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PRIMARY MODULES OVER COMMUTATIVE RINGS

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Abstract. The radical of a module over a commutative ring is the intersection of all prime submodules. It is proved that if R is a commutative domain which is either Noetherian or a *UFD* then R is one-dimensional if and only if every (finitely generated) primary R-module has prime radical, and this holds precisely when every (finitely generated) R-module satisfies the radical formula for primary submodules.

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Prime submodules of modules over commutative rings have been studied by various authors. In particular, a number of papers have been devoted to trying to calculate the radical of a module; see, for example, [**2**]–[**6**]. Only in special cases is there a simple description of the radical. It is natural therefore to ask whether the radical of a primary module has a simple description. McCasland and Moore [**5**, Theorem 2.10] proved that if R is a PID (principal ideal domain) and M is the free R-module $R^{(n)}$, for some positive integer n, then the radical of any primary submodule of M is a prime submodule of M. We shall show that this is not true in general.

Given a commutative ring R, R-modules which satisfy the radical formula were first considered by McCasland and Moore [**5**]. In [**3**], commutative Noetherian rings R such that every R-module satisfies the radical formula, were characterized. In this paper, we characterize all commutative Noetherian rings R such that every R-module satisfies the radical formula for primary submodules and show that these are precisely the commutative Noetherian rings such that every primary R-module has prime radical.

Throughout this note all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. For any submodule N of M let $(N : M) = \{r \in R : rM \subseteq N\}$. A submodule N of M is called *prime* (respectively, *primary*) if $N \neq M$ and whenever $r \in R$, $m \in M$ and $rm \in N$ then $m \in N$ or $r \in (N : M)$ (respectively, $r \in \sqrt{(N : M)}$). Note that for any ideal $\mathbf{a}$ of R,

$$\sqrt{\mathbf{a}} = \{r \in R : r^n \in \mathbf{a} \text{ for some positive integer } n\}.$$

If N is a primary submodule of M and $\mathbf{p} = \sqrt{(N : M)}$, then $\mathbf{p}$ is a prime ideal of R and we shall call N $\mathbf{p}$-primary. The module M will be called *primary* if its zero submodule is primary. For any submodule N of an R-module M, the *radical*, $\mathrm{rad}_M(N)$, of N is defined to be the intersection of all prime submodules of M containing N and $\mathrm{rad}_M(N) = M$ if N is not contained in any prime submodule of M. The *radical of the module M* is defined to be $\mathrm{rad}_M(0)$.

Given a ring R and a submodule N of an R-module M we define

$$E_M(N) = \{rm : r \in R, m \in M \text{ and } r^k m \in N \text{ for some positive integer } k\}.$$

Then $RE_M(N)$ will denote the submodule of M generated by the non-empty subset $E_M(N)$ of M. Note that $RE_M(N)$ consists of all finite sums of elements of $E_M(N)$. McCasland and Moore [**5**] say that the module M *satisfies the radical formula* if $\mathrm{rad}_M(N) = RE_M(N)$ for any submodule N of M. Now we say that the module M *satisfies the radical formula for primary submodules* if $\mathrm{rad}_M(N) = RE_M(N)$ for every primary submodule N of M.

1. Radicals of primary modules. The first result is well known and the proof (which is easy in any case) is omitted.

LEMMA 1.1. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* M *be an* R*-module. Then a proper submodule* N *of* M *is* $\mathbf{p}$*-primary if and only if*

(a) $\mathbf{p} \subseteq \sqrt{(N : M)}$, *and*

(b) $cm \notin N$, *for all* $c \in R\backslash\mathbf{p}$, $m \in M\backslash N$.

It is clear that for any module M, prime submodules of M are primary. In particular, maximal submodules (being prime) are primary. Thus, non-zero modules which are either projective or finitely generated have primary submodules (see [**1**, Proposition 17.14]). The next result shows that free modules have a rich supply of primary submodules.

