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NEUTRON TRANSPORT IN A SPHERICAL REACTOR, A STUDY IN THE APPLICATION OF THE j_N APPROXIMATION OF THE MULTIPLE COLLISION METHOD

by

T. ASAOKA

1966

Joint Nuclear Research Center
Ispra Establishment - Italy

Reactor Physics Department
Reactor Theory and Analysis Service

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1. Introduction

By means of the multiple collision method developed by the author, neutron transport problems for a homogeneous slab have been solved with reasonable accuracy, the solution applying equally to large and small systems (Asaoka et al., 1964). This method is an analytical approach based on a viewpoint different from that of Boltzmann equation, namely, the life-cycle in contrast to the neutron-balance viewpoint.

As was shown in the previous paper, the essential point of the method lies not only in the adoption of a viewpoint different from the usual transport equation, but also in the introduction of discontinuity factors with which one can easily take into account the finiteness of the system and fix the point of measurement. As a result, problems for a finite system can be dealt with in a similar manner to those for an infinite system. In addition, it has been shown that the application of the method is greatly simplified by the appropriate employment of expansions in spherical Bessel functions. When this expansion is truncated beyond the N-th order spherical Bessel function, j_N, the resulting approximation has been called "the j_N approximation".

It was shown in the previous paper that these mathematical techniques arising from the life-cycle approach can be used to solve problems based on the neutron-balance viewpoint more easily.

The present work is concerned with a further development of the multiple collision method. By applying the above-mentioned mathematical techniques directly to an equation governing the balance of neutrons, a theory valid for spherical systems is obtained. A part of this work, connected with the critical condition and flux distribution for a bare sphere (in the constant cross-section approximation), has already been presented in a EURATOM report (Asaoka et al., 1963).

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2. General Formulations

We consider here, within the context of a multigroup (G energy-groups) model, a multiregion (M regions) spherical reactor in which the neutron scattering is spherically symmetric in the L system.

Let r be the radial co-ordinate, μ the cosine of the angle between the neutron velocity and direction of r, Σ_g and v_g the macroscopic total cross-section and speed of the neutrons in the g-th energy group respectively and $c(g'\to g)$ the mean number of secondary neutrons produced in the g-th group as a result of a collision in the g'-th group.

The number of neutrons which, due to collisions in the g'-th group, are born in the g-th group with directions in the range $(\mu, \mu+d\mu)$, positions in the spherical shell of volume $4\pi r'^2 dr'$ around r' and at times in the interval dt' around $t-t'$ is given by

$$\frac{c(g'\to g)}{2}\Sigma_{g'}v_{g'}n_{g'}(r', t-t')4\pi r'^2 dr' dt' d\mu .$$

Assuming that Σ_g and v_g are constant and independent of the spatial region, the probability that these neutrons travel for a time t' without further collision is $\exp(-\Sigma_g v_g t')$ and the radial co-ordinate after this time is expressed by the relation:

$$r^2 = r'^2 + (v_g t')^2 + 2r' v_g t' \mu .$$

Hence, the number of neutrons in the g-th group in the spherical shell $4\pi r^2 dr$ around r at time t can be written as

$$n_g(r,t)4\pi r^2 dr = \sum_{j=1}^{M}\int_{R_{j-1}}^{R_j} dr' 4\pi r'^2 \int_0^t dt' \int_{-1}^{1} \frac{d\mu}{2}\exp(-\Sigma_g v_g t')$$

$$\times\sum_{g'=1}^{G} c_j(g'\to g)\Sigma_{g'}v_{g'}n_{g'}(r', t-t')\left[\delta(\xi_g - r) + \delta(\xi_g + r)\right] dr, \qquad (1)$$

where R_{j-1} and R_j are the inner radius and outer radius of the j-th region, respectively, $C_j(g'\to g)$ stands for $C(g'\to g)$ for this j-th region and $\zeta_g=\sqrt{r'^2+(v_g t')^2+2r' v_g t'\mu}$.

Replacing μ by ζ_g , rewriting $\delta(\zeta_g \mp r)$ in the form of the Fourier representation:

$$\frac{\Sigma_g}{2\pi}\int_{-\infty}^{\infty} dz\, \exp[i\Sigma_g z(\pm\zeta_g - r)],$$

and performing the integration over ζ_g from $|r'-v_g t'|$ to $r'+v_g t'$, we get:

$$r v_g n_g(r,t)=\sum_{j=1}^{M}\left(\int_{-R_j}^{-R_{j-1}} dr' + \int_{R_{j-1}}^{R_j} dr'\right)\int_0^t \frac{dt'}{t'}\frac{1}{2\pi i}\frac{1}{\Sigma_g r}\int_{-\infty}^{\infty} dz\, e^{-i\Sigma_g r z}$$

$$\times\frac{\partial}{\partial z}\left[\frac{\sin(\Sigma_g v_g t' z)}{z} e^{i\Sigma_g r' z}\right] e^{-\Sigma_g v_g t'}\sum_{g'=1}^{G} C_j(g'\to g)\Sigma_{g'} v_{g'} r' n_{g'}(r', t-t'), \tag{2}$$

where the definition of $n_g(r,t)$ has been extended to $r<0$ by putting $n_g(-r,t)=n_g(r,t)$. Equation (2) can be written in the form:

$$r v_g n_g(r,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{z} e^{-i\Sigma_g r z}\int_0^t \frac{dt'}{t'} e^{-\Sigma_g v_g t'}\sin(\Sigma_g v_g z t')$$

$$\times\sum_{j=1}^{M}\left(\int_{-R_j}^{-R_{j-1}} dr' + \int_{R_{j-1}}^{R_j} dr'\right) e^{i\Sigma_g z r'}\sum_{g'=1}^{G} C_j(g'\to g)\Sigma_{g'} v_{g'} r' n_{g'}(r', t-t'). \tag{3}$$

Next, by taking the Laplace transform of $v_g n_g(r,t)e^{\Sigma_1 v_1 t}$, we get:

$$r L_g(r,s) = r\int_0^{\infty} dt\, e^{-st} v_g n_g(r,t) e^{\Sigma_1 v_1 t}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{z} e^{-i\Sigma_g z r}\tan^{-1}\left(\frac{\Sigma_g v_g z}{s-\Sigma_1 v_1+\Sigma_g v_g}\right)\sum_{g'=1}^{G} F\left(g'\to g, \frac{\Sigma_g}{\Sigma_1} z, s\right), \tag{4}$$

where

$$F(g'\to g,y,s)=\sum_{j=1}^{M}\left(\int_{-R_j}^{-R_{j-1}}dr'+\int_{R_{j-1}}^{R_j}dr'\right)e^{i\Sigma_1 y r'}c_j(g'\to g)\Sigma_{g'}r'L_{g'}(r',s)$$

$$=c_M(g'\to g)\Sigma_{g'}\int_{-R_M}^{R_M}dr'r'e^{i\Sigma_1 y r'}L_{g'}(r',s)$$

$$+\sum_{j=M-1}^{1}[c_j(g'\to g)-c_{j+1}(g'\to g)]\Sigma_{g'}\int_{-R_j}^{R_j}dr'r'e^{i\Sigma_1 y r'}L_{g'}(r',s). \tag{5}$$

It follows from equations (4) and (5) that the function $F(g'\to g,y,s)$ satisfies the following integral equation:

$$F\left(g\to g'',\frac{\Sigma_g}{\Sigma_1}y,s\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dz}{z}\tan^{-1}\left(\frac{\Sigma_g v_g z}{s-\Sigma_1 v_1+\Sigma_g v_g}\right)$$

$$\times\Big\{c_M(g\to g'')\Sigma_g R_M j_0[\Sigma_g R_M(y-z)]$$

$$+\sum_{j=M-1}^{1}[c_j(g\to g'')-c_{j+1}(g\to g'')]\Sigma_g R_j j_0[\Sigma_g R_j(y-z)]\Big\}\sum_{g'=1}^{G}F\left(g'\to g,\frac{\Sigma_g}{\Sigma_1}z,s\right), \tag{6}$$

where $j_n(x)$ is the n-th order spherical Bessel function.

