Universal Groups for Right-angled Buildings

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A Félix Costa,

que onde quer que esteja,
continue orgulhoso de mim.

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Ana

2013–2017: Four years in a right-angled mood

After four years of research in a very specific topic, it is a very interesting exercise to come back to the beginning of the story, motivate the research done and highlight the most beautiful and productive moments of my PhD. The solution of that exercise is provided in this introductory chapter that, by lack of a better name, can also be called preface.

The initial project of my PhD, and the reason why I left the beautiful weather of Lisbon in 2013 to move to Gent, was to study groups acting on locally finite trees from a combinatorial, geometrical, topological and algebraic point of view. So many points of view! The possibility of connecting so many areas of mathematics to study structures that at first glance seemed simple really attracted my attention and turned the trip from Lisbon to Gent way shorter than it actually is (and it is already really short).

Reading the title of this thesis, one can think that I engaged in another research project during my PhD. That is really not the case. As soon as I arrived to the department of Mathematics in Gent, buildings were everywhere. And I do not have in mind the "beautiful" location of the department. I am considering the "mathematical structures" called buildings, that were made by gluing together apartments which, in turn, were constructed with chambers, walls, etc. I had never heard of anything like that before and in Gent everyone seemed to be more or less familiar with the notion and to have a few examples of those "buildings" ready at hand.

If my curiosity was already piqued regarding these geometric structures, my mind was blown when I heard the first time that "a tree is a building". What a beautiful and curious world Mathematics is!

So this sentence, that I often use in the introduction of my talks as a catch up phrase, justifies that in fact my PhD involves groups acting on locally finite trees and also on a more general class of structures of which trees are examples.

Framing the research

Totally disconnected locally compact groups

I started my PhD by studying a topology that one considers in the automorphism group of a locally finite tree, called the permutation topology. Endowed with that topology, the automorphism group of a locally finite tree (or of a locally finite graph in general) is a totally disconnected locally compact (t.d.l.c.) group.

The study of locally compact groups G can naturally be split into the connected and totally disconnected cases. This is due to the fact that the connected component of the identity, G_0, is a closed normal subgroup of G and G/G_0 is a totally disconnected group.

The connected case has found a satisfactory answer with the solution of Hilbert's fifth problem.

Theorem (Gleason [Gle52], Montgomery and Zippin [MZ52], Yamabe [Yam53b] and [Yam53a])**.** *Let G be a connected locally compact group and let $\mathcal{O}$ be a neighborhood of the identity. Then there is a compact, normal subgroup $K \leq G$ with $K \subseteq \mathcal{O}$ such that G/K is a Lie group.*

This means roughly that a connected locally compact group can be approximated by Lie groups. Lie groups were investigated by looking separately at the soluble groups and at the simple case. Simple Lie groups were classified first by Killing ([Kil88] and [Kil89]) and Killing's proofs were completed by Cartan in[Car84].

Therefore, a contemporary study of locally compact groups concerns the theory of totally disconnected groups. For a long time, the only structure theorem known regarding t.d.l.c. groups was van Dantzig's theorem:

Theorem ([VD36]). *Every totally disconnected locally compact group contains a compact open subgroup.*

Recently (in fact I was 5 years old at the time and I do not consider myself so old) a program on the study of t.d.l.c. groups was initiated by George Willis in [Wil94] with the concept of the scale function and then continued for instance in [CRW13] and [CRW14].

There are several theorems that relate the structure of compact open subgroups and the global structure of a t.d.l.c. group (see for instance [BEW11] and [Wil07]). These global structure consequences obtained from local properties are often called local-to-global arguments and this designation was introduced by Marc Burger and Shahar Mozes [BM00a] in their study of specific groups of automorphisms of trees.

Universal groups for regular trees

One of the first papers that I read when I started my PhD was the beautiful groundbreaking work of Burger and Mozes ([BM00a]) on groups acting on trees with a prescribed local action (see Section 1.5 for a detailed description of these groups).

They introduced the universal groups $U(F)$ as groups of automorphisms of a regular tree, which are defined by prescribing the local action around every vertex of the tree with a finite permutation group F.

Universal groups form a large class of subgroups of the automorphism group $\mathrm{Aut}(T)$ of a locally finite tree T. This is due to the fact that any closed vertex-transitive subgroup of $\mathrm{Aut}(T)$ whose local action on the vertices is permutationally isomorphic to F can be embedded in a universal group. Moreover, universal groups are examples of compactly generated t.d.l.c. groups and they are non-discrete under mild conditions on the local action. They naturally satisfy Tits's independence property so they have, under some conditions on the local action, an index 2 simple subgroup.

Universal groups are also fundamental in the study of lattices in the automorphism groups of the product of two trees and are a key to prove the normal subgroup theorem, analogous to Margulis's normal subgroup theorem for semisimple Lie groups (see [BM00b]).

Several results on local-to-global arguments in the universal groups were accomplished, considering for instance the case where the local action is 2-transitive [BM00a] or primitive [CD11] (see Section 1.5) and these results show the beauty of the universal groups, in case it was not yet clear.

The universal groups defined by Burger and Mozes are the main motivation for this thesis which, following a suggestion of Pierre-Emmanuel Caprace, generalizes the idea of prescribing a local action to the broader setting of right-angled buildings, for which trees are examples (here is the catchy sentence again).

Right-angled buildings

Buildings were defined by Tits [Tit74] as a way to understand semisimple Lie groups as automorphism groups of some geometric structures. In the afore-mentioned paper he defined buildings as simplicial complexes with some special subcomplexes called apartments where the group acts with some regularity. By gluing the apartments together following a set of axioms, Tits reached the first definition of a building.

A few years later, he rewrote the definition of a building in terms of chamber systems [Tit81] and that is the characterization that we will follow in this thesis. Although not clear at first sight, the two definitions of buildings are equivalent, as explained for instance in [AB08].

Despite the initial motivations of Tits, which led to the classification of spherical buildings [Tit74], these geometric objects became a world of their own and let to innumerous directions of research. The classification of affine buildings ([Tit86]), the study of Moufang polygons [TW02] and Moufang sets (see for instance [DMW06]) which are Moufang buildings of rank 2 and 1, respectively, or the study of groups of Kac-Moody type acting on twin buildings [Tit92] are just a very incomplete list of examples.

There is also another way to look at buildings, which we will use when convenient, that is to regard buildings as metric spaces. This is done by considering geometric realizations of the buildings (see Section 1.4.4 for more details). In the spherical and Euclidean case, this can be done since apartments of such buildings can be seen as groups of reflections of a sphere (in the first case) and tessellations of Euclidean spaces (in the second case). Moussong in his thesis [Mou88a] obtained a general construction, normally called Davis realization, that can be done in general for any building. This construction is developed by Davis who gives an explicit proof that this construction gives rise to CAT(0)-metric spaces (see [Dav98]).

The protagonists of this thesis will be right-angled buildings. Trees are the simplest examples of right-angled buildings, but also some hyperbolic buildings such as Bourdon's buildings (see Example 2.2.5) or some Euclidean buildings, as for instance the product of trees, belong to this class.

There are different lines of research in the world of right-angled buildings. To mention a few, Anne Thomas has been developing a theory of lattices in right-angled buildings (for references see [TW11] and [Tho06]), Dymara, Osajda [DO07] and Clais [Cla16] considered boundaries on these buildings, and there is also a construction by Rémi and Ronan [RR06] of twin buildings of right-angled Coxeter type acted upon Kac-Moody groups.

The class of semi-regular right-angled buildings, i.e., buildings whose panels of the same type have the same cardinality (for a precise characterization, see Definition 2.3.1) will be of main interest for us and it will be the class for which we will generalize the ideas of Burger and Mozes [BM00a].

Semi-regular buildings are interesting objects to look at. Given a Coxeter group W and a set of parameters Q, Haglund and Paulin in [HP03] proved that, up to automorphism, there is a unique right-angled building of type W whose panels of each type have the size of the respective parameter in Q (see the precise statement in Theorem 2.3.2). Furthermore, in the thick irreducible case, that is, when all the panels have size at least three and we cannot decompose the Coxeter group as a direct product, Caprace [Cap14] proved that the automorphism group of such a building is an abstractly simple group.

Main results and methodology

The main results in this dissertation concern groups acting on semi-regular right-angled buildings and are of geometric, topological and group theoretical nature. We will always consider irreducible right-angled buildings, otherwise we can decompose the building as a direct product of irreducible buildings.

We first focus on the full automorphism group of a right-angled building and study its open subgroups. It is known in the case of trees that any open subgroup of the automorphism group is compact (*cf.* [CD11, Theorem A]). We generalize this result to the right-angled buildings setting and we obtain a weaker version of the aforementioned fact.

Theorem 1. *Let Δ be locally finite semi-regular thick right-angled building and let G denote the automorphism group of Δ.*

Then any proper open subgroup of G is contained in the stabilizer in G of a proper residue of Δ.

This result is presented in this thesis as Theorem 3.4.19. The proof relies on considering groups that resemble root groups (namely root wing groups defined in Section 3.3) and, surprisingly, it uses a strategy adopted previously for Kac-Moody groups acting on twin buildings [CM13].

In Chapter 3, as an attempt to obtain the proof of the previous theorem, we also show that the fixator of a ball in the automorphism group acts on the building with a bounded fixed-point set (see Proposition 3.2.6). That implies that an open subgroup of the automorphism group of a thick semi-regular right-angled building is compact if and only if it acts locally elliptically in the Davis realization of the building (see Corollary 3.4.3).

In Chapter 4 we define the universal group for a semi-regular building and, after proving basic (and less basic) initial properties, we reach the main result of the chapter.

Theorem 2. *The universal group for a semi-regular thick right-angled building is simple if and only if the local actions are prescribed by finite groups that are transitive and generated by point stabilizers.*

The proof of the simplicity of these groups, which is stated as Theorem 4.6.7 in the thesis, requires the development of several concepts. The first is a generalization (or better said, an adaptation) of Tits's independence property to the setting of right-angled buildings. This result is proved in [Cap14] for the full automorphism group and we prove it for the universal groups in Proposition 4.4.1. The second is the concept of a tree-wall tree defined in Section 2.2.4 and the study of the action of the universal groups on those trees, which is investigated in Section 4.5.

Once we prove the simplicity of the universal groups, we focus on the structure of the compact open subgroups of the universal group of a locally finite thick semi-regular right-angled building, which is a compactly generated totally disconnected locally compact group.

The maximal compact open subgroups are stabilizers of spherical residues, as proved in Proposition 5.1.2, and we focus on the structure of a chamber stabilizer, which is a finite index subgroup of those. These groups are profinite and hence, in Chapter 5, we describe each of the finite groups appearing in the projective limit through different points of view. These finite groups are, concretely, the induced action of the stabilizer of a chamber in the balls around that chamber.

First we describe an iterated process to construct these groups, which is stated as Theorem 5.2.7 in the thesis, in a similar fashion as Burger and Mozes did for trees in [BM00a, Section 3.2]. Each of these induced actions is constructed through a semi-direct product that resembles complete wreath products in imprimitive action (in the case of trees they are actually complete wreath products). This fact lead us to investigate more deeply the structure of induced actions on spheres through a more directed way. First we consider these groups as subdirect products of the induced actions on w-spheres, for words w of length n, and then we study the structure of the induced action on an w-sphere through a group theoretical construction.

Theorem 3. *The induced action of chamber stabilizers in the universal group on w-spheres is permutationally isomorphic to generalized wreath products constructed using the finite groups that prescribe the local action.*

A more precise statement of this result is presented in Chapter 5 as Proposition 5.3.3.

Once more, we want to spend a couple of lines exhibiting the main ingredients of the proof. The very first one is the way that we regard a right-angled building. Since in this chapter we are considering chamber stabilizers, we want to regard this fixed chamber c_0 as an "initial point" and obtain directions from that chamber. That is achieved in Section 2.4 where we consider a parametrization of the chambers of a thick semi-regular right-angled building using directed colorings of the chambers, introduced in Definition 2.3.14, which are colorings such that, in each panel, the chamber closest to the initial fixed chamber c_0 has color 1.

Generalized wreath products, which are discussed in Section 1.1.2, require a partial order to be defined. We provide a partial order in reduced words of the associated Coxeter group (see Definition 2.1.9) and we use this partial order to construct generalized wreath products of the finite groups that prescribe the local action, each associated to a generator of the Coxeter group.

As most results, the description of the sphere stabilizers in the universal group did not start as beautifully as shown in Theorem 3. Starting from the case of trees, where one gets complete wreath products, the first next step was to consider a few commutation relations for some generators of the right-angled Coxeter group, to see how the situation "evolved". Looking at the different representations of a reduced word, for which one gets a different associated complete wreath product, considering intersections of wreath products seemed to be the solution. The next task was then to realize the different complete wreath products as subgroups of the same symmetric group, which we accomplished in Section 1.1.1 with a clear description for the case of two groups in Proposition 1.1.7. Then, after the use of enough brute force in several concrete examples, we finally realized the w-sphere stabilizers in the universal group as intersections of iterated complete wreath products in imprimitive action.

Then a natural question arose. Are the two concepts the same in our setting? Or in other words, do generalized wreath products and intersections of iterated complete wreath products coincide when one considers a partial order coming from a right-angled Coxeter group? It turns out that it is indeed the case.

Theorem 4. *The intersection of iterated complete wreath products in* *imprimitive action corresponding to distinct reduced representations of an element w of a right-angled Coxeter group is permutationally isomorphic to a generalized wreath product obtained by considering a partial order on the letters of w.*

This result is stated in the thesis as Proposition 5.3.8 and the beauty (and surprise) of it resides in the fact that to obtain the description of the induced action of a chamber stabilizer on a sphere of the building, one does not have to investigate the building itself. It is enough to look at the Coxeter diagram of the associated Coxeter group.

We also succeed in describing the induced action of chamber stabilizers on the whole n-sphere directly through generalized wreath products (see Theorem 5.3.13). For that we consider a new partial order, this time on the tree-walls of a right-angled building.

An open chapter

Although not yet considered a main result at the time of finishing this thesis, but definitely considered as a main idea, is the work developed in the last chapter of this dissertation. In joint work with Anne Thomas during a perfect one month stay in ETH Zurich, we transported the idea of prescribing a local action as Burger and Mozes [BM00a] did for trees, to a world where one does not have the machinery of buildings or the combinatorial properties of Coxeter groups, but where one still has enough regularity to even define what it would be to prescribe a local action. We introduce, in Chapter 6, the concept of a universal group for a particularly regular class of polygonal complexes.

These are complexes, called (Γ, k)-complexes, whose 2-cells are regular k-gons and whose links of vertices are isomorphic to a fixed finite simple graph (see Definition 6.1.6). The automorphism groups of these complexes still belong to the class of topological groups that we are interested in in this thesis, that is, the automorphism group of (Γ, k)-complexes are totally disconnected locally compact groups. For instance Bourdon's buildings, already mentioned as examples of right-angled buildings, are examples of (Γ, k)-complexes, but one of the only examples in the intersection of the two classes of geometric objects.

Ballmann and Brin in [BB94] developed a process to construct CAT(0) (Γ, k)-complexes inductively and moreover they proved that any (Γ, k)-complex can be constructed using that process. Under some graph theoretical conditions on the link (see Theorem 6.2.6), Nir Lazarovich proved in [Laz14] that (Γ, k)-complexes are unique objects, up to isomorphism. A similar situation as in the case of semi-regular right-angled buildings.

Therefore we define the universal group on (Γ, k)-complexes that are unique up to isomorphism of polygonal complexes. In this thesis we focus on the case where the links of the vertices of the complex are isomorphic to a finite cover of the Petersen graph for which all the automorphisms lift.

We define legal colorings on the polygons of a such (Γ, k)-complex and we prove, due to the graph theoretical properties of the Petersen graph and its finite covers, that these legal colorings are unique, up to isomorphism (see Proposition 6.3.2). After the initial setting is layed down, we define in Section 6.4 the universal group $U(F)$ for a (Γ, k)-complex Y by prescribing a local action on the links of vertices of Y with a group F of automorphisms of the Petersen graph. In Section 6.4.2 we prove basic properties of these groups and also provide some local-to-global results for the universal group of a (Γ, k)-complex Y.

Assuming some conditions on the local action F, we are able to prove that these groups are actually universal. This means that if H is a closed subgroup of the automorphism group of Y such that the local action on the links of vertices of Y is permutationally isomorphic to F, then H is embedded in the universal group $U(F)$. This result is presented as Proposition 6.4.11 in the dissertation.

By the time of the conclusion of this PhD, there are plenty of open questions related to these groups. In fact there are more questions than answers. Some of these questions, together with comments, are presented in the last section of the thesis.

What you can find in this thesis

CHAPTER 1

Preliminaries

This thesis is inserted in the area of geometric group theory. As the name suggests, there will be plenty of groups and group constructions acting on geometrical objects. Moreover, the (topological) groups of automorphisms of infinite objects that will be defined will often be studied by investigating their local structure, that is, by using techniques from finite group theory.

Therefore this preliminary chapter will be structured by introducing first the group theoretical constructions of finite groups. Next we present some notation and definitions of graph theory and proceed to the notion and basic properties of topological groups, focusing on the definition of the permutation topology for groups acting on graphs. Then the geometry will take over and we go towards the definition of buildings, through a brief stay in Coxeter groups, and we exhibit some of their properties that will be useful later on.

We finish this introductory chapter by presenting a class of compactly generated totally disconnected and locally compact groups defined by Burger and Mozes in [BM00a] called the universal groups. These groups act on regular trees with local action prescribed by a finite permutation group and they are the motivation for our study of similar groups acting on the broader class of right-angled buildings in Chapter 4 and on polygonal complexes in Chapter 6.

1.1 Group theoretical constructions

We start by introducing the notation on groups and group actions that we will (try to) consistently use throughout this thesis. Our groups will be assumed to act on the left, unless otherwise stated. If A is a set and α is an element of A then the image of α under a permutation g of A will be denoted by $g.\alpha$, or $g\alpha$.

That means that if g and h are permutations then we define composition by the rule that gh means "apply first h and then g". Then $(gh).\alpha = g.(h\alpha)$.

Also, if H and K are subgroups of a group G then we denote conjugation of $k \in K$ by $h \in H$ by

$$^{h}k = hkh^{-1}.$$

When H acts on a group K, that is, $\rho : H \to \mathrm{Aut}(K)$ is a group homomorphism, we denote by ^{h}k the image $\rho(h)k$.

For functions, the notation in use will be the normal composition on the left, that is, $\psi_1 \circ \psi_2$ will mean to first apply ψ_2 and then ψ_1.

If we consider group automorphisms, then the multiplication on those groups will be composition, and here the notation adapted will be as follows. Assume that G is a group of automorphisms of an object Δ. If $g_1, g_2 \in G$ and $c \in \Delta$ then we can either write

$$(g_1 \circ g_2)(c) = g_1(g_2(c)), \;\; g_1 g_2.c \;\; \text{or} \;\; g_1 g_2 c,$$

depending on whether we want to emphasize that the elements of G are functions of the elements of Δ or elements of the symmetric group on the set Δ.

We now establish the notation that we will use for stabilizers and induced actions.

Definition 1.1.1. Let G be a group acting on a set A, or in other words, $G \leq \mathrm{Sym}(A)$. Let B be a subset if A.

1. We denote the *setwise stabilizer* of B in G as

$$\mathrm{Stab}_G(B) = \{g \in G \mid gB = B\},$$

and the *pointwise stabilizer*, also called *fixator*, of B in G as

$$\mathrm{Fix}_G(B) = \{g \in G \mid g\beta = \beta \text{ for all } \beta \in B\}.$$

Observe that if $B = \{\alpha\}$ then these two subgroups coincide and we denote the stabilizer of α in G by G_α, or by $\mathrm{Stab}_G(\alpha)$.

2. One can also look at the action of the group G restricted to the subset B. This is called the *induced action* of G on B, it is denoted by $G|_B$ and it is isomorphic to $\mathrm{Stab}_G(B)/\,\mathrm{Fix}_G(B)$.

With initial notation set up, it is time to present three group theoretical constructions that we will use later on in our study of universal groups.

1.1.1 Wreath products

In this section we define wreath products of finite permutation groups. We will follow the notation of Dixon and Mortimer in [DM96], with the suitable modifications to consider left actions. For the purposes of this dissertation, we will only be interested in considering wreath products in imprimitive action though other actions can be defined in these groups (see loc. cit.). We will first quickly introduce the notation for semidirect products before defining wreath products. In the end of the section we compute intersections of wreath products by realizing them as subgroups of the same symmetric group. This will be useful in Chapter 5 to study compact open groups acting on right-angled buildings.

Let H and K be groups and suppose that we have an action of H on K which respects the group structure on K. In other words, for each $h \in H$ the mapping $k \mapsto {}^{h}k$ is an automorphism of K. Put $G = \{(h,k) \mid h \in H, k \in K\}$ and define a product on G by

$$(h_1,k_1)(h_2,k_2) = (h_1h_2, {}^{h_2^{-1}}k_1k_2),$$

for all $(h_1,k_1),(h_2,k_2) \in G$. This product is associative and hence G is a group under this operation with identity $(1,1)$ and with inverse $(h,k)^{-1} = (h^{-1}, ({}^{h}k)^{-1})$.

We call G the *semidirect product* of K by H and shall use the notation $H \ltimes K$ to denote G. Clearly $|G| = |H||K|$.

The semidirect product G contains subgroups $H' = \{(h,1) \mid x \in H\}$ and $K' = \{(1,k) \mid k \in K\}$ which are isomorphic to H and K respectively, and such that $G = H'K'$ and $K' \cap H' = 1$. Moreover, K' is normal in G and the way that H' acts on K' by conjugation reflects the original action of H on K, namely,

$$(h,1)(1,k)(h,1)^{-1} = (1, {}^{h}k),$$

for all $h \in H$ and $k \in K$.

Sometimes in the literature the semidirect product $H \ltimes K$ is called an external semidirect product and the group $H' \ltimes K'$ is called the internal semidirect product, since both H' and K' are considered already as subgroups of G.

For groups H and K, their semidirect product is not unique as it depends implicitly on the action of H on K even though the action is not specified in the notation. The next example (which can be found for instance in [ST00, Section 2.9]) illustrates the importance of the action of H on K.

Example 1.1.2. Suppose $H \cong \mathbb{Z}_2$ and $K \cong \mathbb{Z}_3$. Then we can consider more than one possibility for the semidirect product $G = H \ltimes K$. The automorphism group of K is H. There are two homomorphisms from H to $H = \mathrm{Aut}(K)$.

1. In the first case, when the whole H is in the kernel of the automorphism, we obtain that H is also normal in G and so $G \cong K \times H$. In this case G is abelian and isomorphic to $\mathbb{Z}_6$.

2. The bijective homomorphism $H \to H$ gives rise to the case when H is not normal in G. In this case we obtain that G is isomorphic to $\mathrm{Sym}(3)$, that $H \cong \mathrm{Alt}(3) = \{(1), (123), (132)\}$ (as a subgroup of $\mathrm{Sym}(3)$) and that $K \cong \{(1), (12)\}$.

We now proceed to the definition of wreath products. If A and B are nonempty sets then we write $\mathrm{Fun}(A, B)$ to denote the set of all functions from A into B. In the case that K is a group, we can turn $\mathrm{Fun}(A, K)$ into a group by defining a product "pointwise":

$$(\phi_1\phi_2)(\alpha) = \phi_1(\alpha)\phi_2(\alpha) \text{ for all } \phi_1, \phi_2 \in \mathrm{Fun}(A, B) \text{ and } \alpha \in A,$$

where the product on the right hand side is in K. In the case that A is finite of size m, say $A = \{\alpha_1, \ldots, \alpha_m\}$, then the group $\mathrm{Fun}(A, K)$ is isomorphic to K^m (a direct product of m copies of K) via the isomorphism $\phi \mapsto (\phi(\alpha_1), \ldots, \phi(\alpha_m))$.

Definition 1.1.3. Let K and H be groups and suppose H acts on the nonempty set A. Then the *(complete) wreath product* of K by H with respect to this action is defined to be the semidirect product $H \ltimes \mathrm{Fun}(A, K)$ where H acts on the group $\mathrm{Fun}(A, K)$ via

$$^{h}\phi(\alpha) = \phi(h^{-1}\alpha) \text{ for all } \phi \in \mathrm{Fun}(A, K), \alpha \in A \text{ and } h \in H. \quad (1.1.1)$$

We denote this group by $H \wr_A K$.

There are two subgroups of the wreath product that one can easily identify

$$P = \{(1, \phi) \mid \phi \in \mathrm{Fun}(A, K)\} \cong \mathrm{Fun}(A, K) \text{ and}$$
$$H^* = \{(h, \mathrm{id}) \mid h \in H\} \cong H,$$

which are called, respectively, the *base group* and the *top group* of the wreath product. We have that the wreath product normalizes the group P.

Again, it is helpful to look at the case where A is finite, say $A = \{1, 2, \ldots, m\}$. In this case we can identify the base group P with the direct product $K \times \cdots \times K$ (m factors), and the action of the top group H^* on P corresponds to permuting the components of the direct product in the same way that H permutes the elements of A:

$$^{h}(k_1, \ldots, k_m) = (k_{1'}, \ldots, k_{m'}) \text{ where } h = \begin{pmatrix} 1 & \cdots & m \\ 1' & \cdots & m' \end{pmatrix},$$

for all $(k_1, \ldots, k_m) \in P$ and $h \in H$. Clearly, $|H \wr_A K| = |H||K|^m$.

Now we will consider the imprimitive action of a wreath product.

Definition 1.1.4. Let $G = H \wr_A K$ be a wreath product as above, in which the group H acts on the set A and there is an action of H on $\mathrm{Fun}(A, K)$ given by Equation (1.1.1). If K acts on a set B then we can define an action of G on $A \times B$ by

$$(h, \phi)(\alpha, \beta) = (h\alpha, \phi(\alpha)\beta) \text{ for all } (\alpha, \beta) \in A \times B, \quad (1.1.2)$$

where $(h, \phi) \in H \ltimes \mathrm{Fun}(A, K) = H \wr_A K$. This is called the *imprimitive action* of the wreath product.

If the set A is finite then this action of the wreath product can be regarded as looking at $A \times B$ as a cover of A by sort of fibres indexed by B, that is,

$$A \times B = \cup_{\alpha \in A} \mathcal{F}_\alpha, \text{ where } \mathcal{F}_\alpha = \{(\alpha, \beta) \in A \times B \mid \beta \in B\}.$$

Then each factor of the direct product $K^{|A|}$ acts on the corresponding fibre as K acts on B and the top group H^* permutes the fibres $\mathcal{F}_\alpha$ in the same way as H acts on the set A. As the name suggests, this action of the wreath product on the set $A \times B$ is imprimitive if $|A|, |B| > 1$. Indeed the relation "being in the same fibre" is a congruence relation for the action of $H \wr K$ on $A \times B$ whose congruence blocks are the fibres.

Example 1.1.5 (Section 2 in [Cam10])**.** Consider the wreath product $\mathrm{Sym}(n) \wr \mathrm{Sym}(2)$, defined with the obvious action on the semidirect product, and also known as the Weyl group of type B_n. It has a normal subgroup $\mathrm{Sym}(2)^n$ that is elementary abelian of order 2^n and which is the base group. The top group is the group $\mathrm{Sym}(n)$.

The imprimitive action is on the vertices of the n-dimensional cross-polytope (or hyper octahedron), consisting of vectors with ± 1 in one coordinate and 0 in all others. The ith factor of the base group changes the signs of the ith basis vector and fixes all other; the top group permutes the coordinates.

Remark 1.1.6. Observe that the wreath product is associative in the sense that if H_1, H_2, H_3 are groups acting on sets A_1, A_2 and A_3, respectively, then the groups $(H_1 \wr H_2) \wr H_3$ and $H_1 \wr (H_2 \wr H_3)$ are isomorphic. Moreover, if we identify the sets $(A_1 \times A_2) \times A_3$ and $A_1 \times (A_2 \times A_3)$ with $A_1 \times A_2 \times A_3$ then the actions actually coincide (see [Hal76, Section 5.9] for more details).

Here $H_1 \wr H_2 \wr \cdots \wr H_n$ denotes the iterated wreath product

$$H_1 \ltimes \mathrm{Fun}(A_1, H_2) \ltimes \cdots \ltimes \mathrm{Fun}(A_1 \times \cdots \times A_{n-1}, H_n),$$

where each of the groups $H_1 \wr \cdots \wr H_i$ acts on the group $\mathrm{Fun}(A_1 \times \cdots \times A_i, H_{i+1})$ as in Equation(1.1.1). If the sets A_i are finite then the action is given by permuting the coordinates of

$$H_{i+1}^{|A_1 \times \cdots \times A_i|} \cong \mathrm{Fun}(A_1 \times \cdots \times A_i, H_{i+1}).$$

The construction above and the definition of imprimitive action showed how wreath products arise as imprimitive groups. Wreath products also play an important role in the study of primitive permutation groups by defining an action on the set $\mathrm{Fun}(A, B)$, normally called product action. More details on the product action of the wreath product can be found for instance in [DM96, Section 2.7].

Intersection of finite wreath products

We finish the section on wreath products with the study of intersections of wreath products in imprimitive action on finite sets. We will do that by regarding all the considered wreath products as subgroups of the same symmetric group.

Let A and B be finite sets. Let H and K be permutation groups on $\mathrm{Sym}(A)$ and $\mathrm{Sym}(B)$, respectively. Consider the wreath products $G_1 = H \wr K$ and $G_2 = K \wr H$ in their imprimitive actions on the sets $A \times B$ and $B \times A$, respectively.

If ρ is a bijection between the sets $\{1, \ldots, |A \times B|\}$ and $A \times B$ then, identifying a point (α, β) in $A \times B$ with the point (β, α) in $B \times A$, then we can consider G_1 and G_2 as subgroups of $\mathrm{Sym}(|A \times B|)$. Hence it makes sense to consider the intersection of such groups.

Proposition 1.1.7. *With the notation above, the group $G_1 \cap G_2$ is isomorphic to $H \times K$.*

Proof. An element $(h, \phi) \in G_1$ (where $\phi : A \to K$) acts on a typical point $(\alpha, \beta) \in A \times B$ by mapping (α, β) to $(h\alpha, \phi(\alpha)\beta)$. We can embed K inside $K^{|A|}$ by defining $\phi_k = k$ for all $\alpha \in A$. Then we get (a copy of) $H \times K$ as a subgroup of $G_1 = H \wr K = H \ltimes \mathrm{Fun}(A, K)$. Using a similar argument to embed H in $H^{|B|}$ we get $K \times H$ as a subgroup of $G_2 = K \wr H$. Therefore $K \times H \subseteq G_1 \cap G_2$.

Now let g be an element in $G_1 \cap G_2$. Then g can be written as $(h, \phi) \in G_1$ and as $(k, \psi) \in G_2$ and the action of these two elements has to be the same, that is, for all $(\alpha, \beta) \in A \times B$, we have

$$(h, \phi)(\alpha, \beta) = (k, \psi)(\beta, \alpha) \Longleftrightarrow (h\alpha, \phi(\alpha)\beta) = (\psi(\beta)\alpha, k\beta).$$

Hence we obtain that $\phi(\alpha) = k$ for all $\beta \in B$ and therefore $\phi = \phi_k$. Similarly $\psi = \psi_h$. Thus the elements in the intersection $G_1 \cap G_2$ are of the form (ψ_h, ϕ_k), with $k \in K$ and $h \in H$. Therefore there is an isomorphism between $H \times K$ and $G_1 \cap G_2$ (given by $(h,k) \mapsto (\psi_h, \phi_k)$) since the action of these groups on the set $A \times B$ is the same, because (ψ_h, ϕ_k) sends (α, β) to $(h\alpha, k\beta)$. □

The identification of the sets $A \times B$ and $B \times A$ with the set $\{1, \ldots, |A \times B|\}$ can be done in a more general setting. Let $H_1 \leq \mathrm{Sym}(A_1), \ldots, H_n \leq \mathrm{Sym}(A_n)$ be groups acting on finite sets, for all $i \in \{1, \ldots, n\}$, and consider the iterated wreath product $G = H_1 \wr \cdots \wr H_n$ with its imprimitive action on the set $A_1 \times \cdots \times A_n$.

Let σ be a permutation of $\mathrm{Sym}(n)$ and consider the group

$$G_\sigma = G_{\sigma.1} \wr \cdots \wr G_{\sigma.n},$$

with its imprimitive action on $A_{\sigma.1} \times \cdots \times A_{\sigma.n}$. Then G and G_σ can be considered as subgroups of $\mathrm{Sym}(|A_1 \times \cdots \times A_n|)$ making similar identifications of the sets $A_1 \times \cdots \times A_n$, $A_{\sigma.1} \times \cdots \times A_{\sigma.n}$ and $\{1, \ldots, |A_1 \times \cdots \times A_n|\}$. Furthermore it is possible to consider their intersection in a similar fashion. However, the outcome is not always so nicely presented as in the case of only two groups, as it is shown in the next example.

Example 1.1.8 (Explaining example of the intersection)**.** Let $H_1 \leq \mathrm{Sym}(A_1), H_2 \leq \mathrm{Sym}(A_2), H_3 \leq \mathrm{Sym}(A_3)$ and $H_4 \leq \mathrm{Sym}(A_4)$ be groups acting on finite sets. Consider the iterated wreath product $G_1 = H_1 \wr H_2 \wr H_3 \wr H_4$ acting with its imprimitive action on the set $S_1 = A_1 \times A_2 \times A_3 \times A_4$.

Consider the permutations

$$\sigma_2 = \begin{pmatrix} 1234 \\ 1243 \end{pmatrix} = (34), \qquad \sigma_3 = \begin{pmatrix} 1234 \\ 2134 \end{pmatrix} = (12),$$

$$\sigma_4 = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix} = (12)(34) \quad \text{and} \quad \sigma_5 = \begin{pmatrix} 1234 \\ 3124 \end{pmatrix} = (132)$$

of $\mathrm{Sym}(4)$. Now take the respective iterated wreath products

$$\begin{array}{ll} G_2 = G_{\sigma_2} = H_1 \wr H_2 \wr H_4 \wr H_3 & \text{acting on } S_2 = A_1 \times A_2 \times A_4 \times A_3 \\ G_3 = G_{\sigma_3} = H_2 \wr H_1 \wr H_3 \wr H_4 & \text{acting on } S_3 = A_2 \times A_1 \times A_3 \times A_4 \\ G_4 = G_{\sigma_4} = H_2 \wr H_1 \wr H_4 \wr H_3 & \text{acting on } S_4 = A_2 \times A_1 \times A_4 \times A_3 \\ G_5 = G_{\sigma_5} = H_2 \wr H_3 \wr H_1 \wr H_4 & \text{acting on } S_5 = A_2 \times A_3 \times A_1 \times A_4 \end{array}$$

We can consider these 5 groups acting on the set $X = \{1, \ldots, |A_1 \times A_2 \times A_3 \times A_4|\}$ so it makes sense to consider their intersection.

Let us identify the sets X and S_1 through f. Then for $(a, b, c, d) \in S_1$ we make the natural identifications with the sets S_j, for $j \in \{2, \ldots, 5\}$, that is,

$$\begin{aligned}(a, b, c, d) \in S_1 &\mapsto (a, b, d, c) \in S_2\\ &\mapsto (b, a, c, d) \in S_3\\ &\mapsto (b, a, d, c) \in S_4\\ &\mapsto (b, c, a, d) \in S_5\end{aligned}$$

We now want to compute the intersection of these 5 wreath products, as subgroups of $\mathrm{Sym}(X)$, which we denote by I. An element in I can be written as an element $g_1 \in G_1$, $g_2 \in G_2, \ldots, g_5 \in G_5$ with the same action on the set X. This means that for all $x \in X$, we can identify x with the element $f(x) = (a, b, c, d) \in S_1$ and we obtain, with slight abuse of notation, that

$$g_1(a, b, c, d) = g_2(a, b, d, c) = g_3(b, a, c, d) = g_4(b, a, d, c) = g_5(b, c, a, d). \tag{1.1.3}$$

Now we have to see how the elements in each of these wreath products look like in order to give a meaning to Equation 1.1.3.

$g_1 = (h_1, \rho_1, \varphi_1, \phi_1)$ in which
$h_1 \in H_1$
$\rho_1 : A_1 \to H_2$
$\varphi_1 : A_1 \times A_2 \to H_3$
$\phi_1 : A_1 \times A_2 \times A_3 \to H_4$

$g_2 = (h_2, \rho_2, \varphi_2, \phi_2)$ in which
$h_2 \in H_1$
$\rho_2 : A_1 \to H_2$
$\varphi_2 : A_1 \times A_2 \to H_4$
$\phi_2 : A_1 \times A_2 \times A_4 \to H_3$

$g_3 = (h_3, \rho_3, \varphi_3, \phi_3)$ in which
$h_3 \in H_2$
$\rho_3 : A_2 \to H_1$
$\varphi_3 : A_2 \times A_1 \to H_3$
$\phi_3 : A_2 \times A_1 \times A_3 \to H_4$

$g_4 = (h_4, \rho_4, \varphi_4, \phi_4)$ in which
$h_4 \in H_2$
$\rho_4 : A_2 \to H_1$
$\varphi_4 : A_2 \times A_1 \to H_4$
$\phi_4 : A_2 \times A_1 \times A_4 \to H_3$

$g_5 = (h_5, \rho_5, \varphi_5, \phi_5)$ in which
$h_5 \in H_2$
$\rho_5 : A_2 \to H_3$
$\varphi_5 : A_2 \times A_3 \to H_1$
$\phi_5 : A_2 \times A_3 \times A_1 \to H_4$

Using the description above, we obtain

$$\begin{cases} g_1(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = & (h_1\,\mathbf{a},\ \rho_1(a)\,\mathbf{b},\ \varphi_1(a,b)\,\mathbf{c},\ \phi_1(a,b,c)\,\mathbf{d}) \\ g_2(\mathbf{a},\mathbf{b},\mathbf{d},\mathbf{c}) = & (h_2\,\mathbf{a},\ \rho_2(a)\,\mathbf{b},\ \varphi_2(a,b)\,\mathbf{d},\ \phi_2(a,b,d)\,\mathbf{c}) \\ g_3(\mathbf{b},\mathbf{a},\mathbf{c},\mathbf{d}) = & (h_3\,\mathbf{b},\ \rho_3(b)\,\mathbf{a},\ \varphi_3(b,a)\,\mathbf{c},\ \phi_3(b,a,c)\,\mathbf{d}) \\ g_4(\mathbf{b},\mathbf{a},\mathbf{d},\mathbf{c}) = & (h_4\,\mathbf{b},\ \rho_4(b)\,\mathbf{a},\ \varphi_4(b,a)\,\mathbf{d},\ \phi_4(b,a,d)\,\mathbf{c}) \\ g_5(\mathbf{b},\mathbf{c},\mathbf{a},\mathbf{d}) = & (h_5\,\mathbf{b},\ \rho_5(b)\,\mathbf{c},\ \varphi_5(b,c)\,\mathbf{a},\ \phi_5(b,c,a)\,\mathbf{d}) \end{cases}$$

Since the action of these elements is the same, we can match the letters of the pair (a,b,c,d) in Equation 1.1.3, that is,

$$\begin{cases} \textbf{(1)}\ h_1\,\mathbf{a} = h_2\,\mathbf{a} = \rho_3(b)\,\mathbf{a} = \rho_4(b)\,\mathbf{a} = \varphi_5(b,c)\,\mathbf{a} \\[2ex] \textbf{(2)}\ \rho_1(a)\,\mathbf{b} = \rho_2(a)\,\mathbf{b} = h_3\,\mathbf{b} = h_4\,\mathbf{b} = h_5\,\mathbf{b} \\[2ex] \textbf{(3)}\ \varphi_1(a,b)\,\mathbf{c} = \phi_2(a,b,d)\,\mathbf{c} = \varphi_3(b,a)\,\mathbf{c} = \phi_4(b,a,d)\,\mathbf{c} = \rho_5(b)\,\mathbf{c} \\[2ex] \textbf{(4)}\ \begin{aligned} \phi_1(a,b,c)\,\mathbf{d} \quad &= \varphi_2(a,b)\,\mathbf{d} = \phi_3(b,a,c)\,\mathbf{d} = \varphi_4(b,a)\,\mathbf{d} \\ &= \phi_5(b,c,a)\,\mathbf{d} \end{aligned} \end{cases}$$

Simplifying these expressions, we obtain that an element in the intersection I of these five wreath products will act in the element (a,b,c,d) as

$$(h_2\,\mathbf{a},\ h_5\,\mathbf{b},\ \rho_5(\mathbf{b})\,\mathbf{c},\ \varphi_2(\mathbf{a},\mathbf{b})\,\mathbf{d})$$

1.1.2 Generalized wreath products

Generalized wreath products, as the name openly suggests, are generalizations of sets of complete wreath products, indexed by a partial ordered set. There is more than one natural way to generalize wreath products and the main first references to such generalizations are done by Holland in [Hol69] and Wells in [Wel76] (the latter concerns semigroup actions). In [BPRS83] the authors analyze the semigroup construction of Wells and prove that if one starts with a group action instead of semigroup actions then the construction gives indeed rise to groups.