PROPOSITION 1.2. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* $\mathbf{a}$ *be a* $\mathbf{p}$*-primary ideal of* R*. Let* M *be a free* R*-module with basis* $\{m_i : i \in I\}$*. Let* $(a_{\lambda i})$ *be an* $\Lambda \times I$ *matrix with entries in* R *such that* $a_{\mu j} \notin \mathbf{a}$ *for some* $\mu \in \Lambda$, $j \in I$*. Let* N *be the subset of* M *consisting of all elements* m *in* M *such that* $m = \sum_{i\in I} r_i m_i$ *for some* $r_i \in R (i \in I)$*, where* $r_i \neq 0$ *for at most a finite number of elements* $i \in I$*, and* $\sum_{i\in I} a_{\lambda i} r_i \in \mathbf{a}$ *for all* $\lambda \in \Lambda$*. Then* N *is a* $\mathbf{p}$*-primary submodule of* M*.*

Proof. It is clear that N is a submodule of M and $m_j \notin N$ so that $N \neq M$. Let $p \in \mathbf{p}$. There exists a positive integer k such that $p^k \in \mathbf{a}$ and hence $p^k M \subseteq \mathbf{a}M \subseteq N$. Let $m \in M$ and $c \in R\backslash\mathbf{p}$ such that $cm \in N$. There exist elements $s_i \in R (i \in I)$ such that $s_i \neq 0$ for at most a finite number of elements $i \in I$ and $m = \sum_{i\in I} s_i m_i$. Then $cm = \sum_{i\in I} cs_i m_i \in N$ implies that $\sum_{i\in I} a_{\lambda i}(cs_i) \in \mathbf{a}$ $(\lambda \in \Lambda)$, i.e. $c(\sum_{i\in I} a_{\lambda i} s_i) \in \mathbf{a}$ $(\lambda \in \Lambda)$ and hence $\sum_{i\in I} a_{\lambda i} s_i \in \mathbf{a}$. Thus $m \in N$. By Lemma 1.1, N is $\mathbf{p}$-primary.

We now turn our attention to the radical of primary modules. Note first the following elementary fact whose proof is omitted.

LEMMA 1.3. *Let* $\mathbf{p}$ *be a prime ideal of a ring* R *and let* N *be a* $\mathbf{p}$*-primary submodule of an* R*-module* M*. Then* $RE_M(N) = N + \mathbf{p}M \subseteq \mathrm{rad}_M(N)$.

LEMMA 1.4. *Let* $\mathbf{m}$ *be a maximal ideal of a ring* R*, let* M *be an* R*-module and let* N *be an* $\mathbf{m}$*-primary submodule of* M*. Then* $\mathrm{rad}_M(N) = N + \mathbf{m}M = RE_M(N)$*. Moreover* $N + \mathbf{m}M$ *is a prime submodule of* M *or* $M = N + \mathbf{m}M$*.*

Proof. Clearly $N \subseteq N + \mathbf{m}M$ and it is easy to check that $N + \mathbf{m}M$ is a prime submodule of M or $M = N + \mathbf{m}M$. Then $\mathrm{rad}_M(N) = N + \mathbf{m}M = RE_M(N)$ by Lemma 1.3.

Corollary 1.5. *Let R be a 0-dimensional ring. Then a primary R-module M has prime radical if and only if M contains a prime submodule.*

Proof. The necessity is clear. Conversely, suppose that M contains a prime submodule K. Let $\mathbf{p} = \sqrt{(0 : M)}$. Because R is 0-dimensional we know that $\mathbf{p}$ is a maximal ideal of R. Now Lemma 1.4 gives that $rad_M(0) = \mathbf{p}M$ and $\mathbf{p}M$ is a prime submodule of M because $\mathbf{p}M \subseteq K \neq M$.

Note that Sharif, Sharifi and Namazi [**6**, Theorem 2.8] proved that if R is a 0-dimensional ring then every R-module satisfies the radical formula. In Lemma 1.4, it is possible for M to equal $N + \mathbf{m}M$ for an $\mathbf{m}$-primary submodule N of M, as the following example shows.

