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A MULTISTAGE SEARCH GAME

by

M. F. Neuts

1963

Report established by
the Catholic University of Louvain, Belgium
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A MULTISTAGE SEARCH GAME

SUMMARY

The problem of optimally searching a set of boxes for a hidden object is formulated as a game. The model is such that parameters account for the possibility of overlooking the object in a single search. We obtain optimal stationary strategies and use them to construct a monotonely converging to Bellman's functional equations. It is felt that this result may be of use in the optimazation of machine searches of data systems.

1. Introduction

In the game which we shall consider Player I hides an object in one of N boxes, labeled from 1 to N. His opponent, Player II, has to search for the object by successive examinations of the boxes. An examination of the i-th box ($i = 1,\dots, N$) can be performed at a cost $t_i > 0$ each time and there is a probability p_i $(0 < p_i \leq 1)$ of finding the object given that the right box is searched.

Upon finding the item Player II receives a reward of $a \geq 0$. We shall solve the game explicitly under the assumption that Player II is restricted to the use of stationary strategies and discuss the solution of Bellman's functional equations for the Bayes risk in the non-stationary case.

Let δ $(0 < \delta \leq 1)$ stand for a discount factor applicable to losses at future stages of the game. The case $\delta = 1$ can be solved using the results on the stationary case.

2. Stationary Minimax Strategies

A stationary strategy for Player II is an N-tuple $y = (y_1,\dots,y_N)$; $y_i \geq 0$, $i = 1,\dots, N$; $\sum_{j=1}^{N} y_j = 1$ which denotes a probability distribution, chosen once and for all, and by which the box to be examined at each stage is selected.

A mixed strategy for Player I is an N-tuple $x = (x_1,\dots,x_N)$ with $x_i \geq 0$; $i = 1,\dots,N$ and $\sum_{j=1}^{N} x_j = 1$. x_i denotes the probability of placing the object in the i-th box.

If Player I uses a mixed strategy x and Player II a stationary strategy y the expected return to Player I for each stage of the game is given by:

$$A(x,y) = \sum_{k=1}^{N} y_k(t_k - ap_k x_k).$$

The probability P(x,y) that the object will be found during a given search equals:

$$P(x,y) = \sum_{k=1}^{N} p_k x_k y_k.$$

The discounted expected return to Player I for the entire search is therefore given by:

$$(1) \qquad F(x,y) = \sum_{r=0}^{\infty} \delta^r [1-P(x,y)]^r \; A(x,y) = \frac{A(x,y)}{1-\delta[1-P(x,y)]}$$

provided $1 - \delta[1 - P(x,y)] \neq 0$.

If we denote by $x^o = (x_1^o, \ldots, x_n^o)$, $y^o = (y_1^o, \ldots, y_n^o)$ and **v** respectively a pair of minimax strategies and the value of the game with payoff (1) then we must have

$$(2) \qquad F(x^o,y) \geq v_\delta \quad \text{for all} \quad y$$

$$F(x,y^o) \leq v_\delta \quad \text{for all} \quad x$$

The relations (2) can be written as:

$$(3) \qquad \sum_{k=1}^{N} t_k y_k - (a+v_\delta.\delta). \sum_{k=1}^{N} p_k x_k^o y_k \geq v_\delta \,.\, (1-\delta)$$

for all y and

$$(4) \qquad \sum_{k=1}^{N} t_k y_k^o - (a+v_\delta.\delta) . \sum_{k=1}^{N} p_k x_k y_k^o \leq v_\delta \, (1-\delta)$$

for all x. Since (3) and (4) are linear inequalities over a simplex we can replace them by the following equivalent systems:

$$(5) \qquad t_j - (a+v_\delta.\delta) \, p_j x_j^o \geq v_\delta \, (1-\delta) \quad \text{for} \quad j=1,\ldots,N$$

and

$$(6) \qquad \sum_{k=1}^{N} t_k y_k^o - (a+v_\delta . \delta)\, p_i y_i^o \leq v_\delta \,(1-\delta) \qquad \text{for} \quad i = 1,\dots,N.$$

The inequalities (5) and (6), together with $\sum_{i=1}^{N} y_i = \sum_{j=1}^{N} x_j = 1$, can all be satisfied with equality. We obtain as the solution of the resulting system:

$$(7) \qquad v_\delta = \left[\sum_{k=1}^{N} \frac{t_k}{p_k} - a \right] \left[\delta + (1-\delta) . \sum_{k=1}^{N} \frac{1}{p_k} \right]^{-1}$$

$$x_j^o = \frac{1}{p_j} \cdot \frac{t_j - (1-\delta) v}{a + v_\delta . \delta} \qquad \text{for} \quad j=1,\dots,N$$

$$y_i^o = \frac{1}{p_i} \left(\sum_{k=1}^{N} \frac{1}{p_k} \right)^{-1} \qquad \text{for} \quad i=1,\dots,N.$$

The independence of y^o of all parameters but the p_i $i=1,\dots,N$ is quite striking. The same holds for the expected duration of the game.

