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ELEMENTARY METHODS IN FAST NEUTRON TRANSPORT THEORY

by

C. SYROS

1967

Joint Nuclear Research Center
Ispra Establishment — Italy

Reactor Physics Department
Reactor Theory and Analysis

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These methods make widely use of integral transform techniques.

One of the main ideas developed in the present report is the expansion of the neutron propagator in an infinite series of which the inverse is summable. An interesting problem in the slowing down theory is the calculation of the

distribution of neutrons, which have been scattered inelastically. This problem has been solved both for discrete and continuous distribution of the nuclear levels. Finally a perturbation theoretical method has been developed and another method allowing for the introduction of the central limit theorem of the theory of random variables into the transport theory has been worked out.

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Summary

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CONTENTS

APPENDIX

INTRODUCTION

The present report* is devoted to the presentation of an analytical method for the calculation of the distribution of inelastically and elastically slowed down fast neutrons in an infinite homogeneous medium. Up to now no attempt has been made to solve the transport equation explicitly by including the inelastic scattering kernel. The corresponding space independent problem for elastic and isotropic slowing down has been solved a long time ago (1). In the recent years some analytical methods have been reported for the solution of the slowing down problem with space dependent sources and for isotropic or anisotropic scattering.

A systematic treatment of the elastic slowing down problem with anisotropic scattering has been given by Kaper (2) in his thesis and more recently (3) in another report. His method is based on the approximate transformation of the transport equation in finite system of differential equations with respect to the lethargy (A^{-N} -approx.). This is obtained by using some properties of the generalized functions and especially of the Dirac -distribution and its derivatives.

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* It has first appeared as EURATOM internal report in June 1966

Manuscript received on February 17, 1967.

A method based on Fourier approach and on the normal mode approach introduced by Case (4) and generalized to anisotropic scattering by several authors Mika (5), Jacobs (6) has been given by Mc Jnerney (7) for elastic scattering.
No of the above mentioned papers contain information about inelastic slowing down.
On the other hand it is not exact, as we know to calculate in general the elastic and the inelastic neutron distributions separately, because there exist mixed terms which get lost if it is done.
The method presented here puts inelastic and elastic scattering on the same footing. This is obtained by separating the neutron distribution according to the number of collisions. For large numbers of collisions, however, the expressions become uncomfortable. We circumvent this difficulty by making use of a theorem from the theory of random variables, the central limit theorem. The significance of this theorem (CLT) throughout the world of random phenomena is well known (8).
It is, therefore, of considerable theoretical interest to give a method which enables us to introduce this remarkable law of random variables into the transport theory.
We give here a short description of this method which can be applied in all cases where n, the number of collisions, is much greater than unity.

An important point of the theory to be developed here is the requirement that in the expansion

$$\sigma_s(E,\mu) = \sum P_j(\mu)\,\sigma_j(E)$$

the functions $\sigma_j(E)\{j = 0, 1, 2, \ldots\}$ are all proportional to $\sigma_t(E)$. This assumption which is in many cases true implies that $\sigma_s(E,\mu)/\sigma_t(E)$ is energy independent.

The main tools of this paper are the decomposition of the distribution in parts according to numbers of collisions, n, and the application of the CLT and integral transform techniques.

In treating transport problems it is convenient to calculate the quantity consisting of the product of the total cross section, $\sigma_t(E)$, and the quantity called the neutron-flux, $\psi(x,\mu,E)$ instead of calculating the latter. It is the product $\sigma_t(E)\psi(x,\mu,E)$, which we calculate throughout this paper and which we call the neutron distribution.

The problems treated in the present report are the following:

Section 1 : Connection between central limit and transport theory.

1. CONNECTION BETWEEN CENTRAL LIMIT AND TRANSPORT THEORY.

Let us consider a function of the form

$$f(k,p,s) = [1-g(k,s)h(p)]^{-1}, \qquad 1.1$$

where (i) g,h, are integral transforms of certain functions and (ii) k,p, and s are the Fourier, or Laplace parameters corresponding to the space, lethargy and time coordinates, respectively.
Forms like Eq. 1.1. occur frequently in transport theory, whenever integral transform techniques have been applied in infinite as well as in finite media (9).
This expression together with factors of the form $[1+ik\mu]^{-1}$, $[(1+ik\mu_0)(1+ik\mu)]^{-1}$ etc., constitutes the neutron propagator in the (k-p-s) image space.
If, now, there exists a domain in which

$$|g(k,s)h(p)| < 1 \qquad 1.2$$

then, from Eq. 1.1 we obtain the absolutely convergent expansion

$$f(k,p,s) = \sum_{n=0}^{\infty} [g(k,s)h(p)]^n. \tag{1.3}$$

The inversion of Eq. 1.3 requires integrations of the type

$$\int_{-\infty}^{\infty} \prod_{j=1}^{n} g_j(k) e^{ik\xi} dk . \tag{1.4}$$

Now, if $g_j(k)$ is the Fourier transform of $G_j(\xi)$, i.e.

$$G_j(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_j(k) e^{ik\xi} dk , \tag{1.5}$$

it follows from the generalized convolution theorem (10) that

$$\begin{aligned} \mathcal{J}_n(\xi) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{n} g_j(k) e^{ik\xi} dk \\ &= \int_{-\infty}^{\infty} d\xi_1 G_1(\xi_1) \cdots \int_{-\infty}^{\infty} d\xi_{n-1} G_n(\xi - \cdots - \xi_{n-1}) \end{aligned} \tag{1.6}$$

The central limit theorem (11) implies that integrals like Eq. 1.6 can be represented approximately by a "normal distribution", if the general conditions of the central limit theorem are satisfied.

For a clear understanding of the conditions under which this result can be obtained we first state the Central Limit Theorem (CLT).

Let there be a sequence of independent random variables. Let us further assume that the random variables are distributed according to the distribution functions f_n (x) (n = 1, 2,...).

Let in addition

$$T_n (k) = \int_{-\infty}^{\infty} e^{ikx} f_n (x) dx \qquad 1.7$$

and $T_n(k)$ be a characteristic function of the distribution $f_n(x)$ (n = 1, 2, 3,...).

The following assumptions are made:

(i) The functions f_n (x) possess finite derivatives, $f_n'(x)$ and there exists a constant K so that

$$\int_{-\infty}^{\infty} |f_n'(x)| dx < K \quad (n = 1, 2, 3, \quad) \qquad 1.8$$

(ii) The functions $f_n(x)$ possess finite moments $M_{\lambda n}$ of the first five orders ($\lambda = 1,2,3,4,5$) whereby $M_{1n} = 0$ without loss of generality.
Then there exist positive constants α, β such that

$$0 < \alpha < M_{2n} < \beta, \quad \overline{M}_{3n} < \beta, \quad M_{4n} < \beta, \quad \overline{M}_{5n} < \beta \quad (n = 1,2,\ldots) \tag{1.9}$$

Here $\overline{M}_{\lambda n}$ is the absolute λ-th moment of the n-th distribution function.
(iii) There exist positive constants a,b, such that for $|k| < a$ holds

$$|T_n(k)| < b; \quad (n = 1,2,\ldots) \tag{1.10}$$

(iv) For every interval (c_1, c_2) with $c_1, c_2 > 0$ there exists a number $\rho(c_1, c_2) < 1$ such that for arbitrary $k \in (c_1, c_2)$

$$|T_n(k)| < \rho(c_1, c_2) \quad (n = 1,2,3,\ldots) \tag{1.11}$$

If now $F_N(x)$ is the distribution function of the sum of the first N members of the sequence
of the random variables, then, for $N \to \infty$ the following equalities hold uniformly

$$F_N(x) = (2\pi B_N)^{-\frac{1}{2}} \exp\left[-\frac{x^2}{2B_N}\right]$$

$$+\begin{cases} \dfrac{S_N + T_N x}{B_N^{5/2}} + 0\left(\dfrac{1+|x|^3}{N^2}\right); & |x| < 2\ln^2 N \\ 0\left(\frac{1}{N}\right); & x = \text{arbitrary.} \end{cases}$$

1.12

In 1.12 $B_N = \sum_{n=1}^{N} M_{2n}$, while S_N and T_N are x-independent and of smaller order than N.

When $M_{1n} \neq 0$, ($n = 1,2,\ldots$), then x^2 in the exponential has to be replaced by $(x - M_1)^2$, where $M_1 = \sum_{n=1}^{N} M_{1n}$.

We omit the proof of the theorem which is a little lengthy (8),(12) and point out that it makes use of the relation

$$F_N(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{ikx}\left[\prod_{\nu=1}^{N} T_{n_\nu}(k)\right]dk, \qquad 1.13$$

from the general theory of characteristic functions. This can also be written as a convolution

$$F_N(x) = \prod_{\nu=1}^{N} * f_\nu(x) \qquad 1.14$$

with the obvious meaning of " $\prod^*$ ".
Clearly, some elements of $\{n_\nu\}$ may be identical (for generality we have put an index ν).
According to the assumptions (i - iv) for the validity of CLT the Fourier partner $f_n(x)$ of $T_n(k)$ should be a semipositive definite function. This condition is not always satisfied by our functions $f_n(x)$.
To show this we consider the integral $\int_{-\infty}^{\infty} f_n(x)\, dx$, which according to 1.7 is given by

$$\int_{-\infty}^{\infty} f_n(x)\, dx = T_n(0) \quad . \qquad 1.15$$

It follows from this that whenever $T_n(0) = 0$, $f_n(x)$ cannot be positive everywhere. Moreover, we shall show that $f_n(x)$ is an odd function of x whenever $n=$ odd.
To prove the above assertion we consider the integral representation

$$T_n(k) = \int_{1}^{1} \frac{\xi^n \, d\xi}{1 + ik\xi} \qquad 1.16$$

Let us now take the Fourier transform of this

$$f_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_n(k) \, e^{ikx} \, dk \qquad 1.17$$

This exists and, obviously, according to Plancherel theorem(13) $f_n(x) \in L^2$.

Let us further consider

$$g_n(|x|) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_n(k) \, e^{ikx} \, dk \qquad 1.18$$

By putting $k \rightarrow -k$ in 1.18 and employing the obvious property of $T_n(k)$

$$T_n(-k) = (-)^n \, T_n(k) \qquad 1.19$$

we deduce

$$f_n(x) = (-)^n g_n(|x|) \tag{1.20}$$

and hence

$$f_n(-x) = -f_n(x) \tag{1.21}$$

for odd n, which proves the assertion.

From 1.21 it follows that $M_{2n} = 0$ for n = odd and, therefore, one cannot immediately apply CLT for the convolution 1.14.

Now we wish to show that a simply modified notation suffices to generalize the applicability of CLT to such cases.

Let us consider for simplicity the convolution product $f_n(x) * f_m(x)$ of only two factors $f_n(x)$, $f_m(x)$ and suppose that n = even, m = odd.

From 1.21 it follows that

$$\mathfrak{f}_m(x) = f_m(x)\,\vartheta(x) - f_m(|x|)\,\vartheta(-x), \tag{1.22}$$

$$\vartheta(x) = \begin{cases} 1 & , \quad x > 0 \\ 0 & , \quad x < 0 \end{cases} \qquad 1.23$$

and $x \in [-\infty, \infty]$.

The convolution product becomes

$$f_n(x) * f_m(x) = f_n(x) * f_m(x) \;\; \vartheta(x) - f_n(x) * f_m(x) \;\; \vartheta(-x) \qquad 1.24$$

Each term on the right-hand side of 1.24 is positive definite, and its factors satisfy the conditions for the validity of CLT.

The generalization of 1.24 to every finite number of factors, both odd and even, or purely odd functions of x is quite trivial.

For the complete solution of our problem we have still to clarify the question of the normalization.

In CLT the functions $f_n(x)$ have to satisfy

$$\int_{-\infty}^{\infty} f_n(x) \; dx = 1 \qquad 1.25$$

This is, however, for our functions $f_n(x) = \mathcal{F}^{-1}\{T_n(k)\}$ not the case.

For definiteness let us consider the general expression occuring in anisotropic transport problems

$$T_{\rho_{j-1}+\lambda_j}(k) \cdot T_{\rho_{j-2}+\lambda_{j-1}}(k) \ldots . T_{\rho_1+\lambda_2}(k) \cdot$$

$$\frac{1}{(1+ik\mu_s)\,(1+ik\mu)} \qquad 1.26$$

Here, the indices $\rho+\lambda$ of $T(k)$ may be even or odd and do not correspond to normalized functions $f_{\rho+\lambda}(x)$. The two last factors in 1.26 correspond

$$f(x,\mu)=\begin{cases} \dfrac{e^{-\frac{x}{\mu}}}{\mu} & ;\ x,\mu>0 \\ 0 & ;\ x<0,\ \mu>0 \\ 0 & ;\ x>0,\ \mu<0 \\ -\dfrac{e^{-\frac{x}{\mu}}}{\mu} & ;x,\mu<0 \end{cases} \qquad 1.27$$

They are automatically normalized

$$\int_{-\infty}^{\infty} f(x,\mu)\,dx = \int_0^{\infty} \frac{e^{-\frac{x}{\mu}}}{\mu}\,dx = 1 \qquad 1.28$$

One finds easily that

$$M_1 = \mu \qquad 1.29$$

and $$M_2 = \mu^2 \qquad 1.30$$

By supposing that all indices of $T_n(k)$ in 1.26 are even we find as normalization factor $1/N_n$

$$N_n = T_n\ (0)$$

$$= \int_{-1}^{1} \xi^n \, d\xi$$

$$= \frac{2}{n+1} \qquad 1.31$$

The first moment of $f_n(x)$ is (n = even)

$$M = \frac{1}{i} \; \frac{n+1}{2} \int_{-1}^{1} \left[\frac{\partial}{\partial k} \; \frac{1}{1+ik\xi} \right]_{k=o} \xi^n \, d\xi$$

$$= \frac{n+1}{2} \cdot 0$$

The second moment (n = even) is

$$M_{2n} = \frac{1}{i^2} \frac{n+1}{2} \int_{-1}^{1} \left[\frac{\partial^2}{\partial k^2} \frac{1}{1+ik\xi} \right]_{k=o} \xi^n \, d\xi$$

$$= \frac{2(n+1)}{n+3} \qquad 1.32$$

From the above and from 1.12 we find for arbitrary x(and every n = even) the result

$$E_j \, (x, \mu/\rho_j, \lambda_j; \ldots \; \rho_1, \lambda_1)$$

$$= (2\pi B_J)^{-1/2} \mu^{\rho_J} \mu_s^{\lambda_1} \prod_{j=1}^{J} \frac{2}{n_j+1} \exp\left[- \frac{(x-\mu-\mu_s)^2}{2B_J} \right] + 0 \left(\frac{1}{J}\right) \qquad 1.33$$

where $n_j = \rho_{j-1} + \lambda_j$ and

$$B_J = \sum_{j=2}^{J} \left(\frac{2(\rho_{j-1} + \lambda_j + 1)}{\rho_{j-1} + \lambda_j + 3} \right) + \mu^2 + \mu_s^2 \qquad 1.34$$

We turn now our attention to the case in which not all indices of T(k) in 1.26 are even numbers.
For simplicity we consider two factors.
According to 1.24 and 1.32 we have for the first term

$$M_{on} = \frac{n+1}{2} ,$$

$$M_{1n} = 0 ,$$

$$M_{2n} = \frac{2(n+1)}{n + 3} ,$$

$$M_{om} = \frac{m+1}{4} ,$$

$$M_{1m} = \frac{m+1}{m+2} ,$$

$$M_{2m} = \frac{m+1}{m+3} , \qquad 1.35$$

For the second term one finds similarly

$$M_{1n} = 0 ,$$

$$M_{2n} = \frac{2(n+1)}{n+3} ,$$

$$M_{1m} = - \frac{n+1}{n+2} ,$$

$$M_{2m} = \frac{n+1}{n +3} , \qquad 1.36$$

From 1.12, 1.24, 1.35, and 1.36 we find finally

$$F_2(x) = (2\pi B_2)^{-1/2}\left[\frac{8}{(n+1)(m+1)}\right]\left\{\exp\left[-\frac{\left(x-\frac{m+1}{m+2}\right)^2}{2B_2}\right] - \exp\left[-\frac{\left(x+\frac{m+1}{m+2}\right)^2}{2B_2}\right]\right\}, \qquad 1.37$$

where

$$B_2 = \frac{2(n+1)}{n+3} + \frac{m+1}{m+3},$$

(n = even, m = odd).

In the same manner one can treat any case of more than two factors of odd order.

Concluding, we remark that the central limit theorem is applicable in all similar cases arising in the theoretical investigation of infinite or finite media transport problems.

In the latter case, however, the formulas become rather complicated.

The validity of these remarkable results for the space, time and energy variables of the neutron derives from the fact that these variables can be interpreted as random ones.

As other fields of application of the CLT we briefly mention here Theory of electronics, Statistical Mechanics and Thermodynamics.

2. TRANSFORMATION OF THE ENERGY DEPENDENT TRANSPORT EQUATION.

The elastic scattering kernel for energy independent cross section depends on $\frac{v}{v'}$, $\frac{E}{E}$, or (u-u').* (App.B)
In the v - or E-representation of the elastic scattering kernel the Mellin transformation allows to apply the convolution theorem to eliminate the energy variable. The same is possible in the u-representation when the Laplace or Fourier transformation is applied. As the lethargy variable takes values from zero to $+\infty$, it is natural to consider the Laplace transformation as the appropriate one for the lethargy dependent transport equation.
Let us consider first the transport equation in the form

$$\vec{\Omega}\cdot\nabla\psi(\vec{r},\vec{\Omega},u) + \sigma_t(u)\psi(\vec{r},\vec{\Omega},u) = \int_{\vec{\Omega}} d\Omega' \int_{u,q,r'} du'\,\sigma(u',u,\vec{\Omega}'\cdot\vec{\Omega})\,\psi(\vec{r},\vec{\Omega}',u') + S(\vec{r},\vec{\Omega},u) \qquad 2.1.$$

* V,E, u are the speed, energy and lethargy variables .

where $\sigma(u', u, \mu_0)$ is given by

$$\sigma(u', u, \mu_0) = \sigma_{el}(u' \to u; \vec{\Omega}' \to \vec{\Omega}) + \sigma_{in}(u' \to u; \vec{\Omega}' \to \vec{\Omega}) , \qquad 2.2$$

and

$$q' = 2 \ln \frac{A+1}{A-1} , \qquad 2.3$$

r' is the nuclear level parameter, and $\mu_0 = \vec{\Omega}' \cdot \vec{\Omega}$.
It is not quite evident that the transport equation containing the inelastic scattering kernel too, is amenable to a transformation of the above kind. However, as it will be shown in Sec. 3 and 4 there exists a good approximation enabling us to reduce Eq. 2.1.

Specifically, one demande the energy independence of the nuclear cross sections, whenever use of integral transformation is made. We whish to point out that this condition is necessary only in space dependent problems.

A somewhat weaker condition, which is useful in space independent problems and which allows the application of the convolution theorem, is to demand the proportionality in energy of the total σ_t (E), and the scattering, σ_s (E, μ_0), cross sections, i.e.,

$$\sigma_s(E,\mu) = \sum_j P_j(\mu)\,\sigma_j(E) \;,$$

where

$$\sigma_j(E)/\sigma_t(E) = \text{Energy independent} \qquad 2.4$$

Condition 2.4 is satisfied at least by the total elastic scattering cross section, σ_s (E), which is also the first coefficient in the Legendre polynomial expansion. To illustrate this fact we give σ_s (E) / σ_t (E) in Fig. 1 for some isotopes as function of the energy, in which very small deviations from constancy are not represented(Macroscopic data taken from Ref. 14).

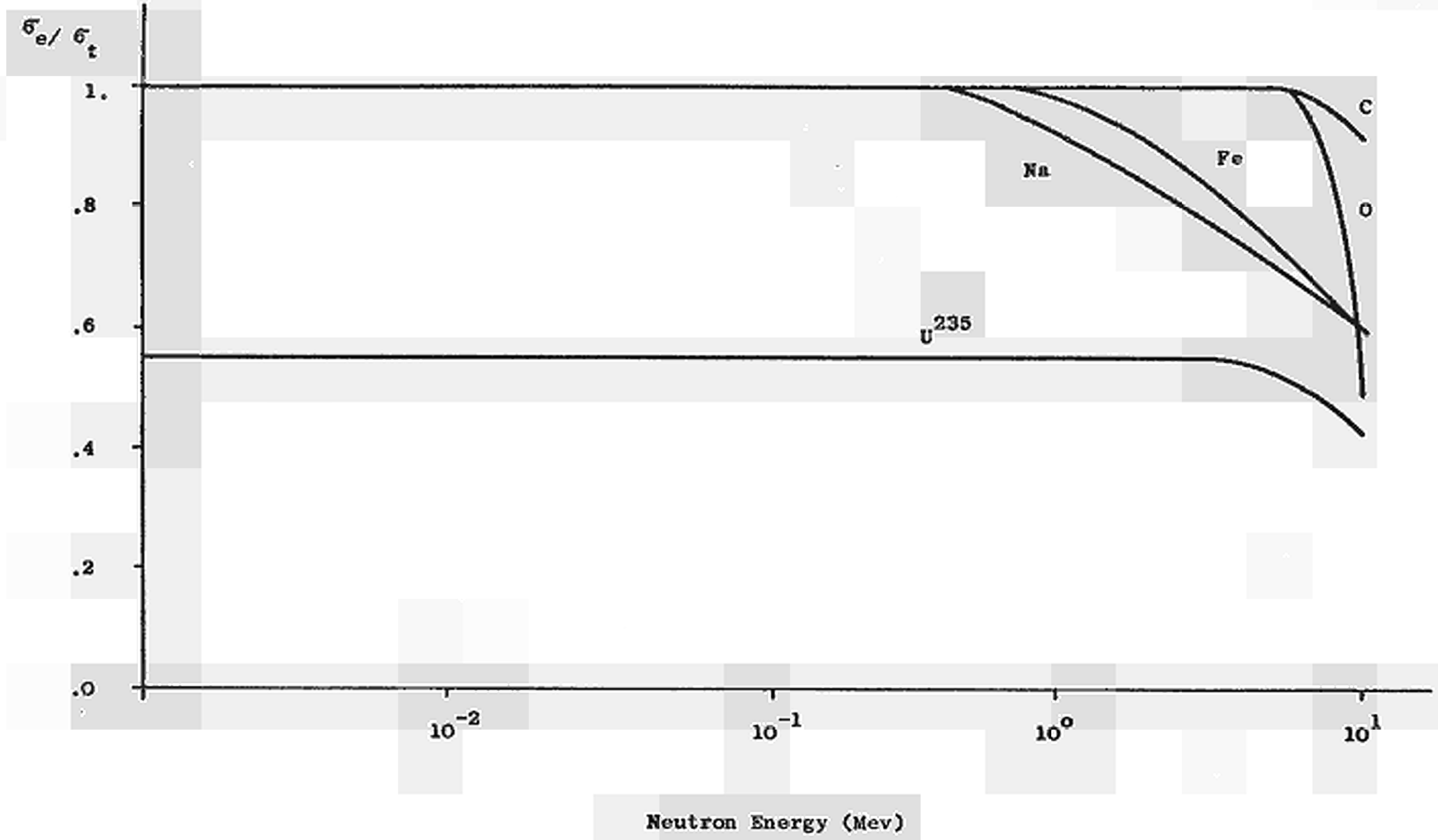

σ_e / σ_t
1.
.8
.6
.4
.2
.0
10^{-2}
10^{-1}
10^{0}
10^{1}
C
Fe
Na
O
U^{235}
Neutron Energy (Mev)

Fig. 1

Let us now consider Eq. 2.1 for plane geometry. For simplicity, we assume only elastic scattering isotropic in CS, and we write the scattering kernel as (App. B).

$$\sigma_s(u' \to u; \mu_0) = \sigma_s \, \delta\left[\mu_0 - \zeta^{el}(u', u)\right], \tag{2.5}$$

where

$$\zeta^{el}(u', u) = \frac{A+1}{2} e^{\frac{u'-u}{2}} - \frac{A-1}{2} e^{\frac{u-u'}{2}}. \tag{2.6}$$

From Eqs. 2.1, 2.4 and 2.5 we obtain for constant cross sections

$$\mu \frac{\partial \phi}{\partial z} + \phi(z, \mu, u) = \int_{\Omega'} d\Omega' \int_{u, q'} du' \, \delta\left[\mu_0 - \zeta^{el}(u, u')\right] \phi(z, \mu', u') + S(z, \mu, u). \tag{2.7}$$

while z has been defined as

$$z = x \, \sigma_t \quad , \tag{2.8}$$

and where x is the space coordinate.

$\phi(z, \mu, u)$ is now given by

$$\phi(z,\mu,u) = \sigma_t \, \psi(x,\mu,u) \quad . \qquad 2.9$$

We introduce now an energy independent parameter k , multiply both sides of Eq. 2.7 by exp $(-ikz)$ and integrate over the interval $-\infty \leq z \leq \infty$

We immediately obtain

$$(1+ik\mu)\varphi(k,\mu,u) = \int_{\Omega'} d\Omega' \int_{u,q'} du' \, \delta\left[\mu_0 - \zeta^{\alpha}(u,u')\right] \varphi(k,\mu',u') \qquad 2.10$$

$$+ S'(k,\mu,u) \, ,$$

where $\varphi(k,\mu,u)$ and $S'(k,\mu,u)$ are the Fourier transforms of $\phi(z,\mu,u)$ and $S(z,\mu,u)$ respectively.

From this equation we can eliminate the variable u also by means of a transformation. This is possible only, because k and c are independent from the variable u.

3. ELASTIC SLOWING DOWN.

In what follows we give the solution of the transport equation for elastic slowing down in an infinite monoisotopic medium of plane symmetry with space independent sources. The method is based on an expansion of the neutron propagator in the lethargy space in a power series of the coupling constant, c , of the neutron field with the interacting medium. We solve this problem treated already by Placzek (1) in order to show the effectiveness of the method. In order to obtain the necessary convergence a further transformation of the transport equation is introduced.

If we assume a monoenergetic source and isotropic scattering in CS Eq. 2.1 becomes

$$\sigma_t(u)\psi(u) = \frac{1}{1-\alpha}\int_{u-q'}^{u} \sigma_{el}(u')\psi(u')e^{u'-u}du' + S\,\delta(u-u_s) \qquad 3.1$$

where u_s is the initial neutron lethargy.

Now we introduce the assumption

$$\sigma_{el}(u) = c \cdot \sigma_t(u) \qquad 3.2$$

where c is a constant.