Example 1.6. *There exists a ring R having a nil idempotent maximal ideal $\mathbf{m}$ so that if M is the R-module $\mathbf{m}$ then 0 is an $\mathbf{m}$-primary submodule of M but $M = \mathbf{m}M$.*

Proof. For any prime p, let F be any field of characteristic p and let G be the Prüfer p-group $C(p^\infty)$. Let R denote the group algebra $F[G]$. Then R is a commutative ring whose augmentation ideal $\mathbf{m}$ has the desired properties; (for more details, see [**8**, Lemma 5.5]).

If R is a Noetherian ring in Lemma 1.4, then $N + \mathbf{m}M$ is a prime submodule of M for any $\mathbf{m}$-primary submodule N of the R-module M. For, in this case, $\mathbf{m}^n \subseteq (N : M)$ for some positive integer n and hence $M = N + \mathbf{m}M$ implies that $M = N + \mathbf{m}^n M = N$, a contradiction. Thus $M \neq N + \mathbf{m}M$. In view of this, to study rings with the property that every primary module has prime radical it is natural to restrict to Noetherian rings.

Let R be any ring and let M be an R-module. If N is a proper submodule of M, then K is a *minimal prime submodule* of N in M if K is a prime submodule of M, $N \subseteq K$ and whenever L is a prime submodule of M with $K \supseteq L \supseteq N$ then $K = L$.

Lemma 1.7. *Let R be any ring, let $\mathbf{p}$ be a prime ideal of R and let N be a $\mathbf{p}$-primary submodule of a finitely generated R-module M. Let $K = \{m \in M : cm \in N + \mathbf{p}M$ for some $c \in R \backslash \mathbf{p}\}$. Then K is a minimal prime submodule of N in M.*

Proof. Clearly K is a submodule of M. Moreover $\mathbf{p}M \subseteq K$ and it is easy to check that the $(R/\mathbf{p})$-module M/K is torsion-free. Suppose that $M = K$. Because M is finitely generated, there exists $a \in R \backslash \mathbf{p}$ such that $aM \subseteq N + \mathbf{p}M$; i.e. $a(M/N) \subseteq \mathbf{p}(M/N)$. Using the usual determinant argument, we find that $b(M/N) = 0$ for some $b \in R \backslash \mathbf{p}$. But this implies that $bM \subseteq N$ and hence $b \in \mathbf{p}$, a contradiction. It follows that $M \neq K$ and K is a prime submodule of M.

Let L be a prime submodule of M such that $K \supseteq L \supseteq N$. Let $\mathbf{q} = (L : M)$. Suppose that $\mathbf{q} \not\subseteq \mathbf{p}$ and let $d \in \mathbf{q} \backslash \mathbf{p}$. Then $dM \subseteq L \subseteq K$ and hence $M = K$, a contradiction. Thus $\mathbf{q} \subseteq \mathbf{p}$. On the other hand, it is clear that $\mathbf{p}M \subseteq L$, so that $\mathbf{p} \subseteq \mathbf{q}$. Hence $\mathbf{p} = \mathbf{q}$. It follows that $N + \mathbf{p}M \subseteq L$ and that $K \subseteq L$. Thus $K = L$.

Let $\mathbf{q}$ be a prime ideal of a ring R and let n be a positive integer. Then the nth *symbolic power* $\mathbf{q}^{(n)}$ of $\mathbf{q}$ is defined by

$$\mathbf{q}^{(n)} = \{r \in R : rc \in \mathbf{q}^n \text{ for some } c \in R\backslash\mathbf{q}\}.$$

It is well known (and easy to check) that $\mathbf{q}^{(n)}$ is a $\mathbf{q}$-primary ideal of R.

