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THEORETICAL FOUNDATIONS
OF PARAMETER OSCILLATION MEASUREMENTS

by

G. BLÄSSER

1963

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REMENTS by G. BLÄSSER

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absorption and production of neutrons to these oscillations will be given.

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SUMMARY

The influence of periodic changes of some localized reactor parameters (cross-sections) on the output signal of a detector placed somewhere in the reactor will be discussed from a general point of view. For the special case of thermal reactors, a procedure of determining independently the contributions of the absorption and production of neutrons to these oscillations will be given.

1 — GREEN'S FUNCTION FOR THE OSCILLATING PART
OF THE NEUTRON DENSITY

In the most general way a reactor characterized by parameters which depend on time can be described by the equations (Ref. 1):

$$\frac{\partial N_t}{\partial t} = (1-\beta)\chi_0(E)J_tN_t + \sum_i \chi_i(E)\lambda_iC_i + S - K_tN_t \tag{1a}$$

$$\frac{\partial C_i}{\partial t} = \beta_iJ_tN_t - \lambda_iC_i \tag{1b}$$

where $N_t(\vec{r},E,\vec{\omega})/4\pi$ is the neutron density at time t in phase space, i.e. per unit volume per unit energy per unit angle of velocity direction $\vec{\omega}$; β_i is the fraction of the fission neutrons emitted by the i-th delayed neutron emitter; $\chi_0(E)$ is the normalized energy spectrum of prompt neutrons arising from fission (the emission of the fission neutrons is assumed to be isotropic in angle); $\chi_i(E)$ is the normalized energy spectrum of the neutrons emitted by the i-th delayed neutron emitter. The operator J_t is defined in such a way that

$$J_tN_t = \int\int \nu(E)\Sigma_f(\vec{r},E,t)vN_t(\vec{r},E,\vec{\omega})\,dE\,d\Omega/4\pi$$

is the production rate of fission neutrons per unit volume, unit time and unit solid angle at the point $\vec{r}$. Similarly, the operator K_t is related to the net rate $K_tN_t/4\pi$ at which neutrons leave the region in phase space as a result of absorption leakage and scattering:

$$K_tN_t = \vec{\omega}\cdot\mathrm{grad}\,(vN_t(\vec{r},E,\vec{\omega})) + \Sigma_{tot}(\vec{r},E,t)vN_t(\vec{r},E,\vec{\omega}) - \int_0^{E_s}dE'\int\Sigma_s(E,E',\vec{\omega}\cdot\vec{\omega}',\vec{r},t)v'N_t(\vec{r},E',\vec{\omega}')\,d\Omega'$$

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where $\Sigma_s(E,E',\vec{\omega}\cdot\vec{\omega}',\vec{r},t)$ is the cross-section at time t for scattering from the element in phase space with coordinates $E',\vec{\omega}',\vec{r}$ to the element in phase space having the coordinates $E,\vec{\omega},\vec{r}$. Finally, $S(t,E,\vec{r},\vec{\omega})$ is the source density per unit volume, energy and solid angle, λ_i^{-1} the mean life of the i-th type delayed neutron emitters and $\beta = \sum_i \beta_i$.

The equations (1) are completed by the boundary conditions, which the solution N_t has to satisfy:

$$N_t(\vec{r},E,\vec{\omega}) \quad \text{continuous in } \vec{r} \tag{2}$$

at all interfaces between different media inside the reactor for all values of $E, \vec{\omega}$ and t.

$$N_t(\vec{r_s},E,\vec{\omega}_-) = 0 \tag{3}$$

for all E and t, where $\vec{r_s}$ are the points of the reactor surface and $\vec{\omega}_-$ are all values of $\vec{\omega}$ for which $\vec{\omega}\cdot\vec{n} < 0$, $\vec{n}$ being the outward normal at the point $\vec{r_s}$.

In general, one must also specify the neutron distribution at an initial instant of time t_0.

Eliminating C_i from equation (1a) by using the integrated form of equation (1b), that is

$$C_i(t) = \beta_i \int_{-\infty}^{t} dt' \, J_{t'} N_{t'} \, e^{-\lambda_i(t-t')} \tag{1b'}$$

we obtain the equation

$$\frac{\partial N_t}{\partial t} = (1-\beta)\chi_0(E)J_t N_t + \sum_i \chi_i(E)\beta_i\lambda_i \int_{-\infty}^{t} dt' J_{t'} N_{t'} \, e^{-\lambda_i(t-t')} + S - K_t N_t \tag{4}$$

