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A LIMIT THEOREM FOR ORTHOGONAL FUNCTIONS

by

P. V. LAMBERT, M. F. NEUTS
(Euratom) (University of Louvain)

1963

Work performed under the Euratom contract No. 011-61-1 CETB (formerly No. 011-61-1 DOB)

Reprinted from JOURNAL D'ANALYSE MATHEMATIQUE Vol. X - 1962/1963

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When a given function F also satisfies analogous integrability requirements, we get similar convergence theorems for the sequence $\{P^F_{p,n}\}$ of its projections $P^F_{p,n}$ on the finite dimensional subspaces $E^{(p)}_n$ subtended by the orthonormal elements $\varphi^{(1)}_n, \varphi^{(2)}_n, \dots \varphi^{(p)}_n$, for each *fixed* $p = 1, 2, \dots$ when $n \to \infty$.

These theorems cover, as special cases, the asymptotic relations between particular orthogonal polynomials discovered by Polyà and Szegö. Cfr. ref. [2].

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JOURNAL D'ANALYSE MATHÉMATIQUE

RÉDIGÉ ET PUBLIÉ
SOUS LES AUSPICES DU
CONSEIL DE RÉDACTION

PAR
BINYAMIN AMIRÀ

EN COLLABORATION AVEC
ZEEV NEHARI et MENAHEM SCHIFFER

EXTRAIT DU
VOLUME X

MARCEL F. NEUTS and POL V. LAMBERT

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JÉRUSALEM
1962/1963

JOURNAL D'ANALYSE MATHÉMATIQUE
(Journal International de Recherches)

Le Journal d'Analyse Mathématique publie des Mémoires originaux d'Analyse Mathématique en français et en anglais.
Les Auteurs reçoivent gratuitement 50 tirés-a-part de leur Mémoire.

Le prix d'abonnement est de 20. — dollars par volume (environ 400 pages). Une remise de 40% est accordée aux Members *individuels* des Sociétés Mathématiques.

Adresse de la rédaction :
B. A. Amirà
13, Rue Abrabanel
Jérusalem
Israël

Agence Générale pour Abonnements
General Agents for Subscriptions :
Stechert-Hafner, Inc.
31 East 10th Street
New-York 3, N.Y., U.S.A.

Imprimé en Israël
Printed in Israel
Imprimerie "Gutenberg", Z. Casap et Fils S.A., Tel-Aviv

A LIMIT THEOREM FOR ORTHOGONAL FUNCTIONS *

By

Marcel F. Neuts and Pol V. Lambert

in Louvain, Belgium

I. Introduction.

We consider a completely additive class M of subsets of a fixed set A of points in an n-dimensional Euclidean space R_n.

Let $\{\mu_k\}$, $k = 1, 2, \ldots,$ be a sequence of measures, each μ_k being defined on M.

Each mesure μ_k permits us to construct an L_k^2 space of functions defined and square-integrable with respect to μ_k on A. Every such L_k^2 space is separable and hence has a countable complete orthonormal system. Suppose we now have a sequence of functions f_i, $i = 0, 1, 2, \ldots,$ satisfying the two following conditions:

a) any f_i belongs to every L_k^2

b) any finite set of these functions is linearly independent in all but a finite number of norms of L_k^2.

Hence, for p fixed, using the Schmidt orthogonalization on the ordered functions $f_0, f_1, \ldots, f_p$, we can construct the uniquely determined orthonormal elements $\varphi_{k,0}, \varphi_{k,1}, \ldots, \varphi_{k,p}$ for each space L_k^2, except perhaps for a finite number of them.

For a given function F belonging to every L_k^2 (except perhaps a finite number of them), we consider the Fourier coefficients:

$$(1.1.1) \qquad a_{k,i}^F = \int_A F \cdot \varphi_{k,i} \cdot \mu_k(dx), \qquad i = 0, 1, \ldots p$$

and the projection

$$(1.1.2) \qquad P_{k,p}^F = \sum_{i=0}^{p} a_{k,i}^F \cdot \varphi_{k,i}$$

of this function F on the $(p+1)$-dimensional subspace $E_{k,p}$ of one space

* This research was sponsored by the European Community for Atomic Energy under contract 011/61/1 DOB.

L^2_k. This can be done for $k = K, K+1, K+2, \ldots$ etc., K being a sufficiently large positive integer.