In this thesis we will follow a more general construction due to Gerhard Behrendt. In [Beh90], Behrendt relates his construction with the ones in [Hol69] and [Wel76] and points out in Section 3 that if the partial ordered set is finite then the 3 constructions coincide.

We remark that this definition of a generalized wreath product requires a systematic subset of a set X (see [Beh90, Section 2]) but in our setting we will always consider the whole set X, which is automatically a systematic subset. Hence we will not be concerned with that concept.

Moreover, in this section we also illustrate a connection between intersections of complete wreath products in imprimitive action and generalized wreath products (see Example 1.1.11). This connection will be completely uncovered in Chapter 5, where we will consider partial orders coming from a right-angled Coxeter group, while studying compact profinite groups

Definition 1.1.9.

1. An *equivalence system* (X, E) is a pair consisting of a set X and a set E of equivalence relations on X. The automorphism group of (X, E) is the set of all permutations of the set X which leave the relations in the set E invariant, i.e.,

$$\begin{aligned}\mathrm{Aut}(X, E) = \{g \in \mathrm{Sym}(X) \mid x \sim y \iff g.x \sim g.y \\ \text{for all } x, y \in X \text{ and for all } \sim \in E\}.\end{aligned}$$

2. Let $(S, \prec)$ be a poset (i.e., a partially ordered set[1]). For each $s \in S$, let G^s be a permutation group acting on a set X_s. Let $X = \prod_{s \in S} X_s$ be the direct product of the sets $\{X_s\}_{s \in S}$.

   For each $s \in S$, we define two equivalence relations on X as follows. For each pair of tuples $x, y \in X$ we define

$$\begin{aligned} x \sim_s y &\iff x_t = y_t \text{ for all } t \succ s, \\ x \simeq_s y &\iff x_t = y_t \text{ for all } t \succeq s. \end{aligned} \tag{1.1.4}$$

   Let $E = \{\sim_s \mid s \in S\} \cup \{\simeq_s \mid s \in S\}$ be the set of all these equivalence relations. Then the *generalized wreath product* $G = X\text{–}\mathrm{WR}_{s \in S} G^s$ is defined as

$$G = \left\{ g \in \mathrm{Aut}(X, E) \;\middle|\; \begin{array}{c} \text{for each } x \in X \text{ and } s \in S \text{ there is} \\ g_{s,x} \in G^s \text{such that } (g.y)_s = g_{s,x}.y_s \\ \text{for all } y \in X \text{ with } y \sim_s x \end{array} \right\}.$$

[1] All partial orders occurring in this thesis are *strict* partial orders, i.e., binary relations that are irreflexive, transitive and antisymmetric. When $\prec$ is a (strict) partial order, we write $\preceq$ for the corresponding non-strict partial order.

We note that $X\text{–}\mathrm{WR}_{s\in S}G^s$ is indeed a group, with $(g^{-1})_{s,x} = (g_{s,g^{-1}.x})^{-1}$ and $(gh)_{s,x} = g_{s,h.x}\, h_{s,x}$.

Remark 1.1.10. There is a close relation between generalized wreath products and wreath products, when one considers different partial orders. If the poset S is a chain, then we get the iterated complete wreath product of the groups G^s with its imprimitive action, as defined in the previous section. If $\prec$ is the empty partial order, then the generalized wreath product is the direct product of the groups G^s.

In the most general case where we will apply this construction, it will, in fact, be possible to view the generalized wreath product in imprimitive action as an *intersection* of wreath products acting on the same product set. This is illustrated in the next example.

Example 1.1.11. Let $S = \{s_1, s_2, s_3, s_4\}$ and define a partial order $\prec$ between the elements of S as follows

$$s_3 \prec s_2,\ s_4 \prec s_1,\ s_4 \prec s_2.$$

Then $(S, \prec)$ is a poset. For $i \in \{1, \dots, 4\}$, consider $A_i = A_{s_i}$ and $H_i = H_{s_i}$ be as in Example 1.1.8, where A_{s_i} is a finite set and H_{s_i} is a permutation group acting on A_{s_i}. Let $X = \prod_{s\in S} A_s$. Consider the partial orders $\sim_s$ and $\simeq_s$ as in Equation (1.1.4), that in this concrete example can be described as:

$$\begin{aligned}
&\sim_{s_1} \text{ and } \sim_{s_2} \text{ are the empty partial relation,}\\
&(a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4}) \sim_{s_3} (b_{s_1}, b_{s_2}, b_{s_3}, b_{s_4}) \Leftrightarrow a_{s_2} = b_{s_2},\\
&(a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4}) \sim_{s_4} (b_{s_1}, b_{s_2}, b_{s_3}, b_{s_4}) \Leftrightarrow a_{s_1} = b_{s_1} \text{ and } a_{s_2} = b_{s_2}.
\end{aligned} \tag{1.1.5}$$

We now consider $E = \{\sim_s\}_{s\in S} \cup \{\simeq_s\}_{s\in S}$ and $\mathrm{Aut}(X, E)$ as in Definition 1.1.9. The generalized wreath product in this case is the group

$$\begin{aligned}
G = \{g \in \mathrm{Aut}(X, E) \mid\ &\text{for each } x \in X \text{ and } s \in S \text{ there is } g_{s,x} \in H_s\\
&\text{such that } (g.y)_s = g_{s,x}.y_s \text{ for all } y \in X \text{ with } y \sim_s x\}.
\end{aligned}$$

Claim. *The group G is isomorphic to the intersection I of wreath products of Example 1.1.8.*

We observe that both groups act on the same set. Moreover, we recall that an element $g \in I$ acts on a typical point $(a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4}) \in X$ as

$$g(a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4}) = (h_1 a_{s_1}, h_2 a_{s_2}, \rho(a_{s_2}) a_{s_3}, \varphi(a_{s_1}, a_{s_2}) a_{s_4}),$$

where $h_1 \in H_{s_1}$, $h_2 \in H_{s_2}$, $\rho : A_{s_2} \to H_{s_3}$ and $\varphi : A_{s_1} \times A_{s_2} \to H_{s_4}$.

Let $g \in I$. It its clear that $g \in \mathrm{Aut}(X, E)$. Let us show that $g \in G$. Consider for instance $x, y \in X$ such that $x \sim_{s_4} y$. Then, using Equation (1.1.5), we know that $x = (a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4})$ and $y = (b_{s_1}, b_{s_2}, b_{s_3}, b_{s_4})$ with $a_{s_1} = b_{s_1}$ and $a_{s_2} = b_{s_2}$. Thus

$$\begin{aligned}&(g(a_{s_1}, a_{s_2}, a_{s_3}, a_{s_4}))_{s_4} = \varphi(a_{s_1}, a_{s_2}) a_{s_4} \text{ and}\\ &(g(b_{s_1}, b_{s_2}, b_{s_3}, b_{s_4}))_{s_4} = \varphi(b_{s_1}, b_{s_2}) b_{s_4} = \varphi(a_{s_2}, a_{s_1}) b_{s_4}.\end{aligned}$$

Hence there is $g_{s_4,x} \in H_{s_4}$ such that $(g.y)_{s_4} = g_{s_4,x} . y_{s_4}$ for all $y \sim_{s_4} x$, namely, $\varphi(a_{s_1}, a_{s_2})$. One can check that the same holds for any s_i. Thus $g \in G$ and hence $I \subseteq G$.

Conversely, let $g \in G$ and let $s \in S$. For each $t \in S$ such that $s \prec t$, one can define a function $f_s : A_{t_1} \times \cdots \times A_{t_n} \to H_s$ by $f_s(a_{t_1}, \ldots, a_{t_n}) = g_{s,x}$ where $x \in X$ has the t_i-th coordinate a_{t_i} for every $i \in \{1, \ldots, n\}$ and $g_{s,x}$ is the element of H_s such that g acts on x_s as $g_{s,x}$. We remark that any element $y \sim_s x$ has also a_{t_i} in its t_i-th coordinate, so the element $g_{s,x}$ is well defined and independent of the choice of x. We can define f_s for every element $s \in S$ and like this we can visualize the action of g on X as an element of I since the functions f_s, for $s \in \{s_1, \ldots s_4\}$, are defined as h_1, h_2, ρ and φ, respectively. Thus the two groups are isomorphic.

In Chapter 5, when we study some compact subgroups of the universal group it will be useful to consider certain quotients of generalized wreath products. That is what we will focus on next. We retain the notation of the previous paragraphs.

Definition 1.1.12. A subset I of the poset $(S, \prec)$ is called an *ideal* of S if for every $s \in I$ and $t \in S$, $t \preceq s$ implies $t \in I$. We define a new equivalence relation on X as

$$x \sim_I y \iff x_s = y_s \text{ for all } s \notin I.$$

As before, let $G = X\text{–}\mathrm{WR}_{s \in S} G^s$. We consider its subset

$$D(I) = \{g \in G \mid x \sim_I g.x \text{ for all } x \in X\}.$$

Proposition 1.1.13 ([Beh90, Proposition 7.1]). *Let I be an ideal of S. Then $D(I)$ is a normal subgroup of G.*

The group $D(I)$ can also be described as a generalized wreath product; see [Beh90, Theorem 7.2]. We will describe such a generalized wreath product in the case that $|I| = 1$. Let $r \in S$ such that $I = \{r\}$ is an ideal of S. Then by definition $t \not\prec r$ for all $t \in S$. We will write $D(r)$ rather than $D(\{r\})$. For any subset $T \subseteq S$, let p_T denote the projection map $p_T \colon X \to \prod_{s \in T} X_s$ defined by $x \mapsto (x_s)_{s \in T}$.

Proposition 1.1.14. *Let $(S, \prec)$ be a poset, and let $I = \{r\}$ be an ideal of S. Consider $d_r = \prod_{t \succ r} |X_t|$ (where $d_r = 1$ if there are no $t \succ r$). Then $D(r)$ is isomorphic to the direct product of d_r copies of G^r. In particular, if the sets X_s are finite, then*

$$|D(r)| = |G^r|^{d_r}.$$

Proof. This follows from [Beh90, Theorem 7.2]. Notice that the general statement from *loc. cit.* requires the definition of an additional partial order, which is empty in our case because we are considering ideals of size 1. $\square$

Lemma 1.1.15. *Let $(S, \prec)$ be a poset, let $I = \{r\}$ be an ideal of S and let $S' = S \setminus \{r\}$. Let $H^s = G^s$ for all $s \in S'$, let $H^r = 1$ and consider the group*

$$H = X\text{–}\mathrm{WR}_{s \in S} H^s \leq G.$$

Then $G = H \ltimes D(r)$.

Proof. Let $X' = \prod_{s \in S'} X_s$ and $X = X' \times X_r$, and consider the generalized wreath product

$$G' = X'\text{–}\mathrm{WR}_{s \in S'} G^s.$$

Since $\{r\}$ is an ideal of S, the group $D(r)$ is a normal subgroup of G by Proposition 1.1.13.

Consider two elements $x_1 = (x', x_r)$ and $x_2 = (x', y_r)$ of the set $X = X' \times X_r$. Since $r \not\succ s$ for all $s \in S'$ (because $\{r\}$ is an ideal of S), we have $x_1 \simeq_s x_2$ for all $s \in S'$. Therefore, by definition of G (see

Definition 1.1.9), we obtain that $(g.x_1)_s = (g.x_2)_s$ for all $s \in S'$ and all $g \in G$.

Consider the projection map $p_{S'}\colon X \to X'$. Then this projection induces a homomorphism $\rho\colon G \to G'$ defined by $(\rho(g).x')_s = (g.(x', x_r))_s$, for $s \in S'$ and for any $x_r \in X_r$ (the choice of the element of X_r is irrelevant by the observation in the previous paragraph). Notice that the kernel of ρ is precisely $D(r)$.

Observe now that for any $h \in G'$, the element $h \times \mathrm{id}$ belongs to G. In particular, ρ is surjective, the map $\sigma\colon G' \to G\colon h \mapsto h \times \mathrm{id}$ is a section for ρ and therefore $\mathrm{im}(\sigma) = H$. We conclude that $G = H \ltimes D(r)$. □

1.1.3 Subdirect products

In Chapter 5 we will encounter groups that are subdirect products of generalized wreath products. We will recall the definition of a subdirect product and then we will exhibit a procedure to realize intransitive groups G as subdirect products of groups acting on disjoint G-invariant sets. This section follows the general ideas in [Hal76, Section 5.5] and [Hul10, Section II.4] and we make the construction explicit when necessary.

Definition 1.1.16. Let $G_1, \ldots, G_n$ be groups. A *subdirect product* of the groups $G_1, \ldots, G_n$ is a subgroup $P \leq G_1 \times \cdots \times G_n$ of the direct product of the groups G_i such that, for each $i \in \{1, \ldots, n\}$, the canonical projections $P \to G_i$ are surjective.

Example 1.1.17.

1. The direct product $G_1 \times G_2$ is always itself a subdirect product.

2. The diagonal group $\{(g, g) \mid g \in G\}$ is a subdirect product of $G \times G$.

3. More generally, if $\rho : G_1 \to G_2$ is a surjective homomorphism, the subgroup of $G_1 \times G_2$ given by $\{(g, \rho(g)) \mid g \in G_1\}$ is a subdirect product.

Now we will consider intransitive groups as subdirect products.

Definition 1.1.18. Let G be a group acting faithfully on a set S, and let

$$S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_n$$

be a decomposition of S into G-invariant subsets, i.e., each set S_i is a union of orbits for the action of G on S. Then we get corresponding homomorphisms

$$\alpha_i \colon G \to \mathrm{Sym}(S_i).$$

Let $G_i = \mathrm{Im}(\alpha_i)$, and define a new homomorphism

$$\phi \colon G \to G_1 \times \cdots \times G_n \colon g \mapsto \big(\alpha_1(g), \ldots, \alpha_n(g)\big). \tag{1.1.6}$$

The homomorphism ϕ is injective. Therefore

$$G \cong \phi(G) \leq G_1 \times \cdots \times G_n,$$

and by definition, $\phi(G)$ surjects onto each G_i. Hence we have realized G as a subdirect product of the groups $G_1, \ldots, G_n$.

For completeness, let us consider the detailed construction of the isomorphism $G \cong \phi(G)$ in Equation (1.1.6), for the case $n = 2$. Let G be a group acting on a set S and let $S = S_1 \sqcup S_2$ be a decomposition of S into G-invariant sets. For $i \in \{1, 2\}$, let $\alpha_i : G \to \mathrm{Sym}(S_i)$ be a homomorphism and $G_i = \mathrm{Im}(\alpha_i)$. Then, with the notation as in Definition 1.1.18, we have $\phi : G \to G_1 \times G_2$ defined by $\phi(g) = (\alpha_1(g), \alpha_2(g))$ and $G \cong \phi(G)$.

Proposition 1.1.19. *Consider $N_1 = \alpha_1(\ker(\alpha_2)) \trianglelefteq G_1$ and $N_2 = \alpha_2(\ker(\alpha_1)) \trianglelefteq G_2$. Let $\varphi : {G_1}/{N_1} \to {G_2}/{N_2}$ to be an isomorphism defined by $N_1 g_1 \mapsto N_2 \alpha_2(g)$, where $g \in G$ is such that $\alpha_1(g) = g_1$. Furthermore, let $\rho_i : G_i \to {G_i}/{N_i}$ be the projection maps, for $i \in \{1, 2\}$. Then $\phi(G) = \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi(\rho_1(g_1)) = \rho_2(g_2)\}$.*

Proof. First we observe that by the isomorphism theorems we have

$$ {G_1}/{N_1} \cong {G}/{\langle \ker(\alpha_1), \ker(\alpha_2) \rangle} \cong {G_2}/{N_2}.$$

Hence the isomorphism φ is well defined.

Let $R = \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi(\rho_1(g_1)) = \rho_2(g_2)\}$. If $(g_1, g_2) \in \phi(G)$ then there exists $g \in G$ such that $g_1 = \alpha_1(g)$ and $g_2 = \alpha_2(g)$.

Hence we can choose g in the definition of φ as $g_1N_1 \mapsto \alpha_2(g)N_2 = g_2N_2$.

On the other hand, if $(g_1, g_2) \in R$ then $\varphi(\rho_1(g_1)) = \rho_2(g_2)$. Take $g \in G$ such that $\alpha_1(g) = g_1$. Hence $g_1N_1 \mapsto \alpha_2(g)N_2$ through φ. By definition of R we obtain that $\alpha_2(g)N_2 = g_2N_2$. Thus $\alpha_2(g)\alpha_2(k_1) = g_2$ for some $k_1 \in \ker(\alpha_1)$. Consider $g' = gk_1 \in G$ then $\alpha_1(g') = g_1$ and $\alpha_2(g') = g_2$. Hence $(g_1, g_2) \in \phi(G)$, which finishes the proof. $\square$

1.2 Graph theoretical notions

The groups that play the main role in this thesis will act on infinite (mostly locally finite) geometrical objects. In the cases that we will be interested in, will it be sufficient to look at the 1-skeleton of the geometrical objects, that is, to look at groups acting on graphs.

In the next section we will define a topology for groups acting on graphs, whose properties we will use to study groups acting on buildings and in other CAT(0)-spaces. Therefore in this section we will gather the basic definitions and notation on graphs that we will need throughout the thesis.

Definition 1.2.1.

1. A *graph* Γ is a pair (V, E) where the set V is called the set of *vertices* of Γ and the set $E \subseteq V\Gamma^{\{2\}}$ is subset of unordered pair of vertices of Γ, called the set of *edges* of Γ. If $e = \{v_1, v_2\}$ is an edge in Γ, we say that v_1 and v_2 are *adjacent* vertices in Γ and that they are the *ends* of e. Furthermore we say that $e \in E$ is *incident* to v_1 (and v_2). We might sometimes write $V\Gamma$ and $E\Gamma$ to emphasize the graph whose vertex and edge-set we are referring to.

2. Given a vertex $v \in V\Gamma$ we call the set
$$\mathrm{St}(v) = \{e \in E\Gamma \mid e \text{ is incident to } v\} \tag{1.2.1}$$
the *star* of the vertex v. If the graph Γ is not clear from the context then we can also denote the star of the vertex $v \in V\Gamma$ by $\mathrm{St}(v, \Gamma)$.

3. The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident to v, that is, $|\mathrm{St}(v)|$.

We call a graph Γ *regular* is all its vertices have the same degree. In particular if that degree is q, we say that Γ is a q-regular graph.

A graph is said to be *locally finite* is each of its vertices has finite degree.

4. A *morphism* between two graphs Γ and Γ' is a map $\phi : V\Gamma \to V\Gamma'$ from the set of vertices of Γ to the set of vertices of Γ' such that ϕ maps edges of Γ to edges of Γ'. A *subgraph* of Γ is a graph Γ' such that the inclusion map $V\Gamma' \to V\Gamma$ is a graph morphism.

We will always consider simple graphs, i.e., graphs Γ that have no loops and that any two edges with the same origin and same terminus are equal. Moreover, we are assuming that the graphs are unoriented but for any edge $e = \{v_1, v_2\}$ we will keep in mind, for when necessary, that there are two oriented edges (v_1, v_2) and (v_2, v_1) associated to e.

The next definition concerns galleries in the graph, a distance defined on galleries and related concepts.

Definition 1.2.2. Let Γ be a graph.

1. A *gallery* of length k in Γ (for some $k \geq 0$) from x to y is a sequence $\gamma = (v_0, v_1, \ldots, v_k)$ of $k+1$ vertices $v_0, v_1, \ldots, v_k$ such that $v_0 = x$, $v_k = y$ and

$$v_{i-1} \text{ is adjacent to } v_i \text{ for all } i \in \{1, \ldots, k\}.$$

   The term gallery will be used in the buildings setting and we observe that a gallery in a general graph is usually called a *walk*.

2. The *discrete distance* from v_1 to v_2 is the length of a shortest gallery from v_1 to v_2 if there are galleries from v_1 to v_2, and ∞ otherwise. We will denote the distance from v_1 to v_2 by $\text{dist}(v_1, v_2)$. Sometimes dist is also called the *discrete metric*. A gallery from v_1 to v_2 is called *minimal* if its length is $\text{dist}(v_1, v_2)$.

3. The graph Γ is *connected* if for any two vertices v_1 and v_2 there exists a gallery from v_1 to v_2. A *connected component* of Γ is the subgraph spanned by an equivalence class with respect to

the equivalence relation "there exists a gallery from v_1 to v_2 on Γ". Hence a graph is connected if and only if it has only one connected component.

4. The *diameter* of Γ is the maximum distance between two vertices of Γ.

5. The *girth* of Γ is the length of a shortest cycle in Γ, that is, the length of a shortest gallery of type $(v_1, v_2, \ldots, v_n, v_1)$.

6. The graph Γ is called a *tree* if Γ is a connected graph without cycles.

   It is clear that if $q \geq 2$, any q-regular tree is infinite, that it, its set of vertices is infinite.

Next we present the definition of graph automorphism and of the automorphism group of a graph. We will, in Section 1.3.2, associate a topology to this group.

Definition 1.2.3. An *automorphism* of a graph Γ is a bijective morphism $V\Gamma \to V\Gamma$. In other words, it is a permutation g of the vertex-set $V\Gamma$ that preserves adjacency, that is,

v_1 and v_2 are adjacent vertices in Γ if and only if gv_1 and gv_2 are adjacent in Γ.

The set of all graph automorphisms under the operation of composition of functions forms a group, called the *automorphism group* of the graph Γ and denoted by $\mathrm{Aut}(\Gamma)$.

Example 1.2.4. 1. Consider the graph Γ as in the figure. The

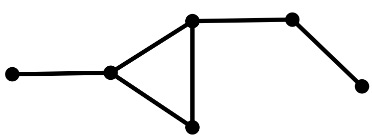


Figure 1.1: A rigid graph.

only permutation of the vertices of Γ preserving adjacency is the identity. Thus $\mathrm{Aut}(\Gamma) = \mathrm{id}$. In this case Γ is called a *rigid* graph.

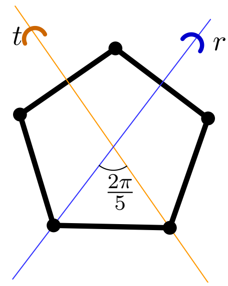


Figure 1.2: Cycle graph with five vertices.

2. Consider the cycle graph Γ with five vertices as in Figure 1.2.

   Any automorphism of Γ is a product of reflections along the axis r and t. Hence the group $\mathrm{Aut}(\Gamma)$ is the dihedral group D_5 with 10 elements.

We finish this section with the notion of cover of a graph. We will consider covers of graphs to define groups acting on regular geometrical objects in Chapter 6.

Definition 1.2.5. A morphism of graphs $f : \Gamma_1 \to \Gamma_2$ is called a *covering* if f maps $V\Gamma_1$ and $E\Gamma_1$ onto $V\Gamma_2$ and $E\Gamma_2$, respectively, in such a way that for every vertex $v \in V\Gamma_1$ the star of v is mapped bijectively to the star of $f(v) \in V\Gamma_2$.

The *fibre* over a vertex $v \in V\Gamma_2$ is the full pre-image of v under f. A similar definition holds for the fibre of an edge $e \in E\Gamma_2$.

The graph Γ_2 is called a *cover* of Γ_1 with *covering map* f.

Example 1.2.6. 1. Consider the graphs Γ_1 and Γ_2, as depicted in Figure 1.3. Considering $f : \Gamma_1 \to \Gamma_2$ defined by $f(x_i) = x$, we have that Γ_1 is a cover of Γ_2 with covering map f.

2. Consider the graphs in Figure 1.4. The Desargues graph is a finite cover of the Petersen graph, obtained by replacing each vertex in the Petersen graph by a pair of vertices and each edge by a pair of crossed edges. The construction is also called the *bipartite double cover*.

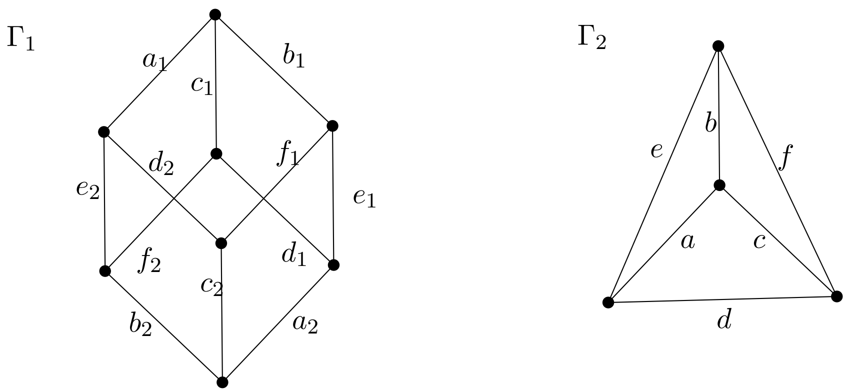
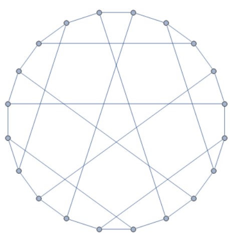
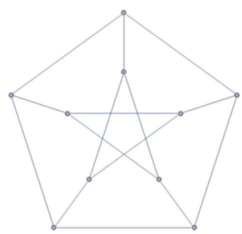


Figure 1.3: Example of a cover.



(a) The Desargues graph.



(b) The Petersen graph.

Figure 1.4: A bipartite double cover.

1.3 Interlude on topology

The groups that we will consider acting on geometric objects will have a topology associated. Therefore we start with some basic definitions regarding general topological groups, mostly following [Dik13] and [HR79].

Then we define the permutation topology for groups acting on graphs (which correspond to 1-skeletons of our geometric objects). In the case that the 1-skeletons of the geometrical objects are locally finite, this topology will turn our groups into totally disconnected locally compact groups.

We end the section with introductory notions on CAT(0)-spaces, for which all the geometric objects that we will consider in this thesis are examples.

1.3.1 Topological groups

Definition 1.3.1. Let G be a group. A topology τ on G is said to be a *group topology* if the map $f : G \times G \to G$ defined by $f(x, y) = xy^{-1}$ is continuous. A *topological group* is a pair (G, τ) of a group G and a group topology τ on G.

Let $x \in G$ and $\mathcal{B}$ be a set of neighborhoods of x. Then $\mathcal{B}$ is a *neighborhood basis* at x if and only if for each neighborhood N of x, there is an $M \in \mathcal{B}$ such that $M \subseteq N$.

Proposition 1.3.2 ((4.5) in [HR79])**.** *The topology of a topological group is completely determined by a neighborhood basis of the identity.*

Once the topology is clear from the context, we will denote the topological group only by G.

Example 1.3.3.

1. Every group G can be considered as a topological group in a trivial way, by considering the discrete topology, where one defines each subset of G to be open. Such groups are called *discrete groups*. If we define the only open sets of G to be the empty set and the whole group then the associated topology is called the *indiscrete topology*.

2. The canonical topology attached to the Euclidean space $\mathbb{R}^n$ $(n \geq 1)$ is defined by the collection of sets U such that, if $x \in U$, then $\{y \in \mathbb{R}^n \mid ||y - x|| < r\} \subseteq U$ for some $r > 0$. Then $\mathbb{R}^n$ with the addition operation becomes a topological group.

3. If G is a topological group and N is a normal subgroup of G then the quotient group G/N is a topological group with the quotient topology, that is, considering the quotient map $\phi : G \to G/N$ a subset $M \subset G/N$ is open if and only if $\phi^{-1}(M)$ is open in G.

Definition 1.3.4. Let G be a topological group.

1. A family $\mathcal{U} = \{U_i \mid i \in I\}$ of non-empty open sets is an *open cover* of G if $G = \bigcup_{i\in I} U_i$. A subfamily $\{U_i \mid i \in J\}$, for $J \subseteq I$, is a *sub-cover* of $\mathcal{U}$ if $G = \bigcup_{i\in J} U_i$.

2. G is called *connected* if whenever $G = W \cup V$, where W and V are non-empty open sets, we have $V \cap W \neq \emptyset$. In other words, G has no proper subsets which are open and closed.

3. For $g \in G$, there is a largest connected subset G_g of G such that $g \in G_g$. Such subset is called the *connected component* of g in G. Normally the connected component of the identity of G is denoted by G_0 and G is connected if and only if G_0 equals G.

Next we define some classes of topological groups.

Definition 1.3.5. Let G be a topological group. Then G is called

1. *compact* if for every open cover of G there exists a finite subcover,

2. *locally compact* if every element of G has a compact neighborhood in G,

3. *totally disconnected* if all the connected subsets are singletons,

4. *compactly generated* if there is a compact subset that generates G.

Proposition 1.3.6. *Let G be a topological group. Then G_0, the connected component of the identity, is a closed normal subgroup and G/G_0 is a totally disconnected group.*

In this thesis we are interested in investigating locally compact groups. Therefore, by the last proposition, the study can be done by splitting it into two cases, the connected and the totally disconnected locally compact (t.d.l.c.) groups.

The connected case has been solved satisfactory with the solution of Hilbert's fifth problem, as stated in the next theorem.

Theorem 1.3.7 (Gleason [Gle52], Montgomery and Zippin [MZ52], Yamabe [Yam53b] and [Yam53a])**.** *Let G be a connected locally compact group and $\mathcal{O}$ be a neighborhood of the identity. Then there is a compact, normal subgroup $K \leq G$ with $K \subseteq \mathcal{O}$ and such that $G \backslash K$ is a Lie group.*

This result is also often stated by saying that connected locally compact groups can be approximated by Lie groups. Hence, trying to characterize the totally disconnected case, i.e., studying t.d.l.c. groups has become a very interesting topic of research in the past years.

1.3.2 Permutation topology

Let Γ be a simple graph. We define a topology in $\mathrm{Aut}(\Gamma)$ so we regard this group as a topological group. The topology that we will consider is normally called permutation topology and, according to [Mol10], which is the reference that we will mostly follow, the earliest references to this topology are made in [Mau55] and [KS56]. Since the topology of a topological group is completely determined by a neighborhood basis of the identity (see Proposition 1.3.2) we will explicitly describe such neighborhood.

Definition 1.3.8. The *permutation topology* on $\mathrm{Aut}(\Gamma)$ is defined by choosing as a neighborhood basis of the identity, the family of pointwise stabilizers of finite subsets of $V\Gamma$, that is, a neighborhood basis of the identity is given by

$$\{\mathrm{Fix}_{\mathrm{Aut}(\Gamma)}(F) \mid F \text{ is a finite subset of } V\Gamma\}.$$

From this definition it follows that a sequence $(g_i)_{i\in\mathbb{N}}$ of elements of $\mathrm{Aut}(\Gamma)$ has an element $g \in \mathrm{Aut}(\Gamma)$ as a limit if and only if for every point $v \in V\Gamma$ there is a number N (possibly depending on v) such that $g_n v = gv$ for every $n \geq N$.

Actually, one could also use the property above describing convergence of sequences as a definition of the topology and then we think of the permutation topology as the topology of pointwise convergence. If we think of $V\Gamma$ as having the discrete topology and the elements of $\mathrm{Aut}(\Gamma)$ as maps $V\Gamma \to V\Gamma$, then the permutation topology is the same as the compact-open topology.

We can characterize now the open subgroups in $\mathrm{Aut}(\Gamma)$. A subgroup $G \leq \mathrm{Aut}(\Gamma)$ is *open* if and only if there is a finite subset F of $V\Gamma$ such that $\mathrm{Fix}_{\mathrm{Aut}(\Gamma)}(F) \subseteq G$.

In the case that Γ is a locally finite and connected graph, the group $\mathrm{Aut}(\Gamma)$, equipped with the permutation topology, becomes a totally disconnected locally compact (t.d.l.c.) group, as the following set of lemmas show.

Lemma 1.3.9 (Lemma 1 of [Woe91]). *Let* Γ *be a locally finite connected graph. Let* v *be a vertex in* $V\Gamma$*.*

Then the stabilizer $\mathrm{Aut}(\Gamma)_v$ *is compact.*

Proof. Let (g_n) be a sequence in $\mathrm{Aut}(\Gamma)_v$ and let $\{v_0 = v, v_1, v_2, \dots\}$ be an enumeration of $V\Gamma$. As $g_n v = v$ for every n, and as Γ is locally finite and connected, the set $\{g_n v_k \mid n \geq 0\}$ is finite for every k. Hence there is a subsequence $(\xi_1(n))$ of (n) such that all $g_{\xi_1(n)} v_1$ coincide: write gv_1 for this common image. Repeating this argument inductively, we get a sub-subsequence $(\xi_k(n))$ of the preceding subsequence $(\xi_{k-1}(n))$, such that all $g_{\xi_k(n)}$, for $n \geq 0$, send v_k to the same element of $V\Gamma$, denoted gv_k. Thus, $g_{\xi_n(n)} \to g \in \mathrm{Aut}(\Gamma)_v$ pointwise. $\square$

Lemma 1.3.10 ([AT08, Corollary 3.1.12]). *In a locally compact topological group* G*, the connected component of* G *is the intersection of all open subgroups of* G*.*

Combining the previous results we obtain that:

Proposition 1.3.11. *The automorphism group of a locally finite connected graph is a totally disconnected locally compact group.*

1.3.3 CAT(0)-spaces

CAT(0)-spaces were introduced by Aleksandrov in [Ale51] and were broadly considered by Gromov in the study of manifolds of non-positive sectional curvature (*cf.* [BGS85]).

These spaces play an important role in geometric group theory as CAT(0)-spaces include trees, buildings (*cf.* [Dav98]) and other cell complexes of non-positive curvature, as the polygonal complexes that we will study in Chapter 6.

Hence in this section we recall the definitions of geodesics and CAT(0)-spaces that we will need throughout the thesis, for which we refer to [BH99].

Definition 1.3.12. Let (X, d_X) be a metric space. A continuous function $\gamma : [a, b] \to X$ (for $a < b$ real numbers) is a *geodesic* if for all $a \leq t < t' \leq b$, we have $d_X(\gamma(t), \gamma(t')) = t' - t$. The metric space (X, d_X) is called *geodesic* if for all $x, y \in X$, there exists a geodesic $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. We denote that geodesic by $[x, y]$.

Observe that there might be more than one geodesic connecting x and y. For instance, in the Euclidean space, each geodesic is a straight line segment and there is a unique geodesic connecting each pair of points. However, on the sphere S^2 with its usual metric, each geodesic is an arc of a great circle, and antipodal points are connected by infinitely many geodesics.

Definition 1.3.13. Let (X, d_X) be a geodesic metric space.

1. A *geodesic triangle* in X is a triple of points x, y, z, together with a choice of geodesics $[x, y], [y, z]$ and $[z, x]$.

2. Given a geodesic triangle $\Delta = \Delta(x, y, z)$, a *comparison triangle* in the Euclidean plane is a triple of points $\bar{x}, \bar{y}, \bar{z}$ such that $d_X(x, y) = d(\bar{x}, \bar{y})$, $d_X(y, z) = d(\bar{y}, \bar{z})$ and $d_X(z, x) = d(\bar{z}, \bar{x})$, where d is the Euclidean metric.

3. For each point $p \in [x, y]$, there is a *comparison point*, denoted $\bar{p}$, in the straight line segment $[\bar{x}, \bar{y}]$, with the comparison point $\bar{p}$ defined by the equation $d_X(x, p) = d(\bar{x}, \bar{p})$.

Definition 1.3.14. A metric space (X, d_X) is called a *CAT(0)-space* if for every geodesic triangle $\Delta = \Delta(x, y, z)$, and every pair of points $p, q \in [x, y] \cup [y, z] \cup [z, x]$, we have $d_X(p, q) \leq d(\bar{p}, \bar{q})$, where d is the Euclidean metric and $\bar{p}$ and $\bar{q}$ are comparison points. This condition says that triangles in a CAT(0)-space are “no fatter” than Euclidean triangles. CAT(0)-spaces are normally called *nonpositively curved.*

Example 1.3.15.

1. A *Banach space* is a complete normed vector space, that is, a vector space with a norm such that the Cauchy sequences converge with respect to that norm. A particular instance of Banach spaces are Hilbert spaces.

A *Hilbert space* is a vector space X with an inner product $\langle f, g\rangle$ such that the norm defined by $||f|| = \langle f, f\rangle$ turns X into a complete metric space.

The real numbers with the standard inner product are an example of a Hilbert space.

The space $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$ with the supremum norm is an example of a Banach space which is not a Hilbert space.

Every Hilbert space is a CAT(0)-space and those are the only examples of Banach spaces which are CAT(0).

2. A metric space X is an $\mathbb{R}$-*tree* if
   - for $x, y \in X$ there is a unique geodesic $[x, y]$;
   - if $[x, y] \cap [y, z] = \{y\}$, then $[x, z] = [x, y] \cup [y, z]$.

   Every $\mathbb{R}$-tree is a CAT(0)-space.
3. The hyperbolic spaces $\mathbb{H}^n$ are also examples of CAT(0)-spaces.
4. The geometric realization of any building is a CAT(0)-space. For a detailed discussion of this result we refer to [Dav98].

We can consider comparison triangles in the hyperbolic plane instead, and define the space X to be CAT(-1) or *negatively curved* if its triangles are "no fatter" than hyperbolic triangles.

