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ON A STOCHASTIC PROBLEM IN NON-LINEAR SYSTEMS WITH A REFERENCE TO CORRELATIONS IN NUCLEAR REACTOR THEORY

by

J. LARISSE

1966

Joint Nuclear Research Center
Ispra Establishment - Italy

Scientific Information Processing Center - CETIS

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SUMMARY

This report describes a stochastic non-linear system, a particular case of which is a nuclear reactor. We derive a functional equation between the statistical properties of the input, the output and the characteristic function of the unit impulse response, from which it is possible to obtain a general method of calculation of correlations.

Introduction

A random function $V(\theta)$ is the input of a system which produces, following a probability law $f[V(\theta)]d\theta$, a sequence of impulses in a linear multiplicative medium. The output of this latter is a random function $N(t)$. The problem is to calculate the relation between the statistical properties of $V(\theta)$ and $N(t)$. We show that this problem is an extension of the so-called random functions derived from Poisson processes.

Two stochastic processes are considered in the sections II and III. The general result is reduced to the calculation of a functional of $V(\theta)$ extensively studied by different authors.

Finally the particular case of the nuclear reactor, suggested by Dr. W. MATTHES (Department of Reactor Physics) is given as illustration of this model.

I Mathematical model

A random function $V(\theta)$ produces in the interval of time $[\theta, \theta+d\theta]$ with the probability $f[V(\theta)]d\theta + O(d\theta)$ at least one impulse and consequently zero impulse with the probability $1 - f[V(\theta)]d\theta + O(d\theta)$. Let us call $dX(\theta)$ the stochastic number of impulses in this interval.

The impulse response of the multiplicative medium is a random function $G(\theta,t)$, if the impulse occurs at time θ and we obtain in the output at time t.

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The hypothesis of linearity leads to the classical equation:

I.1) $$N(t) = \int_{-\infty}^{t} G(\theta,t)\, dX(\theta).$$

In other terms, in the interval $(-\infty, t)$ we observe impulses at time $\ldots\theta_i\ldots\theta_{i+1}\ldots$ The impulse θ_i produces at time t the effect $G(\theta_i,t)$. I.1) represents therefore the infinite sum of random variable $G(\theta_i,t)$. This sum must be almost surely finite (i.e. finite except in cases of probability zero) if we make the necessary assumption of the physical realisability of the system. I.1) is, in fact, the sum

$$N(t) = \sum_{\theta_i} G(\theta_i,t)$$

of independent random variables depending of the parameter t.

We remark that $G(\theta,t)$ may be a impulse random function and consequently also $N(t)$.

Naturally, if the number of impulses grows to infinity such that $dX(\theta)$ becomes, by a suitable normalisation, the differential of a derivable function it is easy to show that I.1) becomes the classical stochastic integral equation

$$N(t) = \int_{-\infty}^{t} G(\theta,t)x(\theta)d\theta.$$

II Stochastic process X(t) induced by V(t)

Between the functions X(t) and $f[v(t)]$ (where f(x) is always a deterministic function) we make the following assumption:

If v(t) is a given value at time t, $f[v(t)]$ the non-negative value corresponding to t, we have:

Probability that at least one impulse occurs in the interval $(t, t+dt) = f[v(t)]dt + 0(dt)$.

Probability that any impulse occurs $= 1-f[v(t)]dt - 0(dt)$.

($0(dt)$ tends to zero with $dt^{1+\alpha}$ $\alpha > 0$).

With this hypothesis, if $p_n(t)$ is the probability to have n impulses in $[0,t)$ we have the equation:

$$\text{II.1)} \qquad p_n(t+dt) = p_n(t)\ [1-f[v(t)]dt] + p_{n-1}(t)f[v(t)]dt + 0(dt).$$

Taking account of the fact that $p_o(0) = 1$, a classical recurrence calculation gives

$$p_n(t) = \frac{\left[\int_o^t f[v(u)]du\right]^n}{n!} \exp\left[-\int_o^t f[v(u)]du\right]$$

If V(t) is a non-random function, $p_n(t)$ is a Poisson distribution.
If V(t) is random, then $p_n(t)$ is a random variable and we have to take the mean value:

$$p_n(t) = \int \text{Prob}\left\{I(t) \leqslant \int_o^t f[v(t)]dt < I(t) + dI(t)\right\} \frac{[I(t)]^n}{n!} e^{-I(t)dI}$$

which is not, in general, a Poisson distribution.