If $F(g'\to g,y,s)$ is now expanded in spherical Bessel functions:

$$F(g\to g'',y,s)=\sum_{m=0}^{\infty}b_m^j(g\to g'',s)j_m(\Sigma_1 R_j y),$$
$$j=1,2,\cdots,M-1 \text{ or } M, \tag{7}$$

$n_g(r,t)$ can be transformed into the following form derived from equation (4):

$$r v_g n_g(r,t)e^{\Sigma_1 v_1 t}$$

$$=\sum_{m=0}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{z}e^{-i\Sigma_g z r}j_m(\Sigma_g R_j z)\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}ds\,e^{st}\tan^{-1}\left(\frac{\Sigma_g v_g z}{s-\Sigma_1 v_1+\Sigma_g v_g}\right)\sum_{g'=1}^{G}b_m^j(g'\to g,s). \tag{8}$$

The asymptotic behaviour of $n_g(r,t)$ as $t\to\infty$ can thus be written:

$$r v_g n_g(r,t) \sim e^{\Sigma_1 \nu_1 (s_1-1)t} \sum_{m=0}^{\infty} B_m^j(g,s_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{z} \tan^{-1} z \, e^{-i\Sigma_g r' z} j_m(\Sigma_g R_j' z), \tag{9}$$

where

$$\Sigma_g r' = \Sigma_g r \left[1 + (s_1-1) \frac{\Sigma_1 \nu_1}{\Sigma_g v_g} \right], \quad \Sigma_g R_j' = \Sigma_g R_j \left[1 + (s_1-1) \frac{\Sigma_1 \nu_1}{\Sigma_g v_g} \right],$$

and $s = \Sigma_1 \nu_1 s_1$ is a pole of the quantity $B_m^j(g,s) = \sum_{g'=1}^{G} b_m^j(g'\to g, s)$ which has the largest real part and $B_m^j(g,s_1)$ is the residue at $s=\Sigma_1\nu_1 s_1$. As is easily seen from equation (9), only the terms with odd values of m remain on the right hand side because $r n_g(r,t)$ is an odd function of r. The explicit expression for the integral (on the right side of equation (9)) with odd values of m is shown in the Appendix.

According to equation (7), a relation between the coefficients $b_m^j(g\to g'', s)$ with different values of j can be obtained in the form:

$$b_m^k(g\to g'', s) = (2m+1) \sum_{n=0}^{\infty} b_n^j(g\to g'', s)\, I_{jk}(n,m), \tag{10}$$

where

$$I_{jk}(n,m) = \frac{\alpha_k}{\pi} \int_{-\infty}^{\infty} dy\, j_m(\alpha_k y) j_n(\alpha_j y)$$

$$= \begin{cases} \dfrac{1}{(\sqrt{2})^{n-m}} \left(\dfrac{\alpha_k}{\alpha_j}\right)^{m+1} \displaystyle\sum_{l=0}^{\frac{n-m}{2}} \frac{(-1)^l (m+n+2l-1)!!}{(2m+2l+1)!! \left(\frac{n-m}{2}-l\right)! \, l!} \left(\frac{\alpha_k}{\alpha_j}\right)^{2l}, \\ \qquad\qquad \alpha_j \geq \alpha_k,\ n \geq m \text{ and } m+n = \text{even}, \\ \dfrac{1}{(\sqrt{2})^{m-n}} \left(\dfrac{\alpha_j}{\alpha_k}\right)^{n} \displaystyle\sum_{l=0}^{\frac{m-n}{2}} \frac{(-1)^l (m+n+2l-1)!!}{(2n+2l+1)!! \left(\frac{m-n}{2}-l\right)! \, l!} \left(\frac{\alpha_j}{\alpha_k}\right)^{2l}, \\ \qquad\qquad \alpha_k \geq \alpha_j,\ m \geq n \text{ and } m+n = \text{even}, \\ 0, \quad \text{otherwise}, \end{cases}$$

in which α_k and α_j stand for $\Sigma_1 R_k$ and $\Sigma_1 R_j$ respectively (the function $I_{jk}(m,n)$ is independent of Σ_1 and when $\alpha_j = \alpha_k$, that is, $j=k$, the expression reduces to $\delta(m-n)/(2m+1)$). Now, by applying Gegenbauer's addition theorem:

$$j_0(y-z) = \sum_{p=0}^{\infty} (2p+1) j_p(y) j_p(z)$$

to the kernel on the right side of equation (6) and using the orthogonality relation for spherical Bessel functions, the following infinite set of linear equations satisfied by the $B_m^j(g, \Sigma_1 \nu_1 \delta_1)$ can be derived

$$\frac{1}{2m+1} B_m^j(g, \Sigma_1 \nu_1 \delta_1) = \sum_{n=0}^{\infty} \sum_{g'=1}^{G} E'_{mnj}(g' \to g) B_n^j(g', \Sigma_1 \nu_1 \delta_1), \tag{11}$$

where

$$E'_{mnj}(g' \to g) = C'_M(g' \to g) \frac{R_j}{R_M} \sum_{p,q=0}^{\infty} (2p+1)(2q+1) I_{jM}(m,p) I_{jM}(n,q) J'_{gM}(p,q)$$
$$+ \sum_{k=M-1}^{1} \left[C'_k(g' \to g) - C'_{k+1}(g' \to g) \right] \frac{R_j}{R_k} \sum_{p,q=0}^{\infty} (2p+1)(2q+1) I_{jk}(m,p) I_{jk}(n,q) J'_{gk}(p,q), \tag{12}$$

in which

$$C'_k(g \to g') = C_k(g \to g') \Big/ \left| 1 + (\delta_1 - 1) \frac{\Sigma_1 \nu_1}{\Sigma_g \nu_g} \right|, \tag{13}$$

$$J'_{gj}(p,q) = \frac{\Sigma_g R_j}{\pi} \int_{-\infty}^{\infty} dy \frac{\tan^{-1} y}{y} j_p(\Sigma_g R_j y) j_q(\Sigma_g R_j y). \tag{14}$$

The explicit expressions for the $J'_{gj}(p,q)$ with p and $q \leq 7$ have been given in the appendix of a previous paper (Asaoka et al., 1964). Since $J'_{gj}(p,q)=J'_{gj}(q,p)$, it follows from equation (12) that $E'_{mnj}(g'\to g) = E'_{nmj}(g'\to g)$.

The condition that $B^i_m(g,\Sigma_i V_i s_i)$ should diverge beyond all bounds is thus

$$\det\left|(2m+1)E'_{mnj}(g'\to g)-\delta_{gg'}\delta_{mn}\right|=0,\quad g,g'=1,2,\cdots,G \text{ for } m,n=1,3,5,\cdots, \tag{15}$$

which gives a relation between the physical properties of a reactor (as contained for example in the parameter C), geometrical dimension R and asymptotic time-constant s_i-1 (see equation (9)).

