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BASIC RELATIONSHIPS IN n-COMPONENT DIABATIC FLOW

by

H. WUNDT

1967

Joint Nuclear Research Center
Ispra Establishment - Italy

Reactor Physics Department
Research Reactors

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differential equation notation. Specialization to one-dimensional two phase flow is made.

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A handy set of two equations is given and repeated in the conclusions which may be useful for calculating transients in heat exchangers and around boiling water reactor fuel elements, stability problems being excluded.

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Preface

Two phase flow has, in particular in connexion with boiling heat transfer, become a field of very intense engineering research during the recent past. The interest has been stimulated by its appearance in boiling water reactors and rocket engines as well as in several types of heat exchangers.

Correlation attempts to interprete numerous experimental results and to predict the behaviour of designed plants are dispersed over well some thousands of papers. A very deserving textbook synopsis has recently appeared [1].

My impression is that, despite of an almost bewildering abundance of data, the common theoretical foundations are to a certain deal fragmentary or even misleading.

In order to fill this gap this paper shall provide research workers with a systematic classification of basic relationships from hydrodynamics and thermodynamics that may be suitable for two phase flow investigations.

As it is irrelevant for our purposes to distinguish between chemically different "components" and chemically identical but physically different "phases" of the same substance in a moving fluid, we speak, for convenience, of components only.

Moreover, most of the used definitions and relations are applicable to more than two components so that the notation is simplified by using the summation symbol over all components and giving the latter ones a current index i. In the case of pure phases, the number of participants is normally restricted to 2 due to GIBBS' phase rule so that the summation notation may be considered as a pure convenience, too.

The rigorous relationships derived in this paper are partly in a striking contrast to what is sometimes taken as a basis in the literature to describe some two phase flow phenomena. It is for this reason that papers which further establish on such relations are intentionally not quoted. References are but made to such papers which deal with the derivation of the fundamental equations and which were thus of utility to prepare this paper itself.

In this sense, the scope of the present paper is limited. It cannot provide any new correlations nor can it finally clarify two phase flow mechanisms. The equations which will be derived show characteristic differences between homogeneous and heterogeneous flow, but are in fact scarcely treated further; it is rather discussed why they cannot be solved reasonably without certain specific experimental informations.

This is to restrain engineers from too a vigorous empirical advance in some cases. The purpose of this paper is to provide research workers with relationships which in no case may be essentially violated, and to suggest how the discussed complex of problems should correctly be attacked.

BASIC RELATIONSHIPS IN n-COMPONENT DIABATIC FLOW

1. General form of balances *)

There are two methods of treatment of two phase flow and boiling heat transfer problems which do not exclude but should complement one another. The one is an empirical approach to explain isolated phenomena such as pressure drop through pipes e.g., by means of dimensional analysis. The other method is the analytical one. This means that first the physical equations governing the system are established. Then, in principle, if also the initial and boundary conditions are given, the solution should supply the complete subsequent system behaviour in space and time. Of course, such a sanguine expectation is, as always, quite academic as sooner or later a point will be reached where the rigorous continuation must be stopped for a serious lack of informations about details.

Nevertheless, just this "rigorous" approach shows in a coherent manner where and possibly how the gaps should be filled by empirical informations of the first kind. True understanding in sciences has almost ever been achieved in this way.

Thus, we try to establish a set of partial differential equations describing the behaviour of a n-component flow with heat addition. This is, at least for two phase flow, not new, but our treatment will be somewhat more general than usual derivations, leading to better understanding.

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*) The "general" method of balances is followed also e.g. in [2] and [3].

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The equations in question are, as known, the balances or conservation equations of mass [*)], energy, and momentum. The thermodynamics of irreversible processes deals also with an entropy balance - the second fundamental law is the central problem of non-equilibrium thermodynamics -, but, as can be seen so far, no practical applicability exists yet for boiling heat transfer problems.

We consider a volume V to which pertain the three extensive quantities M (mass), E (total energy), and $\vec{P}$ (momentum). The amounts of those quantities can simply be added when joining two or more volumes to give the total quantity.

Let Y be any extensive quantity, then the following quantities will also be needed:

- the related "specific" quantity (per mass unit) $y = Y/M$
- the related "Y-density" (per volume unit) ρy (with ρ = mass density),
- the related "Y-current density" $\vec{\Phi}_Y$,
- the Y-flow rate through the closed surface of the given volume Q_Y,
- the Y-production density inside the volume σ_Y.

By definition, Y is given by

$$Y = \int_V \rho y dV . \qquad (1.1)$$

[*)] We prefer the physical notation of mass, not the engineering notation of weight, as for the latter exists no conservation law.

Its time variation is generated by (positive or negative) leakage rate through the(positionally fixed) surface and by the production rate within, thus

$$\frac{dY}{dt} = \frac{d}{dt}\int_V \rho y dV = Q_Y + \int_V \sigma_Y dV . \qquad (1.2)$$

Q_Y is obtained by scalarly multiplying the Y-current density $\vec{\Phi}_Y$ (positive outwards) with the oriented surface $d\vec{A}$ and integrating:

$$Q_Y = -\int_A \vec{\Phi}_Y \cdot d\vec{A} , \qquad (1.3)$$

or, by GAUSS' theorem:

$$Q_Y = -\int_V \mathrm{div}\, \vec{\Phi}_Y \, dV . \qquad (1.4)$$

Substituting (1.4) into (1.2) and cancelling the volume integration yields

$$\frac{\partial}{\partial t}(\rho y) = -\,\mathrm{div}\, \vec{\Phi}_Y + \sigma_Y . \qquad (1.5)$$

This is the differential "local" formulation of the balance of any specific quantity y. It shall in the following be specified to each one of our interesting thermo-hydrodynamic quantities. If y is a scalar, then $\vec{\Phi}_Y$ is a vector; if $\vec{y}$ is a vector, then $\vec{\vec{\Phi}}_{\vec{Y}}$ is a tensor of second order *).

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*) We denote, for typing convenience, scalars with arrowless characters, vectors with one arrow, and tensors with a number of arrows corresponding to their order.

Note that y, ρy, $\vec{\Phi}_Y$, and σ_Y are "locally" defined variables or "field variables", whereas the other ones are "integral variables", no more depending on position coordinates.

We consider now the balance with respect to any velocity field $\vec{\varphi}$, and move the integration volume dV with it. The new current density with respect to this velocity field is $\vec{\Phi}_Y - \rho y \vec{\varphi}$.

If we choose $\vec{\varphi} = \vec{v}$, the "center of mass velocity" which will be defined exactly in chapter 3, the mass ρdV^* remains constant along the path. Instead of (1.2) we get

$$\frac{d}{dt}\int_{V^*} \rho y dV^* = -\int_{A^*} (\vec{\Phi}_Y - \rho y \vec{v}) \cdot d\vec{A}^* + \int_{V^*} \sigma_Y dV^* , \qquad (1.6)$$

where the asterisk suggests the moved volume. As ρdV^* remains unchanged in time, the time derivation acts on y only.

When applying once more GAUSS' theorem and cancelling the volume integration, one gets

$$\rho \frac{Dy}{Dt} = - \operatorname{div} (\vec{\Phi}_Y - \rho y \vec{v}) + \sigma_Y . \qquad (1.7)$$

One calls this the "substantial" formulation of the y-balance *) because the substance (mass) of the integration volume is held constant. The total differential operator d/dt changes to the "substantial differential operator" D/Dt which refers just to the mass velocity $\vec{v}$,

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*) The denotation "flow" formulation is too vague and should not be used.

and no other one. This subtle remark becomes important later on. D/Dt is, as usual,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\mathrm{grad} \,, \qquad (1.8)$$

a relationship which can also be derived by eliminating σ_Y from (1.5) and (1.7). Due to the correction term $\vec{v}\cdot$grad, the independent position coordinates remain the same ones as before, namely fixed.

2. Introduction of thermo-hydrodynamic variables: homogeneous case

After having the general recipe to establish balances, the only task is to specify correctly what is $\vec{\Phi}_Y$ and σ_Y in each case.

As already mentioned, we will substitute mass, total energy, and momentum, resp., for Y. We will treat all these quantities simultaneously and take the volume V itself as fourth extensive variable.

We first give a list of the involved quantities. The nomenclature considers as much as possible the recommandations of the International Union for Pure and Applied Physics (IUPAP), document U.I.P 11 (S.U.N. 65-3) from 1965. Unfortunately, not seldom quantities coïncide which usually have equal characters (e.g. v for specific volume and for velocity) so that side-step notations must be used. The adopted units system is the MKS-system where caloric units have already been transformed (e.g. Joules instead of kilocalories). Conversion factors are so avoided.

For each of the four blocks (see next page), the procedure is the same. In order to go from the basic integral quantity to the corresponding specific quantity, one has to divide by the mass M. The ratios are to be understood as differential quotients. The specific quantities are thus suitable as field variables, and so do also the various densities and current densities. The "specific mass" x for a homogeneous body is, of course, simply unity, but for more than one component the definition becomes non-trivial (see § 3.2).

Quantity	our notation	dimension
mass	M	kg
(specific mass)	$x\ (= M/M) = 1$	-
mass density	$\rho\ (= M/V)$	m^{-3} kg
mass current density (= momentum density)	$\vec{j} = \rho\vec{v}$	m^{-2} kg s^{-1}
volume	V	m^3
specific volume	$1/\rho\ (= V/M)$	m^3 kg^{-1}
(volume density)	$\alpha\ (= V/V) = 1$	-
volume current density (= "volume" velocity)	$\vec{\tilde{v}}$	m s^{-1}
(total) energy	E	m^2 kg s^{-2} ($\hat{=}$ J)
specific energy	$e\ (= E/M)$	m^2 s^{-2} ($\hat{=}$ J kg^{-1})
energy density	$\varepsilon = \rho e\ (= E/V)$	m^{-1} kg s^{-2} ($\hat{=}$ J m^{-3})
energy current density *)	$\vec{q} = \varepsilon\vec{v}$	kg s^{-3} ($\hat{=}$ J m^{-2})
momentum	$\vec{P}$	m kg s^{-1}
specific momentum (= "mass" velocity)	$\vec{v}\ (= \vec{P}/M)$	m s^{-1}
momentum density (= mass current density)	$\vec{j} = \rho\vec{v}\ (= \vec{P}/V)$	m^{-2} kg s^{-1}
momentum current density *)	$\vec{\vec{\Gamma}}$ **)	m^{-1} kg s^{-2}

Table 1. Quantities involved in balance equations

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*) Strictly spoken, the "convective" part of the current densities.

**) The representation of the general tensorial quantity $\vec{\vec{\Gamma}}$ as a dyadic product with one given factor $\vec{j}$, say $\vec{\vec{\Gamma}} = \vec{j}\vec{v}$, with a provisionally undetermined velocity $\vec{v}$, is possible only in particular cases. Every dyadic $\overrightarrow{ab}$ is indeed a tensor, but not every tensor is a dyadic.
The relationship between $\vec{\vec{\Gamma}}$ and $\vec{j}$ becomes clear only in § 3.4.

The respective third lines, the densities, are derived from the integral quantities by division by the volume V. Once more, the ratios should be considered as differential quotients. The direct way to obtain the densities is to multiply the specific quantities with the mass density ρ. Again, the "volume density" α assumes a non-trivial meaning only for more-component flow (see § 3.3).

The various (convective) current densities are obtained from the respective densities by multiplying the latter ones with appropriate velocities. The right velocity is the "mass velocity" $\vec{v}$ in the case of mass, the "volume velocity" $\vec{\tilde{v}}$ for volumes, the "energy velocity" $\vec{\hat{v}}$ for energies, whereas an averaged "momentum velocity" does not exist for more-component flow, unless in the one-dimensional (scalar) case, as shall be shown in chapters 3 and 4. These notions are unusual and shall thoroughly be explained in § 3.4. All velocities are different for more-component flow. The current densities are vectors for mass, volume, and energy, but a tensorial quantity for momentum, which itself is already a vector.

The integral quantities "flow rates" (depending on time only) are obtained according to (1.3) and (1.4) as

$$\left.\begin{aligned}
Q_M &= -\int_A \rho\vec{v}\cdot d\vec{A} = -\int_V \mathrm{div}\,(\rho\vec{v})\,dV && [\mathrm{kg\ s^{-1}}] \\
Q_V &= -\int_A \vec{\tilde{v}}\cdot d\vec{A} = -\int_V \mathrm{div}\,\vec{\tilde{v}}\,dV && [\mathrm{m^3\ s^{-1}}] \\
Q_E &= -\int_A \rho e\vec{\hat{v}}\cdot d\vec{A} = -\int_V \mathrm{div}\,(\rho e\vec{\hat{v}})\,dV && [\mathrm{m^2\ kg\ s^{-3}}]\ (\hat{=}[\mathrm{W}]) \\
\vec{Q}_{\vec{P}} &= -\int_A d\vec{A}\cdot\overset{\rightrightarrows}{\Gamma} = -\int_V \mathrm{Div}\cdot\overset{\rightrightarrows}{\Gamma}\,dV && [\mathrm{m\ kg\ s^{-2}}]
\end{aligned}\right\}\tag{2.1}$$

In the case of momentum, the differential operator "Div" under the volume integral means the tensor divergence, leading to a vectorial quantity, defined by

$$\mathrm{Div}\cdot\vec{\vec{\Gamma}} = \begin{pmatrix} \sum_k \frac{\partial\Gamma^{k1}}{\partial x_1} \\ \sum_k \frac{\partial\Gamma^{k2}}{\partial x_2} \\ \sum_k \frac{\partial\Gamma^{k3}}{\partial x_3} \end{pmatrix} \quad {}^{*)} \qquad (2.2)$$

Concerning GAUSS' theorem for tensorial quantities, one finds in the literature (e.g. [5], p. 134)

$$\int_A d\vec{A}\cdot\vec{\vec{\Gamma}} = \int_V (\mathrm{Div}\cdot\vec{\vec{\Gamma}})\,dV, \qquad (2.3)$$

where the tensor $\vec{\vec{\Gamma}}$ is the postfactor under the surface integral. When changing the factor sequence in this

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*) This holds for rectilinear coordinates only. In the case of curvilinear coordinates additional terms with CHRISTOFFEL symbols occur (cf. e.g. [4], pp. 165 and 177).