Theorem 1.3.16 ([BH99, Theorem II.1.12])**.** *Let X be a metric space. If X is a CAT(-1)-space then X is a CAT(0)-space.*

In fact, the previous theorem states a more general result, but we are only interested in negative and non-positive curvatures.

In Chapter 6 we will study simply-connected CAT(0)-spaces. Thus we finish this section with the definition of a simply-connected topological space.

Definition 1.3.17. Let X be a topological space.

1. X is called *path-connected* if for every two elements $x, y \in G$ there is a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$.

2. The space X is called *simply-connected* if it is path-connected and every loop in the space is null homotopic.

Example 1.3.18. The Euclidean plane $\mathbb{R}^2$ is simply-connected but $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply-connected. For $n > 2$, both $\mathbb{R}^n$ and $\mathbb{R}^n$ minus the origin are simply-connected spaces. A torus is an example of a space that is not simply-connected.

1.4 Coxeter groups and buildings

It is time for the geometry to take over this preliminary chapter. The main goal of this section is to reach the definition of a building.

We will start with the definition of Coxeter groups, which are particular groups generated by reflections, and we will state some properties of those groups which will be useful for us later on.

Then we will define a particular class of edge-colored graphs, called chamber systems which will, equipped with a word distance, give rise to the definition of a building. In the following chapters we will be interested in the right-angled case but in this section every Coxeter group and building will be treated in full generality. For the definitions of Coxeter groups and buildings we will follow [Wei09], [AB08] and [Ron09].

1.4.1 Coxeter groups

The study of finite reflection groups started in the nineteenth century, mainly in two fronts. First, around 1855, Schäfli classified regular polytopes in $\mathbb{R}^{n+1}$ for $n > 2$ and he proved that the symmetry groups of such polytopes were finite groups generated by reflections. Second, around 1890, Killing and Cartan classified complex semisimple Lie algebras in terms of their root systems. Then Weyl showed that the group of symmetries of such a root system was a finite group generated by reflections.

In 1934, H. S. M. Coxeter [Cox34] connected the two lines of research by classifying discrete groups generated by reflections on the n-dimensional sphere or Euclidean plane.

In the second half of the twentieth century, Jacques Tits [Tit64] introduced the notion of an abstract reflection group, which he called a "Coxeter group", due to the fact to the work done by Coxeter in the finite dimensional case.

We will present the definitions and properties of Coxeter groups in a purely group theoretical and combinatorial way. For more details on reflection groups, we point to the references cited above.

Definition 1.4.1 ([Wei09, Definition 2.1])**.**

1. A *Coxeter matrix* is a symmetric array $[m_{ij}]$ with i, j in some index-set I (of arbitrarily cardinality) such that for all $i, j \in I$, the element m_{ij} is either a positive integer or the symbol ∞, and $m_{ij} = 1$ if and only if $i = j$.

2. The *Coxeter diagram* of the Coxeter matrix $[m_{ij}]$ is the graph Σ with vertex-set I and edge-set consisting of unordered pairs $\{i, j\}$ such that $m_{ij} \geq 3$ (including $m_{ij} = \infty$) together with the labeling which assigns the label m_{ij} to each edge $\{i, j\}$. A Coxeter diagram is called *irreducible* if its underlying graph is connected. The *rank* of a Coxeter diagram is the cardinality of its vertex-set.

Definition 1.4.2. For each set I, we denote by M_I the *free monoid* on I, that is, the set of all finite sequences of element of I including the empty sequence, or equivalently, the set of all words in the alphabet I including the empty word, with multiplication given by concatenation.

We are now ready to present the definition of a Coxeter group.

Definition 1.4.3. Let $[m_{ij}]$ be a Coxeter matrix with index-set I and let Σ denote the corresponding Coxeter diagram. The *Coxeter group of type* Σ is the group W having a set of generators $S = \{s_i \mid i \in I\}$ indexed by I such that W is defined by

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I \text{ and } m_{ij} \neq \infty \rangle.$$

In particular $s_i^2 = 1$ for all $i \in I$.

The cardinality $|S|$ is called the *rank* of W. Moreover, if the Coxeter diagram Σ is connected, then W is called an *irreducible* Coxeter group.

Let $\mathbf{s} : f \mapsto s_f$ denote the unique extension of the map $i \mapsto s_i$ to a homomorphism $\mathbf{s}$ from the free monoid M_I to W. (Thus $s_\emptyset = 1$). The pair $(W, \mathbf{s})$ is called the *Coxeter system of type* Σ.

Remark 1.4.4. There is also another convention to associate a graph to a Coxeter system, sometimes in the literature called the defining of the Coxeter system. The vertices of this graph are the elements of I and two vertices i and j are connected by an edge labeled m_{ij} if and only if m_{ij} is *finite* (so in particular if $m_{ij} = \infty$ then the two vertices i and j are disconnected). In this thesis, we will only use Coxeter diagrams.

Theorem 1.4.5 ([Wei09, Theorem 2.3])**.** *Let $[m_{ij}]$ be a Coxeter matrix with index-set I and let $(W, \mathbf{s})$ be the corresponding Coxeter system. Then $|s_i| = 2$ for all $i \in I$ and $|s_i s_j| = m_{ij}$ for all $i, j \in I$.*

By abuse of notation, once the set of generators for W is fixed, we will consider the image of $\mathbf{s}$ instead and denote the Coxeter system by (W, S). We will then consider the Coxeter diagram to have vertex-set $S = \{s_i\}_{i \in I}$ with edges labeled by $m_{s_i s_j}$ (also denoted by m_{ij} for simplicity). Then the free monoid M_I will also be denoted by M_S.

Next we show a few examples of Coxeter groups.

Example 1.4.6.

1. The symmetric group on n letters, denoted by $\mathrm{Sym}(n)$, is a Coxeter group, considering the set of generators

$$S = \{(12), (23), \ldots, (n-1\,n)\}.$$

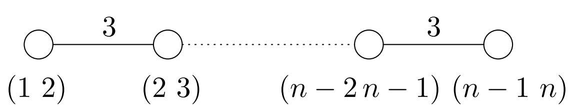
Then $\mathrm{Sym}(n)$ can be described by the following set of relations.

(a) $(i\,i+1)^2 = 1$ for all $i \in \{1, \ldots, n\}$;

(b) $(\,(i\,i+1)(i+1\,i+2)\,)^3 = 1$ for all $i \in \{1, \ldots, n\}$;

(c) $(\,(i\,i+1)(j\,j+1)\,)^2 = 1$, for all $i \in \{1, \ldots, n\}$ and $j \neq \{i-1, i+1\}$.

The corresponding Coxeter diagram of $\mathrm{Sym}(n)$ is

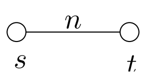


2. The finite dihedral groups $\mathbf{D}_{2n}$ are also Coxeter groups, as they can be generated by two reflections s and t (making an angle

of π/n between them) as in Example 1.2.4(2) and they can be presented as

$$\mathbf{D}_{2n} = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle.$$

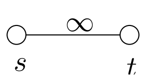
The Coxeter diagram of $\mathbf{D}_{2n}$ with the set of generators s and t is then



3. As in the finite case, the infinite dihedral group $\mathbf{D}_\infty$ is also a Coxeter group considering a set of two parallel reflections r and s as set of generators. A presentation of $\mathbf{D}_\infty$ is as follows.

$$\mathbf{D}_\infty = \langle s, t \mid s^2 = t^2 = 1 \rangle.$$

In this case, since there is no relation between the generators s and t, the respective Coxeter diagram is



By looking at the diagram spanned by a subset of vertices of the Coxeter diagram Σ, one gets the notion of a sub-Coxeter system and parabolic subgroup.

Theorem 1.4.7 ([Wei09, Theorem 4.6])**.** *Let $J \subseteq S$ and let Σ_J denote the subdiagram of Σ spanned by the set J (i.e., the subdiagram obtained from Σ by deleting all the vertices not in J and all the edges containing a vertex not in J). Let $W_J = \langle t \mid t \in J \rangle$. Then (W_J, J) is a Coxeter system of type Σ_J.*

Definition 1.4.8. Let (W, S) be a Coxeter system and let J be a subset of S.

The group W_J as defined in Theorem 1.4.7 is called a *standard parabolic subgroup*. Any of its conjugates by elements of $W = W_S$ will be called a *parabolic subgroup.*

J is called a *spherical set* if the standard parabolic subgroup W_J is finite (also called spherical in the literature). In other words, if $|s_i s_j| \leq 2$ in the respective Coxeter diagram, for all $s_i, s_j \in J$.

Otherwise J is called a *non-spherical set*.

Since any intersection of parabolic subgroups is again parabolic, it makes sense to define the parabolic closure of a subset of W.

Definition 1.4.9. Let E be a subset of W. We define the *parabolic closure* of E, denoted by $Pc(E)$, as the smallest parabolic subgroup of W containing E.

Lemma 1.4.10 ([CM13, Lemma 2.4])**.** *Let $H_1 < H_2$ be subgroups of W. If H_1 is of finite index in H_2 then $Pc(H_1)$ is of finite index in $Pc(H_2)$.*

Reduced words in Coxeter groups

Let (W, S) be a Coxeter system of type Σ and set of generators $S = \{s_i\}_{i \in I}$. Let M_S denote the free monoid on the alphabet S.

Consider a word $s_1 \cdots s_n$ in the monoid M_S. If we assume for instance that $s_n = s_{n-1}$, then, regarding $s_1 \cdots s_n$ in W (as S is its set of generators), the word $s_1 \cdots s_{n-2} s_{n-1} s_n$ and $s_1 \cdots s_{n-2}$ represent the same element of W. Hence it makes sense to define reduced words in the monoid with respect to a Coxeter diagram (or to a Coxeter group). We will present that definition and state some properties of such reduced words.

Definition 1.4.11.

1. Let

$$p(s,t) = \begin{cases} (st)^{m_{st}/2} & \text{if } m_{st} \text{ is even,} \\ t(st)^{(m_{st}-1)/2} & \text{if } m_{st} \text{ is odd} \end{cases}$$

   be a word in the free monoid M_S of length m_{st} ending in t, for all ordered pairs of distinct $s, t \in S$ such that $m_{st} < \infty$. If $m_{st} = \infty$ then the word $p(s,t)$ is not defined.

2. An *elementary homotopy* is a transformation of a word of the form $w_1 p(s,t) w_2$ into the word $w_1 p(t,s) w_2$, where w_1 and w_2 are arbitrary elements of M_S. Sometimes in the literature elementary homotopies are also called braid operations.

3. Two words w_1 and w_2 in M_S are *homotopic* if w_1 can be transformed into w_2 by a sequence of elementary homotopies.

4. A *contraction* is a transformation of a word of the form w_1ssw_2 into the word w_1w_2. The inverse of a contraction is called an *expansion*.

5. A word in M_S is *contractible* if it is of the form w_1ssw_2 for some $s \in S$.

6. An *elementary Σ-operation* on a word in M_S is an elementary homotopy or a contraction. Two words are called *equivalent* if one can be transformed into the other by a sequence of elementary Σ-operations.

7. A word in M_S is called *reduced* if it is not homotopic to a contractible word.

We emphasize that all of these notions depend on the Coxeter diagram Σ.

Next we compile a set of properties that relate reduced words in the free monoid M_S with the elements of W that those words represent. These properties can be found in [Wei09] and we state them without proof.

Proposition 1.4.12. *Let (W, S) be a Coxeter system with Coxeter diagram Σ and consider the free monoid M_S. Then the following hold.*

1. *Two words w_1 and w_2 in M_S are equivalent (with respect to Σ) as defined in Definition 1.4.11 if and only if they represent the same element of W.*

2. *If two words w_1 and w_2 in M_S are such that $w_1 = w_2 \in W$, then the lengths $l(w_1)$ and $l(w_2)$ have the same parity.*

3. *Let $w \in M_S$ and $t \in S$. If w is reduced but wt (respectively tw) is not, then w is homotopic to a word which ends (respectively, begins) with t.*

4. *Let w_1 and w_2 be reduced words on M_S such that they represent the same element of W. Then w_1 is homotopic to w_2.*

5. *Let $w \in W$ and let $s_1 \cdots s_n$ be a representation of w in the monoid M_S. Let $l(w)$ denote the length of w as a group element. Then $l(w) \leq n$ with equality if and only if $s_1 \cdots s_n$ is a reduced word.*

From now on, when we refer to a reduced word in the monoid M_S, we always mean reduced with respect to Σ, as in Definition 1.4.11, that is, a word that cannot be shortened using the relations in the Coxeter diagram, as described in the definition.

Proposition 1.4.13 ([Wei09, Proposition 4.8])**.** *Let $w \in M_S$ be a reduced word, let $J \subset S$ and let W_J as in Theorem 1.4.7. Then $w \in W_J$ if and only if $w \in M_J$, that is, the letters of a reduced representation of w are in J.*

We finish this section by presenting the solution for the word problem in Coxeter groups, which was solved by Tits in 1969 [Tit69]. The word problem is the following:

Given two words $w_1 = r_1 \cdots r_\ell$ and $w_2 = t_1 \cdots t_k$ in the free monoid M_S, decide whether they represent the same element of W.

Theorem 1.4.14 ([Tit69])**.** *Let (W, S) be a Coxeter group with Coxeter diagram Σ. If w_1 and w_2 are reduced words, then they represent the same element if and only if w_1 can be transformed to w_2 by applying a sequence of Σ-elementary homotopies.*

1.4.2 Chamber systems

Chamber systems were defined by Tits in [Tit81]. These will be the geometric objects that, together with a metric coming from a Coxeter group, will give rise to buildings. We will start with some notions of edge-colorings and residues in graphs and then we define a chamber system.

Right after we give some examples of chamber systems of low rank and we define automorphisms of these geometric structures. We stress that we will always consider simple undirected graphs.

Definition 1.4.15. Let $\Gamma = (V, E)$ be a simple graph.

1. An *edge-coloring* of $\Gamma = (V, E)$ is a map from the edge-set E to a set S whose elements we think of as colors. We will always assume that this map is surjective, so that S is unambiguous.
2. An *edge-colored graph* is a graph Γ endowed with an edge-coloring. The image S of the edge-coloring will be called the *index-set* of the edge-colored graph.

A subgraph $\Gamma' = (V', E')$ of an edge-colored graph Γ will always be assumed to have the edge-coloring obtained by restricting the edge-coloring of Γ to E'. The index-set of Γ' is then a subset of the index-set of Γ.

In edge-colored graphs, we can associate a type of adjacency if two vertices share an edge of a certain color. In the next definition we set up notation for such adjacencies in edge-colored graphs.

Definition 1.4.16. Suppose that $\Gamma = (V, E)$ is an edge-colored graph with index-set S. Rather than giving a name to the edge-coloring, we will write, for $v_1, v_2 \in V$ and $s \in S$,

$$v_1 \overset{s}{\sim} v_2 \text{ for '}\{v_1, v_2\} \text{ is an edge of } \Gamma \text{ whose color is } s\text{'}.$$

Two vertices v_1 and v_2 will be called *s-adjacent* (for some $s \in S$) if $v_1 \overset{s}{\sim} v_2$, and two vertices will be called *adjacent* if they are s-adjacent for some $s \in S$. (Since E consists of two-element subsets of V, a vertex is never adjacent to itself).

Next we define types of galleries using the coloring and types of connected components of the graph, called residues.

Definition 1.4.17. Let Γ be an edge-colored graph with index-set S and let $J \subset S$. Consider a gallery $\gamma = (v_0, v_1, \dots, v_k)$ (for some $k \geq 0$) from v_0 to v_k such that

$$v_{j-1} \overset{s_j}{\sim} v_j, \text{ for some } s_j \in S \text{ for all } j \in \{1, \dots, k\}.$$

Then we say that the *type of the gallery* γ is the word $s_1 \cdots s_k$ (an element of the free monoid M_S).

A J-gallery is a gallery whose type is in M_J. The graph Γ is *J-connected* if for any two vertices v_1 and v_2 of Γ there exists a J-gallery from v_1 to v_2. In particular Γ is connected if and only if it is S-connected.

Definition 1.4.18. A *J-residue* $\mathcal{R}$ of Γ is a connected component of the subgraph of Γ obtained from Γ by discarding all the edges whose color is not in J. A *residue* of Γ is a J-residue for some $J \subset S$. If $s \in S$ then an s-residue (that is, a residue where $|J|$ is one) is called an *s-panel*.

If $v \in V\Gamma$ we will denote by $\mathcal{R}_{J,v}$ (sometimes also $\mathcal{R}_J(v)$) the J-residue of Γ containing the vertex v. Normally if $J = \{s\}$ we will denote the s-panel of v in Γ by $\mathcal{P}_{s,v}$ (or $(\mathcal{P}_s(v))$.

Example 1.4.19. In Figure 1.5 we have a connected edge-colored graph with index-set $\{1, 2, 3\}$. The colors highlight examples of a 2-residue, two 1-residues and a $\{2, 3\}$-residue.

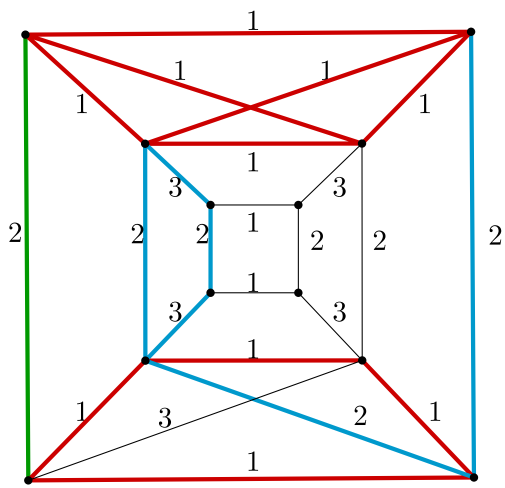


Figure 1.5: Examples of residues.

After the initial concepts we are in a place to define chamber systems.

Definition 1.4.20. A *chamber system* is an edge-colored graph Δ with index-set S such that for each $s \in S$, all s-panels of Δ are complete graphs with at least two vertices. We will refer to the elements of V as *chambers* rather than vertices and we will write $\mathrm{Ch}(\Delta)$ instead of V. The cardinality of S is called the *rank* of Δ.

A chamber system is *thin* if every panel contains exactly two chambers and is *thick* if every panel contains at least three chambers.

From this definition, we infer that, for each $s \in S$, "being in the same s-panel" is an equivalence relation on the chambers where each equivalence class has at least two elements.

We observe that the edge-colored graph of Figure 1.5 is not an example of a chamber system as, for instance, the highlighted bottom red residue of type 1 is not a complete graph.

Example 1.4.21. A chamber system of rank zero is just a collection of chambers with no edges (and no colors). A chamber system of rank one is a graph with edges all of the same color, each of whose connected components is a complete graph with at least two chambers (observe that we are not requiring chamber systems to be connected graphs).

A chamber system of rank two can be viewed as a bipartite graph. In fact, chambers systems of rank two and bipartite graphs are essentially the same thing, as we explain in the next example.

Example 1.4.22 ([Wei09, Example 1.8])**.** Let $\Gamma = (V, E)$ be a bipartite graph (with no edge-coloring). This means that there exists a partition of V into subsets V_1 and V_2 such that each edge joins a vertex of V_1 to a vertex of V_2. Suppose also that every vertex of Γ has at least two neighbors. Let $\Delta_\Gamma = E$ and let $S = \{1, 2\}$. We set $c_1 \overset{s}{\sim} c_2$, for $c_1, c_2 \in \Delta_\Gamma$ and $s \in S$ whenever the edges c_1 and c_2 are distinct but have a vertex in common in V_s. This makes Δ_Γ into a chamber system with index-set S which is connected if and only if Γ is connected. Moreover, it is thin if and only if every vertex of Γ has exactly two neighbors and thick if and only if every vertex of Γ has at least three neighbors. The chamber system Δ_Γ depends on the choice of V_1 and V_2, but if Γ is connected, it is unique up to a relabeling of the index-set.

Conversely, let Δ be a chamber system of rank two and let Γ_Δ denote the graph whose vertices are the panels of Δ, and where two panels are joined by an edge if and only if they have a non-empty intersection. Then Γ_Δ is a bipartite graph (since two panels can have non-empty intersection only if they have different types) all of whose vertices have at least two neighbors. If we restrict our attention to connected bipartite graphs and connected chambers of rank two, this construction is the inverse of the construction above.

Definition 1.4.23. If $J \subset S$ and $\mathcal{R}$ is a J-residue of a chamber system Δ, then each chamber of $\mathcal{R}$ is contained in at least one edge of every color in J and hence J is the index-set of $\mathcal{R}$ and the rank of $\mathcal{R}$ is $|J|$. We call J the *type* of $\mathcal{R}$.

Next we define isomorphisms between chamber systems.

Definition 1.4.24. Two chamber systems Δ and Δ' with index-sets S and S' will be called *isomorphic* if there exist bijections σ from S to S' and ϕ from $\mathrm{Ch}(\Delta)$ to $\mathrm{Ch}(\Delta')$ such that

$$x \overset{s}{\sim} y \text{ if and only if } \phi(x) \overset{\sigma(s)}{\sim} \phi(y)$$

for all $x, y \in \Delta$ and all $s \in S$. If ϕ, σ is such a pair of bijections, we will say that ϕ is a *σ-isomorphism* from Δ to Δ'. By isomorphism we mean a σ-isomorphism for some σ. An isomorphism is *special* (or *type-preserving)* if $S = S'$ and the corresponding map σ from S to S is the identity map.

Since isomorphisms map galleries to galleries of the same length, they preserve the distance between chambers, that is, they are isometries with respect to the discrete distance.

Definition 1.4.25. When $\Delta = \Delta'$ an isomorphism is called an *automorphism*. In this case a special isomorphism is also called a *special automorphism.*

In this thesis we will just be concerned with special automorphisms. Therefore when we refer to automorphisms of a chamber system (or of a building later on), we are implicitly assuming that σ in the Definition 1.4.24 is the identity map.

We denote by $\mathrm{Aut}(\Delta)$ the group of special automorphisms of Δ, sometimes also called the group of *type-preserving automorphisms* of Δ.

Observe that the permutation topology defined in Section 1.3.2 for groups of automorphisms of graphs in general is defined also in the type-preserving case, in a similar way.

Next we present a chamber system Δ_Σ whose chambers are elements of a Coxeter group with Coxeter diagram Σ. This is an important construction since apartments in any building will be isomorphic to Δ_Σ, constructed from the Coxeter group of respective type.

Definition 1.4.26. Let Σ be a Coxeter diagram with vertex-set S and let (W, S) be the Coxeter system of type Σ with set of generators $S = \{s_i\}_{i \in I}$. We can define a thin chamber system with index-set S whose chambers are elements of W by setting

$$x \stackrel{s}{\sim} y \text{ for } s \in S \text{ if and only if } x^{-1}y = s.$$

Since $s \neq 1$, $x \stackrel{s}{\sim} y$ implies that $x \neq y$. As also $s^2 = 1$ for all $s \in S$, the relation $\stackrel{s}{\sim}$ is symmetric and since $s_i \neq s_j$ whenever $i \neq j$, the color of an edge is well defined. We denote this chamber system by Δ_Σ. Thus S is both the vertex-set of Σ and the set of colors of Δ_Σ. A *Coxeter chamber system* of type Σ is a chamber system Δ with index-set S such that there exists a special isomorphism from Δ to Δ_Σ. Thus Δ_Σ is the unique Coxeter chamber system of type Σ up to a special isomorphism.

The Coxeter chamber system Δ_Σ is just the Cayley graph of the group W with respect to the generating set $S = \{s_i \mid i \in I\}$ with edges labeled by the corresponding generator. We emphasize that Δ_Σ is a thin chamber system, i.e., each chamber is $\{s_i\}$-adjacent to exactly one chamber for each $i \in I$.

Proposition 1.4.27 ([Wei09, Proposition 2.5])**.** *Let Σ be a Coxeter diagram with vertex-set S and let (W, S) be the corresponding Coxeter system. Let Δ_Σ be the corresponding Coxeter chamber system. Then for all $c \in \mathrm{Ch}(\Delta_\Sigma)$ and for all reduced words $w \in M_S$, there is a unique gallery of type w in Δ_Σ which begins at c.*

Observation 1.4.28. Left multiplication by an arbitrary element of W is an automorphism of Δ_Σ. Since Δ_Σ is connected and each chamber is s-adjacent to just one chamber for each $s \in S$, the identity is the only automorphism of Δ_Σ which fixes a chamber. It follows that the map which sends $g \in W$ to "left multiplication by g" is an isomorphism from W to $\mathrm{Aut}(\Delta_\Sigma)$. We will, in general, identify W with its image in $\mathrm{Aut}(\Delta_\Sigma)$ under this isomorphism.

1.4.3 Buildings

Buildings were defined by Jacques Tits as a way to understand simple algebraic groups over an arbitrary field. He proved that to every such group one can associate a simplicial complex, called a spherical building, in which the group acts by isometries. The group imposes several regularity conditions on the complex and by taking those conditions as axioms for simplicial complexes, Tits arrived to the definition of a building.

That definition relies on a set of special simplicial subcomplexes, called apartments (see Definition 1.4.38). We will present the original definition of Tits in the end of the section for general historical purposes (*cf.* Definition 1.4.50) but we will use in this thesis a later (equivalent) definition of a building also introduced by Tits in [Tit81] considering the concept of chamber systems. For the remaining of the section we mostly follow [Wei09] and [Ron09].

Definition 1.4.29. Let Σ be a Coxeter diagram with vertex-set S and let (W, S) be the Coxeter system of type Σ. A *building of type* Σ is a pair (Δ, δ), where

1. Δ is a chamber system whose index-set is S, and
2. δ is a function $\delta : \mathrm{Ch}(\Delta) \times \mathrm{Ch}(\Delta) \to W$ such that for each reduced word w in the free monoid M_S (reduced with respect to Σ) and for each pair of chambers $c_1, c_2 \in \mathrm{Ch}(\Delta)$, we have

   $$\delta(c_1, c_2) = s_w \Leftrightarrow \text{there is a gallery in } \Delta \text{ of type } w \text{ from } c_1 \text{ to } c_2.$$

We call the group W the *Weyl group* and the map δ the *Weyl-distance function*. We will sometimes refer to Σ as the *Coxeter diagram of* (Δ, δ). When the Coxeter system (W, S) and the Weyl distance function δ are clear from the context we will only refer to a building as Δ.

Remark 1.4.30. The Weyl-distance function is a map with the following properties:

1. Given a minimal gallery $\gamma = (c_0, \ldots, c_k)$ of type $w \in M_S$, then $\delta(c_0, c_k)$ is the element of W represented by w.

2. Let c_1 and c_2 be chambers. The function $\gamma \mapsto \{\text{type of } \gamma\}$ gives a one-to-one correspondence between minimal galleries from c_1 to c_2 and reduced decompositions of $\delta(c_1, c_2) \in W$ in the monoid M_S. For a proof of this fact we refer to [AB08].

3. We note that $\delta(c_1, c_2)$ should not be confused with the distance $\operatorname{dist}(c_1, c_2)$ from c_1 to c_2 in the sense of Definition 1.2.2. Since the empty word is reduced, $\delta(c_1, c_2) = 1$ if and only if $c_1 = c_2$.

Next we state some properties of buildings, that come as consequences of the definition.

Proposition 1.4.31 ([Ron09, Prosition 3.1])**.** *Let (Δ, δ) be a building. Then the following hold*

1. *δ is surjective and Δ is connected.*

2. *$\delta(c_1, c_2) = \delta(c_2, c_1)^{-1}$ for all chambers $c_1, c_2 \in \mathrm{Ch}(\Delta)$.*

3. *$\delta(c_1, c_2) = s$ if and only if $c_1 \overset{s}{\sim} c_2$.*

4. *If $w \in M_S$ is reduced then a gallery of type w from c_1 to c_2 is unique.*

Proof. For the proof of the first 3 statements we refer to [Wei09, Chapter 7] and for the proof of the last property we point to the afore-mentioned proposition in [Ron09]. □

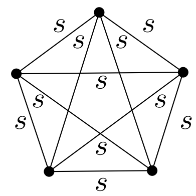
We now present a set of examples of buildings.

Example 1.4.32. Let Σ be a Coxeter diagram, let (W, S) denote the corresponding Coxeter system, let $\Delta = \Delta_\Sigma$ denote the corresponding Coxeter chamber system (see Definition 1.4.26) and let

$$\delta_W : \mathrm{Ch}(\Delta_\Sigma) \times \mathrm{Ch}(\Delta_\Sigma) \to W \text{ be given by } \delta_W(c_1, c_2) = c_1^{-1} c_2,$$

for all c_1, c_2 chambers in Δ. By Proposition 1.4.27, we have that (Δ, δ_W) is a building of type Σ.

Example 1.4.33. Let (W, S) be a Coxeter system where $S = \{s\}$ and $W = \mathbb{Z}_2$. A building Δ of type (W, S) is a complete graph Δ with distance function $\delta(c_1, c_2) = s$ if $c_1 \neq c_2$ and $\delta(c_1, c_2) = 1$ if $c_1 = c_2$. An example is illustrated in the figure on the right. We call such a building a *rank 1 building*.



Example 1.4.34. In this example we will show that a rank-2 building is a generalized m-gon, for some m, and vice versa, that is, any generalized m-gon is a rank-2 building. A detailed proof of this result can be found in [Ron09, Proposition 3.2]. In this example we illustrate how we can regard such an object from the two points of view.

Let $m \geq 2$ or $m = \infty$. A *generalized m-gon* is a connected bipartite graph of diameter m and girth $2m$, in which each vertex is incident to at least two edges (recall the definition of diameter and girth in Definition 1.2.2).

Consider a building Δ of rank 2 and the respective Coxeter group W generated by two elements s and t. The reduced words in W are the alternating sequences $sts...$ of length $\leq m_{st}$ and they give rise to different group elements with the exception of the case when we have equality between $sts...$ and $tst...$, i.e., when both these words have m_{st} letters (and in particular m_{st} is finite). Then it follows that Δ has diameter and girth as required for an m-gon and each vertex is contained in at least two edges.

Conversely, let Γ be a generalized m-gon. Then each vertex of Γ has at least two neighbors. Since Γ is connected and bipartite we can get a chamber system Δ_Γ as in Example 1.4.22. Let (W, S) be a Coxeter system generated by two elements s and t such that $|st| = m$. Then the words $p(s, t)$ and $p(t, s)$, as defined in 1.4.11, are the only words in M_S of length m which are reduced, and any other reduced word of length smaller than m is homotopic only to itself. Thus we can define $\delta : \Delta_\Gamma \times \Delta_\Gamma \to W$ by $\delta(x, y) = w$ if $\mathrm{dist}(x, y) < n$, where w is the type of the only minimal gallery between x and y (which are edges of Γ) in Δ_Γ, and $\delta(x, y) = p(s, t)$ if $\mathrm{dist}(x, y) = n$. Then (Δ_Γ, δ) is a building of type (W, S).

We remark that a generalized ∞-gon is a tree without end points.

Hence the Coxeter group associated to this rank 2 building is the infinite dihedral group, generated by s and t with $m_{st} = \infty$.

Next we present some properties of subbuildings and isometries between buildings.

Theorem 1.4.35 ([Ron09, Theorem 3.5])**.** *Let $J \subseteq S$ and let $\mathcal{R}$ be an J-residue of Δ. Then $\mathcal{R}$ is a building of type (W_J, J), the Coxeter system defined in Lemma 1.4.7.*

Definition 1.4.36. Let (Δ_1, δ_1) and (Δ_2, δ_2) be two buildings of the same type (and thus having the same index-set and the same Weyl group). A map π from a subset $X \subseteq \Delta_1$ to Δ_2 will be called an *isometry* from X to Δ_2 if

$$\delta_2(\pi(c_1), \pi(c_2)) = \delta_1(c_1, c_2), \text{ for all } c_1, c_2 \in X.$$

Proposition 1.4.37 ([Wei09, Proposition 8.2])**.** *Let (Δ_1, δ_1) and (Δ_2, δ_2) be two buildings of the same type Σ and let π be a map from Δ_1 to Δ_2. Then π is an isometry from Δ_1 to Δ_2 if and only if π is a special isomorphism from Δ_1 to $\pi(\Delta_1)$.*

With the definition of isometry and with the last proposition, we can define apartments in a building.

Definition 1.4.38. Let (Δ, δ) be a building of type Σ. An *apartment* of Δ is a subgraph of Δ whose chamber and edge-sets are the images of the chamber and edge-sets of the building Δ_Σ (as in Example 1.4.32) under an isometry from Δ_Σ to Δ.

By Proposition 1.4.37, an isometry π from Δ_Σ to a building of type Σ is a special isomorphism from Δ_Σ to the image of π. Therefore apartments of a building of type Σ are Coxeter chamber systems of type Σ (see Definition 1.4.26).

Next we provide some examples of apartments in buildings. As observed before, Example 1.4.32 is the construction of a thin building, so in this case the building and the apartment coincide.

Example 1.4.39.

1. If Δ is a rank 1 building, whose respective Coxeter group is generated by an element s, then an apartment

in Δ consists of two chambers connected by an edge labelled s, as in Figure 1.7.

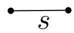


Figure 1.7: A rank-1 apartment.

2. An apartment in a tree without end points, as in Example 1.4.34, considering $m = \infty$, is an infinite ray of chambers connected by edges labelled by the generators of the respective Coxeter group. Denoting such generators by s and t, an apartment is depicted in the Figure 1.8

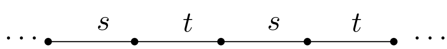


Figure 1.8: An apartment in a tree without end points.

3. We can consider a rank-3 spherical building Δ of type Sym(4), by regarding the symmetric group as a Coxeter group with set of generators

$$\begin{aligned}\mathrm{Sym}(4) = \langle (12), (23), (34) \mid\, & (12)^2 = (23)^2 = (34)^2 = 1 \\ & ((12)(23))^3 = ((23)(34))^3 = 1 \\ & ((12)(34))^2 = 1\rangle\end{aligned}$$

Then an apartment of Δ is depicted in Figure 1.9, which corresponds to the Cayley graph of Sym(4) with respect to the set of generators considered.

4. Consider the Coxeter group

$$W = \langle s, t, r \mid s^2 = t^2 = r^2 = (st)^3 = (sr)^3 = (tr)^3 = 1\rangle.$$

An apartment A of a building of type W is isomorphic to a tessellation of the Euclidean plane by equilateral triangles, considering the barycentric subdivision of the hexagons obtained from

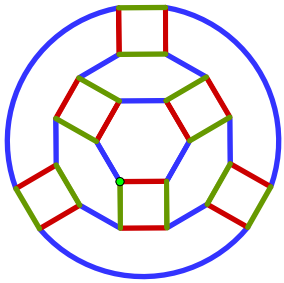


Figure 1.9: Apartment of type Sym(4).

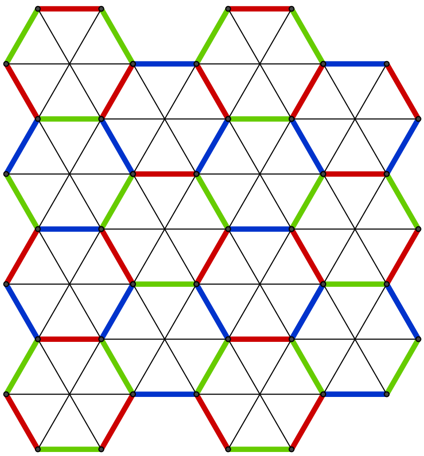


Figure 1.10: An apartment of type $\tilde{A}_2$.

the Cayley graph of W, as partially depicted in Figure 1.10. Each chamber in A is a triangle (maximal dimension simplices) and two chambers are in the same panel if they share an edge. A building Δ of type W is also called a $\tilde{A}_2$-*building*. (Observe that to get this tessellation we are implicitly considering the geometric realization of the building, as it will be explained in Section 1.4.4).

A building whose apartments are isomorphic to tessellations of a Euclidean space is called a *Euclidean building*.

After this first set of examples, we will state some properties of apartments, in particular, that they are combinatorial convex sets.

Proposition 1.4.40 ([Wei09, Corollary 8.6])**.** *Every two chambers of a building are contained in a common apartment.*

Definition 1.4.41. A set of chambers $\mathcal{C}$ of Δ is called *combinatorially convex* if for every pair $c, c' \in \mathcal{C}$, every minimal gallery from c to c' is entirely contained in $\mathcal{C}$.

Proposition 1.4.42 ([Wei09, Corollary 8.9])**.** *Apartments in a building are combinatorially convex.*

Now we will define a stronger version of transitivity on groups of automorphisms of a building.

Definition 1.4.43. Let Δ be a building and let G be a group of automorphisms of Δ. We say that G acts *strongly transitively* on Δ if G is transitive on the set of pairs (A, c) consisting of an apartment A and a chamber $c \in \mathrm{Ch}(A)$.

An equivalent way of defining strong transitivity is to require G to be chamber transitive and $\mathrm{Stab}_G(c)$ to be transitive on the set of apartments containing c.

Remark 1.4.44. This is an empty remark. I promised my non-mathematician friends that I would include in my thesis something that would make them open the book further then page xiv. So they will try to find this remark.

Next we define a wall in an apartment of a building. It is enough to define it in Δ_Σ, since we already know that all apartments are isomorphic to Δ_Σ.

Definition 1.4.45. We identify W with its image under the isomorphism from W to $\mathrm{Aut}(\Delta_\Sigma)$, as described in Observation 1.4.28.

1. A *reflection* is a non-trivial element of W which stabilizes edges (panels) of Δ_Σ. As pointed out in Observation 1.4.28, only the identity fixes a chamber of Δ_Σ. Therefore, if r is a reflection and $\{c_1, c_2\}$ is an edge of Δ_Σ stabilized by r, then r interchanges c_1 and c_2, the order of r is 2 and r is uniquely determined by $\{c_1, c_2\}$. In fact, for each edge $\{c_1, c_2\}$, there exists an $i \in I$ such that $c_2 = c_1 s_i$ and thus there exists a reflection that stabilizes $\{c_1, c_2\}$, namely, the product $c_1 s_i c_1^{-1}$ (observe that $c_1, c_2 \in W$).

2. Let $Ref(W)$ denote the set of reflections in W, which, by the previous paragraph, coincides with the set $\{wsw^{-1} \mid w \in W, s \in S\}$.

3. The set of edges fixed by a reflection r will be called the *wall* of r and denoted by $M(r)$.

4. Let r be a reflection. We will say that a gallery $\gamma = (c_0, \ldots, c_k)$ *crosses the wall* $M(r)$ at the panel $\{c_{i-1}, c_i\}$ for some $i \in \{1, \ldots, k\}$ if the panel $\{c_{i-1}, c_i\}$ is contained in $M(r)$. We will say that γ *crosses* $M(r)$ m *times* if m is the number of indices $i \in \{1, \ldots, k\}$ such that γ crosses $M(r)$ at $\{c_{i-1}, c_i\}$.

Next we present two results concerning walls.

Lemma 1.4.46 ([Wei09, Lemma 3.8]). *A minimal gallery cannot cross a wall more than once.*

The next Lemma is a consequence of Proposition 1.56 of [AB08].