Eq. 3.2 is justified by the observation that σ_{el} (u) and σ_t(u) are almost proportional for some isotopes of interest over a wide energy region (14) (see Fig. 1). We observe that this assumption is very well justified in the case of H, C, and O which are three of the most frequently used materials acting as moderators. The somewhat large deviations from constancy of σ_{el} / σ_t for Na and Fe are mainly due to the inelastic scattering. U^{235} also fulfills Eq. 3.2 quite well. In general we define:

$$c = \frac{\int_{u-q'}^{u} \frac{\sigma_{el}(u')}{\sigma_t(u')} \exp(u')\psi(u')\sigma_t(u')\,du'}{\int_{u-q'}^{u} \exp(u')\psi(u')\sigma_t(u')\,du'} \qquad 3.3$$

which is taken to be constant inside $[u-q', u]$.

By this method the hopeless task of representing by polynomials the rapidly varying cross sections is replaced by the representation of the smooth function $\sigma_{el}(u) / \sigma_t(u)$ (Fig. 1).

We obtain from Eq. 3.1 and 3.2

$$\sigma_t(u)\psi(u') = \frac{c}{1-\alpha}\int_{u-q'}^{u} \sigma_t(u')\psi(u')\exp(u'-u)\,du' + S\cdot\delta(u-u_s) \qquad 3.4$$

Now we introduce a new variable, w, related to the lethargy, u, through

$$w = \frac{c}{1-\alpha}u \equiv u/\tau \qquad 3.5$$

The kernel and $\sigma_t(u)\psi(u)$ transform correspondingly like

$$\exp(u'-u)\,du = \frac{1-\alpha}{c}\exp\left[-\frac{1-\alpha}{c}(w'-w)\right]dw \qquad 3.6$$

and

$$\sigma_t(u)\,\psi(u)\,du = \phi(w)\,dw \qquad 3.7$$

and Eq. 3.4 becomes

$$\phi(w) = \int_{w-q}^{w} \phi(w')\exp[\tau(w'-w)]\,dw' + S\,\delta(w-w_s) \qquad 3.8$$

where

$$q = \frac{c q'}{1-\alpha} = -\frac{c \ln\alpha}{1-\alpha}\,. \qquad 3.9$$

Laplace transformation of Eq. 3.8 and application of the convolution theorem yield

$$f(p) = f(p)\,\frac{1-\exp[-q(\tau+p)]}{\tau+p} + S\exp(-p w_s)$$

or

$$f(p) = \frac{S\exp(-p\,w_s)}{1 - \frac{1-\exp[-q(\tau+p)]}{\tau+p}} \qquad 3.10$$

The denominator of the right-hand side of this equation can be written as

$$\sum_{n=0}^{\infty} \left\{ \frac{1-\exp[-q(\tau+p)]}{\tau+p} \right\}^n \qquad 3.11$$

This series is a geometric progression and converges absolutely whenever

$$\left| \frac{1-2\cos\omega.\exp[-q(\tau+p)]+\exp[-2q(\tau+p)]}{(\tau+p)^2+\omega^2} \right| < 1 \qquad 3.12$$

where

$$\rho = Re(p) \qquad 3.13$$

and

$$\omega = Im(p) \qquad 3.14$$

Now the left-hand side of inequality 3.12 has relative maxima at

$$\omega = (2m+1)\pi \; ; \quad m = 0, 1, 2, \ldots$$

which do not violate ineq. 3.12 provided t $> -\tau$.
At p = 0 inequality 3.12 takes on the form

$$1 - \exp(-q) < 1 \qquad 3.15$$

which is always satisfied.
From Eqs. 3.4 and 3.10 it follows that the introduction of τ is necessary only when

$$\frac{c}{1-\alpha} > 1 \qquad 3.16$$

In every other case the convergence criterium

$$\frac{c}{1-\alpha} \, \frac{1-\exp[-q(\tau+p)]}{\tau+p} < 1 \qquad 3.17$$

is satisfied in a wider p-domain. Inequality 3.17 yields a lower bound for the real part of p for which the absolute convergence of series Eq. 3.11 is assured.

After these considerations we can write Eq. 3.10 in the form

$$f(p) = S\exp(-pw_1)\sum_{n=0}^{\infty}\left\{\frac{1-\exp[-q(\tau+p)]}{\tau+p}\right\}^n \qquad 3.18$$

By applying the inversion operator

$$\mathcal{J} \equiv \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \exp(pw)\, dp \qquad 3.19$$

on both sides of Eq. 3.18 we obtain various results depending on the special value of w. It is pointed out, that because of the absolute convergence of the series Eq. 3.18 we are allowed to reorder the terms of it arbitrarily. This property will be used subsequently in finding the exact form of the collision density first derived by Placzek (1) by a different method.

Let us first consider the term with $n = 0$

$$S \exp(-p w_s) \qquad 3.20$$

From Eqs. 3.19 and 3.20 we obtain

$$\phi_0(w) = \frac{S}{2\pi i} \int_{-i\infty}^{i\infty} \exp[(w-w_s)p]\, dp$$

$$= S \frac{\partial}{\partial(w-w_s)} \left[\vartheta(w-w_s) \right]$$

The definition of $\vartheta(x)$ is

$$\vartheta(x) = \begin{cases} 1; & x > 0 \\ 0; & x < 0 \end{cases}$$

It follows therefore that

$$\phi_0(w) = S\,\delta(w-w_s) \qquad 3.21$$

We consider now a few cases with $n > 0$.

<u>n = 1</u>

From Eqs. 3.18, 3.19 and 3.20 we obtain

$$\phi_1(w) = S\left\{ \mathcal{J}\ \frac{\exp(-p w_s)}{\tau+p} - \exp(-q\tau)\ \mathcal{J}\ \frac{\exp[-(w_s+q)p]}{\tau+p} \right\} \qquad 3.22$$

The first integral in Eq. 3.22 is equal to

$$\phi_{10}(w) = S\exp[-\tau(w-w_s)]\begin{cases} 1\,; & w > w_s \\ 0\,; & w < w_s \end{cases} \qquad 3.23$$

Similary, the second integral is equal to

$$\phi_{11}(w) = S \exp[-\tau(w-w_s)] \begin{cases} 1; & w > w_s+q \\ 0; & w < w_s+q \end{cases} \qquad 3.24$$

From Eqs. 3.22 - 3,23 it follows that

$$\phi_1(w) = S \exp[-\tau(w-w_s)] \begin{cases} 1; & w \in (w_s, w_s+q) \\ 0; & w \notin (w_s, w_s+q) \end{cases} \qquad 3.25$$

Here and in what follows (x,y) is the open set y-x.

<u>n = 2</u>

Similary we see that

$$\phi_2(w) = S\Big\{ \mathcal{J} \frac{\exp(-pw_s)}{(\tau+p)^2} - \exp(-q\tau)\, \mathcal{J} \frac{\exp[-(w_s+q)p]}{(\tau+p)^2} + \exp(-2q\tau)\, \mathcal{J} \frac{\exp[-(w_s+2q)p]}{(\tau+p)^2} \Big\} \qquad 3.26$$

In a way analogous to that used in Eq. 3.22 we obtain from Eq. 3.26.

$$\phi_{20}(w) = S\exp[-\tau(w-w_s)]\begin{cases}(w-w_s); & w > w_s\\ 0; & w < w_s\end{cases} \qquad 3.27$$

$$\phi_{21}(w) = S\exp[-\tau(w-w_s)]\begin{cases}-2(w-w_s-q); & w > w_s+q\\ 0; & w < w_s+q\end{cases} \qquad 3.28$$

and

$$\phi_{22}(w) = S\exp[-\tau(w-w_s)]\begin{cases}(w-w_s-2q); & w > w_s+2q\\ 0; & w < w_s+2q\end{cases} \qquad 3.29$$

From Eqs. 3.27 - 3.29 we find

$$\phi_{2}(w) = S\exp[-\tau(w-w_s)]\begin{cases}(w-w_s); & w \in (w_s, w_s+q)\\ (w_s-w+2q); & w \in (w_s+q, w_s+2q)\\ 0; & w \notin (w_s, w_s+2q)\end{cases} \qquad 3.30$$

For every positive integer n we have

$$\phi_n(w) = S\frac{\exp[-\tau(w-w_s)]}{(n-1)!}\begin{cases} (w-w_s)^{n-1}; & w \in (w_s, w_s+q) \\ (w-w_s)^{n-1} - \binom{n}{1}(w-w_s-q)^{n-1}; & w \in (w_s+q, w_s+2q) \\ (w-w_s)^{n-1} - \binom{n}{1}(w-w_s-q)^{n-1} + \binom{n}{2}(w-w_s-2q)^{n-1}; & w \in (w_s+2q, w_s+3q) \\ \vdots & \\ 0; & w \notin (w_s, w_s+nq) \end{cases} \qquad 3.31$$

Now we observe that according to Eq. 3.5 w is proportional to $\frac{c}{1-\alpha}$. This constant characterizes the coupling of the neutron field with the medium. It is evident that the part of the neutron distribution proportional to the n-th power of the coupling constant of the neutron field with the medium with which it is interacting describes neutrons after n collisions, with arbitrary $w > w_s$.

Keeping this in mind we can easily give the physical interpretation of the various parts of the neutron distribution represented by Eq. 3.31 for n = 0,1,2...
The term (n = 0)

$$\phi_0(w) = S\,\delta(w-w_s) \qquad 3.32$$

is clearly the source term. The term (n = 1)

$$\phi_1(w) = S\exp[-\tau(w-w_s)]\ ;\ w\in(w_s, w_s+q) \qquad 3.33$$

gives the distribution of the simply scattered neutrons which necessarily must be found in the interval given above, However, as it becomes clear from Eq. 3.31 there exist other neutrons having experienced a number $n > 1$ of collisions which have w-values also given by Eq. 3.33.
From Eq. 3.31 we find for these neutrons the expressions:

n = 2 $\phi_{20}(w) = S\exp[-\tau(w-w_s)]\cdot(w-w_s)$

n = 3 $\phi_{30}(w) = S\exp[-\tau(w-w_s)]\,\frac{(w-w_s)^2}{2!}$

n = 4 $\phi_{40}(w) = S\exp[-\tau(w-w_s)]\,\frac{(w-w_s)^3}{3!}$

etc. 3.34

By summing Eqs. 3.33 and 3.34 with $1 \leq n \leq \infty$ we obtain the result

$$F_1(w) = S\exp[(1-\tau)(w-w_s)]; \quad w \in (w_s, w_s+q) \qquad 3.35$$

Eq. 3.35 represents the distribution of the neutrons which have experienced all possible numbers of collisions ($n \geq 1$) and remain in the first interval.

It is seen that

$$F_{1min}(w) = S$$

and

$$F_{1max}(w) = S\exp[(\tau-1)q] \qquad 3.36$$

These two values coincide with those found by Placzek for the case of no absorption ($c = 1$).

In the same way we calculate the neutron distribution in the second interval. The sum now is

$$F_2(w) = S\left[\sum_{n=2}^{\infty}\frac{(w-w_s)^{n-1}}{(n-1)!} - \sum_{n=2}^{\infty}\frac{n(w-w_s-q)^{n-1}}{(n-1)!}\right]\exp[-\tau(w-w_s)]$$

or

$$F_2(w) = S\left\{1-[1+(w-w_s-q)]\exp(-q)\right\}\exp[(1-\tau)(w-w_s)] \qquad 3.37$$

$$w \in (w_s+q,\ w_s+2q)$$

Eq. 3.37 represents the distribution of the neutrons which have experienced all possible numbers of collisions ($2 \leq n \leq \infty$) and did not leave the second interval.

From Eqs. 3.35 and 3.37 we see that $F_1(w)$ and $F_2(w)$, although they are defined in different intervals, are formally related by

$$F_2(w) = F_1(w) - \gamma_2(w) \qquad 3.38$$

where

$$\mathcal{F}_2(w) = S\left[1 + (w - w_s - q)\right] \exp\left[(1-\tau)(w - w_s) - q\right] . \qquad 3.39$$

From Eq. 3.39 we see further that

$$F_1(w - w_s - 0) - F_2(w - w_s + 0) = S \exp(-\tau q) \qquad 3.40$$

This is the discontinuity of the collision densities

$F_1(w)$ and $F_2(w)$ at $w = w_s + q$.

Going over to the energy representation we obtain

$$F_1(E/\alpha + 0) - F_2(E/\alpha - 0) = S \exp(-\tau q) \frac{dw}{dE} \qquad 3.41$$

$$= \frac{S}{E_s} \frac{c}{1-\alpha}$$

This is the well-known Placzek discontinuity. Now we see that the natural interpretation of expansion 3.31 is that it decomposes the neutron distribution according to the number of collisions and energy intervals.

We are now going to show that no other discontinuities exist. First we calculate the collision density $F_3(w)$, $w \in (w_s+2q, w_s+3q)$.
If from Eq. 3.32 we build the sum

$$F_3(w) = \sum_{n=3}^{\infty} \phi_{n3}(w)$$

we find

$$F_3(w) = S\Big\{\Big\langle 1-[1+(w-w_s-q)]\exp(-q) + \frac{w-w_s-2q}{2}[2+(w-w_s-2q)]\exp(-2q)\Big\rangle \cdot \exp[(1-\tau)(w-w_s)]\Big\}$$

$$w \in (w_s+2q, w_s+3q) \qquad 3.42$$

From Eqs. 3.37 and 3.42 it follows that

$$F_2(w_s+2q-0) = F_3(w_s+2q+0)\ , \qquad 3.43$$

if we observe that the last term in Eq. 3.42 vanishes identically at $w = w_s + 2q$, while the other two terms become identical to $F_2(w_s+2q)$. The third term of Eq. 3.42 has a non vanishing first order derivative at $w = w_s + 2q$ so that

$$\left.\frac{dF_2}{dw}\right|_{w=w_s+2q-0} \neq \left.\frac{dF_3}{dw}\right|_{w=w_s+2q+0} \qquad 3.44$$

It follows by induction from Eqs. 3.43 and 3.44 that

$$F_n(w_s+nq-0) = F_{n+1}(w_s+nq+0)$$

and

$$\left.\frac{d^nF_n}{dw^n}\right|_{w=w_s+nq-0} \neq \left.\frac{d^nF_{n+1}}{dw^n}\right|_{w=w_s+nq+0} .$$

4. INELASTIC SLOWING DOWN BY ONE SINGLE LEVEL.

In the preceding section we have given the complete solution of the slowing down problem of neutrons losing energy only by elastic scattering isotropic in CS.

The method developed and applied there allows a clear analysis of the physical properties and the mathematical behaviour of the neutron distribution. Now we are going to apply our method to the slowing down problem of neutrons losing energy only by inelastic scattering isotropic in CS. Such a situation does not occur in neutron physics because there is always elastic scattering present, and, as we shall show, it interferes with inelastic scattering. We, however, consider this extreme case, because, on the one hand, it allows us to extract more clearly the special feature of inelastic scattering, and, on the other hand, because it makes it easier to recognize the appropriate approximations which are necessary for the application of the method. The transport equation in the special case of a single level and a monoenergetic source, $S\,\delta(u-u_s)$, has the form

$$\sigma_t(u)\,\psi(u) = \frac{1}{1-\alpha}\int_{u-r'-q'}^{u-r'} du'\,\frac{\sigma_{in}(u')\,\psi(u')\exp(u'-u)}{\left[1-\frac{A+1}{A}\frac{Q}{E_s}\exp(u')\right]^{1/2}} + S\,\delta(u-u_s)\,, \qquad 4.1$$

where the kernel is different from zero only whenever

$$u' < \ln\frac{E_s}{Q} \equiv \bar{u} \qquad 4.1a$$

In Eq. 4.1 we have defined r' by

$$r' = \ln\frac{E_s}{E_s-Q}\,, \qquad 4.2$$

and Q is the excitation energy of the nuclear level.

In order to find the integration limits in Eq. 4.1, we have used the following model for inelastic scattering.

<u>Step one:</u> Absorption of a neutron of energy E_s (lethargy u_s) and excitation of the target nucleus with excitation energy Q (lethargy r')

<u>Step two:</u> Emission of a neutron as though it had initial energy $(E_s - Q)$ (lethargy u_s+r') and were "elastically" and (LS) isotropically scattered.

It is clear that we cannot assume that the neutron was first scattered "elastically" with initial energy E_s and afterwards had excited the nucleus then it would not have been able to excite the target nucleus, if E is not still larger than Q. Excitation of the inelastically scattering nucleus must precede scattering.

Let us now consider Fig. 2.

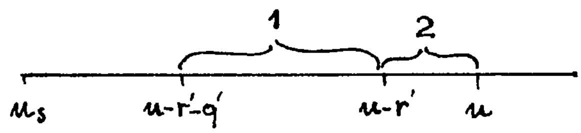


Fig. 2

Only inelastically scattered neutrons having an initial lethargy, u', satisfying $u-r'-q' \leq u' \leq u-r'$ can contribute to ψ(u), i.e. only neutrons from interval 1 can arrive at u. Neutrons scattered inelastically from the interval 2 have to gain first the lethargy r' and then they are scattered "elastically" (according to our model);but meanwhile they have already passed u, and so they cannot reach it at all.

This simplified model implies that the lethargy gain, r', of the neutron per excitation is constant. However, we shall see that we can correct for it by "renormalizing" r'.

To solve Eq. 4.1 we have to make an additional assumption. We first write the integral term of Eq. 4.1 as

$$\frac{1}{1-\alpha}\int_{u-r'-q'}^{u-r'} du' \frac{\sigma_{in}(u')\,\psi(u')\,\sigma_t(u')\exp(u'-u)}{\sigma_t(u')\left[1-\frac{A+1}{A}\frac{Q}{E_s}\exp(u')\right]^{1/2}} \;.$$

Introducing the **distribution** ϕ (w), and using again the variable w previously defined we obtain

$$\phi(w) = \int_{w-r-q}^{u-r} dw' \frac{\sigma_{in}(w')\,\phi(w')\exp[\tau(w'-w)]}{\sigma_t(w')\left[1-\frac{A+1}{A}\frac{Q}{E_s}\exp(\tau w')\right]^{1/2}} + S\,\delta(w-w_s) \qquad 4.3$$

$$0 \leq w' \leq \bar{w}$$

where $\overline{w}$ is defined by

$$Q = E_s \exp(-\tau\bar{w})$$

Now we observe that the first factor of the integrand in Eq. 4.3 is approximately w' - independent for a large number of cases (see Figs. 3,4, and 5).'.
We set therefore

$$c_{in} = \frac{\sigma_{in}(w^*)}{\left[1-\frac{A+1}{A}\frac{Q}{E_s}\exp(\tau w^*)\right]^{1/2}\sigma_t(w^*)} = const., \qquad 4.4.$$

and r, q and τ are given by

$$r = r'/\tau \qquad 4.5$$

$$q = q'/\tau \qquad 4.6$$

$$\tau = \frac{1-\alpha}{C_{in}} \qquad 4.7$$

w^{*} is an appropriate w-value. Some examples of Eq. 4.4. are given in Figs.3,4 and 5 in which the excitation functions have been taken from (15) and (16) .

From Eqs. 4.3 and 4.4. we have

$$\phi(w) = \int_{w-r-q}^{w-r} \phi(w') \exp[\tau(w-w')]\, dw' + S\, \delta(w-w_s) \qquad 4.8$$

Eq. 4.8 is formally identical with Eq. 3.8 for elastic slowing down except that now the integration limits have different values. Taking the Laplace transform of Eq. 4.8 we obtain

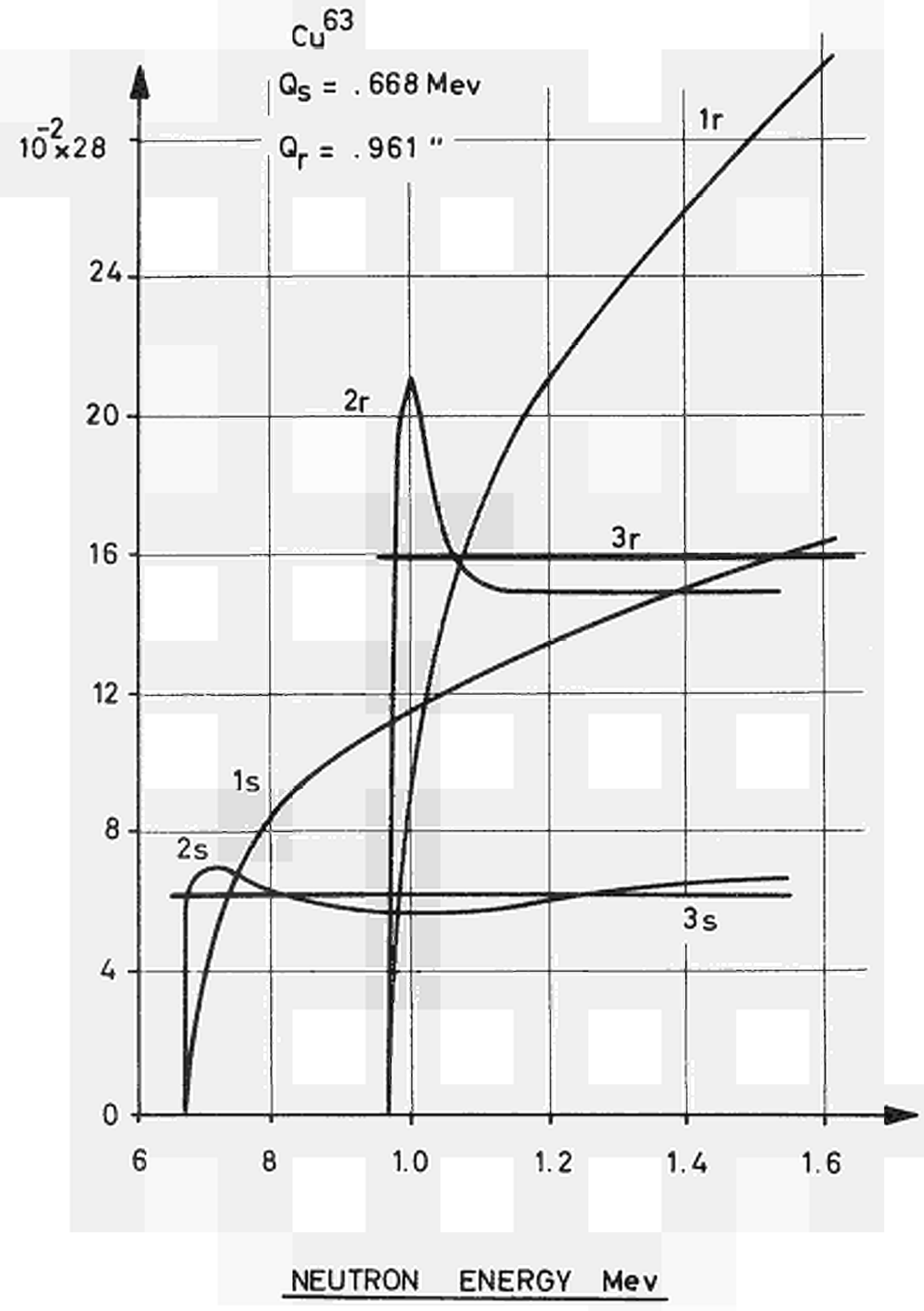
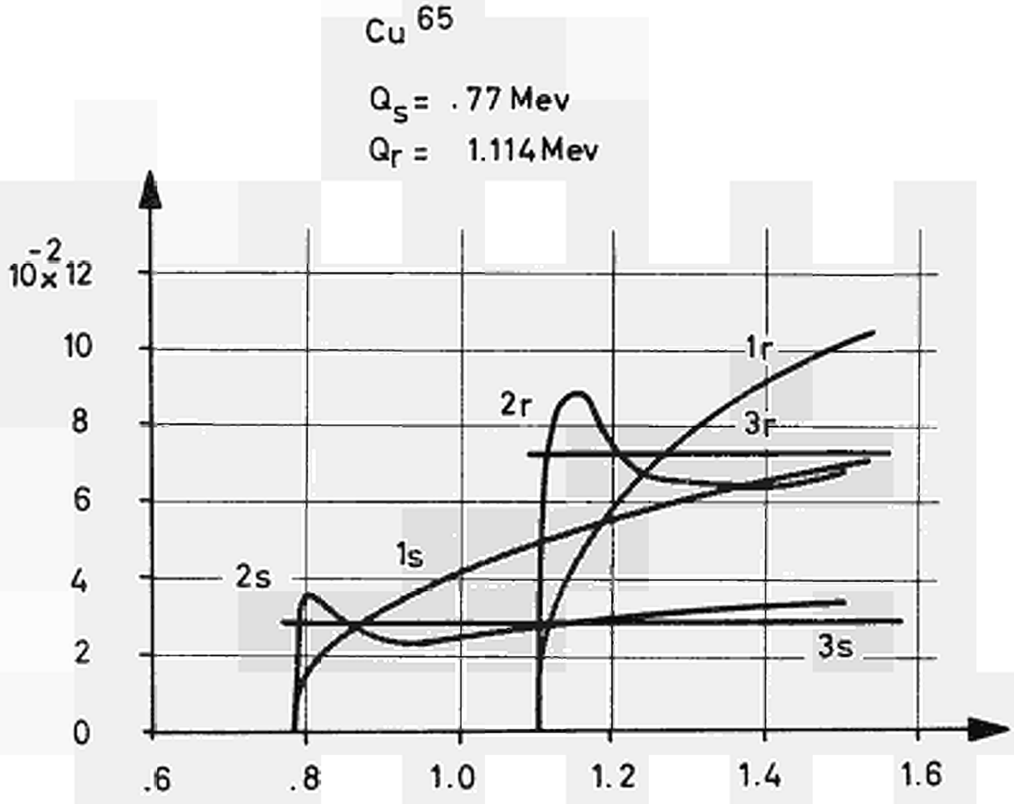

Cu63
Qs = .668 Mev
Qr = .961 "
10-2 x28
24
20
16
12
8
4
0
6
8
1.0
1.2
1.4
1.6
1r
2r
3r
1s
2s
3s
NEUTRON ENERGY Mev