Our divergence has of course nothing to do with the sum at the main diagonal elements, which unfortunately is also sometimes called the tensor divergence. The latter quantity, a scalar invariant, should preferably be called the "trace" of the tensor (not used in this report).

non-commutative "scalar" product, the original tensor $\vec{\vec{\Gamma}}$ must be replaced by the transposed tensor $\vec{\vec{\Gamma}}'$ so that an alternative notation to (2.3) is

$$\int_A \vec{\vec{\Gamma}}\cdot d\vec{A} = \int_V (\mathrm{Div}\cdot\vec{\vec{\Gamma}}')dV = \int_V (\vec{\vec{\Gamma}}\cdot\mathrm{Div})dV. \qquad (2.4)$$

In general, the above defined momentum flow rate $\vec{Q}_{\vec{P}}$ is not parallel to the surface vector $d\vec{A}$. The dimension of $\vec{\vec{\Gamma}}$ is, superficially considered, that of an energy, but has nothing to do with that (different tensor order!).

The various "production densities" σ_Y, which are local quantities, represent in general the unhomogeneous terms in the balances (see eq. (1.5) as well as eq.(1.7)) and do not follow from the already discussed quantities. We denote them as follows:

$$\left.\begin{array}{lll} \text{mass production density} & \sigma_M & [m^{-3}\ kg\ s^{-1}] \\ \text{(volume production density)} & \sigma_V & [s^{-1}] \\ \text{energy production density} & \sigma_E & [m^{-1}\ kg\ s^{-3}](\hat{=}[W\ m^{-3}]) \\ \text{momentum production density} & \vec{\sigma}_{\vec{P}} & [m^{-2}\ kg\ s^{-2}]\ . \end{array}\right\} \qquad (2.5)$$

3. Relationships between thermo-hydrodynamic variables; heterogeneous case

3.1. Integral quantities

The truly interesting case is that our considered volume consists of n components with current index i, which, however, in our model, fill each the whole volume (so as, e.g., air as a mixture of nitrogen and oxygen).

Let us then define "partial" quantities according to

$$\left.\begin{array}{lll} \text{partial masses} & M^{i} = & x^{i}M \\ \text{partial volumes} & V^{i} = & \alpha^{i}V \\ \text{partial energies} & E^{i} = & \gamma^{i}E \\ \text{partial momenta} & \vec{P}^{i} = & \vec{\vec{\chi}}^{i}\cdot\vec{P}\ , \end{array}\right\} \qquad (3.1.1)$$

where the non-dimensional quantities x^{i}, α^{i}, γ^{i}, and $\vec{\vec{\chi}}^{i}$ may be called "mass fractions","volume fractions", "energy fractions", and "momentum fractions", resp. Of course:

$$\sum_{i=1}^{n} x^{i} = 1; \quad \sum_{i=1}^{n} \alpha^{i} = 1; \quad \sum_{i=1}^{n} \gamma^{i} = 1; \quad \sum_{i=1}^{n} \vec{\vec{\chi}}^{i} = \vec{\vec{I}} \text{ (unit tensor).}^{*)} \qquad (3.1.2)$$

---

*) In the two component case (n = 2) with liquid and vapor, we will, in chapter 4, write simply x for x^{v}, and (1-x) for x^{l}; and accordingly also for the other fractions.

Strictly spoken, the denotation of the (generally assymmetric) tensors $\vec{\vec{\chi}}^{i}$ as "fractions" is not good, but shall be kept for the moment. At the end of § 3.4, we will discuss the general suitability of such tensorial quantities.

We will further make the agreement that "partial" quantities shall be denoted by upper indices, i.e. those quantities whose sum gives directly - without any statistical weight - the total quantity *). Thus

$$\sum_{i=1}^{n} M^{i} = M; \sum_{i=1}^{n} V^{i} = V; \sum_{i=1}^{n} E^{i} = E; \sum_{i=1}^{n} \vec{P}^{i} = \vec{P} . \qquad (3.1.3)$$

3.2. Specific quantities

When proceeding to specific quantities, we must distinguish between

- "true" specific quantities, where the starting integral quantities are divided by their own (partial) masses M^{i}, to be denoted by lower indices,
- "partial" specific quantities, where the starting integral quantities are divided by the total mass M.

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*) Provided that a conservation law holds for this kind of quantity. This restriction becomes important when establishing the energy balance (§ 5.4).

This gives

	"true"	"partial"
specific masses	$x_i = M^i/M^i = 1$	$x^i = M^i/M$
specific volumes	$(\frac{1}{\rho})_i = V^i/M^i = \frac{1}{\rho_i}$ *)	$(\frac{1}{\rho})^i = V^i/M$
specific energies	$e_i = E^i/M^i$	$e^i = E^i/M$
specific momenta	$\vec{v}_i = \vec{P}^i/M^i$	$\vec{v}^i = \vec{P}^i/M$

(3.2.1)

As in chapter 2, all quotients should be understood as differential quotients; the above quantities are thus "field" quantities.

For the partial specific quantities holds

$$\sum_{i=1}^{n} x^i = 1; \quad \sum_{i=1}^{n} (\frac{1}{\rho})^i = \frac{1}{\rho}; \quad \sum_{i=1}^{n} e^i = e; \quad \sum_{i=1}^{n} \vec{v}^i = \vec{v} . \qquad (3.2.2)$$

They can also be expressed by means of the fractions:

$$x^i = x^i \cdot 1; \quad (\frac{1}{\rho})^i = \alpha^i \cdot \frac{1}{\rho}; \quad e^i = \gamma^i e; \quad \vec{v}^i = \vec{\chi}^i \cdot \vec{v} , \qquad (3.2.3)$$

similarly to (3.1.1). It can easily be verified that the occurring "fractions" for the specific quantities are just the same ones as for the integral quantities.

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*) The relation $(\frac{1}{\rho})_i = \frac{1}{\rho_i}$ is obtained by comparison with (3.3.1).

Whereas the partial quantities are formal but suitable quantities for computation purposes, the "true" quantities have a physical meaning and numerical values independent of the particular experimental situation. Apart from the trivial quantity $x_i = 1$, $(\frac{1}{\rho})_i$ means the specific volume of each component or phase as it can be found in tables; it is a state variable. For a given pressure e.g., it assumes a well-defined constant value at saturation. This is important, as then, for isobaric processes, $(\frac{1}{\rho})_i$ should not be differentiated in the balance equation whereas $(\frac{1}{\rho})^i$ should be differentiated.

e_i, the specific total energy which includes also the kinetic energy and other part energies, cannot be read from tables, as it is the case for u (internal energy) or for h (enthalpy), which are state variables. The relationships between these energy forms are considered in § 5.4.

The true specific momentum $\vec{v}_i$ is easily understood to be the true mass velocity of the component i.

In order to compute the partial specific quantities from the true ones, one has always to multiply with the mass fractions $x^i = M^i/M$, thus

$$x^i = x^i x_i (= x^i); \quad (\frac{1}{\rho})^i = x^i (\frac{1}{\rho})_i = \frac{x^i}{\rho_i}; \quad e^i = x^i e_i ;$$

$$\vec{v}^i = x^i \vec{v}_i . \qquad (3.2.4)$$

By substituting these relations into (3.2.2), one has (the first expression being trivial):

$$\sum_{i=1}^{n} x^i = 1; \quad \sum_{i=1}^{n} x^i (\frac{1}{\rho})_i = \frac{1}{\rho}; \quad \sum_{i=1}^{n} x^i e_i = e; \quad \sum_{i=1}^{n} x^i \vec{v}_i = \vec{v} .$$

$$(3.2.5)$$

Here we have found the recipe how to compute the quantities $\frac{1}{\rho}$, e, and $\vec{v}$, resp., which we must consider to be representative for the mixture, from the corresponding "true" values of the components.

The second relation of (3.2.5) must, later on, be compared with the averaging rule for the density ρ, to be derived in (3.3.6).

The third equation, applied to two phases, is the energy averaging rule for flowing wet steam. An equivalent rule holds for the entropy s. However, the well-known rules for the internal energy u and for the enthalpy h (cf. [3], p. 163) are restricted to mixtures with no relative motion of their components. This may be understood by considering that each component carries a kinetic energy k_ι which, in irreversible processes, may be partly or completely destroyed leading to supplementary internal energy of the joined body. Thus, for quantities without conservation law, such as for u, h, or k, $\Sigma\, u^\iota = \Sigma\, x^\iota u_\iota \neq u$, etc. Obviously, before establishing averaged quantities, all components must be thought to have already joined one another.

The last equation (3.2.5) finally gives the instruction how to compute the center of mass velocity $\vec{v}$ we had already spoken about in chapters 1 and 2.

By comparing formulas (3.2.3) and (3.2.4), the following relations between the "fractions" α^ι, γ^ι, and $\vec{\chi}^\iota$ with x^ι are obtained:

$$- \;;\quad \alpha^\iota \cdot \frac{1}{\rho} = x^\iota \cdot \left(\frac{1}{\rho}\right)_\iota ;\quad \gamma^\iota e = x^\iota e_\iota ;\quad \vec{\chi}^\iota \cdot \vec{v} = x^\iota \vec{v}_\iota \;. \tag{3.2.6}$$

The last relation indicates that the linear transformation described by $\vec{\chi}^{\iota}$ turns the vector $\vec{v}$ into the direction of $\vec{v}_{\iota}$.

The steam fraction x^{v} of wet steam is also (correctly) called "quality" in thermodynamics. α^{v} is the "void fraction". γ^{v} has no proper name. It is cautioned not to call $\vec{\chi}^{v}$ a quality, not even for one-dimensional flow where it reduces to a scalar, though its value is particularly easy to measure at the end of a test path. "Quality" suggests a material property that $\vec{\chi}^{v}$ is not because of the involved velocity situation.

Further relationships between fractions, specialized to two phase flow, will be given in chapter 4.

3.3. Densities

Let us now proceed to the various densities. Once more, we have to distinguish between "true" and "partial" densities, according to whether we divide the integral quantities by the individual partial volumes V^{ι} or by the total volume V.

	"true"	"partial"
mass densities	$\rho_{\iota} = M^{\iota}/V^{\iota}$	$\rho^{\iota} = M^{\iota}/V$
volume densities	$\alpha_{\iota} = V^{\iota}/V^{\iota} = 1$	$\alpha^{\iota} = V^{\iota}/V$
energy densities	$\varepsilon_{\iota} = E^{\iota}/V^{\iota}$	$\varepsilon^{\iota} = E^{\iota}/V$
momentum densities	$\vec{j}_{\iota} = \vec{P}^{\iota}/V^{\iota}$	$\vec{j}^{\iota} = \vec{P}^{\iota}/V$

(3.3.1)

The sum of the partial densities is the total density in each case:

$$\sum_{i=1}^{n} \rho^{i} = \rho; \quad \sum_{i=1}^{n} \alpha^{i} = 1; \quad \sum_{i=1}^{n} \varepsilon^{i} = \varepsilon; \quad \sum_{i=1}^{n} \vec{j}^{i} = \vec{j} . \qquad (3.3.2)$$

Like (3.1.3), we find

$$\rho^{i} = x^{i}\rho; \quad \alpha^{i} = \alpha^{i}\cdot 1; \quad \varepsilon^{i} = \gamma^{i}\varepsilon; \quad \vec{j}^{i} = \vec{\gamma}^{i}\cdot\vec{j} . \qquad (3.3.3)$$

Once more, the same fractions as for the integral and the specific quantities occur. This is obvious as the partial densities can also be computed from the partial specific quantities by multiplying the latter ones by ρ:

$$\rho^{i} = \rho x^{i}; \quad \alpha^{i} = \rho\left(\frac{1}{\rho}\right)^{i}; \quad \varepsilon^{i} = \rho e^{i}; \quad \vec{j}^{i} = \rho\vec{v}^{i} . \qquad (3.3.4)$$

The partial densities are computed from the known true densities by multiplying the latter ones with the volume fractions $\alpha^{i} = V^{i}/V$:

$$\rho^{i} = \alpha^{i}\rho_{i}; \quad \alpha^{i} = \alpha^{i}\alpha_{i} (= \alpha^{i}); \quad \varepsilon^{i} = \alpha^{i}\varepsilon_{i}; \quad \vec{j}^{i} = \alpha^{i}\vec{j}_{i} . \qquad (3.3.5)$$

This can be summed to yield:

$$\sum_{i=1}^{n} \alpha^{i}\rho_{i} = \rho; \quad \sum_{i=1}^{n} \alpha^{i} = 1; \quad \sum_{i=1}^{n} \alpha^{i}\varepsilon_{i} = \varepsilon; \quad \sum_{i=1}^{n} \alpha^{i}\vec{j}_{i} = \vec{j} . \qquad (3.3.6)$$

By comparing (3.3.5) with (3.3.3), one can relate x^ι, γ^ι, and $\overset{\rightrightarrows}{\chi}^\iota$ to α^ι:

$$x^\iota \rho = \alpha^\iota \rho_\iota \;;\quad - \qquad ;\; \gamma^\iota \varepsilon = \alpha^\iota \varepsilon_\iota \;;\quad \overset{\rightrightarrows}{\chi}^\iota \cdot \vec{J} = \alpha^\iota \vec{J}_\iota . \qquad (3.3.7)$$

The representation of the component densities by the specific quantities and the mass densities of the respective components is given by:

$$- \;;\qquad - \;;\quad \varepsilon_\iota = \rho_\iota e_\iota \;;\qquad \vec{J}_\iota = \rho_\iota \vec{v}_\iota \qquad (3.3.8)$$

$$- \;;\quad \alpha^\iota = \rho^\iota \left(\frac{1}{\rho}\right)_\iota \;;\quad \varepsilon^\iota = \rho^\iota e_\iota \;;\qquad \vec{J}^\iota = \rho^\iota \vec{v}_\iota \,. \qquad (3.3.9)$$

Comparison of the second relation of (3.2.5) with the first relation of (3.3.6) gives

$$\rho = \sum_{\iota=1}^{n} \alpha^\iota \rho_\iota = \frac{1}{\sum\limits_{\iota=1}^{n} \dfrac{x^\iota}{\rho_\iota}} \,. \qquad (3.3.10)$$

In the case of n = 2, this yields a coupling between α^ι and x^ι by means of the known ρ_ι. This shall be considered in more generality in the following chapter. For $n > 2$, similar equations are not obtainable. Notice that eq. (3.3.10) would not be true for the single components.

3.4. Current densities

As in chapter 2, we multiply the various densities with appropriate velocities in order to get the pertaining current densities. From the demand that once more the partial quantities should sum up to the total quantity, those velocities can be computed.

In order to calculate first the "true" component current densities, we multiply the respective true densities with the only available true velocities $\vec{v}_\iota$, namely

$$\vec{j}_\iota = \rho_\iota \vec{v}_\iota \; ; \quad \vec{\tilde{v}}_\iota = \alpha_\iota \vec{v}_\iota = \vec{v}_\iota \; ; \quad \vec{q}_\iota = \varepsilon_\iota \vec{v}_\iota \; ; \quad \overset{\rightrightarrows}{\Gamma}_\iota = \vec{j}_\iota \vec{v}_\iota . \tag{3.4.1}$$

Here, in the case of a homogeneous component, we may indeed represent $\overset{\rightrightarrows}{\Gamma}_\iota$ as a dyadic product *).