Hence:

$$\text{Prob}\,[dX(t)] = k = \int_{v\epsilon V} \frac{[f(v(t)dt]^k}{k!} e^{-f[v(t)]dt}\, p[v(t) = v]dv; \quad k=0,1,2...$$

Suppose now n disjoint intervals. Because the n k_i values are independent we have

$$E[dX(t_1)\dots dX(t_n)] = \sum_{k_1+\dots k_n=0}^{\infty} \text{Prob}\,[dX(t_1) = k_1 \dots dX(t_n) = k_n]\, k_1\, k_2 \dots k_n$$

$$= \sum_{k_1+\dots k_n=0}^{\infty} \int dv_1 \dots dv_n\, p(v_1 \dots v_n) \prod_{i=1}^{n} k_i \frac{[f[v_i]dt]^{k_i}}{k_i!} e^{-f[v_i]dt}$$

Consider then the sequence in m:

$$f_m = \sum_{k_1+\dots k_n=0}^{m} \prod_{i=1}^{n} k_i \frac{[f[v_i]\,dt]^{k_i}}{k_i!} e^{-f[v_i]dt}$$

$$= \prod_{i=1}^{n} \sum_{k_1+\dots k_n=0}^{m} k_i \frac{[f\,[(v_i)]dt]^{k_i}}{k_i!} e^{-f[v_i]\,dt}$$

We have

$$|f| < \prod_{i=1}^{n} \sum_{k_1+\dots k_n=0}^{\infty} k_i \frac{[f[v_i]dt]^{k_i}}{k_i!} e^{-f[v_i]dt} = dt^n f[v_1]\dots f[v_n]$$

If

$$(dt)^n \int dv_1 \dots dv_n\, f[v_1]\dots f[v_n]\, p(v_1 \dots v_n) < \infty$$, by application of the Lebesgue theorem on the inversion of the signs of summation we have:

$$E[dX(t_1)\dots dX(t_n)] = E[f[v(t_1)]\dots f[v(t_n)]]\,dt^n.$$

This expression shows that the independence of the $V(t_i)$ involves the independence of the $dX(t_i)$, in particular when $V(t)$ is a deterministic function.

III Stochastic process induced by the integral transformation

We have seen that the random variable $N_T(t)$ is defined by

III.1) $$N_T(t) = \int_T^t G(t,\theta)dX(\theta)$$ which must be considered as the sum:

$$N_T(t) = \sum_{\theta_i \in [T,t]} G(t,\theta_i)$$

with $G(t,\theta) = 0$ if $\theta > t$. For all finite interval $[T,t)$, each realisation $X(\theta)$ defines a sequence almost surely finite of points θ_i. If we make the assumption, always verified in the applications, that $G(t,\theta)$ is finite except on a set of measure zero, hence almost surely this sum defines a random variable. In addition, we are interested in the behaviour of $N_T(t)$ when $T \to -\infty$ and for the further calculations, how we may deduce the characteristic function

$$\varphi(u,t) = E[\exp iuN(t)] \quad [N(t) = \lim_{T \to -\infty} N_T(t)]$$

from the characteristic function of

$$\varphi_T(u,t) = E[\exp iuN_T(t)].$$

The two following conditions that we state without the proofs, which may be found in classical text books, are the answer to our questions.