LEMMA 1.8. *Let R be any ring such that every primary submodule of the R-module $R \oplus R$ has prime radical. Then $\mathbf{q} = \{r \in \mathbf{q} : rc \in \mathbf{pq} + \mathbf{q}^{(2)}$ for some $c \in R\backslash\mathbf{p}\}$ for all distinct prime ideals $\mathbf{p}$,$\mathbf{q}$ of R with $\mathbf{p} \supseteq \mathbf{q}$.*

Proof. Suppose not and that $\mathbf{p} \supsetneqq \mathbf{q}$ are prime ideals of R such that $\mathbf{q} \supsetneqq \mathbf{a}$, where $\mathbf{a} = \{r \in \mathbf{q} : rc \in \mathbf{pq} + \mathbf{q}^{(2)},$ for some $c \in R\backslash\mathbf{p}\}$. Let $a \in \mathbf{p}\backslash\mathbf{q}$ and $b \in \mathbf{q}\backslash\mathbf{a}$. Set $M = R \oplus R$ and let

$$N = \{m \in M : cm \in R(a, b) + \mathbf{q}^2 M, \text{ for some } c \in R\backslash\mathbf{q}\}.$$

Clearly $\mathbf{q}^2 M \subseteq N$ and N is $\mathbf{q}$-primary, by Lemma 1.1. Set $K = \mathrm{rad}_M(N)$. Then K is a prime submodule of M. By Lemma 1.7,

$$K = \{m \in M : cm \in R(a, b) + \mathbf{q}M \text{ for some } c \in R\backslash\mathbf{q}\}.$$

Note that

$$a(1, 0) = (a, 0) = (a, b) - (0, b) \in R(a, b) + \mathbf{q}M$$

and hence $(1, 0) \in K$. Let $x, y \in R$ such that $(x, y) \in N$. There exist $c \in R\backslash\mathbf{q}$, $r \in R$, $u, v \in \mathbf{q}^2$ such that

$$c(x, y) = r(a, b) + (u, v),$$

and hence

$$cx = ra + u \text{ and } cy = rb + v.$$

Note that $cy \in \mathbf{q}$ and hence $y \in \mathbf{q}$. Moreover

$$c(xb - ya) = ub - va \in \mathbf{q}^2,$$

so that $xb - ya \in \mathbf{q}^{(2)}$ and $xb \in \mathbf{pq} + \mathbf{q}^{(2)}$. By the choice of b, $x \in \mathbf{p}$. Thus $(x, y) \in \mathbf{p} \oplus \mathbf{p} = \mathbf{p}M$. It follows that $N \subseteq \mathbf{p}M$. Clearly $\mathbf{p}M$ is a prime submodule of M. Hence $K = \mathrm{rad}_M(N) \subseteq \mathbf{p}M$. However $(1, 0) \in K$ gives the contradiction $1 \in \mathbf{p}$.

THEOREM 1.9. *The following statements are equivalent for a Noetherian ring R with prime radical $\mathbf{n}$.*

(i) *Every primary R-module has prime radical.*
(ii) *The radical of every primary submodule of the R-module $R \oplus R$ is prime.*
(iii) *For every non-maximal prime ideal $\mathbf{q}$ of R there exists $c \in R\backslash\mathbf{q}$ such that $c\mathbf{q} = 0$.*
(iv) *Every non-maximal prime ideal $\mathbf{q}$ of R is the only $\mathbf{q}$-primary ideal of R.*
(v) *R is Artinian or R is one-dimensional and $\mathbf{n}$ is an Artinian R-module.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii). Let $\mathbf{q}$ be any non-maximal prime ideal of R. There exists a maximal ideal $\mathbf{p}$ of R such that $\mathbf{p} \supsetneqq \mathbf{q}$. By Lemma 1.8, $\mathbf{q} = \{r \in \mathbf{q} : rd \in \mathbf{pq} + \mathbf{q}^{(2)}$ for some $d \in R \backslash \mathbf{p}\}$. The usual determinant argument gives $d'\mathbf{q} \subseteq \mathbf{q}^{(2)}$ for some $d' \in R\backslash \mathbf{p}$. Hence $\mathbf{q} = \mathbf{q}^{(2)}$. By [**7**, Exercise 8.37] $c\mathbf{q} = 0$ for some $c \in R\backslash \mathbf{q}$.

