one test-matrix

Number of correct figures											
7										1	
6		1								2	
5	9	5	1		1	8		4		2	8
4	1	6	8	2	19	12		22	1		2
3			7	4		4	17		5	2	2
2				7			7		7	14	11
1				6					14	9	10
0				1					3		3
order of the matrix	10	12	16	20*	20	24	24	26	30*	30	36
Computing time (s.p.)	.5	.7	1.9	2.4	3.6	3.5	5.5	6.8	8.3	9.0	12.8
(d.p.)	.5	.8	2.1	2.9	4.1	4.1	6.3	7.9	9.1	10.7	

TABLE 13 (Routine EIG4)

Every point of the abscissa characterizes one test-matrix

Number of correct figures											
7											
6		3				1				2	
5	3	5		1		3		3			
4	7	4	3	1	14	18		13	5	1	
3			9	5	5	2	12	10	4	2	10
2			4	3	1		11		6	9	2
1				4					10	8	7
0				6					5	8	17
order of the matrix	10	12	16	20*	20	24	24	26	30*	30	36
Computing time (s.p.)	1.0	1.4	3.5	7.8	5.0	8.8	9.5	10.3	21.0	35.3	31.0
(d.p.)	1.0	1.3	4.2	8.4	6.4	5.0	11.4	12.5	29.1	29.9	

TABLE 14 (Routine AMAT)

Every point on the abscissa characterizes one test-matrix

Number of correct figures																	
7	1			3		1		2		1		4	5		3	17	9
6	7	13	3		3	2		1	23	4		5	1	6	19	8	29
5	2		15	4	4	8	6	6		10	9	12	6	6		2	
4		2	2	4	5	6	7	6	2	3	7	3	13	11	4	6	2
3				5	1	4	2	2		1	3			1	4	2	
2				4	7	3	4	2		3	9	6		3			
1							5	5		8	2		3	2			
0													2	1			
order of matrix	10*	15*	20*	20	20	24	24	24	25*	30	30	30	30	30	30*	35*	40*
Computing time (s.p.)	1.1	2.0	2.3	1.7	2.2	3.3	3.7	3.5	3.4	6.2	6.5	4.9	3.9	3.8	4.2	5.6	7.8
(d.p.)	.1	.4	.8	2.4	2.6	4.2	3.9	4.8	1.5	7.0	7.9	8.2	7.7	7.1	7.7		

TABLE 14 (Routine EIG4)

Every point of the abscissa characteristizes one test-matrix

Number of correct figures																	
7		1	3	1	1		4			1		8	6	6	7	5	5
6			2	1	1											4	1
5	1	2		1	1				2		2	11	1	2	3	9	5
4	5	5	11	4	6	6	2	2	2	13	5	8	8	5	18	16	19
3	4	6		6	5	5	8	9		6	10	1	8	7	2	1	10
2				5	5	5	5	7	4	5	5	1	2	8			
1				1	1		5	6	17	5	8		2	1			
0		1	4	2		8						1	3	1			
order of the matrix	10*	15*	20*	20	20	24	24	24	25*	30	30	30	30	30	30*	35*	40*
Computing time (s.p.)	1.4	3.2	9.7	5.8	6.2	9.8	7.5	11.9	24.8	19.9	23.1	4.8	11.4	11.8	9.7	5.3	24.7
Computing time (d.p.)	.2	.3	.5	2.7	2.0	2.4	2.9	2.5	.8	3.5	7.9	2.5	5.2	3.5	1.2		

The eigenvalues of the above matrices are determinated with the routines AMAT and EIG4 in "single precision" (s.p.) and in "double precision" (d.p.) on IBM 360/65.

The "convergence parameters" for "single precision" calculations are the same as in Study 1. The "convergence parameters" for "double precision" calculations are

$$\varepsilon_1 = \varepsilon_2 = 10^{-15}$$

$$\eta_1 = 10^{-7}, \ \eta_2 = 10^{-3}, \ \eta_3 = 10^{-5}$$

If in this study we consider the "convergence parameters" $\eta_1 = 10^{-10}$, $\eta_2 = 10^{-5}$, $\eta = 10^{-7}$ (see Study 1) for "double precision" calculations, we are unable to find all the eigenvalues.

When the calculations are performed in "double precision", the routine AMAT determines "generally" the eigenvalues with at least 10 exact figures and the routine EIG4 determines the eigenvalues with at least 7 exact figures. If the routine EIG4 were able to calculate all the eigenvalues in "single precision", then the agreement of the sum of the eigenvalues computed with the trace of the original matrix would be good.