FIG. 3



NEUTRON ENERGY Mev

FIG. 4

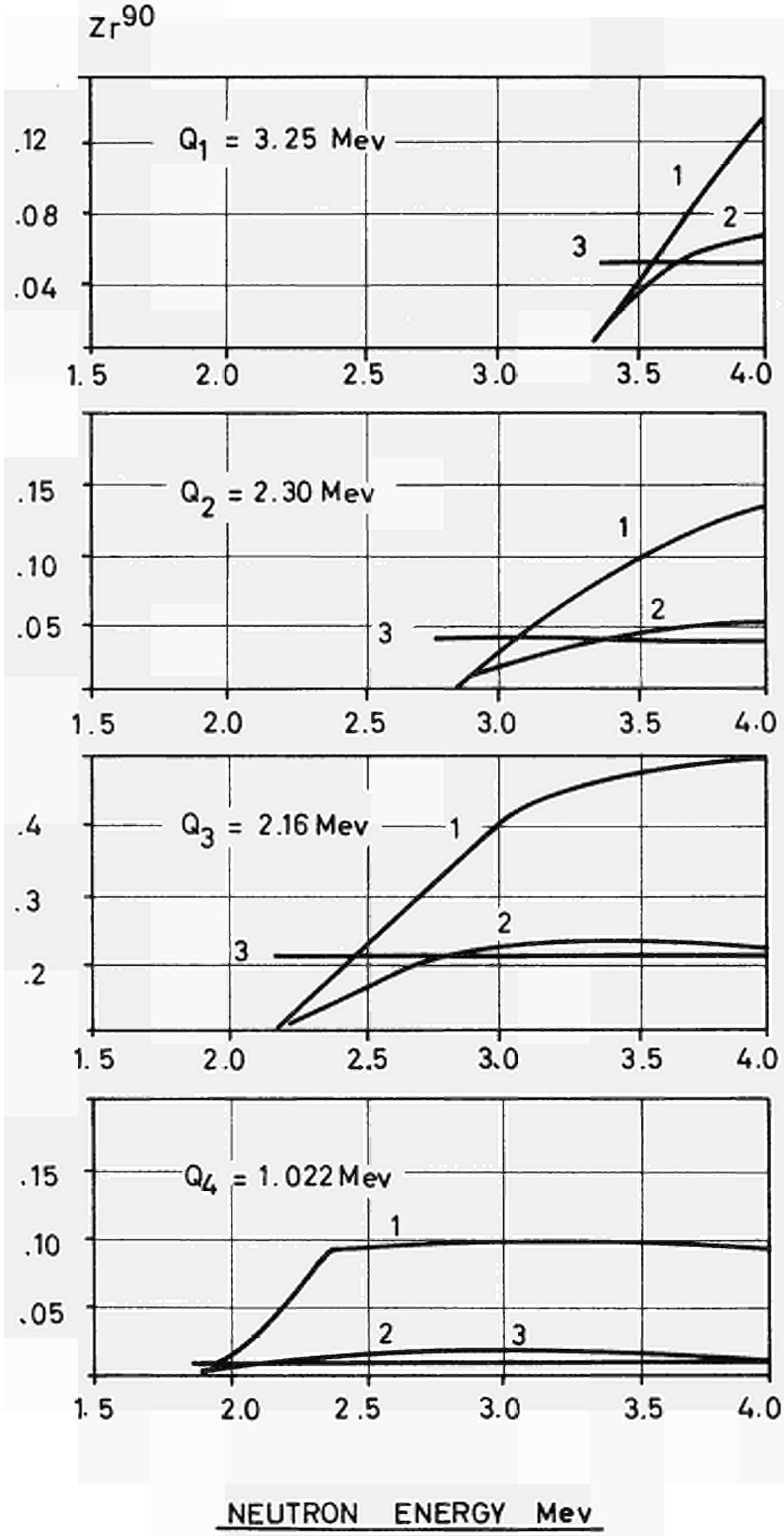

Zr^{90}
Q_1 = 3.25 Mev
.12
.08
.04
1
2
3
1.5 2.0 2.5 3.0 3.5 4.0
Q_2 = 2.30 Mev
.15
.10
.05
1
2
3
1.5 2.0 2.5 3.0 3.5 4.0
Q_3 = 2.16 Mev
.4
.3
.2
1
2
3
1.5 2.0 2.5 3.0 3.5 4.0
Q_4 = 1.022 Mev
.15
.10
.05
1
2
3
1.5 2.0 2.5 3.0 3.5 4.0
NEUTRON ENERGY Mev

FIG. 5

$$f(p) = \frac{S\exp(-pw_s)}{[1-[1-\exp[-q(\tau+p)]\exp[-r(\tau+p)]]/(\tau+p)]]} \qquad 4.9$$

By repeating the argumentation of the preceding sectinn we obtain the expansion

$$f(p) = S\exp(-pw_s)\sum_{n=0}^{\infty}\left\{[1-\exp[-q(\tau+p)]\frac{\exp[-r(\tau+p)]}{\tau+p}]\right\}^n \qquad 4.10$$

We apply the operator Eq. 3.19 on both sides of Eq. 4.10 and we consider the terms of the resulting series corresponding to various progressing n-values.

<u>n= 0</u> This term yields back the source

$$\phi_o(w) = S\cdot\delta(w-w_s) \qquad 4.11$$

<u>n = 1</u> This term depends linearly on the coupling constant and represents the distribution of the simply inelastically scattered neutrons.

$$\phi_1(w)= S\exp[-\tau(w-w_s)]\begin{cases}1 \ ; \ w\in(w_s+r,\ w_s+r+q)\\ 0 \ ; \ w\notin(w_s+r,\ w_s+r+q)\end{cases} \qquad 4.12$$

n = 2 The distribution of the twice inelastically scattered neutrons is given by +

$$\phi_2(w)= S\exp[-\tau(w-w_s)]\begin{cases}(w-w_s-2r)\ ; \ w\in(w_s+2r,\ w_s+2r+q)\\ (w_s+2r+2q-w);\ w\in(w_s+2r+q,\ w_s+2r+2q)\\ 0;\quad w\notin(w_s+2r,\ w_s+2r+2q).\end{cases} \qquad 4.13$$

n = 3 Similarly the distribution of the three times scattered neutrons is given by

$$\phi_3(w)=\frac{1}{2!}S\exp[-\tau(w-w_s)]\begin{cases}(w-w_s-3r)^2;\\ \quad w\in(w_s+3r,\ w_s+3r+q)\\ (w-w_s-3r)^2-3(w-w_s-3r-q)^2;\\ \quad w\in(w_s+3r+q,\ w_s+3r+2q)\\ (w-w_s-3r)^2-3(w-w_s-3r-q)^2+3(w-w_s-3r-2q)^2;\\ \quad w\in(w_s+3r+2q,\ w_s+3r+3q).\\ 0\ ;\quad w\notin(w_s+3r,\ w_s+3r+3q).\end{cases} \qquad 4.14$$

+ The distribution of the twice inelastically slowed dawn neutrons vanishes identically for $w < w_s + r$.

For every integer number, n , we have

$$
\phi_n(w) = \frac{S}{(n-1)!}\exp[-\tau(w-w_s)]
\begin{cases}
(w-w_s-nr)^{n-1}; \\
\quad w \in (w_s+nr,\ w_s+nr+q) \\
(w-w_s-nr)^{n-1} - \binom{n}{1}(w-w_s-nr-q)^{n-1}; \\
\quad w \in (w_s+nr+q,\ w_s+nr+2q) \\
(w-w_s-nr)^{n-1} - \binom{n}{1}(w-w_s-nr-q)^{n-1} + \binom{n}{2}(w-w_s-nr-2q)^{n-1}; \\
\quad w \in (w_s+nr+2q,\ w_s+nr+3q). \\
\quad \vdots \\
0;\ w \notin (w_s+nr,\ w_s+nr+nq).
\end{cases}
\qquad 4.15
$$

For the correct understanding of Eqs. 4.13 - 4.15 we must remember the definition of r'. In Eq. 4.1 r' was the lethargy gain of the neutron due to the first excitation of a target nucleus. When the same neutron excites a second level - if its energy allows it to do so - its lethargy gain is no longer equal to r'. It is given rather by

$$
r_2' = \ln \frac{E_s - Q}{E_s - 2Q} \qquad 4.16
$$

Let us now consider Eq. 4.13. Its right-hand side is different from zero only inside the open interval

$$I \doteq (w_s+2r,\ w_s+2r+2q) \qquad 4.17$$

In Eq. 4.17 2r stands symbolically for $r + r_2$, the second term of which is given by Eq. 4.16 divided by τ . From this we see that the energy of a neutron twice inelastically scattered will lie in the open interval

$$\left(\alpha^2(E_s-2Q),\ (E_s-2Q)\right)$$

This follows immediately from the model for the description of the inelastic slowing down.

In general, the lethargy gain of a neutron after n excitations will be given by

$$\sum_{j=1}^{n} r_j' = \ln \frac{E_s}{E_s - nQ} ,$$

or ,

$$r_n' = \ln \frac{E_s-(n-1)Q}{E_s - nQ} \qquad 4.18$$

for the n-th excitation.

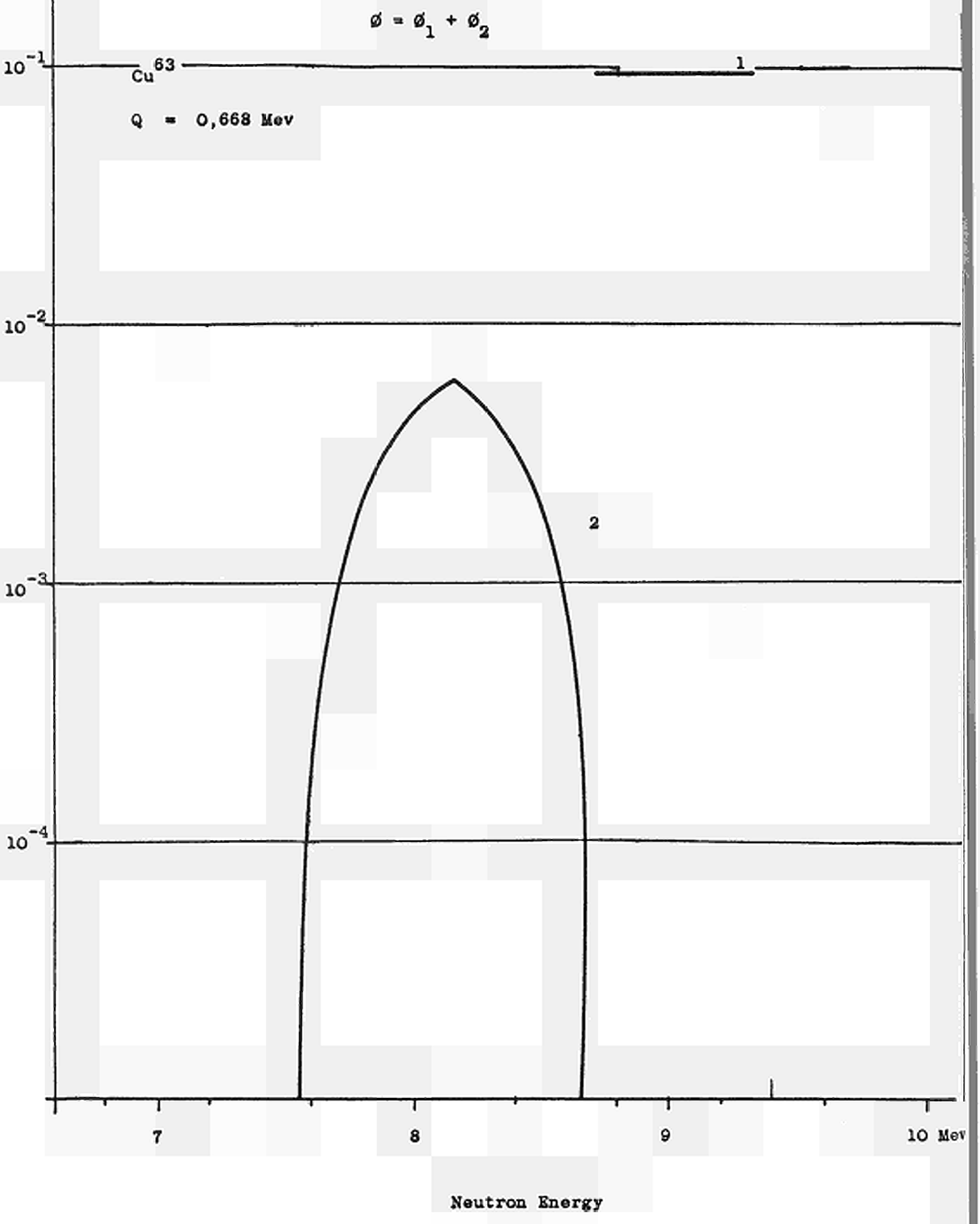
It is understood, that in Eq.4.18 n must satisfy

$$n < \text{integer part of} \left(\frac{E_s}{Q}\right) .$$

In Fig. 6 Eqs. 4.12, 4.13, and 4.14 are shown for a typical example of inelastic scattering by a single level (curve 1 single, curve 2 double scattering).

Now, before considering more complicated situations of inelastic slowing down we observe the following properties.

For $w \in (w_s, w_s + r)$ the distribution function vanishes, i.e. no neutrons corresponding to a w belonging to this interval can be found after inelastic scattering.



$\emptyset = \emptyset_1 + \emptyset_2$
10^{-1}
Cu^{63}
Q = 0,668 Mev
1
10^{-2}
2
10^{-3}
10^{-4}
7
8
9
10 Mev
Neutron Energy

Fig. 6

In terms of energy we may state: If neutrons of initial energy E have been scattered inelastically, their maximal final energy is equal to $E - Q$.

It is pointed out that there cannot be found neutrons scattered inelastically more than once having w-value belonging to $(w_s, w_s + r)$.

By setting $r = 0$, the distribution of the inelastically scattered neutrons reduces to the corresponding distribution of elastically scattered neutrons. In connexion with this we observe that in the inelastic case the sums corresponding to Eqs. 3.35, 3.37, and 3.42 reduce to a single term.

5. INELASTIC SLOWING DOWN BY TWO DISCRETE LEVELS.

Let us now consider the situation in which neutrons are inelastically scattered by two discrete nuclear levels. In principle the method remains the same except for some new aspects which now must be taken into account.

The transport equation for this situation has the form

$$\begin{aligned}\sigma_t(u)\,\psi(u) = {} & \frac{1}{1-\alpha}\int_{u-r'-q'}^{u-r'} du'\,\frac{\sigma_{r,in}(u')\,\psi(u')\exp(u'-u)}{\left[1-\frac{A+1}{A}\,\frac{Q_r}{E_s}\exp(u')\right]^{1/2}} \\ & + \frac{1}{1-\alpha}\int_{u-s'-q'}^{u-s'} du'\,\frac{\sigma_{s,in}(u')\,\psi(u')\exp(u'-u)}{\left[1-\frac{A+1}{A}\cdot\frac{Q_s}{E_s}\exp(u')\right]^{1/2}} \\ & + S\cdot\delta(u-u_s)\;.\end{aligned} \qquad 5.1$$

In Eq. 5.1 the integrals are different from zero only if

$$u' < \ln\frac{E_s}{Q_j}\;;\quad j = r, s\;.$$

r' and s' are defined by equations corresponding to

Eq. 4.2. Proceeding as in Eq. 4.1 we obtain from Eq. 5.1

$$\phi(w) = \sum_{j=r,s} \varkappa_j \int_{u-j-q}^{u-j} \phi(w') \exp[\tau(w'-w)]\,dw' + S\,\delta(w-w_s) \,. \tag{5.2}$$

Here, $c_{r,in}$ and $c_{s,in}$ are defined by equations analogous to Eq. 4.4 but τ , $\varkappa_r$ and $\varkappa_s$ are now defined by

$$\tau = \frac{1-\alpha}{c_{r,in}+c_{s,in}} \tag{5.3}$$

$$\varkappa_j = \frac{c_{j,in}}{c_{r,in}+c_{s,in}} \quad . \tag{5.4}$$

By taking the Laplace transform of Eq. 5.2 and by solving it with respect to the transform we find:

$$f(p) = S\exp(-pw_s)\Big\{1-\Big\langle[g_r-h_r\exp(-qp)]\exp(-rp) + [g_s-h_s\exp(-qp)]\exp(-sp)\Big\rangle\frac{1}{\tau+p}\Big\}^{-1}, \qquad 5.5.$$

where the following definitions have been used

$$g_j = \gamma_j \exp(-\tau\cdot j), \qquad 5.6$$

$$h_j = \gamma_j \exp[-\tau(j+q)], \qquad 5.7$$

$$j = r, s.$$

Expanding the right-hand side of Eq. 5.5 in a series we have

$$f(p) = S\exp(-pw_s)\Big\{1 + \left\langle\frac{g_r\exp(-rp)+g_s\exp(-sp)-h_r\exp[-(r+q)p]-h_s\exp[-(s+q)p]}{\tau+p}\right\rangle$$

$$+\left\langle\frac{g_r\exp(-rp)+g_s\exp(-sp)-h_r\exp[-(r+q)p]-h_s\exp[-(s+q)p]}{\tau+p}\right\rangle^2$$

$$+\;\cdots\cdots\Big\} \qquad 5.8$$

By applying the operator Eq. 3.19 on both sides of Eq. 5.8 we immediately get the distribution of the inelastically scattered neutrons by the two discrete levels.

n = 0

$$\phi_0(w) = S\,\delta(w-w_s) \qquad 5.9$$

n = 1

$$\phi_1(w) = S\exp[-\tau(w-w_s)]\begin{cases}\gamma_r\,; & w\in(w_s+r,\,w_s+r+q)\\ \gamma_s\,; & w\in(w_s+s,\,w_s+s+q)\\ 0\,; & w\notin\{(w_s+r,\,w_s+r+q)\cup(w_s+s,\,w_s+s+q)\}\end{cases} \qquad 5.10$$

From this equation we see that the first collision inelastic distribution is simply the superposition of the two single-level-scattered distributions, i.e. no interference is present.

In Eq. 5.10 we have made use of Eqs. 5.3 and 5.7.

<u>n = 2</u>

$$\phi_2(w) = S\exp[-\tau(w-w_s)]\begin{cases} \gamma_r^2(w-w_s-2r)\,; & w\in(w_s+2r,\, w_s+2r+q) \\ \gamma_r^2(w_s+2r+2q-w)\,; & w\in(w_s+2r+q,\, w_s+2r+2q) \\ 2\gamma_r\gamma_s(w-w_s-r-s)\,; & w\in(w_s+r+s,\, w_s+r+s+q) \\ 2\gamma_r\gamma_s(w_s+r+s+2q-w)\,; & w\in(w_s+r+s+q,\, w_s+r+s+2q) \\ \gamma_s^2(w-w_s-2s)\,; & w\in(w_s+2s,\, w_s+2s+q) \\ \gamma_s^2(w_s-w+2s+2q)\,; & w\in(w_s+2s+q,\, w_s+2s+2q) \\ 0\,; & w\notin\{(w_s+2r,\, w_s+2r+2q)\cup(w_s+2s,\, w_s+2s+2q)\}\,. \end{cases} \qquad 5.11$$

Here and in what follows we define

$$mr' + ks' = \ln\left(\frac{E_s}{E_s - mQ_r - kQ_s}\right)\,; \qquad m,k = \text{posit. int.} \qquad 5.12$$

<u>n = 3</u>

$$\phi_3(w) = \frac{S}{2!}\exp[-\tau(w-w_s)]\begin{cases}
\varkappa_r^3(w-w_s-3r)^2 \ ; \ w \in (w_s+3r, w_s+3r+q) \\
\varkappa_r^3\left[(w-w_s-3r)^2-3(w-w_s-3r-q)^2\right] ; \ w \in (w_s+3r+q, w_s+3r+2q) \\
\varkappa_r^3\left[(w-w_s-3r)^2-3(w-w_s-3r-q)^2+3(w-w_s-3r-2q)^2 ; \right. \\
\qquad\qquad w \in (w_s+3r+2q, w_s+3r+3q). \\
\varkappa_s^3(w-w_s-3s)^2 ; \ w \in (w_s+3s, w_s+3s+q) \\
\varkappa_s^3\left[(w-w_s-3s)^2-3(w-w_s-3s-q)^2\right] ; \ w \in (w_s+3s+q, w_s+3s+2q) \\
\varkappa_s^3\left[(w-w_s-3s)^2-3(w-w_s-3s-q)^2+3(w-w_s-3s-2q)^2 ; \right. \\
\qquad\qquad w \in (w_s+3s+2q, w_s+3s+3q). \\
3\varkappa_r^2\varkappa_s(w-w_s-2r-s)^2 ; \ w \in (w_s+2r+s, w_s+2r+s+q) \\
3\varkappa_r^2\varkappa_s\left[(w-w_s-2r-s)^2-3(w-w_s-2r-s-q)^2\right] ; \\
\qquad\qquad w \in (w_s+2r+s+q, w_s+2r+s+2q). \\
3\varkappa_r^2\varkappa_s\left[(w-w_s-2r-s)^2-3(w-w_s-2r-s-q)^2+3(w-w_s-2r-s-2q)^2\right] ; \\
\qquad\qquad w \in (w_s+2r+s+2q, w_s+2r+s+3q). \\
3\varkappa_r\varkappa_s^2(w-w_s-r-2s)^2 ; \ w \in (w_s+r+2s, w_s+r+2s+q). \\
3\varkappa_r\varkappa_s^2\left[(w-w_s-r-2s)^2-3(w-w_s-r-2s-q)^2\right] ; \\
\qquad\qquad w \in (w_s+r+2s+q, w_s+r+2s+2q) \\
3\varkappa_r\varkappa_s^2\left[(w-w_s-r-2s)^2-3(w-w_s-r-2s-q)^2+3(w-w_s-r-2s-2q)^2\right] ; \\
\qquad\qquad w \in (w_s+r+2s+2q, w_s+r+2s+3q). \\
0 ; \quad w \notin \left\{(w_s+3r, w_s+3r+3q) \cup (w_s+3s, w_s+3s+3q)\right\}.
\end{cases}$$

5.13

n = 4

$$
\phi_4(w) = \frac{S}{3!}\exp[-\tau(w-w_s)]
\begin{cases}
\gamma_r^4(w-w_s-4r)^3; \quad w\in(w_s+4r,\ w_s+4r+q) \\
\gamma_r^4[(w-w_s-4r)^3-4(w-w_s-4r-q)^3]; \quad w\in(w_s+4r+q,\ w_s+4r+2q) \\
\gamma_r^4[(w-w_s-4r)^3-4(w-w_s-4r-q)^3+6(w-w_s-4r-2q)^3]; \quad w\in(w_s+4r+2q,\ w_s+4r+3q). \\
\gamma_r^4[(w-w_s-4r)^3-4(w-w_s-4r-q)^3+6(w-w_s-4r-2q)^3-4(w-w_s-4r-3q)^3]; \\
\qquad w\in(w_s+4r+3q,\ w_s+4r+4q). \\
\gamma_s^4(w-w_s-4s)^3; \quad w\in(w_s+4s,\ w_s+4s+q). \\
\gamma_s^4[(w-w_s-4s)^3-4(w-w_s-4s-q)^3]; \quad w\in(w_s+4s+q,\ w_s+4s+2q). \\
\gamma_s^4[(w-w_s-4s)^3-4(w-w_s-4s-q)^3+6(w-w_s-4s-2q)^3]; \quad w\in(w_s+4s+2q,\ w_s+4s+3q). \\
\gamma_s^4[(w-w_s-4s)^3-4(w-w_s-4s-q)^3+6(w-w_s-4s-2q)^3-4(w-w_s-4s-3q)^3]; \\
\qquad w\in(w_s+4s\,q\,w_s+4s+4q). \\
4\gamma_r^3\gamma_s(w-w_s-3r-s)^3; \quad w\in(w_s+3r+s,\ w_s+3r+s+q). \\
4\gamma_r^3\gamma_s[(w-w_s-3r-s)^3-3(w-w_s-3r-s-q)^3+3(w-w_s-3r-s-2q)^3-(w-w_s-3r-s-3q)^3]; \\
\qquad w\in(w_s+3r+s+3q,\ w_s+3r+s+4q) \\
4\gamma_r^3\gamma_s[(w-w_s-3r-s)^3-3(w-w_s-3r-s-q)^3+3(w-w_s-3r-2q)^3]; \\
\qquad w\in(w_s+3r+s+2q,\ w_s+3r+s+3q) \\
4\gamma_r^3\gamma_s[(w-w_s-3r-s)^3-3(w-w_s-3r-s-2q)]; \quad w\in(w_s+3r+s+q,\ w_s+3r+s+2q). \\
6\gamma_r^2\gamma_s^2(w-w_s-2r-2s)^3; \quad w\in(w_s+2r+2s,\ w_s+2r+2s+q). \\
6\gamma_r^2\gamma_s^2[(w-w_s-2r-2s)^3-4(w-w_s-2r-2s-q)^3]; \quad w\in(w_s+2r+2s+q,\ w_s+2r+2s+2q). \\
6\gamma_r^2\gamma_s^2[(w-w_s-2r-2s)^3-4(w-w_s-2r-2s-q)^3+6(w-w_s-2r-2s-2q)^3]; \\
\qquad w\in(w_s+2r+2s+2q,\ w_s+2r+2s+3q). \\
6\gamma_r^2\gamma_s^2[(w-w_s-2r-2s)^3-4(w-w_s-2r-2s-q)^3+6(w-w_s-2r-2s-2q)^3-4(w-w_s-2r-2s-3q)^3]; \\
\qquad w\in(w_s+2r+2s+3q,\ w_s+2r+2s+4q). \\
4\gamma_r\gamma_s^3(w-w_s-r-3r)^3; \quad w\in(w_s+r+3s,\ w_s+r+3s+q) \\
4\gamma_r\gamma_s^3[(w-w_s-r-3s)^3-3(w-w_s-r-3s-q)^3]; \\
\qquad w\in(w_s+r+3s+q,\ w_s+r+3s+2q) \\
4\gamma_r\gamma_s^3[(w-w_s-r-3s)^3-3(w-w_s-r-3s-q)^3+3(w-w_s-r-3s-2q)^3]; \\
\qquad w\in(w_s+r+3s+2q,\ w_s+r+3s+3q) \\
4\gamma_r\gamma_s^3[(w-w_s-r-3s)^3-3(w-w_s-r-3s-q)^3+3(w-w_s-r-3s-2q)^3-(w-w_s-r-3s-3q)^3]; \\
\qquad w\in(w_s+r+3s+3q,\ w_s+r+3s+4q).
\end{cases}
\tag{5.14}
$$

$\phi_4(w)$ vanishes identically for

$$w \notin \{(w_s+4r, w_s+4r+4q) \cup (w_s+4s, w_s+4s+4q)\}$$

In Eq. 5.14 we see that the inelastic scattering distribution by two discrete levels is not equal to the sum of the inelastic scattering distribution of two single levels, because there exist mixed terms representing neutrons scattered some times by the one and other times by the other level.