We have:

$$\tau = \sum_{k=1}^{\infty} k \left[1 - P(x^o, y^o) \right]^{k-1} = \left(\sum_{k=1}^{N} \frac{1}{p_k} \right)^2$$

Remark

The stationary minimax strategies correspond to the intuitive notion of a memory-less Player II and are therefore of little practical interest. They are essential in our discussion of the case $\delta = 1$ of Bellman's functional equations for the non-stationary case.

3. The functional equation for the Bayes risk.

Let $x = (x_1,\dots,x_N)$ denote an arbitrary mixed strategy for Player I. We can ask for the optimal sequential response for Player II against x and for the minimum expected loss.

We denote by $f_\delta(x)$ the minimum expected loss for Player II against x. An

An application of Bellman's principle of optimality implies that the following functional equation must be satisfied

$$(8) \qquad f_\delta(x) = \min_{1 \leq i \leq N} [t_i - ap_i x_i + \delta(1 - p_i x_i) f_\delta(T_i x)]$$

with $t_i > 0$, $0 < p_i \leq 1$, $0 < \delta \leq 1$, $a \geq 0$ and $T_i x = \xi = (\xi_1, \ldots, \xi_N)$ defined by:

$$(9) \qquad \xi_i = x_i \cdot \frac{1-p_i}{1-p_i x_i}$$

$$\xi_j = x_j \cdot \frac{1}{1-p_i x_i} \quad \text{for} \quad j \neq i.$$

The N-tuple ξ is the a posteriori distribution derived from x, given that one unsuccessful search of the box i was made. We may exclude the case $p_i x_i = 1$ (for some i) as being trivial. The transformation (9) is then always well-defined. When $0 < \delta < 1$ the functional equation (8) is of Type Two in Bellman's terminology [1] p.121. The following theorem of Bellman settles the problem of the existence, uniqueness and continuity of the solution of (8) in this case.

Theorem 1.

Let $f_{\delta,o}(x)$ denote an arbitrary continuous function defined over the set X of all x. Let the sequence of functions $f_{\delta,n}(x)$ $n=1,2,\ldots$ be defined by the recursion formula:

$$(10) \qquad f_{\delta,n+1}(x) = \min_{1 \leq i \leq N} [t_i - ap_i x_i + \delta(1 - p_i x_i) f_{\delta,n}(T_i x)]$$

then the sequence $f_{\delta,n}(x)$ converges to a limit $f_\delta(x)$ for $n \to \infty$. This convergence is uniform in x which implies the continuity of $f_\delta(x)$ in x for each δ. Moreover $f_\delta(x)$ is the unique bounded solution to (8).

Proof:

We refer to the proof given in [1] p.121. The theorem 1 implies that $f_\delta(x)$ is the Bayes risk for the game using sequential strategies.

A particular choice of $f_{\delta,o}(x)$.

• From now on we shall set $f_{\delta,o}(x) = v_\delta$ where v_δ is the value of the game in stationary strategies given by (7). We extend this to $\delta = 1$ and define $f_n(x)$ by:

$$(11) \qquad f_{n+1}(x) = \min_{1 \leq i \leq N} [t_i - ap_i x_i + (1 - p_i x_i) f_n(x)]$$

for n = 0,1,...

It is noteworthy that $f_{\delta,n}(x) \longrightarrow f_n(x)$ for $\delta \longrightarrow 1-$ and for every n. This convergence is uniform in x.

4. Monotonicity properties of the sequence $f_{\delta,n}(x)$

Theorem 2

The sequence $f_n(x)$ n=0,1,... for $\delta = 1$ is monotome decreasing in n for all x in X. This is not true in general for $0 < \delta < 1$ but a sufficient condition for $f_{\delta,n}(x)$ to be monotone decreasing in n for all x and δ is that

$$\sum_{k=1}^{N} \frac{t_k}{p_k} \leq a.$$

Proof:

Since

$$f_{\delta,n+1}(x) - f_{\delta,n}(x) \leq \delta^n [f_{\delta,1}(x) - f_{\delta,o}(x)]$$

for n = 0,1,1.. it is sufficient to study the sign of the difference $f_{1,\delta}(x) - f_{o,\delta}(x)$. We have:

$$(12) \qquad f_{\delta,1}(x) - f_{\delta,o}(x) = \min_{1 \leq i \leq N} [t_i - ap_i x_i + (1 - p_i x_i) v_\delta] - v_\delta$$

$$= \min_{1 \leq i \leq N} [t_i - (a + v_\delta)\, p_i x_i]$$

Let $\rho = \max_{x \in X} \min_{1 \leq i \leq N} [t_i - (a + v_\delta)p_i x_i]$, then it is readily verified that ρ is the value of the N x N-game with pay-off matrix B.