3.3 Discussion of the Test Results

Some pathological examples exist for which convergence will not occur in AMAT (i.e. the Forsythe matrix (chap. II §4). For some test matrices (i.e. the Eberlein's test-matrix) the "number of iterations per eigenvalue" must be "large" in order to obtain the convergence.
The test calculations summarized in Table 11 to 14 give the information that the routine AMAT is very "efficient" with respect to accuracy and computing time for determining the eigenvalues of a complex matrix.
The results contained in Table 14 show the routine AMAT to be slower than the routine EIG4 for test matrices with multiple eigenvalues, when a "convenient" choice of the "convergence parameters" for EIG4 has been made.

§4 DETERMINATION OF THE EIGENVECTORS OF NON-HERMITIAN MATRICES

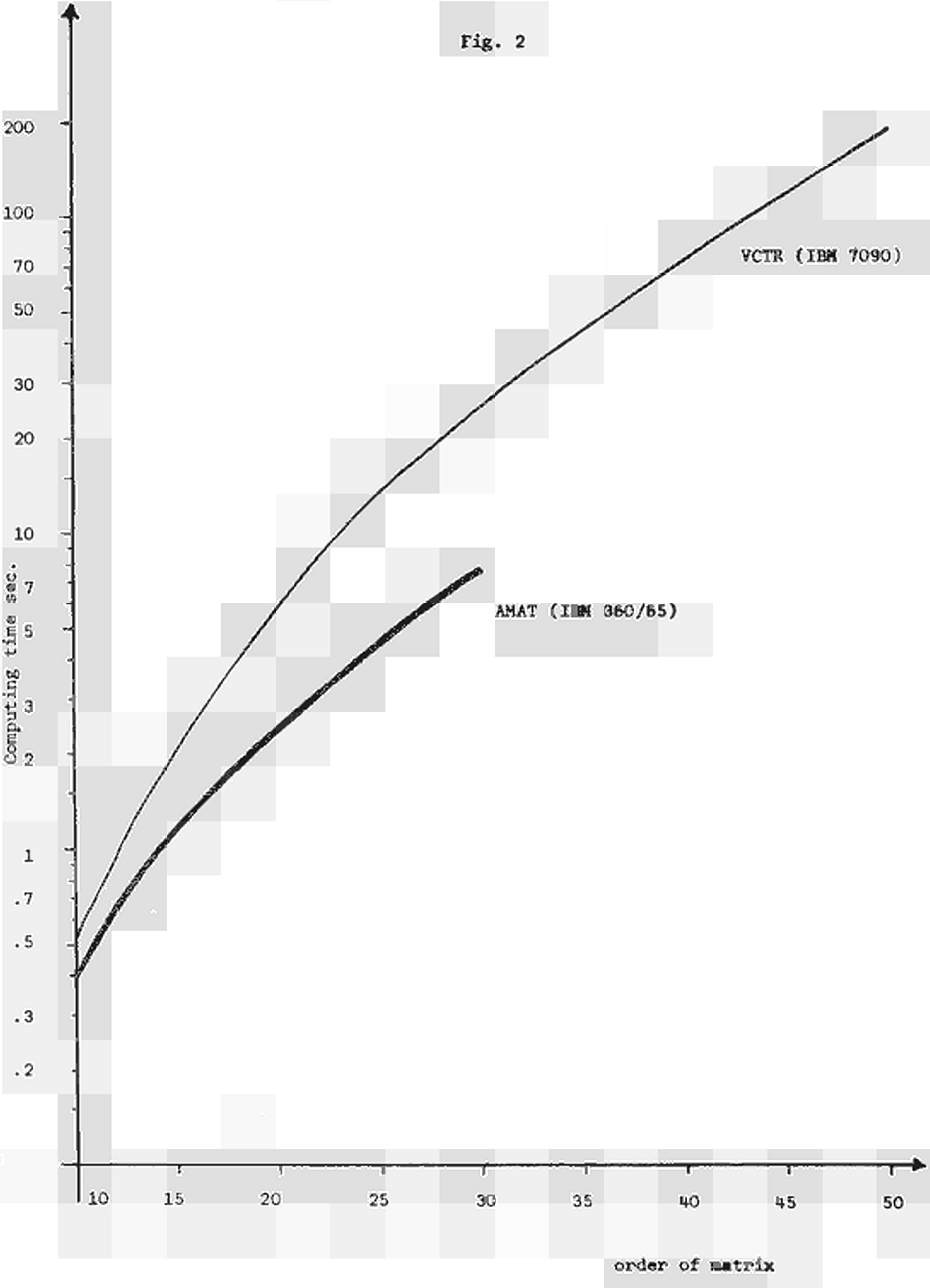
The eigenvectors of real and complex matrices are determined by the iterative method of Wielandt. The routine VCTR (Share program SDA 3053) determines the eigenvector of a real matrix A corresponding to a real eigenvalue λ. The matrix $(A-\lambda I)$ is triangularly decomposed into the product of triangular lower and upper matrices by elementary stabilized matrices of the type M'_{ij} ([1], page 236). Then the eigenvectors are determined by the inverse power method. The routine AMAT (Share program SDA 3441) determines the eigenvectors of a complex matrix A corresponding to a complex eigenvalue. The routine AMAT reduces the given matrix to Hessemberg form H by elementary similarity transformations. The matrix $(H-\lambda I)$ is triangularly decomposed into the product of triangular lower and upper matrices by elementary stabilized matrices of the type N'_{ij} ([1], page 200) and the eigenvectors of H are determined by the inverse power algorithm. Then the elementary similarity transformations are applied in reverse order to obtain the eigenvectors of the original matrix A. The method of Wielandt calculates the normalized eigenvectors of well conditioned real and complex matrices, corresponding to <u>single</u> eigenvalues, with high accuracy. For real matrices ranging in order from 19 to 50, the "absolute error" of each eigenvector is less than 10^{-6} when the calculations are performed on IBM 7090. For complex matrices ranging in order from 5 to 30, the "absolute error" of each eigenvector is less than 10^{-11} when the calculations are performed in "double precision" on IBM 360/65. In the case of a computed eigenvector corresponding to a m-fold eigenvalue λ, we have to test how accurately this eigenvector is lying in the linear space spanned by the "true" eigenvectors corresponding to λ. We study the accuracy of the Wielandt method on real matrices with real eigenvalues.