If the following condition is fulfilled

III.2 $$\int_{-\infty}^{+\infty} E[G(t,\theta)]\, E[dX(\theta)] = \int_{-\infty}^{+\infty} E[G(t,\theta)]\, E[f(v(\theta))]d\theta < \infty$$

then $N(t) = \lim_{T \to -\infty} N_T(t)$ defines almost surely a random variable which is noted:

$$N(t) = \int_{-\infty}^{+\infty} G(t,\theta)dX(\theta).$$

If in addition, we have:

III.3)
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[G(t,\theta) \, . \, G(t',\theta')] \; E[dX(\theta)dX(\theta')] =$$
$$\int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\theta' \; E[G(t,\theta) \; G(t'\theta')] \; E\Big\{f[v(\theta)] \; f[v(\theta')]\Big\} < \infty$$

then $N_T(t)$ converges also in quadratic mean towards the random variable $N(t)$, i.e.

$$\lim_{T \to -\infty} E \; |N(t) - N_T(t)|^2 = 0.$$

In other terms, III.2) ensures that, except in cases theoretically possible, but of which the probability of realisation is zero, $N(t)$ is a finite random variable.

The condition III.3) ensures, in addition, that

i) $E[N(t) \; N(t')]$ = correlation function of $N(t)$ is finite

ii) $\lim_{T \to -\infty} \varphi_T(u,t) = \varphi(u,t)$

Tho physical interpretation of III.2) and III.3) is obvious, but what is less evident, is that they are sufficient to involve the convergence of the characteristic function, important result for the following.

Finally if $f[v(t)] = v(t)$ and $G(t,\theta)$ is stationary in the wide sense, i.e. $G(t,\theta) = G(t-\theta)$, by putting

$$E[V(t) \; V(t')] = \varphi_{vv}(t-t'), \; E[G(t-\theta) \; G(t'-\theta')] = \varphi_{gg}(t'-\theta' - t+\theta)$$

$$E[N(t) \; N(t')] = \varphi_{nn}(t,t')$$

It is well known that, by a suitable change of variables, III.2) is equivalent to

$$\varphi_{nn}(\tau) = \int_{-\infty}^{+\infty} \varphi_{gg}(t) \; \varphi_{vv}(t-\tau)dt.$$

Consider now the K.L. random variables $G_k(t_l,\theta)$ which are, by definition, the responses at time t_l of one impulse at time θ in an elementary volume of the multiplicative medium indexed by K. Put

$$\gamma(\theta) = \sum_{kl} iu_{kl}\, G_k(t_l,\theta) \;;\; \Phi\left\{u_{kl}\right\} = E\left[\exp i \sum_{kl} u_{kl}\, N_k(t_l)\right]$$

$$\phi_N\left\{u_{kl}\right\} = \log \Phi\left\{u_{kl}\right\} = \log E\left[\exp i \sum_{kl} u_{kl}\, N_k(t_l)\right]$$

$$\varphi_G\left\{u_{kl},\theta\right\} = E\left[\exp i \sum_{kl} u_{kl}\, G_k(t_l,\theta)\right] = E\left[\exp \gamma(\theta)\right]$$

It is clear that

$$\phi_{N_T}\left\{u_{kl}\right\} = \log E\left[\exp i \sum_{kl} u_{kl}\int_T^{\infty} G_k(t_l,\theta)\, dX(\theta)\right] = \log E\left[\exp \int_T^{\infty} \gamma(\theta)\, dX(\theta)\right]$$

We investigate now the two most important cases from the practical point of view, i.e. the case where $V(\theta)$ is deterministic and $X(\theta)$ is a random function with uncorrelated increments, and then the case $V(\theta)$ random and $X(\theta)$ with correlated increments. We shall discuss also an example which will show that the statistical properties of $X(t)$ may be quite different from a Poisson law.