(iii) $\Rightarrow$ (iv). This is clear.

(iv) $\Rightarrow$ (i). Let M be any primary R-module; i.e. the zero submodule is $\mathbf{q}$-primary for some prime ideal $\mathbf{q}$. Suppose that $\mathbf{q}$ is a maximal ideal of R. Then $\mathbf{q}M$ is a prime submodule of M and $\text{rad}_M(0) = \mathbf{q}M$, by Lemma 1.4. Now suppose that $\mathbf{q}$ is not maximal. There exists a positive integer n such that $\mathbf{q}^n M = 0$. By (iv) $\mathbf{q} = \mathbf{q}^{(n)}$ and hence $c\mathbf{q} \subseteq \mathbf{q}^n$ for some $c \in R\backslash \mathbf{q}$. Then $c\mathbf{q}M = 0$ and hence $\mathbf{q}M = 0$. By Lemma 1.1, M is a torsion-free $(R/\mathbf{q})$-module and hence 0 is a prime submodule of M. In any case, $\text{rad}_M(0)$ is prime.

(iii) $\Rightarrow$ (v). Suppose that $\mathbf{p}_0 \supsetneqq \mathbf{p}_1 \supsetneqq \mathbf{p}_2$ are distinct prime ideals of R. By (iii), there exists $b \in R\backslash \mathbf{p}_1$ such that $b\mathbf{p}_1 = 0 \subseteq \mathbf{p}_2$ and hence $\mathbf{p}_1 \subseteq \mathbf{p}_2$, a contradiction. Thus R is 0-dimensional, and hence Artinian by [**7**, Proposition 8.38], or R is one-dimensional. Set $\mathbf{a} = \{r \in R : r\mathbf{n} = 0\}$. Let $\mathbf{p}$ be a prime ideal of R such that the ideal $\mathbf{a} \subseteq \mathbf{p}$. If $\mathbf{p}$ is not maximal, then $c\mathbf{p} = 0$ for some $c \in R\backslash \mathbf{p}$. Hence $c\mathbf{n} = 0$ and $c \in \mathbf{a} \subseteq \mathbf{p}$, a contradiction. Thus $\mathbf{p}$ is maximal. It follows that every prime ideal of the ring $R/\mathbf{a}$ is maximal and hence $R/\mathbf{a}$ is an Artinian ring by [**7**, Proposition 8.38] again. Now $\mathbf{n}$ is a finitely generated $(R/\mathbf{a})$-module so that $\mathbf{n}$ is an Artinian $(R/\mathbf{a})$-module and hence also an Artinian R-module.

(v) $\Rightarrow$ (iii). If R is Artinian, then (iii) is clearly true. Now suppose that R is a one-dimensional ring such that $\mathbf{n}$ is an Artinian R-module. Let $\mathbf{q}$ be a non-maximal prime ideal of R. Then $\mathbf{q}/\mathbf{n}$ is a minimal prime ideal of the semiprime Noetherian ring $R/\mathbf{n}$. There exists $g \in R\backslash \mathbf{q}$ such that $g\mathbf{q} \subseteq \mathbf{n}$. Note that the R-module $\mathbf{n}$ has a composition series and hence $\mathbf{p}_1 \dots \mathbf{p}_k \mathbf{n} = 0$ for some positive integer k and maximal ideals $\mathbf{p}_i (1 \leq i \leq k)$ of R. Thus $\mathbf{p}_1 \dots \mathbf{p}_k g\mathbf{q} = 0$ and hence $c\mathbf{q} = 0$ for some $c \in R\backslash \mathbf{q}$ because $\mathbf{p}_i \nsubseteq \mathbf{q}$ $(1 \leq i \leq k)$.

COROLLARY 1.10. *Let R be a semiprime Noetherian ring. Then every primary R-module has prime radical if and only if R is at most one-dimensional.*

Proof. This follows at once from Theorem 1.9.

The following example illustrates the last theorem.