The "true volumetric velocities" $\vec{\tilde{v}}_\iota$ of components are equal to the respective mass velocities $\vec{v}_\iota$.

To convert true densities to partial densities, the former ones are to be multiplied by α^ι. As a consequence, the current densities behave equally:

$$\vec{j}^\iota = \rho^\iota \vec{v}_\iota \; ; \quad \vec{\tilde{v}}^\iota = \alpha^\iota \vec{v}_\iota \; ; \quad \vec{q}^\iota = \varepsilon^\iota \vec{v}_\iota \; ; \quad \overset{\rightrightarrows}{\Gamma}^\iota = \vec{j}^\iota \vec{v}_\iota = \rho^\iota \vec{v}_\iota \vec{v}_\iota . \tag{3.4.2}$$

---

*) The chosen factor sequence is arbitrary as $\vec{j}_\iota \vec{v}_\iota = \rho_\iota \vec{v}_\iota \vec{v}_\iota$ is symmetric, and the dyadic product is exceptionally commutative.

For the mass current density which is identical with the momentum density $\vec{j}$, this was already stated in (3.[illegible]).

But for the first time it comes out that the partial "volumetric" velocities $\vec{\tilde{v}}^{i}$ of the components are not equal to the respective $\vec{v}^{i} = x^{i}\vec{v}_{i}$.

By summing up the partial quantities we get

$$\sum_{i=1}^{n} \vec{j}^{i} = \vec{j} \quad ; \quad \sum_{i=1}^{n} \vec{\tilde{v}}^{i} = \vec{\tilde{v}} \quad ; \quad \sum_{i=1}^{n} \vec{q}^{i} = \vec{q} \quad ; \quad \sum_{i=1}^{n} \vec{\vec{\Gamma}}^{i} = \vec{\vec{\Gamma}} \ . \tag{3.4.3}$$

If we substitute the expressions (3.3.3) into (3.4.2) and add according to (3.4.3), we find

$$\rho\sum_{i=1}^{n} x^{i}\vec{v}_{i} = \rho\vec{v} \quad ; \quad 1\cdot\sum_{i=1}^{n} \alpha^{i}\vec{v}_{i} = \vec{\tilde{v}} \quad ; \quad \varepsilon\sum_{i=1}^{n} \gamma^{i}\vec{v}_{i} = \varepsilon\vec{\hat{v}} \quad ;$$

$$\sum_{i=1}^{n} (\vec{\chi}^{i}\cdot\vec{j})\vec{v}_{i} = \vec{\vec{\Gamma}} \neq \text{any } \vec{j}\vec{v} \ ^{*)} \quad ; \tag{3.4.4}$$

where the right hand sides are taken from table

---

*) As no associative law holds for this kind of multiplication, it is impossible to turn the linear operator $\vec{\chi}^{i}$ from $\vec{j}$ to $\vec{v}_{i}$ so as to isolate $\vec{j}$ and to cancel it from both sides of the equation.

The "mixture" velocities in question are thus found to be:

$$\left.\begin{array}{lll}\text{mass:} & & \vec{v} = \sum_{i=1}^{n} x^{i}\vec{v}_{i} \\[2ex] \text{volume:} & & \vec{\tilde{v}} = \sum_{i=1}^{n} \alpha^{i}\vec{v}_{i} \\[2ex] \text{energy:} & & \vec{\hat{v}} = \sum_{i=1}^{n} \gamma^{i}\vec{v}_{i} \\[2ex] \text{momentum:} & & \text{"}\vec{\ddot{v}}\text{" not separately existing in general.}\end{array}\right\} \qquad (3.4.5)$$

The statistical weights to construct them are just the original ratios known from the integral quantities.

The following relations may also be useful (to deduce from 3.4.5 with 3.2.3):

$$\sum_{i=1}^{n} x^{i}\vec{v}_{i} = \vec{v} \quad ; \quad \sum_{i=1}^{n} \left(\frac{1}{\rho}\right)^{i}\vec{v}_{i} = \frac{1}{\rho}\,\vec{\tilde{v}} \quad ; \quad \sum_{i=1}^{n} e^{i}\vec{v}_{i} = e\vec{\hat{v}} \quad ;$$

$$\text{but} \quad \sum_{i=1}^{n} \vec{\ddot{v}}^{i}\vec{v}_{i} \neq \vec{v}\vec{\ddot{v}} \; . \qquad (3.4.6)$$

It remains still to give relationships between partial and total current densities, for which we write ad hoc

$$\vec{j}^{\,i} = \overset{\rightrightarrows}{\chi}{}^{i} \cdot \vec{j} \quad ; \quad \vec{\tilde{v}}^{\,i} = \overset{\rightrightarrows}{\beta}{}^{i} \cdot \vec{\tilde{v}} \quad ; \quad \vec{\sigma}^{\,i} = \overset{\rightrightarrows}{\delta}{}^{i} \cdot \vec{\sigma} \quad ; \quad \overset{\rightrightarrows}{\Gamma}{}^{i} = \overset{\rightrightarrows}{\overset{\rightrightarrows}{\psi}}{}^{i} \cdot \overset{\rightrightarrows}{\Gamma} \quad {}^{*)}.$$

$$(3.4.7)$$

The first relation is already known from (3.3.3), last equation; $\overset{\rightrightarrows}{\chi}{}^{i}$ could thus directly be introduced.

For the other relations, we substitute (3.3.3) into (3.4.2), leading to

$$x^{i} \rho \vec{v}_{i} = \overset{\rightrightarrows}{\chi}{}^{i} \cdot \rho \vec{v} \quad ; \quad \alpha^{i} \vec{v}_{i} = \overset{\rightrightarrows}{\beta}{}^{i} \cdot \vec{\tilde{v}} \quad ; \quad \gamma^{i} \varepsilon \vec{v}_{i} = \overset{\rightrightarrows}{\delta}{}^{i} \cdot \varepsilon \vec{\hat{v}} \quad ;$$

$$(\overset{\rightrightarrows}{\chi}{}^{i} \cdot \vec{j}) \vec{v}_{i} = \overset{\rightrightarrows}{\overset{\rightrightarrows}{\psi}}{}^{i} \cdot \overset{\rightrightarrows}{\Gamma} \quad , \qquad (3.4.8)$$

with

$$\sum_{i=1}^{n} \overset{\rightrightarrows}{\chi}{}^{i} = \overset{\rightrightarrows}{I} \quad ; \quad \sum_{i=1}^{n} \overset{\rightrightarrows}{\beta}{}^{i} = \overset{\rightrightarrows}{I} \quad ; \quad \sum_{i=1}^{n} \overset{\rightrightarrows}{\delta}{}^{i} = \overset{\rightrightarrows}{I} \quad ; \quad \sum_{i=1}^{n} \overset{\rightrightarrows}{\overset{\rightrightarrows}{\psi}}{}^{i} = \overset{\rightrightarrows}{\overset{\rightrightarrows}{I}} \quad .$$

$$(3.4.9)$$

The significance of $\overset{\rightrightarrows}{\beta}{}^{i}$, $\overset{\rightrightarrows}{\delta}{}^{i}$, and $\overset{\rightrightarrows}{\overset{\rightrightarrows}{\psi}}{}^{i}$ is purely academic, as far as they cannot be computed unambiguously from the various given vectors $\vec{v}$, $\vec{\tilde{v}}$, etc.

---

*) $\overset{\rightrightarrows}{\overset{\rightrightarrows}{\psi}}{}^{i}$ means a tensor of order 4.

This brings us to the problem of the utility of such tensorial quantities in connection with n-component flow, as already mentioned in § 3.1. We do not speak here about the tensors in the general balances treated in chapter 2, such as the momentum current density $\overset{\rightrightarrows}{\Gamma}$, or the dyadic products occurring for $\overset{\rightrightarrows}{P}{}^i$. They all have a real physical meaning.

The question is rather what is about the "fractions" $\overset{\rightrightarrows}{\chi}{}^i$, $\overset{\rightrightarrows}{\beta}{}^i$, $\overset{\rightrightarrows}{\delta}{}^i$, and $\overset{\Rrightarrow}{\psi}{}^i$. Let us discuss e.g. the last equation of (3.2.3), which may be considered as definition for $\overset{\rightrightarrows}{\chi}{}^i$. The vectors $\vec{v}^i$ and $\vec{v}$ are all known. Assume for the moment m-dimensional vectors, then we have just m equations for a certain i, whereas the "unknown" tensor $\overset{\rightrightarrows}{\chi}{}^i$ contains m^2 elements. Thus, equations of this kind are solvable only for $m = 1$, i.e. for scalars. For $m \geq 2$, no unambiguous solution exists. None of the tensors in question has therefore a reasonable meaning except for one-dimensional flow where they reduce to scalars. In more-dimensional formulations, they must completely be avoided.

3.5. Flow rates

Although integral quantities have nothing to do with the differential formulation of balances, we bring them here for completeness. This is in particular in order to show the virtue of χ^i which is widely used as "quality" of a two-phase flow (cf. also eqs. 4.6.3).

Similarly to eqs. (2.1), we get partial flow rates

$$\left.\begin{aligned}
Q_M^i &= -\int_A \vec{j}^{\,i} \cdot d\vec{A} = -\int_A (\overset{\rightrightarrows}{\chi}{}^i \cdot \vec{j}) \cdot d\vec{A} \\
Q_V^i &= -\int_A \vec{\tilde{v}}^{\,i} \cdot d\vec{A} = -\int_A (\overset{\rightrightarrows}{\beta}{}^i \cdot \vec{\tilde{v}}) \cdot d\vec{A} \\
Q_E^i &= -\int_A \vec{q}^{\,i} \cdot d\vec{A} = -\int_A (\overset{\rightrightarrows}{\delta}{}^i \cdot \vec{q}) \cdot d\vec{A} \\
\vec{Q}_{\vec{P}}^{\,i} &= -\int_A d\vec{A} \cdot \overset{\rightrightarrows}{\Gamma}{}^i = -\int_A d\vec{A} \cdot (\overset{\Rrightarrow}{\psi}{}^i \cdot \overset{\rightrightarrows}{\Gamma}) \;,
\end{aligned}\right\} \qquad (3.5.1)$$

where the relations (3.4.7) have been used.

In view of the unprofitableness of $\overset{\rightrightarrows}{\chi}$, $\overset{\rightrightarrows}{\beta}$, $\overset{\rightrightarrows}{\delta}{}^i$, and $\overset{\Rrightarrow}{\psi}{}^i$ (see § 3.4) for more-dimensional flow, we come back to the above relations only in chapter 4.

4. Relationships for one-dimensional two phase flow

When considering two phase flow, in particular water and steam, through a straight pipe, it is rather obvious to simplify the balance equations by restriction to a single axial coordinate. All vectors and tensors reduce to scalar quantities, omitting the components pertaining to the other coordinates. The objections arising from such a neglect shall be treated only in chapter 5.

Here we will compute the formulas of the preceding chapters for n = 2 phases. The fractions of vapor shall be denoted, as usual, by x, α, etc., without index, those of liquid by (1-x), (1-α), etc. For the indices used, v means <u>v</u>apor, l means <u>l</u>iquid. For upper indices we write ″ instead of v, and ′ instead of l.

The elementary volume V is a disk with area A and any (insignificant) thickness dz. All integral quantities can equally be considered as quantities per unit length.

We give formulas for the practical use without much comment, referring simply to the relations of preceding chapters where they have been taken from, in square brackets.

4.1. Integral quantities

[3.1.1]:

$$\left.\begin{aligned} M'' &= xM \quad ; & M' &= (1-x)M \\ V'' &= \alpha V \quad ; & V' &= (1-\alpha)V \\ E'' &= \gamma E \quad ; & E' &= (1-\gamma)E \\ P'' &= \chi P \quad ; & P' &= (1-\chi)P \end{aligned}\right\} \tag{4.1.1}$$

The P are the z-components of the complete momenta.

4.2. Specific quantities

"true" specific quantities [3.2.1]:

$$\left.\begin{aligned} &\left(\frac{1}{\rho}\right)_v = \frac{V''}{M''} = \frac{1}{\rho_v} && \left(\frac{1}{\rho}\right)_l = \frac{V'}{M'} = \frac{1}{\rho_l} \\ &e_v = \frac{E''}{M''} && e_l = \frac{E'}{M'} \\ &v_v = \frac{P''}{M''} && v_l = \frac{P'}{M'} \end{aligned}\right\} \qquad (4.2.1)$$

"partial" specific quantities [3.2.1], [3.2.3], [3.2.4]:

$$\left.\begin{aligned} &x = \frac{M''}{M} && (1-x) = \frac{M'}{M} \\ &\left(\frac{1}{\rho}\right)'' = \frac{V''}{M} = \frac{\alpha}{\rho} = x\left(\frac{1}{\rho}\right)_v && \left(\frac{1}{\rho}\right)' = \frac{V'}{M} = \frac{1-\alpha}{\rho} = (1-x)\left(\frac{1}{\rho}\right)_l \\ &e'' = \frac{E''}{M} = \gamma e = x e_v && e' = \frac{E'}{M} = (1-\gamma)e = (1-x)e_l \\ &v'' = \frac{P''}{M} = \chi v = x v_v && v' = \frac{P'}{M} = (1-\chi)v = (1-x)v_l \end{aligned}\right\}$$

(4.2.2)

specific quantities of mixture [3.2.5]:

$$\left.\begin{aligned} \frac{1}{\rho} &= \frac{x}{\rho_v} + \frac{1-x}{\rho_l} \\ e &= x e_v + (1-x) e_l \\ v &= x v_v + (1-x) v_l \end{aligned}\right\} \qquad (4.2.3)$$

relations between fractions [3.2.6]:

$$\left.\begin{aligned} \frac{\alpha}{x} &= \frac{\rho}{\rho_v} & \qquad \frac{1-\alpha}{1-x} &= \frac{\rho}{\rho_l} \\ \frac{\gamma}{x} &= \frac{e_v}{e} & \qquad \frac{1-\gamma}{1-x} &= \frac{e_l}{e} \\ \frac{\chi}{x} &= \frac{v_v}{v} & \qquad \frac{1-\chi}{1-x} &= \frac{v_l}{v} \end{aligned}\right\} \qquad (4.2.4)$$