In the special case of time independent parameter values, which moreover correspond to the critical condition, we have in the absence of an external source

$$\chi(E)J_c N_c - K_c N_c = 0 \tag{4a}$$

where J_c, K_c are the above-defined operators, in which the parameters are chosen to correspond to criticallity, and

$$\chi(E) = (1-\beta)\chi_0(E) + \sum_i \chi_i(E)\beta_i \tag{5}$$

Let us now consider the case where the parameters (cross-sections) will be changed periodically in a certain region of the reactor, such that

$$\Sigma_f(E,t,\vec{r}) = \Sigma_f^c(E,\vec{r}) + \sigma_f(E,t,\vec{r})$$

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etc. If the amplitudes of these parameter oscillations are small as compared with the mean values of the parameters—which shall correspond to the critical state—then we can write the neutron density in phase space in the form

$$N_t(\vec{r},\vec{E},\omega) = N_c(\vec{r},\vec{E},\omega) + n_t(\vec{r},\vec{E},\omega) \tag{6}$$

For the operators J_t, K_t we can write

$$J_t = J_c + j_t \tag{7}$$

$$K_t = K_c + k_t \tag{8}$$

where the operators j_t and k_t are given by the expressions

$$j_t N_t = \frac{1}{4\pi} \int\!\!\int dE d\Omega\, \nu(E) \sigma_f(E,t,\vec{r}) v N_t(\vec{r},\vec{E},\omega) \tag{9}$$

$$k_t N_t = \sigma_{tot}(\vec{r},E,t) v N_t(\vec{r},\vec{E},\omega) - \int_0^{E_0} dE' \int d\Omega' \sigma_s(E,E',\omega \cdot \omega,\vec{r},t) v' N_t(\vec{r},\vec{E'},\omega') \tag{10}$$

Since we assume the oscillations to be of small amplitude, we can neglect second order terms except in the sample itself, where the flux will always be somewhat disturbed. Thus in general we have to interpret the cross-sections σ as effective cross-sections, differing from the real cross-sections by a self-shielding factor.

We obtain from eqs. (4) and (4a)

$$\frac{\partial n_t}{\partial t} = (1-\beta)\chi_0(E) J_c n_t + \sum_i \chi_i(E)\beta_i\lambda_i \int_{-\infty}^t dt' J_c n_{t'} e^{-\lambda_i(t-t')}$$

$$- K_c n_t + (1-\beta)\chi_0(E) j_t N_c + \sum_i \chi_i(E)\beta_i\lambda_i \int_{-\infty}^t dt' j_{t'} N_c e^{-\lambda_i(t-t')} - k_t N_c \tag{11}$$

Since the changes of the cross-sections are assumed to be periodic in time, we can expand them in a Fourier series. Let T be the period. It is clear that n_t is also a periodic function with the same period T. Since the equations (11) are linear, it is sufficient to know the response n_t to a harmonic oscillation of the parameters, for the general problem can then be solved by a superposition of such solutions. Therefore we assume that

$$j_t = j_\rho e^{i\rho t} \qquad k_t = k_\rho e^{i\rho t} \tag{12}$$

Then we will have a steady state oscillation n_t of the form

$$n_t = n_\rho e^{i\rho t} \tag{13}$$

where n_ρ is in general a complex quantity since the oscillation n_t is not in phase with the parameter oscillations, except in particular cases.

We obtain from (11)

$$(K_c + i\rho - L(E,\rho)J_c)n_\rho = (L(E,\rho)j_\rho - k_\rho)N_c \tag{14}$$

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where

$$L(E,\rho) = (1-\beta)\chi_0(E) + \sum_k \frac{\chi_0(E)\beta_k\lambda_k}{\lambda_k + i\rho} \qquad (15)$$

is a known function.

Let $\Gamma_\rho(\vec{r},E,\vec{\omega};\vec{r}',E'\vec{\omega}')$ be the Green's function of the operator

$$K_c + i\rho - L(E,\rho)J_c$$

i.e. the solution of the equation

$$(K_c + i\rho - L(E,\rho)J_c)\Gamma_\rho = \delta(\vec{r}-\vec{r}')\delta(E-E')\delta(\vec{\omega}-\vec{\omega}') \qquad (16)$$

satisfying the same boundary conditions as the neutron density. Thus $\Gamma_\rho(\vec{r},E,\vec{\omega};\vec{r}',E',\vec{\omega}')$ is the neutron density in phase space due to a monodirectional source at the point $\vec{r}'$. Also it can be interpreted as a measure of importance of a neutron at $(\vec{r}',E',\vec{\omega}')$ to the neutron density at $(\vec{r},E,\vec{\omega})$.