Lemma 1.4.47. *Let Δ be a building and let c_1 and c_2 be two chambers in Δ_Σ. Any two minimal galleries between c_1 and c_2 cross the same set of walls $M(r)$, for a reflection $r \in Ref(W)$.*

The Weyl distance between two chambers c_1 and c_2 in a building, which is an edge-colored graph colored with the generators of the Coxeter group, gives us a type and hence paths between c_1 and c_2 corresponding to the distinct reduced representations of $\delta(c_1, c_2)$.

However, for instance, in inductive arguments, we will only be interested on the length of such galleries and not in their type. Therefore we define a gallery distance between chambers in a building as in [TW11] and spheres and balls around a fixed chamber, with respect to this distance.

Definition 1.4.48. Let Δ be a building and let $c_1, c_2 \in \mathrm{Ch}(\Delta)$. The *gallery distance* between the chambers c_1 and c_2 is defined as

$$\mathsf{d}_W(c_1, c_2) = l(\delta(c_1, c_2)),$$

that is, the length of a minimal gallery between the chambers c_1 and c_2.

For a fixed chamber $c_0 \in \mathrm{Ch}(\Delta)$ we define the spheres at a fixed gallery distance from c_0 as

$$\mathsf{S}(c_0, n) = \{c \in \mathrm{Ch}(\Delta) \mid \mathsf{d}_W(c_0, c) = n\},$$

and the balls as

$$\mathsf{B}(c_0, n) = \{c \in \mathrm{Ch}(\Delta) \mid \mathsf{d}_W(c_0, c) \leq n\}.$$

We finish this section with the original definition of a building given by Tits which was formulated in terms of simplicial complexes. Here the chambers are the maximal dimensional simplices and they really mean "rooms". The inversion of the dimensions makes us consider the definition of a Coxeter complex.

Definition 1.4.49 ([AB08, Definition 3.1])**.** Let (W, S) be a Coxeter system. A *standard coset* of W is a coset of the form wW_J with $w \in W$ and $W_J = \langle J \rangle$, for some subset $J \subseteq S$.

The poset $\Sigma(W, S)$ of standard cosets ordered by reverse inclusion is called the *Coxeter complex* associated to (W, S). We have that $B \leq A$ in $\Sigma(W, S)$ if and only if $A \subseteq B$ as subsets of W. In that case we say that B is a *face* of A.

Coxeter complexes consist of chambers divided by walls and therefore they are referred to as apartments. The axioms in the next definition prescribe the rules to glue the apartments together in order to get a building.

Definition 1.4.50 ([AB08, Definition 4.1])**.** A *building* is a simplicial complex Δ which can be described as the union of subcomplexes, called apartments satisfying the following axioms.

1. Each apartment Λ is a Coxeter complex.

2. For any two simplices $A, B \in \Delta$, there is an apartment Λ containing both of them.

3. If Λ_1 and Λ_2 are two apartments containing A and B then there is an isomorphism $\Lambda_1 \to \Lambda_2$ fixing A and B pointwise, that is, fixing each vertex of A and B.

A map fixes a simplex A pointwise if it fixes every vertex of A.

We can see a building Δ defined above as a chamber system by considering each chamber as a point and by adjacency in the chambers as

$$A \overset{s_i}{\sim} B \text{ if and only if } A \cap B \text{ is a panel of type } s_i.$$

The Weyl distance function is then considered by looking at the distance between two chambers in a common apartment. By Property 3. the distance is independent of the choice of the apartment.

Henceforth, when we refer to buildings, we will keep in mind Definition 1.4.29.

1.4.4 Geometric realizations of buildings

Any building of type (W, S) has a geometric realization as a CAT(0)-space. This means that, given a building Δ of type (W, S), there exists a CAT(0)-space X and a canonical injection $\mathrm{Aut}(\Delta) \to \mathrm{Is}(X)$.

When suitable for our purposes, we will identify the elements in $\mathrm{Aut}(\Delta)$ with their image in $\mathrm{Is}(X)$, that is, we will see $g \in \mathrm{Aut}(\Delta)$ as acting on the geometric realization X. In this section we describe the construction for the standard (Davis) geometric realization, following the notation in [Dav98].

Definition 1.4.51. A *mirror structure* over an arbitrary set S on a space Y is a family of subsets (Y_s) of Y indexed by S. The subsets Y_s are called *mirrors*. A space with a mirror structure is called a *mirror spacemirror space*.

Given a mirror structure on Y, a subspace $Z \subset Y$ inherits a mirror structure by setting $Z_s = Y_s \cap Z$.

Next we define subsets of mirrors and subsets of S that will play an important role in the definition of realizations of buildings.

Definition 1.4.52. Let Y be a mirror space over S. For each non-empty subset $T \subset S$ we define

$$Y_T = \cap_{s \in T} Y_s \text{ and } Y^T = \cup_{s \in T} Y_s.$$

We have that $Y_\emptyset = Y$ and $Y^\emptyset = \emptyset$. Given a subset C of Y or a point $x \in Y$ we define

$$S(C) = \{s \in S \mid C \subset Y_s\} \text{ and } S(x) = \{s \in S \mid x \in Y_s\}.$$

Now we proceed to the definition of the Davis chamber of a Coxeter system.

Definition 1.4.53. Let (W, S) be a Coxeter system.

1. Denote the poset of spherical subsets of S partially ordered by inclusion by $\mathcal{S}(W, S)$ or $\mathcal{S}$ if the Coxeter system is understood, where a spherical set is introduced in Definition 1.4.8.

2. For any $T \subset S$, let $\mathcal{S}_{\geq T}$ be the poset of spherical subsets of S which contain T.

3. Let $K = |\mathcal{S}|$ be the geometrical realization of the poset $\mathcal{S}$. (Recall that the geometric realization of a poset has as n-simplices the chains in $\mathcal{S}$ of cardinality $n + 1$).

   For each $s \in S$, put $K_s = |\mathcal{S}_{\geq \{s\}}|$ and for each $T \subseteq S$ let $K_T = |\mathcal{S}_{\geq T}|$.

   We say that the complex K with this mirror structure is the *Davis chamber* of (W, S).

It follows from the Davis chamber being a realization of a poset that it is a flag complex.

We can also visualize the Davis chamber as the cone of a barycentric subdivision of a poset. We make it more precise in the following definition.

Definition 1.4.54. Let (W, S) be a Coxeter system. The *nerve* of (W, S), denoted by $L(W, S)$, is the poset of non-empty elements of $\mathcal{S}$. It is an abstract simplicial complex whose simplices are spherical subsets of S.

The Davis chamber K can also be defined as the cone of the barycentric subdivision $L(W, S)'$ of $L(W, S)$. For $s \in S$, we define K_s to be the star of s in $L(W, S)'$ and for $x \in K$, consider the set $S(x) = \{s \in S \mid x \in K_s\}$.

These two definitions of the Davis chamber are equivalent. The empty set in the first definition corresponds to the cone point in the second.

Standard geometric realization of a building

Definition 1.4.55. Let Δ be a building of type (W, S) and let Y be a mirror space over S.

1. We define an equivalence relation $\sim$ on $\Delta \times Y$ by
$$(c, x) \sim (d, y) \text{ if and only if } x = y \text{ and } \delta(c, d) \in S(x).$$
The Y*-realization* of Δ is then $\mathcal{U}(\Delta, Y) = (\Delta \times Y)/\sim$. We identify s-mirrors of two chambers when those two chambers are s-adjacent.

2. The realization of a chamber $c \in \mathrm{Ch}(\Delta)$ inside a realization $\mathcal{U}(\Delta, Y)$ is the image of (c, Y) and the realization of a residue $\mathcal{R}$ is the union of the realizations of the chambers contained in $\mathcal{R}$.

3. Two panels $\mathcal{P}_s$ and $\mathcal{P}_t$ in Δ are *adjacent* if they are contained in a spherical residue of type $\{s, t\}$ and the set of distances between chambers in $\mathcal{P}_s$ and $\mathcal{P}_t$ is a coset of $W_{\{s\}}$ in $W_{\{s,t\}}$.

   We extend these adjacencies to equivalence classes and we define a *wall* in Δ as an equivalence class of panels. By the realization of a wall we just mean the union of the mirrors corresponding to the panels in the wall.

We now define the standard realization of a building, using the Davis chamber.

Definition 1.4.56. Let Δ be a building of type (W, S) with Davis chamber K.

1. $X = \mathcal{U}(\Delta, K)$ is called the *standard realization* of Δ.

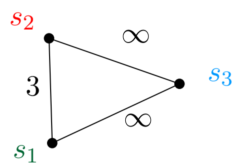
2. If $\Delta = \Delta_\Sigma$ is the thin building of type (W, S) then the standard realization of Δ_Σ is called the *Davis complex* and we have, for $(w, x), (w', x') \in W \times K$, that $(w, x) \sim (w', x')$ if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.

Theorem 1.4.57 ([Dav98, Theorem 11.1])**.** *The standard realization of any building is a complete CAT(0)-space.*

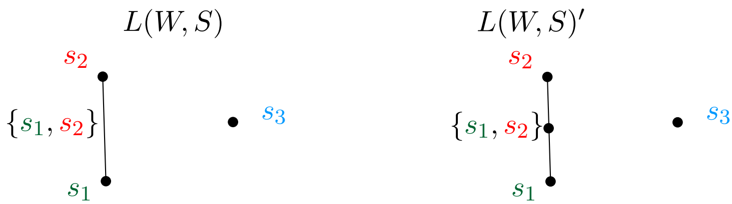
We now present an example of how to construct the Davis chamber and the Davis complex of a building.

Example 1.4.58. Consider the Coxeter system (W, S) given by the following presentation and whose Coxeter diagram is depicted in the figure in the right.

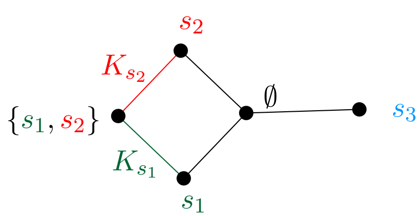
$$W = \langle S = \{s_1, s_2, s_2\} \mid \begin{array}{l} s_1^2 = s_2^2 = s_3^2 = 1 \\ (s_1 s_2)^3 = 1 \rangle. \end{array}$$



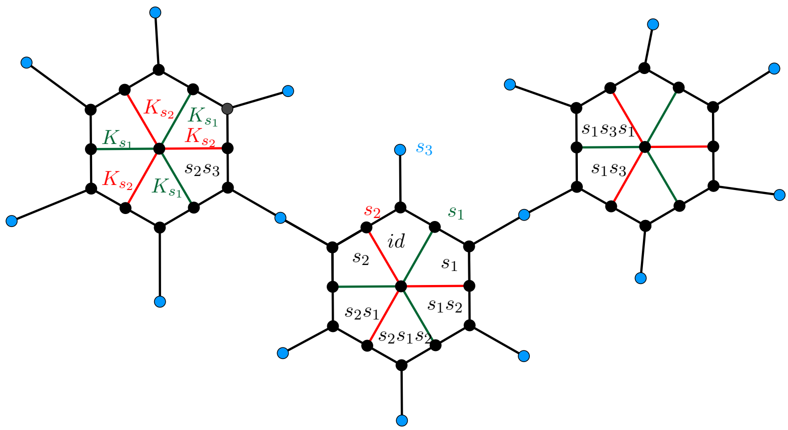
We will construct the Davis chamber and the Coxeter complex of this Coxeter system by following Definition 1.4.54. The nerve of (W, S) and its barycentric subdivision are depicted in the next figure.



The Davis chamber K of (W, S) is the cone of the barycentric subdivision $L(W, S)'$:



One can see the mirrors as in Definition 1.4.53, denoted by K_{s_1} and K_{s_2}. We now construct the Davis complex of (W, S) by using the equivalence relation of Definition 1.4.55 and we obtain the Davis complex for (W, S) as partially shown in the following figure.



Each chamber is a copy of the Davis chamber K, and we labeled some chambers so it is clear how one can construct the Coxeter chamber system (or the Cayley graph) of (W, S). Each hexagon corresponds to a residue of type $\{s_1, s_2\}$ and in the hexagon in the left we highlight the mirrors of type s_1 and s_2.

1.5 Universal groups for regular trees

In this section we focus on groups of automorphisms of locally finite trees, in particular, in a class of groups of automorphisms of regular trees.

In 2000, Burger and Mozes in [BM00a] defined the universal groups, which are defined by prescribing the local action around every vertex of the tree with a finite permutation group. These groups are examples of compactly generated totally disconnected and locally compact groups which are non-discrete under mild conditions on the local action.

Furthermore, they proved, given a permutation group $F \leq \mathrm{Sym}(q)$ with q being the degree of the regular tree, that the universal groups are the largest vertex-transitive closed subgroups of the automorphism group of the tree whose local action in the vertices is permutationally isomorphic to F. Therefore they form a large class of groups of automorphisms of regular trees.

Tits in [Tit70] established a criterion for simplicity of groups acting on trees, known as Tits's independence property (see Definition 1.5.3) and he showed that the group generated by pointwise edge-stabilizers in the automorphism group of a tree is either trivial or simple. Tits called this criterion Property P. By definition, the universal groups satisfy Tits's independence property, so they have, under some condition on the local action, an index-2 simple subgroup. Therefore studying universal groups for regular trees has become a source to find simple compactly generated locally compact totally disconnected groups.

The work of Burger and Mozes is the main motivation in this thesis to study right-angled buildings and to generalize the idea of prescribing a local action in the panels of a right-angled building as we will describe in Chapter 4. Moreover the concept of prescribing a local action in other regular objects (in particular CAT(0)-spaces with some regularity) will be used in Chapter 6 to define universal groups for polygonal complexes.

We will start this section with some considerations on groups acting on trees. Right after we will state Tits's independence property and then present the concepts necessary to define universal groups for regular trees. The section ends with references to several results that show how group theoretical conditions on the local action can be used to derive global topological properties of the universal group.

1.5.1 Groups acting on trees

Let T be a tree and let $\mathrm{Aut}(T)$ denote the full automorphism group of T.

Definition 1.5.1. Let $g \in \mathrm{Aut}(T)$. We call the automorphism g:

1. *elliptic* if it fixes a vertex $v \in VT$.

2. an *inversion* if it inverts an edge $e = \{v_1, v_2\}$, that is, if $gv_1 = v_2$ and $gv_2 = v_1$.

3. *hyperbolic* if does not fix any vertex of the tree and it does not invert any edge of the tree, i.e., it is neither elliptic nor an involution. In this case the vertices with minimal displacement by g form a bi-infinite line graph $A(g)$, called the *axis* of g.

The previous definition actually characterizes all types of automorphisms of a tree, as showed in the next result.

Theorem 1.5.2 ([Tit70, Proposition 3.2])**.** *Let $g \in \mathrm{Aut}(T)$. Then g is an inversion, an elliptic element or a hyperbolic element.*

1.5.2 Tits's independence property

Let T and $\mathrm{Aut}(T)$ as before. Let $\mathrm{Aut}(T)^+$ be the group generated by pointwise stabilizers of edges of T in $\mathrm{Aut}(T)$. Observe that we are considering both groups $\mathrm{Aut}(T)$ and $\mathrm{Aut}(T)^+$ equipped with the permutation topology, as defined in 1.3.8.

Definition 1.5.3. Let $G \leq \mathrm{Aut}(T)$ and let C be a (finite or infinite) chain of T, that is, a path in the tree. Let $F = \mathrm{Fix}_G(C)$ be the pointwise stabilizer of C in G.

For each $v \in VT$, we denote by $\pi(v)$ the vertex of C closest to v. The vertex-sets $\pi^{-1}(c)$, for $c \in VC$, are all invariant under F. We let $F|_{\pi^{-1}(c)}$ denote the permutation group obtained by restricting the action of F to $\pi^{-1}(c)$. Then there is a natural homomorphism

$$\varphi : F \to \prod_{c \in VC} F|_{\pi^{-1}(c)}.$$

The group G satisfies *Tits's independence property* if and only if φ is an isomorphism.

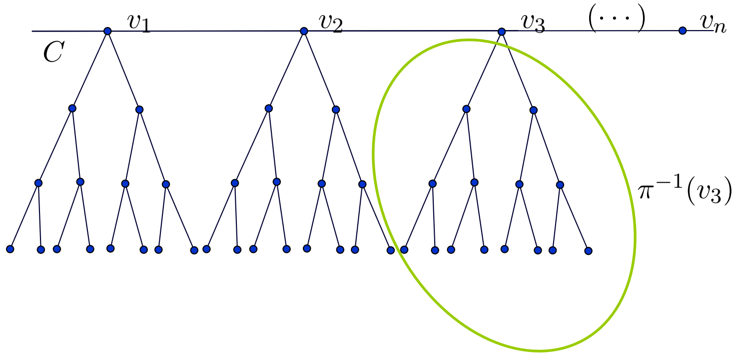


Figure 1.13: Tits's independence property.

The essence of Tits's independence property is that the fixator of a chain F acts on each branch of T at C independently of how it acts on the other branches, as illustrated in Figure 1.13.

Tits used this property to prove simplicity of groups of automorphisms of a tree.

Theorem 1.5.4 ([Tit70, Theorem 4.5]). *Let $G \leq \mathrm{Aut}(T)$ and let G^+ be the subgroup of G generated by pointwise edge-stabilizers. Assume that G does not stabilize a proper subtree and that G does not fix an end of T. If G satisfies Tits's independence property then G^+ is simple or trivial.*

In the case of closed subgroups of $\mathrm{Aut}(T)$, it is enough to look at edges instead on arbitrary chains, that is, it is only necessary to check if the pointwise stabilizer of an edge can be decomposed into 2 independent subgroups.

Proposition 1.5.5 ([Ama03, Lemma10]). *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. Then G satisfies Tits's independence property if and only if for every edge $e \in ET$, the pointwise stabilizer $\mathrm{Fix}_G(e)$ can be decomposed as*

$$\mathrm{Fix}_G(e) = \mathrm{Fix}_G(T_1)\,\mathrm{Fix}_G(T_2),$$

where T_1 and T_2 are the two rooted half-trees emanating from e, i.e., ET is the disjoint union of T_1, e and T_2.

1.5.3 The universal group $U(F)$

Now we will define the universal group for a regular tree with respect to a finite permutation group. We remark that Burger and Mozes (see [BM00a]) considered regular trees with oriented edges and in this dissertation we will consider the unoriented case. Fix $q \in \mathbb{N}$ and let $T = T_q$ be the q-regular tree.

Definition 1.5.6. A *legal coloring* of T is a map

$$h : ET \to \{1, \dots, q\},$$

such that the map $h_{\mathrm{St}(v)} : \mathrm{St}(v) \to \{1, \dots, q\}$ is a bijection, for every $v \in VT$. Recall that $\mathrm{St}(v)$ is the set of edges incident to v, as defined in Equation (1.2.1).

Lemma 1.5.7 ([LMZ94]). *For any two legal colorings h_1 and h_2 and two vertices v_1 and v_2 of T, there exists a unique automorphism $g \in \operatorname{Aut}(T)$ such that $v_2 = gv_1$ and $h_2 = h_1 \circ g$.*

The universal groups for regular trees are defined with respect to a finite permutation group F acting on the set of colors $\{1, \ldots, q\}$ and they consist of the automorphisms of the tree such that, for every vertex $v \in VT$, the permutation of the colors induced from $\operatorname{St}(v)$ to $\operatorname{St}(gv)$ is an element of F. We make this precise in the following definition.

Definition 1.5.8 ([BM00a, Section 3.2]). Let h be a legal coloring of T and let $F \leq \operatorname{Sym}(q)$. The *universal group $U(F)$* with respect to F is defined as

$$U^h(F) = \{\ g \in \operatorname{Aut}(T) \mid h|_{\operatorname{St}(gv)} \circ g \circ (h|_{\operatorname{St}(v)})^{-1} \in F \text{ for all } v \in VT\}.$$

The first thing that we observe is that we can actually drop the upper index h in $U^h(F)$ as a direct consequence of Lemma 1.5.7.

Corollary 1.5.9. *Let h_1 and h_2 be legal colorings of T and let $F \leq \operatorname{Sym}(q)$. Then the universal groups $U^{h_1}(F)$ and $U^{h_2}(F)$ are conjugate in $\operatorname{Aut}(T)$.*

From now on, we will fix a legal coloring h and for a group $F \leq \operatorname{Sym}(q)$ we will denote the respective universal group by $U(F)$.

Example 1.5.10.

1. If $F = \operatorname{Sym}(q)$ then the group $U(F)$ is the whole group $\operatorname{Aut}(T)$.
2. If $F = \operatorname{id}$ then $U(F)$ is isomorphic to the free product of q copies of C_2, where q is the degree of the regular tree. The cyclic group of order 2 comes from the automorphisms which invert edges. More details on this case can be found in [Ama03].

Next we show that the local action of these groups is actually permutationally isomorphic to the finite group F.

Definition 1.5.11. Let $G \leq \operatorname{Aut}(T)$ and fix $v \in VT$. Recall the notation for the stabilizer $G_v = \{g \in G \mid gv = v\}$. The *local action* of G on v is the permutation group formed by restricting the action of G_v to $\operatorname{St}(v)$.

Lemma 1.5.12. *The local action of $U(F)$ on v is permutationally isomorphic to F, for all $v \in VT$.*

We prove a similar lemma later in Chapter 4 (see Lemma 4.2.2) for universal groups acting on right-angled buildings. The proof can naturally be adapted for trees, since those are an instance of right-angled buildings.

We state now a set of properties of the universal groups that can be found on [BM00a]. For detailed proofs of those properties we refer to [GGT16].

Lemma 1.5.13. *Let $F \leq \mathrm{Sym}(q)$ and consider the universal group $U(F)$. Then the following hold:*

1. *$U(F)$ is a closed subgroup of $\mathrm{Aut}(T)$.*
2. *$U(F)$ is vertex-transitive.*
3. *$U(F)$ is compactly generated.*
4. *$U(F)$ is edge-transitive if and only if F is a transitive subgroup of $\mathrm{Sym}(q)$.*
5. *$U(F)$ is discrete in $\mathrm{Aut}(T)$ if and only if F is free in its action on $\{1, \dots, q\}$.*

The previous lemma shows the importance of the universal groups as topological groups since if F is not free then $U(F)$ is a non-discrete compactly generated totally disconnected locally compact group.

Next we present a lemma that justifies the name of these groups, i.e., we present a universality condition.

Proposition 1.5.14 ([BM00a, Proposition 3.2.2])**.** *Let $F \leq \mathrm{Sym}(q)$ be a transitive group. Let $H \leq \mathrm{Aut}(T)$ be a vertex-transitive group whose local action on every vertex of T is permutationally isomorphic to F.*

Then there is a legal coloring h of T such that $H \leq U^h(F)$.

Another very important characteristic of these groups is that they satisfy Tits's independence property defined in Section 1.5.2 and so they have rather large simple subgroups.

Proposition 1.5.15 ([BM00a, Proposition 3.2.1]). *Let $U(F)^+$ be the group generated by pointwise stabilizers of edges in $U(F)$.*

1. *The group $U(F)^+$ is simple or trivial.*
2. *The group $U(F)^+$ is of finite index in $U(F)$ if and only if F is transitive and generated by point stabilizers. In this case, $U(F)^+ = U(F) \cap \mathrm{Aut}(T)^+$ and it has index 2 in $U(F)$.*

Burger and Mozes state this result without proof. For a proof of this proposition we refer to [GGT16].

1.5.4 Vertex stabilizers in $U(F)$

Burger and Mozes in [BM00a, Section 3.2] also describe the structure of maximal compact open subgroups of the universal group when the local action is prescribed by a transitive permutation group.

Let $F \leq \mathrm{Sym}(q)$ be a transitive group and consider the universal group $U(F)$. Fix $v_0 \in VT$. We will describe the structure of the maximal compact open subgroup $U(F)_{v_0}$. Since it is a compact totally disconnected group, $U(F)_{v_0}$ is a profinite group ([Sha72, Theorem 2]). Therefore the description will be done through a projective limit of finite groups. Consider the sets

$$A = \{1, \dots, q\} \text{ and } B = \{2, \dots, q-1\}$$

and let F_1 denote the stabilizer of the element 1 in F. Recall that the choice of the element 1 is irrelevant since as F is transitive, all the point stabilizers are conjugate. Moreover, we consider the group F_1 as acting on the set B. Consider the sets

$$A_n = A \times B^{n-1}.$$

We now define bijections b_n between the n-spheres $\mathsf{S}(v_0, n)$ around v_0 and the sets A_n. We will use the legal coloring h in the definition of $U(F)$, as follows:

$$\begin{aligned} &b_1 : \mathsf{S}(v_0, 1) \to A \text{ is defined by } b_1(v) = h(\{v_0, v\}), \\ &b_{n+1} : \mathsf{S}(v_0, n+1) \to A_{n+1} \text{ is s.t. } \pi_n \circ b_{n+1} = b_n \circ p_{n+1}, \end{aligned} \tag{1.5.1}$$

where $\pi_n : A_{n+1} = A_n \times B \to A_n$ is the projection map on the first n coordinates and $p_n : \mathsf{S}(v_0, n) \to \mathsf{S}(v_0, n-1)$ maps $v \in \mathsf{S}(v_0, n)$ to the unique vertex in $\mathsf{S}(v_0, n-1)$ that is adjacent to v (recall that we are working with trees).

In this way, if $v \in \mathsf{S}(v_0, n)$ is identified with $(a_1, \dots, a_{n-1}, a_n) \in A_n$ through b_n then $(a_1, \dots, a_{n-1}) \in A_{n-1}$ corresponds to the unique vertex in $\mathsf{S}(v_0, n-1)$ that is connected to v. Further, for each element $a_{n+1} \in B$, the vertex $v' \in \mathsf{S}(v_0, n+1)$ mapped through b_{n+1} to $(a_1, \dots, a_{n-1}, a_n, a_{n+1})$ is connected to v.

We now define inductively $F(n)$ as follows:

$$\begin{aligned} F(1) &= F \leq \mathrm{Sym}(A_1), \\ F(n+1) &= F(n) \ltimes F_1^{A_n} \leq \mathrm{Sym}(A_{n+1}) \end{aligned} \tag{1.5.2}$$

where the wreath products $F(n)$ are considered with their imprimitive action on A_n as described in Section 1.1.1 (observe that $F(n+1) \cong F(n) \ltimes \mathrm{Fun}(A_n, F_1) = F(n) \wr F_1$).

The groups $F(n)$ will grasp the action of $U(F)_{v_0}$ on $\mathsf{S}(v_0, n)$ as we show in the next lemma.

Lemma 1.5.16 ([BM00a, Section 3.2])**.** *Let F be a transitive permutation group acting on $\{1, \dots, q\}$ and let $U(F)$ be the respective universal group. Fix a vertex v_0 of the q-regular tree.*

Then the stabilizer $U(F)_{v_0}$ is isomorphic (as a topological group) to the inverse limit of wreath products $\varprojlim_n F(n)$, where the groups $F(n)$ are defined in Equation (1.5.2).

In particular the induced action of $U(F)_{v_0}$ on the set of vertices at distance n from v is permutationally isomorphic to $F(n)$.

Proof. The bijection b_n as in Equation (1.5.1) induces a surjective homomorphism

$$\varphi_n : U(F)_{v_0} \to F(n) \text{ defined by } g \mapsto b_n \circ g \circ b_n^{-1},$$

with kernel

$$\ker \varphi_n = \{g \in U(F)_{v_0} \mid g|_{\mathsf{S}(v_0,n)} = \mathrm{id}\} = \{g \in U(F)_{v_0} \mid g|_{\mathsf{B}(v_0,n)} = \mathrm{id}\}.$$

Considering $\rho_n : F(n) \to F(n-1)$ the natural projection, we have hence that the map $\varphi = (\varphi_n)_{n \in \mathbb{N}} : U(F)_{v_0} \to \varprojlim_n F(n)$ is an isomorphism of topological groups, where

$$\varprojlim_n F(n) = \{(f_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F(n) \mid \rho_n f_n = f_{n-1} \text{ for all } n \in \mathbb{N}\}.$$

Then it means that, for all $n \in \mathbb{N}$, the induced action of $U(F)_{v_0}$ on $\mathsf{S}(v_0, n)$, namely $U(F)_{v_0}/(\ker \varphi_n)$, is isomorphic to $F(n)$. $\square$

1.5.5 Local to global structure of $U(F)$

A lot of progress has been made in investigating the universal groups through their local structure. It is very interesting that in several cases one can deduce topological properties in the universal group by assuming group theoretical conditions on the finite group that prescribes the local action.

In this section we present a medley of results that illustrate the local to global arguments in the study of universal groups for regular trees.

We start by considering the case when the finite permutation group F is 2-transitive, which was investigated in [BM00a].

Proposition 1.5.17 ([BM00a, Proposition 3.3.1]). *Let F be a 2-transitive permutation group on the set $\{1, \dots, q\}$ and such that F_1 is simple and non-abelian. Let $H \leq \mathrm{Aut}(T)$ be a closed vertex-transitive group whose local action on every vertex is permutationally isomorphic to F. Let $v \in VT$.*

Then $\mathrm{Fix}_H(\mathsf{S}(v,1))/\mathrm{Fix}_H(\mathsf{S}(v,2)) \cong F_1^a$, *where* $a \in \{0,1,q\}$. *Moreover,*

$$a \in \{0,1\} \Leftrightarrow H \text{ is discrete.}$$
$$a = q \Leftrightarrow H = U(F).$$

Caprace and De Medts in [CD11] considered the case where the prescribed local action is primitive. Recall that $U(F)^+$ denotes the subgroup of $U(F)$ generated by the pointwise stabilizers of edges. By Lemma 1.5.13 if F is transitive and generated by point stabilizers then $U(F)^+$ has index 2 in $U(F)$.

Proposition 1.5.18 ([CD11, Proposition 4.1]). *Assume that $F \leq$* $\mathrm{Sym}(q)$ *is transitive and generated by point stabilizers. Let $H = U(F)^+$. Then the following hold.*

1. *F is primitive if and only if every proper open subgroup of H is compact.*

2. *Suppose that F is imprimitive, with maximal blocks of imprimitivity with size k. Assume moreover that F acts regularly in each of such blocks. Let $\mathcal{N}_H(\mathrm{Fix}_H(e)) = \{h \in H \mid h(\mathrm{Fix}_H(e)) = (\mathrm{Fix}_H(e))h\}$ be the normalizer in H of $\mathrm{Fix}_H(e)$.*

   Then for every $e \in ET$, the quotient $\mathcal{N}_H(\mathrm{Fix}_H(e))/\,\mathrm{Fix}_H(e)$ is virtually free, that is, it has a finite-index subgroup which is free.

   Furthermore, if $k \geq 3$ then H has open subgroups which are not compact.

The next result, which is stated in [Ama03, Proposition 59] and whose simpler proof is recently presented in [BM17], concerns topological properties of a vertex-stabilizer in the universal group assuming some conditions in the point stabilizers of the finite group F. We recall first the necessary definitions to state the proposition.

Definition 1.5.19. Let G be a group. The *commutator subgroup* of G, normally denoted by G', is the group generated by the commutators of its elements, i.e., the subgroup generated by $[g, h] = ghg^{-1}h^{-1}$, for every $g, h \in G$.

The group G is called *perfect* if $G = G'$.

Definition 1.5.20. A topological group G is called *topologically finitely generated* if it has a dense finitely generated subgroup, that is, if G has a finitely generated subgroup H whose closure is the whole G.

Proposition 1.5.21 ([Ama03, Proposition 59])**.** *Let $F \leq \mathrm{Sym}(q)$ be a transitive permutation group on $\{1, \dots, q\}$ and assume that the stabilizer F_1 is non-trivial. Let $v \in VT$.*

The $U(F)_v$ is topologically finitely generated if and only if F_1 is perfect and equal to its normalizer.

CHAPTER 2

Right-angled buildings

This chapter is devoted to the study of right-angled buildings, which are the main geometric objects of this thesis.

We start by presenting the definition of a right-angled Coxeter group and then we show how the relations between the generators behave nicely, allowing us to have some control over the reduced representations of group elements. With that at hand, we will define a partial order on the letters of reduced representations of group elements and we prove some properties of the partial order using the relations of the associated Coxeter group. This tool will be useful later on in Chapter 5 to describe compact open subgroups of the automorphism group of a right-angled building through generalized wreath products.

We also define the concept of firm words in right-angled Coxeter groups (see Definition 2.1.12) and connect them with the partial order previously defined. This concept will be used often in Chapter 3.

In Section 2.2 we arrive to right-angled buildings. After the definition and first examples, we will present properties related with the distance between residues in the building. Then we will focus on the concept of a tree-wall (see Definition 2.2.22) and we will define the tree-wall tree (Definition 2.2.37). This tree will provide us with a distance between tree-walls of the same type which will be used very often for inductive arguments.

We finish this chapter by defining distinct types of colorings in the chambers of semi-regular right-angled buildings. These colorings will be useful not only to define the universal group for a right-angled building in the next chapter, but also to describe a right-angled building in a standard (and "directed") way in Section 2.4. This directed description will be done through a standard parametrization of the chambers of the building.

2.1 Right-angled Coxeter groups

We now present the definition of right-angled Coxeter groups and henceforth we will assume that we are always in the right-angled case, unless otherwise stated.

Definition 2.1.1. A Coxeter group W is called *right-angled* if the entries of the Coxeter matrix are 1, 2 and ∞. In other words, if the group W can be presented as

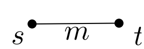
$$W = \langle \{s_i\}_{i \in I} \mid (s_i s_j)^{m_{ij}} \rangle, \quad \begin{aligned} &\text{with } m_{ij} \in \{2, \infty\} \text{ for all } i \neq j \\ &\text{and } m_{ii} = 1 \text{ for all } i \in I. \end{aligned}$$

In this case, we call the Coxeter diagram Σ of W a *right-angled* Coxeter diagram.

Since one grasps better a definition with examples, we present some right-angled Coxeter groups.

Example 2.1.2. The rank 2 right-angled Coxeter groups are

1. $W = \langle s, t \mid s^2 = t^2 = (st)^2 = 1 \rangle$, which is finite, and
2. $W = \langle s, t \mid s^2 = t^2 = 1 \rangle$, which is an infinite Coxeter group.

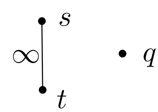


In both cases these groups will give rise, respectively, to generalized di-gons and trees (*cf.* Example 1.4.34) and the Coxeter diagram associated to these groups is as showed in the figure, with $m = 2$ and $m = \infty$, respectively.

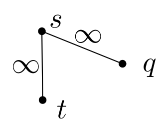
Normally, if $m = 2$ then we present the Coxeter diagram as a disconnected graph with two vertices. If $m = \infty$ then the right-angled buildings associated to W will be trees without end-points.

Example 2.1.3. Let us present now two examples of rank 3 right-angled Coxeter groups which are generated by 3 elements s, t and q, with respective Coxeter diagrams depicted on the right of the group presentation.

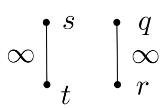
$$W_1 = \langle s, t, q \mid s^2 = t^2 = q^2 = (sq)^2 = (tq)^2 = 1 \rangle$$



$$W_2 = \langle s, t, q \mid s^2 = t^2 = q^2 = (tq)^2 = 1 \rangle$$



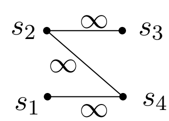
Example 2.1.4. The next example of a right-angled Coxeter group will give rise to buildings which are infinite direct product of trees.



$$W_3 = \langle s, t, q, r \mid \begin{array}{l} s^2 = t^2 = q^2 = r^2 = 1 \\ (sq)^2 = (sr)^2 = (tq)^2 = (tr)^2 = 1 \rangle \end{array}$$

As last example we consider the group

$$W_4 = \langle s_1, s_2, s_3, s_4 \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1 \\ (s_1 s_2)^2 = (s_1 s_3)^2 = (s_3 s_4)^2 = 1 \rangle$$



2.1.1 A poset of reduced words

Next we move towards the definition of a partial order on the letters of a reduced representation of an element of the Coxeter group. We start with some considerations about elementary operations and reduced words in right-angled Coxeter groups.

The elementary operations from Definition 1.4.11 become easier to describe in the right-angled case since the generators either commute or are not related (the cases $m = 2$ or $m = \infty$, respectively).

Definition 2.1.5. Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ and set of generators $S = \{s_i \mid i \in I\}$. We define a Σ-*elementary operation* as an operation of the following two types:

(1) Delete a subword of the form ss, with $s \in S$.

(2) Replace a subword st by ts if $m_{ts} = 2$.

A word in the free monoid M_S is then *reduced* if it cannot be shortened by a sequence of Σ-elementary operations.

Moreover, by Lemma 1.4.14, two reduced words represent the same element of W if and only if one can be obtained from the other by a sequence of elementary operations of type (2).

In particular, we observe that if w_1 is a reduced word with respect to Σ and w_2 is a word obtained from w_1 by applying one Σ-elementary operation of type (2), then w_2 is also a reduced word and w_1 and w_2 represent the same element of W. Furthermore, the words w_1 and w_2 only differ in two consecutive letters that have been switched, let us say,

$$w_1 = s_1 \cdots s_i s_{i+1} \cdots s_\ell \text{ and } w_2 = s_1 \cdots s_{i+1} s_i \cdots s_\ell, \\ \text{with } |s_i s_{i+1}| = 2 \text{ in } \Sigma. \quad (2.1.1)$$

If $\sigma \in \mathrm{Sym}(\ell)$ then we denote by $\sigma.w_1$ the word obtained by permuting the letters in w_1 according to the permutation σ, that is,

$$\sigma.w_1 = s_{\sigma(1)} \cdots s_{\sigma(r)} s_{\sigma(r+1)} \cdots s_{\sigma(\ell)}.$$

Hence, if w_1 and w_2 are as in Equation (2.1.1) and $\sigma = (i\ i+1) \in \mathrm{Sym}(\ell)$, then $\sigma.w_1 = w_2$.

Definition 2.1.6. Let $w = s_1 \cdots s_\ell$ be a reduced word in M_S with respect to a right-angled Coxeter diagram Σ. Let $\sigma = (i\ i+1)$ be a transposition in $\mathrm{Sym}(\ell)$, with $i \in \{1, \ldots, \ell - 1\}$. We call σ a w-*elementary transposition* if s_i and s_{i+1} commute in W.

In this way, we can associate an elementary transposition to each Σ-elementary operation of type (2). Using Lemma 1.4.14, we obtain that two reduced words w_1 and w_2 represent the same element of W if and only if

$$w_2 = (\sigma_n \cdots \sigma_1).w_1, \text{ where each } \sigma_i \text{ is a } (\sigma_{i-1} \cdots \sigma_1).w_1\text{-elementary transposition,}$$

i.e., if w_2 is obtained from w_1 by a sequence of elementary transpositions.