A) Case non-random $V(\theta)$ and $X(\theta)$ is a process with uncorrelated increments

We follow here a procedure described by R. FORTET [1]. Divide the interval $[T,\infty]$ (which is, in fact, the interval $[T, \max_l t_l]$ because $G_k(t_l-\theta) = 0$ for $\theta > t_l$), in intervals $d\theta$. In each of these $d\theta_j$ interval there are $dX(\theta_j)$ random variables $\gamma(\theta_j)$ depending on the same probability law with parameter θ, of which the characteristic function is $\varphi_G\left\{u_{kl},\theta\right\}$. Recall us that these $d\theta_j$ are independent , then it follows that each $d\theta_j$ gives the contribution

$\gamma_1(\theta_j) + \gamma_2(\theta_j) + \ldots\ldots \gamma_{dX(\theta_j)}(\theta_j)$ and the characteristic function of this sum is

$$\varphi_G^{dX(\theta_j)}\left\{u_{kl}, \theta_j\right\}$$

It follows that

$$\psi_{N_T}\left\{u_{kl}\right\} = \log E\left[\exp \sum_{(\theta)} \gamma(\theta_j)\, dX(\theta_j)\right] = \log E\left[\prod_{(\theta)} \exp \gamma(\theta_j)\, dX(\theta_j)\right]$$

$$\psi_{N_T}\left\{u_{kl}\right\} = \log \prod_{(\theta_j)} E[\exp \gamma(\theta_j)\, dX(\theta_j)] = \sum_{(\theta_j)} \log E\left[\exp \gamma(\theta_j) dX(\theta_j)\right]$$

By putting

$$\psi_{N_T}\left\{u_{kl}, \theta\right\} = \sum_{T}^{\theta} \log E\left[\exp \gamma(\theta_j)\, dX(\theta_j)\right]$$

$$\psi_{N_T}\left\{u_{kl}, \theta+d\theta\right\} = \sum_{T}^{\theta+d\theta} \log E\left[\exp \gamma(\theta_j)\, dX(\theta_j)\right]$$

we see that

$$\psi_{N_T}\left\{u_{kl}, \theta+d\theta\right\} - \psi_N\left\{u_{kl}, \theta\right\} = d\, \psi_N\left\{u_{kl}, \theta\right\} = \log E\left[\exp \gamma(\theta) dX(\theta)\right]$$

and

$$\psi_{N_T}\left\{u_{kl}\right\} = \int_T^{\infty} d\, \psi_{N_T}\left\{u_{kl}, \theta\right\}.$$

On the other part, by taking successively the mean values on $\gamma(\theta)$ and $dX(\theta)$ we have:

$$\log E\left[\exp \gamma(\theta)\, dX(\theta)\right] = \log E_{dX}\left[\varphi_G^{dX(\theta)}\left\{u_{kl}, \theta\right\}\right]$$

Since

$$\text{Prob}\left\{dx(\theta) = k\right\} = \frac{[f[v(\theta)]\ d\theta]^k}{k!}\ e^{-f[v(\theta)]d\theta}$$

we obtain

$$E_{dX}\left[\varphi_G^{dX(\theta)}\left\{u_{kl},\theta\right\}\right] = \sum_{k=0}^{\infty} \frac{[\varphi_G\ f[v(\theta)]d\theta]^k}{k!}\ e^{-f[v(\theta)]d\theta}$$

$$= \exp\left[(\varphi_G\left\{u_{kl},\theta\right\} - 1)\ f[v(\theta)]d\theta\right].$$

Hence

$$\phi_{N_T}\left\{u_{kl}\right\} = \int_T^{\infty}\left[\varphi_G\left\{u_{kl},\theta\right\} - 1\right]\ f[v(\theta)]d\theta\ .$$

If now the two conditions III.2) and III.3) are verified, i.e.

$$\int_{-\infty}^{+\infty} E\left[\sum_{kl} G_n(t_l,\theta)\right] f[v(\theta)]\ d\theta < \infty\ ,$$

$$\int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\theta'\ E\left[\sum_{kl} G_k(t_l,\theta) \sum_{kl} G_k(t_l,\theta')\right] f[v(\theta)]\ f[v(\theta')] < \infty$$

we may pass to the limit and

$$\text{III.5)}\qquad \phi_N\left\{u_{kl}\right\} = \int_{-\infty}^{+\infty}\left[\varphi_G\left\{u_{kl},\theta\right\} - 1\right] f[v(\theta)]\ d\theta$$