If this function were known, we could calculate n_ρ directly as a function of $j_\rho N_c$ and $k_\rho N_c$:

$$n_\rho(\vec{r},E,\vec{\omega}) = \iiint d^3r'\,d\Omega'\,dE'\,\Gamma_\rho(\vec{r},E,\vec{\omega};\vec{r}',E',\vec{\omega}')\{L(E',\rho)j_\rho N_c' - k_\rho N_c'\} \qquad (17)$$

where the prime at $j_\rho N_c$ and $k_\rho N_c$ indicates that these expressions have to be taken at the point $(\vec{r}',E',\vec{\omega}')$.

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2 — DEFINITION OF THE RESPONSE FUNCTIONS
FOR ABSORPTION AND PRODUCTION AND OF THE RESPECTIVE INTEGRALS

Let a detector be placed at a point $\vec{r}$ in the reactor. This detector can be assumed to have a large bandwidth, so that its response can be taken independent of the frequency ρ with which the flux oscillates at the point $\vec{r}$. But in general the response will be a function of the neutron energy and neutron direction. Let the response function be $R(E,\vec{\omega})$, that is, the signal at the detector output is

$$S(t) = S_c + s_\rho e^{i\rho t} = \int\int dE d\Omega R(E,\vec{\omega}) N_c(\vec{r},E,\vec{\omega}) + e^{i\rho t}\int\int dE d\Omega R(E,\vec{\omega}) n_\rho(\vec{r},E,\vec{\omega}) \quad (18)$$

We are mainly interested in the periodic signal s_ρ. Inserting for $n_\rho(\vec{r},E,\vec{\omega})$ the expression eq. (17) we get

$$s_\rho = \int\int dE' d\Omega' d^3 r' C_\rho(\vec{r};\vec{r}',E',\vec{\omega}')\{L(E',\rho)j_\rho N_c' - k_\rho N_c'\} \quad (19)$$

where

$$C_\rho(\vec{r};\vec{r}',E',\vec{\omega}') = \int\int dE d\Omega R(E,\vec{\omega})\Gamma_\rho(\vec{r},E,\vec{\omega};\vec{r}',E',\vec{\omega}') \quad (20)$$

can be interpreted as the contribution of a neutron at r' with energy E' and direction $\vec{\omega}'$ to the counting rate of the detector at $\vec{r}$.

From now on we shall restrict our consideration to samples occupying only a small volume V around the point r_0, i.e. we put $\sigma_f(E,t,\vec{r}) \approx \sigma_f(E) V e^{i\rho t} \delta(\vec{r}-\vec{r}_0)$ etc. Using the expressions (9) and (10), eq. (19) becomes

$$s_\rho = G_f(\vec{r},\vec{r}_0,\rho) P(\vec{r}_0) - V\int\int dE' d\Omega' C_\rho(\vec{r};\vec{r}_0,E',\vec{\omega}')\sigma_a(E')v'N_c(\vec{r}_0,E',\vec{\omega}') + VG_s(\vec{r},\vec{r}_0,\rho) \quad (21)$$

where

$$G_f(\vec{r},\vec{r}_0,\rho) = \frac{1}{4\pi}\int\int dE' d\Omega' C_\rho(\vec{r};\vec{r}_0,E',\vec{\omega}')L(E',\rho) \quad (22)$$

$$P(\vec{r}_0) = V\int\int dE d\Omega \nu(E)\sigma_f(E)vN_c(\vec{r}_0,E,\vec{\omega}) \quad (23)$$

$$G_s(\vec{r},\vec{r}_0,\rho) = \int\int dE' d\Omega' C_\rho(\vec{r};\vec{r}_0,E',\vec{\omega}')$$
$$\left\{\int_0^{E_0} dE'' \int d\Omega'' \sigma_s(E',E'',\vec{\omega}'\cdot\vec{\omega}'')v''N_c(\vec{r}_0,E'',\vec{\omega}'') - \sigma_s(E')v'N_c(\vec{r}_0,E',\vec{\omega}')\right\} \quad (24)$$

If the oscillation is done with a non-moderating sample, then $\sigma_s(E',E'',\vec{\omega}'\cdot\vec{\omega}'') = \sigma_s(E',\vec{\omega}'\cdot\vec{\omega}'')\delta(E'-E'')$ and if furthermore the neutron density at $\vec{r}_0$ is isotropic in the velocity space (which is the case in the center of the reactor) then $G_s(\vec{r},\vec{r}_0,\rho)$ vanishes for all $\vec{r}$ and ρ.