4.3. Densities

"true" densities [3.3.1], [3.3.8]:

$$\left.\begin{aligned} \rho_v &= \frac{M''}{V''} & & \qquad \rho_l &= \frac{M'}{V'} & \\ \varepsilon_v &= \frac{E''}{V''} &= \rho_v e_v & \qquad \varepsilon_l &= \frac{E'}{V'} &= \rho_l e_l \\ j_v &= \frac{P''}{V''} &= \rho_v v_v & \qquad j_l &= \frac{P'}{V'} &= \rho_l v_l \end{aligned}\right\} \qquad (4.3.1)$$

"partial" densities [3.3.1], [3.3.3], [3.3.4], [3.3.5]:

$$\left.\begin{array}{ll}
\rho'' = \dfrac{M''}{V} = x\rho = \alpha\rho_v & \rho' = \dfrac{M'}{V} = (1-x) = (1-\alpha)\rho_l \\[2ex]
\varepsilon'' = \dfrac{E''}{V} = \gamma\varepsilon = \alpha\varepsilon_v = \rho e'' = \rho'' e_v & \varepsilon' = \dfrac{E'}{V} = (1-\gamma) = (1-\alpha)\varepsilon_l = \rho e' = \rho' e_l \\[2ex]
j'' = \dfrac{P''}{V} = \chi j = \alpha j_v = \rho v'' = \rho'' v_v & j' = \dfrac{P'}{V} = (1-\chi) j = (1-\alpha) j_l = \rho v' = \rho' v_l
\end{array}\right\} \qquad (4.3.2)$$

mixture densities [3.3.6]:

$$\left.\begin{array}{l}
\rho = \alpha\rho_v + (1-\alpha)\rho_l \\[2ex]
\rho e = \varepsilon = \alpha\varepsilon_v + (1-\alpha)\varepsilon_l = \alpha\rho_v e_v + (1-\alpha)\rho_l e_l \\[2ex]
\rho v = j = \alpha j_v + (1-\alpha) j_l = \alpha\rho_v v_v + (1-\alpha)\rho_l v_l
\end{array}\right\} \qquad (4.3.3)$$

relations between fractions [3.3.7]:

$$\left.\begin{array}{ll}
\frac{x}{\alpha} = \frac{\rho_v}{\rho} & \frac{1-x}{1-\alpha} = \frac{\rho_l}{\rho} \\[2ex]
\frac{\gamma}{\alpha} = \frac{\varepsilon_v}{\varepsilon} = \frac{\rho_v e_v}{\rho e} & \frac{1-\gamma}{1-\alpha} = \frac{\varepsilon_l}{\varepsilon} = \frac{\rho_l e_l}{\rho e} \\[2ex]
\frac{\chi}{\alpha} = \frac{j_v}{j} = \frac{\rho_v v_v}{\rho v} & \frac{1-\chi}{1-\alpha} = \frac{j_l}{j} = \frac{\rho_l v_l}{\rho v}
\end{array}\right\} \qquad (4.3.4)$$

4.4. Current densities (now scalars)

"true" current densities [3.4.1]:

$$\left.\begin{array}{ll}
j_v = \rho_v v_v & j_l = \rho_l v_l \\[1ex]
\tilde{v}_v = v_v & \tilde{v}_l = v_l \\[1ex]
q_v = \varepsilon_v v_v = \rho_v e_v v_v & q_l = \varepsilon_l v_l = \rho_l e_l v_l \\[1ex]
\Gamma_v = j_v v_v = \rho_v v_v^2 & \Gamma_l = j_l v_l = \rho_l v_l^2
\end{array}\right\} \qquad (4.4.1)$$

"partial" current densities [3.4.2], [3.4.7]:

$$\left.\begin{array}{ll}
j'' = \rho'' v_v = \chi j = \alpha j_v & j' = \rho' v_l = (1-\chi) j = (1-\alpha) j_l \\[1ex]
\tilde{v}'' = \alpha v_v = \beta \tilde{v} = \alpha \tilde{v}_v & \tilde{v}' = (1-\alpha) v_l = (1-\beta)\tilde{v} = (1-\alpha)\tilde{v}_l \\[1ex]
q'' = \varepsilon'' v_v = \delta q = \alpha q_v & q' = \varepsilon' v_l = (1-\delta) q = (1-\alpha) q_l \\[1ex]
\Gamma'' = j'' v_v = \psi \Gamma = \alpha \Gamma_v & \Gamma' = j' v_l = (1-\psi)\Gamma = (1-\alpha)\Gamma_l
\end{array}\right\}$$

$$(4.4.2)$$

mixture current densities [3.4.4]:

$$\left.\begin{aligned}
j &= \alpha j_v + (1-\alpha) j_l = \alpha\rho_v v_v + (1-\alpha)\rho_l v_l = \rho[x v_v + (1-x) v_l] = \rho v \\
\tilde{v} &= \alpha\tilde{v}_v + (1-\alpha)\tilde{v}_l \\
q &= \alpha q_v + (1-\alpha) q_l = \alpha\rho_v e_v v_v + (1-\alpha)\rho_l e_l v_l = \rho e[\gamma v_v + (1-\gamma) v_l] = \rho e\hat{v} \\
\Gamma &= \alpha\Gamma_v + (1-\alpha)\Gamma_l = \alpha\rho_v v_v^2 + (1-\alpha)\rho_l v_l^2 = \rho v[\chi v_v + (1-\chi) v_l] = \rho v\ddot{v} \;{}^{*)}
\end{aligned}\right\} \qquad (4.4.3)$$

definition of velocities [3.4.5]:

$$\left.\begin{aligned}
v &= x v_v + (1-x) v_l \\
\tilde{v} &= \alpha v_v + (1-\alpha) v_l \\
\hat{v} &= \gamma v_v + (1-\gamma) v_l \\
\ddot{v} &= \chi v_v + (1-\chi) v_l
\end{aligned}\right\} \qquad (4.4.4)$$

---

*) Here, and only in this one-dimensional case, $\ddot{v}$ gets a meaning: $\ddot{v} = \sum_{i=1}^{n} \chi^i v_i$

4.5. Relationships between fractions

In contrast to formulas (4.2.4) and (4.3.4), the relations can also be given without recourse to mixture properties, but with the "true" phase properties only.

For this purpose, we define the following ratios between "true" specific quantities

$$\left.\begin{array}{lll} \text{- the specific volume ratio} & \xi & = \dfrac{(1/\rho)_v}{(1/\rho)_l} = \dfrac{\rho_l}{\rho_v} \\ \text{- the specific energy ratio} & \eta & = \dfrac{e_v}{e_l} \\ \text{- the specific momentum ratio} & S & = \dfrac{v_v}{v_l} \; . \\ \text{(= velocity ratio or \underline{slip} \underline{ratio})} & & \end{array}\right\} \quad (4.5.1)$$

For a given saturation pressure, ξ is numerically known. The same is but not true for the slip ratio S, as it is not built by state variables, nor for η, as e is no state variable.

Let

$$e_v = u_v + k_v = h_v - \left(\frac{p}{\rho}\right)_v + k_v ; \; e_l = u_l + k_l = h_l - \left(\frac{p}{\rho}\right)_l + k_l \; ^{*)} \qquad (4.5.2)$$

where $k_v = \frac{v_v^2}{2}$ and $k_l = \frac{v_l^2}{2}$ are the specific kinetic energies and u_v, u_l the specific internal energies of the components. Then

$$\eta = \frac{u_v + k_v - (u_l + k_l)}{u_l + k_l} - 1 \approx \frac{u_v - u_l}{u_l} - 1 \; . \qquad (4.5.3)$$

---

*) $h = u + \frac{p}{\rho}$ denotes a specific enthalpy

$u_v - u_l$ is the evaporation heat at constant volume (constant ρ). In most cases, k_l is negligible against u_l; nevertheless, (4.5.3) is rigorous only for the components at rest.

Similarly,

$$\eta = \frac{h_v-(p/\rho)_v+k_v-[h_l-(p/\rho)_l+k_l]}{h_l-(p/\rho)_l+k_l} - 1 \approx \frac{h_v-h_l}{h_l} - 1, \tag{4.5.4}$$

where $h_v - h_l$ is the evaporation heat at constant pressure p, and the so-called expansion energy p/ρ is omitted.

It is further to be noticed that both h_l and u_l are known except for an additive constant. For saturated water of 0°C, h_l is, by convention, equated to zero. The enthalpy difference h_{lv} (and u_{lv}, too) is free from such arbitrariness and thus by far more suitable than η for theoretical purposes.

From (4.2.3), (4.2.4), (4.3.3), and (4.3.4), the following relationships are easily obtainable

$$\left.\begin{aligned}
x &= \frac{\alpha}{\alpha + \xi(1-\alpha)} = \frac{\gamma}{\gamma + \eta(1-\gamma)} = \frac{\chi}{\chi + S(1-\chi)}\\
\alpha &= \frac{\xi x}{\xi x + (1-x)} = \frac{\xi\gamma}{\xi\gamma + \eta(1-\gamma)} = \frac{\xi\chi}{\xi\chi + S(1-\chi)}\\
\gamma &= \frac{\eta x}{\eta x + (1-x)} = \frac{\eta\alpha}{\eta\alpha + \xi(1-\alpha)} = \frac{\eta\chi}{\eta\chi + S(1-\chi)}\\
\chi &= \frac{Sx}{Sx + (1-x)} = \frac{S\alpha}{S\alpha + \xi(1-\alpha)} = \frac{S\gamma}{S\gamma + \eta(1-\gamma)}
\end{aligned}\right\} \tag{4.5.5}$$

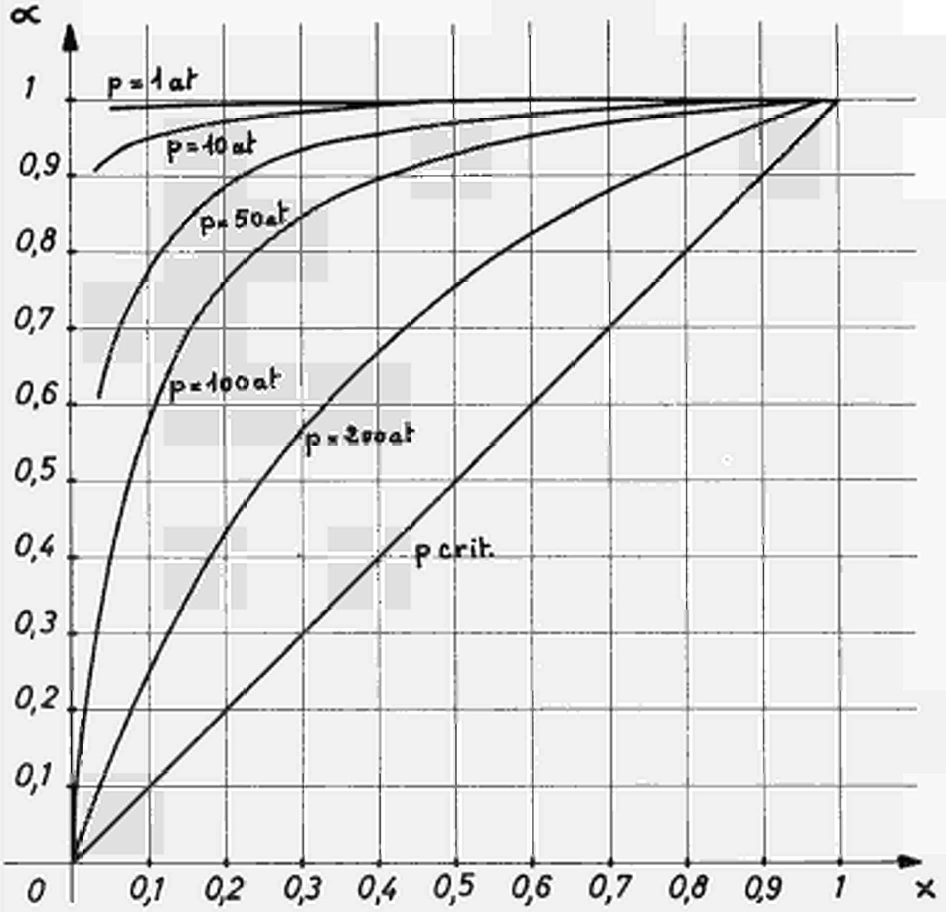

Figure 1

„Void fraction" α vs. „mass. fraction" x (eq. 4.5.5,2)

(Saturation pressure p as parameter)

4.6. Flow rates

"partial" flow rates [3.5.1]:

$$\left.\begin{aligned}
Q''_M &= -\int_A \chi j\,dA & Q'_M &= -\int_A (1-\chi)j\,dA \\
Q''_V &= -\int_A \beta\tilde{v}\,dA & Q'_V &= -\int_A (1-\beta)\tilde{v}\,dA \\
Q''_E &= -\int_A \delta q\,dA & Q'_E &= -\int_A (1-\delta)q\,dA \\
Q''_P &= -\int_A \psi\Gamma\,dA & Q'_P &= -\int_A (1-\psi)\Gamma\,dA\,.
\end{aligned}\right\} \quad (4.6.1)$$

The integrations must be extended over flow cross sections A perpendicular to the flow direction. But as a variability of the integrands normal to the flow axis cannot be considered in a consistent one-dimensional theory (unique position coordinate is z), the above expressions degenerate to

$$\left.\begin{aligned}
Q''_M &= -\chi jA & Q'_M &= -(1-\chi)jA \\
Q''_V &= -\beta\tilde{v}A & Q'_V &= -(1-\beta)\tilde{v}A \\
Q''_E &= -\delta qA & Q'_E &= -(1-\delta)qA \\
Q''_P &= -\psi\Gamma A & Q'_P &= -(1-\psi)\Gamma A\,.
\end{aligned}\right\} \quad (4.6.2)$$

If the left hand expressions are divided by the total flow rates Q_M, Q_V, Q_E, Q_P, resp., (see 2.1., specialized to one dimension), the area A and the current densities j, $\tilde{v}$, q, and Γ, cancel in all cases so that follows

$$\left.\begin{aligned}
\chi &= \frac{Q''_M}{Q_M} && \text{mass flow rate ratio} \\
\beta &= \frac{Q''_V}{Q_V} && \text{volumetric flow rate ratio} \\
\delta &= \frac{Q''_E}{Q_E} && \text{energy flow rate ratio} \\
\psi &= \frac{Q''_P}{Q_P} && \text{momentum flow rate ratio.}
\end{aligned}\right\} \qquad (4.6.3)$$

All these quantities are integral quantities, thus foreign to a differentially formulated theory. They are results of the processes inside a test distance and have a meaning only at its end.