We note that if the $G_k(t_k,\theta)$ are non-random then

$$\varphi_G\left\{u_{kl},\theta\right\} = \exp \gamma(\theta)$$

and we may demonstrate that if the process is such that $dX(\theta)$ becomes very large (e.g. for high values of $f[v(\theta)]$, by a suitable normalization, $\phi_N\left\{u_{kl}\right\}$ tends towards the characteristic function of a gaussian law. In this case we meet again the classical transformation of a gaussian law to another gaussian law by a linear system.

B) <u>Case $V(\theta)$ random and $dX(\theta)$ correlated</u>

In this case, the preceding method is not applicable until the end of the calculation, and we must proceed in the following manner:

With the same notation:

$$\phi_{N_T}\left\{u_{kl}\right\} = \log E\left\{\exp \sum_{(\theta)} \gamma(\theta)\ dX(\theta)\right\}$$

$$= \log E_{dX}\ E_{\gamma}\left\{\exp \sum_{(\theta)} \gamma(\theta)\ dX(\theta)\right\}$$

$$= \log E_{dX}\left\{\prod_{(\theta)} \phi_G^{dX(\theta)}\left\{u_{kl}\right\}\right\}.$$

By dividing the interval $[T, \infty] = [T, \max t_l]$ in n sub-intervals, the conditions III.2) and III.3) give that

$\int_{-\infty}^{+\infty} \gamma(\theta) dX(\theta)$ exists with probability one, and is the limit in

quadratic mean, for $T \to -\infty$ and $n \to +\infty$, the maximum module $|\theta_{n+1} - \theta_n| \to 0$ of the random variable

$$\sum_{T}^{\infty} \sum_{j=1}^{n} \gamma(\theta_j) dX(\theta_j)$$

Hence, if $p(v_1, v_2 \ldots v_n)$ is the joint frequency function and $k_i = dX(\theta_i)$ we have:

$$U_n = \log \int_v dv_1 \ldots dv_n\ p(v_1 \ldots v_n) \sum_{k_i = 0}^{\infty} \prod_{j=1}^{n} \phi_G^{k_i}(\theta_i) \frac{\left[\int v(\theta_i)\ d\theta_i\right]^{k_i}}{k_i!}$$

$$\exp\left[-\int v(\theta_i)\ d\theta_i\right] = \log \int_v dv_1 \ldots dv_n\ p(v_1 \ldots v_n) \prod_{i=1}^{n} \exp\left[(\phi_G(\theta_i) - 1) \int [v(\theta_i)] d\theta_i\right]$$

We may calculate explicitly U_n and if the limit $U = \lim_{n \to \infty} U_n$ exists and is independent of the subdivision we define

III.6) $$U = \phi_N \left\{u_{kl}\right\} = \log E \left\{\exp \int_{-\infty}^{+\infty} \left[\varphi_G \left\{u_{kl}, \theta\right\} - 1\right] f \left[v(\theta)\right] d\theta\right\}$$

For the calculation of this expression we refer to D.A. DARLING and A.J.F. SIEGERT [2] who have extensively studied the most general form

III.7) $$r(v_o t_o | v, t, \lambda) = \frac{E\left[\exp\left\{-\lambda \int_{t_o}^{t} \Phi(V(\theta), \theta) d\theta\right\} \; V(t_o) = v_o, \; V(t) = v\right]}{p(v_o t_o | v, t)}.$$

where $p(v_o, t_o | v, t)dv$ is the probability that $V(t)$ is in the interval $(v, v+dv)$, if $V(t_o) = V_o$; $r(v_o t_o | v, t, \lambda)$ is, consequently, the conditional characteristic function of our formula.