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Up to now, we did not introduce any simplifying assumptions except for the neglect of higher order terms in eq. (11). Therefore, eq. (21) is quite general and can be used for any type of reactors — fast, intermediate or thermal. In the particular case of thermal reactors, almost all absorptions occur in the thermal range or slightly above — let us say between the energies zero and 5 kT. If the point $\vec{r}$ is not very close to $\vec{r}_0$, the function $C_\rho(\vec{r},\vec{r}_0,E,\vec{\omega})$ should not vary much, if E' is varied within the thermal range, especially in a heterogeneous reactor, since a neutron originating at $\vec{r}$ with a particular energy E' will alter this energy very soon as a result of collisions with moderator nuclei, so that its energy will fluctuate within the thermal range. Also the angular direction, in which the neutron is emitted, will not have a large influence, if $\vec{r}$ is far enough from $\vec{r}_0$. Therefore, for thermal reactors we can assume approximately that

$$C_\rho(\vec{r},\vec{r}_0,E',\vec{\omega}') \approx G_a(\vec{r},\vec{r}_0,\rho) \equiv \frac{1}{5kT}\int_0^{5kT} dE \int \frac{d\Omega}{4\pi} C_\rho(\vec{r};\vec{r}_0,E,\vec{\omega}) \qquad if \ 0 \leqslant E' \leqslant 5kT \qquad (25)$$

Thus we can write for thermal reactors and non-moderating samples

$$s_\rho = G_f(\vec{r},\vec{r}_0,\rho)P(\vec{r}_0) - G_a(\vec{r},\vec{r}_0,\rho)A(\vec{r}_0) \qquad (26)$$

where

$$A(\vec{r}_0) = V \int\int dE d\Omega \sigma_a(E) v N_r(\vec{r}_0,E,\vec{\omega}) \qquad (27)$$

that is, $A(\vec{r}_0)$—the "absorption integral"— is the total neutron absorption per unit time in the sample (under the irradiation given by $v N_r(\vec{r}_0,E,\vec{\omega})$), and $P(\vec{r}_0)$—the "production integral"—is the number of fission neutrons produced per unit time by the sample. The functions $G_f(\vec{r},\vec{r}_0,\rho)$ and $G_a(\vec{r},\vec{r}_0,\rho)$, which we shall call "production response function" and "absorption response function" respectively, do not depend any more on the sample characteristics but only on the properties of the reactor and the detector in question.

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3 — DETERMINATION OF THE RESPONSE FUNCTIONS
AND THE RELATED INTEGRALS

In order to calculate these functions, one has first to calculate the neutron distribution N_c in the critical state of the reactor and then one must determine Γ_ρ as a solution of eq. (16). From Γ_ρ one obtains the response functions after some integrations as indicated in eqs. (20), (22) and (25). Of course, the exact solution of the equation (4a) for N_c and of eq. (16) for Γ_ρ is not possible, but if one uses standard approximation methods as multigroup diffusion theory, Monte-Carlo- or S_N-calculations, one can obtain an answer with a high degree of accuracy. Knowing the functions G_f and G_a we can—in the case of a well-thermalized reactor—determine the production and absorption integrals from the signal of the detector by using eq. (26). In order to separate the two integrals in question, one could examine the response signal s_ρ for two frequencies ρ_1 and ρ_2 and solve the two corresponding equations for P and A, obtaining

$$P = \{ G_a(\vec{r},\vec{r}_0,\rho_2)s_{\rho_1} - G_a(\vec{r},\vec{r}_0,\rho_1)s_{\rho_2} \}/\Delta(\vec{r},\vec{r}_0,\rho_1,\rho_2) \tag{28}$$

$$A = \{ G_f(\vec{r},\vec{r}_0,\rho_2)s_{\rho_1} - G_f(\vec{r},\vec{r}_0,\rho_1)s_{\rho_2} \}/\Delta(\vec{r},\vec{r}_0,\rho_1,\rho_2) \tag{29}$$

where

$$\Delta(\vec{r},\vec{r}_0,\rho_1,\rho_2) = \begin{vmatrix} G_f(\vec{r},\vec{r}_0,\rho_1) & G_f(\vec{r},\vec{r}_0,\rho_2) \\ G_a(\vec{r},\vec{r}_0,\rho_1) & G_a(\vec{r},\vec{r}_0,\rho_2) \end{vmatrix} \tag{30}$$

Examining approximate expressions for G_a and G_f as can be found using two-group diffusion theory, one finds that a determination of P and A on the basis of the frequency response is possible only if one locates the detector very close to the sample (i.e. $|\vec{r}-\vec{r}_0|$ smaller than a relaxation length). Otherwise the determinant Δ becomes too small.