Some convergence relations of the sequence $\{\varphi_{k,i}\}$ (i fixed, $k = 1, 2, \ldots$) have been established by Szegö [2] pp. 33—37, in particular cases, using the explicit form of the element $\varphi_{k,i}$.

The object of this paper is to establish general convergence relations of the sequences $\{\varphi_{k,i}\}$, $\{a^p_{k,i}\}$, $\{P^F_{k,p}\}$ (i, p, F fixed) essentially using convergence requirements on the sequence of measures $\{\mu_k\}$.

II. Basic theorems.

We state the following known results for further reference. Munroe [1] p. 107 and p. 173.

Theorem 2.1. If $\{\mu_k\}$ is a sequence of everywhere finite measures on a completely additive class M and if for each $E \in M$

$$0 \leq \mu_0(E) = \lim_{k\to\infty} \mu_k(E) < \infty , \tag{2.1.1}$$

then μ_0 is a measure defined on M.

Theorem 2.2. If 1°) $\{\mu_k\}$ is a sequence of finite measures on a completely additive class M and if for each $E \in M$

$$0 \leq \mu_0(E) = \lim_{k\to\infty} \mu_k(E) < \infty ,$$

2°) f is a function defined, (M) measurable and bounded on E for which $\int_E f \cdot d\mu_k$ exists for each k,

then $\int_E f \cdot d\mu_0$ exists and

$$\lim_{k\to\infty} \int_E f \cdot d\mu_k = \int_E f \cdot d\mu_0 . \tag{2.2.2}$$

If the function f is not bounded, we can easily derive the following corollary of Theorem 2.2 :

Theorem 2.3. If 1°) $\{\mu_k\}$ is a sequence of finite measure-functions on a completely additive class M and if for each $E \in M$

$$0 \leqq \mu_0(E) = \lim_{k\to\infty} \mu_k(E) < \infty$$

2°) f is a function defined and (M)-measurable on E satisfying the following conditions:

a) The integrals $\int_E f \cdot d\mu_0$ and

$$\int_E f \cdot d\mu_k \text{ for each } k \text{ exist.}$$

b) For every $\varepsilon > 0$, we can find a non decreasing sequence $\{E_j\}$ of subsets of E in the class M for which f is bounded on each E_j and

$$\mu_0(E - E_j) < \varepsilon, \quad \text{for } j > J(\varepsilon, E, f) \tag{2.3.1}$$

$$\begin{cases} \lim\limits_{j\to\infty} \int\limits_{E-E_j} f \cdot d\mu_k = 0, \text{ uniformly in } k, \\ \lim\limits_{j\to\infty} \int\limits_{E-E_j} f \cdot d\mu_0 = 0 \end{cases} \tag{2.3.2}$$

then we have

$$\lim_{k\to\infty} \int_E f \cdot d\mu_k = \int_E f \cdot d\mu_0 . \tag{2.3.3}$$

Remarks.

1) By Theorem 2.1, μ_0 is a finite measure on M, hence the assumption that $\int_E f \cdot d\mu_0$ exists is a restriction on f and not on the sequence $\{\mu_k\}$.

2) One can easily verify that a bounded and (M)-measurable function on E for which the integral $\int_E f \cdot d\mu_k$ exists for each k satisfies all the requirements of this theorem, so that 2.3 is an extension of Theorem 2.2.

3) Finally we note that the Theorems 2.2 and 2.3 are in no way restricted to L^2-spaces of functions.

III. Convergence theorems.

We now consider the L_0^2-space, corresponding to the limit measure defined by Theorem 2.1. Suppose we have a sequence $\{f_i\}$, $i = 0, 1, 2, \ldots$ of (M)-measurable functions, bounded on A, or at least such that the squares f_i^2 satisfy the conditions of Theorem 2.3, the sequence of sets E_j used in that theorem being the same for every finite number of functions f_i^2 of the sequence.

For two arbitrarily chosen functions f_{i_1}, f_{i_2} in $\{f_i\}$, we have the convergence relation

$$\lim_{k \to \infty} \int_A f_{i_1} \cdot f_{i_2} \cdot d\mu_k = \int_A f_{i_1} \cdot f_{i_2} \cdot d\mu_0 . \tag{3.1.1}$$

To prove this if f_{i_1} and f_{i_2} are both bounded, we apply Theorem 2.2, and if not, by our assumptions, we can find a non decreasing sequence of sets $\{E_j\}$ which satisfies the condition (2.3.1) and (2.3.2) with respect to the set A and the function $f_{i_1}^2$ and $f_{i_2}^2$.