Definition 2.1.7. If w is a reduced word of length ℓ with respect to Σ, then we define

$$\mathrm{Rep}(w) = \{\sigma \in \mathrm{Sym}(\ell) \mid \sigma = \sigma_n \cdots \sigma_1, \text{ where each } \sigma_i \text{ is a } (\sigma_{i-1} \cdots \sigma_1).w\text{-elementary transposition}\}.$$

The set $\mathrm{Rep}(w)$ is formed by the permutations of ℓ letters which give rise to reduced representations of w, according to the relations in the right-angled Coxeter diagram Σ.

Observation 2.1.8. Let $w = s_1 \cdots s_i \cdots s_j \cdots s_\ell$ be a reduced word in M_S with respect to a right-angled Coxeter diagram Σ.

1. Let $\sigma \in \mathrm{Rep}(w)$ such that σ can be written as a product of elementary transpositions $\sigma_n \cdots \sigma_1$. Then for each $k \in \{1, \ldots, n\}$, the word $(\sigma_k \cdots \sigma_1).w$ is also a reduced representation of w.

2. Assume that σ_1 is a w-elementary transposition switching two generators s_i and s_j and σ_2 is a $\sigma_1.w$-elementary transposition switching two generators $s_{i'}$ and $s_{j'}$ such that $\{i, j\} \cap \{i', j'\} = \emptyset$. Then σ_2 is also a w-elementary transposition.

Now we define a partial order $\prec_w$ on the letters of a reduced word w in M_S with respect to Σ.

Definition 2.1.9. Let $w = s_1 \cdots s_\ell$ be a reduced word of length ℓ in M_S with respect to Σ. Let $I_w = \{1, \ldots, \ell\}$. We define a new partial order "$\prec_w$" on I_w as follows:

$$i \prec_w j \iff \sigma(i) > \sigma(j) \text{ for all } \sigma \in \mathrm{Rep}(w).$$

Note that $i \prec_w j$ implies that $i > j$. As a mnemonic, one can regard $i \prec_w j$ as "$i \leftarrow j$", that is, the generator s_i comes always after the generator s_j regardless of the reduced representation of w.

We remark a couple of basic but enlightening consequences of the definition of this partial order.

Observation 2.1.10. Let $w = s_1 \cdots s_i \cdots s_j \cdots s_\ell$ be a reduced word in M_S with respect to a right-angled Coxeter diagram Σ.

1. If $|s_i s_j| = \infty$ in Σ, then $j \prec_w i$.

2. If $j \not\prec_w i$ then by (1), it follows that $|s_i s_j| = 2$ and, moreover, for each $k \in \{i+1, \ldots, j-1\}$, either $|s_i s_k| = 2$ or $|s_j s_k| = 2$ (or both).

3. If $|s_i s_j| = \infty$ then $j \prec_w i$ but the converse is not true.

   Suppose there is $i < k < j$ such that $|s_i s_k| = \infty$ and $|s_k s_j| = \infty$. Then $j \prec_w i$, independently of whether $|s_i s_j| = 2$ or not.

4. However, let s_j and s_{j+1} be consecutive letters in w. Then $|s_j s_{j+1}| = 2$ if and only if $j+1 \not\prec_w j$.

The next lemma describes some conditions on the existence and structure of distinct reduced representations of elements in right-angled Coxeter groups.

Lemma 2.1.11. *Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ. Let $w = w_1 s_i \cdots s_j w_2$ be a reduced word in M_S with respect to Σ. If $j \not\prec_w i$ then there exist two reduced representations of w of the form*

$$w_1 \cdots s_i s_j \cdots w_2 \quad \text{and} \quad w_1 \cdots s_j s_i \cdots w_2,$$

i.e., one can exchange the positions of s_i and s_j using only elementary operations on the generators in the set $\{s_i, s_{i+1}, \ldots, s_{j-1}, s_j\}$, without changing the prefix w_1 and the suffix w_2, and still obtain the same element of W.

Proof. We will prove the result by induction on the number N of letters between s_i and s_j. If $N = 0$ then $w = w_1 s_i s_j w_2$, and the result follows from Observation 2.1.10(2).

Assume by induction hypothesis that if $n \leq N$ then the result holds. Consider $w = w_1 s_i \cdots s_j w_2$ with $N+1$ letters between s_i and s_j in w, and let $\sigma \in \mathrm{Rep}(w)$ such that $\sigma(j) < \sigma(i)$.

If $|s_i s_{i+1}| = 2$ then $(i\ i+1) \in \mathrm{Rep}(w)$ and

$$\widetilde{w} = (i\ i+1).w = w_1 s_{i+1} s_i \cdots s_j w_2$$

is a reduced representation of w with N letters between s_i and s_j. Moreover the permutation $(i\ i+1)\sigma \in \mathrm{Rep}(\widetilde{w})$ satisfies the conditions of the lemma. Thus the result follows from the induction hypothesis.

Assume now that $|s_i s_{i+1}| = \infty$. By Observation 2.1.10(2) we have $|s_{i+1} s_j| = 2$. Furthermore from Observation 2.1.10(1) we obtain that $\sigma(i) < \sigma(i+1)$. Hence by assumption $\sigma(j) < \sigma(i+1)$ and we can apply the induction hypothesis to the generators s_{i+1} and s_j since the number of letters between them is less than or equal to N. Thus we obtain that

$$w' = w_1 s_i \cdots s_{i+1} s_j \cdots w_2 \quad \text{and} \quad w^* = w_1 s_i \cdots s_j s_{i+1} \cdots w_2$$

are reduced representations of w.

Let $\tau \in \mathrm{Rep}(w)$ such that $\tau.w = w^*$. The number of letters between s_i and s_j in w^* is less than or equal to n. Therefore we can apply the induction hypothesis to w^* with $\sigma\tau^{-1} \in \mathrm{Rep}(w^*)$ and we obtain that

$$w_1 \cdots s_i s_j \cdots w_2 \quad \text{and} \quad w_1 \cdots s_j s_i \cdots w_2$$

are two reduced representations of w^* and hence of w. □

2.1.2 Firm words in right-angled Coxeter groups

In this section we define firm reduced words in Coxeter groups (see Definition 2.1.12). The characterization of being firm will be done by means of the combinatorics of the right-angled Coxeter group and through the poset of reduced words described in Definition 2.1.9. This concept will be of relevance since, for a fixed chamber v, it will allow us to prove that the fixator of any ball in a right-angled building around v acts on the building with a bounded fixed-point set (see Proposition 3.2.6).

Definition 2.1.12. Let w be a reduced word in M_S with respect to Σ.

1. We say that w is *firm* if $w = s_1 \cdots s_k$ is such that for all $i \in \{1, \ldots, k-1\}$, we have
$$s_1 \cdots s_i \cdots s_k \not\sim s_1 \cdots s_k s_i.$$
2. Let $F^{\#}(w)$ be the largest k such that w can be transformed by elementary operations into a word in the form
$$s_1 \cdots s_k t_{k+1} \cdots t_\ell, \text{ with } s_1 \cdots s_k \text{ firm.}$$
Moreover, let $F(w)$ be the set of such elements s_k.
Observe, using the notation above, that if $s'_k \in F(w)\backslash\{s_k\}$ then s'_k is an element of the set $\{t_{k+1}, \ldots, t_\ell\}$.

Observation 2.1.13. Let $w = s_1 \cdots s_k t_{k+1} \cdots t_\ell$ be a reduced word such that $s_1 \cdots s_k$ is firm and $F^{\#}(w) = k$. Then the following hold.

1. $|s_k t_i| = 2$ in Σ, for all $i \in \{k+1, \ldots, \ell\}$.
Indeed, take j minimal such that $|s_k t_j| = \infty$. Using elementary operations to swap t_j to the left in w as much as possible, we obtain that
$$w \sim s_1 \cdots s_k t'_1 \cdots t'_p t_j \cdots$$
is a word with $s_1 \cdots s_k t'_1 \cdots t'_p t_j$ firm, which is a contradiction to the maximality of k.
2. Let $r \in S$. If $l(wr) > l(w)$ then $F^{\#}(wr) \geq F^{\#}(w)$. In particular, if $F^{\#}(w) = F^{\#}(wr)$ then we have that $F(w) \subseteq F(wr)$.

We now connect the definition of firm reduced words with the partial order that we have on such words. This will be a useful tool to identify which letters of the word appear in a firm subword.

Definition 2.1.14. Let $w = s_1 \cdots s_n$ be a reduced word in M_S and consider the poset $(I_w, \prec_w)$ as in Definition 2.1.9. For any $i \in \{1, \ldots, n\}$, we define
$$I_w(i) = \{ \ j \in \{1, \ldots, n\}\backslash\{i\} \mid i \prec_w j \ \}.$$
In words, $I_w(i)$ is the set of indices j such that s_j comes at the left of s_i in any reduced representation of the element $w \in W$.

We combine Definitions 2.1.12 and 2.1.14 in the following observation, in order to help to grasp the meaning of these concepts, which are defined making use of technical notation.

Observation 2.1.15. Let $w = s_1 \cdots s_n$ be a reduced word in M_S with respect to Σ.

1. As $\prec_w$ is a partial order, for each $i \in \{1, \dots, n\}$, we can perform elementary operations on w so that

   $$w \sim s_{j_1} \cdots s_{j_{k-1}} s_i t_1 \cdots t_\ell, \text{ with } j_p \in I_w(i) \text{ and } j_p < j_{p+1}.$$

   Then the word $s_{j_1} \cdots s_{j_{k-1}} s_i$ is firm.

   In particular, if $I_w(i) = \emptyset$ then we can rewrite w as $s_i w_1$.

2. If $|s_i s_{i+1}| = 2$ for some i then $i \notin I_w(i+1)$. This means that if we can rewrite w as $s_{j_1} \cdots s_{j_{k-1}} s_{i+1} w_1$ with $s_{j_1} \cdots s_{j_{k-1}} s_{i+1}$ firm then $i \notin \{j_1, \dots, j_{k-1}\}$.

3. Consider $r \in S$ such that $l(w) < l(wr)$. Then, for any $i \in \{1, \dots, n\}$, we have $I_w(i) = I_{wr}(i)$.

4. Let $r \in S$ such that $l(w) < l(wr)$. If $|s_i r| = \infty$ for some $i \in \{1, \dots, n\}$ then $I_w(i) = I_{wr}(i) \subseteq I_{wr}(n+1)$ (the letter r corresponds to the index $n+1$).

   In general, if $j \prec_w i$ then $I_w(i) \subsetneq I_w(j)$.

5. Consider $F^{\#}(w)$ as in Definition 2.1.12.

   Then $F^{\#}(w) = \max_{i \in \{1,\dots,n\}} |I_w(i)| + 1$.

   Therefore $F^{\#}(w) = |I_w(i)| + 1$ if and only if $s_i \in F(w)$.

   Hence $F(w) = \{s_i \mid |I_w(i)| + 1 = F^{\#}(w)\}$.

6. If $s_k \in F(w)$, then we can apply elementary operations on w to rewrite it as $w_1 s_k$. This is the same conclusion as in Observation 2.1.13(1) using the poset $(I_w, \prec_w)$.

Remark 2.1.16. If the Coxeter system (W, S) is spherical then we have that $F^{\#}(w) = 1$ for all reduced words w in M_S with respect to Σ. Indeed, if W is finite then, as each pair of distinct generators commute, we have that $I_w(i) = \emptyset$ for any reduced word and any letter s_i on it.

Definition 2.1.17. Let $w = s_1 \cdots s_n$ be a reduced word in M_S with respect to Σ. We define $R(w) \subseteq S$ to be the set of elements $s \in S$ such that

$$|rs_i| \leq 2 \text{ for all } s_i \in F(w).$$

We remark also that by definition $F(w) \subseteq R(w)$.

Observe that if $l(wr) > l(w)$ for some $r \in R(w)$ with $F^{\#}(wr) = F^{\#}(w)$, then $r \in R(wr)$.

Remark 2.1.18. Let $r \in S$ with $l(w) < l(wr)$. We observe that the condition $r \in R(w)$ is not sufficient for the equality $F(w) = F(wr)$. Suppose that

$$w \sim s_1 \cdots s_{k-1} s_k t_{k+1} \cdots t_\ell \text{ with } s_1 \cdots s_k \text{ firm and } F^{\#}(w) = k.$$

If $|rt_i| = 2$ for all $i \in \{k+1, \ldots, \ell\}$ and $|rs_k| = \infty$ then $r \in F(wr)$ but $r \notin F(w)$.

However, if we assume that $F^{\#}(wr) = F^{\#}(w)$ then we can conclude further conditions regarding the connection between the sets $F(w), R(w), F(wr)$ and $R(wr)$.

Lemma 2.1.19. *Let w be a reduced word in M_S with respect to Σ. Let $r \in S$ be such that $l(w) < l(wr)$. Assume that $F^{\#}(wr) = F^{\#}(w)$. Then the following hold:*

1. *$r \in R(w)$;*
2. *If $r \notin F(wr)$ then $R(wr) = R(w)$;*
3. *If $r \in F(wr)$ then $R(wr) = R(w) \setminus \{r' \in R(w) \mid |rr'| = \infty\}$;*
4. *In particular, $R(wr) \subseteq R(w)$ and if $r' \in R(w)$ with $|rr'| \leq 2$ then $r' \in R(wr)$.*

Proof. Let us prove Statement 1. Assume that $r \notin R(w)$. Then, using elementary transformations, we can write w as

$$s_1 \cdots s_k t_{k+1} \cdots t_\ell r,$$

with $s_1 \cdots s_k$ firm and such that $|s_k r| = \infty$. Swapping r as much as possible to the left in wr we obtain a rewriting of this word as

$$s_1 \cdots s_k t'_{k+1} \cdots r \cdots t'_\ell$$

and the word $s_1 \cdots s_k t'_{k+1} \cdots r$ is firm. Hence $F^{\#}(wr) > k$.

We observe now that if $F^{\#}(w) = F^{\#}(wr)$ then we have

$$F(w) \subseteq F(wr) \subseteq F(w) \cup \{r\}.$$

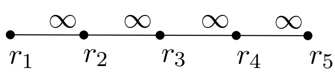
If $r \notin F(wr)$ then $F(w) = F(wr)$ and therefore $R(w) = R(wr)$. If $r \in F(wr)$ then $F(wr) = F(w) \cup \{r\}$ so $R(wr)$ is constructed from $R(w)$ by removing the elements that don't commute with r. Hence Statements 2 and 3 are proved.

Statement 4 follows from Statements 2 and 3. $\square$

Remark 2.1.20. The converse of Lemma 2.1.19(1) is not true. Consider, as a counter-example, the right-angled Coxeter group

$$W = \langle r_1, \ldots, r_5 \mid (r_1r_3)^2 = (r_1r_4)^2 = (r_1r_5)^2 = (r_2r_4)^2 = 1 \\ (r_2r_5)^2 = (r_3r_5)^2 = 1 \rangle,$$

with Coxeter diagram depicted in the following figure.



Consider the word $w = r_2r_1r_4r_5$. We have $F^{\#}(w) = 2$ and $F(w) = \{r_1, r_5\}$.

Then $l(w) < l(wr_3)$ and $r_3 \in R(w)$. However $wr_3 \sim r_2r_4r_3r_1r_5$ and $r_2r_4r_3$ is firm. Hence $r_3 \in R(w)$ but $F^{\#}(w) < F^{\#}(wr_3)$.

In the next definition we set up notation that will be used very often henceforth.

Definition 2.1.21. A sequence of letters $r_1, r_2, \ldots \in S$ such that $l(r_1 \cdots r_i) < l(r_1 \cdots r_i r_{i+1})$ for all i will be called a *reduced increasing sequence* in S.

Lemma 2.1.22. *Let $\alpha = r_1, r_2, \ldots$ be a reduced increasing sequence in S. Assume that each subsequence of α of the form*

$$(r_{a_1}, r_{a_2}, \ldots) \text{ with } |r_{a_i} r_{a_{i+1}}| = \infty \text{ for all } i$$

has $k \leq b$ elements. Then there is $f(b)$ depending only on b (and on the Coxeter system (W, S)), such that α has $n \leq f(b)$ elements.

Proof. We will prove this result by induction on $|S|$. If $|S| = 1$ then it is obvious. If $|S| = 2$ then either (W, S) is spherical and each such sequence has length at most 2 or (W, S) is an infinite dihedral group generated by two elements r and t. In the latter case, the reduced increasing sequences in S are of the form $r_{a_i} = r$ if a_i is even and $r_{a_i} = t$ if a_i is odd (or vice versa). Hence the result also follows because $f(b) = b$ in this case.

Suppose now that $|S| > 3$. If (W, S) is a spherical Coxeter group then it is obvious since there are no infinite such sequences. Assume then that there is $s \in S$ such that s doesn't commute with some other generator in $S \backslash \{s\}$.

Observe that since we are considering an increasing sequence α of reduced words, in between any two s's there is t_i such that $|st_i| = \infty$. Consider the subsequence of α given by

$$(s, t_1, s, t_2, \ldots).$$

This subsequence has $\leq b$ elements by assumption and between any two generators s in this subsequence we only use letters in $S \backslash \{s\}$. Therefore the result follows by induction hypothesis. □

Lemma 2.1.23. *Let w be a reduced word in M_S with respect to Σ. There is $f(w) \in \mathbb{N}$, depending only on w, such that, for every reduced increasing sequence $r_1, r_2, \ldots$ in S, we have*

$$F^{\#}(wr_1 \cdots r_{f(w)}) > F^{\#}(w).$$

Proof. Assume that there is a reduced increasing sequence $\alpha = r_1, r_2, \ldots$ in S such that:

$$(*)\ F^{\#}(wr_1 \cdots r_i) = F^{\#}(w) \text{ for all } i.$$

Define $w_0 = w$, $w_i = w_{i-1}r_i$ and denote $R_i = R(w_i)$ and $I_i = I_{w_i}(i)$.

Let $b = F^{\#}(w) - 1$. By assumption (*), for each i, $|I_i| \leq b$. Moreover, by Lemma 2.1.19, we have that

a) each $r_i \in R_{i-1}$;

b) $R_0 \supseteq R_1 \supseteq \cdots$;

c) $R_{i-1} \subsetneq R_i$ if $|I_i| = b$;

d) if $i < j$ with $|r_i r_j| = \infty$ then $I_i \subsetneq I_j$.

If there is i such that R_i is a spherical set (see Definition 1.4.8) then all the elements r_k of α with $k > i$ are in a spherical subset. Therefore the sequence α is finite since it is a reduced increasing sequence.

Assume that for all i, the set R_i is non-spherical. If there is a sequence $(r_{a_1}, r_{a_2}, \ldots)$ of elements r_i, with $a_i < a_{i+1}$ and such that $|r_{a_i} r_{a_{i+1}}| = \infty$ then, as $I_{a_i} \subsetneq I_{a_{i+1}}$ and $0 \leq |I_j| \leq b$ for all j, such a sequence has $k \leq b+1$ elements. Hence by Lemma 2.1.22 that sequence α must have $n \leq f(b)$ elements, with $f(b)$ only depending on b.

Moreover, the first pair of non-commuting elements r_{a_1}, r_{a_2} is found after at most $|R_0 = R(w)|$ indices. Then

1. either we increase this subsequence by finding another element r_{a_3} with $|r_{a_3} r_{a_2}| = \infty$, after at most $|R(w)| - 1$ indices,

2. or there is no such element r_{a_3}. In that case $|r_k r_{a_2}| = 2$ for all $k > a_2$. Consider then set $P = \{s \in S \mid |s r_{a_2}| = 2\}$ and let $P_{a_2} = R_{a_2} \cap P$. Then $r_i \in P_{a_2}$ for all $i > a_2$. Observe that $P_{a_2} \subsetneq R_0$ as $r_{a_1} \in R_0$ but $r_{a_1} \notin P_{a_2}$. If P_{a_2} is a spherical set then we are done using the same reasoning for when R_i is spherical.

   If P_{a_2} is a non-spherical set then we look for a new pair of non-commuting elements r_{b_1}, r_{b_2}, which we find after at most $|P_{a_2}| < |R(w)|$ indices, and we try to increase this new subsequence.

Therefore

$$f(b) \leq \underbrace{(|R(w)| + (|R(w)| - 1) + \cdots + (|R(w)| - i))}_{1)} + \underbrace{(|R(w)| - (i+1)) \times b}_{2)},$$

where $i \in \{0, \ldots, |R(w)| - 2\}$ denotes the number of times that we have to decrease the set $R(w)$ until we find a non-spherical set containing the reduced increasing sequence $(r_{a_1}, r_{a_2}, \ldots)$ we are looking for.

The first pair r_{a_1}, r_{a_2} is found after at most $|R(w)|-i$ indices and each of the (at most b) next elements is found after $|R(w)|-(i+1)$ indices after the previous one, as described in part 2) of the equation. The sum in part 1) has a maximum of $|R(w)|-1$ factors, all bounded by $|R(w)|$. Moreover $(|R(w)|-(i+1)) \leq |R(w)|-1$. Hence it follows that

$$f(b) \leq (|R(w)|-1) \times (|R(w)|+b).$$

Therefore each reduced increasing sequence of α satisfying Condition (*) is finite of length at most $f(b)$. Thus, for any sequence α of length $f(w) = f(b)+1$ in the conditions of the Lemma, we have that $F^{\#}(wr_1 \cdots r_{f(w)}) > F^{\#}(w)$. □

Lemma 2.1.24. *Assume that the Coxeter system (W, S) is non-spherical. Then for all $n \geq 1$ there is $d(n)$ depending only on n, such that $F^{\#}(w) > n$ for all reduced words w in M_S with $l(w) > d(n)$.*

Proof. We prove the lemma by induction on n. If $n = 1$ then as (W, S) is non-spherical, there exists a pair of non-commuting elements. Therefore, if $w = s_1 \cdots s_{|S|}$ is a reduced word of length $|S|$ then there are $i < j$ such that $|s_i s_j| = \infty$. Therefore we have that

$$0 \leq I_w(i) < I_w(j) \leq F^{\#}(w) - 1,$$

which implies that $F^{\#}(w) \geq 2 > 1$. Thus if $n = 1$ then $d(n) = |S|$.

Assume by induction hypothesis that if $n \leq N$ then there exists a $d(n)$ satisfying the conditions of the lemma. Then all the reduced words w of length $d(N)$ have $F^{\#}(w) > N$. If w is one of those words, then by Lemma 2.1.23 there is a constant $f(w)$ such that $F^{\#}(w) < F^{\#}(wr_1 \cdots r_{f(w)})$, for all reduced increasing sequence $r_1, r_2, \ldots$ of elements.

The set of reduced words of length $d(N)$ is finite because W is finitely generated. Therefore we can consider

$$f(N) = \max_{w \in W(d(N))} f(w).$$

Thus for every reduced word w of length $d(N) + f(N)$ it follows that

$$F^{\#}(w) > F^{\#}(w_1) > N,$$

for some $w_1 \in W(d(N))$. Hence $F^{\#}(w) > N + 1$ for each $w \in W(d(N) + f(N))$. Therefore $d(N+1) = d(N) + f(N)$ exists and depends only on N. □

2.2 Right-angled buildings

In this section we will present the definition and first examples of right-angled buildings and then we deduce properties of their residues, walls and wings. The section proceeds with the definition of a tree-wall tree and with some remarks on the particular case of semi-regular right-angled buildings.

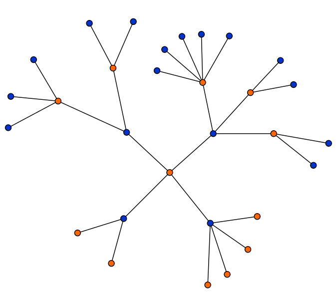
From now on, we will always assume the following notation. Let (W, S) be a right-angled Coxeter system with set of generators $S = \{s_i\}_{i \in I}$ and with Coxeter diagram Σ.

Definition 2.2.1. A *right-angled building* Δ is a building of type Σ, where Σ is a right-angled Coxeter diagram.

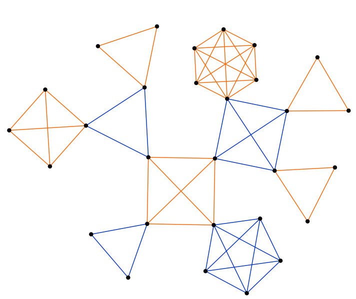
We present right away a couple of examples of right-angled buildings using the examples of Coxeter groups in previous section as the groups prescribing the type.

Example 2.2.2. As mentioned before, a tree without end points is a building. Since the Coxeter group associated to it is the infinite dihedral group (*cf*. Example 2.1.2) is a right-angled Coxeter group, a tree is a right-angled building.

We observe that if we want to regard a tree as a chamber system, the graphic representation is slightly different, since the pictures we normally see from trees correspond to the geometric realization of a tree. In the geometric realization, the chambers are the edges of the tree and the panels correspond to stars of vertices. To visualize a tree as a chamber system, one has to draw the line graph of the geometric realization, as show in the following pictures.



(a) Geometric realization of a tree



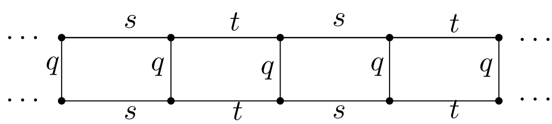
(b) A tree as a chamber system

Example 2.2.3. In this example we illustrate buildings that arise from the right-angled Coxeter groups in Example 2.1.3. We will present partial pictures of an apartment in the building (i.e., a thin right-angled building) of those types by impossibility of drawing the thick cases.

1. A building of type

$$W_1 = \langle s, t, q \mid s^2 = t^2 = q^2 = (sq)^2 = (tq)^2 = 1 \rangle$$

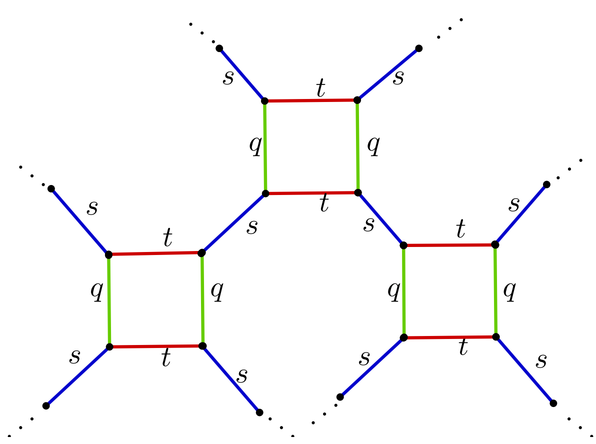
will give rise to a thin building which is the direct product of 2 trees. A partial representation of the Coxeter chamber system associated to such a right-angled Coxeter group is



2. The group

$$W_2 = \langle s, t, q \mid s^2 = t^2 = q^2 = (tq)^2 = 1 \rangle,$$

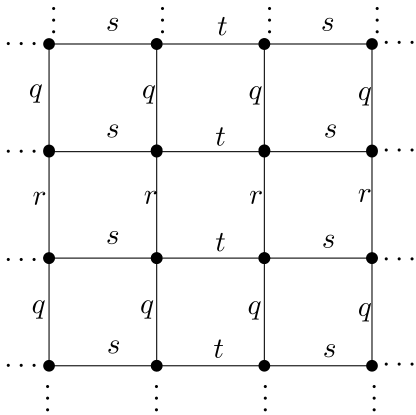
which has only one commutation relation between two distinct generators, gives rise to a completely distinct type of buildings. A thin building of type W_2 is partially represented in the next figure.



Example 2.2.4. Consider the Coxeter group

$$W_3 = \langle s,t,q,r \mid \begin{array}{l} s^2 = t^2 = q^2 = r^2 = 1 \\ (sq)^2 = (sr)^2 = (tq)^2 = (tr)^2 = 1 \rangle \end{array}$$

from Example 2.1.4. A building Δ associated to W_3 is an infinite direct product of trees. The figure partially represents the Coxeter chamber system associated to W_3.



An apartment of Δ is isomorphic to a tessellation of the Euclidean plane by squares. Therefore Δ is also an example of a Euclidean building as in Example 1.4.39(4).

Example 2.2.5. Let p and q be integers such that $p \geq 5$ and $q \geq 2$. Consider the Coxeter group

$$W = \langle S = \{s_1, \dots s_p\} \mid (s_i)^2 = (s_i s_{i+1})^2 = 1 \text{ for all } i \in \{1, \dots, p\} \rangle,$$

with cycling indexing, meaning that $(s_p s_1)^2 = 1$. *Bourdon's buildings* $I_{p,q}$, defined in [Bou97], are buildings of type (W, S) whose panels all have size q. The Bourdon's building $I_{5,2}$ is depicted in Figure 2.7.

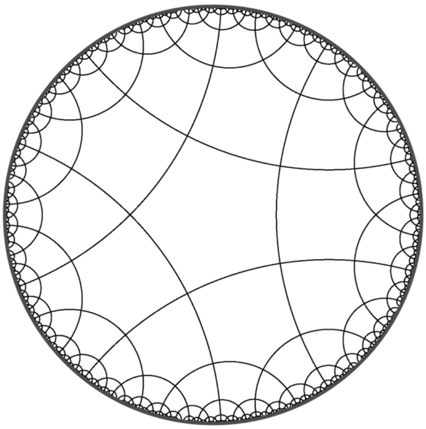


Figure 2.7: Bourdon's building $I_{5,2}$.

It is the simplest example of a *hyperbolic building*, i.e., a building whose apartments are isomorphic to tessellations of a hyperbolic space. Bourdon's buildings are *Fuchsian buildings* which are hyperbolic buildings of dimension 2.

2.2.1 Minimal galleries in right-angled buildings

We will now present two results that can be used in right-angled buildings to modify minimal galleries using the commutation relations of the Coxeter group. We will refer to these results as the "Closing Squares Lemmas" (see also Figure 2.8 below).

Lemma 2.2.6 (Closing Squares 1)**.** *Let c_0 be a fixed chamber in a right-angled building Δ. Let $c_1, c_2 \in \mathsf{S}(c_0, n)$ and $c_3 \in \mathsf{S}(c_0, n+1)$ such that*

$$c_1 \overset{t}{\sim} c_3 \quad \textit{and} \quad c_2 \overset{s}{\sim} c_3$$

for some $s \neq t$. Then $|st| = 2$ in Σ and there exists $c_4 \in \mathsf{S}(c_0, n-1)$ such that

$$c_1 \overset{s}{\sim} c_4 \quad \textit{and} \quad c_2 \overset{t}{\sim} c_4.$$

Proof. Let w_1 and w_2 be reduced representations of $\delta(c_0, c_1)$ and $\delta(c_0, c_2)$, respectively. Then $w_1 t$ and $w_2 s$ are two reduced representations of $\delta(c_0, c_3)$ and thus $w_1 t = w_2 s$ in W. Hence $|st| = 2$. Furthermore, $l(w_1 s) < l(w_1)$ and thus $l(w_1 s) = n - 1$. Let c_4 be the chamber in $\mathsf{S}(c_0, n-1)$ that is s-adjacent to c_1. Then $w_1 st = w_1 ts = w_2 ss = w_2$ in W. Therefore $c_4 \overset{t}{\sim} c_2$. $\square$

Lemma 2.2.7 (Closing Squares 2)**.** *Let c_0 be a fixed chamber in a right-angled building Δ. Let $c_1, c_2 \in \mathsf{S}(c_0, n)$ and $c_3 \in \mathsf{S}(c_0, n-1)$ such that*

$$c_1 \overset{s}{\sim} c_2 \quad \textit{and} \quad c_2 \overset{t}{\sim} c_3$$

for some $s \neq t$. Then $|st| = 2$ in Σ and there exists $c_4 \in \mathsf{S}(c_0, n-1)$ such that

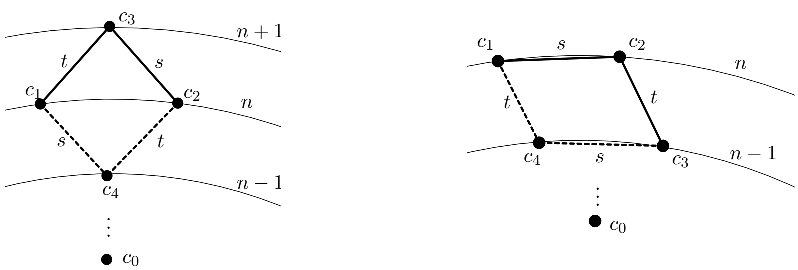
$$c_1 \overset{t}{\sim} c_4 \quad \textit{and} \quad c_3 \overset{s}{\sim} c_4.$$

Proof. Let w_1 and w_2 be reduced representations of $\delta(c_0, c_1)$ and $\delta(c_0, c_2)$. As c_1 and c_2 are s-adjacent and are both in $\mathsf{S}(c_0, n)$, we know that $w_1 = w_2 = s_1 \cdots s_{n-1} s$ in W. Let $v_1 \in \mathsf{S}(c_0, n-1)$ be the chamber s-adjacent to c_1 (and c_2) at Weyl distance $s_1 \cdots s_{n-1}$ from c_0.

Applying Lemma 2.2.6 to v_1 and c_3 we obtain $v_2 \in \mathsf{S}(c_0, n-2)$ such that

$$v_2 \overset{t}{\sim} v_1 \quad \text{and} \quad v_2 \overset{s}{\sim} c_3.$$

Furthermore $|st| = 2$ in Σ. Since there is a minimal gallery of type ts between v_2 and c_1, there must be one of type st. Hence there is a chamber $c_4 \in \mathsf{S}(c_0, n-1)$ that is t-adjacent to c_1 and s-adjacent to v_2. Since v_2 is s-adjacent to c_3 and to c_4, we conclude that c_3 and c_4 are s-adjacent. $\square$



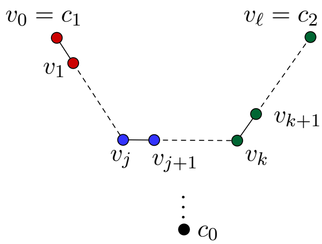
(a) Lemma 2.2.6 (b) Lemma 2.2.7

Figure 2.8: Closing squares Lemmas.

As a consequence of the closing squares lemmas, we are able to transform minimal galleries into "concave" minimal galleries. Recall the gallery distance, denoted by d_W, which was presented in Definition 1.4.48.

Lemma 2.2.8. *Let c_1 and c_2 be two chambers in Δ. There exists a minimal gallery $\gamma = (v_0, \dots, v_\ell)$ in Δ between $c_1 = v_0$ and $c_2 = v_\ell$ such that there are numbers $0 \leq j \leq k \leq \ell$ satisfying the following:*

1. $\mathsf{d}_W(c_0, v_i) < \mathsf{d}_W(c_0, v_{i-1})$ *for all $i \in \{1, \dots, j\}$;*
2. $\mathsf{d}_W(c_0, v_i) = \mathsf{d}_W(c_0, v_{i-1})$ *for all $i \in \{j+1, \dots, k\}$;*
3. $\mathsf{d}_W(c_0, v_i) > \mathsf{d}_W(c_0, v_{i-1})$ *for all $i \in \{k+1, \dots, \ell\}$.*



Proof. Let $(v_0, \dots, v_\ell)$ be a minimal gallery from c_1 to c_2 in Δ. We will essentially prove the result by closing squares whenever possible.

Let $h(\gamma) := \sum_{i=0}^{\ell} \mathsf{d}_W(c_0, v_i)$ be the "total height" of the gallery with respect to c_0. Observe that the gallery γ is of the required form if and only if it does not contain length 2 subgalleries of any of the following form (see also Figure 2.10):

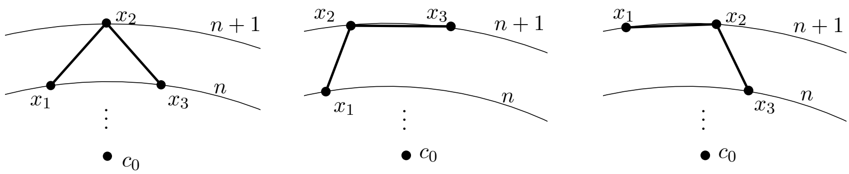
(a) (x_1, x_2, x_3) with $\mathsf{d}_W(c_0, x_1) = n$, $\mathsf{d}_W(c_0, x_2) = n+1$, $\mathsf{d}_W(c_0, x_3) = n$;

(b) (x_1, x_2, x_3) with $\mathsf{d}_W(c_0, x_1) = n$, $\mathsf{d}_W(c_0, x_2) = n+1$, $\mathsf{d}_W(c_0, x_3) = n+1$;

(c) (x_1, x_2, x_3) with $\mathsf{d}_W(c_0, x_1) = n+1$, $\mathsf{d}_W(c_0, x_2) = n+1$, $\mathsf{d}_W(c_0, x_3) = n$.

Indeed, the exclusion of galleries of type (a) and (b) says that once we start going up, we have to continue going up, and the exclusion of galleries of type (a) and (c) says that once we stop going down, we can never go down again.

We will now show that if γ contains a length 2 subgallery of any of the forms above, then we can replace γ by another minimal gallery γ'



(a) case (b) case (c) case

Figure 2.10: The forbidden cases in a "concave" gallery.

from c_1 to c_2 for which $h(\gamma') < h(\gamma)$. Since the height of the gallery $h(\gamma)$ is a natural number, this process has to stop eventually, and we will be left with a minimal gallery of the required form.

If we have a subgallery of type (a), then we can apply Lemma 2.2.6 to replace x_2 by some chamber x_2' with $\mathsf{d}_W(c_0, x_2') = n-1$. If we have a subgallery of type (b) or type (c), then we can apply Lemma 2.2.7 to replace x_2 by some chamber x_2' with $\mathsf{d}_W(c_0, x_2') = n$. In all cases, we have replaced one chamber in γ by a chamber which is closer to c_0, and hence we have indeed decreased the value of $h(\gamma)$, as claimed. □

Corollary 2.2.9. *Let c_0 be a fixed chamber in Δ. If $c_1, c_2 \in \mathsf{B}(c_0, n)$, then there exists a minimal gallery from c_1 to c_2 inside $\mathsf{B}(c_0, n)$.*

Proof. This follows directly from Lemma 2.2.8 by transforming a minimal gallery into a "concave" minimal gallery. □

2.2.2 Projections and parallel residues

Let Σ be a right-angled Coxeter diagram with vertex-set S and let (W, S) be the Coxeter system of type Σ with set of generators $S = (s_i)_{i \in I}$. Let Δ be a right-angled building of type Σ.

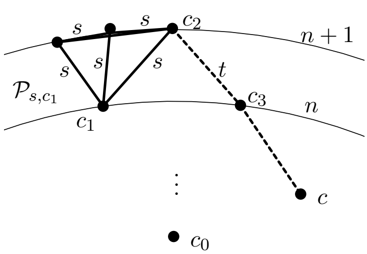
Definition 2.2.10. Let c be a chamber in Δ and $\mathcal{R}$ be a residue in Δ. The *projection* of c on $\mathcal{R}$ is the unique chamber in $\mathcal{R}$ that is closest to c and it is denoted by $\mathrm{proj}_{\mathcal{R}}(c)$.

The next fact is usually called the gate property and can be found, for instance, in [AB08, Proposition 3.105].