Usually it is preferable to use a spatial discrimination instead of a frequency discrimination. For this purpose one records the same (usually the fundamental) harmonic in two different detectors 1 and 2 at $\vec{r}_1$ and $\vec{r}_2$. Since the response functions $R^1(E,\omega)$ and $R^2(E,\omega)$ of the two detectors will, in general, not be the same, one has to calculate separately the functions G_a^1, G_f^1 and G_a^2, G_f^2. Once these functions are known, one can determine A and P from the signals of the two detectors:

$$P = \{ G_a^2(\vec{r}_2,\vec{r}_0,\rho)s_\rho^1 - G_a^1(\vec{r}_1,\vec{r}_0,\rho)s_\rho^2 \}/\Theta_{12}(\vec{r}_1,\vec{r}_2,\vec{r}_0;\rho) \tag{28'}$$

$$A = \{ G_f^2(\vec{r}_2,\vec{r}_0,\rho)s_\rho^1 - G_f^1(\vec{r}_1,\vec{r}_0,\rho)s_\rho^2 \}/\Theta_{12}(\vec{r}_1,\vec{r}_2,\vec{r}_0;\rho) \tag{29'}$$

where

$$\Theta_{12}(\vec{r}_1,\vec{r}_2,\vec{r}_0;\rho) = \begin{vmatrix} G_f^1(\vec{r}_1,\vec{r}_0,\rho) & G_f^2(\vec{r}_2,\vec{r}_0,\rho) \\ G_a^1(\vec{r}_1,\vec{r}_0,\rho) & G_a^2(\vec{r}_2,\vec{r}_0,\rho) \end{vmatrix} \tag{30'}$$

Again it is necessary to place one of the two detectors very close to the sample, the other rather far away in order to get a large value for the determinant.

It is even possible to make relative measurements of the production and absorption integrals without knowing the functions G_a and G_f [3]. In order to do so, one first has to oscillate a standard sample of some suitable $1/v$-absorber (for instance boron).

One then obtains (the index "0" referring to the standard sample)

$$s_\rho^0 = -G_a(\vec{r},\vec{r}_0,\rho)A_0(\vec{r}_0) \tag{31}$$

Afterwards one oscillates the (non-fissionable) absorber under exactly the same conditions as the standard sample. The ratio of the two absorption integrals is then simply given by the ratio of the two signals:

$$A/A_0 = s_\rho/s_\rho^0 \tag{32}$$

The situation becomes more difficult in the case of a fissionable sample. Here it is necessary to introduce a second standard sample, containing uranium 235. Since the absorption cross-section of uranium 235 has an almost pure $1/v$-behaviour [2], the ratio

$$Z = A_{25}/A_0$$

will be independent of the position in the reactor and can be obtained by a simple calculation.

Eq. (26) can now be written in the following form

$$G_f(\vec{r},\vec{r}_0,\rho)P_{25} = s_\rho^{25} - s_\rho^0 Z$$

For an arbitrary sample containing fissionable material we can now write

$$P/P_{25} = \frac{s_\rho - s_\rho^0(A/A_0)}{s_\rho^{25} - s_\rho^0 Z} \tag{33}$$

Defining the functions a and b by

$$a = s_\rho^{25} - s_\rho^0 Z$$
$$b = -s_\rho^0$$

we can rewrite eq. (33) in the simple form

$$s_\rho = a \cdot (P/P_{25}) + b \cdot (A/A_0) \tag{34}$$

It is to be kept in mind, that the functions a and b are determined by measurements with the standards only. They are functions of detector location and frequency.

Eq. (34) is analogous to eq. (26) except that instead of the functions G_a and G_f there appear the experimentally determined coefficients a and b and instead of the absolute production and absorption integrals there intervene the relative ones. But it is clear, that the same methods for separating A and P that we have discussed above can be applied here too for the determination of A/A_0 and P/P_{25}.

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Appendix

CALCULATION OF THE FUNCTIONS G_a, G_f,
FOR AN OSCILLATOR IN THE CENTER OF A SPHERICAL BARE REACTOR,
USING TWO-GROUP DIFFUSION THEORY [4]

In the two-group diffusion approximation we have two possible energy values, E_1 and E_2, which we shall indicate simply in form of indices at the physical quantities, i.e. $\phi_1 = \phi(E_1)$ etc. We can write

$$\chi_{i,1} = \chi_{0,1} = 1 \qquad \chi_{i,2} = \chi_{0,2} = 0 \tag{35}$$

$$J_c N = k\Sigma_a \phi_2 \tag{36}$$

$$K_c N = \begin{pmatrix} -D_1\nabla^2 + \Sigma_{12} & 0 \\ -\Sigma_{12} & -D_2\nabla^2 + \Sigma_a \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{37}$$

and eq. (4a) simply becomes

$$D_1\nabla^2\phi_1 - \Sigma_{12}\phi_1 + k\Sigma_a\phi_2 = 0$$
$$D_2\nabla^2\phi_2 - \Sigma_a\phi_2 + \Sigma_{12}\phi_1 = 0 \tag{38}$$