By Schwarz' inequality, we have for $k = 0, 1, 2, 3, \ldots$

$$\left| \int_{A-E_j} f_{i_1} \cdot f_{i_2} \, d\mu_k \right| \leqq \left| \int_{A-E_j} f_{i_1}^2 \cdot d\mu_k \right|^{1/2} \cdot \left| \int_{A-E_j} f_{i_2}^2 \cdot d\mu_k \right|^{1/2} . \tag{3.1.2}$$

This, by Theorem 2.3, implies the relation (3.1.1).

From (3.1.1) follows for finite p

$$\lim_{k \to \infty} \int_A \left(\sum_{i=0}^{p} \lambda_i \cdot f_i \right)^2 \cdot d\mu_k = \int_A \left(\sum_{i=0}^{p} \lambda_i \cdot f_i \right)^2 \cdot d\mu_0 . \tag{3.1.3}$$

We now require that every finite set of functions f_i are linearly independent with respect to the norm of L_0^2.

From (3.1.3) follows that they are also linearly independent in the norms L_k^2 for all but a finite number of indices k.

We now call (3.1.1) the relation of convergence of moments. This being established, suppose we fix i and, using the Schmidt orthogonalization on the first $(i+1)$ functions $f_0, f_1, \ldots, f_i$ of $\{f_j\}$ (in that order), so construct the uniquely determined orthonormal elements $\varphi_{k,0}, \varphi_{k,1}, \ldots, \varphi_{k,i}$ subtending a subspace $E_{k,i}$ of dimensions $i+1$ in the space L_k^2.

We can do this for $k=0$ and all but a finite number of indices k. We then have the following convergence theorems.

Theorem 3.1. $\lim\limits_{k\to\infty} \varphi_{k,i}(x) = \varphi_{0,i}(x)$, *for every x in A where the functions $f_0, f_1, \ldots, f_i$ are finite, and the convergence is uniform in each subset of A in which these functions are bounded.*

Proof: For $k=0$ and $k \geqq K$, K large enough, the functions $f_0, f_1, \ldots, f_i$ are linearly independent in the norm of L_k^2.

For such a k, the expression of $\varphi_{k,i}$ is given by

$$\varphi_{k,i}(x) = \sum_{j=0} \lambda_{k,j}^{(i)} \cdot f_j(x). \tag{3.1.4}$$

For the coefficients $\lambda_{k,j}^{(i)}$ we have Szegö [2] pp. 23—28

$$\varphi_{k,i}(x) = (D_{k,i-1} \cdot D_{k,i})^{-1/2} \cdot D_{k,i}(x) \tag{3.1.5}$$

where

(3.1.6) a) $$D_{k,i}(x) = \frac{1}{i!} \underbrace{\int_A \cdots \int_A}_{i} \begin{vmatrix} f_0(x_0) & f_1(x_0) & \ldots & f_i(x_0) \\ f_0(x_1) & f_1(x_1) & \ldots & f_i(x_1) \\ \vdots & \vdots & & \vdots \\ f_0(x_{i-1}) & f_1(x_{i-1}) & \ldots & f_i(x_{i-1}) \\ f_0(x) & f_1(x) & \ldots & f_i(x) \end{vmatrix} .$$

$$\begin{vmatrix} f_0(x_0) & f_1(x_0) & \ldots & f_{i-1}(x_0) \\ f_0(x_1) & f_1(x_1) & \ldots & f_{i-1}(x_1) \\ \vdots & \vdots & & \vdots \\ f_0(x_{i-1}) & f_1(x_{i-1}) & \ldots & f_{i-1}(x_{i-1}) \end{vmatrix} \cdot \mu_k(dx_0) \cdot \mu_k(dx_1) \ldots \mu_k(dx_{i-1})$$

for $i \geqq 1$.

b) $D_{k,0}(x) = f_0(x)$.

$$(3.1.7) \quad \text{a)} \; D_{k,i} = \frac{1}{(i+1)!} \underbrace{\int_A \cdots \int_A}_{i+1} \begin{vmatrix} f_0(x_0) & f_1(x_0) & \dots & f_i(x_0) \\ f_0(x_1) & f_1(x_1) & \dots & f_i(x_1) \\ \vdots & \vdots & & \vdots \\ f_0(x_i) & f_1(x_i) & \dots & f_i(x_i) \end{vmatrix}^2 \cdot \mu_k(dx_0) \cdot \mu_k(dx_1) \dots d_k(dx_i)$$

for $i \geqq 0$.

b) $D_{k,-1} = +1$.