In particular, the relatively easily measurable quantity χ, frequently called "quality" in the literature, is not suitable to be used in our theory. Obviously, any correlation between integral and locally defined (field-) quantities is without meaning, e.g. between χ and α, or similar ones, as it can be stated between two local quantities resp., like between x and α (see § 4.5).

5. The balance equations

5.1. Mass balance

After the lengthy preparations of chapters 2, 3, and 4, we will now use the general balances, either in "local" form (1.5) or in substantial form (1.7), and specify the general current densities $\vec{\Phi}_Y$ and the production densities σ_Y.

In the case of mass, y is the "specific mass", i.e. unity (see table 1). The current density $\vec{\Phi}_M$ is $\vec{J} = \rho\vec{v}$, and, without mass sources, $\sigma_M = 0$.

This gives

$$\frac{\partial\rho}{\partial t} = - \operatorname{div} \rho\vec{v}. \qquad (5.1.1)$$

The substantial formulation (1.7) is not directly suitable in the mass case. But by applying (1.8) on the density ρ and considering (5.1.1) one finds

$$\frac{D\rho}{Dt} = - \rho \operatorname{div} \vec{v} , \qquad (5.1.2)$$

another well-known formulation of the continuity equation.

We now proceed to the balance of partial masses. y is here x^i (partial mass per total mass, cf. 3.2.1), and $\vec{\Phi}^i_M = \vec{J}^i = \rho^i\vec{v}_i$ (cf. 3.4.2). This can all be chosen quite schematically.

Thus, according to (1.5):

$$\frac{\partial(\rho x^{i})}{\partial t} = \frac{\partial \rho^{i}}{\partial t} = - \operatorname{div} \rho^{i}\vec{v}_{i} + \sigma_{M}^{i} , \qquad (5.1.3)$$

or substantially (1.7):

$$\rho \frac{Dx^{i}}{Dt} = - \operatorname{div} (\rho^{i}\vec{v}_{i} - \rho^{i}\vec{v}) + \sigma_{M}^{i} , \qquad (5.1.4)$$

where $\sum_{i=1}^{n} \sigma_{M}^{i} = 0$.

The expression $\rho^{i}(\vec{v}_{i}-\vec{v})$ will also appear later on. It is the excess of the partial current density over the correction due to the movement of observation point and may be called "(mass-) diffusion current density"

$$\vec{J}^{i} = \rho^{i}(\vec{v}_{i}-\vec{v}) \qquad (5.1.5)$$

of the component i.

Of course, $\sum_{i=1}^{n} \vec{J}^{i} = 0$.

Eq. (5.1.3) may herewith also be written as

$$\frac{\partial \rho^{i}}{\partial t} = - \operatorname{div} \rho^{i}\vec{v} - \operatorname{div} \vec{J}^{i} + \sigma_{M}^{i}. \qquad (5.1.6)$$

When specializing the formulas to one-dimensional two phase flow, we have

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial z} + \rho \frac{\partial v}{\partial z} = 0 \tag{5.1.7}$$

$$\left.\begin{aligned} \frac{\partial \rho'}{\partial t} + v_l \frac{\partial \rho'}{\partial z} + \rho' \frac{\partial v_l}{\partial z} &= \sigma'_M(z,t) \\ \frac{\partial \rho''}{\partial t} + v_v \frac{\partial \rho''}{\partial z} + \rho'' \frac{\partial v_v}{\partial z} &= \sigma''_M(z,t) \; . \end{aligned}\right\} \tag{5.1.8}$$

The equations are not independent one from the others, as the sum of eqs. (5.1.8) gives just eq. (5.1.7).

In the one-dimensional treatment, the overall suitability of which for boiling phenomena investigations is not assured, the partial densities ρ^i can be replaced by the known true densities ρ_i. At the same time the vapor velocity v_v may be reduced to the liquid velocity v_l by introducing the slip ratio $S(z,t)$. The suitability of this measure in view of a solution of the system is also doubtful.

Eqs. (5.1.8) transform to

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left[\rho_l(1-\alpha)\right] + v_l \frac{\partial}{\partial z}\left[\rho_l(1-\alpha)\right] + \rho_l(1-\alpha)\frac{\partial v_l}{\partial z} &= \sigma'_M(z,t) \\ \frac{\partial}{\partial t}(\rho_v \alpha) + S(z,t) v_l \frac{\partial}{\partial z}(\rho_v \alpha) + \rho_v \alpha \frac{\partial}{\partial z}\left[S(z,t) v_l\right] &= -\sigma'_M(z,t) \end{aligned}\right\} \tag{5.1.9}$$

or, if ρ_l and ρ_v are constant for constant saturation pressure,

$$\left.\begin{aligned} -\frac{\partial\alpha}{\partial t} - v_l\frac{\partial\alpha}{\partial z} + (1-\alpha)\frac{\partial v_l}{\partial z} &= \frac{\sigma_M'(z,t)}{\rho_l} \\ +\frac{\partial\alpha}{\partial t} + S(z,t)v_l\frac{\partial\alpha}{\partial z} + \alpha\left[v_l\frac{\partial S}{\partial z} + S(z,t)\frac{\partial v_l}{\partial z}\right] &= -\frac{\sigma_M'(z,t)}{\rho_v}\,. \end{aligned}\right\} \qquad (5.1.10)$$

The state variable α acts as a dimensionless mass surrogate. Emphasis is to be laid on the fact that $S(z,t)$ enters into the equation as a foreign body. The replacement of v_v by Sv_l reveals to be only a pseudo-simplification as long as the slip ratio S is not known as function of z and t *)

The right-hand inhomogeneous terms must be otherwise procured (from the energy equation, as we shall see).

5.2. Volume balance

Though resulting only in trivialities, the formal procedure shall also be applied to our volume quantities, in order to show the overall consistency.

The specific volume is $y = 1/\rho$, and the volume current density is $\vec{\Phi}_V = \vec{v}$, according to table 1.

---

*) This question is illustrated more clearly at the end of § 5.3.

Thus, the local formulation yields

$$\frac{\partial}{\partial t}\,"1" = - \operatorname{div} \vec{\vec{v}} + \sigma_V = 0 \qquad (5.2.1)$$

which means that no geometrical volume storage is possible. The substantial formulation reads

$$\rho \frac{D}{Dt}\left(\frac{1}{\rho}\right) = - \operatorname{div} (\vec{\vec{v}} - \vec{v}) + \sigma_V \;^{*)} \qquad (5.2.2)$$

or

$$- \frac{1}{\rho}\frac{D\rho}{Dt} = + \operatorname{div} \vec{v}, \qquad (5.2.3)$$

when cancelling div $\vec{\vec{v}}$ due to (5.2.1). This is just eq. (5.1.2).

Notice the important difference between $\vec{v}$ and the special velocity $\vec{\vec{v}}$.

The partial volume balances may be omitted.

5.3. Momentum balance

As the energy balance requires some additional thermodynamic considerations, we will first investigate the momentum balance.

This balance is called NAVIER-STOKES' equation of movement in hydrodynamics. Thus, in the case of homogeneous one-component flow, we should arrive at some well-known notation, but the heterogeneous flow will lead to a more sophisticated formulation, as we shall see.

---

*) Besides, the source can be evaluated to be $\sigma_V = \sum_{i=1}^{n} \sigma_V^i = \sum_{i=1}^{n} \sigma_M^i/\rho_i$ by comparison with the partial mass balances.

According to our general procedure, we choose for y the specific momentum $\vec{v}$, a vector; see table 1.

The momentum current density, to be read from eq. (3.4.4), is

$$\vec{\vec{\Gamma}} = \sum_{\iota=1}^{n} (\vec{\vec{y}}^{\iota} \cdot \vec{j})\vec{v}_{\iota} = \sum_{\iota=1}^{n} \rho^{\iota}\vec{v}_{\iota}\vec{v}_{\iota} .$$

This is the pure convection part and is not complete. As well known, particles of real fluids interact mutually by pressure and viscous forces resulting in a momentum transfer. This additional (and preponderant) momentum current density is described by the stress tensor

$$\vec{\vec{\Pi}} = \left\| \begin{matrix} p+p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p+p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p+p_{zz} \end{matrix} \right\|^{*)} . \quad [\mathrm{kg\ m^{-1}\ s^{-2}}] \qquad (5.3.1)$$

The normal pressure $p^{**)}$ is superimposed by "viscous pressures" $p_{j\iota}$ which, according to STOKES, are given by

$$p_{j\iota} = -\mu \left(\frac{\partial v_{(j)}}{\partial l} + \frac{\partial v_{(\iota)}}{\partial j} \right) - \mu' \delta_{j\iota} \operatorname{div} \vec{v} \qquad \begin{matrix} j = x,y,z \\ \iota = x,y,z . \end{matrix}$$

$$(5.3.2)$$

---

*) For the description in non-Cartesian coordinates, cf. [7].

**) This pressure $p(t,\vec{r})$ is one of our system variables and not equal to the "hydrostatic" pressure.

The tensor $\overset{\rightrightarrows}{\Pi}$ is symmetric: $p_{jl} = p_{lj}$. μ is the usual dynamic viscosity and μ' is the so-called "second viscosity parameter" which is sometimes specified to $-\mu' = \frac{2}{3}\mu$. The latter relation is however rigorously valid only for monoatomic gases (ENSKOG 1917). We shall make no use of the definition (5.3.2). It is further convenient to use the viscous part of $\overset{\rightrightarrows}{\Pi}$ separately by subtracting the scalar pressure p from the diagonal elements:

$$\overset{\rightrightarrows}{\Pi}{}^{*} = \overset{\rightrightarrows}{\Pi} - p\overset{\rightrightarrows}{I} = \left\| \begin{matrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{matrix} \right\| \qquad (5.3.3)$$

It may be anticipated that $\mathrm{Div}\cdot\overset{\rightrightarrows}{I} = 0$ so that

$$\mathrm{Div}\cdot(p\overset{\rightrightarrows}{I}) = \mathrm{grad}\ p. \qquad (5.3.4)$$

A momentum production $\vec{\sigma}_{\vec{P}}$ may be caused by external forces proportional to mass, such as gravity. Let $\vec{f}_l$ [m/s²] be this force (per unit mass) on the i^{th} component, then

$$\vec{\sigma}_{\vec{P}} = \sum_{l=1}^{n} \rho^l \vec{f}_l \qquad [\mathrm{kg\ m^{-2}\ s^{-2}}] \ . \qquad (5.3.5)$$

In the case of gravity which we will consider exclusively, $\vec{f}_l$ is equal to $\vec{g}$, the free-fall acceleration vector, so that

$$\vec{\sigma}_{\vec{P}} = \rho\vec{g}. \qquad (5.3.6)$$

Substitution of these expressions into eq. (1.5) gives the local formulation

$$\frac{\partial}{\partial t}(\rho\vec{v}) = -\mathrm{Div}\cdot(\vec{\vec{\Gamma}}+\vec{\vec{\Pi}}) + \rho\vec{g}\ , \qquad (5.3.7)$$

or

$$\frac{\partial}{\partial t}(\rho\vec{v}) = -\mathrm{Div}\cdot\vec{\vec{\Gamma}} - \mathrm{grad}\ p - \mathrm{Div}\cdot\vec{\vec{\Pi}}^{*} + \rho\vec{g}. \qquad (5.3.8)$$

In order to reduce the current density to the given velocity field $\vec{v}$, we must subtract $\rho\vec{v}\vec{v}$ from $\vec{\vec{\Phi}}_{p} = \vec{\vec{\Gamma}}+\vec{\vec{\Pi}}$ (see eq. (1.6)). Thus, the substantial formulation according to eq. (1.7) is

$$\rho\frac{D\vec{v}}{Dt} = -\mathrm{Div}\cdot\vec{\vec{\Gamma}} + \mathrm{Div}\cdot(\rho\vec{v}\vec{v}) - \mathrm{Div}\cdot\vec{\vec{\Pi}} + \rho\vec{g}, \qquad (5.3.9)$$

or

$$\rho\frac{D\vec{v}}{Dt} = -\mathrm{Div}\cdot(\vec{\vec{\Gamma}}-\rho\vec{v}\vec{v}) - \mathrm{grad}\ p - \mathrm{Div}\cdot\vec{\vec{\Pi}}^{*} + \rho\vec{g}\ {}^{*)}. \qquad (5.3.10)$$

Unless for homogeneous (one-component) flow, $\vec{\vec{\Gamma}}$ is not equal to $\rho\vec{v}\vec{v}$, as can be seen from the last expression in (3.4.4).

---

*) The term $\mathrm{Div}\cdot(\vec{\vec{\Gamma}}-\rho\vec{v}\vec{v})$ can also be written as $\mathrm{Div}\cdot\sum_{i=1}^{n}\vec{v}_i\vec{J}^i$ with $\vec{J}^i$ defined by (5.1.5). This illustrates the origin from more-component diffusion phenomena.

The vector $\mathrm{Div}\cdot(\rho\vec{v}\vec{v})$ is parallel to $\vec{v}$, as every expression of this kind is parallel to the last vector in the dyadic product:

$$\mathrm{Div}\cdot(\rho\vec{a}\vec{b}) = (\mathrm{grad}\ \rho\cdot\vec{a})\vec{b} + (\rho\ \mathrm{div}\ \vec{a})\vec{b} + \rho\ (\vec{a}\cdot\mathrm{Grad})\vec{b}. \qquad (5.3.11)$$

In contrast to this, $\mathrm{Div}\cdot\vec{\vec{\Pi}}$ is in general not parallel to $\vec{v}$, because it cannot be expressed in the form $\mathrm{Div}\cdot(\rho\vec{v}\vec{v})$ as we have seen. Thus, the additional term $\mathrm{Div}\cdot(\vec{\vec{\Pi}}-\rho\vec{v}\vec{v})$ changes both magnitude and direction of the time derivative vector $D\vec{v}/Dt$ against the case of homogeneous flow.

The above formulations 5.3.7 to 5.3.10 may be considered as an extension of the usual NAVIER-STOKES' equation to heterogeneous flow.

The elements of the tensor $\vec{\vec{\Pi}}^{*}$ are, as shown in eq. (5.3.2), themselves functions of the velocity components and a material property μ, the viscosity. This is already true for the simpler case of laminar motion. For a turbulent flow, the stress tensor is superimposed by the tensor $\vec{\vec{\Pi}}_{turb}$ of turbulent "apparent" viscosity, the element a_{jk} of which is $\rho\ \overline{v'_j v'_k}$. v'_j etc. are Cartesian components of the turbulent oscillation velocity $\vec{v}'$, and the bar means a time average *). Their computation as well as their measurement are extremely difficult; the theoretical statements (PRANDTL's mixing length etc.) are by far unable to predict correct results in complex cases like ours.