EXAMPLE 1.11. *Let R be the polynomial ring $\mathbb{Z}[X]$, M the R-module $R \oplus R$ and N the submodule $R(2, X) + R(X, 0)$. Then N is a $\mathbf{p}$-primary submodule of M, where $\mathbf{p}$ is the prime ideal RX, and* $\text{rad}_M(N) = K \cap \mathbf{m}M$, *where K is the prime submodule $R \oplus RX$ and* $\mathbf{m}$ *the maximal ideal $R2 + RX$.*

Proof. It is easy to check that N is $\mathbf{p}$-primary, because $N = \{(u, v) \in M : Xu - 2v \in \mathbf{p}^2\}$ and Proposition 1.2 applies, K and $\mathbf{m}M$ are prime submodules of M and $K = \{m \in M : cm \in N + \mathbf{p}M$ for some $c \in R\backslash \mathbf{p}\}$. Let L be a prime submodule of M such that $N \subseteq L$. Let $\mathbf{q} = (L : M)$. Note that $\mathbf{p} \subseteq \mathbf{q}$. If $\mathbf{p} = \mathbf{q}$, then $K \subseteq L$. Suppose that $\mathbf{p} \neq \mathbf{q}$. Then $\mathbf{q} = RX + Rq$ for some prime q in $\mathbb{Z}$. Next note that

$$K \cap \mathbf{m}M = \mathbf{m} \oplus RX \subseteq N + \mathbf{q}M$$

because if $\mathbf{q} \neq \mathbf{m}$ then $N + \mathbf{q}M = R \oplus \mathbf{q}$. It follows that $\mathrm{rad}_M(N) = K \cap \mathbf{m}M$.

Among non-Noetherian rings, the case of UFD's is of interest. For UFD's we have the following result.

THEOREM 1.12. *The following statements are equivalent for a UFD R.*

(i) *R is a PID.*

(ii) *Every primary R-module has prime radical.*

(iii) *The radical of every primary submodule of the R-module $R \oplus R$ is prime.*

Proof. (i) $\Rightarrow$ (ii). This follows from Corollary 1.10.

(ii) $\Rightarrow$ (iii). This is clear.

(iii) $\Rightarrow$ (i). It is sufficient to prove that R is one-dimensional. (See [**7**, Exercise 14.21].) Suppose not. Let $\mathbf{p} \supsetneq \mathbf{q} \supsetneq 0$ be distinct prime ideals of R, where $\mathbf{q}$ has height 1. Then $\mathbf{q} = Rx$ for some element x. By Lemma 1.8, there exists $c \in R \backslash \mathbf{p}$ such that $cx \in \mathbf{pq} + \mathbf{q}^{(2)} = \mathbf{p}x + Rx^2$ and hence $c \in \mathbf{p} + Rx \subseteq \mathbf{p}$, a contradiction. Thus R is one-dimensional.

2. The radical formula. In this section, our concern is with rings R such that every R-module satisfies the radical formula for primary submodules. An R-module M satisfies the radical formula for primary submodules if and only if $\mathrm{rad}_M(N) = N + \mathbf{p}M$ for every prime ideal $\mathbf{p}$ of R and $\mathbf{p}$-primary submodule N of M (Lemma 1.3).

LEMMA 2.1. *Let R be a ring such that for every non-maximal prime ideal* $\mathbf{q}$, $c\mathbf{q} = 0$ *for some element* $c \in R \backslash \mathbf{q}$. *Then every R-module satisfies the radical formula for primary submodules.*

Proof. Let M be an R-module and let N be a $\mathbf{p}$-primary submodule of M, for some prime ideal $\mathbf{p}$ of R. Suppose first that $\mathbf{p}$ is a maximal ideal of R. By Lemma 1.4, $\mathrm{rad}_M(N) = N + \mathbf{p}M = RE_M(N)$. Now suppose that $\mathbf{p}$ is not maximal. There exists $c \in R \backslash \mathbf{p}$ such that $c\mathbf{p} = 0$ and hence $c\mathbf{p}M = 0 \subseteq N$. It follows that $\mathbf{p}M \subseteq N$ and M/N is a torsion-free $(R/\mathbf{p})$-module. Thus N is a prime submodule of M and $\mathrm{rad}_M(N) = N = RE_M(N)$. Therefore M satisfies the radical formula for primary submodules.