It is known that $D_{k,j} > 0$ for every k and j.

The coefficient $A^{(i)}_{k,j}$ of $f_j(x)$ in the expression (3.1.6) for $D_{k,i}(x)$ is given by

(3.1.8)

$$A^{(i)}_{k,j} = \frac{(-1)^{i+j}}{i!} \underbrace{\int_A \cdots \int_A}_{i} \begin{vmatrix} f_0(x_0) & \dots & f_{j-1}(x_0) & f_{j+1}(x_0) & \dots & f_i(x_0) \\ f_0(x_1) & \dots & f_{j-1}(x_1) & f_{j+1}(x_1) & \dots & f_i(x_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_0(x_{i-1}) & \dots & f_{j-1}(x_{i-1}) & f_{j+1}(x_{i-1}) & \dots & f_i(x_{i-1}) \end{vmatrix} \cdot$$

$$\begin{vmatrix} f_0(x_0) & f_1(x_0) & \dots & f_{i-1}(x_0) \\ f_0(x_1) & f_1(x_1) & \dots & f_{i-1}(x_1) \\ \vdots & \vdots & & \vdots \\ f_0(x_{i-1}) & f_1(x_{i-1}) & \dots & f_{i-1}(x_{i-1}) \end{vmatrix} \cdot \mu_k(dx_0) \cdot \mu_k(dx_1) \dots \mu_k(dx_{i-1}).$$

The coefficient $\lambda^{(i)}_{k,j}$ of $f_j(x)$ in the expression (3.1.4) for $\varphi_{k,i}(x)$ is then given by

$$(3.1.9) \qquad \lambda^{(i)}_{k,j} = \frac{A^{(i)}_{k,j}}{(D_{k,i-1} \cdot D_{k,i})^{1/2}}.$$

The integrand in (3.1.8) is the product of two determinants with i lines and i columns.

So letting the index v run through the values $0, 1, \dots, j-1, j+1, \dots i$ and the index w through the values $0, 1, \dots, (i-1)$, putting $m = i - 1$ and designating the different values of v by $v_0, v_1, \dots, v_m$ and the different values of w by $w_0, \dots, w_m$ we can put $A^{(i)}_{k,j}$ into the form

$$(3\text{-}1.10) \qquad A^{(i)}_{k,j} = \frac{(-1)^{i+j}}{i!} \sum_{\text{perm. on } v} \sum_{\text{perm. on } w}$$

$$\operatorname{sign}\begin{pmatrix} 0, & 1, & \dots, & m \\ v_0, & v_1, & \dots, & v_m \end{pmatrix} \cdot \operatorname{sign}\begin{pmatrix} 0, & 1, & \dots, & m \\ w_0, & w_1, & \dots, & w_m \end{pmatrix} \cdot \int_A f_{v_0}(x_0) \cdot f_{w_0}(x_0) \cdot \mu_k(dx_0) \cdot$$

$$\int_A f_{v_1}(x_1) \cdot f_{w_1}(x_1) \cdot \mu_k(dx_1) \dots \int_A f_{v_m}(x_m) \cdot f_{w_m}(x_m) \cdot \mu_k(dx_m).$$

Applying the relation (3.1.1) to (3.1.10) we obtain :

$$\lim_{k\to\infty} A^{(i)}_{k,j} = A^{(i)}_{0,j}.$$

The same reasoning applies to expressions $D_{k,i-1}$ and $D_{k,i}$ in the denominator of (3.1.9) so that we have

$$\lim_{k\to\infty} D_{k,i-1} \cdot D_{k,i} = D_{0,i} \cdot D_{0,i-1}.$$

Since $D_{k,i-1} \cdot D_{k,i} > 0$ for each $k = K,\ K+1,\ K+2, \dots$ and $D_{0,i} \cdot D_{0,i-1} > 0$, the coefficients in (3.1.4), converge, i.e.

$$(3.1.11) \qquad \lim_{k\to\infty} \lambda^{(i)}_{k,j} = \lambda^{(i)}_{0,j}.$$

Taking now into account the expression (3.1.4) for $\varphi_{k,i}(x)$ we get the proof of the theorem.