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*) The method applies to quasi-stationary turbulence. For macroscopically rapidly changing velocity fields, the above time averages loose their meaning.

What is still more serious is the fact that the elements of both tensors $\vec{\vec{\Pi}}^*$ and $\vec{\vec{\Pi}}_{turb.}$ refer of course to a homogeneous fluid. The important open question is how to incorporate two or more entirely different viscosities μ_ι into the stress tensor, and which velocities should reasonably be applied.

Here the complete-mixing-model suggests some idea but not yet a solution. It is obvious that only a unique pressure field $p(t,\vec{r})$ is physically possible, and not a set of different but completely overlapping (true) pressure fields $p_\iota(t,\vec{r})$ *). The same must be true for the tensor $\vec{\vec{\Pi}}^*(t,\vec{r})$ which is a unique one for the mixture. No different tensor components $p_{xy,i}$ can occur because the constituents are undistinguishible as concerns their position.

Hence the "mixture" friction tensor $\vec{\vec{\Pi}}^*$ must entirely refer to mixture velocity components and a "mixture viscosity" μ. If μ is a material property for homogeneous fluids, the same can hardly be true for a mixture, as the effect of friction between - really - separated components ι and κ must be considered in any way. This latter cross-effect however depends on both velocity fields $\vec{v}_\iota$ and $\vec{v}_\kappa$, strictly spoken on $\vec{v}_\iota - \vec{v}_\kappa$.

This reasoning reveals that the relatively simple STOKES' statement (5.3.2) is insufficient for more-component flow and may not be used. The effective viscosity for slip-flow must be higher than that computed from molecular-statistical theories for gas and liquid mixtures at rest (see e.g. [8]).

---

*) "Partial" pressures $p^\iota = \alpha^\iota p$ can of course be considered, if suitable.

Obviously, the same physical situation exists already for a unique (averaged) momentum balance like eq. (5.3.10) above, so that, simply spoken, the substance of the stress tensor elements remains undetermined for more-component flow, because statements of type (5.3.2) are no longer applicable.

As emphasized in the introduction, we want, in this report, to collect basic and assured relationships about more-component flow. But we do not wish to enter the field of semi-empirical speculations. Here we have arrived at a wall, but it is important to know where it is. To integrate the momentum equation for two phase flow, say by programming it for a computer, as has occasionally been done, reveals to be useless, as the most important data are simply unknown.

After these distressing remarks we may look on the slip ratio S(z,t) of the preceding paragraph. Apart from the justification of one-dimensional treatment, S(z,t) follows directly from the main result of an integration of the one-dimensional momentum equation, namely from the velocities $v_l(z,t)$*). As this is not possible, efforts should be concentrated to obtain S(z,t)**) experimentally under various conditions. This would be the only way to bypass a solution of the unattackable momentum equation.

This remark should be understood as a suggestion only; no details how to perform such experiments can be given so far.

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*) Obviously, in order to obtain both velocities $v_l(z,t)$ at the same time, also two coupled momentum equations (for two components) should be solved.

**) or equivalent momentum equation surrogates.

5.4. Energy balance

The establishment of an energy balance for a heterogeneous flow reveals to be unexpectedly complicated if asked in coordinate-invariant partial differential equation notation. Classical thermodynamics show the balance written with differentials only, the significance of which remains undetermined as soon as all independent position and time variables come into play. The formulations found in various reports or textbooks differ considerably and are sometimes incorrect.

In order to keep lucidity we will treat the subject in different steps.

5.4.1. Homogeneous flow

In this case the flow is composed by a single component with unique velocity $\vec{v}$.

The total specific energy e is composed by four parts, namely

- u	the "internal energy" which considers the energy due to microscopic molecular movement, a state variable,
- $\frac{p}{\rho}$	sometimes called "flow energy", better "expansion energy", also a state variable *),
- $\vec{v}^2/2$	the kinetic energy, no state variable,
- $-\vec{g}\cdot\vec{r}$	the potential energy due to the gravity field, no state variable.

---

*) The distinction between "energy" and "work" becomes somewhat academic by our way of treatment.

$\vec{g}$ is the free fall acceleration vector with Cartesian components (0, 0, -g). $\vec{r} = (x,y,z)$ is the position vector, with $\partial\vec{r}/\partial t = 0$, but $D\vec{r}/Dt \equiv \vec{v}$. For $z = 0$, the potential energy may arbitrarily be equated to zero, since it is anyhow only defined except a constant amount. For $z > 0$, $-\vec{g}\cdot\vec{r}$ is of course positive.

For pure (isothermal) fluid mechanics, BERNOULLI's law gives the energy balance in integral form:

$$\frac{p}{\rho} + \frac{\vec{v}^2}{2} - \vec{g}\cdot\vec{r} = \text{constant.} \qquad (5.4.1.1)$$

This is true for viscous-free flow, where the internal energy u is unaffected, but if friction and other heat sources come into play, u must be included so that

$$e = u + \frac{p}{\rho} + \frac{\vec{v}^2}{2} - \vec{g}\cdot\vec{r} = \text{constant} + F(\rho,T) \ . \qquad (5.4.1.2)$$

The internal energy u **is**, by experience, a function of any two basic state variables for which we have provisionally chosen the density ρ and the temperature T.

We apply the total differential operator D/Dt on e, and multiply by ρ:

$$\rho \frac{De}{Dt} = f(\rho,T) \ , \qquad (5.4.1.3)$$

where $f(\rho,T)$ is a function to specify. It signifies the divergence of the non-convective energy flow rate $\vec{\Phi}_E$ through the surface of our elementary volume, because σ_E is in any case zero.

As known, such a heat transport can be caused by (molecular) conduction and by radiation. We confine us to conduction and write according to FOURIER

$$\vec{\Phi}_E = - \lambda \text{ grad } T, \qquad (5.4.1.4)$$

thus

$$\rho \frac{De}{Dt} = + \text{div}(\lambda \text{ grad } T) , \qquad (5.4.1.5)$$

or, according to (1.5)

$$\frac{\partial}{\partial t}(\rho e) = - \text{div}(\rho e \vec{v} - \lambda \text{ grad } T) . \qquad (5.4.1.6)$$

Heat conduction depends on T only, and not on ρ. Eq. (5.4.1.4) is one of the "phenomenological equations" of thermodynamics of irreversible processes. T is no new variable but can, in principle, be computed from p and ρ by the state equation $\varphi(p, \rho, T) = 0$.

Nevertheless, if, later on, boiling two phase flow shall be considered, where the heat, by rights, is supplied only from the walls, it is obviously practically impossible to follow the heat propagation through the inside with its chaotic and non-stationary mechanical microstructure. It is for this reason that we replace the conduction and radiation flow yields by a volumetric energy source q per unit time, the position and time dependence of which must be properly "assumed".

Hence

$$\rho \frac{De}{Dt} = q \,, \qquad (5.4.1.7)$$

and

$$\frac{\partial}{\partial t}(\rho e) = - \operatorname{div}(\rho e \vec{v}) + q \,. \qquad (5.4.1.8)$$

We emphasize that this notation is only a convenience which saves the trouble to compute the temperature field. Strictly spoken, we have no other choice because the heat propagation depends on the mechanical distribution of particles of the various phases (the "flow pattern") which is just not computable with our model where the position of the phases is not specified. Each of the phases virtually fills the whole space with its proper partial density.

Eqs. (5.4.1.7) and (5.4.1.8) are not very useful, as the total energy e is not yet related with the variables p, ρ,and $\vec{v}$, encountered up to now. On the contrary, the energy balance should just supply the fifth needed equation without introducing a new system variable e or u.

For this reason, we must successively subtract balances of the part energies $\frac{p}{\rho}$, $\frac{\vec{v}^2}{2}$, and $-\vec{g}\cdot\vec{r}$, in order to keep finally the balance for the internal energy u, expressed by known quantities.

We begin with the specific potential energy which temporarily shall be called l. According to (1.5), we have with $\vec{\Phi}_L = \rho l \vec{v}$:

$$\frac{\partial}{\partial t}(\rho l) = - \operatorname{div}(\rho l \vec{v}) + \sigma_{K\to L} \,. \qquad (5.4.1.9)$$

The first index K indicates that the potential energy source σ_L is supplied by an equal sink of the kinetic energy k.

Now,

$$\frac{\partial}{\partial t}(\rho l) = \frac{\partial}{\partial t}(-\rho\,\vec{g}\cdot\vec{r}) = -\,\vec{g}\cdot\vec{r}\,\frac{\partial\rho}{\partial t} \qquad (5.4.1.10)$$

and

$$-\,\mathrm{div}(\rho l\vec{v}) = +\,\mathrm{div}[\rho(\vec{q}\cdot\vec{r})\vec{v}] = +\,\vec{g}\cdot\vec{r}\,\mathrm{div}(\rho\vec{v}) + \rho\,\vec{g}\cdot\vec{v}\ ^{*)}\ . \qquad (5.4.1.11)$$

We obtain, as the number of terms of (5.4.1.9) is complete, purely formally, by considering also the mass continuity eq. (5.1.1):

$$\sigma_{K\to L} = -\,\rho\,\vec{q}\cdot\vec{v}\ . \qquad (5.4.1.12)$$

Alternatively, the substantial formulation is

$$\rho\,\frac{Dl}{Dt} = -\,\rho\,\vec{g}\cdot\vec{v}\ , \qquad (5.4.1.13)$$

an almost trivial result which can also be obtained directly.

---

*) $\mathrm{grad}(\vec{g}\cdot\vec{r}) = \vec{g}\cdot\mathrm{Grad}\,\vec{r} = \vec{g}\cdot\overset{\rightrightarrows}{I} = \vec{g}$

The second part energy balance we establish is that of the specific kinetic energy k. This can formally be achieved by scalarly multiplying the momentum balance by $\vec{v}$. For this end, we choose eq. (5.3.10) where, in the homogeneous case, $\vec{\vec{\Gamma}} = \rho\vec{v}\vec{v}$:

$$\rho \frac{D}{Dt} \frac{\vec{v}^2}{2} = - \vec{v}\cdot\mathrm{grad}\, p - \vec{v}\cdot(\mathrm{Div}\cdot\vec{\vec{\Pi}}^*) + \rho\, \vec{g}\cdot\vec{v}\ , \qquad (5.4.1.14)$$

or, in local formulation

$$\frac{\partial}{\partial t} (\rho \frac{\vec{v}^2}{2}) = - \mathrm{div}(\rho \frac{\vec{v}^2}{2} \vec{v}) - \vec{v}.\mathrm{grad}\, p - \vec{v}.(\mathrm{Div}.\vec{\vec{\Pi}}^*) + \rho\, \vec{g}.\vec{v}. \qquad (5.4.1.15)$$

From this, one can clearly see that there are three source terms of kinetic energy, namely

$$\sigma_{L\to K} = - \sigma_{K\to L} = + \rho\, \vec{g}\cdot\vec{v}\ , \qquad (5.4.1.16)$$

and

$$\sigma_{D\to K} = - \vec{v}\cdot\mathrm{grad}\, p\ , \qquad (5.4.1.17)$$

where D suggests the specific expansion energy $d = \frac{p}{\rho}$, and $\sigma_{D\to K}$ signifies the amount converted from expansion energy to kinetic energy.

Thirdly, there is

$$\sigma_{F\to K} = - \vec{v}\cdot(\mathrm{Div}\cdot\vec{\vec{\Pi}}^*)\ , \qquad (5.4.1.18)$$

a term that gives the kinetic energy loss in favour of friction energy (denoted by F). The latter one is an intermediate energy form, which is converted into internal energy as soon as it originates from kinetic or expansion energy. Thus, the sum of all source terms involving F must vanish

$$\sigma_{F\rightarrow K} + \sigma_{F\rightarrow D} + \sigma_{F\rightarrow U} = 0 \; . \qquad (5.4.1.19)$$

The next balance is that of specific expansion energy d. Here we have again three source terms, namely first

$$\sigma_{K\rightarrow D} = - \sigma_{D\rightarrow K} = + \vec{v}\cdot \mathrm{grad}\; p \; . \qquad (5.4.1.20)$$

The other terms, $\sigma_{F\rightarrow D}$, the gain from friction energy, and $\sigma_{U\rightarrow D}$, the source from internal energy, are not so easy to overlook.

The term $\sigma_{F\rightarrow U}$, the internal energy produced by friction, is however known (see [9], p.77):

$$\sigma_{F\rightarrow U} = - \vec{\vec{\Pi}}{}^{*} : \mathrm{Grad}\; \vec{v} \; . \qquad (5.4.1.21)$$

Here ":" means the double scalar product of two second order tensors. This scalar quantity is obtained by simply summing all products of inversely indexed tensor components:

$$\vec{\vec{A}} : \vec{\vec{B}} = \sum_{l,k} A_{lk} \, B_{kl} \; . \qquad (5.4.1.22)$$

Now we deduce from the friction balance (5.4.1.19):

$$\sigma_{F\rightarrow D} = + \vec{v}\cdot(\mathrm{Div}\cdot\vec{\vec{\Pi}}{}^{*}) + \vec{\vec{\Pi}}{}^{*} : \mathrm{Grad}\; \vec{v} = + \mathrm{div}(\vec{\vec{\Pi}}{}^{*}\cdot\vec{v}) \; . \qquad (5.4.1.23)$$

According to the structure of $\sigma_{F\to U}$, the look of the source term $\sigma_{D\to U}$ can be found by replacing the $\vec{\vec{\Pi}}^*$ by $p\vec{\vec{I}}$, leading to

$$\sigma_{D\to U} = - p \operatorname{div} \vec{v} \; . \qquad (5.4.1.24)$$

By collecting $\sigma_{K\to D}$, $\sigma_{F\to D}$, and $\sigma_{U\to D} = -\sigma_{D\to U}$, we get for the expansion energy balance

$$\begin{aligned} \rho \frac{D}{Dt}\left(\frac{p}{\rho}\right) &= + \operatorname{div}(\vec{\vec{\Pi}}^* \cdot \vec{v}) + p \operatorname{div} \vec{v} + \vec{v} \cdot \operatorname{grad} p \\ &= + \operatorname{div}(\vec{\vec{\Pi}}^* \cdot \vec{v}) + \operatorname{div}(p\vec{v}) \qquad (5.4.1.25) \\ &= + \operatorname{div}(\vec{\vec{\Pi}} \cdot \vec{v}) \; , \quad ^{*)} \end{aligned}$$

or, in local formulation

$$\frac{\partial p}{\partial t} = + \operatorname{div}(\vec{\vec{\Pi}}^* \cdot \vec{v}) \; . \quad ^{*)} \qquad (5.4.1.26)$$

The last expression shows illustratively how local pressure changes are generated by the work flow due to friction. For stationary processes, there is no exchange $\sigma_{F\to D}$ between friction and expansion energies.