Trying a solution of the form

$$\phi_1 = C_1 X, \quad \phi_2 = C_2 X$$

where X is a solution of

$$\nabla^2 X + B_0^2 X = 0$$

vanishing at the extrapolated surface of the reactor, leads to the well-known critical condition

$$\begin{vmatrix} D_1 B_0 + \Sigma_{12} & -k\Sigma_a \\ -\Sigma_{12} & D_2B_0 + \Sigma_a \end{vmatrix} = 0 \tag{39}$$

Equation (39) has two values of B_0^2 as roots of which the larger will be denoted by μ_0^2, the smaller by $-\nu_0^2$. Writing

$$L(E_1,\rho) = f(\rho) \qquad L(E_2,\rho) = 0 \tag{40}$$

we have

$$f(\rho) = 1 - \beta + \sum_k \frac{\beta_k \lambda_k}{\lambda_k + i\rho} = 1 - i\rho \sum_k \frac{\beta_k}{\lambda_k + i\rho} \tag{40a}$$

which can be written in terms of its real and imaginary parts

$$f(\rho) = 1 - \psi_1(\rho) - i\psi_2(\rho) \tag{41}$$

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where

$$\psi_1(\rho) = Re\left\{ \iota\rho \sum_k \frac{\beta_k}{\lambda_k + i\rho} \right\} = \sum_k Re\,\frac{i\rho\beta_k}{\lambda_k + i\rho} = \sum_k \frac{\rho^2\beta_k}{\lambda_k^2 + \rho^2} \tag{42}$$

$$\psi_2(\rho) = Im\left\{ i\rho \sum_k \frac{\beta_k}{\lambda_k + i\rho} \right\} = \sum_k Im\,\frac{i\rho\beta_k}{\lambda_k + i\rho} = \sum_k \frac{\rho\beta_k\lambda_k}{\lambda_k^2 + \rho^2} \tag{43}$$

The functions $\psi_1(\rho)$ and $\psi_2(\rho)$ are given in table I.

Eq. (16) now becomes

$$\begin{pmatrix} D_1\nabla^2 - \Sigma_{12} - i\rho/v_1 & f(\rho)k\Sigma_a \\ \Sigma_{12} & D_2\nabla^2 - \Sigma_a - i\rho/v_2 \end{pmatrix} \begin{pmatrix} \Gamma_{11}(\vec{r},\vec{r}',\rho) \\ \Gamma_{21}(\vec{r},\vec{r}',\rho) \end{pmatrix} = -\begin{pmatrix} \delta(\vec{r}-\vec{r}') \\ 0 \end{pmatrix} \tag{44}$$

$$\begin{pmatrix} D_1\nabla^2 - \Sigma_{12} - i\rho/v_1 & f(\rho)k\Sigma_a \\ \Sigma_{12} & D_2\nabla^2 - \Sigma_a - i\rho/v_2 \end{pmatrix} \begin{pmatrix} \Gamma_{12}(r,r',\rho) \\ \Gamma_{22}(\vec{r},\vec{r}',\rho) \end{pmatrix} = -\begin{pmatrix} 0 \\ \delta(\vec{r}-\vec{r}') \end{pmatrix} \tag{45}$$

where all $\Gamma_{ik}(\vec{r},\vec{r}',\rho)$ have to vanish for values of $\vec{r}$ that correspond to the extrapolated surface of the reactor.

For a spherical reactor and a source in the center, i.e. $\vec{r}' = 0$, it is clear from symmetry considerations, that the $\Gamma_{ik}(\vec{r},0,\rho)$ depend only on r (and ρ). We can construct these solutions of eqs. (44) and (45) by superposition of the solutions u and w of the equations

$$\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \mu^2 u = 0 \qquad \frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\frac{dw}{dr}\right) - v^2 w = 0 \tag{46}$$

which vanish at the extrapolated boundaries of the reactor, i.e. at $r = R$. The quantities μ^2 and $-v^2$ are the roots of the equation

$$\begin{vmatrix} D_1 B^2 + \Sigma_{12} + i\rho/v_1 & -f(\rho)k\Sigma_a \\ -\Sigma_{12} & D_2 B^2 + \Sigma_a + i\rho/v_2 \end{vmatrix} = 0$$

which can be written in the more familiar form

$$\begin{vmatrix} 1 + L_1^2 B^2 + i\rho\tau_1 & -f(\rho)k \\ -1 & 1 + L_2^2 B^2 + i\rho\tau_2 \end{vmatrix} = 0 \tag{47}$$