Proposition 2.2.11 (Gate property)**.** *Let c be a chamber in Δ and $\mathcal{R}$ be a residue in Δ. For any chamber c' in $\mathcal{R}$, there is a minimal gallery* *from c to c' passing through* $\mathrm{proj}_{\mathcal{R}}(c)$*, and such that the subgallery from* $\mathrm{proj}_{\mathcal{R}}(c)$ *to c' is contained in $\mathcal{R}$.*

Next we present applications of the projection map that will be useful for us later on. The first shows how the projections to panels (residues of rank 1) are related to the structure of the Coxeter diagram of the building.

Lemma 2.2.12. *Let c_0 be a fixed chamber of Δ and let $s \in S$. Let $c_1 \in \mathsf{S}(c_0, n)$ and $c \in \mathsf{B}(c_0, n+1) \setminus \mathrm{Ch}(\mathcal{P}_{s,c_1})$. If* $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c) = c_2 \in \mathsf{S}(c_0, n+1)$ *then c_2 is t-adjacent to some $c_3 \in \mathsf{S}(c_0, n)$ with $t \neq s$ and $st = ts$ in W.*



Proof. By Lemma 2.2.8 we can take a concave minimal gallery between c and c_2. Consider $w = s_1 \cdots s_\ell$ to be the corresponding reduced representation of $\delta(c, c_2)$. Let x be the chamber s_ℓ-adjacent to c_2 that is at Weyl distance $s_1 \cdots s_{\ell-1}$ from c. We have $\mathsf{d}_W(c_0, c_2) \geq \mathsf{d}_W(c_0, x)$, because we took a concave gallery.

We know that $s_\ell \neq s$ because $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c) = c_2$. Thus $l(w) < l(ws)$. If $x \in \mathsf{S}(c_0, n)$ then the result follows from Lemma 2.2.6 with $c_3 = x$ since we obtain that $|s_\ell s| = 2$ in W. If $v_1 \in \mathsf{S}(c_0, n+1)$ then the desired adjacency follows from Lemma 2.2.7. $\square$

The next result will allow us to extend a permutation of an s-panel to an automorphism of the whole building in a useful way, i.e., in a way that we have control over a specific set of chambers of the building.

Proposition 2.2.13 ([Cap14, Proposition 4.2])**.** *Let Δ be a semi-regular right-angled building of type Σ. Let $s \in S$ and $\mathcal{P}$ be an s-panel. Given any permutation $\theta \in \mathrm{Sym}(\mathrm{Ch}(\mathcal{P}))$ there is $\widetilde{\theta} \in \mathrm{Aut}(\Delta)$ stabilizing $\mathcal{P}$ satisfying the following two conditions:*

1. $\widetilde{\theta}|_{\mathrm{Ch}(\mathcal{P})} = g$;

2. $\widetilde{\theta}$ *fixes all chambers of* Δ *whose projection to* $\mathcal{P}$ *is fixed by* θ.

Our last application of the projection map gives a consequence of combinatorial convexity (*cf.* Definition 1.4.41), in terms of the projection map.

Proposition 2.2.14 ([Cap14, Section 2])**.** *If a set of chambers* $\mathcal{C}$ *is combinatorially convex then for every* $c \in \mathcal{C}$ *and every residue* $\mathcal{R}$ *of* Δ *with* $\mathrm{Ch}(\mathcal{R}) \cap \mathcal{C} \neq \emptyset$ *we have* $\mathrm{proj}_{\mathcal{R}}(c) \in \mathcal{C}$.

Proof. Let $c_1 \in \mathrm{Ch}(\mathcal{R}) \cap \mathcal{C}$ and $c \in \mathcal{C}$. We know that there is a minimal gallery from c to c_1 passing through $\mathrm{proj}_{\mathcal{R}}(c)$. As $\mathcal{C}$ is combinatorially convex the result follows. $\square$

Assuming commutation between two generators, one can also obtain similar results to the Closing Squares Lemmas 2.2.6 and 2.2.7, that allow us to close squares "up".

Lemma 2.2.15. *Let* c_0 *be a fixed chamber in a right-angled building* Δ. *Let* c_1 *and* c_2 *be chambers in* $\mathsf{S}(c_0, n)$.

1. *Assume that there exists* $c_3 \in \mathsf{S}(c_0, n-1)$ *such that* $c_1 \overset{s}{\sim} c_3$ *and* $c_2 \overset{t}{\sim} c_3$ *for some* $s \neq t$ *with* $|st| = 2$.

   Then there exists $c_4 \in \mathsf{S}(c_0, n+1)$ *such that* $c_1 \overset{t}{\sim} c_4$ *and* $c_2 \overset{s}{\sim} c_4$.

2. *Suppose that there is* $c_3 \in \mathsf{S}(c_0, n+1)$ *such that* $c_1 \overset{s}{\sim} c_2$ *and* $c_2 \overset{t}{\sim} c_3$ *for some* $s \neq t$ *with* $|st| = 2$.

   Then there exists $c_4 \in \mathsf{S}(c_0, n+1)$ *such that* $c_1 \overset{t}{\sim} c_4$ *and* $c_3 \overset{s}{\sim} c_4$.

Proof. We prove the first statement and the second follows a similar reasoning. Let $c_4 = \mathrm{proj}_{\mathcal{P}_{t,c_1}}(c_2)$. We have that $c_4 \neq c_1$ because $\delta(c_2, c_1) \sim st \sim ts$. Hence $\delta(c_2, c_4) \sim tst \sim t$ and by definition c_4 is s-adjacent to c_1. $\square$

One can get a graphical visualization of Lemma 2.2.15 by interchanging the dotted edges with the full edges in Figure 2.8, taking in account that the commutation relation between the generators is part of the initial assumptions in this lemma.

We end this subsection by defining the concept of parallel residues and by collecting some facts from [Cap14, Section 2] about this notion. The definition of parallelism, when considered in residues of a right-angled building is an equivalence relation. Restricting to the case of panels, this equivalence relation will allow us to define tree-walls in the next section.

Definition 2.2.16. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two residues, then

$$\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) = \{\mathrm{proj}_{\mathcal{R}_1}(c) \mid c \in \mathrm{Ch}(\mathcal{R}_2)\}$$

is the set of chambers of a residue contained in $\mathcal{R}_1$. This is again a residue (*cf.* [Cap14, Section 2]) and the rank of $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2)$ is bounded above by the ranks of both $\mathcal{R}_1$ and $\mathcal{R}_2$.

The residues $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *parallel* if $\mathrm{proj}_{\mathcal{R}_1}(\mathcal{R}_2) = \mathcal{R}_1$ and $\mathrm{proj}_{\mathcal{R}_2}(\mathcal{R}_1) = \mathcal{R}_2$.

In particular, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two parallel panels, then the chamber sets of $\mathcal{P}_1$ and $\mathcal{P}_2$ are mutually in bijection under the respective projection maps (*cf.* [Cap14, Section 2]).

The next lemma relates residues whose types commute.

Lemma 2.2.17 ([Cap14, Lemma 2.2])**.** *Let $J_1, J_2 \subset S$ be two disjoint subsets with $[J_1, J_2] = 1$. Let $c \in \mathrm{Ch}(\Delta)$. Then*

$$\mathrm{Ch}(\mathcal{R}_{J_1 \cup J_2, c}) = \mathrm{Ch}(\mathcal{R}_{J_1, c}) \times \mathrm{Ch}(\mathcal{R}_{J_2, c}),$$

where $\mathcal{R}_{J_1 \cup J_2, c}$ is the $J_1 \cup J_2$-residue of c.

Moreover, for $i \in \{1, 2\}$, the canonical projection map

$$\mathrm{Ch}(\mathcal{R}_{J_1 \cup J_2, c}) \to \mathrm{Ch}(\mathcal{R}_{J_i, c})$$

coincides with the restriction of $\mathrm{proj}_{\mathcal{R}_{J_i, c}}$ to $\mathrm{Ch}(\mathcal{R}_{J_1 \cup J_2, c})$. In particular, any two J_i-residues contained in $\mathcal{R}_{J_1 \cup J_2, c}$ are parallel.

Lemma 2.2.18 ([Cap14, Lemma 2.5])**.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be panels in Δ. If there are two chambers of $\mathcal{P}_2$ having distinct projections on $\mathcal{P}_1$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel.*

Definition 2.2.19. Let $J \subseteq S$. We define the set

$$J^\perp = \{t \in S \setminus J \mid ts = st \text{ for all } s \in J\}.$$

If $J = \{s\}$ then we denote the set $J^\perp$ by $s^\perp$.

Proposition 2.2.20 ([Cap14, Proposition 2.8])**.** *Let Δ be a right-angled building of type (W, S).*

1. *Any two parallel residues have the same type.*
2. *Let $J \subseteq S$. Given a residue $\mathcal{R}$ of type J, a residue $\mathcal{R}'$ is parallel to $\mathcal{R}$ if and only if $\mathcal{R}'$ is of type J, and $\mathcal{R}$ and $\mathcal{R}'$ are both contained in a common residue of type $J \cup J^{\perp}$.*

Proposition 2.2.21 ([Cap14, Corollary 2.9])**.** *Let Δ be a right-angled building. Parallelism of residues of Δ is an equivalence relation.*

Observe that two parallel panels have the same number of chambers. Therefore, to each equivalence class of parallel panels, we can associate a number q, which is the number of chambers of a panel in that equivalence class.

2.2.3 Tree-walls and wings

We want to describe the equivalence classes of parallelism of panels in right-angled buildings. It turns out that these classes are the so called tree-walls, initially defined in [Bou97] for Fuchsian buildings and taken over in [TW11]. The motivation for the name comes from the fact that the intersection of a tree-wall with an apartment of the building is actually a wall in that apartment.

Each panel $\mathcal{P}$ separates the building into combinatorially convex components, which will be called wings as in [Cap14]. Moreover, this partition can be described using only the tree-wall of the same type containing the panel $\mathcal{P}$.

At the end of this section, we will construct trees out of tree-walls of the same type and we will use those trees to define a distance between tree-walls in the building. We keep the notation of the previous sections.

Definition 2.2.22. Let $s \in S$. An *s-tree-wall* in Δ is an equivalence class of parallel s-panels of Δ.

Using Proposition 2.2.20(2) we know exactly how to describe the tree-walls in a right-angled building.

Corollary 2.2.23. *Let Δ be a right-angled building of type (W, S) and let $s \in S$. Then two s-panels $\mathcal{P}_1$ and $\mathcal{P}_2$ belong to the same s-tree-wall if and only if they are both contained in a common residue of type $s \cup s^\perp$.*

In other words, the s-tree-walls are the sets of s-panels contained in a residue of type $s \cup s^\perp$. Thomas and Wortman in [TW11] prove that the possibilities for tree-walls can actually be described.

Corollary 2.2.24 ([TW11, Corollary 3])**.** *Let $s \in S$ and let $\mathcal{T}$ be an s-tree-wall of Δ. Then only one of the following possibilities can occur:*

1. *$\mathcal{T}$ is reduced to a panel if and only if $\langle s^\perp \rangle$ is trivial.*
2. *$\mathcal{T}$ is finite but not reduced to a panel if and only if $\langle s^\perp \rangle$ is finite but non-trivial.*
3. *$\mathcal{T}$ is infinite if and only if $\langle s^\perp \rangle$ is infinite.*

Next we present some illustrative examples of Corollary 2.2.24

Example 2.2.25.

1. Assume that Δ is a tree without end points, whose associated Coxeter group is generated by s and t. The s-tree-walls are the s-panels of Δ and the t-tree-walls are the t-panels of Δ.
2. Let $W_1 = \langle s, t, q \mid s^2 = t^2 = r^2 = (tq)^2 = 1 \rangle$ as in Example 2.1.3 and let Δ be a right-angled building of type W.
   - The s-tree-walls are the sets of s-panels of Δ.
   - The t-tree-walls are the sets of t-panels in a common residue of type $\{t, q\}$ in Δ.
   - The q-tree-walls are the sets of q-panels in a common residue of type $\{t, q\}$ in Δ.
3. As last example, consider the group

$$W_4 = \langle s_1, s_2, s_3, s_4 \mid \begin{array}{l} s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1 \\ (s_1 s_2)^2 = (s_1 s_3)^2 = (s_3 s_4)^2 = 1 \rangle. \end{array}$$

   as in Example 2.1.4 and let Δ be a right-angled building of type W_4.

- The s_1-tree-walls are the sets of s_1-panels in a common residue of type $\{s_1, s_2, s_3\}$ in Δ.
- The s_2-tree-walls are the sets of s_2-panels in a common residue of type $\{s_1, s_2\}$ in Δ.
- The s_3-tree-walls are the sets of s_3-panels in a common residue of type $\{s_1, s_3, s_4\}$ in Δ.
- The s_4-tree-walls are the sets of s_4-panels in a common residue of type $\{s_1, s_3, s_4\}$ in Δ.

By some slight abuse of notation, we will write $\mathrm{Ch}(\mathcal{T})$ for the set of all chambers contained in some s-panel belonging to the s-tree-wall $\mathcal{T}$, and we will refer to these chambers as the *chambers of* $\mathcal{T}$.

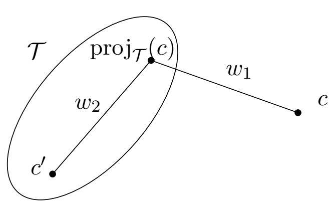
Corollary 2.2.26. *Let $\mathcal{T}$ be an s-tree-wall in Δ, let $\mathcal{P}$ be an s-panel in $\mathcal{T}$, and let $\mathcal{R}$ be the residue of type $s \cup s^\perp$ containing $\mathcal{P}$. Then* $\mathrm{Ch}(\mathcal{T}) = \mathrm{Ch}(\mathcal{R})$.

By Corollary 2.2.26, it makes sense to define projections on tree-walls.

Definition 2.2.27. Let $s \in S$, let $\mathcal{T}$ be an s-tree-wall of Δ, and let $c \in \mathrm{Ch}(\Delta)$. We define the *projection of c on $\mathcal{T}$* as $\mathrm{proj}_{\mathcal{T}}(c) := \mathrm{proj}_{\mathcal{R}}(c)$, where $\mathcal{R}$ is the residue of type $s \cup s^\perp$ containing the s-panels of $\mathcal{T}$.

Lemma 2.2.28. *Let $s \in S$, let $\mathcal{T}$ be an s-tree-wall of Δ, let $c \in$ $\mathrm{Ch}(\Delta)$ and let $c' \in \mathrm{Ch}(\mathcal{T})$. Let w_1 and w_2 be reduced representations of $\delta(c, \mathrm{proj}_{\mathcal{T}}(c))$ and $\delta(\mathrm{proj}_{\mathcal{T}}(c), c')$, respectively. Then $w_1 w_2$ is a reduced representation of $\delta(c, c')$.*

Proof. This follows immediately from the gate property (Proposition 2.2.11). ☐



Let $\mathcal{T}$ be an s-tree-wall and let $q_{\mathcal{T}}$ be the number of chambers in an s-panel of $\mathcal{T}$. Then $\mathcal{T}$ yields a partition of the building into

$q_{\mathcal{T}}$ combinatorially convex components which are called wings. We present the definition of wings in a building and state some results that connect wings, projections and tree-walls.

Definition 2.2.29. Let $c \in \mathrm{Ch}(\Delta)$ and $s \in S$. Then the set of chambers

$$X_s(c) = \{x \in \mathrm{Ch}(\Delta) \mid \mathrm{proj}_{\mathcal{P}_{s,c}}(x) = c\}$$

is called the *s-wing* of c.

We note that we consider wings with respect to panels only since it is sufficient for our purposes. However, this concept can be generalized to residues of any type (see [Cap14]).

Notice that if $\mathcal{P}$ is any s-panel, then the set of s-wings of each of the q_s different chambers of $\mathcal{P}$ forms a partition of $\mathrm{Ch}(\Delta)$ into q_s subsets. Moreover, these subsets are combinatorially convex, as the next proposition states.

Proposition 2.2.30 ([Cap14, Proposition 3.2])**.** *In a right-angled building, wings are combinatorially convex.*

The next lemma, which we will only state for panels, presents a connection between wings and projections to different panels in a common tree-wall.

Lemma 2.2.31 ([Cap14, Lemma 3.1])**.** *Let $s \in S$ and $\mathcal{T}$ be an s-tree-wall. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be s-panels of $\mathcal{T}$ and let $c_1 \in \mathcal{P}_1$ and $c_2 \in \mathcal{P}_2$. Then $X_s(c_1) = X_s(c_2)$ if and only if $c_2 = \mathrm{proj}_{\mathcal{P}_2}(c_1)$, i.e., if and only if $c_1 \in X_s(c_2)$.*

In particular, projection between parallel s-panels induces an equivalence relation on $\mathrm{Ch}(\mathcal{T})$, as the next proposition states.

Proposition 2.2.32. *Let $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ be s-panels in a common s-tree-wall of Δ. Let $c_1 \in \mathcal{P}_1$, $c_2 = \mathrm{proj}_{\mathcal{P}_2}(c_1)$ and $c_3 = \mathrm{proj}_{\mathcal{P}_3}(c_1)$. Then $\mathrm{proj}_{\mathcal{P}_3}(c_2) = c_3$.*

Proof. By Lemma 2.2.31, $c_2 = \mathrm{proj}_{\mathcal{P}_2}(c_1)$ implies $X_s(c_1) = X_s(c_2)$, and $c_3 = \mathrm{proj}_{\mathcal{P}_3}(c_1)$ implies $X_s(c_1) = X_s(c_3)$. Hence $X_s(c_2) = X_s(c_3)$, and therefore $\mathrm{proj}_{\mathcal{P}_3}(c_2) = c_3$. $\square$

The next proposition presents relations between wings of different types in right-angled buildings.

Proposition 2.2.33 ([Cap14, Lemma 3.4])**.** *Let $s, t \in S$ and $c_1, c_2 \in \mathrm{Ch}(\Delta)$. Suppose that $c_1 \in X_t(c_2)$ and $c_2 \notin X_s(c_1)$ and, moreover, $s = t$ or $m_{st} = \infty$. Then $X_s(c_1) \subseteq X_t(c_2)$.*

By Lemma 2.2.31, it makes sense to define a partition of $\mathrm{Ch}(\Delta)$ into s-wings with respect to an s-tree-wall.

Definition 2.2.34. Let $s \in S$ and $\mathcal{T}$ be an s-tree-wall. Let $\mathcal{P}$ be an arbitrary s-panel of $\mathcal{T}$. Then $\mathcal{P}$ induces a partition

$$\{X_s(c) \mid c \in \mathcal{P}\}$$

of $\mathrm{Ch}(\Delta)$ into $q_{\mathcal{T}}$ subsets, where $q_{\mathcal{T}}$ is the number of chambers in $\mathcal{P}$, which we call the *partition of* $\mathrm{Ch}(\Delta)$ *into s-wings with respect to* $\mathcal{T}$. By Lemma 2.2.31, this partition is independent of the choice of $\mathcal{P}$ in $\mathcal{T}$.

We will now study the interaction between different s-tree-walls.

Definition 2.2.35. Let $s \in S$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two s-tree-walls in Δ. We define the set

$$\mathrm{proj}_{\mathcal{T}_1}(\mathcal{T}_2) = \{\mathrm{proj}_{\mathcal{T}_1}(c) \mid c \in \mathcal{T}_2\}.$$

The next proposition states some technical properties of projections to tree-walls and shows how one can frame distinct tree-walls of the building in the partition by the wings with respect to a specific tree-wall.

Proposition 2.2.36. *Let $s \in S$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be distinct s-tree-walls. Let $c_2 \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$ and $c_1 = \mathrm{proj}_{\mathcal{T}_1}(c_2)$. Let w be a reduced representation of $\delta(c_1, c_2)$. Then the following hold.*

1. *If $t \in S$ with $|ts| \leq 2$ then $l(tw) = l(wt) = l(w) + 1$.*
2. *$c_2 = \mathrm{proj}_{\mathcal{T}_2}(c_1)$.*
3. *$\mathrm{Ch}(\mathcal{T}_1) \subseteq X_s(c_2)$.*
4. *If $c_2' \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$ then $\mathrm{proj}_{\mathcal{P}_{s,c_2}}(c_2') = c_2$.*

Proof. 1. Let $t \in S$ such that $|ts| \leq 2$, that is, $t = s$ or $|ts| = 2$. Let c be a chamber t-adjacent to c_1 (and hence $c \in \mathrm{Ch}(\mathcal{T}_1)$). Since $c_1 = \mathrm{proj}_{\mathcal{T}_1}(c_2)$, we can apply the gate property (Proposition 2.2.11) to find a minimal gallery from c to c_2 passing through c_1, of type tw. In particular, $l(w) < l(tw)$.

As $c_2 \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$, there is a chamber $d_1 \in \mathcal{T}_1$ such that $\mathrm{proj}_{\mathcal{T}_2}(d_1) = c_2$. Let w_1 be a reduced representation of $\delta(d_1, c_1)$. By the gate property again, $w_1 w$ is a reduced representation of $\delta(d_1, c_2)$. Let d_2 be a chamber t-adjacent to c_2 (and therefore in $\mathrm{Ch}(\mathcal{T}_2)$). The gate property with respect to $\mathrm{proj}_{\mathcal{T}_2}(d_1) = c_2$ now implies that $l(w_1 wt) > l(w_1 w)$ and hence $l(wt) > l(w)$ as well.

2. By the gate property, w can be written as $w_1 w_2$ where w_2 is a reduced representation of the subgallery from $\mathrm{proj}_{\mathcal{T}_2}(c_1)$ to c_2 inside $\mathcal{T}_2$. Hence $|ts| \leq 2$ for all $t \in w_2$. By Statement 1 we get $l(w_2) = 0$ and thus $c_2 = \mathrm{proj}_{\mathcal{T}_2}(c_1)$.

3. Let $c \in \mathrm{Ch}(\mathcal{T}_1)$ and let $w_1 \in W$ be a reduced representation of $\delta(c, c_1)$. By Lemma 2.2.28, the word $w_1 w$ is a reduced representation of $\delta(c, c_2)$. Statement 1 now gives, $l(w_1 w) < l(w_1 ws)$ and which implies that $\mathrm{proj}_{\mathcal{P}_{s,c_2}}(c) = c_2$.

4. By Statement 3, $\mathrm{Ch}(\mathcal{T}_1) \subseteq X_s(c_2)$ and $\mathrm{Ch}(\mathcal{T}_1) \subseteq X_s(c_2')$. Since the s-wings with respect to $\mathcal{T}_2$ form a partition of $\mathrm{Ch}(\Delta)$ (see Definition 2.2.34), this implies $X_s(c_2) = X_s(c_2')$, and hence $\mathrm{proj}_{\mathcal{P}_{s,c_2}}(c_2') = c_2$ by Lemma 2.2.31. $\square$

2.2.4 The tree-wall tree

We finish this section by defining tree-wall trees and a distance between tree-walls of the same type.

Let (W, S) be a right-angled Coxeter system with set of generators $S = \{s_i\}_{i \in I}$ and Coxeter diagram Σ. Let Δ be a right-angled building of type (W, S).

Definition 2.2.37. Let $s \in S$. Let V_1 be the set of all s-tree-walls of Δ and let V_2 be the set of all residues of type $S \backslash \{s\}$ of Δ. Consider the bipartite graph Γ_s with vertex-set $V_1 \sqcup V_2$, where an s-tree-wall $\mathcal{T} \in V_1$ is adjacent to a residue $\mathcal{R} \in V_2$ in Γ_s if and only if $\mathrm{Ch}(\mathcal{T}) \cap \mathrm{Ch}(\mathcal{R}) \neq \emptyset$.

The graph Γ_s will be called the *tree-wall tree of type s.*

Notice that each s-tree-wall $\mathcal{T}$ in Γ_s has precisely $q_{\mathcal{T}}$ neighbors, corresponding to each of the residues of type $S \setminus \{s\}$ lying in a distinct part of the partition of $\mathrm{Ch}(\Delta)$ induced by the s-wings with respect to $\mathcal{T}$ (see Definition 2.2.34).

Moreover, if there is a minimal path

$$\mathcal{T}_1 - \mathcal{R}_1 - \cdots - \mathcal{R}_2 - \mathcal{T}_2$$

in the graph Γ_s and we consider two chambers $c_1 \in \mathrm{Ch}(\mathcal{T}_1) \cap \mathrm{Ch}(\mathcal{R}_1)$ and $c_2 \in \mathrm{Ch}(\mathcal{T}_2) \cap \mathrm{Ch}(\mathcal{R}_2)$, then by defintion $c_2 \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$ and $c_1 \in \mathrm{proj}_{\mathcal{T}_1}(\mathcal{T}_2)$. Therefore Proposition 2.2.36 implies that $\mathrm{Ch}(\mathcal{T}_2) \subseteq X_s(c_1)$ and $\mathrm{Ch}(\mathcal{T}_1) \subseteq X_s(c_2)$.

Proposition 2.2.38. *Let $s \in S$. The tree-wall tree Γ_s is a tree.*

Proof. By definition, the graph Γ_s is connected as Δ is connected. Therefore it is enough to show that there are no cycles in Γ_s. If there were a non-trivial cycle in Γ_s, say

$$\mathcal{T}_1 - \mathcal{R}_1 - \mathcal{T}_2 - \cdots - \mathcal{R}_n - \mathcal{T}_1,$$

then the chambers of $\mathcal{T}_2$ would be contained in two distinct s-wings with respect to $\mathcal{T}_1$, namely the ones corresponding to $\mathcal{R}_1$ and $\mathcal{R}_n$, which is a contradiction. We conclude then that Γ_s is a tree. $\square$

Remark 2.2.39. Observe that if Δ is a spherical building, that is, if its Coxeter group associated is finite, then every s-tree-wall tree is reduced to a vertex, for every $s \in S$.

This tree is constructed using the data from the building. Therefore it provides us with a natural distance that will allow us to produce inductive arguments on the right-angled building using the distance between the tree-walls.

Definition 2.2.40. Let $s \in S$ and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two s-tree-walls. We define the *s-tree-wall distance*, denoted by dist_{TW}, as

$$\mathrm{dist}_{TW}(\mathcal{T}_1, \mathcal{T}_2) = \frac{1}{2}\,\mathrm{dist}(\mathcal{T}_1, \mathcal{T}_2)_{\Gamma_s},$$

where $\mathrm{dist}(\mathcal{T}_1, \mathcal{T}_2)_{\Gamma_s}$ denotes the discrete distance in the tree-wall tree Γ_s of type s.

2.3 Colorings in semi-regular right-angled buildings

In this section we will define different types of colorings of the chambers of semi-regular right-angled buildings. This class of buildings is the class where we will define the universal group and we are interested in such regularity mainly because semi-regular right-angled buildings are unique up to isomorphism. Moreover, its automorphism group is simple. The colorings on the chambers will be necessary to define the universal group later in Chapter 4.

Definition 2.3.1. Let Δ be a right-angled building of type (W, S). Then Δ is called *semi-regular* if the s-panels of Δ all have the same number of chambers, for every $s \in S$.

If Δ is semi-regular and q_s denote the cardinality of the s-panels of Δ, then the building is said to have *prescribed thickness* $(q_s)_{s\in S}$ in its panels.

Theorem 2.3.2 ([HP03, Proposition 1.2])**.** *Let (W, S) be a right-angled Coxeter group and $(q_s)_{s\in S}$ be a family of cardinal numbers with $q_s \geq 2$. There exists a right-angled building of type (W, S) such that for every $s \in S$, each s-panel has size q_s. This building is unique, up to isomorphism.*

This was proved for the right-angled Fuchsian case in [Bou97]. According to Haglund and Paulin [HP03] this result was proved by M. Globus and was also known by M. Davis, T. Januszkiewicz and J. Światkowski.

Theorem 2.3.3 (Theorem 1.1 in [Cap14])**.** *Let Δ be a thick semi-regular building of right-angled type (W, S). Assume that (W, S) is irreducible and non-spherical. Then the group* $\mathrm{Aut}(\Delta)$ *of type-preserving automorphisms of Δ is abstractly simple and acts strongly transitively on Δ.*

Remark 2.3.4. If Δ is a semi-regular right-angled building then the tree-wall trees associated to Δ will be regular in the type of vertices associated to the tree-walls. In other words, if each s-panel of Δ has q_s chambers, then the vertices of V_1 (see Definition 2.2.37) all have q_s neighbors.

After initial considerations on semi-regular right-angled buildings, we focus on colorings of the chambers of these objects. We will use the following notation throughout the section.

Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ with vertex-set S and set of generators $S = \{s_i\}_{i \in I}$. Let $(q_s)_{s \in S}$ be a set of cardinal numbers with $q_s \geq 2$ for all $s \in S$. Let Δ be the unique right-angled building of type (W, S) with parameters $(q_s)_{s \in S}$, as showed in Theorem 2.3.2. For each $s \in S$, let Y_s be a set of cardinality q_s. We will refer to Y_s as the set of *s-colors*.

Definition 2.3.5. Let $s \in S$. A map $h_s \colon \mathrm{Ch}(\Delta) \to Y_s$ is called an *s-coloring*Δ if

(C) for every s-panel $\mathcal{P}$ there is a bijection between the colors in Y_s and the chambers in $\mathcal{P}$.

2.3.1 Legal colorings

To define the universal group in the next chapter, we will need a set of s-colorings, one for each generators of the Coxeter group. Moreover, we have to assure that these colorings are consistent with each other, in the sense that chambers in a common s-panel have the same t-color for $t \neq s$.

For that we define in this section legal s-colorings and we prove that a set of legal s-colorings is unique, up to building automorphism.

Definition 2.3.6. Let $s \in S$. An s-coloring $h_s \colon \mathrm{Ch}(\Delta) \to Y_s$ is called a *legal s-coloring* if it satisfies

(L) for every $S \setminus \{s\}$-residue $\mathcal{R}$ and for all $c_1, c_2 \in \mathrm{Ch}(\mathcal{R})$, $h_s(c_1) = h_s(c_2)$.

In particular, if $\mathcal{P}_t$ is a t-panel then for every $s \in S \setminus \{t\}$ one can consider the s-color of the panel $\mathcal{P}_t$, denoted by $h_s(\mathcal{P}_t)$, since all the chambers in $\mathcal{P}_t$ have the same s-color. Similarly, if $\mathcal{T}$ is a t-tree-wall with $|st| = \infty$ in Σ, then by Corollary 2.2.23, $h_s(c_1) = h_s(c_2)$ for all $c_1, c_2 \in \mathrm{Ch}(\mathcal{T})$, and hence the color $h_s(\mathcal{T})$ is well-defined.

Proposition 2.3.7. *Let $c_0 \in \mathrm{Ch}(\Delta)$ and let $(h^1_s)_{s \in S}$ and $(h^2_s)_{s \in S}$ be two sets of legal colorings of Δ. Then there exists $c \in \mathrm{Ch}(\Delta)$ such that $h^1_s(c) = h^2_s(c_0)$ for all $s \in S$.*

Proof. We will prove the result recursively. For each $s \in S$, let $\mathcal{P}_{s,c_0}$ be the s-panel that contains c_0. By the definition of a legal coloring, we know that there exists $c_1 \in \mathcal{P}_{s,c_0}$ such that $h_s^1(c_1) = h_s^2(c_0)$. Moreover, $h_t^1(c_1) = h_t^1(c_0)$ for all $t \neq s$. Repeating this procedure for each $s \in S$ we find a chamber c such that $h_s^1(c) = h_s^2(c_0)$ for all $s \in S$. $\square$

Next we show that we can use automorphisms of the building to map one set of legal colorings to another.

Proposition 2.3.8. *Let $(h_s^1)_{s\in S}$ and $(h_s^2)_{s\in S}$ be two sets of legal colorings of Δ. Let $c_0, c_0' \in \mathrm{Ch}(\Delta)$ be such that $h_s^1(c_0') = h_s^2(c_0)$ for all $s \in S$. Then there exists $g \in \mathrm{Aut}(\Delta)$ such that $g(c_0) = c_0'$ and $h_s^2 = h_s^1 \circ g$, for all $s \in S$.*

Proof. Consider the set

$$G_n = \{g \in \mathrm{Aut}(\Delta) \mid c_0' = g(c_0) \text{ and } h_s^1 \circ g|_{\mathsf{B}(c_0,n)} = h_s^2|_{\mathsf{B}(c_0,n)}, \forall s \in S\}.$$

We will recursively construct a sequence of elements g_i ($i \in \mathbb{N}$) such that for every $i \in \mathbb{N}$, we have $g_i \in G_i$, and if $j \in \mathbb{N}$ is larger than i, then g_i and g_j agree on the ball $\mathsf{B}(c_0, i)$.

Since $\mathrm{Aut}(\Delta)$ is chamber-transitive, the set G_0 is non-empty and we can pick a $g_0 \in G_0$ at random.

Let us assume that we already have constructed automorphisms g_i for every $i \leq n$ with the right properties. In particular $g_n(c_0) = c_0'$ and $h_s^1 \circ g_n(c) = h_s^2(c)$ for all $c \in \mathsf{B}(c_0, n)$. Therefore without loss of generality we can assume that $c_0' = c_0$ and that

$$h_s^1(c) = h_s^2(c) \text{ for all } s \in S \text{ and for all } c \in \mathsf{B}(c_0, n). \tag{2.3.1}$$

We will construct an element g_{n+1} of G_{n+1} by modifying g_n (which now acts trivially on $\mathsf{B}(c_0, n)$) step by step along $\mathsf{S}(c_0, n+1)$.

Let $v \in \mathsf{S}(c_0, n)$ and fix some $s \in S$. Let θ_s be a permutation of the chambers of $\mathcal{P}_{s,v}$ such that $h_s^2(c) = h_s^1(\theta_s c)$ for all $c \in \mathcal{P}_{s,v}$. By Proposition 2.2.13, θ_s extends to an automorphism $\widetilde{\theta_s}$ such that $\widetilde{\theta_s}$ stabilizes $\mathcal{P}_{s,v}$ and fixes all the chambers of Δ whose projection on $\mathcal{P}_{s,v}$ is fixed by θ_s.

We claim that $\widetilde{\theta_s}$ fixes $\mathsf{B}(c_0, n+1) \setminus \mathrm{Ch}(\mathcal{P}_{s,v})$. Consider a chamber $c \in \mathsf{B}(c_0, n+1) \setminus \mathrm{Ch}(\mathcal{P}_{s,v})$, and let $c_2 = \mathrm{proj}_{\mathcal{P}_{s,v}}(c)$. If $c_2 \in \mathsf{B}(c_0, n)$, then d is fixed by θ_s, and this already implies that c_2 is fixed by $\widetilde{\theta_s}$.

Suppose now that $c_2 \in \mathsf{S}(c_0, n+1)$. By Lemma 2.2.12 there exists $c_3 \in \mathsf{S}(c_0, n)$ such that c_2 is t-adjacent to c_3 with $t \neq s$ (and $ts = st$ in W). The definition of a legal coloring together with (2.3.1) now implies

$$h^1_s(c_2) = h^1_s(c_3) = h^2_s(c_3) = h^2_s(c_2), \tag{2.3.2}$$

so θ_s must fix c_2. Hence the automorphism $\widetilde{\theta_s}$ fixes c also in this case.

We have thus constructed, for each $s \in S$, an automorphism $\widetilde{\theta_s} \in \mathrm{Aut}(\Delta)$, with the property that all elements of $\mathsf{B}(c_0, n+1)$ that are moved by $\widetilde{\theta_s}$ are contained in $\mathsf{S}(c_0, n+1) \cap \mathcal{P}_{s,v}$. We now vary s, and we consider the element

$$\theta_v = \prod_{s \in S} \widetilde{\theta_s} \in \mathrm{Aut}(\Delta)$$

where the product is taken in an arbitrary order (observe that the sets $\mathsf{S}(c_0, n+1) \cap \mathcal{P}_{s,v}$ are disjoint for any two distinct s). Even though the element θ_v might depend on the chosen order, its action on $\mathsf{B}(c_0, n+1)$ does not because the sets of chambers of $\mathsf{B}(c_0, n+1)$ moved by the elements $\widetilde{\theta_s}$ are disjoint, for distinct s.

Now we have an automorphism θ_v, for each chamber $v \in \mathsf{S}(c_0, n)$, fixing the chambers in $\mathsf{B}(c_0, n+1) \setminus \mathsf{S}(v, 1)$. Next, we want to vary v along $\mathsf{S}(c_0, n)$. We claim that if $v_1, v_2 \in \mathsf{S}(c_0, n)$ then θ_{v_1} and θ_{v_2} restricted to $\mathsf{B}(c_0, n+1)$ have disjoint support.

The only case that remains to be checked is when $\mathsf{S}(v_1, 1)$ and $\mathsf{S}(v_2, 1)$ have a chamber $c \in \mathsf{S}(c_0, n+1)$ in common, as depicted in Figure 2.12.

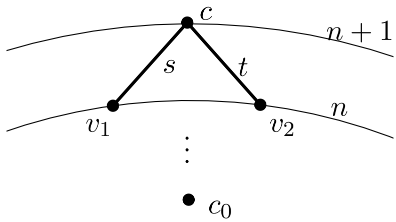


Figure 2.12: $\mathrm{St}(v_1) \cap \mathrm{St}(v_2) \neq \emptyset$.

We want to show that in this case both θ_{v_1} and θ_{v_2} fix c. By Lemma 2.2.6, there are $s \neq t$ in W (with $st = ts$) such that $c \overset{s}{\sim} v_1$

and $c \overset{t}{\sim} v_2$. By the definition of a legal coloring together with (2.3.1) above, we have

$$h_t^1(c) = h_t^1(v_1) = h_t^2(v_1) = h_t^2(c). \tag{2.3.3}$$

Therefore $\theta_{v_2}(c) = \widetilde{\theta_{v_2,t}}(c)$ must fix c. Similarly $h_s^1(c) = h_s^2(c)$, and so θ_{v_1} fixes c. This proves our claim, and hence θ_{v_1} and θ_{v_2} restricted to $\mathsf{B}(c_0, n+1)$ have disjoint support for any two chambers v_1 and v_2 in $\mathsf{S}(c_0, n)$.

We can now consider the product

$$g_{n+1} = \prod_{v \in \mathsf{S}(c_0,n)} \theta_v = \prod_{v \in \mathsf{S}(c_0,n)} \prod_{s \in S} \widetilde{\theta_{v,s}} \in \mathrm{Aut}(\Delta),$$

where the product is again taken in an arbitrary order. By the previous paragraph, the action of g_{n+1} on $\mathsf{B}(c_0, n+1)$ is independent of the chosen order, and the sets of chambers of $\mathsf{B}(c_0, n+1)$ moved by distinct elements $\widetilde{\theta_{v,s}}$ are disjoint. Since every $\widetilde{\theta_{v,s}}$ has the property that $h_s^2(c) = h_s^1(\widetilde{\theta_{v,s}}(c))$ for all $c \in \mathcal{P}_{s,v}$, we conclude that $h_s^2(c) = h_s^1(g_{n+1}(c))$ for all $c \in \mathsf{S}(c_0, n+1)$, and therefore $g_{n+1} \in G_{n+1}$.