For a critical reactor, s_1 must be equal to unity and equation (15) with $s_1=1$ therefore gives the critical condition, that is a relation between the physical properties and geometrical dimension of a critical reactor. In order to obtain the value of the effective multiplication factor k_{eff} for a given reactor, $C_k(g\to g')$ is divided into two parts; the scattering part $C_{sk}(g\to g')=\Sigma_{sk}(g\to g')/\Sigma_g$ and the fission part $C_{fk}(g\to g') = \chi_{g'}(\nu\Sigma_f)_g/\Sigma_g$. Using this separation the value of k_{eff} is obtained by solving equation (15) with $s_1=1$ and

$$C_k(g\to g')=C_{sk}(g\to g')+C_{fk}(g\to g')/k_{eff}.$$

The ratios between the $B^i_m(g,\Sigma_i V_i s_i)$'s can now be determined by the use of equations (11) and (15) for any of the above-mentioned problems, that is, the evaluation of the time-constant, critical condition or the value of k_{eff}. Having thus obtained the B^i_m's, the flux distribution or neutron spectrum can be obtained from equation (9) for each problem.

3. Numerical Results for a Bare Sphere and Discussion

For a bare sphere (M = 1), equations (9), (11) and (15) reduce to

$$r v_g n_g(r,t) \sim e^{\Sigma_1 v_1 (s_1 - 1) t} \sum_{m=0}^{\infty} B_m(g, s_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{z} \tan^{-1} z \, e^{-i \Sigma_g r' z} j_m(\Sigma_g R' z), \tag{9'}$$

$$\frac{1}{2m+1} B_m(g, \Sigma_1 v_1 s_1) = \sum_{n=0}^{\infty} \sum_{g'=1}^{G} c'(g' \to g) J'_{g'}(m,n) B_n(g', \Sigma_1 v_1 s_1), \tag{11'}$$

$$\det \left| (2m+1) c'(g' \to g) J'_{g'}(m,n) - \delta_{g'g} \delta_{mn} \right| = 0, \quad g, g' = 1, 2, \cdots, G \text{ for } m, n = 1, 3, 5, \cdots. \tag{15'}$$

In a one-group model (G = 1), these equations reduce even further to the following results respectively:

$$r v n(r,t) \sim e^{\Sigma v (s_1 - 1) t} \sum_{m=0}^{\infty} B_m(s_1) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{z} \tan^{-1} z \, e^{-i \Sigma r' z} j_m(\Sigma R' z), \tag{9''}$$

$$\frac{1}{2m+1} B_m(\Sigma v s_1) = \sum_{n=0}^{\infty} c' J'(m,n) B_n(\Sigma v s_1), \tag{11''}$$

$$\det \left| (2m+1) c' J'(m,n) - \delta_{mn} \right| = 0, \quad m, n = 1, 3, 5, \cdots, \tag{15''}$$

which for a critical system ($s_1 = 1$) have already been derived in a previous report (Asaoka et al., 1963).

3.1. Numerical results in a one-group model

The numerical values of c for critical bare spheres with various values of ΣR between 0.005 and 50 are given in Table 1. Even the results of the j_3 approximation (containing just two terms on the right hand side of equation (9″)) have shown that the difference between Carlson's result and ours is indistinguishable on a figure (Asaoka et al., 1963). By using a quadratic trial function in the single-iteration moments method, Carlson calculated the critical radius for various values of c between 1.1 and 3.0 (Carlson, 1949). In Table 1 are also shown the values of the so-called extrapolation distance d, the distance between the boundary and the point at which the asymptotic neutron flux expressed in the form $\sin[\pi r/(R+d)]/r$ would extrapolate to zero. These are calculated from the following equation (see Asaoka, 1961):

$$\Sigma d = \frac{\pi}{\sqrt{3(1-1/c)}} - \Sigma R. \tag{16}$$

In the j_3 approximation, the asymptotic expressions for these quantities are

$$\frac{1}{c} \sim \begin{cases} \frac{5}{144}(15+\sqrt{57})\alpha - \frac{2}{9}\left(1+\frac{1226}{11875}\sqrt{57}\right)\alpha^2, & \alpha \ll 1, \\ 1-\frac{7}{2\alpha^2}+\frac{1799}{240\alpha^3}, & \alpha \gg 1, \end{cases} \tag{17}$$

$$\Sigma d \sim \begin{cases} \frac{\pi}{\sqrt{3}} - \left(1-5\pi\frac{5\sqrt{3}+\sqrt{19}}{288}\right)\alpha, & \alpha \ll 1, \\ -\left(1-\frac{2\pi}{\sqrt{42}}\right)\alpha + \frac{\pi}{\sqrt{42}}\frac{257}{120}, & \alpha \gg 1, \end{cases} \tag{18}$$

where $\alpha = \Sigma R$. For large α, the corresponding expressions in the j_5 approximation are as follows:

$$\begin{aligned}
&\frac{1}{c} \sim 1 - \frac{10-3\sqrt{5}}{\alpha^2} + \frac{6975-2769\sqrt{5}}{168\alpha^3}, \\
&\Sigma d \sim -\left(1 - \frac{\pi}{\sqrt{30-9\sqrt{5}}}\right)\alpha + \frac{\pi}{\sqrt{30-9\sqrt{5}}}\,\frac{171-41\sqrt{5}}{112} \\
&\qquad = -0.00029288\alpha + 0.7080177.
\end{aligned} \tag{19}$$

Table 2 shows the numerical values of the flux distribution in critical spheres with ΣR = 0.005, 0.5 and 10. For large ΣR , the results obtained from elementary diffusion theory by using Σd = 0.71 show good agreement with ours except in the region within about a mean free path from the boundary. The asymptotic expressions for the flux distribution in the j_3 approximation are given by

$$\frac{2i\alpha R}{B_1} v n(\xi) \sim \begin{cases}
\frac{5}{18}(\sqrt{57}-3)\alpha\left[1 - \frac{25+3\sqrt{57}}{8}\left(\frac{\xi}{\alpha}\right)^2\right], & \xi, \alpha \ll 1, \\
\frac{5}{144}(21-\sqrt{57})\alpha + \frac{5}{24}(9-\sqrt{57})\left[\ln\frac{2\alpha}{\xi_0} + \frac{\sqrt{57}-1}{4}\right]\xi_0, & \xi_0 = \alpha - \xi,\ \alpha \ll 1, \\
\frac{5}{2} - \frac{2}{\alpha} - \frac{5}{\alpha^2}\left(1 + \frac{1}{2}\xi^2\right), & \xi \ll 1,\ \alpha \gg 1, \\
\frac{23}{12\alpha} + \frac{2}{3\alpha}\left(\ln\frac{1}{\xi_0} - \gamma + \frac{17}{2}\right)\xi_0, & \xi_0 = \alpha - \xi \ll 1,\ \alpha \gg 1,
\end{cases} \tag{20}$$

where $\xi = \Sigma r$ and γ is the Euler-Mascheroni constant. For large α , the asymptotic expressions in the j_5 approximation can be written in the forms:

$$\frac{2i\alpha R}{B_1} v n(\xi) \sim \begin{cases}
\frac{189\sqrt{5}-140}{88} - \frac{36090-4881\sqrt{5}}{3872\alpha} - \frac{315\sqrt{5}-490}{22\alpha^2}\left(1 + \frac{1}{2}\xi^2\right), & \xi \ll 1, \\
\frac{235-67\sqrt{5}}{44\alpha} + \frac{15-\sqrt{5}}{11\alpha}\left(\ln\frac{1}{\xi_0} - \gamma + \frac{23-7\sqrt{5}}{2}\right)\xi_0, & \xi_0 \ll 1.
\end{cases} \tag{21}$$

Furthermore, the extrapolated end-point, that is, the distance beyond the boundary of the medium at which the asymptotic flux (due to the pole $z = 0$ in equation (9")) vanishes (see Appendix), takes the value $(3+\sqrt{5})/7 = 0.7480$ in the j_5 approximation for infinite α, this being measured in units of the total mean free path.