Let $x_1\ x_2\ \dots\ x_m$ be the eigenvectors corresponding to the m-fold eigenvalue λ. Let x be the computed eigenvector corresponding to the "true" eigenvalue λ. Let $\tilde{x}$ be the approximation of x in the sense of least-squares. The vector $\tilde{x}$ is given by the expression $\tilde{x} = \sum_{k=1}^{m} c_k x_k$ where the coefficients c_k are determined by the system

$$\sum_{k=1}^{m} c_k (x_j^T x_k) = x_j^T x \qquad (j = 1,2,..,m)$$

The quantity $|x-\tilde{x}|_\infty$ is the "absolute error" of the vector x.
The test-matrices A of order **n** are generated by similarity transformations: $A = B\Lambda B^{-1}$, where B is the Bremner's matrix $B = \alpha I+\beta Q$ (α and β real non-zero numners; I is the nxn identity matrix and Q is the nxn matrix whose entries are all 1's), and Λ is a diagonal matrix which contains m-fold eigenvalue λ. The other elements on the diagonal of Λ are pseudo random numbers generated in the interval [0,1] with uniform distribution.
For these matrices the "absolute error" of the eigenvectors computed with the routine VCTR on IBM 7090 is less than 10^{-4}, for $10 \leq n \leq 50$ and $2 \leq m \leq 5$.
In fig. 2 we give the behaviour of the "computation time" taken by the routines VCTR for determining all the eigenvectors of real matrices of order $n(n \leq 50)$ on IBM 7090 and by the routine AMAT for determining all the eigenvectors of complex matrices of order n ($n \leq 30$) on IBM 360/65 ("double precision" arithmetic).

Fig. 2


Computing time sec.
200
100
70
50
30
20
10
7
5
3
2
1
.7
.5
.3
.2
VCTR (IBM 7090)
AMAT (IBM 360/65)
10
15
20
25
30
35
40
45
50
order of matrix

BIBLIOGRAPHY

Chapter I

[1] E. Durand : Solutions numériques des équations algébriques (Tome II). Masson & C.ie édit, Paris, 1961.

[2] A.J. Fox, F.A. Johnson : On finding the eigenvalues of real symmetric tridiagonal matrices. The Comp. Journ. 9 (1966) 98.

[3] J.B. Rosser et al. : Separation of close eigenvalues of a real symmetric matrix. J. Res. Nat. Bur. Standards 47 (1951).

[4] M.D. Sawhney : The computation of eigenvalues. I. Real symmetric matrices. J. Soc. Indust. Appl. Math. 12 (1964) 726.

[5] J.H. Wilkinson : The calculation of the eigenvectors of codiagonal matrices. The Comp. Journ. 1 (1958) 90.

[6] J.H. Wilkinson : Householder's method for the solution of the algebraic eigenproblem. The Comp. Journ. 3 (1960) 23.

[7] J.H. Wilkinson : Householder's method for symmetric matrices. Numer. Math. 4 (1962) 354.

[8] P.J. Eberlein : A Jacobi-like method for the automatic computation of eigenvalues and eigenvectors of an arbitrary matrix. J. Soc. Industr. Appl. Math. 10 (1962) 74.

[9] M. Lotkin : Determination of characteristic values. Quart. Appl. Math. 17 (1959) 237.