COROLLARY 2.2. *Let R be a one-dimensional domain. Then every R-module satisfies the radical formula for primary submodules.*

Proof. By Lemma 2.1.

Leung and Man [**3**, Theorem 1.1] showed that a Noetherian domain R is Dedekind if and only if every R-module satisfies the radical formula. In view of Corollary 2.2, we see that any one-dimensional Noetherian domain R that is not Dedekind (for example, $R = \mathbb{Z}[\sqrt{5}]$ or more generally $R = \mathbb{Z}[\sqrt{d}]$ for any square-free integer d with $d \equiv 1 (\mathrm{mod}\ 4)$) has the property that every R-module satisfies the radical formula for primary submodules but the R-module $R \oplus R$ does not satisfy the radical formula (see [**3**, Corollary 5.2]).

We now turn our attention to trying to prove a converse of Lemma 2.1.

LEMMA 2.3. *Let R be a ring such that the R-module $R \oplus R$ satisfies the radical formula for primary submodules. Let $\mathbf{q}$ be any prime ideal of R. Then $a\mathbf{q} \subseteq a^2\mathbf{q} + \mathbf{q}^{(2)}$ for any element $a \in R\backslash\mathbf{q}$.*

Proof. Let $a \in R\backslash\mathbf{q}$, $b \in \mathbf{q}$. Let $M = R \oplus R$ and let

$$N = \{(x, y) \in M : a^2x - by \in \mathbf{q}^{(2)}\}.$$

By Proposition 1.2, N is a $\mathbf{q}$-primary submodule of M. Note that if $(x, y) \in N$ then $a^2x \in by + \mathbf{q}^{(2)} \subseteq \mathbf{q}$, so that $x \in \mathbf{q}$. Let K be any prime submodule of M such that $N \subseteq K$. Then $\mathbf{q}^2M \subseteq N$ gives that $\mathbf{q}M \subseteq K$. Since $(b, a^2) \in N$ it follows that $(0, a^2) = (b, a^2) - (b, 0) \in K$, and hence $(0, a) \in K$. Thus $(0, a) \in \mathrm{rad}_M(N) = N + \mathbf{q}M$ by Lemma 1.3

There exist $x, y \in R$, $u, v \in \mathbf{q}$ such that $(0, a) = (x, y) + (u, v)$, where $(x, y) \in N$; i.e. $a^2x - by \in \mathbf{q}^{(2)}$. Now $by = a^2x + z$ for some $z \in \mathbf{q}^{(2)}$ and $a = y + v$, so that

$$ab = yb + vb = a^2x + z + vb \in a^2\mathbf{q} + \mathbf{q}^{(2)}$$

because $x \in \mathbf{q}$ (see above).

THEOREM **2.4**. *The following statements are equivalent for a Noetherian ring R.*
(i) *Every R-module satisfies the radical formula for primary submodules.*
(ii) *The R-module $R \oplus R$ satisfies the radical formula for primary submodules.*
(iii) *For every non-maximal prime ideal $\mathbf{q}$ of R there exists $c \in R\backslash\mathbf{q}$ such that $c\mathbf{q} = 0$.*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii). Let $\mathbf{q}$ be any non-maximal prime ideal of R. There exists a maximal ideal $\mathbf{p}$ of R such that $\mathbf{q} \subsetneqq \mathbf{p}$. Let $a \in \mathbf{p}\backslash\mathbf{q}$. By Lemma 2.3, $a\mathbf{q} \subseteq a^2\mathbf{q} + \mathbf{q}^{(2)}$ and hence $M = aM$, where M is the finitely generated R-module $(a\mathbf{q} + \mathbf{q}^{(2)})/\mathbf{q}^{(2)}$. Hence there exists $b \in R$ such that $(1 - ab)M = 0$; i.e. $(1 - ab)a\mathbf{a} \subseteq \mathbf{q}^{(2)}$. It follows that $\mathbf{q} = \mathbf{q}^{(2)}$ and hence $c\mathbf{q} = 0$ for some $c \in R\backslash\mathbf{q}$, by [**7**, Exercise 8.37].