We now study the convergence in the L^2 norms.

We write $\|\cdot\|_k$ for the norm in L^2_k and $\|\cdot\|_0$ for the norm in L^2_0.

Theorem 3.2. For every fixed i

$$(3.2.1) \qquad \lim_{k\to\infty} \|\varphi_{k,i}\|_k = \|\varphi_{0,i}\|_0$$

and for each $m = 1, 2, \dots$ or $m = 0$ and each i (m and i fixed)

$$(3.2.2) \qquad \lim_{k\to\infty} \|\varphi_{k,i}\|_m = \|\varphi_{0,i}\|_m.$$

Proof : We take $k \geq K(i)$, K being large enough for the functions $f_0, f_1, \dots, f_i$, to be linearly independent in all L^2_k. We have

$$\|\varphi_{k,i}\|^2_m = \int_A \left[\sum_{j=0}^{i} \lambda^{(i)}_{k,j} \cdot f_j(x)\right]^2 \cdot \mu_m(dx)$$

$$= \sum_{r=0}^{i} \sum_{s=0}^{i} \lambda_{k,s}^{(i)} \cdot \lambda_{k,r}^{(i)} \cdot \int_A f_r(x) \cdot f_s(x) \cdot \mu_m(dx)$$

which implies (3.2.2) by application of (3.1.11).

We now let m vary, take $m = k$, and (3.2.1) follows from (3.1.1).

We now take an (M)-measurable function F in A which is either bounded in A or at least is such that the square F^2 satisfies the requirements of Theorem 2.3 in such a way that the same sequence of sets $\{E_j\}$ can be used for the functions F^2 and an arbitrary finite set of functions from the sequence $\{f_j^2\}$.

First of all, it follows (cfr. Theorem 2.2 or Theorem 2.3), by using Schwarz's inequality, that the Fourier coefficients $a_{k,i}^F$ are well defined for all but a finite number of indices k.

We now have the following convergence theorem.

Theorem 3.3. For each i fixed, we have

$$\lim_{k \to \infty} a_{k,i}^F = a_{0,i}^F .$$

Proof: For $k \geq K$ we have:

(1)
$$|a_{0,i}^F - a_{k,i}^F| = \left| \int_A F(x) \cdot \varphi_{0,i}(x) \cdot \mu_0(dx) - \int_A F(x) \cdot \varphi_{k,i}(x) \cdot \mu_k(dx) \right| \leq \left| \int_A F(x) \cdot \varphi_{0,i}(x) \cdot \mu_0(dx) - \int_A F(x) \cdot \varphi_{0,i}(x) \cdot \mu_k(dx) \right| + \left| \int_A F(x) \cdot \varphi_{0,i}(x) \cdot \mu_k(dx) - \int_A F(x) \cdot \varphi_{k,i}(x) \cdot \mu_k(dx) \right| .$$

First of all, we see, using Schwarz's inequality, that the integral $\int_A F(x) \cdot \varphi_{0,i}(x) \cdot \mu_k(dx)$ exists.

We examine separately the two terms on the right of the above inequality. For the first term we have,

$$F(x) \cdot \varphi_{0,i}(x) = \sum_{j=0}^{i} \lambda_{0,j}^{(i)} \cdot F(x) \cdot f_j(x).$$

If $F \cdot f_j$ is bounded, Theorem 2.2 yields:

$$\lim_{k\to\infty} \int_A F(x) \cdot f_j(x) \cdot \mu_k(dx) = \int_A F(x) \cdot f_j(x) \cdot \mu_0(dx). \tag{2}$$

If $F \cdot f_j(x)$ is not bounded, by our assumptions for F we can use a sequence of sets $\{E_j\}$ as in the proof of relation (3.1.1) and using the same argument we see that the product $F \cdot f_j$ satisfies all conditions of Theorem 2.3. This shows that the first term vanishes in the limit.

For the second term we have,

$$\left| \int_A F(x) \cdot (\varphi_{0,i}(x) - \varphi_{k,i}(x)) \cdot \mu_k(dx) \right|$$
$$\leq \left| \int_A [F(x)]^2 \cdot \mu_k(dx) \right|^{1/2} \cdot \left| \int_A [\varphi_{0,i}(x) - \varphi_{k,i}(x)]^2 \cdot \mu_k(dx) \right|^{1/2}.$$

By our assumptions for F,

$$\lim_{k\to\infty} \int_A [F(x)]^2 \cdot \mu_k(dx) = \int_A [F(x)]^2 \cdot \mu_0(dx) < \infty .$$

The second factor vanishes in the limit by (3.2.1), (3.1.1) and (3.1.11). This implies the theorem.