As can be seen from the above-mentioned results, the infinite series on the right hand side of equation (9") converges faster for smaller ΣR and hence the results obtained from the j_3 approximation can be regarded as accurate for the smaller values of ΣR. On the other hand, for large ΣR, the infinite series does not converge very quickly but the j_5 approximation will give a sufficiently accurate result for practical purposes.

Additional test calculations with a one-group model were performed on several bare spheres and the results were compared with the exact ones and with those of the S_N calculations. In Table 3 are shown the results for bare spheres with c equal to 1.02, 1.2 or 1.8. The exact critical radius ΣR_c is given in the first column of the table and the critical radius obtained from the S_N approximation with various values of N is shown in the second column. The deviation from the exact result is shown in parenthesis. All these values are obtained from a paper by C.E. Lee (1962). In the third column are shown the critical radii obtained from the j_N approximation with several values of N, the deviation from the exact result being given in parenthesis. It will be noted from these results that the j_3 approximation gives a reasonably accurate result while the j_5 approximation is exact for all systems of interest. In the fourth and fifth columns of Table 3 are shown respectively the values of the time-constant $\lambda_1 - 1$ (see equation (9")) and k_{eff} for a fixed value of the radius. These were calculated by means of the j_N approximation.

The asymptotic expressions for the critical radius $\alpha_c = \Sigma R_c$ are, in the j_3 approximation,

$$\alpha_c \sim \begin{cases} \dfrac{6(15-\sqrt{57})}{35c}\left[1+\left(1-\dfrac{1753\sqrt{57}}{208715}\right)\dfrac{105456}{153125c}\right], & c \gg 1, \\ \sqrt{\dfrac{7/2}{1-1/c}}\left[1-\dfrac{257}{840}\sqrt{\dfrac{7}{2}}\sqrt{1-\dfrac{1}{c}}\right], & 1-\dfrac{1}{c} \ll 1, \end{cases} \tag{22}$$

and in the j_5 approximation,

$$\alpha_c \sim \sqrt{\frac{10-3\sqrt{5}}{1-1/c}}\left[1-\frac{1881-451\sqrt{5}}{1232\sqrt{10-3\sqrt{5}}}\sqrt{1-\frac{1}{c}}\right], \qquad 1-\frac{1}{c} \ll 1. \tag{23}$$

In addition, the asymptotic expressions in the j_3 approximation for the time-constant and k_{eff} of a sphere with a radius $\alpha=\alpha_c(1+\varepsilon)$ with $|\varepsilon| \ll 1$ are

$$\lambda_1 - 1 \sim \begin{cases} \frac{3261171875}{2236912896}\left(1+\frac{1753}{208715}\sqrt{57}\right)c\varepsilon = 1.550336\, c\varepsilon, & c \gg 1, \\ 2(c-1)\varepsilon\left[1-2\varepsilon(1-2\varepsilon)+2(c-1)(1-6\varepsilon)+4(c-1)^2\right], & \\ & c-1 \ll 1, \end{cases} \tag{24}$$

$$k_{eff} - 1 \sim \begin{cases} \varepsilon\frac{c}{c_f}\left[1+\varepsilon\frac{c_s}{c_f}-\frac{105456}{153125}\left(1-\frac{1753\sqrt{57}}{208715}\right)\frac{1+\varepsilon}{c}\right], & c \gg 1, \\ 2\varepsilon(c-1)\frac{c}{c_f}\left[1-\frac{3}{2}\varepsilon-\frac{257}{240}\sqrt{\frac{2}{7}}\sqrt{1-\frac{1}{c}}(1-3\varepsilon)-\frac{66049}{33600}\left(1-\frac{1}{c}\right)\left(1-\frac{9}{4}\varepsilon\right)\right. & \\ \qquad \left. -\frac{16974593}{3456000}\sqrt{\frac{2}{7}}\left(1-\frac{1}{c}\right)^{3/2}+2\varepsilon\frac{c_s c}{c_f}\left(1-\frac{1}{c}\right)\right], & c-1 \ll 1, \end{cases} \tag{25}$$

where $(c-1)c/c_f$ can be rewritten in terms of $k_\infty=\nu\Sigma_f/\Sigma_a$ in the form:

$$(c-1)\frac{c}{c_f} = c\,\frac{k_\infty - 1}{k_\infty}.$$

For small $c-1$, the asymptotic expression for k_{eff} in the j_5 approximation is:

$$k_{eff}-1 \sim 2\varepsilon(c-1)\frac{c}{c_f}\left[1-\frac{3}{2}\varepsilon-\frac{171-41\sqrt{5}}{112\sqrt{10-3\sqrt{5}}}\sqrt{1-\frac{1}{c}}(1-3\varepsilon)\right.$$

$$\left.-\frac{249195-40923\sqrt{5}}{172480}\left(1-\frac{1}{c}\right)\left(1-\frac{9}{4}\varepsilon\right)-\frac{4250130-1434569\sqrt{5}}{689920\sqrt{10-3\sqrt{5}}}\left(1-\frac{1}{c}\right)^{3/2}+2\varepsilon\frac{c_s c}{c_f}\left(1-\frac{1}{c}\right)\right], \tag{26}$$

while the expression for $\Delta_1 - 1$ in the j_5 approximation is the same as the second of equations (24).

From equations (24) and (25), the mean lifetime ℓ of neutrons can be obtained in the following form by using the relation $\ell = (k_{eff} - 1)/[\Sigma\nu(\Delta_1 - 1)]$:

$$
\ell \sim \begin{cases}
\frac{1}{\Sigma\nu C_f}\frac{105456}{153125}\left(1-\frac{1753\sqrt{57}}{208715}\right)\left[1+\varepsilon\frac{C_4}{C_f}-\frac{105456}{153125}\left(1-\frac{1753\sqrt{57}}{208715}\right)\frac{1+\varepsilon}{c}\right] \\
\quad = \frac{0.64502148}{\nu\Sigma_a k_\infty}\left[1+\varepsilon\frac{C_4}{C_f}-0.64502148\frac{1+\varepsilon}{c}\right], \quad c \gg 1, \\
\frac{1}{\Sigma\nu C_f/c}\left[1+\frac{1}{2}\varepsilon(1-6\varepsilon)-\frac{257}{240}\sqrt{\frac{2}{7}}\sqrt{1-\frac{1}{c}}(1-\varepsilon)-\frac{66049}{33600}\left(1-\frac{1}{c}\right)\left(1-\frac{\varepsilon}{4}\right)\right. \\
\quad \left.-2c\left(1-\frac{1}{c}\right)\left(1-\frac{7}{2}\varepsilon\right)-\frac{16974593}{3456000}\sqrt{\frac{2}{7}}\left(1-\frac{1}{c}\right)^{3/2}+\frac{257}{120}\sqrt{\frac{2}{7}}c\left(1-\frac{1}{c}\right)^{3/2}+2\varepsilon\frac{C_4 c}{C_f}\left(1-\frac{1}{c}\right)\right] \\
\quad = \frac{1}{\nu\Sigma_a k_\infty/c}\left[1+\frac{\varepsilon}{2}(1-6\varepsilon)-0.57238449\sqrt{1-\frac{1}{c}}(1-\varepsilon)-1.96574405\left(1-\frac{1}{c}\right)\left(1-\frac{\varepsilon}{4}\right)\right. \\
\quad \left.-2c\left(1-\frac{1}{c}\right)\left(1-\frac{7}{2}\varepsilon\right)-2.62537663\left(1-\frac{1}{c}\right)^{3/2}+1.14476898\,c\left(1-\frac{1}{c}\right)^{3/2}+2\varepsilon\frac{C_4 c}{C_f}\left(1-\frac{1}{c}\right)\right], \\
\qquad 1-\frac{1}{c} \ll 1,
\end{cases} \tag{27}
$$