(iii) $\Rightarrow$ (i). This follows from Lemma 2.1.

We have already remarked that there exist Noetherian domains R for which every R-module satisfies the radical formula for primary submodules but not every R-module satisfies the radical formula. The following result shows that this cannot happen for UFD's.

THEOREM 2.5. *The following statements are equivalent for a UFD R.*
(i) *R is a PID.*
(ii) *Every R-module satisfies the radical formula.*
(iii) *Every R-module satisfies the radical formula for primary submodules.*
(iv) *The R-module $R \oplus R$ satisfies the radical formula for primary submodules.*

Proof. (i) $\Rightarrow$ (ii). See [**2**, Theorem 9].

(ii)$\Rightarrow$ (iii) $\Rightarrow$ (iv). These implications are clear.

(iv) $\Rightarrow$ (i). Let $\mathbf{q}$ be any height 1 prime ideal of R. Then $\mathbf{q} = Rq$ for some (prime) element q of R. Moreover, $\mathbf{q}^{(2)} = Rq^2$. Let $a \in R\backslash\mathbf{q}$. By Lemma 2.3, $a\mathbf{q} \in a^2\mathbf{q} + \mathbf{q}^{(2)} =$

$Ra^2q + Rq^2$ and hence $a \in Ra^2 + Rq$. There exist $r, s \in R$ such that $a = ra^2 + sq$ and hence $(1 - ra)a = sq \in \mathbf{q}$. It follows that $1 - ra \in \mathbf{q}$; i.e. $R = Ra + \mathbf{q}$. Therefore $\mathbf{q}$ is a maximal ideal of R. Hence R is a PID.

Combining Theorems 1.9, 1.12, 2.4 and 2.5 we have the following result without further proof.

COROLLARY 2.6. *Let R be a ring which is either Noetherian or a UFD. Then the following statements are equivalent.*

(i) *Every primary R-module has prime radical.*

(ii) *Every R-module satisfies the radical formula for primary submodules.*

It is natural to ask whether (i) and (ii) in Corollary 2.6 are always equivalent. We have the following special case.

PROPOSITION 2.7. *Let S be a domain and let R be the polynomial ring* $S[X]$. *Then the following statements are equivalent for R.*

(i) *R is a PID*

(ii) *Every primary R-module has prime radical.*

(iii) *Every R-module satisfies the radical formula (for primary submodules).*

(iv) *S is a field.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 1.12.

(ii) $\Rightarrow$ (i). Let $\mathbf{q}$ denote the prime ideal RX of R. By the proof of (iii) $\Rightarrow$ (i) of Theorem 1.12, $\mathbf{q}$ is a maximal ideal of R. Hence S is a field and R is a PID.

(i) $\Rightarrow$ (iii). This follows from Theorem 2.5.

(iii) $\Rightarrow$ (i). Again using $\mathbf{q} = RX$, the proof of Theorem 2.5, (iv) $\Rightarrow$ (i), shows that $\mathbf{q}$ is a maximal ideal and hence R is a PID.

(i) $\Rightarrow$ (iv). This result is well-known.

Let R denote the polynomial ring $\mathbb{Z}[X]$ and $\mathbf{q}$ the prime ideal RX of R. Let $M = R \oplus R$ and let

$$N = \{(r, s) \in M : 4r - Xs \in RX^2\}.$$

By Proposition 1.2, N is a $\mathbf{q}$-primary submodule of M. It can easily be checked that $N = R(X, 4) + R(0, X) + X^2M$. In this case, $RE_M(N) = N + XM = R(0, 4) + XM$. However, it is also easy to check that $\text{rad}_M(N) = R(0, 2) + XM \neq RE_M(N)$.

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