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*) The appearance of the source term $\sigma_{F\to D}$ in form of a divergence suggests the question if it should not rather be considered as a current term through the surface. This is but only a matter of definition and does not change the results. The formal application of (1.5) with $y = \frac{p}{\rho}$ gives indeed $\frac{\partial p}{\partial t} = - \operatorname{div}(p\vec{v}) + \operatorname{div}(\vec{\vec{\Pi}} \cdot \vec{v})$ if we add, like in the momentum equation, the total "work flow rate" $\vec{\vec{\Pi}} \cdot \vec{v}$ to the formal convective term $p\vec{v}$. This is just eq. (5.4.1.26).

We give now the balance of the specific internal energy u ($\sigma_E = q = \sigma_U$, as q is pure heat):

$$\left.\begin{aligned} \rho \frac{Du}{Dt} &= \sigma_{D\to U} + \sigma_{F\to U} + \sigma_U \\ &= - p \operatorname{div} \vec{v} - \overset{\rightrightarrows}{\Pi}{}^{*} : \operatorname{Grad} \vec{v} + q \\ &= - \overset{\rightrightarrows}{\Pi} : \operatorname{Grad} \vec{v} + q \quad , \end{aligned}\right\} \qquad (5.4.1.27)$$

or

$$\frac{\partial}{\partial t} (\rho u) = - \operatorname{div}(\rho u \vec{v}) - \overset{\rightrightarrows}{\Pi} : \operatorname{Grad} \vec{v} + q \quad . \qquad (5.4.1.28)$$

Internal energy is produced, besides the heat supply q, by conversion of convective energy through expansion and friction.

By summing up all four part energy balances, of course the total energy balance is regained.

Of special interest may be to consider the sum of internal energy u and expansion energy $\frac{p}{\rho}$, which is called the enthalpy h:

$$\left.\begin{aligned} \rho \frac{Dh}{Dt} &= - \overset{\rightrightarrows}{\Pi} : \operatorname{Grad} \vec{v} + \operatorname{div}(\overset{\rightrightarrows}{\Pi} \cdot \vec{v}) + q \\ &= + \vec{v} \cdot (\operatorname{Div} \cdot \overset{\rightrightarrows}{\Pi}) + q \quad , \end{aligned}\right\} \qquad (5.4.1.29)$$

and, locally formulated,

$$\frac{\partial}{\partial t} (\rho h) = - \operatorname{div}(\rho h \vec{v}) + \vec{v} \cdot (\operatorname{Div} \cdot \overset{\rightrightarrows}{\Pi}) + q \quad . \qquad (5.4.1.30)$$

By considering again (5.4.1.26), an alternative form to (5.4.1.29) is

$$\rho \frac{Dh}{Dt} = - \frac{Dp}{Dt} - \vec{\vec{\Pi}}^{*}:\mathrm{Grad}\,\vec{v} + q \, . \qquad (5.4.1.31)$$

5.4.2. Heterogeneous flow

When considering the heterogeneous case, we must about all bear in mind that the stress tensor of a mixed multicomponent flow is, till now, not reasonably defined from the respective component stress tensors, as already emphasized in § 5.3. So we are indeed unable to compute the correct pressure drop from our equations. They remain, unfortunately, rather academic for all cases where friction plays an important part, e.g. for natural circulation loops. Here, the equilibrium between friction and buoyancy forces determines the flow rate, and, by consequence, the amount of transportable heat.

The second drawback lies in the overall concept of the model. As all phases are considered to be arbitrarily well mixed and to fill each one the whole volume, the phases may indeed have different velocities; this applies however not for individually taken subvolumes such as bubbles and droplets. Their buoyancy speeds against the surrounding main medium depend on the flow resistances at their proper surfaces and are thus from the first out of consideration.

This shows clearly the limitation of the applicability of our model. Nevertheless, we will give the equations, as they may be useful for forced convection processes where the pressure field is nearly entirely dictated from outside (by pumps).

All phenomena which basically outgrow from the micro-structure (flow pattern, boundary layer whirling by bubbles), such as pressure drop, heat transfer rate from walls, conditions of pattern transition including boiling crisis (bourn-out), cannot be learned from our model. Empirical correlations therefore fully keep their importance. Though it cannot be expected that our equations give sufficient indications on the structure of dimensionless parameters of relevance, they may give necessary conditions for what kind of interrelationships are imaginable and which not. A striking example is: The slip ratio S cannot depend on the "flow quality" χ (see 4.5.5); respective experiments are useless *). The basis from which parameters for two phase flow pressure drop are usually calculated should also be revised.

On the other hand, an entirely satisfying, i.e. physically correct model of boiling flow is practically hopeless, as the inner boundary conditions at all bubble surfaces can certainly never be considered. Not even a stationary description of what is commonly called "steady boiling" is possible, since bubble growth, detachment, and migration are a priori non-stationary so that indeed a "steady two phase flow" does not exist.

After these preliminaries, we generalize our homogeneous flow results by taking into account the different velocities we derived in preceding chapters.

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*) Whenever giving graphical or numerical "correlations" between any two quantities, at least the pertaining correlation coefficient should be computed according to statistical principles. In case it is too poor, the necessary conclusions should be drawn.

First we define formally for all part energies of component ι the "true" specific energies, namely

- u_ι "true" specific internal energy
- $(\frac{p}{\rho})_\iota$ " " expansion energy
- $\frac{1}{2}\vec{v}_\iota^{\,2}$ " " kinetic energy
- $-\vec{g}\cdot\vec{r}_\iota$ " " potential energy

where $\vec{r}_\iota$ is the position vector of the center of mass of component ι.

The "partial" specific energies are, also formally, defined by

- $u^\iota = x^\iota u_\iota$
- $(\frac{p}{\rho})^\iota = x^\iota(\frac{p}{\rho})_\iota$
- $\frac{1}{2}(\vec{v}^2)^\iota = x^\iota\cdot\frac{1}{2}\vec{v}_\iota^{\,2}$
- $-\vec{g}\cdot\vec{r}^\iota = -x^\iota\vec{g}\cdot\vec{r}_\iota$.

The essential feature is that not for all of these part energies the sum over all components gives the respective part energy of the mixture, e.g. $\sum_{\iota=1}^{n} u^\iota \neq u$. Such a relationship is a priori correct only for the total energy e which is conserved.

The consequence is that individual definitions of mixture internal energy u, or mixture enthalpy energy h, based on those of the constituents, have no meaning as long as transitions between the energy forms are not excluded.

Such an exclusion is however often possible. Consider the case that the components show no relative motion one against the other so that kinetic energy can neither be produced nor destroyed. If no inner friction exists (ideal fluid), no additional supply of internal energy occurs.

For a water-steam mixture, $\sum_{i=1}^{n} (\frac{1}{\rho})^i = \frac{1}{\rho}$ (cf. eq.3.2.2) means, with the terminology of chapter 4:

$$\frac{1}{\rho} = (1-x) \frac{1}{\rho_l} + x \frac{1}{\rho_v} \quad . \tag{5.4.2.1}$$

With the above restrictions, $\sum_{i=1}^{n} u^i = u$ gives

$$u = (1-x)u_l + xu_v \quad . \tag{5.4.2.2}$$

If also the pressure is equal for all constituents *), eq. (5.4.2.1) may be extended to $\frac{p}{\rho}$. Then $\sum_{i=1}^{n} h^i = h$, which reads

$$h = (1-x)h_l + xh_v \quad . \tag{5.4.2.3}$$

All three relationships are well known for wet steam (cf. [3], p.163), but their validity limitation should

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*) It is emphasized that the complete-mixing-model implies this equality, as already discussed in the momentum balance's paragraph.

carefully be kept in mind. In particular, by solving (5.4.2.2) for x, the "definition" of x

$$x = \frac{h-h_l}{h_v-h_l} = \frac{h-h_l}{h_{lv}} \qquad (5.4.2.4)$$

where h_{lv} is the latent evaporation heat is restricted to isobaric processes and both phases at rest. Though this relationship may be suitable to be extended towards negative x for subcooled boiling where the mixture enthalpy h is below the saturation liquid enthalpy h_l (but note that the phases are not at rest!), it is better to remember that x is originally a mass fraction.

Now, with a single pressure field p for all components, we notice that, among the part energies, $l = -\vec{g}\cdot\vec{r}$ and $d = \frac{p}{\rho}$ behave "additively":

$$\sum_{i=1}^{n} l^i = \sum_{i=1}^{n} -\vec{g}\cdot x^i \vec{r}_i = l \;{}^{*)} \; ; \sum_{i=1}^{n} \left(\frac{p}{\rho}\right)^i = p \sum_{i=1}^{n} \frac{x^i}{\rho_i} = \frac{p}{\rho} \; , \qquad (5.4.2.5)$$

whereas the kinetic energy does not:

$$\sum_{i=1}^{n} k^i = \sum_{i=1}^{n} \frac{1}{2}\, \vec{v}^i \cdot \vec{v}_i \neq k = \frac{1}{2}\, \vec{v}^2 \; . \qquad (5.4.2.6)$$

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*) $\sum_{i=1}^{n} x^i \vec{r}_i = \vec{r}$ is the usual rule to compute the center of mass vector $\vec{r}$ from those of the constituents.

As a consequence, as $\sum\limits_{\iota=1} e^{\iota} = e$, the fourth part energy $u = e - (1+\frac{p}{\rho}+k)$ is also not additive in the normal case with $\vec{v}_{\iota} \neq \vec{v}_{k}$.

With the models of eqs. (5.4.1.13), (5.4.1.25), (5.4.1.14), and (5.4.1.27) the components' energy balances are in substantial notation:

$$\rho \frac{Dl^{\iota}}{Dt} = - \operatorname{div}[\rho^{\iota} l_{\iota}(\vec{v}_{\iota}-\vec{v})] - \rho\, \vec{g}\cdot\vec{v}^{\iota} \quad , \tag{5.4.2.7}$$

$$\rho \frac{D}{Dt} \left(\frac{p}{\rho}\right)^{\iota} = - \operatorname{div}[\rho^{\iota} \left(\frac{p}{\rho}\right)_{\iota}(\vec{v}_{\iota}-\vec{v})] + \operatorname{div}(\vec{\vec{\Pi}}\cdot\vec{v}^{\iota}) \quad , \tag{5.4.2.8}$$

$$\rho \frac{D}{Dt} \frac{\vec{v}^{\iota}\cdot\vec{v}_{\iota}}{2} = - \operatorname{div}[\rho^{\iota} \frac{\vec{v}_{\iota}^{\,2}}{2} (\vec{v}_{\iota}-\vec{v})] - \vec{v}^{\iota}\cdot(\operatorname{Div} \vec{\vec{\Pi}}) + \rho\, \vec{g}\cdot\vec{v}^{\iota}, \tag{5.4.2.9}$$

$$\rho \frac{Du^{\iota}}{Dt} = - \operatorname{div}[\rho^{\iota} u_{\iota}(\vec{v}_{\iota}-\vec{v})] - \vec{\vec{\Pi}}:\operatorname{Grad} \vec{v}^{\iota} + q^{\iota}. \tag{5.4.2.10}$$

All four lines can be added to give

$$\rho \frac{De^{\iota}}{Dt} = - \operatorname{div}[\rho^{\iota} e_{\iota}(\vec{v}_{\iota}-\vec{v})] + q^{\iota}, \;^{*)} \tag{5.4.2.11}$$

as it should be.

The problem is how to eliminate e^{ι} and e_{ι}, resp., in order to get an internal energy balance for a more-phase flow.

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*) The term in square brackets is just $\rho e(\vec{\hat{v}}-\vec{v})$, with $\vec{\hat{v}}$ introduced in table 1, when summed over all ι .

When adding eqs. (5.4.2.7) and (5.4.2.8), resp., over all i, it is convenient to apply once more the diffusion current densities $\vec{J}^i = \rho^i(\vec{v}_i - \vec{v})$ (cf. 5.1.5). This gives:

$$\rho \frac{Dl}{Dt} = - \operatorname{div} \sum_{i=1}^{n} l_i \vec{J}^i - \rho \, \vec{g}\cdot\vec{v} \quad , \qquad (5.4.2.12)$$

$$\rho \frac{D}{Dt}\left(\frac{p}{\rho}\right) = - \operatorname{div} \sum_{i=1}^{n} \left(\frac{p}{\rho}\right)_i \vec{J}^i + \operatorname{div}(\vec{\vec{\Pi}}\cdot\vec{v}) \quad , \text{*)} \qquad (5.4.2.13)$$

but for the third line, the summation is not allowed, thus

$$\rho \frac{Dk}{Dt} \neq - \operatorname{div} \sum_{i=1}^{n} k_i \vec{J}^i - \vec{v}\cdot(\operatorname{Div}\cdot\vec{\vec{\Pi}}) + \rho \, \vec{g}\cdot\vec{v} \ .$$

To get the correct expression, we go back to eq. (5.3.10) and multiply scalarly with $\vec{v}$, considering the pertaining footnote:

$$\rho \frac{Dk}{Dt} = - \vec{v}\cdot\left(\operatorname{Div}\cdot\sum_{i=1}^{n} \vec{v}_i \vec{J}^i\right) - \vec{v}\cdot(\operatorname{Div}\cdot\vec{\vec{\Pi}}) + \rho \, \vec{g}\cdot\vec{v} \ . \qquad (5.4.2.14)$$

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*) The term $\sum_{i=1}^{n} \left(\frac{p}{\rho}\right)_i \vec{J}^i$ may also be written $p(\vec{\tilde{v}} - \vec{v})$, as can easily be verified.

Of course, all terms containing $\vec{J}^{\iota}$ vanish for single-phase flow.

By considering the total energy balance

$$\rho \frac{De}{Dt} = - \operatorname{div} \sum_{\iota=1}^{n} e_{\iota} \vec{J}^{\iota} + q \ , \tag{5.4.2.15}$$

the result for $u = e - 1 - \frac{p}{\rho} - k$ is

$$\rho \frac{Du}{Dt} = - \operatorname{div} \sum_{\iota=1}^{n} u_{\iota} \vec{J}^{\iota} - \operatorname{div} \sum_{\iota=1}^{n} k_{\iota} \vec{J}^{\iota} + \vec{v} \cdot (\mathrm{Div} \cdot \sum_{\iota=1}^{n} \vec{v}_{\iota} \vec{J}^{\iota}) - \overset{\rightrightarrows}{\Pi} : \mathrm{Grad}\, \vec{v} + q \ . \tag{5.4.2.16}$$

There is a remarkable residuum (giving the u-gain from kinetic energy and indirectly from potential energy) which distinguishes more-phase flow from the simple single-phase flow (5.4.1.27).