(13)
$$B \equiv \begin{pmatrix} t_1 - p_1(a+v_\delta) & t_2 & \cdots & t_N \\ t_1 & t_2 - p_2(a+v_\delta) & \cdots & t_N \\ \vdots & \vdots & & \vdots \\ t_1 & t_2 & & t_N - p_N(a+v_\delta) \end{pmatrix}$$

Under the assumptions on t_i and p_i the game with matrix B is completely mixed and the optimal strategies for the players I^* and II^* are given by:

(14)
$$x_s^* = \frac{1}{p_s(a+v_\delta)} \left[t_s - \left(\sum_{k=1}^{N} \frac{1}{p_k} - v_\delta - a \right) \left(\sum_{k=1}^{N} \frac{1}{p_k} \right)^{-1} \right]$$

$$s = 1,\dots,N$$

$$z_r^* = \frac{1}{p_r} \left(\sum_{k=1}^{N} \frac{1}{p_k} \right)^{-1}$$

$$r = 1,\dots,N.$$

The value ρ is given by:

(15)
$$\rho = (1-\delta) \left(\sum_{k=1}^{N} \frac{t_k}{p_k} - a \right) \left(\sum_{k=1}^{N} \frac{1}{p_k} \right)^{-1} \left(\sum_{k=1}^{N} \frac{1}{p_k} - 1 \right) \left[\delta + (1-\delta) \sum_{k=1}^{N} \frac{1}{p_k} \right]^{-1}$$

We have

$$\rho = 0 \quad \text{for} \quad \delta = 1 \quad \text{or} \quad a = \sum_{k=1}^{N} \frac{t_k}{p_k}$$

$$\rho < 0 \quad \text{for} \quad a > \sum_{k=1}^{N} \frac{t_k}{p_k}$$

and since

$$f_{\delta,1}(x) - f_{\delta,o}(x) \leq \rho \quad \text{for all} \quad x$$

the result follows.

5. The case $\delta = 1$

Theorem 3

There exists a bounded concave solution f(x) to equation (8) for $\delta = 1$.

Proof: (*)

There exists a bounded solution to equation (8). The functions $f_n(x)$ are uniformly bounded below by $-a$ as is seen from the recurrence relation. Since the sequence $f_n(x)$ is monotone decreasing in n for every x the result follows. Let f(x) denote the limit of the sequence $f_n(x)$.

The limit-function f(x) is a concave function of x over X. The functions $f_n(x)$ are indeed concave as an induction argument readily shows. Therefore so is the limit.

Remark:

In many cases $f_\delta(x)$ will be an increasing function of δ for all x. This is true in particular if $t_i - a\,p_i \geq 0$ for $i = 1, \ldots, N$. In this case we have

$$f_\delta(x) \leq f(x) \leq f_n(x)$$

for all δ and n.

We were unable to prove a relationship between the limit of $f_\delta(x)$ and f(x).

6. Some gen ralizations

Results similar to those above can be obtained for the following generalized models. We define the game as above, but Player I is no longer compelled to put the object in one of the N boxes. Player II can now, before each search, make the claim that the object is not in any of the boxes. If his claim is correct he receives a reward $b > 0$. If his claim is wrong he pays a penalty $c > 0$ and there are two alternative models depending on whether

(*) The author thanks Mister Pol V. Lambert for pointing out an oversight in an earlier version of this theorem.

this ends the game or whether the search should be continued with the knowledge that the object is hidden. Let us refer to these alternative models as Model A and Model B, respectively.

The functional equation for the Bayes risk $\varphi(x)$ in Model A is given by:

$$\varphi(x) = \min \begin{cases} c(1 - x_0) - bx_0 \\ \min\limits_{1 \le i \le N} [t_i - a\, p_i x_i + \delta(1 - p_i x_i)\varphi(T_i x)] \end{cases}$$

in which $b > 0$, $c > 0$, $a \geq 0$, $t_i > 0$, $0 < p_i \leq 1$ and $0 < \delta < 1$. Here $x = (x_0, x_1, \ldots, x_N)$ denotes the a priori distribution for Player I. x_0 is the probability of not putting the object in any of the N boxes. $T_i x = \xi = (\xi_0, \xi_1, \ldots, \xi_N)$ is defined by

$$\xi_i = x_i \cdot \frac{1-p_i}{1-p_i x_i}$$

$$\xi_j = x_j \cdot \frac{1}{1-p_i x_i} \qquad j = 0, 1, \ldots, i-1, i+1, \ldots, N.$$

The fundamental equation for the Bayes risk $\varphi(x)$ in Model B is given by:

$$\varphi(x) = \min \begin{cases} c(1-x_0) - b\, x_0 + \delta(1-x_0)\, f_\delta(T_0\, x) \\ \min\limits_{1 \le i \le N} [t_i - a\, p_i x_i + \delta(1 - p_i x_i)\varphi(T_i\, x)] \end{cases} \tag{16}$$

where $T_0\, x = \xi = (\xi_1, \ldots, \xi_N)$ with $\xi_i = \dfrac{x_i}{1-x_0}$ $i = 1, \ldots, N$, and $f_\delta(x)$ is the solution of equation (8).

B I B L I O G R A P H Y

[1] BELLMAN, R. Dynamic Programming. Princeton University Press, 1957.