For an arbitrary fixed p, we now study the convergence (on k) of the projections $P_{k,p}^F$ defined in (1.1.2) of the function F on the $(p+1)$-dimensional subspaces $E_{k,p}$ of the spaces L_k^2 in the sequence of spaces $\{L_k^2\}$. As in the case of the elements $\varphi_{k,i}$ we examine the point-wise convergence, the uniform convergence and the convergence in the norms L_k^2. We always take $k \geqq K$, K being large enough in order that the functions $f_0, f_1, \dots, f_p$ be linearly independent in the norms of L_k^2 for $k \geqq K$. So the orthonormal functions $\varphi_{k,0}, \varphi_{k,1}, \dots, \varphi_{k,p}$ are always defined. We have the following convergence theorems.

Theorem 3.4. *For fixed p, we have*

$$\lim_{k\to\infty} P_{k,p}^F(x) = P_{0,p}^F(x),$$

in each point x of A where the functions $f_0, f_1, \dots, f_p$ are finite and the convergence is uniform on each subset of A where these functions are bounded.

Proof: We have by (1.1.2) for $k \geqq K$ and for $k = 0$

$$P^F_{k,p}(x) = \sum_{i=0}^{p} a^F_{k,i} \cdot \varphi_{k,i}(x),$$

so we just have to apply the Theorems 3.1 and 3.3 in order to prove this theorem.

Theorem 3.5. For fixed p, we have

$$\lim_{k\to\infty} \| P^F_{k,p} \|_k = \| P^F_{0,p} \|_0 \tag{3.5.1}$$

and for $m = 0$ or $m = 1, 2, 3, \ldots$ $(m, p$ fixed), we have

$$\lim_{k\to\infty} \| P^F_{k,p} \|_m = \| P^F_{0,p} \|_m . \tag{3.5.2}$$

Proof: For $k \geqq K$ and for $k = 0$ we have,

$$\| P^F_{k,p} \|^2_m = \sum_{i=0}^{p} \sum_{j=0}^{p} a^F_{k,i} \cdot a^F_{k,j} \cdot \int_A \varphi_{k,i}(x) \cdot \varphi_{k,j}(x) \cdot \mu_m(dx) \tag{1}$$

$$= \sum_{i=0}^{p} \sum_{j=0}^{p} a^F_{k,i} \cdot a^F_{k,j} \cdot \sum_{r=0}^{i} \sum_{s=0}^{j} \lambda^{(i)}_{k,r} \cdot \lambda^{(j)}_{k,s} \cdot \int_A f_r(x) \cdot f_s(x) \cdot \mu_m(dx).$$

Now the application of formula (3.1.11) and Theorem 3.3 to (1) proves (3.5.2), while if, in addition, we let m vary, take $m = k$ and apply (3.1.1) to (1) we get (3.5.1).

IV. Examples.

Let now the linearly independent functions $f_0(x)$, $f_1(x)$, $f_2(x)$, ... be specified by the polynomials 1, x, x^2, ... We then get four examples of Section III applied to orthogonal polynomials.

1. As a first example we have the known asymptotic relation between Fischer's polynomials and Legendre's polynomials (Szegö [2] pp. 33—34). Fischer's poylnomials of order k are associated with the step distribution function $\alpha_k(x)$ with jumps of one unit at the points $x = 0, 1, 2, \ldots, k-1$.

Hence, except for a constant factor, they can be represented by the polynomials $t_n^{(k)}(x)$ and $t_m^{(k)}(x)$ of degree n and m respectively defined by

$$(4.1.1)\qquad \int_{-\infty}^{+\infty} t_n^{(k)}(x)\cdot t_m^{(k)}(x)\cdot d\alpha_k(x) = \frac{k\cdot(k^2-1^2)\cdot(k^2-2^2)\ldots(k^2-n^2)}{(2n+1)}\,\delta_{n,m}$$

$$n\,,\,m = 0,\,1,\,2,\,\ldots,\,k-1\,.$$

In the same way Legendre's polynomials $P_n(x)$ and $P_m(x)$ of degree n and m respectively are associated with the Lebesgue measure in $[-1\,,\,+1)$ and hence can be defined, except for a constant factor, by:

$$(4.1.2)\qquad \int_{-1}^{+1} P_n(x)\cdot P_m(x)\cdot dx = \frac{2}{(2n+1)}\cdot\delta_{n,m}$$

$$n\,,\,m = 0,\,1,\,2,\,\ldots\,.$$

After suitable normalization one can verify the conditions of Theorems 2.1 and 2.2 which implies Szegö's result i.e.