We have given sofar some expressions for the distribution of the inelastically scattered neutrons generated by a strictly monoenergetic source.

Eqs. 5.10 and 5.11 are shown in Fig. 7 for the levels Q_r = 0,668Mev and Q_s = 0,961 Mev of Cu^{63} (15).

6. ELASTIC AND INELASTIC SLOWING DOWN. TWO DISCRETE LEVELS.

In the preceding section we have discussed the case in which the scattering kernel consisted of only two terms corresponding to two discrete levels. As such a case is only approximately realizable we give now a discussion of the more realistic situation in which elastic and inelastic scattering have not been separated. This is in as much interesting as the sum of the elastic and inelastic scattered distributions does not equal the actually scattered distribution.

Using the results of the previous sections we write

$$\begin{aligned}\phi(w) = \; & \varkappa \int_{w-q}^{w} \exp(w'-w)\,\phi(w')\,dw' \\ & + \varkappa_r \int_{w-r-q}^{w-r} \exp[(w'-w)]\,\phi(w')\,dw' \\ & + \varkappa_s \int_{w-s-q}^{w-s} \exp[(w'-w)]\,\phi(w')\,dw' \\ & + S\,\delta(w-w_s) \end{aligned} \qquad 6.1.$$

where

$$\tau = \frac{1-\alpha}{C + C_{r,in} + C_{s,in}} \quad ,$$

$$\varkappa = \frac{C}{C + C_{r,in} + C_{s,in}}$$

and $\varkappa_r$, $\varkappa_s$ are defined by equations analogous to 5.4 and satisfy $\varkappa + \varkappa_r + \varkappa_s \leq 1$.

The Laplace transform is now given by

$$f(p) = S \exp(-pw_s) \sum_{n=0}^{\infty} \left\{ \frac{g - h \exp(-qp)}{\tau + p} + \frac{g_r \exp(-rp) - h_r \exp[-(r+q)p]}{\tau + p} + \frac{g_s \exp(-sp) - h_s \exp[-(s+q)p]}{\tau + p} \right\}^n \qquad 6.2$$

where $g = \varkappa$ and $h = \varkappa \exp(-q\tau)$. g_i, h_i have been defined analogously to Eqs. 5.6 and 5.7.

The various contributions to the distribution of the scattered neutrons are

<u>n = 0</u>

This case corresponds again to the source term Eq. 3.21 discussed in Sec . 3.

$$\phi_0(w) = S\,\delta(w-w_s) \qquad 6.3$$

<u>n = 1</u>

The neutrons have been scattered only once, therefore there cannot exist any interference of the elastic and inelastic scattering. Hence, we have

$$\phi_1(w) = S\exp[-\tau(w-w_s)]\begin{cases} \gamma & ; w \in (w_s, w_s+q) \\ \gamma_r & ; w \in (w_s+r, w_s+r+q) \\ \gamma_s & ; w \in (w_s+s, w_s+s+q) \\ 0 & ; w \notin \{(w_s+r, w_s+r+q) \cup (w_s+s, w_s+s+q)\} \end{cases} \qquad 6.4$$

<u>n = 2</u>

For n = 2 we have the first term in which interference of the various modes of scattering occurs.

The physical reason is that neutrons may have been scattered by one of the 9 modes:

1	elastic	-	elastic	
2	elastic	-	r-inel	
3	elastic	-	s-inel	
4	r-inel	-	r-inel	
5	r-inel	-	s-inel	
6	r-inel	-	elastic	
7	s-inel	-	s-inel	
8	s-inel	-	r-inel	
9	s-inel	-	elastic	6.5

From these nine combinations of scattering modes the first, fourth, fifth, seventh, and eight are commutative; that means, it does not matter how the neutron was scattered the first and how the second time. The remaining cases from 6.5, however are not commutative. In order to show it, let us consider a neutron of initial energy E.

Let us further suppose that it has been scattered once by the combination 2 and the second time by the combination 6.

We have the minimum final energies

$$\text{combination } 2 \; : \quad E_{f2} = \alpha(\alpha E - Q)$$

$$6 \; : \quad E_{f6} = \alpha^2(E - Q)$$

whence it follows that

$$E_{f2} \neq E_{f6}$$

After these considerations we can write the unsymmetrized distribution of the twice scattered neutrons as it follows from the inversion of the third term of the series Eq. 6.2

$$\phi_2(w) = S \exp[-\tau(w-w_s)] \begin{cases} \gamma^2(w-w_s) & ; w \in (w_s, w_s+q) \\ \gamma^2(w_s-w+2q) & ; w \in (w_s+q, w_s+2q) \\ 2\gamma\gamma_r(w-w_s-r) & ; w \in (w_s+r, w_s+r+q) \\ 2\gamma\gamma_r(w_s+r+2q-w) & ; w \in (w_s+r+q, w_s+r+2q) \\ 2\gamma\gamma_s(w-w_s-s) & ; w \in (w_s+s, w_s+s+q) \\ 2\gamma\gamma_s(w_s+s+2q-w) & ; w \in (w+s+q, w_s+s+2q) \end{cases} \qquad 6.6$$

$$\begin{cases} \gamma_r^2(w-w_s-2r) ; & w \in (w_s+2r, w_s+2r+q) \\ \gamma_r^2(w-w_s-2r-q) ; & w \in (w_s+2r+q, w_s+2r+2q) \\ 2\gamma_r\gamma_s(w-w_s-r-s) ; & w \in (w_s+r+s, w_s+r+s+q) \\ 2\gamma_r\gamma_s(w_s-w+r+s+2q) ; & w \in (w_s+r+s+q, w_s+r+s+2q) \\ \gamma_s^2(w-w_s-2s) ; & w \in (w_s+2s, w_s+2s+q) \\ \gamma_s^2(w-w_s-2s-q) ; & w \in (w_s+2s+q, w_s+2s+2q) \end{cases}$$

vanishes identically for

$$w \notin \left\{(w_s, w_s+2q) \cup (w_s+2r, w_s+2r+2q) \cup (w_s+2s, w_s+2s+2q)\right\}$$

The first and second lines of Eq. 6.6 represent the distributions of neutrons which have suffered two elastic collisions. They are identical to the corresponding expressions of Eq. 3.31. The third and fourth terms, proportional to $\gamma\gamma_r$, represent the distribution of the neutrons which have been scattered once elastically and once by the r-level. The order of succession, however, is fixed by the definition of r', the lethargy gain during r-inelastic scattering. There exist two possibilities:

$$r' = \ln \frac{E_s}{E_s - Q_r} \qquad 4.2$$

This definition of r' implies that the energy of the neutron after the scattering will be confined between $\alpha(E_s-Q_r)$ and (E_s-Q_r).
From this it follows that inelastic scattering has first taken place.

$$r' = \ln \frac{\alpha E_s}{\alpha E_s - Q_r} \qquad 6.7$$

From this definition of r' it follows that the energy of the neutron after scattering will lie between $\alpha(\alpha E_s-Q_r)$ and (αE_s-Q_r). Obviously in this event elastic scattering has first taken place.
We therefore may state:
Definition Eq. 4.2 of r' corresponds to combination 6 while definition Eq. 6.7 of r' corresponds to combination 2 of the scheme 6.5.

Eq. 6.6 has therefore to be complemented by two lines which make the expression symmetric with respect to the succession of $\{(r - \text{inel.}) - \text{el.}\}$ and $\{\text{el.} - (r - \text{inel.})\}$ of the scattering.

For the fifth and sixth lines of Eq. 6.6 there holds exactly the same, if we replace s' by r'. They correspond to combinations 3 and 9 of scheme 6.5.

The seventh and eight lines of Eq. 6.6 represent the distribution of neutrons scattered twice by the r-level. Lines ninth and tenth represent the distribution of neutrons scattered once by the r-level, and once by the s-level. The order, here, does nor play any part. Finally the last two lines represent the distribution of the neutrons scattered twice by the s-level. A graphical representation of Eq. 6.6 is given in Figs. 7,8 for Cu^{63} and Cu^{65}. It is instructive to compare Fig. 7 with Fig. 8. For three collisions we obtain similarily the distribution.

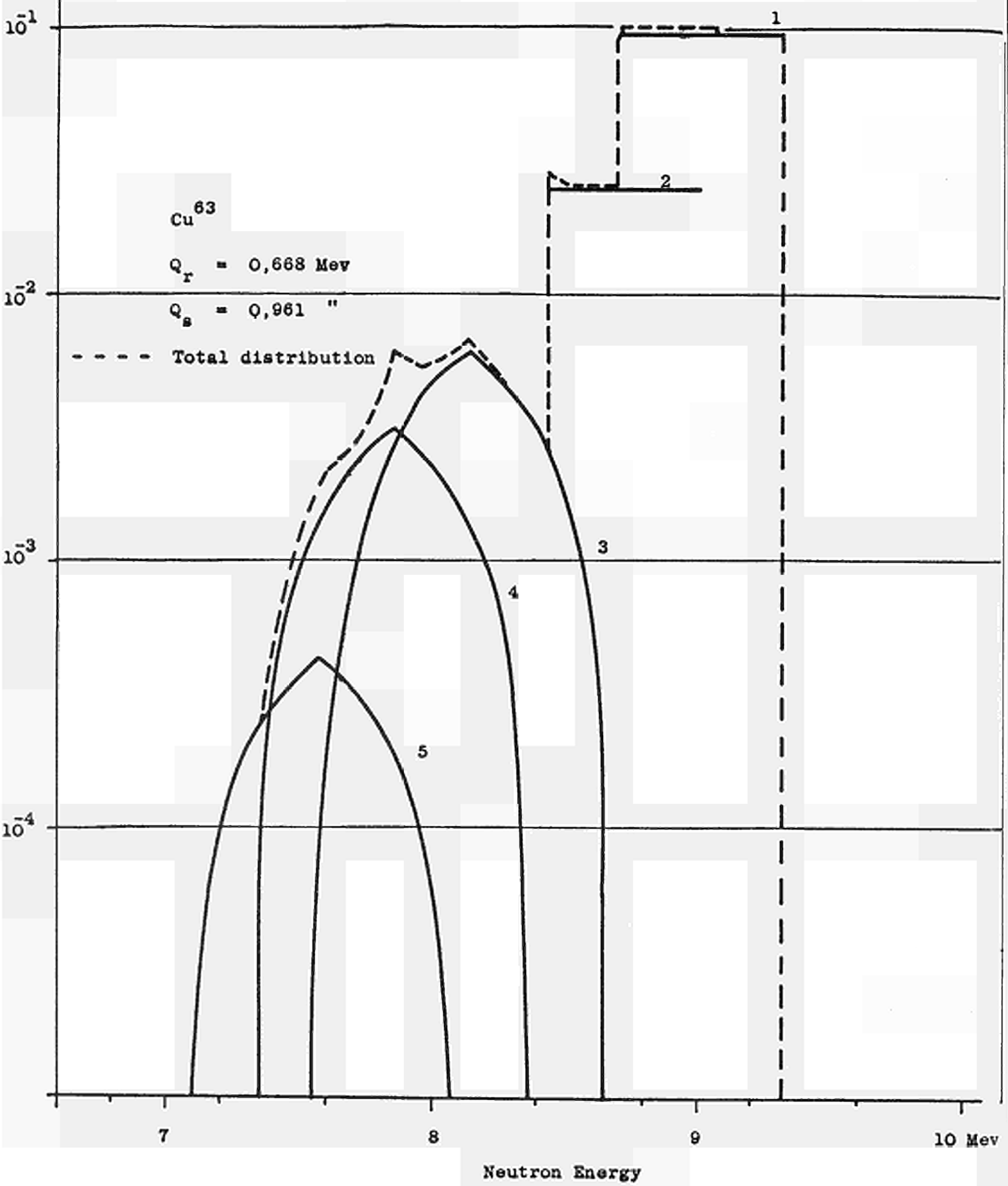


Fig. 7

Energy distribution of neutrons scattered once (curves 1,2) and twice (curves 3,4,5) by the two levels of Cu^{63}. The dashed line represents the total distribution.

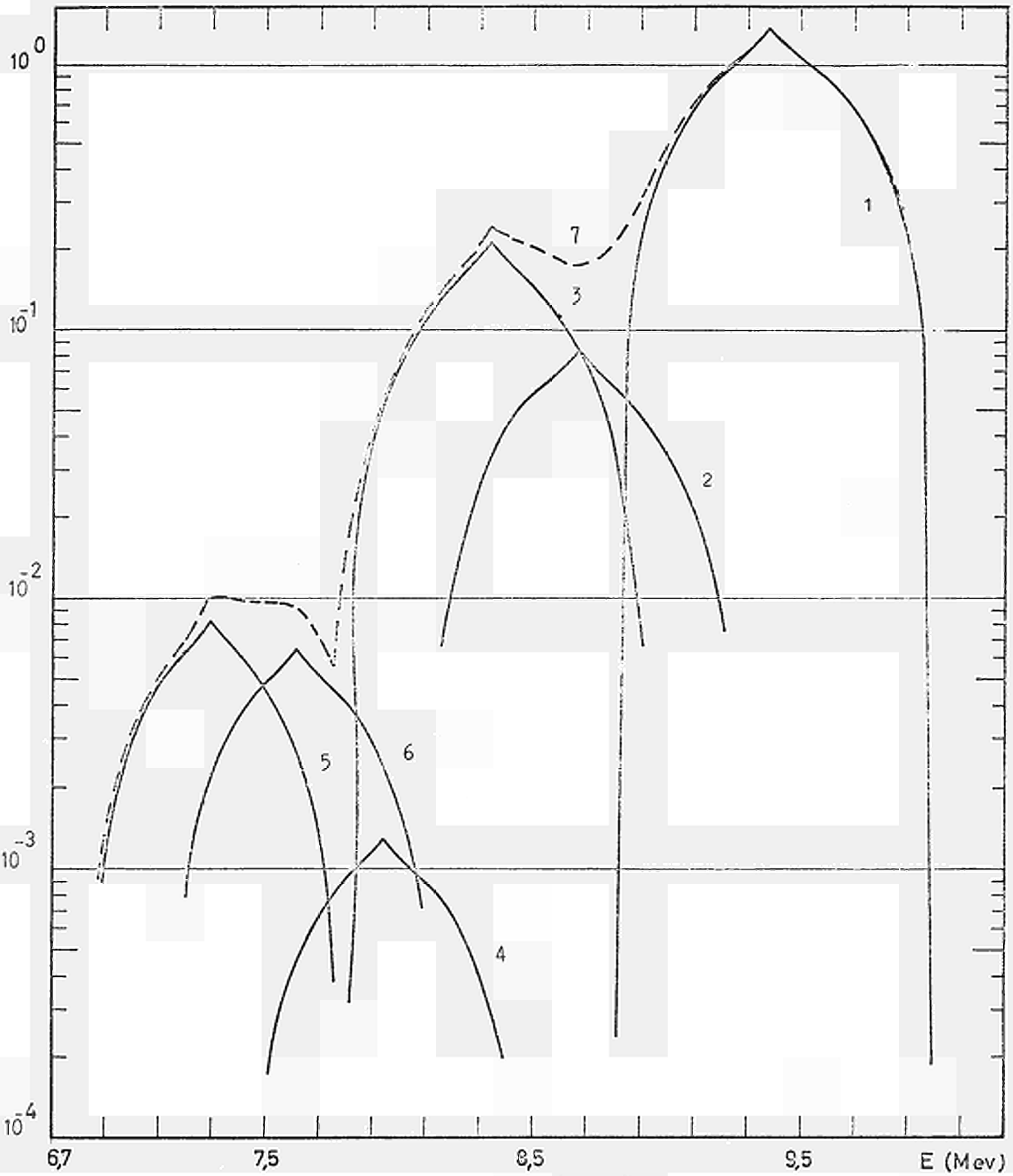


Fig. 8

Energy distribution of neutrons scattered twice by the levels of Cu^{65}. Curves 2 and 6 represent the distributions of neutrons scattered once by the one and once by the other level. The dotted line is the total distribution.

$$
\phi_3(w) = \frac{S}{2!}\exp[-\tau(w-w_s)]
\begin{cases}
\gamma^3(w-w_s)^2 \; ; & w\in(w_s,\, w_s+q) \\
\gamma^3[(w-w_s)^2-3(w-w_s-q)^2+3(w-w_s-2q)^2]; & w\in(w_s+2q,\, w_s+3q) \\
\gamma^3[(w-w_s)^2-3(w-w_s-q)^2]; & w\in(w_s+q,\, w_s+2q) \\
\gamma_r^3(w-w_s-3r)^2 \; ; & w\in(w_s+3r,\, w_s+3r+q) \\
\gamma_r^3[(w-w_s-3r)^2-3(w-w_s-3r-q)^2]; & w\in(w_s+3r+q,\, w_s+3r+2q) \\
\gamma_r^3[(w-w_s-3r)^2-3(w-w_s-3r-q)^2+3(w-w_s-3r-2q)^2]; & w\in(w_s+3r+2q,\, w_s+3r+3q) \\
\gamma_s^3(w-w_s-3s)^2 \; ; & w\in(w_s+3s,\, w_s+3s+q) \\
\gamma_s^3[(w-w_s-3s)^2-3(w-w_s-3s-q)^2]; & w\in(w_s+3s+q,\, w_s+3s+2q) \\
\gamma_s^3[(w-w_s-3s)^2-3(w-w_s-3s-q)^2+3(w-w_s-3s-2q)^2]; & w\in(w_s+3s+2q,\, w_s+3s+3q) \\
3\gamma^2\gamma_r(w-w_s-r)^2 \; ; & w\in(w_s+r,\, w_s+r+q) \\
3\gamma^2\gamma_r[(w-w_s-r)^2-3(w-w_s-r-q)^2]; & w\in(w_s+r+q,\, w_s+r+2q) \\
3\gamma^2\gamma_r[(w-w_s-r)^2-3(w-w_s-r-q)^2+3(w-w_s-r-2q)^2]; & w\in(w_s+r+2q,\, w_s+r+3q). \\
3\gamma^2\gamma_s(w-w_s-s)^2 \; ; & w\in(w_s+s,\, w_s+s+q) \\
3\gamma^2\gamma_s[(w-w_s-s)^2-3(w-w_s-s-q)^2]; & w\in(w_s+s+q,\, w_s+s+2q) \\
3\gamma^2\gamma_s[(w-w_s-s)^2-3(w-w_s-s-q)^2+3(w-w_s-s-2q)^2]; & w\in(w_s+s+2q,\, w_s+s+3q) \\
3\gamma\gamma_r^2(w-w_s-2r)^2 \; ; & w\in(w_s+2r,\, w_s+2r+q) \\
3\gamma\gamma_r^2[(w-w_s-2r)^2-3(w-w_s-2r-q)^2]; & w\in(w_s+2r+q,\, w_s+2r+2q) \\
3\gamma\gamma_r^2[(w-w_s-2r)^2-3(w-w_s-2r-q)^2+3(w-w_s-2r-2q)^2]; & w\in(w_s+2r+2q,\, w_s+2r+3q) \\
3\gamma\gamma_s^2(w-w_s-2s)^2 \; ; & w\in(w_s+2s,\, w_s+2s+q) \\
3\gamma\gamma_s^2[(w-w_s-2s)^2-3(w-w_s-2s-q)^2]; & w\in(w_s+2s+q,\, w_s+2s+2q) \\
3\gamma\gamma_s^2[(w-w_s-2s)^2-3(w-w_s-2s-q)^2+3(w-w_s-2s-2q)^2]; & w\in(w_s+2s+2q,\, w_s+2s+3q) \\
3\gamma_r^2\gamma_s(w-w_s-2r-s)^2 \; ; & w\in(w_s+2r+s,\, w_s+2r+s+q) \\
3\gamma_r^2\gamma_s[(w-w_s-2r-s)^2-3(w-w_s-2r-s-q)^2]; & w\in(w_s+2r+s+q,\, w_s+2r+s+2q) \\
3\gamma_r^2\gamma_s[(w-w_s-2r-s)^2-3(w-w_s-2r-s-q)^2+3(w-w_s-2r-s-2q)^2]; & w\in(w_s+2r+s+2q,\, w_s+2r+s+3q) \\
3\gamma_r\gamma_s^2(w-w_s-r-2s)^2 \; ; & w\in(w_s+r+2s,\, w_s+r+2s+q) \\
3\gamma_r\gamma_s^2[(w-w_s-r-2s)^2-3(w-w_s-r-2s-q)^2]; & w\in(w_s+r+2s+q,\, w_s+r+2s+2q) \\
3\gamma_r\gamma_s^2[(w-w_s-r-2s)^2-3(w-w_s-r-2s-q)^2+3(w-w_s-r-2s-2q); & w\in(w_s+r+2s+2q,\, w_s+r+2s+3q) \\
6\gamma\gamma_r\gamma_s(w-w_s-r-s)^2 \; ; & w\in(w_s+r+s,\, w_s+r+s+q) \\
6\gamma\gamma_r\gamma_s[(w-w_s-r-s)^2-2(w-w_s-r-s-q)^2]; & w\in(w_s+r+s+q,\, w_s+r+s+2q). \\
6\gamma\gamma_r\gamma_s[(w-w_s-r-s)^2-2(w-w_s-r-s-q)^2+2(w-w_s-r-s-2q)^2], & w\in(w_s+r+s+2q,\, w_s+r+s+3q).
\end{cases}
$$

6.8.

It can easily be seen that $\phi_3(w)$ vanishes identically for

$$w \notin \{(w_s, w_s+3q) \cup (w_s+3r, w_s+3r+3q) \cup (w_s+3s, w_s+3s+3q)\}$$

Formulas for $\phi_n(w)$ with $n > 3$ can easily be derived.
In this section we have considered the combined elastic and two-level inelastic slowing down of neutrons. The case of one level can immediately be derived from the above formulas by setting $\gamma r_s =0$.

The case of the combined elastic and inelastic one-level slowing down is especially interesting for the derivation of formulas describing the combined elastic and inelastic infinite -unresolved-level slowing down of neutrons which is going to be discussed in the following section.

7. ELASTIC AND INELASTIC SLOWING DOWN. CONTINUOUSLY DISTRIBUTED NUCLEAR LEVELS.

The transition from the discrete to the continuous nuclear level distribution can now easily be effected if the level distribution is given. Let us suppose that in the energy region (E, E_s) there exist M discrete levels with excitation energies $Q_1, Q_2 \ldots Q_L$ and with partitions $\rho_1, \rho_2, \ldots \ldots \rho_L$ respectively

$$\left(\sum_{\ell=1}^{L} \rho_\ell = M \right).$$

The total distribution of the simply scattered neutrons is given by

$$\varphi_1(E) = \varphi_{1,el}(E) + \sum_{\ell=1}^{L} \varphi_{1,in}(E, Q_\ell) \rho_\ell \qquad 7.1$$

where the summation is extended only over the Q's which lead to E.

In Eq. 7.1 and in what follows $\phi_{n,el}(E)$ and $\phi_{n,in}(E)$ denote the n-th collision distribution.

To simplify the notation we introduce the operator $\mathscr{E}\,(E_s, E,\ Q_1)$ which multiplies by unity the partitions of Q contributing to the distribution at E and by zero all others. Eq. 7.1 can now be written

$$\varphi_1(E) = \phi_{1,el}(E) + \sum_Q \phi_{1,in}(E,Q)\,\mathscr{E}(E_s,E,Q)\,\rho(Q) \qquad 7.2$$

If we now let $M \to \infty$ the right-hand side of the above equation becomes an integral over Q, and Eq. 7.2 takes the form

$$\varphi_1(E) = \phi_{1,el}(E) + \int_{-\infty}^{\infty} \phi_{1,in}(E,Q)\,\mathscr{E}(E_s,E,Q)\,\rho(Q)\,dQ \qquad 7.3$$

We have extended the region of integration by defining $\phi_{1,in}(E,-Q)$, $\rho(-Q) = 0$. Here, $\rho(Q)$ is nothing but the density of the nuclear levels which can be excited by neutrons of energy E_s. If E is the neutron energy after inelastic scattering it is seen that the maximum Q leading to E will be $Q'' \lesssim (E_s-E)$, and the minimum $Q' \gtrsim (E_s-E/\alpha)$.

Every $Q \notin (Q', Q'')$ cannot lead the neutron from initial energy E_s to final energy E.

To carry out the integration in Eq. 7.3 we have to express $\mathscr{E}(E, E_s, Q)$ in definite form. It is convenient to express $\mathscr{E}(E, E_s, Q)$ in its integral representation.

$$\mathscr{E} = \frac{1}{2\pi i}\int_{\Gamma} \frac{\exp[-ik(Q-\lambda E_s+E/\alpha)]-\exp[-ik(Q-\lambda E_s+E)]}{k}\,dk \qquad 7.4$$

The integration path, Γ , is defined by the real k-axes and by boundary of the lower part of the k-plane (positively oriented). For convenience we have introduced the parameter λ which will be put equal to unity afterwards.

From Eqs. 7.4 and 7.5 we obtain

$$\varphi_1(E) = \phi_{1,el}(E) + \frac{1}{2\pi i}\int_{\Gamma} \frac{\exp[i(\lambda E_s-E/\alpha)k]-\exp[i(\lambda E_s-E)k]}{k}\,dk \qquad 7.5$$
$$\times\int_{-\infty}^{\infty} \phi_{1,in}(E,Q)\rho(Q)\exp(-ikQ)\,dQ$$

If we interprete the last integral as a Fourier transformation we have

$$\varphi_1(E) = \phi_{1el}(E) + \frac{1}{i}\int_{\Gamma}\left\{\frac{\exp[i(\lambda E_s - E/\alpha)k] - \exp[i(\lambda E_s - E)k]}{k}\right\}\left\{F(E,k)\right\}dk \tag{7.6}$$

The second bracket in Eq. 7.6 represents the Fourier transform of $\left\{\phi_{1,in}(E,Q)\,\rho(Q)\right\}$ the existence of which we suppose.