We note that, in the work cited above, it is assured that $V(t)$ is markovian, $r(v_o t_o | v, t, \lambda)$ is then the solution of two integral equations

III.8) $$\left\{\dot{\dot{\mathcal{L}}} - \lambda \, \Phi(V, t) - \frac{\partial}{\partial t}\right\} r(v_o t_o | v+\lambda) = 0.$$

$$\left\{\dot{\dot{\dot{\mathcal{L}}}}{}_o^+ - \lambda \, \Phi(V, t) + \frac{\partial}{\partial t_o}\right\} r(v_o t_o | v+\lambda) = 0 .$$

in the case of practical interest where $p(v_o t_o | v, t)$ is the solution of the Fokker-Plank-Kolmogoroff operator $\mathcal{L} - \frac{\partial}{\partial t}$ and its adjoint $\mathcal{L}_o^+ + \frac{\partial}{\partial t_o}$; III.8) may be, then resolved by a perturbation formalism.

Following that $V(\theta)$ is brownian [3] or markov [1] processes some particular methods are suitable, and the particular forms for $f[v(\theta)]$: $V(\theta)$; $V^2(\theta)$; $|V(\theta)|$; $1 + \text{sign } V(\theta)/2$ have been particularly investigated.

Finally we remark that from III.6) we may obtain the two first moments

$$\bar{N} = \int_{-\infty}^{+\infty} d\theta \; E\left\{G(t-\theta)\right\} E\left\{f\left[V(\theta)\right]\right\}$$

$$\overline{N^2} = \int_{-\infty}^{+\infty} E\left\{G^2(t-\theta)\right\} E\left\{f\left[V(\theta)\right]\right\} + \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\theta' \; E\left\{G(t-\theta)\; G(t-\theta')\right\}$$

$$E\left\{f\left[V(\theta)\right] f\left[V(\theta')\right]\right\}$$

and also the correlation forms if we introduce in the preceding expressions the $G_k(t_1-\theta)$.

IV Examples

We shall give here some examples with additional physical assumptions to simplify the calculation.

A) We take $V(\theta)$ as a WIENER-LEVY process (brownian motion, i.e. once integrated white noise). On the interval of length T we calculate

$$U_T = \int_0^T V^2(t)dt \quad \text{with} \quad V(0) = 0$$

And these successive values are used as input of our system. It is known that, by the theory of markov processes additive functionals, the **characteristic** function of U_T is

$$\Phi_{U_T}(u) = \sec\,(2iu)^{\frac{1}{2}}\,T \quad (\sec = \text{trigonometric function}).$$

For one particular interval $[nT, (n+1)T]$ we have

$$\text{Prob}\left[X(T) = \int_{nT}^{(n+1)T} dX(t) = k\right] = \frac{[U_T]^k}{k!}\, e^{-U_T}$$

For the total interval $(0,\infty)$, by the result of section II, we obtain

$$\Phi_k(z) = E\left[e^{\,izk}\right] = \sum_{k=0}^{\infty} e^{izk} \int_0^{\infty} e^{-U_T}\, \frac{(U_T)^k}{k!}\, dF(U_T)$$

where $F(U_T)$ is the distribution function of U_T.

We remark that

$$f_n = \sum_0^n e^{\,ivk}\, e^{-U_T}\, \frac{(U_T)^k}{k!} < 1$$

This sequence is absolutely bounded by the function 1 integrable for the measure dF, and therefore we may reverse the two signs of summation, which gives finally

$$\Phi_k(v) = \int_0^{+\infty} \sum_{k=0}^{\infty} e^{-U_T} e^{ivk} \frac{(U_T)^k}{k!} \, dF(U_T)$$

$$= \int_0^{\infty} \left[\exp \; U_T(e^{iv} - 1) \right] dF(U_T) = \Phi_{U_T} \left(\frac{e^{iv} - 1}{i} \right)$$

$$= \sec \left\{ \left[2(e^{iv} - 1)^{\frac{1}{2}} \, T \right\}^{1/2} \right.$$ which is not at all a Poisson distribution.