and the expression for small $1 - 1/c$ in the j_5 approximation is given by

$$
\begin{aligned}
\ell \sim \frac{1}{\Sigma\nu C_f/c}&\left[1+\frac{1}{2}\varepsilon(1-6\varepsilon)-\frac{171-41\sqrt{5}}{112\sqrt{10-3\sqrt{5}}}\sqrt{1-\frac{1}{c}}(1-\varepsilon)-\frac{249175-40923\sqrt{5}}{172480}\left(1-\frac{1}{c}\right)\left(1-\frac{\varepsilon}{4}\right)\right. \\
&-2c\left(1-\frac{1}{c}\right)\left(1-\frac{7}{2}\varepsilon\right)-\frac{4250130-1434569\sqrt{5}}{689920\sqrt{10-3\sqrt{5}}}\left(1-\frac{1}{c}\right)^{3/2}+\frac{171-41\sqrt{5}}{56\sqrt{10-3\sqrt{5}}}c\left(1-\frac{1}{c}\right)^{3/2} \\
&\left.+2\varepsilon\frac{C_4 c}{C_f}\left(1-\frac{1}{c}\right)\right] \\
= \frac{1}{\nu\Sigma_a k_\infty/c}&\left[1+\frac{\varepsilon}{2}(1-6\varepsilon)-0.39035061\sqrt{1-\frac{1}{c}}(1-\varepsilon)-0.91424159\left(1-\frac{1}{c}\right)\left(1-\frac{\varepsilon}{4}\right)\right. \\
&\left.-2c\left(1-\frac{1}{c}\right)\left(1-\frac{7}{2}\varepsilon\right)-0.83270777\left(1-\frac{1}{c}\right)^{3/2}+0.78070122\,c\left(1-\frac{1}{c}\right)^{3/2}+2\varepsilon\frac{C_4 c}{C_f}\left(1-\frac{1}{c}\right)\right].
\end{aligned} \tag{28}
$$

The expressions for $1 - 1/c \ll 1$ show that, as expected, $\ell \sim 1/(\nu\Sigma_a)$ for an infinite system.

Tables 4 and 5 show the flux distributions in spheres calculated by means of the j_3 and j_7 approximations respectively. In each case, C takes on the values 1.02 and 1.8 and the flux is evaluated three times on the basis of criticality, time-constant and k_{eff} calculations. An interesting feature to be seen from these tables is that the flux obtained from the time-constant calculation with a negative value of $s_1 - 1$ (subcritical system) decreases more slowly as the radial co-ordinate increases than that obtained from the k_{eff} calculation. This tendency can be demonstrated analytically for large $\alpha = \Sigma R$ by using equation (20) or (21).

The flux distribution in the time-constant calculation is given by equation (20) or (21) by replacing α and ξ by $\alpha' = \alpha[1+(s_1-1)]$ and $\xi' = \xi[1+(s_1-1)]$, respectively, while that in the k_{eff} calculation is given by the unmodified equation. Hence, in the time-constant calculation the ratio of the flux value at the outer boundary of the sphere with a large radius α to that at the centre is given by

$$\frac{n(\alpha)}{n(0)} \sim \begin{cases} \dfrac{23}{30\alpha(1+(s_1-1))}\left[1+\dfrac{4}{5\alpha(1+(s_1-1))}\right], & \text{in the } j_3 \text{ approximation,} \\[2ex] \dfrac{182\sqrt{5}-158}{413\alpha(1+(s_1-1))}\left[1+\dfrac{1143+15942\sqrt{5}}{18172\alpha(1+(s_1-1))}\right], & \text{in the } j_5 \text{ approximation.} \end{cases} \tag{29}$$

This means that, for large α, the ratio in the time-constant calculation is obtained approximately by dividing that in the k_{eff} calculation by a factor of $1+(s_1-1)$. On the other hand, for small α, the first term of the asymptotic expression for this ratio is a constant. $(1+3\sqrt{57})/64 = 0.369523949$, in the j_3 approximation (see equation (20)) and hence the flux distribution stays nearly the same independently of whether the distribution is obtained from the time-constant or k_{eff} calculation.

3.2. Numerical examples in a multigroup model

As numerical examples of a multigroup model, calculations were performed on two fast neutron critical assemblies, Godiva and Jezebel. The numerical results are summarised in Table 6. For all these calculations, the 18-group (10 MeV-thermal) set of cross-sections of LASL (Mills, 1959) is adopted in the transport approximation. A 10-group model has been constructed by extracting just the higher 10 energy-groups out of the 18 groups. Since the contribution of slow neutrons to the reactor behaviour can be neglected, it is better to reduce the number of energy-groups G by cutting out the lower energies so that the rounding error in the evaluation of a determinant of order $G(N+1)/2$ can be reduced (see equation (15')), N being the order of j_N approximation.

Since the radii of these assemblies are equivalent to 2-3 fast neutron mean free paths, the S_4 approximation will overestimate k_{eff} slightly while the j_3 approximation should give an accurate value with a very slight underestimation (see Table 3). Although this tendency cannot be seen clearly from the results shown in Table 6 because of the transport approximation and the rounding error in the j_N approximation (the transport approximation has resulted in an overestimate of the number of secondary neutrons per collision $C(g \to g')$), all the calculated values except those of the j_1 approximation agree quite well with the experimental results (Hansen, 1958 and Jarvis et al., 1960). The mean lifetime of prompt neutrons in the j_N approximation was calculated by using the formula $\ell = (k_{eff} - 1)/[\Sigma_1 v_1 (\lambda_1 - 1)]$ (see equation (9')), while that in the S_4 approximation is given by the total importance divided by the rate of destruction of importance (see Goertzel, 1955).

In Table 7, our results for the number of leakage neutrons (the total number of fission neutrons produced in the reactor has been normalized to unity) are compared with those of the S_4 calculation and with the experimental values (Stewart, 1960). The experimentally observed total number of leakage neutrons with energies between 3 and 0.4 MeV has been normalized to the value of the S_4 calculation and the error shown is estimated from the values given by Stewart as a result of counting statistics alone. The experimental values are approximately extrapolated to 10 MeV and to 0.1 MeV (for Godiva) by the author, though the observed upper limit was about 9 MeV and the lower limit was 0.2 MeV (for Godiva). As is seen from this table, the j_N results coincide quite well with those of the S_4 calculation for both the assemblies. In addition, they agree satisfactorily with the observed values in the case of Godiva, though the calculated values depend on the adopted nulcear cross-sections. For Jezebel, the calculated values for the highest energy-group are too small but the agreement is, on the whole, reasonably good (the j_5 approximation has failed to give satisfactory values because of the rounding error arising in the course of calculating the flux distribution (see Table 8)).

Table 8 shows the calculated neutron spectra (the total number of fission neutrons produced being normalized to unity) at the centre of the assemblies (r/R = 0) and near the boundary (r/R = 0.95). It is seen here again that the j_N results coincide very well with those obtained from the S_4 calculation, if we exclude the j_5 values for Jezebel as mentioned already.

4. Conclusion

The neutron transport problems for a spherical reactor dealt with in this report have been solved satisfactorily by the j_N approximation (or the multiple collision method). In particular, it has been shown that the j_3 approximation (keeping just first two terms of expansions in spherical Bessel functions) gives results (critical condition, k_{eff}, mean lifetime of neutrons, neutron spectrum etc.) comparable in accuracy to the S_4 approximation of transport theory.