where

$$L_1^2 = D_1/\Sigma_{12} \qquad L_2^2 = D_2/\Sigma_a$$

and

$$\tau_1 = \frac{1}{v_1\Sigma_{12}} \qquad \tau_2 = \frac{1}{v_2\Sigma_a}$$

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For a reactor of ORGEL type, τ_1 is of the order of some fractions of a micro-second and can therefore be neglected, since we consider only small frequencies. For small frequencies we can write

$$\mu^2 = \mu_0^2 + \delta\mu^2 \qquad \nu = \nu_0^2 + \delta\nu^2 \qquad (48)$$

Taking into account eq. (39) and neglecting higher order terms, we obtain

$$\delta\mu^2 = -\frac{\psi_1 k + i(\psi_2 k + \rho\tau_2\,[1 + L_1^2\mu_0^2])}{L_1^2 + L_2^2 + 2\,L_1^2 L_2^2 \mu_0^2}$$

(49)

$$\delta\nu^2 = +\frac{\psi_1 k + i(\psi_2 k + \rho\tau_2\,[1 - L_1^2\nu_0^2])}{L_1^2 + L_2^2 - 2\,L_1^2 L_2^2 \nu_0^2}$$

After this calculation of ν^2 and μ^2, we proceed to the solution of the eqs. (46) in which the substitutions $u = U/r$, $w = W/r$ lead to the simple equations

$$\frac{d^2 U}{dr^2} + \mu^2 U = 0 \qquad \frac{d^2 W}{dr^2} - \nu^2 W = 0 \qquad (50)$$

The solutions of the eqs. (50) that vanish at $r = R$ are (normalized to 1 for $r = 0$)

$$U = \frac{\sin \mu(R - r)}{\sin \mu R}$$

(51)

$$W = \frac{\sinh \nu(R - r)}{\sinh \nu R}$$

We can now try a solution of eq. (44) in the following form:

$$r\Gamma_{11}(r,0,\rho) = AU + BW$$

(52)

$$r\Gamma_{21}(r,0,\rho) = ASU + BS'W$$

where

$$S = \frac{\Sigma_{12}}{\Sigma_a} \cdot \frac{1}{1 + L_2^2\mu^2 + i\rho\tau_2} \approx S_0\left(1 - \frac{L_2^2\delta\mu^2 + i\rho\tau_2}{1 + L_2^2\mu_0^2}\right)$$

(53)

$$S' = \frac{\Sigma_{12}}{\Sigma_a} \cdot \frac{1}{1 - L_2^2\nu^2 + i\rho\tau_2} \approx S'_0\left(1 - \frac{L_2^2\delta\nu^2 - i\rho\tau_2}{1 - L_2^2\nu_0^2}\right)$$

with

$$S_0 = \frac{\Sigma_{12}}{\Sigma_a} \cdot \frac{1}{1 + L_2^2\mu_0^2} \qquad S'_0 = \frac{\Sigma_{12}}{\Sigma_a} \cdot \frac{1}{1 - L_2^2\nu_0^2} \qquad (54)$$

Since Γ_{21} is everywhere regular, $r\Gamma_{21}(r,0,\rho)$ must vanish for $r = 0$, which yields the condition

$$AS + BS' = 0 \qquad (55)$$

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Because of the δ-function there will be a singular behaviour of Γ_{11} at $r=0$, which is such that

$$\left[-D_1 \cdot 4\pi r^2 \frac{d}{dr}\Gamma_{11}(r,0) \right]_{r=0} = 1$$

This gives the condition

$$4\pi D_1(A + B) = 1 \tag{56}$$

We thus obtain

$$A = -\frac{S'}{4\pi D_1(S-S')} \qquad B = \frac{S}{4\pi D_1(S-S')} \tag{57}$$

and therefore

$$\Gamma_{11}(r,0,\rho) = \frac{1}{4\pi r D_1(S-S')}\{ -S'U + SW \} = \frac{1}{4\pi r D_1}\{ \beta U + \alpha W \}$$

$$\Gamma_{21}(r,0,\rho) = \frac{-SS'}{4\pi r D_1(S-S')}\{ U - W \} \tag{58}$$

where

$$\alpha = \frac{S}{S-S'}, \quad \beta = \frac{-S'}{S-S'}.$$

By the same procedure we find

$$\Gamma_{12}(r,0,\rho) = \frac{1}{4\pi r D_2(S-S')}\{ U - W \}$$

$$\Gamma_{22}(r,0,\rho) = \frac{1}{4\pi r D_2(S-S')}\{ SU - S'W \} = \frac{1}{4\pi r D_2}\{ \alpha U + \beta W \} \tag{59}$$