$$(4.1.3)\qquad \lim_{k\to\infty} k^{-n}\cdot t_n^{(k)}(k\cdot x) = P_n(2x-1)\,.$$

2. A second example is given by the known asymptotic relation of normalized Krawtchouk polynomials to the Poisson-Charlier polynomials. (Szegö [2], pp. 34—37). Krawtchouk's polynomials $p_n^{(k)}(x)$ and $p_m^{(k)}(x)$ of order k and degree n and m respectively are defined by:

$$(4.2.1)\qquad \int_{-\infty}^{+\infty} p_n^{(k)}(x)\cdot p_m^{(k)}(x)\cdot d\alpha_k(x) = \delta_{n,m}$$

$$n\,,\,m = 0,\,1,\,2,\,\ldots,\,k,$$

where $\alpha_k(x)$ is a step distribution function with the jump at the point x of

$$(4.2.2)\qquad j(x) = \binom{k}{x}\cdot p^x\cdot(1-p)^{k-x},\qquad x = 0,\,1,\,2,\,\ldots,\,k.$$

$$0<p<1$$

Poisson-Charlier's polynomials $s_n(x)$ and $s_m(x)$ of degree n and m respectively are defined by

$$\int_{-\infty}^{+\infty} s_n(x) \cdot s_m(x) \cdot d\alpha(x) = \delta_{n,m} \tag{4.2.3}$$

where $\alpha(x)$ is a step distribution function with the jump at the point x of

$$i(x) = e^{-a} \cdot a^x \cdot (x!)^{-1}; \qquad x = 0, 1, 2, \ldots . \quad a > 0 \tag{4.2.3}$$

After the normalization $p \cdot k = a$, a fixed, letting $k \to \infty$ (and hence $p \to 0$), one can verify the conditions of Theorems 2.1 and 2.3, which implies Szegö's result, i.e. for a fixed integer $x \geqq 0$,

$$\lim_{k\to\infty} p_n^{(k)}(x) = s_n(x). \tag{4.2.4}$$

3. A third example is given by the known asymptotic relation of normalized Krawtchouk polynomials to Hermite's polynomials.

Hermite's polynomials $H_i(x)$ of degree i are defined by:

$$\frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} e^{-x^2} \cdot H_n(x) \cdot H_m(x) \cdot dx = 2^n \cdot (n!) \cdot \delta_{n,m} \tag{4.3.1}$$

$$n, m = 0, 1, 2, \ldots .$$

A verification of the conditions of Theorems 2.1 and 2.3, after normalization, yields Szegö's result, i.e. letting z be real and letting x denote the greatest integer $\leqq p \cdot k + z\, [2p \cdot (1-p) \cdot k]^{1/2}$ where p and z are fixed and letting $k \to \infty$, this gives us for a fixed n,

$$\lim_{k\to\infty} p_n^{(k)}(x) = (2^n \cdot n!)^{-1/2} \cdot H_n(z). \tag{4.3.2}$$

4. As fourth and last example we quote the convergence relation of normalized Hahn polynomials to renormalized Jacobi polynomials.

For brevity we refer to [3] pp. 1—6 for the definition of Hahn polynomials of degree n

$$Q_n(x; \alpha, \beta, k), \ \alpha, \beta > -1 \ ; \ n = 0, 1, 2, \ldots, k-1;$$

and of renormalized Jacobi polynomials $J_n(z; \alpha, \beta)$ of degree n,

$$n = 0, 1, 2, \ldots; \alpha, \beta > -1 .$$

The verification of the conditions of Theorems 2.1 and 2.2 implies Erdelyi and Weber's result i.e.

(4.4.1) $$\lim_{k\to\infty} Q_{,}^{(k)}[(k-1)\cdot x\,;\alpha\,,\beta\,,k] = J_n[(1-2x)\,;\alpha\,,\beta]\,.$$

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University of Louvain
December 1961

(Received August 2, 1962)

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