The computer code which has been used is designed to obtain the results of an (up to) 18 energy-group model in (up to) the j_5 approximation for a bare sphere. As will be clear from the formalism presented above, this code calculates, in contrast to the S_N code, first the eigenvalue (that is the critical radius, k_{eff} or time-constant) and then the flux distribution (or neutron spectrum) corresponding to this eigenvalue. A typical running time on the IBM-7090 is nearly 15 min. to obtain all the three eigenvalues by the use of the j_3 approximation and 18-group model. The k_{eff} and the corresponding flux calculations take about 10 min.

A computer code for a two-region sphere is now under test running which seems to show that the infinite series in spherical Bessel functions converges rather slowly in this case. Later, it is hoped to extend the method to more general problems.

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Appendix

Explicit expression for $\frac{1}{\pi}\int_{-\infty}^{\infty} dz\, e^{-i\xi z} \frac{\tan^{-1} z}{z} j_{2m+1}(\alpha z)$

Reforming the integration, we get:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dz\, e^{-i\xi z} \frac{\tan^{-1} z}{z} j_{2m+1}(\alpha z)$$

$$= i\mathcal{R}\left[\left(e^{i(\alpha-\xi)z} - e^{i(\alpha+\xi)z}\right)\frac{\tan^{-1} z}{z} j^{1}_{2m+1}(\alpha z)\right]_{z=0}$$

$$+ \int_{1}^{\infty} \frac{dz}{z}\left(e^{-(\alpha-\xi)z} - e^{-(\alpha+\xi)z}\right) j^{1}_{2m+1}(i\alpha z),$$

where

$$j_{2m+1}(z) = j^{1}_{2m+1}(z)e^{iz} + j^{2}_{2m+1}(z)e^{-iz}$$

$$\left(\text{note that } j^{2}_{2m+1}(z) = -j^{1}_{2m+1}(-z)\right),$$

and $\mathcal{R}\left(f(z)\right)_{z=0}$ stands for the residue of $f(z)$ at $z = 0$.

Hence, the explicit expression can be obtained in the following form by introducing the abbreviation $E_n' = E_n(\alpha-\xi) - E_n(\alpha+\xi)$, where $E_n(x) = \int_1^{\infty} dz\, e^{-xz} z^{-n}$:

$$m=0;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(1-\frac{\alpha}{2\xi}E_2'-\frac{1}{2\xi}E_3'\right),$$

$$m=1;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(\frac{3}{2}-\frac{5}{2}\frac{\xi^2}{\alpha^2}-\frac{5}{\alpha^2}+\frac{\alpha}{2\xi}E_2'+\frac{3}{\xi}E_3'+\frac{15}{2\alpha\xi}E_4'+\frac{15}{2\alpha^2\xi}E_5'\right),$$

$$m=2;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(\frac{15}{8}-\frac{35}{4}\frac{\xi^2}{\alpha^2}+\frac{63}{8}\frac{\xi^4}{\alpha^4}-\frac{35}{2\alpha^2}+\frac{105}{2\alpha^2}\frac{\xi^2}{\alpha^2}+\frac{189}{\alpha^4}\right.$$
$$\left.-\frac{\alpha}{2\xi}E_2'-\frac{15}{2\xi}E_3'-\frac{105}{2\alpha\xi}E_4'-\frac{210}{\alpha^2\xi}E_5'-\frac{945}{2\alpha^3\xi}E_6'-\frac{945}{2\alpha^4\xi}E_7'\right),$$

$$m=3;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(\frac{35}{16}-\frac{315}{16}\frac{\xi^2}{\alpha^2}+\frac{693}{16}\frac{\xi^4}{\alpha^4}-\frac{429}{16}\frac{\xi^6}{\alpha^6}-\frac{315}{8\alpha^2}+\frac{1155}{4\alpha^2}\frac{\xi^2}{\alpha^2}-\frac{3003}{8\alpha^2}\frac{\xi^4}{\alpha^4}+\frac{2079}{2\alpha^4}\right.$$
$$-\frac{9009}{2\alpha^4}\frac{\xi^2}{\alpha^2}-\frac{19305}{\alpha^6}+\frac{\alpha}{2\xi}E_2'+\frac{14}{\xi}E_3'+\frac{189}{\alpha\xi}E_4'+\frac{1575}{\alpha^2\xi}E_5'+\frac{17325}{2\alpha^3\xi}E_6'$$
$$\left.+\frac{31185}{\alpha^4\xi}E_7'+\frac{135135}{2\alpha^5\xi}E_8'+\frac{135135}{2\alpha^6\xi}E_9'\right).$$

When α is small, these expressions can be reduced to the following asymptotic formulae:

$$m=0;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(\frac{\alpha^2-\xi^2}{4\xi}\ln\frac{\alpha+\xi}{\alpha-\xi}+\frac{\alpha}{2}-\frac{\alpha^2}{2}\left(1-\frac{1}{3}\frac{\xi^2}{\alpha^2}\right)\right),$$

$$m=1;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{\alpha^2-\xi^2}{16\xi}\left(1-5\frac{\xi^2}{\alpha^2}\right)\ln\frac{\alpha+\xi}{\alpha-\xi}+\frac{13}{24}\alpha\left(1-\frac{15}{13}\frac{\xi^2}{\alpha^2}\right)-\frac{\alpha^2}{8}\left(1-\frac{\xi^2}{\alpha^2}\right)^2\right],$$

$$m=2;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{\alpha^2-\xi^2}{32\xi}\left(1-14\frac{\xi^2}{\alpha^2}+21\frac{\xi^4}{\alpha^4}\right)\ln\frac{\alpha+\xi}{\alpha-\xi}+\frac{113}{240}\alpha\left(1-\frac{420}{113}\frac{\xi^2}{\alpha^2}+\frac{315}{113}\frac{\xi^4}{\alpha^4}\right)\right.$$
$$\left.-\frac{\alpha^2}{16}\left(1-\frac{\xi^2}{\alpha^2}\right)^2\left(1-3\frac{\xi^2}{\alpha^2}\right)\right],$$

$$m=3;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{5}{256}\frac{\alpha^2-\xi^2}{\xi}\left(1-27\frac{\xi^2}{\alpha^2}+99\frac{\xi^4}{\alpha^4}-\frac{429}{5}\frac{\xi^6}{\alpha^6}\right)\ln\frac{\alpha+\xi}{\alpha-\xi}\right.$$
$$+\frac{1873}{4480}\alpha\left(1-\frac{14273}{1873}\frac{\xi^2}{\alpha^2}+\frac{27335}{1873}\frac{\xi^4}{\alpha^4}-\frac{15015}{1873}\frac{\xi^6}{\alpha^6}\right)$$
$$\left.-\frac{5}{128}\alpha^2\left(1-\frac{\xi^2}{\alpha^2}\right)^2\left(1-\frac{22}{3}\frac{\xi^2}{\alpha^2}+\frac{143}{15}\frac{\xi^4}{\alpha^4}\right)\right].$$