The enthalpy balance is obtained formally:

$$\rho \frac{Dh}{Dt} = - \operatorname{div} \sum_{\iota=1}^{n} h_{\iota} \vec{J}^{\iota} - \operatorname{div} \sum_{\iota=1}^{n} k_{\iota} \vec{J}^{\iota} + \vec{v} \cdot (\mathrm{Div} \sum_{\iota=1}^{n} \vec{v}_{\iota} \vec{J}^{\iota}) + \vec{v} \cdot (\mathrm{Div}\, \overset{\rightrightarrows}{\Pi}) + q \ . \tag{5.4.2.17}$$

The corresponding local formulations are omitted here; they are easily obtainable by replacing $\rho \frac{Dy}{Dt}$ by $\frac{\partial}{\partial t} (\rho y) + \operatorname{div}(\rho y \vec{v})$, where y stands for any specific part energy.

5.4.3. Specialization to one-dimensional two phase flow

This sub-paragraph is added in view of practical applications, as, up to now, the equations might appear to be little transparent. In contrast with the preceding text, we will accept here certain approximations to which we are practically obliged.

One-dimensional flow parallel to the walls, with a single independent coordinate z, does not allow to consider the adhesion conditions for anyone of the velocity components. Therefore, the external friction effect which gives an appreciable part of the total pressure drop, is not computable.

The pressure field p(t,z) must be inserted from outside and is no longer a system variable. The pertinent knowledge may be percured from semi-empirical correlations about pressure drops of two phase flow. In the simplest case that we will adopt here, p may be assumed constant in position.

The positional constancy is justified if the static pressure is sufficiently high so that the pressure drop through the pipe is of little influence on the state equation. Then, under boiling conditions, also the temperature and the "true" densities of each phase are constant.

The time variation of p depends mainly on what we suppose at the boundaries of the system, i.e. on the inlet pressure $p_{in}(t)$, supplied by a pump, and on the outlet pressure $p_{out}(t)$, depending on the subsequent hydraulic resistances, e.g. through a turbine. In order to keep a "mathematically closed" system *), namely without simultaneous description of pumps etc., we assume a "perfect pressure regulation" that keeps p constant or guides it according to a given law p(t)**). More complicated assumptions are of course possible.

The terms involving the tensor $\overset{\rightrightarrows}{\Pi}{}^{*}$ are anyhow not applicable for turbulent and more-phase flow, as already broadly discussed. This is so more true as soon as the component structure of $\overset{\rightrightarrows}{\Pi}{}^{*}$ is drastically mangled by unidimensional treatment. As a consequence, effects of (inner) viscosity can also not be considered. Their contribution to the pressure drop could anyhow not be utilized (see above), but now their energy aspects can equally not be taken into account. This is acceptable as soon as the friction heat source $\overset{\rightrightarrows}{\Pi}:\mathrm{Grad}\,\vec{v}$ is indeed negligible against the "external" heat supply q.

In the same sense the enthalpy transformation to kinetic energy may be neglected. Its amount is assumed to be small against q***); the component kinetic energies can anyhow not correctly be calculated if viscous effects are omitted.

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*) Although a system like ours is called "physically open", if there is a through flow rate.

**) This regulation may however not influence the other system variables, in particular the flow rate, because otherwise the regulation should be simultaneously described. Here the model must be adapted to the various engineering arrangements.

***) Applications on turbines, nozzles, rocket engines etc. are excluded after this assumption.

The remainder of eq. (5.4.2.17) is therefore

$$\rho \frac{Dh}{Dt} + \text{div} \sum_{\iota=1}^{n} h_\iota \vec{J}^\iota = q. \qquad (5.4.3.1)$$

For two phase flow, $J' = -J''$, so that

$$\sum_{\iota=1}^{2} h_\iota J^\iota = h_v J'' + h_l J' = (h_v - h_l) J'' = h_{lv} J'' , \qquad (5.4.3.2)$$

where h_{lv} is the latent evaporation heat at constant pressure. With our assumptions, it is independent of z.

Hence, (5.4.3.1) becomes for two phases

$$\rho \frac{Dh}{Dt} + h_{lv} \frac{\partial J''}{\partial z} = q . \qquad (5.4.3.3)$$

Now, we look on equation (5.1.4) with the diffusion current density of steam (5.1.5) already inserted. This reads

$$\rho \frac{Dx}{Dt} + \frac{\partial J''}{\partial z} = \sigma_M'' . \qquad (5.4.3.4)$$

By applying the substantial differential operator on expression (5.4.2.4) we find (h_l and h_{lv} being both constant):

$$\frac{Dx}{Dt} = \frac{1}{h_{lv}} \frac{Dh}{Dt} . \qquad (5.4.3.5)$$

When dividing (5.4.3.3) by h_{lv} and comparing it with (5.4.3.4), one gets:

$$\sigma_M'' = \frac{q}{h_{lv}} = -\sigma_M' \; . \qquad (5.4.3.6)$$

This seemingly blamably meagre result means that the steam production rate is equal to the heat added per volume unit, divided by the latent heat. For bodies at rest, this relationship pertains to the simplest physical conceptions.

Emphasis is, however, not laid on the result itself, but on the series of restrictive assumptions which allowed us to arrive, also for the very complex boiling two phase flow, at relation (5.4.3.6). It is shown that it is an approximation only, and one might have an idea how good it is.

The entire energy balance was utilized only to determine the source terms for the partial mass balances. Its function is herewith fulfilled, because (5.1.10) is already a mathematically closed system of two equations with the two variables α and v_l.

There still remain two points to discuss. σ_M' is a function of z and t because of $q = q(z,t)$. For a given wall heating in terms of energy per unit length, e.g. electrically, there is no problem. The z- and t-dependence is then obvious as one can divide the latter one by the pipe cross section.

If, however, in the case of a heat exchanger, q is given by the wall superheat $\Delta T(z,t)$ (against a proper fluid temperature), times the heating surface, divided by the pertaining fluid volume, and multiplied by a heat transfer coefficient, the question is how to determine the latter one. The main difficulty is indeed shifted to the solution of this problem.

The heat transfer coefficient, involving a mixture of material property and flow pattern influences, is only a surrogate conception for not rigorously describable thermo-hydrodynamic phenomena. Thus, its determination is from the first thrown upon correlation techniques. In particular, for boiling flow, its magnitude may vary over a very wide range according to the flow pattern. When, e.g., passing from a more or less compact liquid boundary layer to a gaseous one, it shows a sudden break-down which is responsible for the "boiling crisis" in the case of energy-steered heating (in the case of temperature-steered heating a crisis cannot occur). All these questions are however by far beyond the scope of this report.

The second open point is the slip problem. As had been pointed out in § 5.1., we were obliged to introduce $S(z,t)$ as a coefficient in order to reduce the number of dependent variables from 3 to 2. At the same time, the one-dimensional (scalar) momentum equation proved to be not reasonably attackable in § 5.3. This means that $S(z,t)$ is a surrogate for the solution of the momentum equation. As there is no hope to obtain $S(z,t)$ from the complete-mixing model, we must once more apply correlation techniques to determine $S(z,t)$ as a function of proper parameters, e.g. of α (not χ), of p, etc. Also this problem is beyond the frame of the article.

6. Conclusions

The "complete-mixing model" has been applied to establish a "field" description of n-phase flow with heat addition.

The advantage of the model, where each of the phases fills the whole volume but, nevertheless, may have a proper velocity $\vec{v}_i$, lies in the fact that in this manner, and only so, the complicated internal boundary conditions, at bubble or droplet surfaces, can be avoided.

The disadvantage is that certain effects, in particular those of viscosity and improved heat transfer through violently whirling boundary layers, are due to the microstructure and are thus irrecoverably lost.

Moreover, nothing occurs about overpressure in bubbles so that fundamental phenomena like that of surface tension are systematically disregarded.

On the other hand, if the microstructure should - hypothetically - appear in some advanced discontinuous description, the mathematical solution would become absolutely impracticable.

The compromise of this report is the proposal to apply the complete-mixing model and to add from empirical correlations knowledges at two points, namely for the slip ratio and for the heat transfer coefficient. Existing correlations are not yet fully appropriate to be inserted into the proposed equation set.

From the rather general formulation in three coordinates, the finally given equation set is a specialization to one-dimensional, strictly isobaric flow. Other specializations are possible.

The equation set is:

$$\left.\begin{aligned} -\frac{\partial\alpha}{\partial t} - v_l\frac{\partial\alpha}{\partial z} + (1-\alpha)\frac{\partial v_l}{\partial z} &= -\frac{q(z,t)}{\rho_l h_{lv}} \\ +\frac{\partial\alpha}{\partial t} + S(z,t)v_l\frac{\partial\alpha}{\partial z} + \alpha\, S(z,t)\frac{\partial v_l}{\partial z} + \alpha\, v_l\frac{\partial S}{\partial z} &= +\frac{q(z,t)}{\rho_v h_{lv}} \end{aligned}\right\} \qquad (6.1)$$

with

$$\left.\begin{array}{rl} \alpha &= \text{local void fraction} \\ v_l &= \text{liquid velocity} \end{array}\right\} \text{dependent variables}$$

$$\left.\begin{array}{rl} S(z,t) &= v_v/v_l = \text{slip ratio} \\ q(z,t) &= \text{energy supply per unit volume and unit time} \end{array}\right\} \text{given functions}$$

ρ_l, ρ_v = liquid, vapor densities

h_{lv} = latent evaporation heat at constant pressure.

Initial and boundary conditions may be given at choice. Perturbations may be introduced either by time variations of q or by changing the boundary conditions with time.

$S(z,t)$ should rather be given as function of α and v_l, thus following automatically the flow-pattern evolution. The equations are anyhow highly non-linear.

List of symbols

symbol	dimension	signification	explanation see page
A	m^2	surface, pipe cross section	9
d	$m^2 \; s^{-2}$	specific expansion energy	58
E	$m^2 \; kg \; s^{-2}$	total energy	8, 13, 17
e	$m^2 \; s^{-2}$	specific total energy	13, 19
$\vec{g}$	$m \; s^{-2}$	free fall acceleration vector	48
h	$m^2 \; s^{-2}$	specific enthalpy	21, 38
h_{lv}	$m^2 \; s^{-2}$	latent evaporation heat at constant pressure	38
$\overset{\Rightarrow}{I}$	-	unit tensor	17
$\overset{\Rightarrow}{J}$	$m^{-2} \; kg \; s^{-1}$	mass diffusion current density	43
$\vec{j}$	$m^{-2} \; kg \; s^{-1}$	mass current density = momentum density	13, 22, 25
k	$m^2 \; s^{-2}$	specific kinetic energy	37
l	$m^2 \; s^{-2}$	specific potential energy	56
M	kg	mass	8, 13, 17
$\vec{P}$	$m \; kg \; s^{-1}$	momentum	8, 13, 17
p	$m^{-1} \; kg \; s^{-2}$	pressure	47
p_{jl}	$m^{-1} \; kg \; s^{-2}$	elements of $\overset{\Rightarrow}{\Pi}{}^*$	47
Q_Y	various	flow rate of Y through a surface	8, 14, 30
q	$m^{-1} \; kg \; s^{-3}$	heat source per unit volume and unit time	55
$\vec{q}$	$kg \; s^{-3}$	total energy current density	13, 25
$\vec{r}$	m	position vector	53, 54
S	-	slip ratio v_v/v_l	37
T	deg	temperature	55
t	s	time	9
u	$m^2 \; s^{-2}$	specific internal energy	21, 37
V	m^3	volume	8, 13, 17
$\vec{v}$	$m \; s^{-1}$	center of mass velocity = specific momentum	10, 19, 27

List of symbols (continued)

symbol	dimension	signification	explanation see page
$\vec{\tilde{v}}$	$\mathrm{m\ s^{-1}}$	"volume" velocity	13, 25, 27
$\vec{\hat{v}}$	$\mathrm{m\ s^{-1}}$	"energy" velocity	13, 27
$\ddot{v}$	$\mathrm{m\ s^{-1}}$	"momentum" velocity (in one-dimensional case)	13, 27, 36
$\vec{v}'$	$\mathrm{m\ s^{-1}}$	turbulent oscillation velocity	50
x	-	mass fraction	13, 17, 19, 31
Y	various	extensive quantity	8
y	various	"specific" quantity to Y	8
z	m	axial coordinate	44
α	-	volume fraction	13, 17, 22, 31
$\vec{\vec{\beta}}$	-	"volume current density fraction"	28, 41
$\vec{\vec{\Gamma}}$	$\mathrm{m^{-1}\ kg\ s^{-2}}$	momentum current density	13, 25
γ	-	total energy fraction	17, 31
$\vec{\vec{\delta}}$	-	"energy current density fraction"	28, 41
δ_{jl}	-	$= 0$ if $j \neq l$; $= 1$ if $j = l$	47
ε	$\mathrm{m^{-1}\ kg\ s^{-2}}$	total energy density	13, 22
η	-	ratio of true specific total energies e_v/e_l	37
λ	$\mathrm{m\ kg\ s^{-3}(^{o})^{-1}}$	thermal conductivity	55
μ	$\mathrm{m^{-1}\ kg\ s^{-1}}$	dynamic viscosity	47
μ'	$\mathrm{m^{-1}\ kg\ s^{-1}}$	"second" viscosity parameter	47
ξ	-	ratio of true specific volumes ρ_l/ρ_v	37
$\vec{\vec{\Pi}}$	$\mathrm{m^{-1}\ kg\ s^{-2}}$	(complete) stress tensor	47
$\vec{\vec{\Pi}}^*$	$\mathrm{m^{-1}\ kg\ s^{-2}}$	friction tensor	48
$\vec{\vec{\Pi}}_{turb}$	$\mathrm{m^{-1}\ kg\ s^{-2}}$	tensor of turbulent "apparent" viscosity	50
ρ	$\mathrm{m^{-3}\ kg}$	mass density	8, 13, 22

List of symbols (continued)

symbol	dimension		explanation see page
σ_Y	various	production density of Y	8, 16
$\vec{\Phi}_Y$	various	Y-current density	8
$\overset{\rightrightarrows}{\chi}$	-	"mass current density fraction"	17, 28 31, 41
$\overset{\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}}{\psi}$	-	"momentum current density fraction"	28, 41

<u>indices:</u>

lower	(in general "ι")	refer to "true" quantities	18
upper		refer to "partial" quantities	18
$'$, l		refer to liquid	31
$''$, v		refer to vapor	31

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> Alfred Nobel

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