We differentiate the exponetials in Eq. 7.6 with respect to λ, we integrate symbolically the same expression over λ, and we interchange the λ and k-integrations, and then we have

$$\varphi_1(E) = \phi_{1,el}(E) + E_s\int d\lambda \times\int_{\Gamma}\left[\exp[i(\lambda E_s - E/\alpha)k] - \exp[i(\lambda E_s - E)k]\right]F(E,k)\,dk \tag{7.8}$$

The second integral in this equation can again be understood as a Fourier transformation. Assuming that the integrand vanishes on the boundary of the lower k-plane we obtain:

$$\varphi_1(E) = \phi_{1,el}(E) + E_s\int d\lambda\, \phi_{1,in}(E, \lambda E_s - E)\,\rho(\lambda E_s - E)$$

$$- E_s\int d\lambda\, \phi_{1,in}(E, \lambda E_s - E/\alpha)\,\rho(\lambda E_s - E/\alpha) \qquad 7.9$$

We have put a (-) in front of the integral in order to take into account the orientation of the path . This is equivalent to

$$\varphi_1(E) = \int_{E_s - E/\alpha}^{E_s - E} \phi_{1,in}(E,Q)\,\rho(Q)\,dQ \quad ; E \in (0, \alpha E_s) \qquad 7.10$$

and

$$\varphi_1(E) = \phi_{1,el}(E) + \int_0^{E_s - E} \phi_{1,in}(E,Q)\,\rho(Q)\,dQ; \; E \in (\alpha E_s, E_s). \qquad 7.11$$

The results obtained above can be generalized for any number of collisions. We give here the distribution of the doubly scattered neutrons in the energy representation for later use.

$$\phi_{2,in}(E,Q_r) = \frac{S}{E_s(1-\alpha)^2} \begin{cases} c^2 \ln \frac{E_s}{E} \, ; \; E \in (\alpha E_s, E_s) \\ c^2 \ln \frac{E}{E_s \alpha^2} \, ; \; E \in (\alpha^2 E_s, \alpha E_s) \\ 2 c c_r \ln \frac{E_s - Q_r}{E} \, ; \; E \in (\alpha(E_s - Q_r), (E_s - Q_r)) \\ 2 c c_r \ln \frac{E}{(E_s - Q_r)\alpha^2} \, ; \; E \in (\alpha^2 (E_s - Q_r), \alpha (E_s - Q_r)) \\ c_r^2 \ln \frac{E_s - 2Q_r}{E} \, ; \; E \in (\alpha(E_s - 2Q_r), (E_s - 2Q_r)) \\ c_r^2 \ln \frac{E}{\alpha(E_s - 2Q_r)} \, ; \; E \in (\alpha^2 (E_s - Q_r), \alpha (E_s - Q_r)) \end{cases} \qquad 7.12$$

The first and second terms are Q-independent and represent pure elastic scattering. The third and fourth terms are mixed terms arising from one elastic and one inelastic scattering. In order to keep these expressions simple we disregard for the moment symmetrization with respect to the succession of elastic and inelastic scattering.

The transition from discrete to continuous level distribution is now effected by means of the same arguments used in the case of simple inelastic scattering, and Eq. 7.10 applies.

Let us consider the terms corresponding to double inelastic scattering.

If the initial and final energies of the neutrons are fixed, the maximum energy loss, Q_1, during the first inelastic collision cannot exceed $(E_s - E - Q_2)$, if Q_2 is the energy loss during the second collision.

From the above said and from Eq. 7.12 it is easily seen that the total distribution will be

$$\begin{aligned}
\varphi_2(E) = \; & \phi_{2,el}(E) \qquad ; \; E \in (\alpha^2 E_s, E_s) \\
& + \int_0^{E_s-E} dQ_2\, \rho(Q_2) \int_0^{E_s-E-Q_2} dQ_1\, \rho(Q_1)\, \phi_{2,in}(E, Q_2, Q_1) \; ; \; E \in (0, E_s) \\
& + \int_{E_s-E/\alpha}^{E_s-E} dQ\, \rho(Q)\, \phi_{2,el,in}(E, Q) \\
& - \int_0^{E_s-E/\alpha} dQ_2\, \rho(Q_2) \int_0^{E_s-E-Q_2} dQ_1\, \rho(Q_1)\, \phi_{2,in}(E, Q_2, Q_1) \; ; \; E \in [0, \alpha E_s)
\end{aligned} \tag{7.13}$$

In the case of

<u>n = 3</u>

we have three possibilities.

i) One inelastic and two elastic collisions; formula Eq. 6.7 applies.

ii) Two inelastic and one elastic collisions; formula Eq. 7.2 applies.

iii) Three inelastic collisions.

Arguing analogously we find for case iii)

$$\begin{aligned}
\varphi_3(E) = \; & \phi_{3,el}(E) \qquad ; \quad E \in (\alpha^3 E_s, E_s) \\
& + \int_0^{E_s-E} d\varphi_3\, \rho(\varphi_3) \int_0^{E_s-E-\varphi_3} d\varphi_2\, \rho(\varphi_2) \int_0^{E_s-E-\varphi_3-\varphi_2} d_1\, \rho(\varphi_1) \\
& \times \phi_{3,in}(E,\varphi_3,\varphi_2,\varphi_1) \qquad ; \quad E \in (0, E_s) \\
& - \int_0^{E_s-E/\alpha} d\varphi_3\, \rho(\varphi_3) \int_0^{E_s-E-\varphi_3} d\varphi_2\, \rho(\varphi_2) \int_0^{E_s-E-\varphi_3-\varphi_2} d\varphi_1\, \rho(\varphi_1) \\
& \times \phi_{3,in}(E,\varphi_3,\varphi_2,\varphi_1] \qquad ; \quad E \in (0, \alpha E_s) .
\end{aligned} \qquad 7.14$$

+ mixed terms

In the general case the integrals become

$$\int_0^{E_s-E} d\varphi_n \rho(\varphi_n) \int_0^{E_s-E-\varphi_n} d\varphi_{n-1} \rho(\varphi_{n-1}) \quad \cdots$$

$$\times \int_0^{E_s-E-\varphi_n \cdots \varphi_2} d\varphi_1 \rho(\varphi_1) \phi_{n,in}(E,\varphi_n,\cdots\varphi_1) \quad ; \; E \in (0,E_s)$$

$$-\int_0^{E_s-E/\alpha} d\varphi_n \rho(\varphi_n) \int_0^{E_s-E/\alpha-\varphi_n} d\varphi_{n-1} \rho(\varphi_{n-1}) \quad \cdots$$

$$\times \int_0^{E_s-E/\alpha-\varphi_n-\cdots\varphi_2} d\varphi_1 \rho(\varphi_1) \phi_{n,in}(E,\varphi_n,\cdots\varphi_1) \quad ; \; E \in (0,\alpha E_s) \qquad 7.15$$

+ mixed terms

8. THE STATISTICAL METHOD FOR THE NUCLEAR LEVEL DISTRIBUTION

The general formulas given in the preceding section are now applied. Let us first consider Eq. 6.7 for n = 1. To carry out the integration it is required to make some specifications concerning the distribution of the nuclear levels. As a first approximation we shall use the expression for the level density given first by Weisskopf (17) According to the statistical model of the nuclear reactions.

$$\rho(Q) = D\exp(\sqrt{4aQ}) \qquad 8.1$$

where D and a are nuclear parameters depending on A. (the mass number).

From Eqs. 6.7 and 8.1 we have

$$\varphi_1(E) = \frac{S}{E_s}\begin{cases} \varkappa & ; \quad E \in (\alpha E_s, E_s) \\ \mathcal{D}\int_0^{E_s - E} \varkappa_{in}(Q)\exp(\sqrt{4aQ})\,dQ & ; \quad E \in (0, E_s) \\ -\mathcal{D}\int_0^{E_s - E/\alpha} \varkappa_{in}(Q)\exp(\sqrt{4aQ})\,dQ & ; \quad E \in (0, \alpha E_s) \end{cases}$$

Introducing the explicit forms of $\varkappa$ and $\varkappa_{in}$ we can write

$$\varphi_1(E) = \frac{S\,\mathcal{D}}{(1-\alpha)E_s}\begin{cases} \frac{C}{\mathcal{D}} & ; \quad E \in (\alpha E_s, E_s) \\ \int_0^{E_s - E} \frac{\sigma_{in}(E^*, Q)\exp(\sqrt{4aQ})\,dQ}{\sigma_t(E^*, Q)\sqrt{1 - \frac{A+1}{A}\frac{Q}{E^*}}} & ; \quad E \in (0, E_s) \\ -\int_0^{E_s - E/\alpha} \frac{\sigma_{in}(E^*, Q)\exp(\sqrt{4aQ})}{\sigma_t(E^*, Q)\sqrt{1 - \frac{A+1}{A}\frac{Q}{E^*}}}\,dQ & ; \quad E \in (0, \alpha E_s) \end{cases} \qquad 8.2$$

Here E^* is a convenient value for which the expression takes its mean value.

Now we introduce the assumption

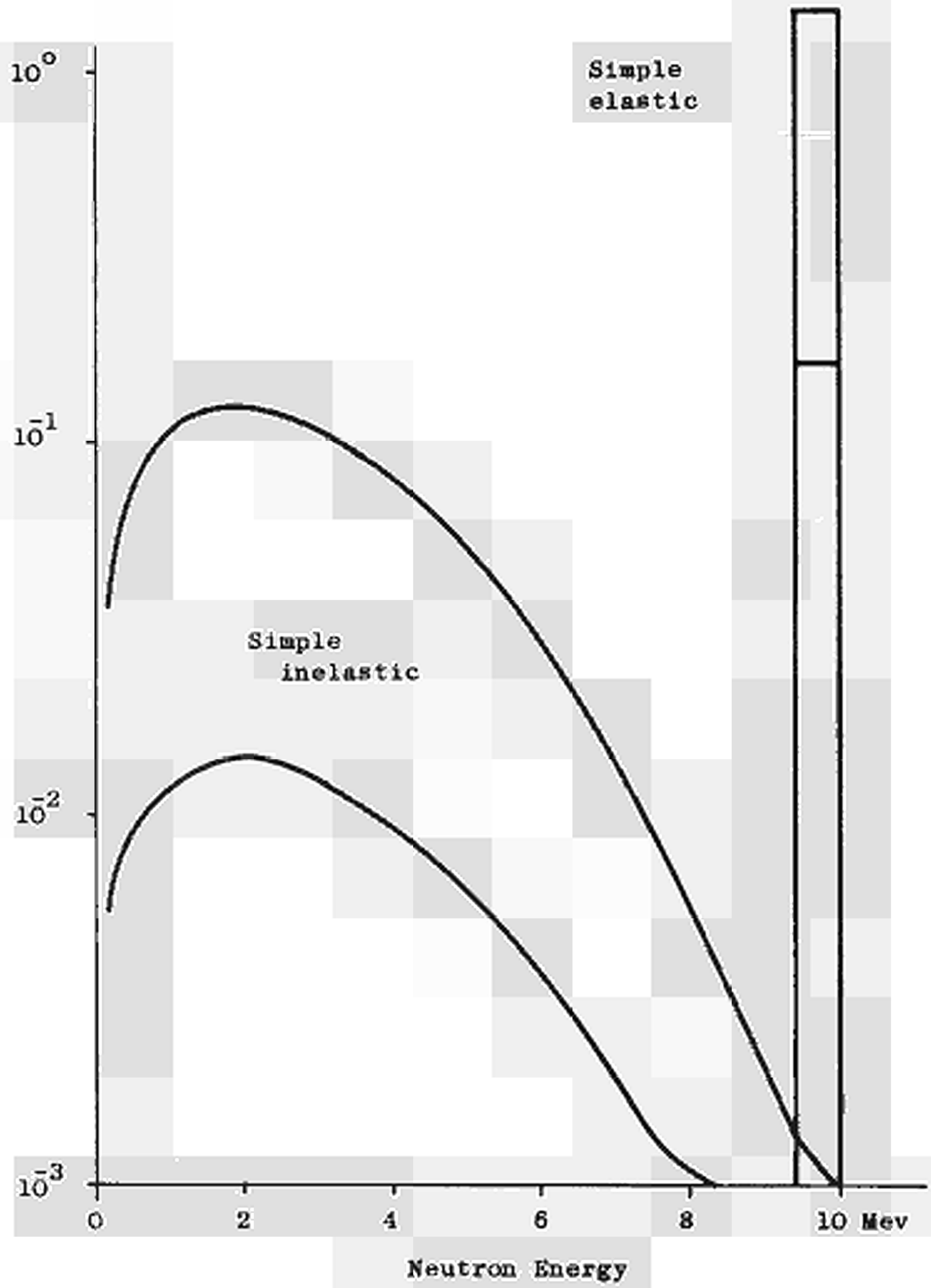
$$\frac{\sigma_{in}(E^*, Q)}{\sigma_t(E^*, Q)\sqrt{1 - \frac{A+1}{A}\frac{Q}{E^*}}} = \text{constant} = c_{in} \qquad 8.3$$

From Eqs. 8.2 and 8.3 it follows that

$$\varphi_1(E) = \frac{S\,D}{(1-\alpha)E_s} \begin{cases} \dfrac{C}{D} & ; \quad E \in (\alpha E_s, E_s) \\ C_{im} \exp\left(\sqrt{4a(E_s-E)}\right)\left(\sqrt{4a(E_s-E)}-1\right) & ; \quad E \in (0, E_s) \\ C_{im} \exp\left(\sqrt{4a(E_s-E/\alpha)}\right)\left(\sqrt{4a(E_s-E/\alpha)}-1\right) & ; \quad E \in (0, \alpha E_s) \end{cases} \qquad 8.4$$

The right-hand side of Eq. 8.4 vanishes identically unless E belongs to corresponding intervals mentioned.

As an illustration we give in Fig. 9 the graphical representation of Eq. 8.4 where we have used the following set of parameters (17)



10^0
10^{-1}
10^{-2}
10^{-3}
Simple elastic
Simple inelastic
0
2
4
6
8
10 Mev
Neutron Energy

Fig. 9

$\mathcal{S} = 1\ n/cm^3\ sec$

$A = 63$

$\alpha = 0{,}938$

$a = 2\ Mev^{-1}$

$D = 0{,}3\ Mev^{-1}$

$E_s = 10\ Mev$

$c = 0{,}9$

$c_{in} = 0{,}1$

As a further application we give now the energy distribution of the twice scattered neutrons. Again we assume here the validity of Eq. 8.1. The expression is not, however, convenient for the analytical integrations in Eq. 7.13, and we therefore approximated it by the simpler one

$$\rho(Q) \simeq D(1 + K_1 Q + K_2 Q^2) \qquad 8.5$$

where $K_1 = 6a$

$$K_2 = \frac{28a^2}{25}$$

Expression 8.5 is a good approximation of the formula 8.1 in the interval of most interest $0 \leq \sqrt{4\alpha Q} \leq 15$, as it can be seen from Fig. 10.

From Eqs. 7.12, 7.13 and 8.5 we obtain

$$\varphi_2(E) = \frac{S}{(1-\alpha)^2 E_s} \begin{cases} c^2 \ln \frac{E_s}{E} \quad ; \quad E \in (\alpha E_s, E_s) \\ c^2 \ln \frac{E}{\alpha^2 E_s} \quad ; \quad E \in (\alpha^2 E_s, \alpha E_s) \\ \frac{D^2 c_{in}^2}{3} \Big\{ K_1 (E_s - E)^3 + \left(\frac{7K_1^2}{2} - \frac{5K_2}{4}\right)(E_s - E)^4 \\ \quad + \frac{K_1 K_2}{10}(E_s - E)^5 + \frac{13 K_2^2}{30}(E_s - E)^6 \Big\} ; \; E \in (0, E_s) \\ - \frac{D^2 c_{in}^2}{3} \Big\{ K_1 (E_s - E/\alpha)^3 + \left(\frac{7K_1^2}{2} - \frac{5K_2}{4}\right)(E_s - E/\alpha)^4 \\ \quad + \frac{K_1 K_2}{10}(E_s - E/\alpha)^5 + \frac{13 K_2^2}{30}(E_s - E)^6 \Big\} ; \; E \in (0, \alpha E_s) \\ + c c_{in} D \Big\{ \left[\sqrt{4a(E_s - E)} - 1\right] \exp\left(\sqrt{4a(E_s - E)}\right) ; \; E \in (0, E_s) \\ - c c_{in} D \Big\{ \left[\sqrt{4a(E_s - E/\alpha)} - 1\right] \exp\left(\sqrt{4a(E_s - E/\alpha)}\right) ; \; E \in (0, \alpha E_s) \end{cases}$$

8.6

This equation is represented in Fig. 11 for two values of c/c_{in}.

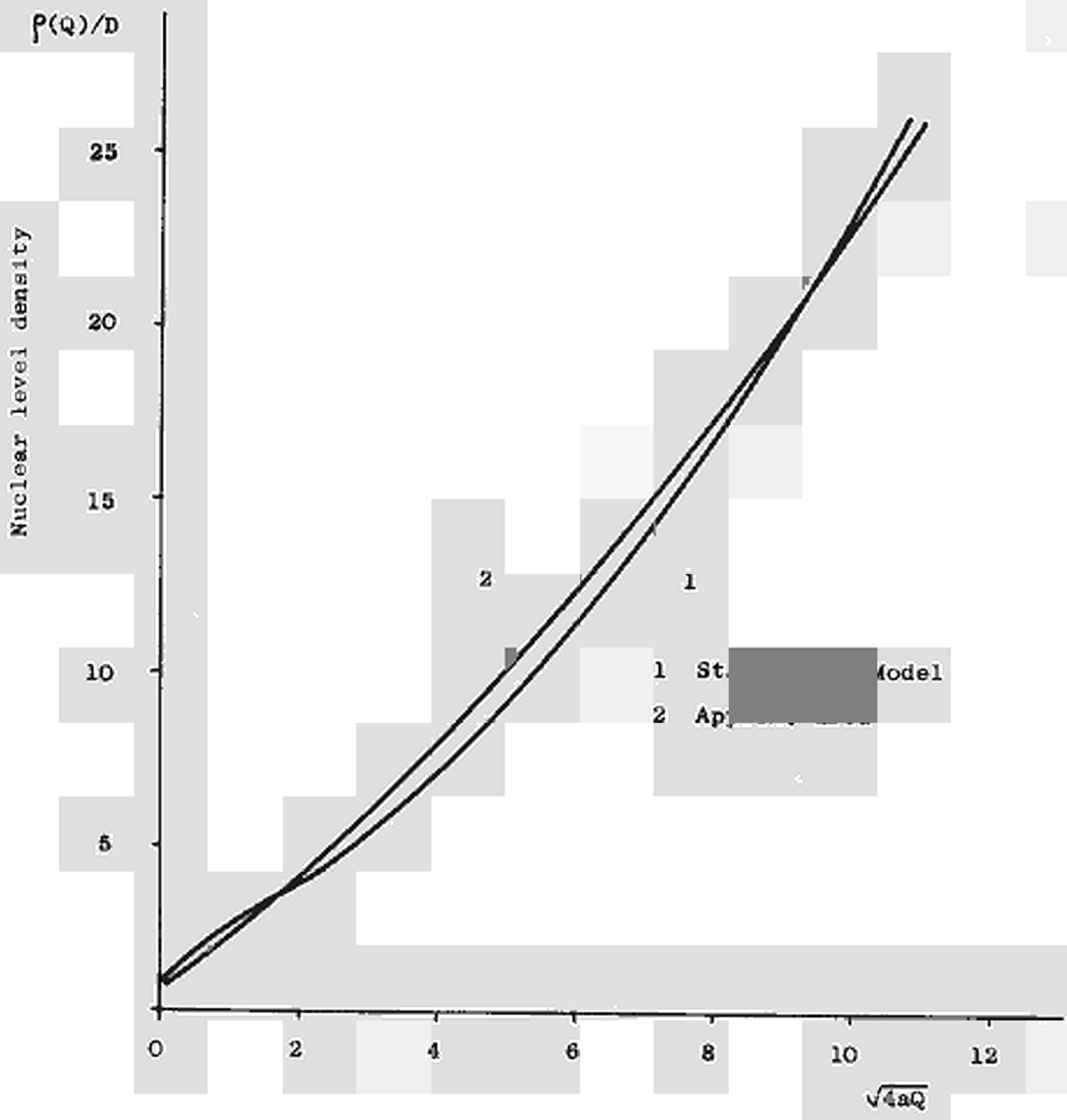
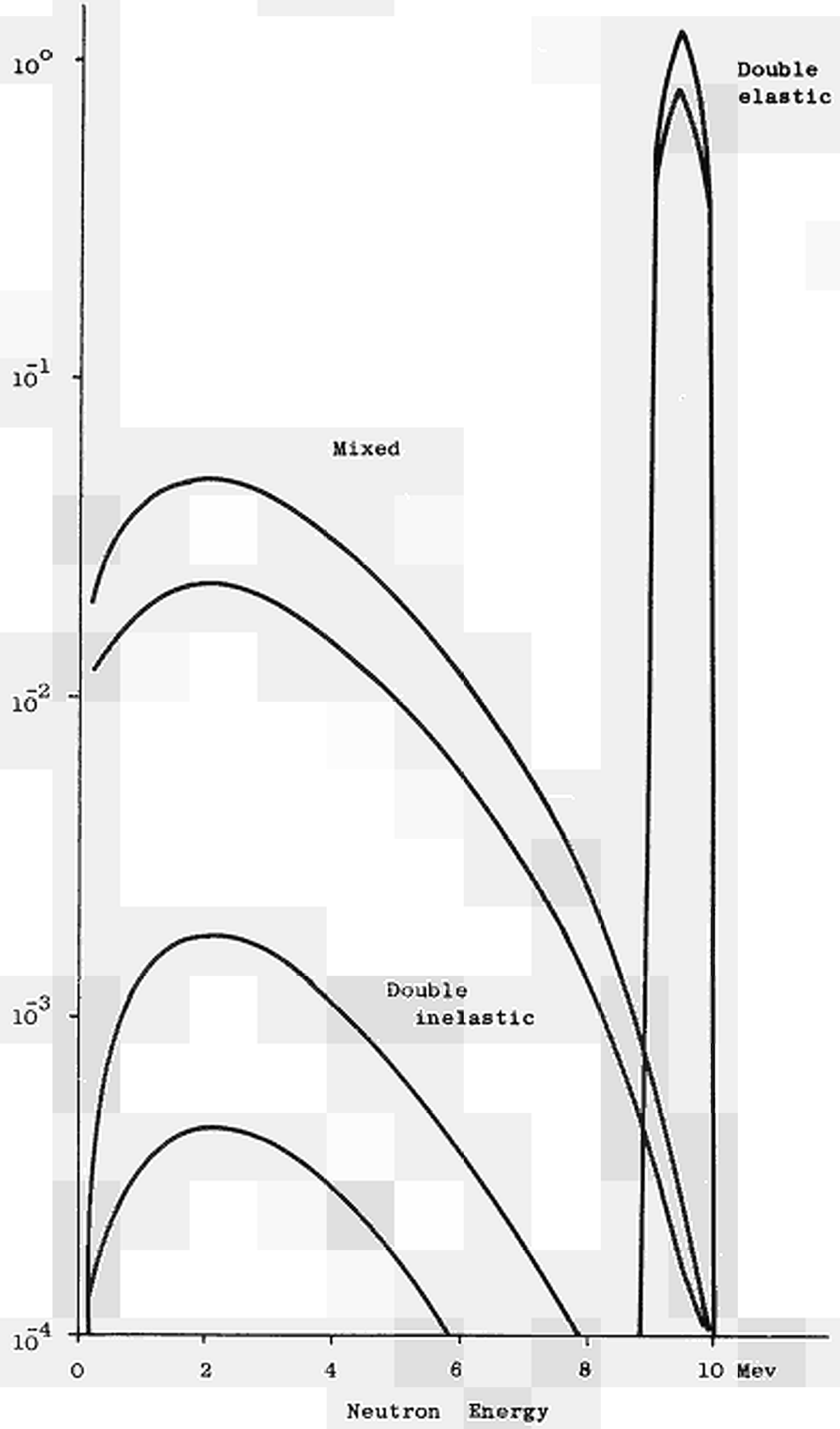

ρ(Q)/D
Nuclear level density
25
20
15
10
5
0
2
4
6
8
10
12
√4aQ
2
1
1 St.[redacted]Model
2 Ap[redacted]

Fig. 10



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Double elastic
Mixed
Double inelastic
0
2
4
6
8
10 Mev
Neutron Energy

Fig. 11

9. CENTRAL LIMIT THEOREM IN ENERGY DISTRIBUTIONS.

For the application of the CLT in the calculation of the convolution integrals we observe, that our function are of a special character. They vanish for negativ arguments and are zero for arguments greater as a given number.
We call such functions energy limited or grid functions.
Of this type are the functions occuring in elastic and inelastic scattering. They are constant inside (r, r+q) and vanish outside this interval. The height and width of each member of the grid depends on the characteristics of each nucleus contributing to the scattering.

Let us suppose we have L discret nuclear levels. The distribution of the simply scattered neutrons will have the form (Fig. 11).

To apply the CLT we have first to normalize the distributions (Fig. 11), so that the grid attaines constant height.

The distribution of the n times inelastically slowed down neutrons is given by a convolution integral

$$\phi_n(\xi) \simeq P\int_{-\infty}^{\infty} \phi_1(\xi-\xi_1-\xi_2-\cdots-\xi_n)\phi^{n_1-1}(\xi_1)\cdots \qquad 9.1$$

$$\times\cdots\phi^{n_L}(\xi_n)\,d\xi_1\cdots d\xi_n$$

where

$$\sum_{\ell=1}^{L} n_\ell = n \qquad 9.2$$

r_1 is given by

$$r_1 = \ln\frac{E_s}{E_s-Q_\ell} \qquad 4.2$$

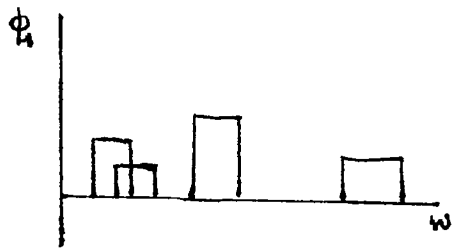


Fig. 12

$$Q_1 = \mathrm{Min}\{Q_\ell ; \ell = 1,2 \cdots L\}, \qquad 9.3$$

and $\phi_1(\xi)$ is the distribution of the simply (by the l-th level) scattered neutrons.

P in the front of the above integral signifies the permutation operator which permuts the exponents n_ℓ with the indices of the functions $\phi_j^{n_\ell}(\xi')$ and sums over all possible n_ℓ such, that Eq. 9.2 is satisfied.