Furthermore, the asymptotic expressions for small ξ are:

$$m=0;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left(1-e^{-\alpha}-\frac{2+\alpha}{6\alpha}e^{-\alpha}\xi^2-\frac{8+8\alpha+4\alpha^2+\alpha^3}{120\alpha^3}e^{-\alpha}\xi^4\right),$$

$$m=1;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{3}{2}-\frac{5}{\alpha^2}-\frac{5}{2}\frac{\xi^2}{\alpha^2}+\left(1+\frac{5}{\alpha}+\frac{5}{\alpha^2}\right)e^{-\alpha}+\frac{15+7\alpha+\alpha^2}{6\alpha^2}e^{-\alpha}\xi^2\right.$$
$$\left.+\frac{48+33\alpha+9\alpha^2+\alpha^3}{120\alpha^3}e^{-\alpha}\xi^4\right],$$

$$m=2;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{15}{8}-\frac{35}{2\alpha^2}+\frac{189}{\alpha^4}-\frac{35}{4}\left(1-\frac{6}{\alpha^2}\right)\frac{\xi^2}{\alpha^2}+\frac{63}{8}\frac{\xi^4}{\alpha^4}-\left(1+\frac{14}{\alpha}+\frac{77}{\alpha^2}+\frac{189}{\alpha^3}+\frac{189}{\alpha^4}\right)e^{-\alpha}\right.$$
$$\left.-\frac{315+315\alpha+105\alpha^2+16\alpha^3+\alpha^4}{6\alpha^4}e^{-\alpha}\xi^2-\frac{945+351\alpha+141\alpha^2+18\alpha^3+\alpha^4}{120\alpha^4}e^{-\alpha}\xi^4\right],$$

$$m=3;\quad \frac{1}{i\alpha}\frac{\xi}{\alpha}\left[\frac{35}{16}-\frac{315}{8\alpha^2}+\frac{2079}{2\alpha^4}-\frac{19305}{\alpha^6}-\frac{315}{16}\left(1-\frac{44}{3\alpha^2}+\frac{1144}{5\alpha^4}\right)\frac{\xi^2}{\alpha^2}\right.$$
$$+\frac{693}{16}\left(1-\frac{26}{3\alpha^2}\right)\frac{\xi^4}{\alpha^4}-\frac{429}{16}\frac{\xi^6}{\alpha^6}+\left(1+\frac{27}{\alpha}+\frac{324}{\alpha^2}+\frac{2178}{\alpha^3}+\frac{8613}{\alpha^4}+\frac{19305}{\alpha^5}+\frac{19305}{\alpha^6}\right)e^{-\alpha}$$
$$+\frac{27027+27027\alpha+11781\alpha^2+2772\alpha^3+378\alpha^4+29\alpha^5+\alpha^6}{6\alpha^6}e^{-\alpha}\xi^2$$
$$\left.+\frac{45045+45045\alpha+17325\alpha^2+3590\alpha^3+440\alpha^4+31\alpha^5+\alpha^6}{120\alpha^6}e^{-\alpha}\xi^4\right].$$

Table 1 Mean number of secondaries per collision and extrapolation distance for critical spheres of radius R

Radius ΣR	Mean number of secondaries, C			Extrapolation distance, Σd		
	j_3 approx.	j_5 approx.	j_7 approx.	j_3 approx.	j_5 approx.	j_7 approx.
0.005	256.0795	256.0748	256.0727	1.8124	1.8124	1.8124
0.05	26.1915	26.1909	26.1907	1.7995	1.7995	1.7995
0.25	5.77735	5.77764	5.77759	1.7446	1.7446	1.7446
0.5	3.23740	3.23729	3.23728	1.6818	1.6818	1.6818
1	1.988696	1.988413	1.988391	1.5724	1.5726	1.5726
2	1.396344	1.395896	1.395876	1.4045	1.4059	1.4059
5	1.096470	1.095779	1.095763	1.1149	1.1350	1.1355
10	1.028710	1.028160	1.028150	0.8572	0.9598	0.9616
50	1.001342	1.001279	1.001278	-0.4602	0.7506	0.7607

Table 2 Scalar flux distribution in critical spheres of radius R (normalized to unity at the centre, $r = 0$)

	Radius $\Sigma R = 0.005$			$\Sigma R = 0.5$			$\Sigma R = 10$			
r/R	j_3	j_5	j_7	j_3	j_5	j_7	Diff.	j_3	j_5	j_7
0	1	1	1	1	1	1	1	1	1	1
0.2	0.97282	0.97211	0.97127	0.96962	0.96843	0.96787	0.9436	0.96142	0.94670	0.94261
0.4	0.89270	0.89032	0.88795	0.88323	0.87872	0.87713	0.7859	0.84572	0.79480	0.78298
0.6	0.76379	0.76000	0.75732	0.73651	0.72990	0.72815	0.5581	0.65315	0.56828	0.55480
0.8	0.59244	0.58892	0.58743	0.54740	0.54144	0.54052	0.3042	0.38588	0.30713	0.30177
0.9	-	-	-	-	-	-	0.1821	0.22857	0.18022	0.17847
0.96	-	-	-	-	-	-	0.1136	0.13120	0.10880	0.10731
1	0.36879	0.36722	0.36611	0.30810	0.30579	0.30515	0.0705	0.06528	0.05698	0.05571

Table 3 Critical radius, time-constant and k_{eff} by the j_N approximation and the comparison with the S_N and exact results

	S_N approximation		j_N approximation			
	N	Critical radius ΣR (% error)	N	Critical radius ΣR (% error)	Time-const. s_1-1	k_{eff}
	2	11.9168(-0.916)			for ΣR = 15.0	
C = 1.02	4	12.0203(-0.056)				
(C_s = 0.82	6	12.0310(+0.033)	1	38.2852(+218.3)	-0.0326058	0.86014
C_f = 0.2)	8	12.0334(+0.053)	3	12.1821(+1.290)	0.0064922	1.03314
ΣR_c=12.0270	12	12.0345(+0.062)	5	12.0291(+0.017)	0.0068721	1.03515
	16	12.0346(+0.063)	7	12.0266(-0.003)	0.0068786	1.03518
	2	3.0606(-3.512)			for ΣR = 3.3	
C = 1.2	4	3.1426(-0.927)				
(C_s = 0.8	6	3.1591(-0.407)	1	4.3831(+38.181)	-0.0807481	0.83841
C_f = 0.4)	8	3.1639(-0.255)	3	3.1783(+0.199)	0.0126375	1.02909
ΣR_c= 3.1720	12	3.1679(-0.129)	5	3.1722(+0.006)	0.0133343	1.03071
	16	3.1695(-0.079)	7	3.1719(-0.003)	0.0133546	1.03076
	2	1.1173(-5.578)			for ΣR = 1.15	
	4	1.1622(-1.783)	1	1.33370(+12.710)	-0.177468	0.89668
	6	1.1732(-0.854)	3	1.18369(+0.033)	-0.038497	0.97755
C = 1.8	8	1.1767(-0.558)	5	1.18334(-)	-0.038117	0.97777
(C_s = 0.5	12	1.1796(-0.313)	7	1.18332(-)	-0.038090	0.97779
C_f = 1.3)	16	1.1809(-0.203)			for ΣR = 1.10	
ΣR_c= 1.1833			1	-	-0.237721	0.86668
			3	-	-0.101117	0.94352
			5	-	-0.100767	0.94372
			7	-	-0.100747	0.94374

Table 4 Flux distributions obtained from three different j_3 calculations (normalized to unity at the centre of the sphere, $r = 0$)

c	1.02			1.8		
Radius ΣR	12.1821	15.0		1.18369	1.10	
r/R	Critical	$s_1-1=0.0064922$	$k_{eff}=1.03314$	Critical	$s_1-1=-0.101117$	$k_{eff}=0.94352$
0	1	1	1	1	1	1
0.2	0.96127	0.96110	0.96111	0.96655	0.96729	0.96684
0.4	0.84509	0.84441	0.84442	0.86796	0.87092	0.86912
0.5	0.75797	0.75689	0.75692	0.79583	0.80044	0.79763
0.6	0.65154	0.64994	0.64998	0.70986	0.71645	0.71245
0.7	0.52596	0.52361	0.52367	0.61153	0.62035	0.61501
0.8	0.38181	0.37821	0.37829	0.50260	0.51379	0.50704
0.9	0.22160	0.21548	0.21562	0.38462	0.39807	0.39000
0.95	0.13831	0.13016	0.13037	0.32171	0.33607	0.32749
1	0.05509	0.04559	0.04584	0.24811	0.26307	0.25418