[10] J. Greenstadt : Some numerical experiments in triangularizing matrices. Numer. Math. 4 (1962) 187.

[11] D.J. Muller : Householder's method for complex matrices and eigensystems of Hermitian matrices. Numer. Math. 8 (1966) 72.

[12] J.H. Wilkinson : Error analysis of floating point computation. Numer. Math. 2 (1960) 319.

[13] W.L. Frank : Computing eigenvalues of complex matrices by determinant evaluation and by methods of Danilevski and Wielandt. J. Soc. Indust. Appl. Math. 6 (1958) 378.

[14] Comm. Ass. Comp. Mach. Collected Algorithms.

[15] H.E. Fettis, J.C. Caslin : Eigenvalues and eigenvectors of Hilbert matrices of order 3 through 10. Math. of Comp. 21 (1967) 431.

[16] P.J. Eberlein : A two parameter test matrix. Math. of Comp. 18 (1964) 296.

[17] J.L. Brenner : A set of matrices testing computer programs. Comm. A.C.M. 5 (1962) 443.

[18] M. Newman, J. Todd : The evaluation of matrix inversion programs. J. Soc. Indust. Appl. Math. 6 (1958) 466.

[19] E.R. Hansen : On the Danilewski method. Journ. Ass. Comp. Mach. 10 (1963) 102.

[20] E. Bodewig : Matrix calculus. North-Holland Publ. Comp., Amsterdam 1959.

[21] A.H. Stroud, Don Secrest : Gaussian Quadrature Formulas. Prentice Hall, Inc. Englewood Cliffs, N.J. 1966.

[22] G.N. Polozhii : The method of Summary Representation for numerical solution of problems of Mathematical Physics. Pergamon Press, Oxford 1965.

[23] J.H. Wilkinson : The calculation of Lamé polynomials. Comput. Journ. 8 (1965) 273.

[24] P.A. White : The computation of eigenvalues and eigenvectors of a matrix. J. Soc. Indust. Appl. Math. 6 (1958) 393.

[25] B. Parlett : Laguerre's method applied to the matrix eigenvalue problem. Math. Comput. 18 (1964) 464.

[26] J.M. Ortega : Generation of test matrices by similarity transformations. Comm. A.C.M. 7 (1964) 377.

Chapter II

[1] J.H. Wilkinson : The algebraic eigenvalue problem. Clarendon Press, Oxford 1965.

[2] F.L. Bauer, C.T. Fike : Norms and Exclusion theorems - Numer. Math. 2 (1960) 137.

[3] A.S. Householder : The theory of matrices in Numerical Analysis. Blaisdell Publ. Comp. New York 1964.

[4] R.A. Smith : The Condition Numbers of the Matrix Eigenvalue Problem. Numer. Math. 10 (1967) 232.

[5] N.P. Zhidkov : Some comments concerning the conditionality of systems of linear algebraic equations. U.S.S.R. Comp. Math. Math. Physics 3 (1963) 1095.

[6] P. Henrici : Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices. Numer. Math. 4 (1962) 24.

[7] E. Isaacson, H.B. Keller : Analysis of numerical methods. J. Wiley & Sons, Inc., New York 1966.

[8] A. Ralston, H.S. Wilf (editors): Mathematical methods for digital computers, vol. II. J. Wiley & Sons, Inc., New York 1966.

[9] J.H. Wilkinson : Rigorous Error Bounds for Computed Eigensystems. The Comput. Journ. 4 (1961) 230.

Chapter III

[1] J.H. Wilkinson : The algebraic eigenvalue problem. Claredon Press, Oxford 1965.

[2] F.J. Corbato : On the coding of Jacobi's method for computing eigenvalues and eigenvectors of real symmetric matrices. Journ. Ass. Comp. Mach. 10 (1963) 123.

[3] A. Ralston, H.S. Wilf (editors) : Mathematical methods for digital computers. J. Wiley and Sons, Inc., New York 1960.

[4] A. Rolston, H.S. Wilf (editors) : Mathematical methods for digital computers, vol. II. J. Wiley & Sons, Inc., New York 1966.

[5] H. Rutishauser, H.R. Schwarz: The LR transformation method for symmetric matrices. Numer. Math. 5 (1963) 273.

[6] J.M. Ortega : An error analysis of Householder's method for the symmetric eigenvalue problem. Numer. Math. 5 (1963) 211.

[7] A.J. Fox, F.A. Johnson : On finding the eigenvalues of real symmetric tridiagonal matrices. The Comp. Journ. 9 (1966) 98.

[8] B. Parlett : Laguerre's method applied to the matrix eigenvalue problem. Math. Comput. 18 (1964) 464.

[9] P. Henrici : Bounds for iterates, inverses spectral variation and fields of values of non-normal matrices. Numer. Math. 4 (1962) 24.

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