Now according to a modified version of the CLT valid for grid functions, the integral 9.1 is given approximately by

$$\phi_n(W) \simeq PN(E_s, Q_1 \cdots Q_L, n_1 \cdots n_L, n)\, G\, \xi^{\beta}(1-\xi)^{\gamma} \qquad 9.4$$

where $$\xi = \frac{W - a(n_1 \cdots n_L, n)}{b(n_1 \cdots n_L, n) - a(n_1 \cdots n_L, n)}\,. \qquad 9.5$$

G is given by:

$$G_n = \frac{\Gamma(\beta_n+\gamma_n+2)}{\Gamma(\beta_n+1)\,\Gamma(\gamma_n+1)} \qquad 9.6$$

with

$$\beta_n = \frac{\eta_n(\eta_n-\eta_n^2-\delta_n^2)}{\delta_n^2} - 1\,, \qquad 9.7$$

and

$$\gamma_n = \frac{(1-\eta_n)(\beta_n+1)}{\eta_n} \qquad 9.8$$

δ_n^2, η_n, a_L and b_L are defined as

$$\delta_n^2 = \frac{\delta^2}{n}\,, \qquad 9.9$$

$$\eta_n = \eta\,, \qquad 9.10$$

$$a_L(n_1,\dots n_L) = \sum_{l=1}^{L} n_l r_l \qquad 9.11$$

and

$$b_L(n_1,\dots n_L) = nq + a_L(n_1\dots n_L)\,. \qquad 9.12$$

It is the convolution of such functions which yields the neutron distribution of neutrons n times scattered inelastically by the L discrete nuclear levels. The convolution integral in question is approximately represented by Eq. 9.4. Now, the quantities $\tilde{\eta}$ and $\tilde{\delta}^2$ are given by:

$$\tilde{\eta} = 1/\ln(1/\alpha) - \alpha/(1-\alpha) \qquad 9.16$$

and

$$\tilde{\delta}^2 = 2/(\ln 1/\alpha)^2 - \alpha(1+2/(\ln 1/\alpha))/(1-\alpha) \qquad 9.17$$

It might appear surprising that $\tilde{\eta}$ and $\tilde{\delta}^2$ are independent from the level characteristics.

This is, however, only the consequence of the normalization. The nuclear characteristics determine the factor $N(E_s, Q_1, \ldots Q_L, n_1, \ldots n_L, n)$ and the interval

$$(a_L, a_L + nq) \qquad 9.18$$

η_n and δ_n^2 are the first moment and the variance of the n-th distribution respectively.

The distribution of the neutrons which have been scattered once by the l-th nuclear level is according to Sec. 4.

$$\phi_l(w) = \varkappa_l \exp(-\tau w)\, ; \; w \in (\eta_l, \eta_l + q) \qquad 4.12$$

with the normalization factor $N_l = \frac{\exp(\tau \eta_l)}{C_l}$,

and where now $\varkappa_l$ and τ are defined by

$$\varkappa_l = C_l / \sum_l C_l \qquad 9.13$$

and

$$\tau = \left(\sum_l^L C_l\right) / (1-\alpha) \qquad 9.14$$

By normalizing and using the linear transformation Eq. 9.5 , $\phi_1(w)$ takes on the form

$$\phi_l(\xi) = q / \left(\sum_l^L C_l\right) \cdot \exp(-\tau q \xi). \qquad 9.15$$

$(0 < \xi < 1)$ is the interval

inside which expression Eq. 9.4 does not vanish identically.

The normalisation factor can now be written in the form

$$N = \prod_{l=1}^{L} \left[\frac{\exp(\tau n_l r_l)}{C_l^{n_l}} \right] \qquad 9.19$$

$$= \prod_{l=1}^{L} \left[\frac{E_s}{(E_s - Q_l) C_l} \right]^{n_l}$$

The last equation can be corrected by taking into account the fact that the maximum energy after the n_l-th collision by the l-th level is equal to $(E_s - n_l \cdot Q_l)$. After this the corrected normalisation constant, N^* be comes

$$N^* = \frac{E_s R_L(n_1 \ldots n_L)}{E_s - S_L(n_1, \ldots n_L)} ; (S_L < E_s) \qquad 9.20$$

where

$$R_L(n_1,\dots,n_L) = \prod_{l=1}^{L} C_l^{-n_l}$$

and

$$S_L(n_1,\dots,n_L) = \sum_{l=1}^{L} Q_l n_l \;;(< E_s)$$

From Eq. 9.4 and 9.20 we obtain finally for the distribution the expression

$$\phi_n(w) = PN^*\left(\frac{w-a_L}{nq}\right)^{\beta}\left(\frac{b_L-w}{nq}\right)^{\gamma} \qquad 9.21$$

where according to Eq. 3.5

$$w = \frac{S_L}{1-\alpha} \ln \frac{E_s}{E} \qquad 9.22$$

The operator P operates on all quantities depending on the partition $(n_1, n_2,\dots,n_L)$.

For the right understanding of the Eq. 9.21 it is necessary to elucidate the nature of P.

The quantities in which P acts, are

$$a_L(n_1,..,n_L)\ ,\ b_L(n_1,..,n_L)\ ,\ N^*(n_1,..,n_L)$$

Let us suppose we are given L arbitrary quantities $C_1 \dots C_L$ and the integers $n_1^j, \dots n_L^j$ and n, such that $\sum_{\ell=1}^{L} n_\ell^j = n$

If we, now, constract an expression $f(n_1^j c_1, \cdots n_L^j c_L)$ and let P operate on it, we have the result

$$P f(n_1 c_1, \cdots n_L c_L) = \frac{1}{L_n} \sum_{j=1}^{L_n} f(n_1^j c_1, \cdots n_L^j c_L) \qquad 9.23$$

The calculation of L_n requires the solution of the following problem: given two integers (L, n), to combine each of the partitions $(n_1^j, .. n_L^j)$ with a set of numbers $(C_1 \dots C_L)$, i.e. to find the number of the combinations $(n_1^j c_1, \cdots n_L^j c_L)$

The solution can be found by the following arrangement.

j	n
1	(n-1,1)
2	(n-2,2), (n-2,1,1)
3	(n-3,3),(n-3,2,1), (n-3,1,1,1)
	(n-4,4), (n-4,3,1),(n-4,2,11)
	(n-4,1,1,1,1)

$$(n-\lambda, \lambda), \ (n-\lambda, \lambda-1, 1) \tag{9.24}$$

where

$$\lambda \leq \frac{n+\varepsilon}{2} \tag{9.25}$$

and

$$\lambda \leqslant n-\lambda$$

In the inequality 9.25, ε is defined by

$$\varepsilon = \begin{cases} 0; & n = \text{even} \\ 1; & n = \text{odd} \end{cases} \qquad 9.26$$

Now, the number of combinations of the first line of the arrangement 9.24 with $c_1, \ldots c_L$ is equal to $\binom{L}{1}$.

The second line yields $\binom{L}{2}$ combinations, and any of them belongs to 2! permutations.

We have, therefore, $2!\binom{L}{2}$ total.

The third line yields again $2!\binom{L}{2}$ plus $\frac{3!}{2!}\binom{L}{3}$, where the divisor 2! takes into account the identical permutations of the unities. Continuing in the same way one obtains the result

$$L_n = \binom{L}{1} + 2!\binom{L}{2} + 2!\binom{L}{2} + \frac{3!}{2!}\binom{L}{3} + \cdots \text{etc.} \qquad 9.27$$

Eq. 9.21 is valid both for elastic and inelastic scattering the difference consisting in that $r = 0$ for this case, and that the energy after the n-th collision is $\alpha^n E_S$.

As an illustration of the above theory we give an exemple in order to check the quality of the approximations

$n_1 = 1$

$n_2 = 1$

$L = 2$

$r_1 = 0{,}118$

$r_2 = 0{,}108$

$C_r = 7.10^{-2}$

$C_s = 2{,}8.10^{-2}$

$\alpha = 0{,}938$

$E_s = 10$ Mev

$Q_1 = 1{,}114$ Mev

$Q_2 = 0{,}77$ Mev

$+ 1 = 3{,}0325$

$+ 1 = 3{,}0818$

G 1

The results obtained from Eq, 9.21 is shown in Figs. 13 and 14.

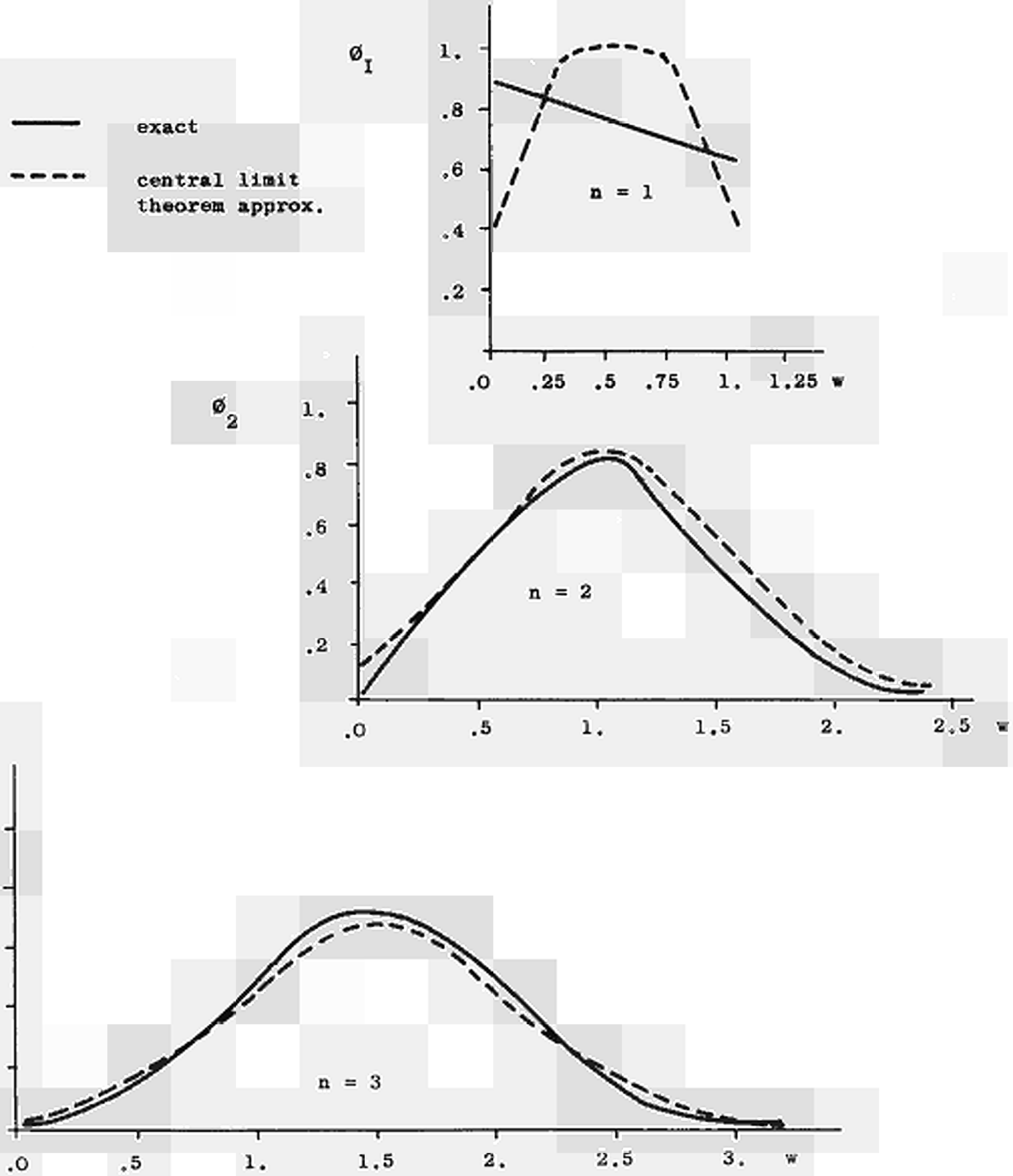

exact
central limit
theorem approx.
ϕ_1
1.
.8
.6
.4
.2
n = 1
.0
.25
.5
.75
1.
1.25
w
ϕ_2
n = 2
1.5
2.
2.5
n = 3
3.

Fig. 13

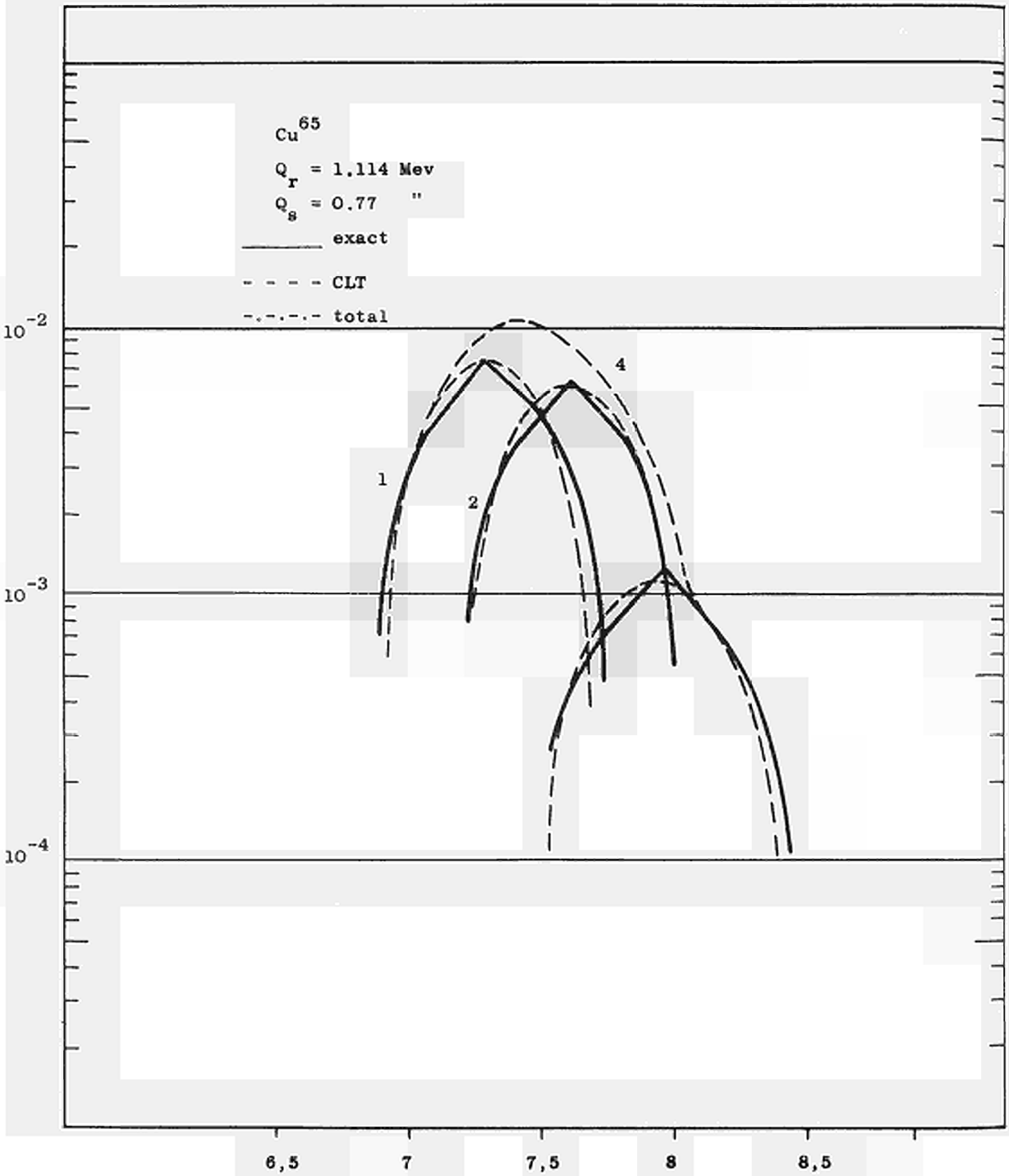

Cu^{65}
Q_r = 1.114 Mev
Q_s = 0.77 "
exact
CLT
total
10^{-2}
10^{-3}
10^{-4}
1
2
4
6,5
7
7,5
8
8,5

Fig. 14

10. PERTURBATION METHOD FOR NON-VANISHING ABSORPTION CROSS SECTION

The break down of the proportionality between the total cross section and the scattering cross section is due, generally, to the appearance of inelastic scattering and absorption.
As a matter of fact, it is more comfortable to represent the ratios of the cross sections by polynomials of small degree than to do this for the cross sections themselves. This observation allows us to construct a perturbation method for the solution of the transport equation just in the energy region where $\frac{\sigma_{el}}{\sigma_t} \neq$ const.
Let us consider the slowing down equation (Sec. 3)

$$\phi(w) = \int_{w-q}^{w} \frac{\sigma_{el}(w')}{\sigma_t(w')} \phi(w') e^{-\tau(w-w')} \cdot dw' + \delta(w-w_s), \qquad 10.1$$

where

$$\phi(w) = \varphi(w) \cdot \sigma_t(w), \qquad 10.2$$

By studying the experimental neutron data one easily confirms that the deviation, $R(w;w_o)$, from constancy of the ratio $\sigma(w)_{el}/\sigma_t(w)$ can be approximately represented by a low degree polynomial.

We write it here as

$$\frac{\sigma_{el}(w)}{\sigma_t(w)} = c\left[1-\lambda R(w;w_o)\right], \qquad 10.3$$

where

$$R(w;w_o) = \begin{cases} \sum_{n=0}^{N} A_n \left(\frac{w}{w_o}\right)^n ; & w<w_o \\ 0 ; & w>w_o \end{cases}.$$

In the above expansion A_o, A_1 are given for a linear approximation ($N = 1$) by

$$A_o = -A_1 = \frac{\sigma_a(w_s)+\sigma_{in}(w_s)}{\sigma_t(w_s)\cdot c}\cdot\frac{w_o-w_s}{w_c}. \qquad 10.4$$

Let us consider the perturbed distribution $\phi(w)$. Following the standard methods of the stationary perturbation theory of Quantum Mechanics (18) we write

$$\phi(w) = \sum_{m=0}^{\infty} \lambda^m \phi^{(m)}(w). \qquad 10.5$$

In Eqs. 10.3 and 10.5 λ is a continuous parameter which we take afterwards to be equal to unity, and $\phi^{(0)}(w)$ is the solution for $\sigma_e(w) = c.\sigma_t(w)$.

From Eqs. 10.1 10.3 and 10.5 it follows by equating coefficients of equal power of λ that

$$\phi^{(0)}(w) = \mathcal{J}\,\phi^{(0)}(w) + \delta(w-w_s)\,,$$

$$\phi^{(1)}(w) = \mathcal{J}\,\phi^{(1)}(w) - \mathcal{J}\,R(w;w_0)\,\phi^{(0)}(w')\,,$$

$$\phi^{(2)}(w) = \mathcal{J}\,\phi^{(2)}(w) - \mathcal{J}\,R(w;w_0)\,\phi^{(1)}(w')\,,$$

etc. 10.6

In Eqs. 10.6 the integral operator $\mathcal{J}$ has been defined by

$$\mathcal{J} = \int_{w-q}^{w} \exp[-\tau(w-w')]\,dw' \qquad 10.7$$

Here τ is equal to $(1-\alpha)/c$, where c is defined by $\sigma_{el}(w)/\sigma_t(w) = c$ in the range of its validity; $\alpha = \left(\frac{A-1}{A+1}\right)^2$.

The first equation from Eqs. 10.6 gives the unperturbed solution. and has already been discussed in (Sec. 3).

All subsequent equations can be written as

$$\phi^{(m)}(w) = \mathcal{J}\,\phi^{(m)}(w) - S_m(w),$$

$$m = 1, 2, 3 \cdots \qquad 10.8$$

In Eqs. 10.8 $S_m(w)$'s have been defined by

$$S_m(w) = \mathcal{J}\,R(w;w_c)\,\phi^{(m-1)}(w) \quad , \qquad 10.9$$

and can be calculated progressively from Eq. 10.6
The physical interpretation of Eqs. 10.8 is quite obvious. According to Eqs. 10.9 we have now modified sources which are determined by the perturbing deviation from the proportionality between $\sigma_{el}(W)$ and $\sigma_t(W)$, i.e. by the presence of absorption and / or inelastic scattering. These sources cause a deformation of the initial distribution $\phi^{(o)}(w)$, which in the presence of non-elastic processes represents the PWP-approximation. Using the formulas 10.8 and 10.9 we wish now to carry out the first-order perturbation.

First we calculate from Eq.10.9 the source $S_1(w)$ which actually is a sink.
To do this, a definite form of the (non-elastic) ratio, R $(w\ ;\ w_0)$, is required. However, in order to keep the formulas as general as possible, we retain the integral representation of the sinks.
In what follows we discuss the solution of the first-order perturbation equation

$$\phi^{(1)}(w) = \mathcal{J}\,\phi^{(1)}(w) - S_1(w) \qquad 10.10$$

From this we have after Laplace transformation

$$f^{(1)}(p) = -\frac{\mathcal{L}\{S_1(w)\}}{1-[1-\exp[-q(\tau+p)]]/(\tau+p)}\,, \qquad 10.11$$

where

$$f^{(1)}(p) = \mathcal{L}\{\phi^{(1)}(w)\}.$$

By expanding the right-hand side of Eq.10.11 in series we obtain

$$f^{(1)}(p) = -\sum_{n=0}^{\infty}\left[\frac{1-e^{-q(\tau+p)}}{\tau+p}\right]^{n}\cdot\mathcal{L}\{S_1(w)\}. \qquad 10.12$$

From this we have (15)

$$\phi^{(1)}(w) = -\sum_{n=0}^{\infty}\sum_{m=0}^{M}\binom{n}{m}(-e^{-q\tau})^{m}\cdot G_n^{(1)}(w-w_s-mq), \qquad 10.13$$

where

$$G_n^{(1)}(w) = \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{S_1(w)\}}{(z+p)^n}\right\}, \qquad 10.14$$

and

$M =$ integral part of $\frac{w-w_s}{q}$.

Noting that the integral 10.9 representing the sink is of the convolution type we can immediately write

$$\mathcal{L}\{S_1(w)\} = \mathcal{L}\{e^{-zw}\}\cdot\mathcal{L}\{R(w;w_0)\phi^{(0)}(w)\}. \qquad 10.15$$

Using the convolution theorem for Laplace transforms (19) we obtain from Eqs. 10.14 and 10.15

$$G_n^{(1)}(w) = \mathcal{L}^{-1}\left\{\mathcal{L}\left\{\frac{w^{n-1}}{(n-1)!}e^{-zw}\right\}\mathcal{L}\{e^{-zw}\}\mathcal{L}\{R(w;w_0)\phi^{(0)}(w)\}\right\}, \qquad 10.16$$

where (20)

$$\mathcal{L}\left\{\frac{w^{n-1}}{(n-1)!}e^{-zw}\right\} = \frac{1}{(z+p)^n} \qquad 10.17$$

From Eq.10.16 we obtain finally

$$G_n^{(1)}(w) = \iint R(w-\xi_1-\xi_2; w_0)\,\phi^{(0)}(w-\xi_1-\xi_2)\,e^{-\tau\xi_1} \times \frac{\xi_2^{n-1}}{(n-1)!}\,e^{-\tau\xi_2}\,d\xi_1 d\xi_2 \qquad 10.18$$

From Sec. 3 we obtain the explicit expression for $\phi^{(0)}(w)$; it is

$$\phi^{(0)}(w) = e^{-\tau(w-w_s)} \sum_{n=0}^{\infty} \sum_{m'=0}^{[M']} (-)^{m'} \binom{n}{m'} \frac{(w-w_s-m'q)^{n-1}}{(n-1)!}, \qquad 10.19$$

where $[M']$ is the integer part of $M' = \frac{u-u_s}{q}$ and $w \in (w_s, w_s + m'q)$. For all other w-values the expression vanishes identically.

By changing variables and remembering that

$$R(w; w_c) \equiv 0 \;;\; w \notin (w_s, w_0)$$

we conclude from Eqs.10.18 and 10.19 that

$$G_n^{(1)}(w) = \frac{e^{-\tau(w-w_s)}}{n!} \sum_{n'=0}^{\infty} \int_{w_s}^{w_0} \sum_{m'=0}^{\left[\frac{w'-w_s}{q}\right]} (-)^{m'} \binom{n'}{m'}$$

$$\times R(w'; w_c) \frac{(w-w_s-m'q)^{n'-1}}{(n'-1)!} (w-w'-q)^{n} dw' \qquad 10.20$$

The integration in Eq.10.20 seems to be very complicated, because the integration variable is involved in the summation index.
Fortunately this is not the case, and the integral can be written as

$$\sum_{m'} (-)^{m'} \binom{m'}{n'} I_{m'n'n}(w), \qquad 10.21$$

where

$$m' \in \left\{0, \left[\frac{n'q+w_s}{q}\right]-1 \ (\leqslant w \leqslant w_0)\right\}, \qquad 10.22$$

and

$$I_{m'n'n}(w) = \int_{m'q+w_s}^{(m'+1)q+w_s} R(w; w_0)(w'-w_s-m'q)^{n'-1}(w-w'-q)^{n}\,dw'. \qquad 10.23$$

Collecting the results we have

$$G_n^{(4)}(w) = \frac{e^{-\tau(w-w_s)}}{n!} \sum_{n'=0}^{\infty} \sum_{m'} \frac{I_{m'n'n}(w)}{(n'-1)!} \qquad 10.24$$

From Eqs.10.13 and 10.24 it follows that

$$\phi^{(1)}(w) = -\sum_{n=0}^{\infty}\sum_{m=0}^{M}(e^{-sq})^{m}\binom{n}{m}G_n^{(1)}(w-mq-w_s) . \qquad 10.25$$

By correcting the PWP-approximation, $\phi^{(o)}(w)$, in the first-order perturbation we obtain the distribution

$$\phi(w) = \phi^{(o)}(w) + \phi^{(1)}(w) . \qquad 10.26$$

As we see from Eq.10.25 , $\phi^{(1)}(w)$ is negative, and, therefore, $\phi(w) \leq \phi^{(o)}(w)$.

This was to be expected, because $\phi^{(o)}(w)$ has been calculated with the PWP-c-value, which by definition is $c \geq \sigma_{el}(w) / \sigma_t(w)$.

11. COLLISION PROBABILITIES AND GREEN'S FUNCTION.

In what follows we consider ahomogeneous monoisotopic infinite medium of plane symmetry with energy independent cross sections and isotropic scattering. The method is rigorous when $\sigma_t = \sigma_{el} = \text{const.}$ Other cases, however, where this assumption does not hold, can be treated by the perturbation method discussed in Sec. 10.