<u>Table 5</u> Flux distributions obtained from three different j_7 calculations (normalized to unity at the centre of the sphere, $r = 0$)

c	1.02			1.8		
Radius ΣR	12.0266	15.0		1.18332	1.10	
r/R	Critical	$s_1 - 1 = 0.0068786$	$k_{eff} = 1.03518$	Critical	$s_1 - 1 = -0.100744$	$k_{eff} = 0.94374$
0	1	1	1	1	1	1
0.2	0.94149	0.94049	0.93960	0.95990	0.96304	0.96263
0.4	0.77887	0.77506	0.77249	0.84768	0.85754	0.85578
0.5	0.66880	0.66312	0.66009	0.77042	0.78327	0.78041
0.6	0.54698	0.53933	0.53643	0.68262	0.69744	0.69323
0.7	0.41932	0.40989	0.40777	0.58637	0.60209	0.59638
0.8	0.29139	0.28071	0.27976	0.48275	0.49878	0.49161
0.9	0.16792	0.15690	0.15686	0.37118	0.38782	0.37955
0.95	0.10892	0.09829	0.09833	0.31101	0.32819	0.31959
1	0.04685	0.03772	0.03773	0.23978	0.25706	0.24829

Table 6 Numerical results obtained from the j_N approximation and the comparison with S_4 and experimental values

		Godiva	Jezebel
Core	Composition Density (g/cm^3)	U(93.8% U-235) 18.75	Pu(4.5% Pu-240) 15.66
Observed critical core for ideal homog. sphere	Mass (kg) Radius (cm) Volume (ℓ)	52.04 U 8.717 2.774	16.28±0.05 Pu 6.284 1.040
Calculated k_{eff}	18-group S_4 18-group j_3	1.0046 1.0037	0.9916 0.9935
	10-group j_1 10-group j_3 10-group j_5	0.8490 0.9934 0.9933	0.8824 0.9965 0.9973
Calculated time-constant s_1-1	18-group j_3	0.001000	-0.004198
	10-group j_1 10-group j_3 10-group j_5	-0.03687 -0.002071 -0.001812	-0.04340 -0.002135 -0.002143
Mean lifetime of prompt neutrons ℓ(10^{-8} sec)	Experiment	0.60	0.298
	18-group S_4 18-group j_3	0.588 0.632	0.358 0.324
	10-group j_1 10-group j_3 10-group j_5	0.707 0.547 0.642	0.567 0.342 0.264
Calculated critical radius (cm)	18-group S_4 18-group j_3	8.669 8.680	6.346 6.340
	10-group j_1 10-group j_3 10-group j_5	11.014 8.788 8.770	7.406 6.323 6.320

Table 7 Leakage spectra obtained from the j_N approximation and the comparison with S_4 and experimental results

	Godiva				Jezebel			
Energy group	S_4 18-group	j_3 10-group	j_5 10-group	Experiment	S_4 18-group	j_3 10-group	j_5 10-group	Experiment
1 10-3 MeV	0.07861	0.07928	0.07900	0.0812±15%	0.11036	0.11439	-	0.1455±17%
2 3-1.4	0.14450	0.14574	0.14539	0.1391± 8%	0.19179	0.19860	-	0.1945± 9%
3 1.4-0.9	0.09012	0.09096	0.09094	0.0902± 5%	0.10686	0.11033	-	0.0987± 7%
4 0.9-0.4	0.14631	0.14763	0.14793	0.1516± 4%	0.13541	0.13929	-	0.1407± 6%
5 0.4-0.1	0.09181	0.09232	0.09268	0.1042± 4%	0.08669	0.08861	-	-
6 100-17 keV	0.01052	0.01050	0.01056	-	0.01198	0.01220	-	-
7 17-3	$0.0^{4}7665$	$0.0^{4}7704$	$0.0^{4}7816$	-	$0.0^{4}1472$	$0.0^{4}1489$	-	-
8 3-0.454	$0.0^{6}2786$	$0.0^{6}2782$	$0.0^{6}2832$	-	$0.0^{7}1726$	$0.0^{7}1745$	-	-
9 454-61.44 eV	$0.0^{9}4301$	$0.0^{9}4290$	$0.0^{9}4545$	-	$0.0^{11}4991$	$0.0^{12}3297$	-	-
10 61.44-	$0.0^{12}3196$	-	-	-	$0.0^{14}3049$	-	-	-

Table 8 Neutron spectra at the core centre and near the core boundary obtained from the j_N and S_4 calculations

	Energy group	Godiva			Jezebel		
		S_4 18-group	j_3 10-group	j_5 10-group	S_4 18-group	j_3 10-group	j_5 10-group
	1	$0.0^{3}6232$	$0.0^{3}6140$	$0.0^{3}5553$	0.001278	0.001377	0.001518
	2	0.0011713	0.0011520	0.0010500	0.002257	0.002428	0.002632
	3	$0.0^{3}7377$	$0.0^{3}7239$	$0.0^{3}6705$	0.001280	0.001370	0.001400
Spectrum	4	0.0012554	0.0012258	0.0011625	0.001718	0.001823	0.001713
at the	5	$0.0^{3}9783$	$0.0^{3}9460$	$0.0^{3}9150$	0.001292	0.001351	0.001119
core	6	$0.0^{3}1426$	$0.0^{3}1374$	$0.0^{3}1322$	$0.0^{3}2130$	$0.0^{3}2214$	$0.0^{3}1874$
centre	7	$0.0^{5}1244$	$0.0^{5}1190$	$0.0^{5}1182$	$0.0^{6}3096$	$0.0^{6}3129$	$0.0^{6}1815$
(r/R=0)	8	$0.0^{8}5575$	$0.0^{8}5303$	$0.0^{8}5340$	$0.0^{9}3946$	$0.0^{9}3907$	$0.0^{9}1876$
	9	$0.0^{10}1135$	$0.0^{10}1078$	$0.0^{10}1084$	$0.0^{12}1636$	$0.0^{12}2604$	$0.0^{12}2874$
	10	$0.0^{13}1130$	-	-	$0.0^{16}9648$	-	-
	1	$0.0^{3}1708$	$0.0^{3}1685$	$0.0^{3}1695$	$0.0^{3}4154$	$0.0^{3}4146$	-
	2	$0.0^{3}3178$	$0.0^{3}3134$	$0.0^{3}3156$	$0.0^{3}7271$	$0.0^{3}7265$	-
	3	$0.0^{3}1996$	$0.0^{3}1965$	$0.0^{3}1982$	$0.0^{3}4086$	$0.0^{3}4089$	-
Spectrum	4	$0.0^{3}3323$	$0.0^{3}3261$	$0.0^{3}3294$	$0.0^{3}5306$	$0.0^{3}5327$	-
near the core	5	$0.0^{3}2315$	$0.0^{3}2272$	$0.0^{3}2297$	$0.0^{3}3632$	$0.0^{3}3674$	-
boundary	6	$0.0^{4}3059$	$0.0^{4}3000$	$0.0^{4}3042$	$0.0^{4}5434$	$0.0^{4}5517$	-
(r/R=0.95)	7	$0.0^{6}2390$	$0.0^{6}2339$	$0.0^{6}2373$	$0.0^{7}7021$	$0.0^{7}7212$	$0.0^{8}4778$
	8	$0.0^{9}9892$	$0.0^{9}9633$	$0.0^{9}9756$	$0.0^{10}8352$	$0.0^{10}8594$	$0.^{11}9405$
	9	$0.0^{11}1908$	$0.0^{11}1839$	$0.0^{11}1889$	$0.0^{13}3132$	-	-
	10	$0.0^{14}1846$	-	-	$0.0^{16}1740$	-	-

To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

Alfred Nobel

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