B) We consider now a nuclear reactor of which the source is an accelerator with an input fluctuating voltage $V(\theta)$. The function $f(v)$ is taken as equal to v. The set reactor and detector are supposed to be linear and the characteristic function of the impulse response is $\varphi_G(u,\theta)$. Unfortunatly this function is not known and its calculation is not easy, perhaps some experimental data or some analytical properties deduced from stochastic Boltzmann equation may be used for further calculations. Nevertheless, for lack of additional information on φ_G we may go on with some particular forms for $V(\theta)$. We suppose first, $V(\theta)$ a gaussian stationary random function of mean value m and correlation matrix Q. We know that

$$p(v_1 \; v_2 \ldots v_n) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\Lambda}} \exp \left\{ - \frac{1}{2\Lambda} \sum_{i,j=1}^{n} \Lambda_{jk} \, (v_j - m) \; (v_i - m) \right\}$$

where

$$\begin{cases} m = E\left[v_i\right] \\ \Lambda_{ik} = \left[Q^{-1}\right]_{jk} \quad = (j,k) \text{ element of the inverse matrix } Q^{-1} \end{cases}$$

$$\left[Q_{jk}\right] = E\left[v_i \; v_k\right] = \left[Q\right]_{j,k} .$$

We have:

IV.2) $$\phi_{N_T}(u) = \lim_{n\to\infty} \log \int dv_1 \ldots dv_n \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\sqrt{\Lambda}} \exp - \left\{ \frac{1}{2\Lambda} \sum_{j,k=1}^{n} \Lambda_{jk} (v_j-m)(v_k-m) \right\}$$

$$\exp \sum_{i=1}^{n} \left[\varphi_G(\theta_i)-1 \right] v_i \, d\theta$$

Since the characteristic function of the n-dimensional gaussian variate is

IV.3) $$\Phi(u_1 \ldots u_n) = \int dv_1 \ldots dv_n \; p(v_1 \ldots v_n) \exp i(v_1u_1 \ldots v_nu_n) =$$

$$= \exp im \sum_j u_j - \frac{1}{2} \sum_{i,k=1}^{n} \lambda_{ik} u_i u_k ; \text{ where } \lambda_{ik} = E\left[v_i v_k \right] = R(\theta_i-\theta_k)$$

We pass from IV.3) to IV.2) noting that

$$\Phi \left[\frac{\varphi_G(\theta_1)-1}{i} d\theta \ldots \frac{\varphi_G(\theta_n)-1}{i} d\theta \right] = \int dv_1 \ldots dv \; p(v_1 \ldots v_n)$$

$$\exp \left\{ \sum_{i=1}^{n} \left[\varphi_G(\theta_i)-1 \right] v_i \, d\theta_i \right\}$$

$$= \exp m \sum_{k=1}^{n} \left[\varphi_G(\theta_k)-1 \right] d\theta + \frac{1}{2} \sum_{i,k=1}^{n} \lambda_{ik} \left[\varphi_G(\theta_i)-1 \right] \left[\overline{\varphi_G(\theta_k)-1} \right] d\theta_i \, d\theta_k$$

$$= \exp \left\{ m \sum_{k=1}^{n} \left[\varphi_G(\theta_k)-1 \right] d\theta \right\} + \frac{1}{2} \sum_{i,k=1}^{n} R(\theta_i-\theta_k) \left[\varphi_G(\theta_i)-1 \right] \left[\overline{\varphi_G(\theta_k)-1} \right],$$

where $\varphi_G(\theta)$ replaces $\varphi_G(u_{kl},\theta)$ for simplification.