So we have extended the $g \in G_n$ to an element g_{n+1} in G_{n+1} agreeing with g_n on the ball $\mathsf{B}(c_0, n)$. The sequence g_0, g_1, $\ldots$ obtained by repeating this procedure hence converges to an element $g \in \mathrm{Aut}(\Delta)$ (with respect to the permutation topology). From the construction and the definition of the sets G_i, the automorphism g has the desired properties. $\square$

2.3.2 Weak legal colorings

We consider now a weaker version of the legal colorings of Definition 2.3.6. The goal will be to prove later that these two types of colorings play a similar role in the definition of the universal group. Hence we will be able to use the stronger version when needed and the weaker version to make constructions of colorings. We keep the notation from the previous section.

Definition 2.3.9. Let $s \in S$. An s-coloring h_s is called a *weak legal s-coloring* if the following holds:

(W) if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two s-panels in a common s-tree-wall then for all $c \in \mathcal{P}_1$, we have $h_s(c) = h_s(\mathrm{proj}_{\mathcal{P}_2}(c))$.

A legal coloring is in particular a weak legal coloring. Conversely, the restriction of a weak legal coloring to a tree-wall is a legal coloring.

Lemma 2.3.10. *Let $s \in S$. A weak legal s-coloring restricted to an s-tree-wall $\mathcal{T}$ is a legal coloring of* $\mathrm{Ch}(\mathcal{T})$.

Proof. If $c_1, c_2 \in \mathrm{Ch}(\mathcal{T})$ are two t-adjacent chambers with $t \in S \setminus \{s\}$ then c_1 and c_2 lie in distinct parallel s-panels of $\mathcal{T}$ and moreover $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c_2) = c_1$. Hence $h_s(c_1) = h_s(c_2)$. □

Next we define equivalent colorings up to a permutation of the set of colors and we prove a result that already slightly uncovers the connection between weak legal colorings and legal colorings.

Definition 2.3.11. Let $s \in S$ and let $G \leq \mathrm{Sym}(Y_s)$. Two s-colorings h_s^1 and h_s^2 are said to be *G-equivalent* if for every s-panel $\mathcal{P}$ there is $g \in G$ such that $h_s^1|_{\mathcal{P}} = g \circ h_s^2|_{\mathcal{P}}$.

Proposition 2.3.12. *Let $s \in S$ and $G \leq \mathrm{Sym}(Y_s)$ be a transitive permutation group. Then every weak legal s-coloring is G-equivalent to some legal s-coloring.*

Proof. Let h_s be a weak legal coloring of Δ. We want to show that there is a legal coloring h_s^ℓ that is G-equivalent to h_s.

Let c_0 be a fixed chamber of Δ. We will define h_s^ℓ recursively using the s-tree-wall distance (see Definition 2.2.40). Let $\mathcal{T}_0 = \mathcal{T}_{s,c_0}$. For each chamber $c \in \mathrm{Ch}(\mathcal{T}_0)$, we define

$$h_s^\ell(c) = h_s(c) = \mathrm{id}_G \circ h_s(c).$$

By Lemma 2.3.10, the restriction of h_s^ℓ to $\mathrm{Ch}(\mathcal{T}_0)$ is a legal coloring.

Assume that we have defined h_s^ℓ for all chambers of every s-tree-wall of Δ at tree-wall distance $\leq n$ from $\mathcal{T}_0$. Let $\mathcal{T}_2$ be an s-tree-wall at tree-wall distance $n+1$ from $\mathcal{T}_0$. By Proposition 2.2.38 there is a unique s-tree-wall $\mathcal{T}_1$ at tree-wall distance 1 from $\mathcal{T}_2$ that is at tree-wall distance n from $\mathcal{T}_0$. By our recursion assumption, h_s^ℓ is already defined for all chambers of $\mathcal{T}_1$. Pick some $c_2 \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$ (*cf.*

Definition 2.2.35), and let $c_1 = \mathrm{proj}_{\mathcal{T}_1}(c_2)$. Fix a $g \in G$ such that $g \circ h_s(c_2) = h_s^\ell(c_1)$ (which exists by transitivity) and define

$$h_s^\ell(c) = g \circ h_s(c), \text{ for all } c \in \mathrm{Ch}(\mathcal{T}_2).$$

That is, we set $h_s^\ell(c_2) = h_s^\ell(c_1)$ and carry out the same permutation of the s-colors on each s-panel of $\mathcal{T}_2$.

We claim that

$$h_s^\ell(c) = h_s^\ell\big(\mathrm{proj}_{\mathcal{T}_1}(c)\big) \quad \text{for all } c \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1). \tag{2.3.4}$$

So let $c_2' \in \mathrm{proj}_{\mathcal{T}_2}(\mathcal{T}_1)$ and $c_1' = \mathrm{proj}_{\mathcal{T}_1}(c_2')$ as in Figure 2.13.

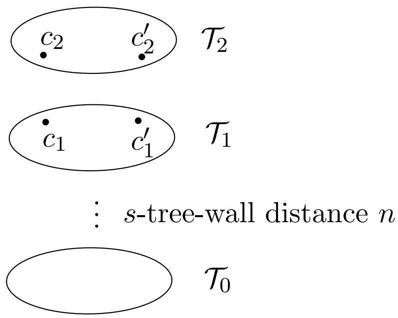


Figure 2.13: The tree-walls $\mathcal{T}_1$ and $\mathcal{T}_2$.

By Proposition 2.2.36(2) we conclude that $c_2 = \mathrm{proj}_{\mathcal{T}_2}(c_1)$ and $c_2' = \mathrm{proj}_{\mathcal{T}_2}(c_1')$. Hence we apply Statement 4 of the same proposition to conclude that $\mathrm{proj}_{\mathcal{P}_{s,c_2}}(c_2') = d$ and that $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c_1') = c_1$. Since h_s is a weak legal coloring, we have $h_s(c_2) = h_s(c_2')$, and hence by construction $h_s^\ell(c_2) = h_s^\ell(c_2')$. Moreover, since we already know that h_s^ℓ is a legal coloring on $\mathcal{T}_1$, we also have $h_s^\ell(c_1) = h_s^\ell(c_1')$. Since $h_s^\ell(c_2) = h_s^\ell(c_1)$ and g is fixed in each tree-wall, we conclude that $h_s^\ell(c_2') = h_s^\ell(c_1')$, proving the claim (2.3.4).

With this procedure, we recursively define the map h_s^ℓ for all the chambers of Δ. Since h_s^ℓ was defined in each panel of Δ just by permuting the colors of h_s, it follows that h_s^ℓ is a coloring of the chambers of Δ.

It remains to show that h_s^ℓ is a legal coloring, i.e, that it satisfies (L) from Definition 2.3.6. So let c_1 and c_2 be chambers in a residue $\mathcal{R}$

of type $S \setminus \{s\}$. We have to show that $h_s^\ell(c_1) = h_s^\ell(c_2)$. Since residues are combinatorially convex, the minimal galleries from c_1 to c_2 do not contain s-adjacent chambers. In particular, $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c_2) = c_1$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the s-tree-walls of c_1 and c_2, respectively. If $\mathcal{T}_1$ coincides with $\mathcal{T}_2$ then we know that $h_s(c_1) = h_s(c_2)$ and thus

$$h_s^\ell(c_1) = g \circ h_s(c_1) = g \circ h_s(c_2) = h_s^\ell(c_2),$$

where $g \in G^s$ was the permutation element used to define h_s^ℓ in $\mathcal{T}_1 = \mathcal{T}_2$.

Suppose now that $\mathcal{T}_1 \neq \mathcal{T}_2$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are both adjacent to the vertex in the s-tree-wall tree Γ_s that corresponds to the residue $\mathcal{R}$. Thus $\mathrm{dist}_{TW}(\mathcal{T}_1, \mathcal{T}_2) = 1$. Assume without loss of generality that $n = \mathrm{dist}_{TW}(\mathcal{T}_0, \mathcal{T}_1) = \mathrm{dist}_{TW}(\mathcal{T}_0, \mathcal{T}_2) - 1$, where $\mathcal{T}_0$ is the s-tree-wall containing the base chamber c_0. Then $\mathcal{T}_1$ is the unique s-tree-wall at tree-wall distance n from $\mathcal{T}_0$ that is at tree-wall distance 1 from $\mathcal{T}_2$. Therefore h_s^ℓ has been defined on $\mathcal{T}_2$ using the coloring in $\mathcal{T}_1$.

Let $c_2' = \mathrm{proj}_{\mathcal{T}_2}(c_1)$, and let $c_1' = \mathrm{proj}_{\mathcal{T}_1}(c_2')$. By the gate property (Proposition 2.2.11), there is a minimal gallery from c_1 to c_2 through c_1' and c_2', and the subgalleries from c_1 to c_1' and from c_2' to c_2 are completely contained in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. These subgalleries do not contain s-adjacent chambers because c_1 and c_2 are contained in $\mathcal{R}$ of type $S \setminus \{s\}$. Hence $h_s^\ell(c_1) = h_s^\ell(c_1')$ and $h_s^\ell(c_2') = h_s^\ell(c_2)$. Finally, by Equation (2.3.4), we also have $h_s^\ell(c_1') = h_s^\ell(c_2')$. Therefore $h_s^\ell(c_1) = h_s^\ell(c_2)$. We conclude that h_s^ℓ is a legal coloring, and by construction it is G-equivalent to h_s. $\square$

2.3.3 Directed legal colorings

Now we define a particular set of weak legal colorings. As before, let Y_s (for each $s \in S$) be the set of s-colors. We additionally assume that each Y_s contains a distinguished element 1_s, or 1 for short if no confusion can arise.

The key point of directed colorings is to get a set of colorings such that in every s-panel $\mathcal{P}$, the chamber of $\mathcal{P}$ closest to a fixed chamber has s-color 1. This will be particularly useful for studying chamber stabilizers in groups of automorphisms of the building.

First we show that we can always modify weak legal colorings in order that in every panel the closest chamber to a fixed chamber has color 1. That is, we can always impose some sort of direction in a weak legal coloring, using a transitive permutation group acting on the set of colors.

Proposition 2.3.13. *Let $s \in S$ and let $c_0 \in \mathrm{Ch}(\Delta)$ be a fixed chamber. Let h_s be a weak legal s-coloring of Δ and $G \leq \mathrm{Sym}(Y_s)$ be a transitive permutation group.*

Then there exists a weak legal s-coloring f_s of Δ which is G-equivalent to h_s, such that $f_s(\mathrm{proj}_{\mathcal{P}}(c_0)) = 1_s$ for every s-panel $\mathcal{P}$.

Proof. For each s-tree-wall $\mathcal{T}$, we fix an element $g_{\mathcal{T}} \in G$ such that

$$g_{\mathcal{T}}\big(h_s(\mathrm{proj}_{\mathcal{T}}(c_0))\big) = 1.$$

Notice that such an element exists because we assume that G is transitive. We now define a coloring $f_s \colon \mathrm{Ch}(\Delta) \to Y_s$ by

$$f_s(c) := g_{\mathcal{T}_{s,c}}(h_s(c)) \text{ for all } c \in \mathrm{Ch}(\Delta),$$

where $\mathcal{T}_{s,c}$ is the unique s-tree-wall containing c. Let $\mathcal{P}$ be an arbitrary s-panel and let $c = \mathrm{proj}_{\mathcal{P}}(c_0)$; we claim that $f_s(c) = 1$. Indeed, let $\mathcal{T}$ be the s-tree-wall containing $\mathcal{P}$, and let $c_1 = \mathrm{proj}_{\mathcal{T}}(c_0)$. Then $\mathrm{proj}_{\mathcal{P}_{s,c_1}}(c) = c_1$, so $h_s(c) = h_s(c_1)$, and hence

$$f_s(c) = g_{\mathcal{T}}(h_s(c)) = g_{\mathcal{T}}(h_s(c_1)) = 1.$$

Next, we claim that f_s is a weak legal s-coloring. Indeed, fix some s-tree-wall $\mathcal{T}$; then by definition, $g_{\mathcal{T}}$ induces the same permutation on each s-panel of $\mathcal{T}$. Since h_s is a weak legal s-coloring, it satisfies property (W) from Definition 2.3.9 for each s-tree-wall $\mathcal{T}$, and hence the same holds for f_s. Thus we conclude that also f_s is a weak legal s-coloring. □

Definition 2.3.14. Let $s \in S$. A weak legal s-coloring as in Proposition 2.3.13 is called a *directed legal s-coloring* of Δ *with respect to* c_0.

In other words, if f_s is a weak legal s-coloring and $c_0 \in \mathrm{Ch}(\Delta)$ then f_s is called a directed legal s-coloring with respect to c_0 if for every chamber $c \in \mathrm{Ch}(\Delta)$ at Weyl distance w from c_0 with $l(w) < l(ws)$, we have $f_s(c) = 1$.

Observation 2.3.15. By definition, one can construct a weak legal coloring on any right-angled building Δ. Therefore Proposition 2.3.13 implies that, given a chamber c in Δ, there exists a directed legal coloring of Δ with respect to c.

We also remark that if $\{f_s^1\}_{s\in S}$ and $\{f_s^2\}_{s\in S}$ are two sets of directed legal colorings of Δ with respect to a fixed chamber c_0 then $f_s^i(c_0) = 1$ for all $s \in S$ and $i \in \{1, 2\}$. Moreover c_0 is the only chamber whose colors are all 1. Therefore we can conclude that sets of directed legal colorings with respect to the same chamber are also unique up to building automorphism. This is stated in the next lemma.

Lemma 2.3.16. *Let $\{f_s^1\}_{s\in S}$ and $\{f_s^2\}_{s\in S}$ be two sets of directed legal colorings of Δ with respect to a fixed chamber c_0. Then there exists $g \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ such that $f_s^2 = f_s^1 \circ g$ for all $s \in S$.*

Proof. The construction of the automorphism g follows the same reasoning as in Proposition 2.3.8. Indeed, Equations (2.3.2) and (2.3.3) also hold when we consider weak legal colorings since the chambers c_2 and c_3, and c and v_1, are in the same s-tree-wall, respectively for each equation. □

2.4 Directed semi-regular right-angled buildings

We finish this chapter by giving a description of the chambers of a semi-regular right-angled building in a standard way using the description of the (directed) colorings provided in the previous section.

Since a right-angled building with prescribed thickness in the panels is unique up to isomorphism, as proved in [HP03], we will describe such a building by means of reduced words in the Coxeter group and the sets of colors, providing us with a concrete model for the objects we work with.

Throughout the section we will retain the same notation, that is, let (W, S) be a right-angled Coxeter system with set of generators $S = \{s_i\}_{i\in I}$ and Coxeter diagram Σ. For each $s \in S$, let Y_s be a set with cardinality q_s (with $q_s \geq 2$) with a distinguished element 1.

Definition 2.4.1. We define an edge-colored graph Δ_D as follows.

The vertex-set of Δ_D, denoted by $\mathrm{Ch}(\Delta_D)$, is

$$\left\{ \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix} \;\middle|\; \begin{array}{l} s_1 \cdots s_n \text{ is a reduced word in } M_S \text{ w.r.t. } \Sigma, \\ \text{and } \alpha_i \in Y_{s_i} \setminus \{1\} \text{ for each } i \in \{1, \dots, n\} \end{array} \right\}$$

(where we allow n to be zero, yielding the empty matrix), modulo the equivalence relation defined by

$$\begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix} \sim \begin{pmatrix} s'_1 & \cdots & s'_n \\ \alpha'_1 & \cdots & \alpha'_n \end{pmatrix}$$

if there exists an element $\sigma \in \mathrm{Rep}(s_1 \cdots s_n)$ (see Definition 2.1.7) such that

$$s_{\sigma(1)} \cdots s_{\sigma(n)} = s'_1 \cdots s'_n \text{ and } \alpha_{\sigma(j)} = \alpha'_j, \text{ for all } j \in \{1, \dots n\},$$

i.e., $s'_1 \cdots s'_n$ is obtained from $s_1 \cdots s_n$ by performing elementary operations of type (2) (*cf.* Definition 2.1.5) and the colors associated to the generators after performing the elementary operations are the same. (We are denoting the vertex-set of Δ_D by $\mathrm{Ch}(\Delta_D)$ because we will prove later that Δ_D is a right-angled building. We will already call the elements of $\mathrm{Ch}(\Delta_D)$ chambers.)

We will now define adjacency in this graph. Let $c = \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}$ be an arbitrary chamber. Then the neighbors of c are:

1. all chambers of the form $c' = \begin{pmatrix} s_1 & \cdots & s_n & s_{n+1} \\ \alpha_1 & \cdots & \alpha_n & \alpha_{n+1} \end{pmatrix}$. In this case, we declare c and c' to be s_{n+1}-adjacent.

2. all chambers of the form $c' = \begin{pmatrix} s_1 & \cdots & s_{n-1} & s_n \\ \alpha_1 & \cdots & \alpha_{n-1} & \alpha'_n \end{pmatrix}$, where α'_n takes any value in $Y_{s_n} \setminus \{1, \alpha_n\}$. In this case, we say that c and c' are s_n-adjacent.

3. the unique chamber $c' = \begin{pmatrix} s_1 & \cdots & s_{n-1} \\ \alpha_1 & \cdots & \alpha_{n-1} \end{pmatrix}$. In this case, we declare c and c' to be s_n-adjacent.

Observe that the chambers of Δ_D are equivalence classes. Therefore the definition of adjacency above is unambiguous.

Proposition 2.4.2. *The edge-colored graph Δ_D is a chamber system with index-set S and with prescribed thickness $(q_s)_{s\in S}$ in the panels. Let $s \in S$ and let $\mathcal{P}$ be an s-panel of Δ_D. Then the chambers of $\mathcal{P}$ are of the form*

$$\left\{\begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} s_1 & \cdots & s_n & s \\ \alpha_1 & \cdots & \alpha_n & \alpha_s \end{pmatrix} \mid \alpha_s \in Y_s \setminus \{1\}\right\}.$$

Proof. It is clear from the definition of s-adjacency that the relation "s-adjacent or equal" is an equivalence relation on the set of chambers of Δ_D. Hence Δ_D is a chamber system. The equivalence classes of this relation are precisely the s-panels, which are therefore of the required form. Clearly, each s-panel has cardinality $|Y_s| = q_s$. □

Now we want to define colorings on the chamber system Δ_D.

Definition 2.4.3. Let Δ_D be as in Definition 2.4.1, and let $s \in S$. We define a map $F_s\colon \mathrm{Ch}(\Delta_D) \to Y_s$ as follows. Let $c = \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}$ be a chamber of Δ_D.

1. If $l(s_1 \cdots s_n s) > l(s_1 \cdots s_n)$, then $F_s(c) := 1$.
2. If $l(s_1 \cdots s_n s) < l(s_1 \cdots s_n)$, there exists a $\sigma \in \mathrm{Rep}(s_1 \cdots s_n)$ such that $s_{\sigma(n)} = s$, i.e.,

   $$s_1 \cdots s_n = s_{\sigma(1)} \cdots s_{\sigma(n-1)} s \text{ in } W.$$

   Then

   $$F_s(c) := F_s \begin{pmatrix} s_{\sigma(1)} & \cdots & s_{\sigma(n)} \\ \alpha_{\sigma(1)} & \cdots & \alpha_{\sigma(n)} \end{pmatrix} := \alpha_{\sigma(n)} \in \{2, \ldots, q_s\}.$$

We call F_s the *standard s-coloring* of Δ_D.

Our next goal is to prove that given a semi-regular right-angled building Δ with a set of directed legal colorings with respect to a fixed chamber, one can construct a color-preserving isomorphism to Δ_D equipped with its standard colorings. In particular, it implies that a pair consisting of a semi-regular right-angled building and a set of directed legal colorings is unique up to isomorphism.

Proposition 2.4.4. *Let Δ be a right-angled building of type (W, S) with parameters $(q_s)_{s\in S}$, and let $(f_s)_{s\in S}$ be a set of directed colorings of Δ with respect to a fixed chamber $c_0 \in \mathrm{Ch}(\Delta)$. Let Δ_D be the corresponding chamber system as in Definition* 2.4.1, *and let $(F_s)_{s\in S}$ be its standard s-colorings as in Definition* 2.4.3.

Then there is an isomorphism $\psi\colon \Delta \to \Delta_D$ such that $f_s(c) = F_s(\psi(c))$ for all $s \in S$ and all $c \in \mathrm{Ch}(\Delta)$. In particular, Δ_D is a right-angled building of type (W, S), with prescribed thickness $(q_s)_{s\in S}$.

Proof. We start by setting $\psi(c_0) = (\ ) \in \mathrm{Ch}(\Delta_D)$. Let $c \in \mathrm{Ch}(\Delta)$ be an arbitrary chamber. Let $w = s_1 \cdots s_n$ be a reduced word in M_S representing $\delta(c_0, c)$ and let $\gamma = (c_0, c_1, \dots, c_n)$ be the minimal gallery of type $s_1 \cdots s_n$ connecting the chambers c_0 and $c_n = c$. Then we define

$$\psi(c) = \left[\begin{pmatrix} s_1 & \cdots & s_n \\ f_{s_1}(c_1) & \cdots & f_{s_n}(c_n) \end{pmatrix}\right]_{\sim}.$$

We claim that ψ is well-defined, i.e. that it is independent of the choice of the reduced representation of w. Let w' be another reduced representation of $\delta(c_0, c)$. We know that there is an element $\sigma \in \mathrm{Rep}(w)$ such that $\sigma.w = w'$ and, moreover, that σ can be written as a product of elementary transpositions $\sigma = \sigma_k \cdots \sigma_1$ such that, for all $i \in \{1, \dots, n\}$, σ_i is a $(\sigma_{i-1} \cdots \sigma_1).w$-elementary transposition.

Let $i \in \{1, \dots, n\}$. Consider the reduced words

$$w_1 = (\sigma_{i-1} \cdots \sigma_1).w \text{ and } w_2 = \sigma_i.w_1.$$

These two words only differ in two generators r_j and r_{j+1} which are switched by σ_i. Let $\gamma_1 = (x_0, \dots, x_n)$ and $\gamma_2 = (y_0, \dots, y_n)$ be the minimal galleries between $c_0 = x_0 = y_0$ and $c = x_n = y_n$ corresponding to the word w_1 and w_2 respectively, as in Figure 2.14.

We now prove that the colors $f_{r_k}(x_k)$ and $f_{r_k}(y_{\sigma_i(t)})$ agree for all $k \in \{0, \dots, n\}$. Note that the galleries γ_1 and γ_2 only differ in the chamber x_j, and that σ_i is a transposition switching j and $j+1$. Therefore we only need to check the cases $k = j$ and $k = j+1$. We have

$$x_{j-1} \overset{r_j}{\sim} x_j \overset{r_{j+1}}{\sim} x_{j+1} \text{ and } x_{j-1} = y_{j-1} \overset{r_{j+1}}{\sim} y_j \overset{r_j}{\sim} y_{j+1} = x_{j+1}.$$

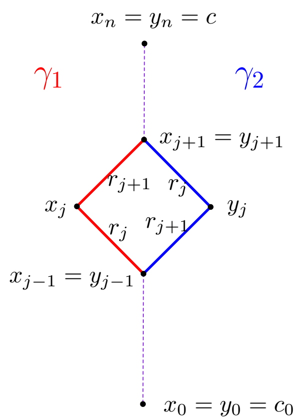


Figure 2.14: The galleries γ_1 and γ_2.

As $(f_s)_{s\in S}$ is a set of direct colorings with respect to c_0 we infer that

$$f_{r_j}(x_j) = f_{r_j}(y_{j+1}) = f_{r_j}(y_{\sigma_i(j)}) \text{ and}$$

$$f_{r_{j+1}}(x_{j+1}) = f_{r_{j+1}}(y_j) = f_{r_{j+1}}(y_{\sigma_i(j+1)}),$$

which implies that the colors indeed agree. From this we deduce that

$$\begin{pmatrix} & w_1 & \\ f_{r_1}(x_1) & \cdots & f_{r_n}(x_n) \end{pmatrix} \sim \begin{pmatrix} & w_2 & \\ f_{r_1}(y_1) & \cdots & f_{r_n}(y_n) \end{pmatrix},$$

which allows us to conclude that

$$\begin{pmatrix} s_1 & \cdots & s_n \\ f_{s_1}(c_1) & \cdots & f_s(c_n) \end{pmatrix} \sim \begin{pmatrix} s_{\sigma 1} & \cdots & s_{\sigma n} \\ f_{s_{\sigma 1}}(c'_1) & \cdots & f_{s_{\sigma n}}(c'_n) \end{pmatrix},$$

where $(c'_0, \ldots, c'_n)$ is the minimal gallery between c_0 and c corresponding to w'. This proves that ψ is independent of the choice of the reduced representation of w and, in particular, that it is well defined.

We claim that ψ is color-preserving, i.e., that $f_s(c) = F_s(\psi(c))$ for all $s \in S$ and all for $c \in \mathrm{Ch}(\Delta)$. Let c be a chamber of Δ at Weyl distance w from c_0, and let $s_1 \cdots s_n$ be a reduced word in M_S with respect to Σ representing the element w. By construction of ψ and definition of F_{s_n}, we have $f_{s_n}(c) = F_{s_n}(\psi(c))$.

Let $s \in S \setminus \{s_n\}$. If $l(s_1 \cdots s_n) < l(s_1 \cdots s_n s)$, then by definition of a directed coloring and of F_S we have $f_s(c) = 1$ and $F_s(\psi(c)) = 1$. If $l(s_1 \cdots s_n) > l(s_1 \cdots s_n s)$, then we could have picked a reduced representation $s_1 \cdots s_n$ of w where $s_n = s$, which reduces the problem to the case handled in the previous paragraph.

Next, we show that ψ is a bijection. It is clear from the definition that ψ is surjective. It remains to prove that it is injective. Let $c_1, c_2 \in \mathrm{Ch}(\Delta)$ be distinct chambers. If c_1 and c_2 are at distinct Weyl distances from c_0 then by definition $\psi(c_1) \neq \psi(c_2)$.

Assume in the other case that $s_1 \cdots s_n$ is a reduced representation of the Weyl distance from c_0 to both c_1 and c_2.

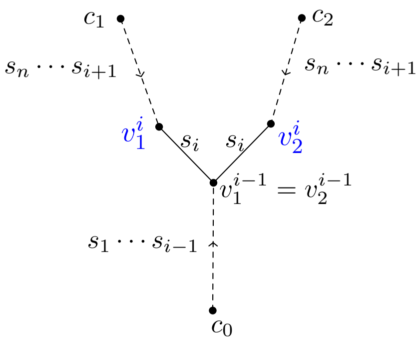


Figure 2.15: The chambers c_1 and c_2.

For $k \in \{1, \dots, n\}$ and for $j \in \{1, 2\}$, let v_j^k be the unique chamber in Δ at Weyl distance $s_1 \cdots s_k$ from c_0 and at Weyl distance $s_n \cdots s_{k+1}$ from c_j, as in Figure 2.15. Let i be the minimal number in $\{1, \dots, n\}$ such that $v_1^i \neq v_2^i$. Then $v_1^{i-1} = v_2^{i-1}$, which means that v_1^i and v_2^i are in the same i-panel of Δ. So $f_{s_i}(v_1^i) \neq f_{s_i}(v_2^i)$. This implies, by definition of the map ψ, that $\psi(c_1) \neq \psi(c_2)$. Therefore ψ is a bijection.

Finally, we show that ψ is a homomorphism. If c_1 and c_2 are s-adjacent chambers in Δ, for $s \in S$, then by Proposition 2.4.2 two distinct cases can happen:

1. either c_1 is at Weyl distance $s_1 \cdots s_n$ from c_0 and c_2 is at Weyl distance $s_1 \cdots s_n s$ from c_0, or vice-versa. Then by definition of ψ and point 1 of the definition of adjacency in Δ_D, the chamber

$\psi(c_1)$ is s-adjacent to $\psi(c_2)$.

2. or c_1 and c_2 are both at Weyl distance $s_1 \cdots s_n s$ from c_0 and $f_s(c_1) \neq f_s(c_2)$. Again we conclude that $\psi(c_1)$ is s-adjacent to $\psi(c_2)$, this time by the second point of the definition of adjacency in Δ_D.

This shows that ψ is an isomorphism from Δ to Δ_D respecting the set of colorings $(f_s)_{s\in S}$, and completes the proof of the proposition. $\square$

Definition 2.4.5. We call the building Δ_D the *directed right-angled building* of type (W, S) with prescribed thickness $(q_s)_{s\in S}$.

We will denote directed right-angled buildings by Δ dropping the D when it is clear from the context.

Remark 2.4.6. The directed legal coloring F_s is not a legal s-coloring unless s commutes with all elements of S (in which case every weak legal s-coloring is a legal s-coloring by Lemma 2.3.10).

To conclude this section (and the chapter), we construct automorphisms of a right-angled building that interchange the chambers in an s-tree-wall according to a prescribed permutation of the set Y_s. We will regard a right-angled building in a directed way in order to make use of the matrix description of its chambers.

Proposition 2.4.7. *Let Δ be the directed right-angled building with respect to a chamber c_0 of type (W, S) with prescribed thickness $(q_r)_{r\in S}$. Let $s \in S$ and fix an s-tree-wall $\mathcal{T}$ in Δ. Let g be a permutation of $Y_s \setminus \{1\}$.*

Consider the following map $g_{\mathcal{T}}$ on the set of chambers of Δ. Let c be a chamber represented by the matrix

$$\begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}.$$

If there is an $i \in \{1, \ldots, n\}$ such that $s_i = s$ and the chambers represented by

$$\begin{pmatrix} s_1 & \cdots & s_{i-1} \\ \alpha_1 & \cdots & \alpha_{i-1} \end{pmatrix} \text{ and } \begin{pmatrix} s_1 & \cdots & s_i \\ \alpha_1 & \cdots & \alpha_i \end{pmatrix} \text{ are in } \mathcal{T}, \tag{2.4.1}$$

then $g_{\mathcal{T}}$ maps c to the chamber represented by

$$\begin{pmatrix} s_1 & \cdots & s_{i-1} & s & s_{i+1} & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_{i-1} & g\alpha_i & \alpha_{i+1} & \cdots & \alpha_n \end{pmatrix}. \tag{2.4.2}$$

If there is no such i, then $g_{\mathcal{T}}$ fixes c.

Then the map $g_{\mathcal{T}}$ constructed in this way is an automorphism of Δ. Moreover, if g, h are two permutations of $Y_s \setminus \{1\}$, then we have $g_{\mathcal{T}} h_{\mathcal{T}} = (gh)_{\mathcal{T}}$.

Proof. Notice that there can be at most one index i with $s_i = s$ for which (2.4.1) holds, since $s_1 \cdots s_n$ is a reduced word and $\mathcal{T}$ is an s-tree-wall. Indeed, two chambers in $\mathcal{T}$ are at Weyl distance $w = r_1 \cdots r_t$ from each other, with $r_j \in s^{\perp}$ for all $j \in \{1, \dots, t\}$.

We start by showing that $g_{\mathcal{T}}$ is well defined, i.e., that our description of $g_{\mathcal{T}}$ is independent of the representation of the chamber c. Assume that c is represented as in the statement of the proposition. It suffices to look at equivalence by a single elementary transposition. Assume then that there is a $j \in \{1, \dots, n-1\}$ such that s_j and s_{j+1} commute. The only non-trivial cases are when condition (2.4.1) is satisfied and either $j = i$ or $j = i - 1$. Assume that $j = i$ (the other case can then be handled analogously); so $s_i = s$ commutes with s_{i+1}.

Note that the chambers represented by

$$\begin{pmatrix} s_1 & \cdots & s_{i-1} & s_{i+1} \\ \alpha_1 & \cdots & \alpha_{i-1} & \alpha_{i+1} \end{pmatrix} \text{ and } \begin{pmatrix} s_1 & \cdots & s_{i-1} & s_{i+1} & s_i \\ \alpha_1 & \cdots & \alpha_{i-1} & \alpha_{i+1} & \alpha_i \end{pmatrix}$$

are still contained in $\mathrm{Ch}(\mathcal{T})$ by Corollary 2.2.23. Hence the image under $g_{\mathcal{T}}$ of c using its representation obtained by applying the elementary transposition is therefore represented by

$$\begin{pmatrix} s_1 & \cdots & s_{i-1} & s_{i+1} & s_i & s_{i+2} & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_{i-1} & \alpha_{i+1} & \alpha_i g & \alpha_{i+2} & \cdots & \alpha_n \end{pmatrix},$$

which is an equivalent representation of the image $g_{\mathcal{T}} c$ as in Equation (2.4.2). This proves that the map $g_{\mathcal{T}}$ is indeed well defined.

In order to show that $g_{\mathcal{T}}$ is a homomorphism of Δ, we have to show that t-panels are mapped to t-panels, for all $t \in S$. However, this is now immediately clear from the description of a t-panel in Proposition 2.4.2.

Finally observe that each $g_{\mathcal{T}}$ is invertible, with inverse $(g^{-1})_{\mathcal{T}}$, which implies that $g_{\mathcal{T}}$ is an automorphism of Δ. The last statement is also clear by the way that the map was defined. $\square$

Definition 2.4.8. Let $s \in S$ and let $\mathcal{T}$ be a tree-wall of Δ. An element $g_{\mathcal{T}} \in \mathrm{Aut}(\Delta)$ as defined in Proposition 2.4.7 is called a *tree-wall automorphism*.

The tree-wall automorphisms will generate very useful subgroups of the universal group of a right-angled building whose support is contained in a single wing with respect a tree-wall. We will discuss these subgroups in the next chapter (see the notion of tree-wall group in Definition 4.2.4).

CHAPTER 3

Automorphism group of a right-angled building

In this chapter we study the group $\mathrm{Aut}(\Delta)$ of type-preserving automorphisms of a thick semi-regular right-angled building. We look at open subgroups of the automorphism group and show that any proper open subgroup is contained with finite index in the stabilizer in $\mathrm{Aut}(\Delta)$ of a proper residue, if we consider the building to be locally finite.

In the non-compact open case we will make use of groups that resemble root groups, defined in Section 3.3, and we show that an open subgroup of $\mathrm{Aut}(\Delta)$ contains sufficiently many of those groups.

When the open subgroup is compact we will show that compactness is equivalent to being locally X-elliptic on the action of the Davis realization X of the building (see the definition of the Davis realization in Section 1.4.4). This result can be deduced in the locally finite case from the fact that any non-compact open subgroup of $\mathrm{Aut}(\Delta)$ must have a hyperbolic element (Lemma 3.4.9) but we will show it in general by proving that the fixator in $\mathrm{Aut}(\Delta)$ of any ball in Δ acts on the building with a bounded fixed-point set (see Proposition 3.2.6). That result is of independent interest and the proof really highlights the beauty of the geometry of a right-angled building with all the squares associated to the respective commuting generators of the Coxeter group.

3.1 Sets of chambers closed under squares

We will start by describing a procedure in a right-angled building, called closing squares, and we define the square closure of a set of chambers. Then we describe the square closure of a ball in the building.

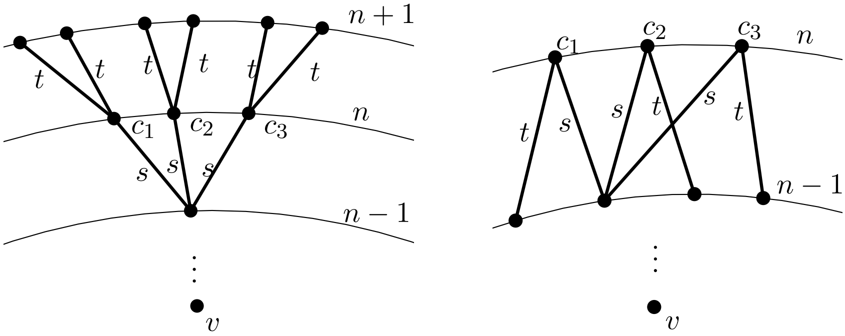
We will use the following notation throughout the section. Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ. Let $(q_s)_{s\in S}$ be a set of cardinal numbers with $q_s \geq 3$. Consider the thick semi-regular right-angled building Δ of type (W, S) and cardinality (q_s) on its s-panels, for every $s \in S$, which is unique by Theorem 2.3.2.

Definition 3.1.1. Let $n \in \mathbb{N}$ and let v be a fixed chamber of Δ.

1. For any $w \in W$, let
$$L(w) = \{s \in S \mid l(ws) < l(w)\}, \tag{3.1.1}$$
that is, the set of generators which added to a reduced representation for w do not form a new reduced word. Let $W(n)$ denote the set of elements $w \in W$ of length n. Define
$$W_1(n) = \{w \in W(n) \mid |L(w)| = 1\},$$
$$W_2(n) = \{w \in W(n) \mid |L(w)| \geq 2\}.$$

2. We will create a partition of the sphere $\mathsf{S}(v, n)$ by defining, for each $i \in \{1, 2\}$,
$$A_i(n) = \{c \in \mathsf{S}(v, n) \mid \delta(v, c) \in W_i(n)\}. \tag{3.1.2}$$
as in Figure 3.1.

3. Let $c \in \mathsf{S}(v, k)$ for some $k > n$. We say that c is of *type A_2 with respect to* $\mathsf{B}(v, n)$ if for each $d \in \mathsf{S}(v, n)$ such that $\mathsf{d}_W(v, c) = k - n$ and for each minimal gallery γ between d and c, all the chambers in γ are in $A_2(n + i)$, for $i \in \{1, \ldots, k\}$.



(a) $c_i \in A_1(n)$: for all $t \neq s$, $l(\delta(c_0, c_i)t) > l(\delta(c_0, c_i))$.

(b) $c_i \in A_2(n)$: for some $t \neq s$, $l(\delta(c_0, c_i)t) < l(\delta(c_0, c_i))$.

Figure 3.1: Partition of $\mathsf{S}(v, n)$.

4. We define the set A_2 *with respect to* $\mathsf{B}(v, n)$, sometimes denoted by $A_2(\mathsf{B}(v, n))$, to be the set of all chambers of type A_2 with respect to $\mathsf{B}(v, n)$.

We observe that if two chambers c_1 and c_2 in $\mathsf{S}(v, n)$ are s-adjacent for some $s \in S$, then there is a unique chamber c in their s-panel that is in $\mathsf{S}(v, n-1)$. Therefore $\delta(v, c_1) = \delta(v, c)s = \delta(v, c_2)$ and hence c_1 and c_2 are in the same part $A_i(n)$. Thus $A_1(n)$ and $A_2(n)$ are mutually disconnected parts of $\mathsf{S}(v, n)$.

Definition 3.1.2. 1. We say that a subset $T \subseteq W$ is *closed under squares* if the following holds:

if ws_i and ws_j are in T for some $w \in T$ with $|s_i s_j| = 2$, $s_i \neq s_j$ and $l(ws_i) = l(ws_j) = l(w) + 1$ then $ws_i s_j = ws_j s_i$ is an element of T.

2. Let v be a fixed chamber of Δ. We say that a set of chambers $\mathscr{C} \subseteq \mathrm{Ch}(\Delta)$ is *closed under squares* with respect to v if for any $n \in \mathbb{N}$ the following holds:

if for any two distinct chambers $c_1, c_2 \in \mathscr{C} \cap \mathsf{S}(v, n)$ such that there exist $c_3 \in \mathscr{C} \cap \mathsf{S}(v, n-1)$ with $c_3 \overset{s_i}{\sim} c_1$ and $c_3 \overset{s_j}{\sim} c_2$ for some $|s_is_j| = 2$ with $s_i \neq s_j$ then the chamber $c_4 \in \mathsf{S}(v, n+1)$ such that $c_4 \overset{s_j}{\sim} c_1$ and $c_4 \overset{s_i}{\sim} c_2$ is also in the set $\mathscr{C}$.