Let us assume that our detector responds only to thermal neutrons and that its thermal counting efficiency is unity, i.e. $R_1 = 0$, $R_2 = 1$. Then we get from eq. (20):

$$C_1(r,0,\rho) = \Gamma_{21}(r,0,\rho) \qquad C_2(r,0,\rho) = \Gamma_{22}(r,0,\rho) \tag{60}$$

Finally we obtain

$$G_f(r,0,\rho) = C_1(r,0,\rho)f(\rho) = \frac{-S'Sf(\rho)}{4\pi r D_1(S-S')}\{ U - W \}$$

$$G_a(r,0,\rho) = C_2(r,0,\rho) = \frac{1}{4\pi r D_2}\{ \alpha U + \beta W \} \tag{61}$$

Numerical example:

We take a spherical unreflected core with the following properties:

$$k = 1.08, \ L_1 = 120 \text{ cm}^2, \ L_2^2 = 200 \text{ cm}^2, \ D_1 = 1.2 \text{ cm}, \ D_2 = 1 \text{ cm},$$

thus $\Sigma_{12} = 10^{-2} \text{ cm}^{-1}, \Sigma_a = 5 \times 10^{-3} \text{ cm}^{-1}, \ \tau_1 = 0, \tau_2 = 0.91 \times 10^{-3} \text{ sec.}$

We then obtain:

$$\mu_0^2 = 2.452 \times 10^{-4} \text{ cm}^{-2}$$

$$\nu_0^2 = 1.358 \times 10^{-2} \text{ cm}^{-2}$$

and

$$R = \frac{\pi}{\mu_0} = 2.008 \text{ m}$$

We consider two oscillation frequencies, $\rho_1 = 0.314 \text{ sec}^{-1}$ and $\rho_2 = 3.14 \text{ sec}^{-1}$, corresponding to the periods $T_1 = 20 \text{ sec}$ and $T_2 = 2 \text{ sec}$, respectively. We use the following convenient abbreviations:

$$M_f(r,\rho) = 10^2 \times (\text{Magnitude of } G_f(r,0,\rho))$$
$$\varphi_f(r,\rho) = -\text{Phase of } G_f(r,0,\rho) \text{ in degrees}$$
and similarly for G_a.

As a result of the indicated numerical calculations we obtain the values of table II.

TABLE I

THE FUNCTIONS $\psi_1(\rho)$ AND $\psi_2(\rho)$

T(sec)	100	50	20	10	5	2	1	0.5	0.1
$\rho(\text{sec}^{-1})$	0.0628	0.126	0.314	0.628	1.26	3.14	6.28	12.6	62.8
$\psi_1 \times 10^3$	1.750	2.622	4.088	5.067	5.695	6.163	6.331	6.378	6.398
$\psi_2 \times 10^3$	1.681	1.955	2.018	1.639	1.180	0.667	0.385	0.202	0.041

TABLE II

MAGNITUDE AND PHASE OF G_f AND G_a AS FUNCTIONS OF r

r (cm)	10.8	20.8	40.8	60.8	80.8	100.8	125.8	150.8	175.8	200.8
$M_f(r,\rho_1)$	0.994	0.879	0.750	0.662	0.568	0.439	0.343	0.217	0.100	0.000
$\varphi_f(r,\rho_1)$	21.5	24.2	27.1	28.5	29.3	29.8	30.3	30.5	30.7	—
$M_f(r,\rho_2)$	0.752	0.640	0.526	0.452	0.383	0.314	0.228	0.143	0.0654	0.000
$\varphi_f(r,\rho_2)$	18.5	21.7	25.4	27.3	28.4	29.1	29.8	30.1	30.4	—
$M_a(r,\rho_1)$	1.224	0.983	0.764	0.687	0.582	0.450	0.351	0.223	0.103	0.000
$\varphi_a(r,\rho_1)$	19.0	23.3	27.2	28.4	29.3	29.8	30.2	30.5	30.7	—
$M_a(r,\rho_2)$	0.966	0.688	0.540	0.463	0.392	0.325	0.233	0.147	0.0678	0.000
$\varphi_a(r,\rho_2)$	14.7	20.6	25.3	27.3	28.4	29.1	29.8	30.1	30.4	—

LITERATURE

[1] Reactor Handbook - Physics, McGraw Hill - New York (1955), p. 532.

[2] BNL 325.

[3] V. RAJEVSKI, personal communication.

[4] V. RAJEVSKI, *Physique des Piles Atomiques*, Paris 1960, Chap. VII.