We shall give here a method for calculating the energy, space, and angle dependent collision probabilities and the corresponding Green's function. First we wish to define the perturbation method mentioned above.

Let us consider the transport equation in the form

$$\left(\mu \frac{\partial}{\partial x} + \sigma_t\right) \psi(x,\mu,u) = \frac{1}{2} \mathcal{R} \psi + \delta(\mu-\mu_0)\,\delta(x)\,\delta(u-u_s), \tag{11.1}$$

where

$$\mathcal{R} = \mathcal{R}_{el} + \mathcal{R}_{in}$$

$$= \int_{-1}^{1} d\mu' \left\{ \int_{u-q'}^{u} e^{-(u-u')} \sigma_{el}(u')\,du' + \sum_{l=1}^{L} \int_{u-q'-r_l'}^{u-r_l} \frac{e^{-(u-u')} \sigma_{in}^{l}(u')\,du'}{\sqrt{1-\frac{A+1}{A}\frac{Q_l}{E_s} e^{u'}}} \right\}. \tag{11.2}$$

σ_t and σ_{el} are the total and the elastic cross sections. The sum in Eq. 11.12 orresponds to L discrete nuclear levels (For more details Sec. 6) .

As we shall show subsequently a separation of the space and energy variables is possible. Since this separation is not affected by the presence of inelastic scattering we shall consider for simplicity only elastic scattering. In the region in which inelastic scattering occurs σ_t may be constant but not $\sigma_{el}(u)$. If $\psi \cong \psi_0 + \psi_1$ where ψ_0 and ψ are the unperturbed solution and the first-order-perturbation corrected solution respectively then ψ_1 satisfies the equation

$$\left(\mu \frac{\partial}{\partial x} + \sigma_t\right)\psi_1 = \frac{1}{2}\mathcal{R}_0 \, \psi_1 + Q(x,\mu,u) \qquad 11.3$$

where

$$\mathcal{R}_0 = \sigma_t c \int_{-1}^{1} d\mu' \int_{u-q}^{u} e^{-(u-u')} du' \qquad 11.4$$

and

$$Q(x,\mu,u) = -\frac{\sigma_t}{2}\int_{-1}^{1} d\mu' \int_{u-q}^{u} e^{-(u-u')} \sigma_1(u') \, \psi_0(x,\mu',u') du'. \qquad 11.5$$

The perturbing quantity $\sigma_1(u)$ has been defined by

$$\sigma_1(u) = c - \frac{\sigma_{el}(u)}{\sigma_t} , \qquad 11.6$$

and c is the constant value of $\sigma_{el}(u)/\sigma_t$ in the absence of non-elastic processes.

The source term $Q(x,\mu,u)$ corresponds to a sink. In the collision number representation of $\psi_o(x,\mu,u)$ space and lethargy are separated, and the u-integration in Eq 11.5 can be carried out as it is done in Sec. 10.

We can therefore consider $\sigma_{el}(u)$ as constant in Eq. 11.2 and treat the inelastic terms in the fashion exposed in Sec. 6 . Under these special conditions we can use the well-known integral transform approach. We apply Laplace and Fourier transformations for the u and z ($z = \sigma_t . x$) variables respectively. Before doing this we introduce a new variable w defined by $w = u/\tau$, where

$$\tau = \left[\frac{c}{1-\alpha} + \sum_{l=1}^{L} \frac{c_l}{1-\alpha}\right]^{-1} \qquad 11.7$$

and c_l are constants defined by

$$c_\ell \cong \frac{\sigma_{in}^{\ell}(u)}{\sigma_t\sqrt{1-\frac{A+1}{A}\frac{Q_\ell}{E_s}e^{u}}} \qquad 11.8$$

By introducing the Fourier and Laplace transform of $\sigma_t \cdot \psi$; $\varphi(k,\mu,p) = \mathcal{L}_w\{\mathcal{F}_z\{\sigma_t\psi(z,\mu,w)\}\}$ we obtain from Eq. 11.1

$$(1+ik\mu)\varphi(k,\mu,p) = \frac{1}{2}F(p)g(k,p)+\delta(\mu-\mu_0)e^{pu_s}, \qquad 11.9$$

where

$$g(k,p) = \int_{-1}^{1}\varphi(k,\mu,p)\,d\mu \, . \qquad 11.10$$

The factor

$$F(p) = \frac{1-e^{-q(\tau+p)}}{\tau+p}\left[\varkappa+\sum_{\ell=1}^{L}\varkappa_\ell\, e^{-r_\ell(\tau+p)}\right] \qquad 11.11$$

$$\left(\varkappa = \frac{\tau c}{1-\alpha} \;,\quad \varkappa_\ell = \frac{\tau c_{i\ell}}{1-\alpha}\right)$$

is the Laplace transform of the scattering kernel.

The separation of the variables p and k in Eq. 11.9 implies the separation of the energy and space coordinates in the number-of-collisions representation of the distribution, when the cross sections are considered as independent from energy.

In other words, the solution of Eq. 11.1 will be of the form

$$\psi(z,\mu,w) = \sum_{n=0}^{\infty} \phi_n(w)\ \psi_n(z,\mu), \tag{11.12}$$

where $\psi_n(z,\mu)$ and $\phi_n(w)$ are the distributions in z-and w-space respectively of the neutrons scattered n_1 times elastically and n_2 times inelastically, n_1 and n_2 satisfying the equality $n_1 + n_2 = n$.

The possibility to represent the solution in this way yields as a by-product a quantity which sometimes may be useful- the collision probability.

The n-th order collision probability, P_n, is defined by the ratio

$$P_n(z,\mu,w) = \frac{\phi_n(w)\,\psi_n(z,\mu)}{\psi(z,\mu,w)} \tag{11.13}$$

Eq. 11.13 gives the probability that after n collisions a neutron will have the energy w, the direction μ, and the space coordinate $x = \dfrac{z}{\sigma_t}$,

If P_n^{el}, P_n^{in} are the partial probabilities for elastic and inelastic collision respectively, then we have

$$P_n^{el}(z,\mu,w) = \frac{\sigma_{el}}{\sigma_t} P_n(z,\mu,w), \tag{11.14}$$

and

$$P_n^{in}(z,\mu,w) = \frac{\sigma_{in}}{\sigma_t} P_n(z,\mu,w) . \tag{11.15}$$

The probability that the n-th collision will result to an absorption is, obviously,

$$P_n^a(z,\mu,w) = P_n - P_n^{el} - P_n^{in} . \tag{11.16}$$

From Eq. 11.1 we obtain

$$\varphi(k,\mu,p) = \frac{1}{2}\frac{F(p)}{1+ik\mu}\, g(k,p) + e^{-pw_s} \cdot \frac{\delta(\mu-\mu_0)}{1+ik\mu} , \tag{11.17}$$

where $g(k,p)$ is defined by

$$g(k,p) = \int_{-1}^{1} \varphi(k,\mu,p)\, d\mu . \tag{11.18}$$

From Eqs. 11.17 and 11.18 it follows that

$$g(k,p) = \frac{\exp[-pw_s]}{[1-D(k,p)](1+ik\mu_0)} . \tag{11.19}$$

The right-hand side of Eq.11.19 is nothing but the neutron propagator in space and energy.

In Eq.11.19 we have defined

$$D(k,p) = \frac{1}{2} F(p) T(k) \tag{11.20}$$

where

$$T(k) = \frac{1}{ik} \ln \frac{1+ik}{1-ik} \; . \tag{11.21}$$

The second factor of the right-hand side in Eq.11.20 satisfies

$$\frac{1}{2} T(k) < 1 \tag{11.22}$$

for all real k-values different from zero.

Therefore, in order to keep

$$|D(k,p)| < 1 \tag{11.23}$$

it is required that

$$|F(p)| < 1 \; , \tag{11.24}$$

which is possible by choosing the integration contour for the Laplace inversion conveniently. The procedure is the same as in the space independent case.
From Eq.11.17 and from conditions11.22 and11.14 we obtain the expansion

$$\varphi(k,\mu,p) = \frac{\exp[-pw_s]}{(1+ik\mu)(1+ik\mu_0)} \sum_{n=1}^{\infty} 2^{-n} F^n(p) T^{n-1}(k)$$

$$+ \frac{\delta(\mu-\mu_0)}{1+ik\mu} \exp[-pw_s]. \qquad 11.25$$

This expansion converges absolutely for $c < 1$ and $p \in \mathfrak{D}$. $\mathfrak{D}$ is the appropriate integration contour.
By carrying out the Fourier and the Laplace inversions we obtain the expression

$$\psi(z,\mu,w) = \delta(\mu-\mu_0)\,\delta(w-w_s)\,\frac{e^{-\frac{z}{\mu}}}{\mu}$$

$$= \sum_{n=1}^{\infty} 2^{-n} \psi_n(z,\mu)\,\phi_n(w)$$

11.26

$\psi_n(z,\mu)$ and $\phi_n(w)$ have been defined by

$$\psi_n(z,\mu) = \mathcal{F}^{-1}\left\{\frac{T^{n-1}(k)}{(1+ik\mu)(1+ik\mu_0)}\right\}, \qquad 11.27$$

and

$$\phi_n(w) = \mathcal{L}^{-1}\left\{ F^n(p)\exp[-pw_s]\right\}. \qquad 11.28$$

From Eq. 11.26 we see that by decomposing the neutron distribution according to the numoers of collisions the separation of the energy and space coordinates arises in quite a natural way when σ_t is constant.
In what follows we do not consider any more the w-dependent factors in Eq. 11.26. They have been givenin previous Secs.
We focus now our attention on the z and u dependent distributions.
The first term in Eq. 11.26 represents obviously the distribution of the unscattered neutrons, because it is independent from the coupling constant, c, of the neutron field with the medium.
Let us now consider the inversion in Eq. 11.27 for n = 1.

As this term linearly depends on c it represents the distribution of the neutrons having undergone one collision only. By carrying out the Fourier inversion we find the following expressions corresponding to positive and negative values of μ, z.

$$\psi_1(z,\mu) = \frac{1}{\mu_0-\mu}\begin{cases} e^{-\frac{z}{\mu_0}} - e^{-\frac{z}{\mu}} ; & z>0,\ \mu>0 \\ 0 ; & z<0,\ \mu>0 . \end{cases} \qquad 11.29$$

For negative μ-values we find

$$\psi_1(z,\mu) = \frac{1}{\mu_0-\mu}\begin{cases} e^{-\frac{z}{\mu}} ; & z<0,\ \mu<0 \\ e^{-\frac{z}{\mu_0}} ; & z>0,\ \mu<0 \end{cases} . \qquad 11.30$$

From the above equations we conclude that for $z>0$ all $\mu \gtrless 0$ are allowed.

For $z<0$, however, no $\mu>0$ is allowed. This is physically clear for once scattered neutrons.

For n = 2 we have to find the inverse

$$\psi_2(z,\mu) = \mathcal{F}^{-1}\left\{\frac{T(k)}{(1+ik\mu)(1+ik\mu_0)}\right\}. \qquad 11.31$$

which is as follows

$$\psi_2(z>0,\mu>0)=\frac{1}{4(\mu_0-\mu)}\left\{\left[e^{-\frac{z}{\mu_0}}\ln\frac{1+\mu_0}{1-\mu_0}+E_1(z)+e^{-\frac{z}{\mu_0}}\overline{E}_1\left(\frac{1-\mu_0}{\mu_0}z\right)\right]\mu_0\right.$$
$$\left.-\left[e^{-\frac{z}{\mu}}+E_1(z)+e^{-\frac{z}{\mu}}\overline{E}_1\left(\frac{1-\mu}{\mu}z\right)\right]\mu\right\}, \qquad 11.32$$

$$\psi_2(z>0,\mu<0)=\frac{1}{4(\mu_0-\mu)}\left\{e^{-\frac{z}{\mu_0}}\ln\frac{1+\mu_0}{1-\mu_0}+E_1(z)+e^{-\frac{z}{\mu_0}}\overline{E}_1\left(\frac{1-\mu_0}{\mu_0}z\right)\right]\mu_0 \qquad 11.33$$
$$\left.+\left[E_1(|z|)-e^{-\frac{z}{|\mu|}}E_1\left(\frac{1+|\mu|}{|\mu|}z\right)\right]|\mu|\right\},$$

$$\psi_2(z<0,\mu>0)=\frac{1}{4(\mu_0-\mu)}\left\{\left[E_1(|z|)-e^{-\frac{|z|}{\mu_0}}E_1\left(\frac{1+\mu_0}{\mu_0}|z|\right)\right]\mu_0\right. \qquad 11.34$$
$$\left.-\left[E_1(|z|)-e^{-\frac{|z|}{\mu}}E_1\left(\frac{1+\mu}{\mu}|z|\right)\right]\mu\right\},$$

$$\psi_2(z<0,\mu<0)=\frac{1}{4(\mu_0-\mu)}\left\{\left[E_1(|z|)-e^{-\frac{|z|}{\mu_0}}\overline{E}_1\left(\frac{1+\mu_0}{\mu_0}|z|\right)\right]\mu_0\right. \qquad 11.35$$
$$\left.+\left[e^{-\frac{|z|}{|\mu|}}\ln\frac{1+|\mu|}{1-|\mu|}+E_1(|z|)+e^{-\frac{|z|}{|\mu|}}\overline{E}_1\left(\frac{1-|\mu|}{|\mu|}|z|\right)\right]|\mu|\right\}.$$

In the above equations $E_1(z)$ and $\overline{E}_1(z)$ have been defined by $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$, and $\overline{E}_1(x) = \int_{-\infty}^{x} \frac{e^{t}}{t}dt$ (21)

For higher values of n the expressions for $\psi_n(x,\mu)$ become lenghty. We prefer, therefore, to give an approximate method based on the CLT for the inversion of Eq.11.27. The functions of interest here are the Fourier transforms of $T(k)$, $(1 + ik\mu)^{-1}$ and $(1 + ik\mu_o)^{-1}$.

These are in general equal to $E_1(|z|)$, $\frac{e^{-\frac{z}{\mu}}}{\mu}$ and $\frac{e^{-\frac{z}{\mu_o}}}{\mu_o}$ respectively.

These functions are all normalized to unity.

Their first moments for $\mu > 0$ and $\mu_o > 0$ are easily found to be

$$m_1 = 0 ,$$
$$m_1' = \mu ,$$
$$m_1'' = \mu_o . \qquad 11.36$$

The second moments are

$$m_2 = 2/3 ,$$
$$m_2' = \mu^2 ,$$
$$m_2'' = \mu_o^2 . \qquad 11.37$$

The corresponding variances are

$$\delta^2 = 2/3 \,,$$

$$\delta'^2 = \mu^2 ,$$

$$\delta''^2 = \mu_0^2 . \qquad 11.38$$

From Eqs. 11.36-11.38 we find for the general case according to Eqs. 1.7 and 1.9

$$\delta_n^2 = \mu_0^2 + \mu^2 + 2(n-1)/3 \,, \qquad 11.39$$

$$m_n = \mu_0 + \mu \,, \qquad 11.40$$

where n takes the values 3,4,5,..,.

From Eqs. 1.8, 11.39 and 11.40 we find the approximate expression

$$\psi_n(z,\mu) = 2^{n-1} \frac{\exp\left[-\frac{(z-\mu_0-\mu)^2}{2(\mu_0^2+\mu^2+2(n-1)/3)}\right]}{\sqrt{2\pi(\mu_0^2+\mu^2+2(n-1)/3)}} \,, \qquad 11.41$$

for $n > 2$.

If we replace the transform of the scattering kernel in Eq 11.1 by c then we have the energy independent case with (LS) isotropic scattering.

In this case the total distribution of the scattered neutrons is given by

$$\psi(z,\mu) = \frac{e^{-\frac{z}{\mu}}}{\mu}\delta(\mu-\mu_0) + \frac{c}{2}\psi_1(z,\mu) + \left(\frac{c}{2}\right)^2\psi_2(z,\mu) + \sum_{n=3}^{\infty}\left(\frac{c}{2}\right)^n\psi_n(z,\mu) \qquad 11.42$$

In this formula which is valid for $z > 0$ and $\mu > 0$ only the three first terms are exact.

However, the approximation is fairly good as the comparison (Fig.15) with results obtained by numerical methods (22) reveals.

For $z < 0$ an analogous formula can be obtained in the same way.

The angle independent distribution is obtained from Eq.11.27 by integrating it over μ. The result of the integration $\chi_n(z)$ is given by

$$\chi_n(z) = \mathcal{F}^{-1}\left\{ \frac{T^n(k)}{1+ik\mu_0} \right\} . \qquad 11.43$$

Reasoning similarly as in the case of the angle dependent distribution, we find for the distribution of the n-times scattered neutrons the approximate expression

$$\chi_n(z) = \frac{e^{-\frac{z}{\mu_0}}}{\mu_0} + \frac{c}{2}\chi_1(z) + \sum_{n=2}^{\infty}\left(\frac{c}{2}\right)^n \chi_n(z) \qquad 11.44$$

In Eq.11.44 $\chi_n(z)$ is given by

$$\chi_1(z) = e^{-\frac{z}{\mu_0}}\left[\ln\frac{1+\mu_0}{1-\mu_0} + \bar{E}_1\left(\frac{1-\mu_0}{\mu_0}z\right)\right] + E_1(z); \qquad 11.45$$

$$z > 0 \ , \ n = 1$$

and

$$\chi_n(z) = 2^n \frac{\exp\left[-\frac{(z-\mu_0)^2}{2(\mu_0^2 + 2(n-1)/3)}\right]}{\sqrt{2\pi(\mu_0^2 + 2(n-1)/3)}} ;$$

$$z > 0 \ , \ n \geq 2 . \qquad 11.46$$

Analogous formulas can be obtained for $z < 0$.

For the special case of $\mu_0 = 1$, $\chi_1(z)$ is given by

$$\chi_1(z) = e^{-z} \cdot \ln(2z) + E_1(z) . \qquad 11.47$$

The machine time for corresponding calculations reported in (22) was of the order of 100 h. It should be pointed out that, if a still better accuracy is desired than that provided by the CLT in Eqs. 11.42 **and** 11.44 **one can use more than one exact terms in the evaluation of the sums involved.**

For the illustration of the method we give here the angular distribution of the scattered neutrons for a plane monodirectinal source (c=0.5, z 0, = 1) with z as parameter. At small distances from the source plane the CLT approximation becomes bad, because there small numbers of collisions contribute mainly to the total scattered distribution(dotted lines taken from Ref. 13). In these results only the distribution of the once scattered neutrons has been taken exactly into account.

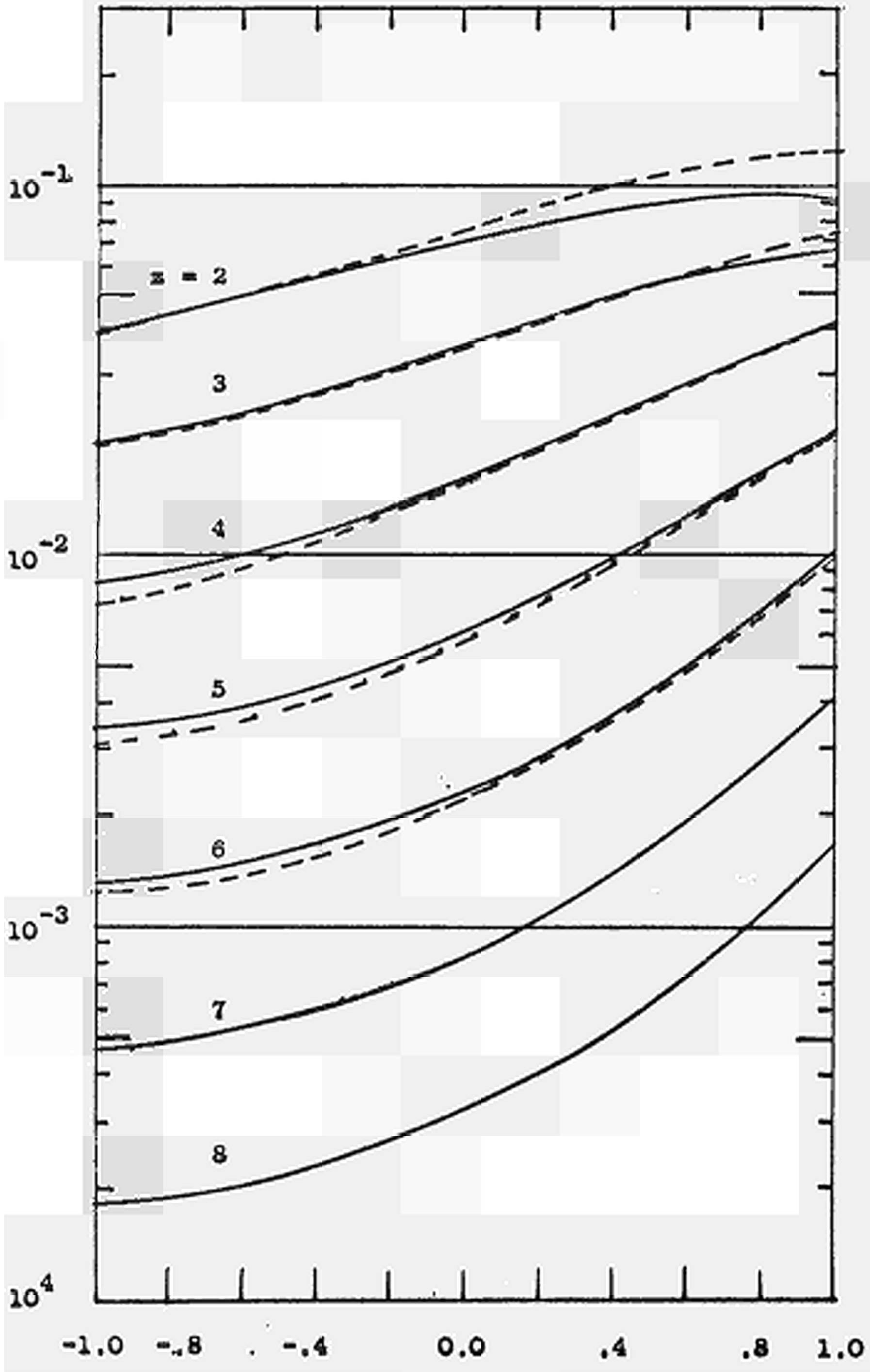


Fig. 15

12. CONCLUSIONS.

In the preceding sections we have exposed some elementary methods for treating the basic problems in Fast Neutron Transport Theory of infinite media in the isotropic scattering approximation.

As the reader has already observed, the principal tools in obtaining our results have been the Laplace transformation and the decomposition of the distribution in parts according to the numbers of collisions.

In those cases, in which portionality between $\sigma_s(E)$ and $\sigma_t(E)$ holds (i.e., no absorption or inelastic scattering), the method yields the exact solution of the slowing down problem in the space independent problems .

As a special case we have obtained in Sec. 3 the Placzek distributions in a slightly more general form by including energy dependent cross sections. A useful observation is that the n-th order distributions can be summed exactly over n (from 1 to ∞).

As a result the total distribution in the intervals ($\alpha^m E_s$, $\alpha^{m-1} E_s$) for every positive $m \geq 1$ is obtained.

When, however, absorption is present some approximations are required. The most useful method for treating such cases is the piece-wise-proportionality approximation. In the energy region in which the absorption cross section does not vanish, the relation $\sigma_s(E) = c_\lambda \sigma_t(E)$ can again be applied, where c_λ is a constant corresponding to the λ-th energy interval. By making this assumption one can solve the transport equation in each interval separately, by determining the source in the $(\lambda + 1)$-th interval from the solution in the λ-th interval.

For the deviation of $\sigma_s(E) / \sigma_t(E)$ from c_λ inside the λ-th interval the perturbation method developed in Sec. 10 can be applied.

Sofar no inelastic scattering has been considered. For the solution of the slowing down problem with inelastic scattering an approximate model has been given in Sec. 4.

For the application of the model use has been made of the observation, that the expression $\dfrac{\sigma_{in}(E)}{\sqrt{1-\frac{A+1}{A}\frac{Q}{E}}\,\sigma_t(E)}$ is almost constant for many isotopes.

This fact implies that the elastic and inelastic distributions differ only in that the latter is shifted toward lower energy by an amount proportional to Q.

In the case of many discrete levels the method remains the same.

The one-level slowing down distribution constitutes the basis for the calculation of the distribution, when a continuous distribution of nuclear levels is given.

This method is exposed in Sec.8.
Finally in Sec. 11 a space dependent problem has been considered. In connection with Sec. 11 one important fact must be pointed out.
The integral transform techniques applied and the expansion according to powers of c (the coupling constant of the neutron field with the medium) has yielded the separation of energy and space coordinates. This extrordinary result allows to use immediately the space independent solutions in order to construct the space dependent ones, if the monoenergetic solution is known.
Of course, this can be done immediatly only for the infinite medium.
For finite media we have developed an analogous method discussed in another paper.

A P P E N D I X A

INELASTIC SCATTERING KERNELS. ENERGY INDEPENDENT CROSS SECTIONS.

It has become the custom to discuss elastic and inelastic scattering quite independently from each other. This procedure although some times meaningful is not justified by the formal relationship of these two kinds of scattering.
To make it clear we observe that the only difference between them consists in the Q-value of the collision. In the elastic scattering case we have

$$Q = 0 \qquad \text{A1}$$

while in the inelastic nuclear collision Q is not identical zero; it takes values depending on the corresponding energy states of the nucleus before and after the collision. On the other hand, in systems in which every neutron suffers at least two collisions it is impossible to separate the elastically scattered neutrons from the inelastically scattered ones. A large part of neutrons has been scattered both elastically and inelastically. According to this observation it appears quite natural to relate inelastic and elastic scattering as closely as possible and to consider the latter as a particular case of the former in which $Q = 0$. We proceed now to the derivation of the kernel for inelastic scattering. (A derivation and compilation of the most useful forms of the scattering kernels for neutrons does not seem to exist in the literature.)

a. The v-representation.

We consider the collision of two particles. The first particle of the mass equal to 1 has a speed v' in the laboratory system (LS).

The second particle of mass A is initially at rest in the same system. If we introduce the center of mass system (CS), the first particle will have a speed given by

$$v_1 = \frac{A}{A+1} v' \qquad \text{A2}$$

The second particle will have the speed

$$v_2 = \frac{1}{A+1} v' \qquad \text{A3}$$

Eqs. A2 and A3 give the speeds before collision.
From the kinematics of the collision we have

$$v_1 \cos\vartheta + v_2 = v \cos\vartheta_0 \qquad \text{A4}$$

$$v_1 \sin\vartheta = v \sin\vartheta_0 \qquad \text{A5}$$

$$\varphi = \varphi_0 \qquad \text{A6}$$

In Eqs. A4 and A5 v is the speed of the first particle in LS after collision; ϑ_0, ϑ are the scattering angles in LS and CS respectively.