In a finite interval $[T,t]$ if $n \to \infty$, $d\theta \to 0$, since $\varphi_G(\theta)$ is always continuous, and also $R(\theta_j-\theta_k)$ (stationarity of gaussian process) we have:

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left[\varphi_G(\theta_k) - 1 \right] d\theta = \int_T^t \left[\varphi_G(\theta) - 1 \right] d\theta$$

$$\lim_{n \to \infty} \sum_{i,k=1}^{n} R(\theta_i - \theta_k) \left[\varphi_G(\theta_i) - 1 \right] \left[\varphi_G(\theta_k) - 1 \right] d\theta_j \, d\theta_j =$$

$$= \iint_T^t R(\theta - \theta') \left[\varphi_G(\theta) - 1 \right] \left[\varphi_G(\theta') - 1 \right] d\theta \, d\theta'$$

We may pass to $T = -\infty$ by conditions III.2) and III.3) and putting $\theta - \theta' = \tau$, $\theta = \theta$ we obtain

IV.4) $$\varphi_N(u,m) = \exp \left[m \int_{-\infty}^{+\infty} \left[\varphi_G(u,\theta) - 1 \right] d\theta + \frac{1}{2} \int_{-\infty}^{+\infty} d\theta \int_{-\infty}^{+\infty} d\tau \, R(\tau) . \right.$$
$$\left. \left[\varphi_G(u,\theta) - 1 \right] \left[\varphi_G(u,\theta+\tau) - 1 \right] \right]$$

When the contribution of the $V(\theta) < 0$ becomes small as m increases, the error could be explicitly calculated, this expression may be taken as a good approximation for the case $f\left[V(\theta) \right] = |V(\theta)|$.

We see that for the most part of the correlation functions which have a known analytical expression, e.g. gaussian markov process, brownian motion with respectively $R(\tau) = e^{-\tau}$, $R(\theta_1, \theta_2) = \text{Min}\,(\theta_1, \theta_2)$ this formula is not easily computable. But for pure white noise with $R(\tau) = \delta(\tau)$ (of which the use may be justified by distribution theory), we have immediately

IV.5) $$\varphi_N(u,m) = \exp \left\{ +m \int_{-\infty}^{+\infty} \left[\varphi_G(u,\theta) - 1 \right] d\theta + \frac{1}{2} \int_{-\infty}^{+\infty} \left[\varphi_G(u,\theta) - 1 \right]^2 d\theta \right\} .$$

Nevertheless, we note that in this simple case a direct procedure is suitable. In fact we have to calculate the integral:

$$I = \int_{-\infty}^{t} V(\theta)\left[1-\varphi_G(u,\theta)\right] d\theta = m\int_{-\infty}^{t}\left[1-\varphi_G(u,\theta)\right] d\theta + \int_{-\infty}^{t} v(\theta)\left[1-\varphi_G(u,\theta)\right] d\theta.$$

We know that this kind of white noise may be represented by

$$v(\theta)d\theta = dX(\theta) = \xi\sqrt{d\theta}$$

where $X(\theta)$ is the random function of the brownian motion and ξ a laplacian variate with mean value zero, standard deviation 1.

Now we sketch the calculation. We have:

$$I_2 = \int_{-\infty}^{t}\left[1-\varphi_G(u,\theta)\right] v(\theta)d\theta = \lim_{|d\theta|\to 0} \text{p.s.} \sum_i \left[1-\varphi_G(u,\theta_i)\right] dX(\theta_i)$$

But

$$\sum_i \left[1-\varphi_G(u,\theta_i)\right] dX(\theta_i) = \sum_i \xi_i \sqrt{\left[1-\varphi_G(u,\theta_i)\right]^2 d\theta_i}$$

$$= \xi\sqrt{\sum_i \left|1-\varphi_G(u,\theta)\right|^2 d\theta}$$

by application of the theorem on sum of Laplacian variates, and by passing to the limit:

$$I_2 = \xi\sqrt{\int_{-\infty}^{t}\left[1-\varphi_G(u,\theta)\right]^2 d\theta} \ .$$

Hence

$$\varphi_N(u,m) = \left\{\exp m\int_{-\infty}^{t}\left[\varphi_G(u,\theta)-1\right] d\theta\right\} E_\xi \exp\left\{+\xi\sqrt{\int_{-\infty}^{t}\left[1-\varphi_G(u,\theta)\right]^2 d\theta}\right\},$$

and using the distribution function of ξ we find again IV.5).

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To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

Alfred Nobel

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