From Eqs. A4 and A5 we have

$$v_1^2 + 2v_1 v_2 \cos \;\; + v_2^2 = v'^2 \qquad \text{A7}$$

Introducing Eqs. A2 and A3 into Eq.A7 we obtain

$$\left(\frac{v}{v'}\right)^2 = \frac{A^2 + 2A\mu + 1}{(A+1)^2} \qquad \text{A8}$$

Eqs. A4 and A5 yield

$$\tan \vartheta_o = \frac{\sin \vartheta}{\gamma + \cos \vartheta} \qquad \text{A9}$$

where γ is given (14) by

$$\gamma = \frac{1}{A} \frac{1}{\sqrt{1 - \frac{A+1}{A} \frac{2Q}{v'^2}}} \qquad \text{A10}$$

In Eq. A10 Q ($\geqslant$ 0) is the Q-value of the collision.

From Eqs. A 9 and A 10 we obtain a relation between the cosines of the scattering angles in LS and CS.

For the case of inelastic scattering collision we have

$$\cos \vartheta_0 = \frac{1 + A\mu \sqrt{1 - \frac{A+1}{A} \frac{2Q}{v'^2}}}{\sqrt{1 + 2A\mu \sqrt{1 - \frac{A+1}{A} \frac{2Q}{v'^2}} + A^2 - \frac{A+1}{A} \frac{2Q}{v'^2}}} \qquad \text{A11}$$

This relation includes the conservation laws for momentum and energy and permits to construct the scattering kernels.

The scattering kernel $\sigma(v' \to v; \Omega' \to \Omega)$ determines, according to its definition, the probability that a scattered particle will undergo a definite change of its coordinates in phase space during collision.

The change of coordinates takes place according to the conservation laws mentioned above.
This probability is proportional to the product of two factors. The one factor $q_v^{in}(v', v)$ determines the probability that the scattering from v' to v will take place. The other factor represents the probability that the collision obeys the conservation laws of momentum and energy.
Then it follows that

$$\sigma_{\nu}^{in}\left(\nu'\to\nu;\underline{\Omega}'\to\underline{\Omega}\right)=q_{\nu}^{in}(\nu',\nu)\,\delta\left[\mu_0-\zeta_{\nu}^{in}\left(\nu',\nu\right)\right] \tag{A12}$$

In this equation δ (x) is the Dirac delta function; $\underline{\Omega}'$, $\underline{\Omega}$ are the two directions which determine μ_0 , the cosine of the scattering angle in LS

$$\mu_0 \equiv \underline{\Omega}'\cdot\underline{\Omega} = \mu_1\mu_2+\sqrt{(1-\mu_1^2)(1-\mu_2^2)}\cos(\varphi_1-\varphi_2) \tag{A13}$$

where $\mu_i, \varphi_i\ (i=1,2)$ determine the coordinates of the unit vectors $\underline{\Omega}', \underline{\Omega}$; $\zeta_{\nu}^{in}(\nu',\nu)$ has yet to be determined.

Now we want to express $\zeta_{\nu}^{in}(\nu',\nu)$ as a function only of variables defined in LS.

From the same conservation laws it follows that

$$\left(\frac{v}{v'}\right)^2 = 1 - \frac{2A\left(1-\frac{Q}{v'^2}\right)}{(A+1)^2} + \frac{2A}{(A+1)^2}\sqrt{1-\frac{A+1}{A}\,\frac{2Q}{v'^2}}\,\mu \quad \text{A14}$$

From Eq. A 14 we obtain the cosine, μ, of the scattering angle in CS

$$\mu = \frac{\left(\frac{A+1}{2A}\right)^2\left(\frac{v^2}{v'^2}-1\right)+\left(1-\frac{Q}{v'^2}\right)}{\sqrt{1-\frac{A+1}{A}\,\frac{2Q}{v'^2}}} \quad \text{A15}$$

Elimination of μ from Eqs. A11 and A15 yields the desired relation between μ_0, v', v

$$\mu_0 = \frac{1+A\left[\frac{(A+1)^2}{2A}\left(\frac{v^2}{v'^2}-1\right)+1-\frac{Q}{v'^2}\right]}{\sqrt{1+2A\left[\frac{(A+1)^2}{2A}\left(\frac{v^2}{v'^2}-1\right)+1-\frac{Q}{v'^2}\right]+A^2-\frac{A+1}{A}\,\frac{2Q}{v'^2}}} \quad \text{A16}$$

Eq. A16 states that only those speeds v', v are allowed which satisfy $-1 \leq \mu_0 < 1$. From this we have

$$\zeta_v^{in}(v',v) \equiv \frac{1 + A\left[\frac{(A+1)^2}{2A}\left(\frac{v^2}{v'^2}-1\right)+1-\frac{Q}{v'^2}\right]}{\sqrt{1+2A\left[\frac{(A+1)^2}{2A}\left(\frac{v^2}{v'^2}-1\right)+1-\frac{Q}{v'^2}\right]+A^2-\frac{A+1}{A}\frac{2Q}{v'^2}}}$$

A17

For the complete determination of Eq. A12 we have yet to define a normalization factor. This can be done by using the normalization condition

$$\iiint \sigma_{as}^{in}\left(v'\rightarrow v\,;\,\underline{\Omega}'\rightarrow\underline{\Omega}\right) d\Omega\, dv = 1$$

A18

From the assumption of isotropy in CS and from Eq. A18 it follows that this factor is

$$q_v^{in}(v',v) = \frac{(A+1)^2\, v\, \sigma_{in}}{4\pi\, A\, v'^2 \sqrt{1-\frac{A+1}{A}\,\frac{2Q}{v'^2}}} \qquad \text{A19}$$

In this case Eq. A12 can be written in the form

$$\sigma_v^{in}(v'\to v;\underline{\Omega}'\to\underline{\Omega})=\frac{(A+1)^2 v\quad \sigma_{in}}{4\pi A v'^2\sqrt{1-\frac{A+1}{A}\frac{2Q}{v'^2}}}\,\delta\left[\underline{\Omega}'\cdot\underline{\Omega}-\zeta_v^{in}(v',v)\right] \tag{A20}$$

This expression vanishes identically except for the v-values satisfying the inequalities

$$v'\sqrt{\left[1-\frac{2A}{(A+1)^2}\left(1-\frac{Q}{v'^2}+\sqrt{1-\frac{A+1}{A}\frac{2Q}{v'^2}}\right)\right]}\leqslant v\leqslant v'\sqrt{\left[1-\frac{2A}{(A+1)^2}\left(1-\frac{Q}{v'^2}-\sqrt{1-\frac{A+1}{A}\frac{2Q}{v'^2}}\right)\right]}, \tag{A21}$$

and when $\zeta_v^{in}(v',v)$ is given by Eq. A17.

It is sometimes convenient to expand the δ - function in a series of Legendre polynomials

$$\delta\left[\underline{\Omega}'\cdot\underline{\Omega}-\zeta_v^{in}(v',v)\right]=\sum_\ell\frac{2\ell+1}{2}\,P_\ell\left(\underline{\Omega}'\cdot\underline{\Omega}\right)P_\ell\left(\zeta_v^{in}\right). \tag{A22}$$

This form is particularly interesting in solving the transport equation by approximating the scattering kernel. With Eq. A22 we obtain from Eq. A20 the useful form

$$\sigma_v^{in}(v' \to v; \underline{\Omega}' \to \underline{\Omega}) = \frac{(A+1)^2 \, v \, \sigma_{in}}{4\pi A v'^2 \sqrt{1 - \frac{A+1}{A} \frac{2Q}{v'^2}}} \sum_{\ell} \frac{2\ell+1}{2} P_\ell(\underline{\Omega}' \cdot \underline{\Omega}) P_\ell(\zeta_v^{in}) \qquad \text{A23}$$

b. <u>The E-representation</u>

After having derived the inelastic scattering kernel in the v-representation it is easy to transform it in the E-representation.

In section Ia it has been assumed that the nuclear collision takes place between two particles. The particle of mass A was at rest in LS. The other particle (a neutron) only possessed translational energy corresponding to speed v'. To carry out the desired transformation we only need to require the conservation of the elementary probabilities, i.e.

$$\left|\sigma_v^{in}\left(v'\to v;\ \underline{\Omega}'\to\underline{\Omega}\right)dv\,d\Omega\right| = \left|\sigma_E^{in}\left(E'\to E;\ \underline{\Omega}'\to\underline{\Omega}\right)dE\,d\Omega\right| \qquad \text{A24}$$

From Eqs. A20, A24 and from $E = \frac{v^2}{2}$ we find that

$$\sigma_E^{in}\left(E'\longrightarrow E;\ \underline{\Omega}'\to\underline{\Omega}\right) = \frac{(A+1)^2\,\sigma_{in}}{8\pi A E'\sqrt{1-\frac{A+1}{A}\frac{Q}{E'}}} \times \delta\left[\underline{\Omega}\cdot\underline{\Omega} - \zeta_E^{in}(E',E)\right] \qquad \text{A25}$$

or

$$\sigma_E^{in}\left(E'\longrightarrow E;\ \underline{\Omega}'\to\underline{\Omega}\right) = \frac{(A+1)^2\,\sigma_{in}}{8\pi A E'\sqrt{1-\frac{A+1}{A}\frac{Q}{E'}}} \qquad \text{A26}$$

$$\delta\left[\underline{\Omega}'\cdot\underline{\Omega} - \zeta_E^{in}(E',E)\right]$$

This expression vanishes identically except for the E-values satisfying the inequalities

$$E'\left[1-\frac{2A}{(A+1)^2}\left(1-\frac{Q}{2E'}+\sqrt{1-\frac{A+1}{A}\,\frac{Q}{E'}}\right)\right] \leqslant E \leqslant E'$$

$$\left[1-\frac{2A}{(A+1)^2}\left(1-\frac{Q}{2E'}-\sqrt{1-\frac{A+1}{A}\,\frac{Q}{E'}}\right)\right] \quad \text{A27}$$

and $\zeta_E^{in}(E',E)$ is now given by

$$\zeta_E^{in}(E',E)=\frac{1+A\left[\frac{(A+1)^2}{2A}\left(\frac{E}{E'}-1\right)+1-\frac{Q}{2E'}\right]}{\sqrt{1+2A\left[\frac{(A+1)^2}{2A}\left(\frac{E}{E'}-1\right)+1-\frac{Q}{2E'}\right]+A^2-\frac{A+1}{A}\,\frac{Q}{E'}}} \quad \text{A28}$$

c. <u>The u-Representation</u>

Another representation of the scattering kernel which is frequently used is the u-representation.

According to the definition of the lethargy

$$E = E_o e^{-u} \qquad A29$$

(E_o is an arbitrary energy) we have

$$\left|\frac{dE}{du}\right| = E_o e^{-u}$$

Now Eq. A25 takes on the form

$$\sigma_u^{in}(u' \to u; \underline{\Omega}' \to \underline{\Omega}) = \frac{(A+1)^2}{8\pi A} e^{u'-u} \frac{\sigma_{in}}{\sqrt{1 - \frac{A+1}{A}\frac{Q_e^{u''}}{E_o}}}$$

$$\delta\left[\underline{\Omega}'\cdot\underline{\Omega} - \zeta_u^{in}(u',u)\right] \qquad A30$$

This expression is different from zero only for u-values defined by the condition

$$e^{-u'}\left[1 - \frac{A}{(A+1)^2}\left(2 - \frac{Q}{E_o}e^{u'} + 2\sqrt{1 - \frac{A+1}{A}\frac{Q}{E_o}e^{u'}}\right)\right]$$

$$\leq e^{-u} \leq e^{-u'}\left[1 - \frac{A}{(A+1)^2}\left(2 - \frac{Q}{E_o}e^{u'} - 2\sqrt{1 - \frac{A+1}{A}\frac{Q}{E_o}e^{u'}}\right)\right]$$

A31

Now the function $\zeta_u^{in}(u',u)$ is given by

$$\zeta_u^{in}(u',u) =$$

$$= \frac{1 + A\left[\frac{(A+1)^2}{2A}\left(e^{u'-u}-1\right)+1-\frac{Q}{2E_0}e^{u'}\right]}{\sqrt{1+2A\left[\frac{(A+1)^2}{2A}\left(e^{u'-u}-1\right)+1-\frac{Q}{2E_0}e^{u'}\right]+A^2-\frac{A+1}{A}\,\frac{Q}{E_0}e^{u'}}} \qquad \text{A32}$$

APPENDIX B

ELASTIC SCATTERING KERNELS. ENERGY INDEPENDENT CROSS SECTIONS.

From Appendix A we can immediately derive as a special case the corresponding formulas for the elastic scattering.

a. The v-Representation

From Eq. A20 we obtain the elastic scattering kernel in the v-representation by setting Q = 0

$$\sigma_v^{el}\left(v' \rightarrow v;\ \underline{\Omega}' \rightarrow \underline{\Omega}\right) = \frac{(A+1)^2\, v\, \sigma_s}{4\pi A\, v'^2}\ \delta\left[\underline{\Omega}\cdot\underline{\Omega} - \zeta_v^{el}(v',v)\right] \qquad \text{B1}$$

Now condition A21 becomes

$$v'\frac{A-1}{A+1} \leqslant v \leqslant v'$$

B2

In Eq. B1 $\zeta_v^{el}(v',v)$ is obtained from Eq. A17 as

$$\zeta_v^{el}(v',v) = \left(\frac{A+1}{2}\right)\left(\frac{v}{v'}\right) - \left(\frac{A-1}{2}\right)\left(\frac{v'}{v}\right) \qquad \text{B3}$$

b. <u>The E-Representation</u>

By setting Q = 0 we obtain from Eqs. A26 and A 27

$$\sigma_E^{el}(E'\rightarrow E; \underline{\Omega}'\rightarrow\underline{\Omega}) = \frac{(A+1)^2\sigma_s}{8\pi A E'}\,\delta\left[\underline{\Omega}'\cdot\underline{\Omega} - \zeta_E^{el}(E',E)\right]$$

B4

Again we have the condition

$$E'\frac{(A-1)^2}{(A+1)^2} \leqslant E \leqslant E' \qquad \text{B5}$$

and

$$\zeta_E^{el}(E',E) = \frac{A+1}{2}\sqrt{\frac{E}{E'}} - \frac{A-1}{2}\sqrt{\frac{E'}{E}} \qquad \text{B6}$$

c. <u>The u-Representation</u>

Finally we obtain from Eqs. A30 and A32 by setting Q = 0.

$$\sigma_u^{el}(u' \to u; \underline{\Omega}' \to \underline{\Omega}) = \frac{(A+1)^2 \sigma_s}{8\pi A} e^{u'-u}$$

$$\delta\left[\underline{\Omega}' \underline{\Omega} - \zeta_u^{el}(u', u)\right] \qquad \text{B 7}$$

The region in which the kernel does not vanish identically follows from condition A31

$$u' \leqslant u \leqslant u' - \ln\left(\frac{A-1}{A+1}\right)^2, \qquad \text{B 8}$$

or,
$$\left(1 - \frac{4A}{(A+1)^2}\right) e^{-u'} \leqslant e^{-u} \leqslant e^{-u'}$$

Finally

$$\zeta_u^{el}(u', u) = \frac{A+1}{2} e^{\frac{1}{2}(u'-u)} - \frac{A-1}{2} e^{\frac{1}{2}(u-u')} \qquad \text{B 9}$$

APPENDIX C

INELASTIC SCATTERING. ANISOTROPIC SCATTERING IN CS. ENERGY DEPENDENT CROSS SECTIONS.

In the preceding Appendices the assumption of isotropic scattering in CS has been made. In the present section we give the correspon-ding formulas for the general case of anisotro-pic scattering in CS. The main idea for the construction of the scattering kernels in this case remains the same. The scattering kernel is nothing but the product of two probabilities (or of two quantities proportional to them): The probability that the collision results to scattering, and the probability that conserva-tion of energy and momentum holds. The first factor is the differential scattering cross section. The second factor is again a δ - function guaranteeing the conservation of energy and momentum in CS.

a. <u>The v-Representation</u>

Under the circumstances stated above the inelastic scattering kernel can be written as

$$\sigma_v^{in}\left(v' \to v; \underline{\Omega}' \to \underline{\Omega}\right) = \sigma_v^{in}\left(v', \mu_0\right) \qquad \text{C1}$$
$$\delta\left[v^2 - v'^2 + \frac{2A\left(v'^2 - Q\right)}{(A+1)^2} - \frac{2Av'^2}{(A+1)^2}\sqrt{1 - \frac{A+1}{A}\,\frac{2Q}{v'^2}}\,\mu\right]$$

Suppose, now, we want to expand the scattering kernel in a series of Legendre polynomials with arguments μ_0.

$$\sigma_v^{in}\left(v' \to v; \underline{\Omega}' \to \underline{\Omega}\right) = \sum_{\ell} \sigma_{v,\ell}^{in}\left(v' \to v\right) P_\ell\left(\mu_0\right) \qquad \text{C2}$$

Here we have

$$\mu_0 = \underline{\Omega}' \, \underline{\Omega} \qquad \text{A13}$$

and

$$\sigma^{in}_{v,\ell}(v' \to v) = \frac{2\ell+1}{2} \int_{-1}^{1} v \sigma^{in}_{v}(v' \to v; \underline{\Omega}' \to \underline{\Omega}) P_\ell(\mu_0) d\mu_0 \qquad \text{C3}$$

Introducing Eq. A12 into Eq. C3 and using Eq. A17 we obtain

$$\sigma^{in}_{v,\ell}(v' \to v) = \frac{2\ell+1}{2} \int_{-1}^{1} v \sigma^{in}_{v}(v', \mu_0) \delta\Big[v^2 - v'^2 + \frac{2v'^2 A\left(1 - \frac{Q}{v'^2}\right)}{(A+1)^2} + \frac{2Av'^2}{(A+1)^2}\sqrt{1 - \frac{A+1}{A}\frac{2Q}{v'^2}}\mu\Big] \qquad \text{C4}$$

$$P_\ell(\mu_0)\, d\mu_0$$

In Eq. C4 $\sigma_v^{in}(v', \mu_0)$ is the differential inelastic scattering cross section in LS.

As a matter of fact most nuclear data are given in CS. It seems therefore reasonable to transform the integral Eq. C4 so that the nuclear data appear in CS.

This transformation is easily effected by introducing the transformation formula for cross sections in the special form

$$\sigma_v^{in}(v, \mu_0)\, d\mu_0 = \sigma_v^{in}(v', \mu)\, d\mu \qquad \text{C5}$$

The evaluation of the integral in Eq. C4 can be done in two different forms.

$\underline{\alpha}$ The first method can be applied when the nuclear data are given in an analytical form. In this case we can carry out the integration. Using the properties of the δ-function we immediately deduce from Eqs. C4 and C5 that

$$\sigma^{in}_{v,\ell}(v' \to v) = \frac{2\ell+1}{2} \frac{(A+1)^2 v}{2Av'^2} \frac{\sigma_v^{in}(v',\mu)}{\sqrt{1-\frac{A+1}{A}\frac{2Q}{v'^2}}} P_\ell(\mu_0(\mu)) \qquad \text{C6}$$

In Eq. C6 μ is defined by Eq. A15, $\mu(\mu)$ is defined by Eq. A11, and $\sigma^{in}_v(v,\mu)$ is the differential inelastic scattering cross section in CS.

The right-hand side from Eq. C6 is different from zero only under the condition A21.

β It happens some times that the measured differential scattering cross sections are given as a Legendre series. In this case we can calculate the integral in Eq. C4 after having expanded the differential scattering cross section in a Legendre series.
Then we have

$$\sigma_{v}^{in}(v',\mu) = \sum_{j} \sigma_{v,j}^{in}(v')\, P_j(\mu) \qquad C7$$

where

$$\sigma_{v}^{in}(v',\mu) = \sum_{j} \sigma_{v,j}^{in}(v')\, P_j(\mu) \qquad C8$$

In particular $\sigma^{in}_{v,0}(v')$ is the total inelastic scattering cross section in CS.

From Eqs. C4 and C7 we obtain the second expression for the coefficients of the inelastic scattering kernel Eq. C2.

$$\sigma^{in}_{v,\ell}(v' \to v) = \frac{2\ell+1}{2} \frac{P_\ell(\mu_0(\mu))}{\sqrt{1-\frac{A+1}{A}\frac{2Q}{v'^2}}} \frac{(A+1)^2 v}{2Av'^2} \sum_j \sigma^{in}_{v,j}(v') P_j(\mu) \qquad C9$$

Here μ and $\mu_0(\mu)$ are defined as in case (α); the right-hand side of Eq. C9 vanishes identically for v not satisfying condition A21.

b. The E-Representation

Continuing we give here the scattering kernels for non isotropic scattering in CS in the energy representation. The way is exactly the same as in the isotropic case. The expansion coefficient of the scattering kernel corresponding to Eq. C6 is

α)

$$\sigma^{in}_{E,\ell}\,(E' \to E) = \frac{2\ell+1}{2}\,\frac{(A+1)^2}{4AE'}\,\frac{\sigma^{in}_{E}(E',\mu)}{\sqrt{1-\frac{A+1}{A}\,\frac{Q}{E'}}}\;P_\ell\big(\mu_0(\mu)\big) \qquad \text{C10}$$

where μ is given by

$$\mu = \frac{\frac{(A+1)^2}{2A}\left(\frac{E}{E'}-1\right)+\left(1-\frac{Q}{2E'}\right)}{\sqrt{1-\frac{A+1}{A}\,\frac{Q}{E'}}} \qquad \text{C11}$$

and the right-hand side of Eq. C10 vanishes identically for E not satisfying condition Eq. A27. $\mu_0(\mu)$ is given by Eq. A11.

β)

$$\sigma^{in}_{E,\ell}\left(E' \rightarrow E\right) = \frac{2\ell+1}{2} \frac{(A+1)^2}{4AE'} \frac{P_\ell\left(\mu_0(\mu)\right)}{\sqrt{1-\frac{A+1}{A}\frac{Q}{E'}}} \sum_j \sigma^{in}_{E,j}(E')\, P_j(\mu) \qquad \text{C12}$$

The same conditions as for Eq. C10 hold for the right-hand side of these equations and definitions.

c) The u-Representation

Finally we give here the expansion coefficients of the scattering kernel in the lethargy representation.

α)

$$\sigma^{in}_{\mu,\ell}(u' \to u) = \frac{2\ell+1}{2} \, \frac{(A+1)^2}{4A} \, e^{u} \, \frac{\sigma^{in}_{u}(u',\mu)}{\sqrt{1-\frac{A+1}{A}\frac{Q}{E_0}e^{u'}}} \, P_\ell\left(\mu_0(\mu)\right) \qquad \text{C13}$$

where μ is defined by

$$\mu = \frac{\frac{(A+1)^2}{2A}\left(e^{u'-u}-1\right)+1-\frac{Q}{2E_0}\,e^{u'}}{\sqrt{1-\frac{A+1}{A}\,\frac{Q}{E_0}\,e^{u'}}} \qquad \text{C14}$$

β)

$$\sigma^{in}_{u,\ell}(u' \to u) = \frac{2\ell+1}{2} \frac{(A+1)^2}{4A} e^{u'-u} \frac{P_\ell(\mu_0(\mu))}{\sqrt{1-\frac{A+1}{A}\frac{Q}{E_0}e^{u'}}}$$

$$\sum_j \sigma^{in}_{u,j}(u') \; P_j(\mu) \qquad \text{C15}$$

The right-hand sides of Eqs. C13 and C15 are different from zero only for u satisfying condition A31.

APPENDIX D

ELASTIC SCATTERING KERNELS.ANISOTROPIC SCATTERING IN CS. ENERGY DEPENDENT CROSS SECTIONS.

Here we deduce the elastic scattering kernel from the inelastic one in the manner mentioned above for the case of anisotropic scattering in CS.

a. The v-Representation

For this case the coefficients of the expansion Eq. C2 follow immediately from Eqs. C6 and C9. We have respectively α)

$$\sigma^{el}_{v,\ell}\left(v' \longrightarrow v\right) = \frac{2\ell+1}{2}\,\frac{(A+1)^2 v}{2Av'^2}\;P_\ell\left(\mu_0(\mu)\right)\sigma^{el}_{v}\left(v',\mu\right)$$

D1

and

β)

$$\sigma^{el}_{v,\ell}\left(v' \longrightarrow v\right) = \frac{2\ell+1}{2}\,\frac{(A+1)^2 v}{2Av'^2}\;P_\ell\left(\mu_0(\mu)\right)$$

$$\sum_j \sigma^{el}_{v,j}\left(v'\right)\;P_j\left(\mu\right)$$

D2

The right-hand sides of Eqs. D1 and D2 are different from zero only for v satisfying condition B2.

b. The E-Representation

In this case we obtain from Eqs. C10 and C12 respectively

α)

$$\sigma^{el}_{E,l}\left(E' \rightarrow E\right) = \frac{2l+1}{2}\,\frac{(A+1)^2}{4AE'}\,\sigma^{el}_{u}\left(E',\mu\right) P_l\left(\mu_0(\mu)\right)$$

D3

and

β)

$$\sigma^{el}_{E,\ell}\left(E'\to E\right)=\frac{2\ell+1}{2}\ \frac{(A+1)^2}{4AE'}\ P_\ell\left(\mu_0(\mu)\right)$$

$$\sum_j \sigma^{el}_{E,j}\left(E'\right) P_j\left(\mu\right) \qquad \text{D4}$$

The right-hand sides of Eqs. D3 and D4 do not vanish identically if E satisfies the condition B5.

c. The u-Representation

Finally we obtain from Eqs. C13 and C15

α)

$$\sigma^{el}_{u,\ell}\left(u'\to u\right)=\frac{2\ell+1}{2}\ \frac{(A+1)^2}{4A}\, e^{u'-u}\, \sigma^{el}_{u}\left(u',\mu\right) P_\ell\left(\mu_0(\mu)\right)$$

D5

and

β)

$$\sigma_{u}^{el}(u' \rightarrow u) = \frac{2\ell+1}{2} \frac{(A+1)^2}{4A} e^{u'-u} P_\ell(\mu_0(\mu)) \sum_j \sigma_{u,j}^{el}(u') P_j(\mu)$$

D6

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To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

Alfred Nobel

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