In particular, if $\mathscr{C}$ is closed under squares with respect to v, then the set of Weyl distances $\{\delta(v, c) \mid c \in \mathscr{C}\} \subseteq W$ is closed under squares.

3. Let $v \in \mathrm{Ch}(\Delta)$. For a subset $\mathscr{C} \subseteq \mathrm{Ch}(\Delta)$, we define the *square closure* $\overline{\mathscr{C}}$ of $\mathscr{C}$ with respect to v to be the smallest subset of $\mathrm{Ch}(\Delta)$ closed under squares with respect to v such that $\mathscr{C} \subseteq \overline{\mathscr{C}}$.

Lemma 3.1.3. *The set* $\mathsf{B}(v, n) \cup A_2(\mathsf{B}(v, n))$ *is closed under squares with respect to* v.

Proof. Let $\mathscr{C}$ denote $\mathsf{B}(v, n) \cup A_2(\mathsf{B}(v, n))$. Let c_1 and c_2 be chambers in $\mathscr{C}$ at Weyl distance ws_i and ws_j from v, respectively, such that $|s_is_j| = 2$ and $l(ws_i) = l(ws_j) = l(w) + 1$. Let c_3 be the chamber at Weyl distance ws_is_j from v that is s_j-adjacent to c_1 and s_i-adjacent to c_2. We want to prove that c_3 is an element of the set $\mathscr{C}$. If $l(ws_is_j) \leq n$ then it is clear that $c_3 \in \mathscr{C}$ as $\mathsf{B}(v, n) \subseteq \mathscr{C}$. Assume that $l(ws_is_j) > n$.

Let d be a chamber in $\mathsf{S}(v, n)$ at minimal distance from c_3 (with respect to the other chambers in $\mathsf{S}(v, n)$) and take $\gamma = (v_0, \ldots, v_k)$ to be a minimal gallery between d and c_3 as in Figure 3.2. It is clear that $c_3 \in A_2(l(w) + 2)$ for some $m \in \mathbb{N}$. Now we have to prove that each $v_i \in A_2(\ell)$ for some ℓ.

If $v_{k-1} \in \{c_1, c_2\}$ then the result follows because $c_1, c_2 \in \mathscr{C}$ and $(v_0, \ldots, v_{k-1})$ is a minimal gallery between d and c_1 (or c_2). Assume that v_{k-1} is distinct from c_1 and c_2 and is s_{k-1}-adjacent to c_3. Then using closing squares (Lemma 2.2.6), we obtain that $|s_js_{k-1}| = 2$ and there is a chamber $c^j_{k-1} \in \mathsf{S}(v, l(w))$ that closes a square, that is, such that $c^j_{k-1} \overset{s_j}{\sim} v_{k-1}$ and $c^j_{k-1} \overset{s_{k-1}}{\sim} c_1$. Analogously, there is a chamber $c^i_{k-1} \in \mathsf{S}(v, l(w))$ such that $c^i_{k-1} \overset{s_i}{\sim} v_{k-1}$ and $c^i_{k-1} \overset{s_{k-1}}{\sim} c_2$. Hence $v_{k-1} \in A_2(l(ws_i))$.

Continuing this argument inductively (see Figure 3.2), we conclude that all the chambers in γ are in $A_2(\ell)$ for some ℓ. Hence $c_3 \in A_2$ with respect to $\mathsf{B}(v, n)$. Thus $\mathscr{C} = \mathsf{B}(v, n) \cup A_2(\mathsf{B}(v, n))$ is closed under squares with respect to v. $\square$

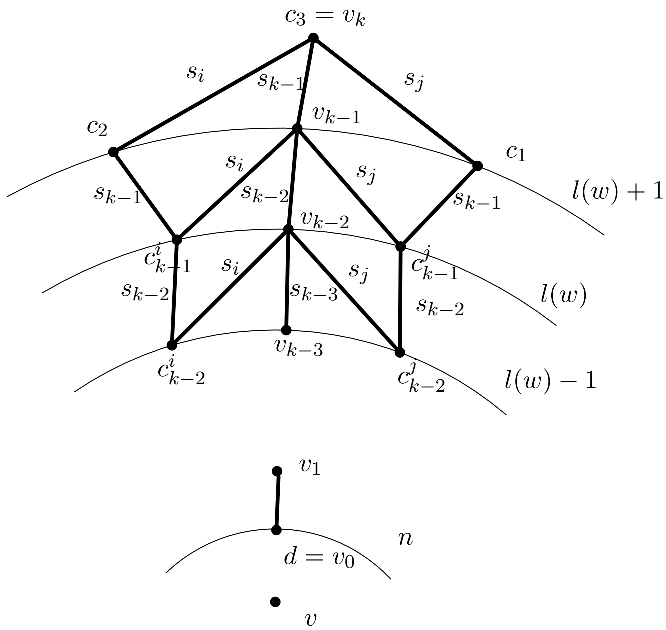


Figure 3.2: Proof of Lemma 3.1.3.

Lemma 3.1.4. *Let* $v \in \mathrm{Ch}(\Delta)$ *and* $n \in \mathbb{N}$. *The square closure of the ball* $\mathsf{B}(v,n)$ *with respect to* v *is* $\mathsf{B}(v,n) \cup A_2(\mathsf{B}(v,n))$.

Proof. Let $\mathscr{C}$ denote $\mathsf{B}(v,n) \cup A_2(\mathsf{B}(v,n))$. The set $\mathscr{C}$ is closed under squares by Lemma 3.1.3.

Let $\mathscr{C}'$ be a set of chambers closed under squares that contains $\mathsf{B}(v,n)$. We have to prove that $\mathscr{C} \subseteq \mathscr{C}'$.

We will show that

$$\mathsf{S}(v,k) \cap A_2(\mathsf{B}(v,n)) \subseteq \mathsf{S}(v,k) \cap \mathscr{C}' \text{ for all } k > n. \tag{3.1.3}$$

If $k = n+1$ then it follows by definition of closing squares. Assume by induction hypothesis that Equation (3.1.3) holds for every N such that $n+1 \leq N \leq k$.

Let $c \in \mathsf{S}(v,k+1) \cap A_2(\mathsf{B}(v,n))$. Let $c_1, c_2 \in \mathsf{S}(v,k)$ such that

$$c_1 \overset{s_1}{\sim} c \text{ and } c_2 \overset{s_2}{\sim} c.$$

By Lemma 2.2.6 we have that $|s_1s_2| = 2$ and there is $d \in \mathsf{S}(v, k-1)$ such that

$$d \overset{s_2}{\sim} c_1 \text{ and } d \overset{s_1}{\sim} c_2.$$

For $i \in \{1,2\}$, for any minimal gallery $\gamma = (v_1, \dots, v_k)$ of length k between c_i and a chamber in $\mathsf{S}(v,n)$, we have that $(c,\gamma) = (c, v_1, \dots, v_k)$ is a minimal gallery of length $k+1$ between c and $\mathsf{S}(v,n)$. Hence all the chambers in γ are in $A_2(\ell)$ for some ℓ because $c \in A_2(\mathsf{B}(v,n))$.

Thus c_1 and c_2 are in $A_2(\mathsf{B}(v,n)) \cap \mathsf{S}(v,k)$. Therefore, using the induction hypothesis, $c_1, c_2 \in \mathscr{C}'$. With a similar reasoning we obtain that $d \in \mathscr{C}'$. As $\mathscr{C}'$ is closed under squares, we obtain that $c \in \mathscr{C}'$. Therefore $\mathsf{B}(v,n) \cup A_2(\mathsf{B}(v,n))$ is the square closure of $\mathsf{B}(v,n)$ with respect to v. □

3.2 The action of the fixator of a ball in Δ

In this section we study the action of the fixator K in $\mathrm{Aut}(\Delta)$ of a ball $\mathsf{B}(v,n)$ of radius n around a chamber v. The final goal will be to prove that the fixed point set Δ^K is bounded for any $n \in \mathbb{N}$.

We will prove that Δ^K coincides with the square closure of the ball with respect to v and we will have a closer look at this square closure using firm words in right-angled Coxeter groups, as defined in Section 2.1.2, to show that this set indeed is bounded.

We start by making a remark on the local action of $\mathrm{Aut}(\Delta)$ on panels of the building.

Remark 3.2.1. Let $s \in S$ and let $\mathcal{P}$ be an s-panel. We define the *local action* of $\mathrm{Aut}(\Delta)$ on $\mathcal{P}$ to be the induced action $\mathrm{Aut}(\Delta)|_{\mathcal{P}}$ of the group $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{P})$ on the panel $\mathcal{P}$, which is isomorphic to $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{P}) / \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathcal{P})$.

For any s-panel $\mathcal{P}$ we have that $\mathrm{Aut}(\Delta)|_{\mathcal{P}} \cong \mathrm{Sym}(q_s)$. Since $q_s \geq 3$ this induced action is 2-transitive and therefore is generated by any two point stabilizers in $\mathrm{Sym}(q_s)$, which are distinct because $\mathrm{Sym}(q_s)$ is primitive.

Theorem 3.2.2. *Let v be a fixed chamber of Δ and $n \in \mathbb{N}$. Consider the pointwise stabilizer $K = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n))$ in $\mathrm{Aut}(\Delta)$ of the ball $\mathsf{B}(v,n)$.*

Then the fixed-point set Δ^K *is the square closure of* $\mathsf{B}(v,n)$ *with respect to* v*, i.e., the set* $\mathsf{B}(v,n) \cup A_2(\mathsf{B}(v,n))$*.*

Proof. By definition $\mathsf{B}(v,n) \subseteq \Delta^K$. We will prove that $A_2(\mathsf{B}(v,n)) \subseteq \Delta^K$ and that any chamber in $(\mathrm{Ch}(\Delta)\backslash\mathsf{B}(v,n)) \cap \Delta^K$ is in $A_2(\mathsf{B}(v,n))$.

Let $c \in \mathrm{Ch}(\Delta)\backslash\mathsf{B}(v,n)$ and let $N = \mathrm{dist}(c, \mathsf{B}(v,n))$. We will show both inclusions by induction on N.

Suppose first that c is of type A_2. If $d = 1$ then $c \overset{s_1}{\sim} c_1$ and $c \overset{s_2}{\sim} c_2$ with $c_1, c_2 \in \mathsf{S}(v,n)$. We know that K fixes c_1 and c_2. If there were another chamber c' s_1-adjacent to c_1 and s_2-adjacent to c_2 then c and c' would be both s_1- and s_2-adjacent, which is impossible. Therefore c is the only chamber s_1-adjacent to c_1 and s_2-adjacent to c_2. As K fixes c_1 and c_2, it implies that K must fix c.

Assume by induction hypothesis that if $N \leq k$ then it holds that if c is of type A_2 then K fixes c. Let c be a chamber of type A_2 at distance $k+1$ from $\mathsf{B}(v,n)$. By definition of square closure, we have $c \overset{s_1}{\sim} c_1$ and $c \overset{s_2}{\sim} c_2$ with $|s_1 s_2| = 2$ and $c_1, c_2 \in A_2(k)$. By induction hypothesis, the group K fixes c_1 and c_2. Thus K fixes the chamber c, using an analogous argument as in the case $N = 1$.

We prove now the other inclusion, also by induction on N. Assume that c is not of type A_2. In this case we show that there is $g \in K$ such that $gc \neq c$. Suppose $N = 1$. Let $c \in A_1(n+1)$ and let e be the unique chamber in $\mathsf{S}(v,n)$ such that $e \overset{s}{\sim} c$. We claim that $\mathsf{B}(v,n) \subseteq X_s(e)$.

Suppose that there exists $d \in \mathsf{B}(v,n)$ such that $\mathrm{proj}_{\mathcal{P}_{s,e}}(d) = c' \in \mathsf{S}(v,n+1)$. Then, by Lemma 2.2.12, there is $t \in S\backslash\{s\}$ such that $|ts| = 2$ and $d' \in \mathsf{S}(v,n)$ such $c' \overset{t}{\sim} d' \in \mathsf{S}(v,n)$. Using Lemma 2.2.7 we find $d'' \in \mathsf{S}(v,n)$ such that $c \overset{t}{\sim} d''$ which is a contradiction to the fact that c is not of type A_2. Hence $\mathsf{B}(v,n) \subseteq X_s(e)$.

Take $g \in \mathrm{Aut}(\Delta)|_{\mathcal{P}_{s,e}}$ fixing the chamber of e and mapping c to a chamber $c' \in \mathrm{Ch}(\mathcal{P})\backslash\{e,c\}$ and extend it to an element $g \in \mathrm{Aut}(\Delta)$ fixing $X_s(e)$ by Proposition 2.2.13. Then $g \in K$ and g does not fix the chamber c. Observe that here we use the fact that Δ is thick and that the induced action $\mathrm{Aut}(\Delta)|_{\mathcal{P}_{s,e}}$ is permutationally isomorphic to $\mathrm{Sym}(q_s)$ by Remark 3.2.1. Moreover, for $\alpha \in Y_s$, the group $\mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$ only fixes α. Therefore, if $N = 1$ then the result holds.

Consider now a chamber $c \in \mathsf{S}(v, n+N+1)$ not of type A_2 with respect to $\mathsf{B}(v,n)$. Then two cases can occur.

1. $c \in A_1(n+N+1)$. In this case we use a similar argument as above because $\mathsf{B}(v,n) \subseteq X_s(e)$, where e is the unique chamber in $\mathsf{S}(v, n+N)$ adjacent to c.

2. $c \in A_2(n+N+1)$ but there exists a minimal gallery $\gamma = (v_0, \dots, v_{k+1})$ between $c = v_0$ and a chamber $e = v_{k+1} \in \mathsf{S}(v,n)$ such that $v_i \in A_1(i)$ for some $i \in \{1, \dots, k\}$.

   Let $j \in \{1, \dots, k\}$ such that $v_j \in A_1(j)$ with j minimal. Observe that the gallery $(v_1, \dots, v_j, \dots, v_{k+1})$ is a minimal gallery of length k between v_1 and e. Therefore $v_1 \in \mathsf{S}(v, n+N)$ is not of type A_2 with respect to $\mathsf{B}(v,n)$. Hence, by the induction hypothesis, there is $g \in K$ such that $gv_1 \neq v_1$. Moreover, $\mathrm{dist}(gv_1, v) = n+N$ and v_1 is the only chamber in the s-panel $\mathcal{P}$ of c in $\mathsf{S}(v, n+N)$ (assuming $v_1 \overset{s}{\sim} c$). So $\mathcal{P}$ is not stabilized by g. Thus $g \in K$ does not fix c.

This finishes the proof that $(\mathrm{Ch}(\Delta)\backslash\mathsf{B}(v,n)) \cap \Delta^K \subseteq A_2(\mathsf{B}(v,n))$. Therefore the fixed-point set Δ^K coincides with $\mathsf{B}(v,n) \cup A_2(\mathsf{B}(v,n))$. $\square$

Next we will analyze the set $A_2(\mathsf{B}(v,n))$ with the goal of proving that, with respect to any ball, this set is bounded. As a consequence, denoting $K = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n))$, we obtain that Δ^K is bounded for any $n \in \mathbb{N}$.

We start by clarifying the connection between firm reduced words, introduced in Definition 2.1.12, and the study of the set A_2 with respect to a ball around a fixed chamber.

Lemma 3.2.3. *Let v be a fixed chamber in Δ. Let $s_1 \cdots s_k$ be a reduced word in the monoid M_S with respect to Σ.*

The chambers at Weyl distance $s_1 \cdots s_k$ from v are in the set $A_1(k)$ if and only if $s_1 \cdots s_k$ is firm.

Proof. Let c be a chamber at Weyl distance $s_1 \cdots s_k$ from v.

If $s_1 \cdots s_k$ is firm then the only chamber in $\mathsf{S}(v, k-1)$ adjacent to c is the chamber at Weyl distance $s_1 \cdots s_{k-1}$ from v that is s_k-adjacent to c. Hence $c \in A_1(k)$.

Conversely, if $c \in A_2(k)$ then there exist $r_i, r_j \in S$ with $|r_i r_j| = 2$ and such that

$$s_1 \cdots s_k \sim w' r_i r_j \sim w' r_j r_i.$$

This description implies that $s_1 \cdots s_k$ is not firm. □

We recall the concept of $F^{\#}(w)$ in Definition 2.1.12, as the size of a largest firm prefix of w.

Lemma 3.2.4. *Let $v \in \mathrm{Ch}(\Delta)$ be a fixed chamber. If $c \in A_2(\mathsf{B}(v, n))$ then we have that $F^{\#}(\delta(v, c)) \leq n$.*

Proof. Let $c \in \mathrm{Ch}(\Delta) \backslash \mathsf{B}(v, n)$ such that $F^{\#}(\delta(v, c)) > n$. Assume that $\delta(v, c)$ can be written as

$$s_1 \cdots s_n \cdots s_k t_{k+1} \cdots t_\ell, \text{ with } s_1 \cdots s_n \cdots s_k \text{ firm.}$$

Let $d \in \mathsf{S}(v, n)$ be the chamber at Weyl distance $s_1 \cdots s_n$ from v that is at distance $t_\ell \cdots t_{k+1} s_k \cdots s_{n+1}$ from c. Moreover let d' be the chamber at Weyl distance $s_1 \cdots s_n \cdots s_k$ from v that is at Weyl distance $t_\ell t_{\ell-1} \cdots t_{k+1}$ from c.

Then d' is a chamber in a minimal gallery between d and c, with $d \in \mathsf{S}(v, n)$ at minimal distance to c among the chambers in $\mathsf{S}(v, n)$. As $d' \in A_1(k)$ by Lemma 3.2.3, we obtain that $c \notin A_2(\mathsf{B}(v, n))$. □

Proposition 3.2.5. *The set A_2 with respect to $\mathsf{B}(v, n)$ is bounded for any $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and v be a fixed chamber of Δ. By Lemma 2.1.24, there is $d(n)$ such that $F^{\#}(w) > n$ for all reduced words w of length $d(n)$. Therefore, using Lemma 3.2.4, we then obtain that chambers at distance $d(n)$ from v are not in $A_2(\mathsf{B}(v, n))$. Hence $A_2(\mathsf{B}(v, n)) \subset \mathsf{B}(v, d(n))$. □

The next Proposition, and the goal of the section, now follows directly from Theorem 3.2.2 and Proposition 3.2.5.

Proposition 3.2.6. *Let v be a fixed chamber of Δ and $n \in \mathbb{N}$. Let $K = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v, n))$. Then the fixed-point set Δ^K is bounded.*

3.3 Root wing groups

In this section we define groups that resemble root groups, using the partition of the chambers of a right-angled building by wings. If we consider an apartment A in the building, then the partition of the chambers of A according to the wings corresponds to half-spaces, i.e., roots.

The root wing groups will fix pointwise walls of a fundamental apartment. In the next section we will prove that a non-compact open subgroup of the automorphism group contains sufficiently many root wing groups and that will be the key to prove that such a subgroup is contained in the stabilizer of a residue.

After the definition of a root wing group we prove, for a ball $\mathsf{B}(c_0, n)$ around a fixed chamber c_0, that the root wing groups corresponding to roots far enough away from $\mathsf{B}(c_0, n)$ are contained in the fixator of $\mathsf{B}(c_0, n)$ in the automorphism group.

Moreover, using the fact that $\mathrm{Aut}(\Delta)$ acts strongly transitively on the chambers of Δ, and hence $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ acts transitively on the set of apartments of Δ containing c_0, we can also relate the fixator of a ball with root wing groups by first considering roots in another apartment than the fundamental one.

Fix a chamber $c_0 \in \mathrm{Ch}(\Delta)$ and an apartment A_0 containing c_0 (which can be considered as the fundamental chamber and the fundamental apartment). Let Φ denote the set of roots of A_0. For any other apartment A containing c_0 we will denote its set of roots by Φ_A.

Definition 3.3.1. For $\alpha \in \Phi$ of type s, let $c \in \alpha$ whose s-panel $\mathcal{P}$ is in $\partial\alpha$. We define the *root wing group* U_α as

$$U_\alpha = U_s(c) = \{g \in \mathrm{Aut}(\Delta) \mid gd = d \text{ for all } d \in X_s(c)\}.$$

Let c' be the unique chamber in $\mathrm{Ch}(\mathcal{P})\backslash\{c\}$ that lies in the apartment A_0. We define $U_{-\alpha} = U_s(c')$.

Observe that U_α (and $U_{-\alpha}$) does not depend of the choice of the chamber c as panels in the wall $\partial\alpha$ are parallel. Therefore the partition determined by the s-wings of these panels is the same.

Next we present a property similar to FPRS property introduced in [CR09] for the setting of right-angled buildings. It is the analogous statement of Lemma 3.8 in [CM13], but in the right-angled case we can determine the constant.

Lemma 3.3.2. *For every root $\alpha \in \Phi$ with* $\mathrm{dist}(c_0, \alpha) > r$ *the group $U_{-\alpha}$ is contained in $K_r = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(c_0, r))$.*

Proof. Let α be a root at distance $r+1$ from c_0 and let s be the type of the panels in the wall $\partial\alpha$. Let $c = \mathrm{proj}_\alpha(c_0)$ and let c' be the other chamber in $\mathcal{P}_{c,s} \cap A_0$. Then $c' \in \mathsf{S}(c_0, r)$ since $\mathrm{dist}(c_0, c) = r + 1$.

We claim that for every $d \in \mathsf{B}(c_0, r)$ we have $\mathrm{proj}_{\mathcal{P}_{c,s}}(d) = c'$. Assume that there is $d \in \mathsf{B}(c_0, r)$ such that $\mathrm{proj}_{\mathcal{P}_{c,s}}(d) = c'' \neq c'$. Without loss of generality, take $d \in \mathsf{S}(c_0, r)$. Then, using closing squares (Lemma 2.2.6), we find $e \in \mathsf{S}(c_0, r-1)$ such that

$$e \overset{t}{\sim} c' \text{ and } e \overset{s}{\sim} d \text{ with } |st| = 2.$$

However, in this case, the s-panel $\mathcal{P}'$ of e is parallel to $\mathcal{P}_{c,s}$ and thus it belongs to the s-tree-wall of $\mathcal{P}_{c,s}$. Hence there exists $e' \in \mathcal{P}' \cap A_0$ such that $e' \in \alpha$ and we have $\mathrm{dist}(c_0, e') < \mathrm{dist}(c_0, c)$, which contradicts the fact that $\mathrm{dist}(c_0, \alpha) = r+1$. Thus $\mathsf{B}(c_0, r) \subseteq X_s(c')$ which implies that $U_{-\alpha} = U_s(c')$ is contained in K_r. □

The next set of results relate the apartment A_0 with the other apartments containing the chamber c_0. The proofs rely on the constant of Lemma 3.3.2 and follow exactly the same reasoning as in the proofs of Lemmas 3.9 and 3.10, respectively, in [CM13], so we present them without proof.

Lemma 3.3.3. *Let $g \in \mathrm{Aut}(\Delta)$ and let $A \in \mathcal{A}_{\geq c_0}$ containing the chamber $d = gc_0$. Let $b \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ such that $A = bA_0$, and let $\alpha = b\alpha_0$ be a root of A with $\alpha_0 \in \Phi$.*

If $\mathrm{dist}(d, -\alpha) > r$ *then $bU_{\alpha_0}b^{-1} \subseteq gK_rg^{-1}$.*

Proof. Analogous to the proof of [CM13, Lemma 3.9]. □

The next lemma will be very useful in Section 3.4.2 when we describe hyperbolic isometries through reflections along essential walls. We first define those.

Definition 3.3.4. Let $w \in W$.

1. A root $\alpha \in \Phi$ is called *w-essential* if either $w^n\alpha \subsetneq \alpha$ or $w^{-n}\alpha \subsetneq \alpha$, for some $n \in \mathbb{N}$.

2. A wall is called *w-essential* if it bounds a w-essential root. We denote by $Ess(w)$ the set of w-essential walls. $Ess(w)$ is empty if w is of finite order.

Lemma 3.3.5. *Let $A \in \mathcal{A}_{\geq c_0}$ and let $b \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ such that $A = bA_0$. Also, let $\alpha = b\alpha_0$ (with $\alpha_0 \in \Phi$) be a w-essential root for some $w \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A)/\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(A)$. Let $g \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A)$ be a representative of w.*

Then there exists $n \in \mathbb{Z}$ such that for $\epsilon \in \{+, -\}$ we have $U_{\epsilon\alpha_0} \subseteq b^{-1}g^{\epsilon n}K_r g^{-\epsilon n}b$.

Proof. Similar to the arguments used in [CM13, Lemma 3.10]. □

3.4 Open subgroups of Aut(Δ)

We now focus on open subgroups of the automorphism group of Δ. The main result of this section is that any proper open subgroup of the automorphism group of a locally finite thick semi-regular right-angled building Δ is contained with finite index in the stabilizer in $\mathrm{Aut}(\Delta)$ of a proper residue of Δ.

We will split the proof in the cases where the open subgroup is compact and non-compact. Before we reach the main result of the section, which is Theorem 3.4.19, we study these two cases separately.

3.4.1 The compact case

Using the work developed in Section 3.2 we can prove that an open subgroup of $\mathrm{Aut}(\Delta)$ which is locally X-elliptic on the Davis realization X of the building must be compact.

Definition 3.4.1. A group acting continuously on a space X is called *locally X-elliptic* if every compactly generated subgroup of $\mathrm{Aut}(\Delta)$ fixes a point in X.

Lemma 3.4.2. *Let Δ be a thick irreducible semi-regular right-angled building, not necessarily locally finite. Let H be an open subgroup of*

$\mathrm{Aut}(\Delta)$ *and let* X *denote the Davis realization of* Δ*. If* H *is locally* X*-elliptic then* H *is contained with finite index in the stabilizer of a spherical residue of* Δ *and so it is compact.*

Proof. If H is locally X-elliptic then H has a global fixed point on X or H fixes an end of X (see [CL10]). Assume first that H has a global fixed point on X. Then by definition, the group H is contained in the stabilizer of a spherical residue, which is a compact subgroup of $\mathrm{Aut}(\Delta)$. As H is open, the containment is with finite index.

Now assume that H fixes an end of X. As H is open, we have $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n)) \subseteq H$ for some $v \in \mathrm{Ch}(\Delta)$ and some $n \in \mathbb{N}$. Moreover, for each $h \in H$, the group $H_h = \langle h, \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n))\rangle$ is open and compactly generated. Therefore H_h has a global fixed point, that is, $X^{H_h} \neq \emptyset$, for each h because H is locally X-elliptic by assumption.

Hence $H = \cup H_h$ with H_h open and compactly generated and we can take this union to be countable because we are dealing with second-countable groups. We want to prove that $\cap X^{H_h} \neq \emptyset$. Observe that, for each $h \in H$, we have $X^{H_h} \subseteq X^{\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n))}$.

The fixed-point set $X^{\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(v,n))}$ is bounded for any $v \in \mathrm{Ch}(\Delta)$ and $n \in \mathbb{N}$ by Proposition 3.2.6. The countable intersection of compact bounded sets is non-empty. Therefore H has a global fixed point in its action on X, for which we already proved the claim in the previous paragraph. Hence H is compact in both cases. □

Corollary 3.4.3. *For open subgroups of the automorphism group of a thick irreducible semi-regular right-angled building, compactness and local* X*-ellipticity on the Davis realization of the building are equivalent.*

Proof. Observe that if H is a compact open subgroup of $\mathrm{Aut}(\Delta)$ then it is contained in the stabilizer of a spherical residue of Δ, which is a maximal compact open subgroup of $\mathrm{Aut}(\Delta)$. Hence, by definition, H fixes a point in X, meaning that it is X-locally elliptic. □

3.4.2 The non-compact case

Let Δ be a thick, semi-regular, irreducible, locally finite right-angled building. Consider an open subgroup H of $\mathrm{Aut}(\Delta)$. Assume that H is non-compact.

The goal of this section is to prove that H is contained in the stabilizer of a residue of the building.

We fix a fundamental chamber c_0 and an apartment A_0 containing c_0 which will be regarded as the fundamental apartment and can be identified with W as a Coxeter chamber system. We denote by Φ the set of roots of A_0. Given a root $\alpha \in \Phi$, let $r_\alpha = r_{\partial\alpha}$ denote the unique reflection of W setwise stabilizing the panels in $\partial\alpha$ and recall the definition of the root wing group U_α in 3.3.1.

Remark 3.4.4. Consider $a \in \{1, \dots, q_s\}$. For any $g \in \mathrm{Stab}_{\mathrm{Sym}(q_s)}(a)$ we can consider the respective permutation g_a of the chambers of $\mathcal{P}$ which fixes a chamber c_a (we can consider a coloring in Δ to see this better). By Lemma 2.2.13, we can extend g_a to an element $\widetilde{g_a}$ fixing $X_s(c_a)$ and therefore $\widetilde{g_a} \in U_s(c_a)$.

Therefore $U_s(c_a)|_{\mathcal{P}} \cong \mathrm{Stab}_{\mathrm{Sym}(q_s)}(a)$. Moreover, for any two distinct elements $a, b \in \{1, \dots q_s\}$ we have $\langle U_s(c_a), U_s(c_b)\rangle|_{\mathcal{P}} \cong \mathrm{Sym}(q_s)$, which is also isomorphic to $\mathrm{Aut}(\Delta)|_{\mathcal{P}}$ by Remark 3.2.1.

Let $\alpha \in \Phi$. Using the remark above, we can choose an element $n_\alpha \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A_0) \cap \langle U_\alpha \cup U_{-\alpha}\rangle$ which is mapped onto r_α under the quotient map

$$\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A_0) \to \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(A_0)/\,\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(A_0) \cong W.$$

Let $J \subseteq S$ be minimal amongst the subsets $L \subseteq S$ such that there is $g \in \mathrm{Aut}(\Delta)$ such that $H \cap g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{L,c_0})g$ has finite index in H.

For such a g, we set $H_1 = gHg^{-1} \cap \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$. Thus H_1 stabilizes $\mathcal{R}_{J,c_0}$ and it is an open subgroup of $\mathrm{Aut}(\Delta)$ contained in gHg^{-1} with finite index. Moreover, since H is non-compact, so is H_1.

The idea will be to prove that H_1 contains a hyperbolic element h_{A_1} for which the chamber c_0 belongs to the set of chambers at minimal displacement. Moreover, we can find the element h_{A_1} in the stabilizer in H_1 of an apartment A_1 containing c_0. Thus we can identify it with an element $\overline{h_{A_1}}$ of W and consider its parabolic closure (see Definition 1.4.9). The key point will be to prove that the type of $Pc(\overline{h_{A_1}})$ is J which will be achieved in Lemma 3.4.13.

We will at the same time conclude that H_1 acts transitively on the chambers of $\mathcal{R}_{J,c_0}$ and this will allow us to conclude that any open subgroup of $\mathrm{Aut}(\Delta)$ containing H_1 as a finite index subgroup is contained in the stabilizer of $\mathcal{R}_{J\cup J',c_0}$ with J' a spherical subset of $J^\perp$ (Lemma 3.4.15).

This strategy is analogous to the construction of the proof of [CM13, Lemma 3.19]. As the arguments are of geometric nature, we will adapt them to our setting. The root groups associated with the Kac-Moody group in that paper can be replaced by the root wing groups defined in the previous section and in our setting we have that $\mathrm{Aut}(\Delta)$ is strongly transitive on its action on Δ ([Cap14, Proposition 6.1]). When a proof goes through exactly as a result in [CM13] we will point a precise reference and only present the main ideas behind the proof and the adaptation to our setting.

We start by showing that we are choosing a good set J to start with.

Definition 3.4.5. A subset $T \subseteq S$ is called *essential* if each irreducible component of T is non-spherical (see the notion of spherical set in Definition 1.4.8).

Lemma 3.4.6 ([CM13, Lemma 3.4])**.** *The set J is essential.*

Let $\mathcal{A}_{\geq c_0}$ be the set of apartments of Δ containing c_0. For $A \in \mathcal{A}_{\geq c_0}$ we denote $N_A = \mathrm{Stab}_{H_1}(A)$ and $\overline{N_A} = N_A/\,\mathrm{Fix}_{H_1}(A)$, which we identify with a subgroup of W. For $h \in N_A$, let $\overline{h}$ denote its image in $\overline{N_A} \leq W$.

Lemma 3.4.7 ([CM13, Lemma 3.5])**.** *For all $A \in \mathcal{A}_{\geq c_0}$, there exists $h \in N_A$ such that*

$$Pc(\overline{h}) = \langle r_\alpha \mid \alpha \textit{ is an } \overline{h}\textit{-essential root of } \Phi\rangle$$

and is of finite index in $Pc(\overline{N_A})$.

Proof. Recall the concept of essential root in Definition 3.3.1. This result is a consequence of the work in [CM13] done for general Coxeter groups, namely Corollary 2.17 and Lemma 2.7. $\square$

Lemma 3.4.8 ([CM13, Lemma 3.6])**.** *Let $(g_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of H_1. Then there is an apartment $A \in \mathcal{A}_{\geq c_0}$, a* *subsequence* $(g_{\psi(n)})_{n\in\mathbb{N}}$ *and elements* $z_n \in H_1$, *with* $n \in \mathbb{N}$, *such that for all* $n \in \mathbb{N}$ *we have*

1. $h_n = z_0^{-1} z_n \in N_A$,
2. $\mathrm{dist}(c_0, z_n\mathcal{R}) = \mathrm{dist}(c_0, g_{\psi(n)}\mathcal{R})$ *for every residue* $\mathcal{R}$ *containing* c_0 *and*
3. $|\mathrm{dist}(c_0, h_n c_0) - \mathrm{dist}(c_0, g_{\psi(n)} c_0)| < \mathrm{dist}(c_0, z_n c_0)$.

Proof. Since H_1 is open, it has a subgroup $K = \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(c_0, r))$. The group K has finite index in $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ because Δ is locally finite. On the other hand, as $\mathrm{Aut}(\Delta)$ is strongly transitive, the group $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ is transitive on $\mathcal{A}_{\geq c_0}$. Hence K has finite orbits in $\mathcal{A}_{\geq c_0}$. The proof now follows exactly in the same lines as in the proof of [CM13, Lemma 3.6]. □

Lemma 3.4.9 ([CM13, Lemma 3.7]). *There exists an apartment* $A \in \mathcal{A}_{\geq c_0}$ *such that the orbit* $N_A.c_0$ *is unbounded. In particular, the parabolic closure in* W *of* $\overline{N_A}$ *is non-spherical.*

Proof. Since H_1 is non-compact, the orbit $H_1.c_0$ is unbounded in Δ. For $n \in \mathbb{N}$ we can then chose $g_n \in H_1$ such that $\mathrm{dist}(c_0, g_n c_0) > n$. By Lemma 3.4.8 we can find an apartment $A \in \mathcal{A}_{\geq c_0}$ and a unbounded sequence of elements $h_n \in N_A$ such that $\mathrm{dist}(c_0, h_n c_0)$ is arbitrary large along n. Then $Pc(\overline{N_A})$ is non-spherical as desired. □

Remark 3.4.10. Observe that Lemmas 3.4.9 and 3.4.7 show that any non-compact open subgroup of the automorphism group of Δ contains a hyperbolic element. This also proves, in the locally finite case, that non-compact open subgroups of $\mathrm{Aut}(\Delta)$ are not locally X-elliptic in their action on the Davis realization of the building.

Let $A_1 \in \mathcal{A}_{\geq c_0}$ be an apartment such that the product of the non-spherical irreducible components of $Pc(\overline{N_{A_1}})$ is non-empty and maximal with respect to this property. Such an apartment exists by Lemma 3.4.9. Choose $h_{A_1} \in N_{A_1}$ as in Lemma 3.4.7. In particular h_{A_1} is a hyperbolic element of H_1.

Up to conjugating H_1 by $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$, we can assume without loss of generality that $Pc(\overline{h_{A_1}})$ is a standard parabolic subgroup that is non-spherical and has essential type I ($\neq \emptyset$). Moreover, the type I is maximal in the following sense: if $A \in \mathcal{A}_{\geq c_0}$ is such that $Pc(\overline{N_A})$ contains a parabolic subgroup of essential type I_A with $I \subseteq I_A$ then $I = I_A$.

For $T \subset S$, let

$$\begin{aligned} &\Phi_T = \{\alpha \in \Phi \mid \text{ there exists } v \in W_T \text{ and } s \in T \text{ such that } \alpha = v\alpha_s\} \\ &\text{and } L_T^+ = \langle U_\alpha \mid \alpha \in \Phi_T \rangle, \end{aligned}$$

where U_α is the root wing group introduced in Definition 3.3.1. Observe that $L_S^+ = \langle U_\alpha \mid \alpha \in \Phi \rangle$ and if $U_\alpha \subseteq L_T^+$ for some $\alpha \in \Phi$ then $U_{-\alpha} \subseteq L_T^+$.

The goal is now to prove that H_1 contains L_J^+ because we will prove in Lemma 3.4.11 that this fact is equivalent to H_1 being transitive on the chambers of $\mathcal{R}_{J,c_0}$.

We will need the results in Section 3.3 regarding fixators of balls and root wing groups. Since H_1 is open we fix, for the remaining of the section, $r \in \mathbb{N}$ such that $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathsf{B}(c_0, r)) \subseteq H_1$.

Lemma 3.4.11. *Let $T \subseteq S$ be essential and let $A \in \mathcal{A}_{\geq c_0}$. The the following are equivalent:*

1. *H_1 contains L_T^+;*
2. *H_1 is transitive on $\mathcal{R}_{c_0,T}$;*
3. *N_A is transitive on $\mathcal{R}_{c_0,T} \cap A$;*
4. *$\overline{N_A}$ contains the standard parabolic subgroup W_T of W.*

Proof. The equivalence between statements 3. and 4. is clear.

To prove that 1. implies 2. (and that 1. implies 3. in an analogous way) we observe that if c_1 and c_2 are s-adjacent chambers in $\mathcal{R}_{T,c_0}$ for some $s \in S$, then there are $c_1', c_2' \in \mathcal{P}_{s,c_1} \cap A_0$ such that $U_s(c_1'), U_s(c_2') \in L_T^+$.

Then we consider a permutation of $\mathrm{Ch}(\mathcal{P}_{s,c_1})$ fixing c_1' and mapping c_1 to c_2 and we can extend it by Lemma 2.2.13 to an element of $U_s(c_1') \subseteq H_1$ (see Remark 3.4.4).

In the case, without loss of generality, that $(c_1, c_2) = (c_1', c_2')$, since Δ is thick, we know that there exists $c_3 \in \mathcal{P}_{s,c_1} \backslash \{c_1, c_2\}$. We consider then $g_1 \in U_s(c_1' = c_1)$ such that $g_1(c_1, c_3) = (c_1, c_2)$ and $g_2 \in U_s(c_2' = c_2)$ such that $g_2(c_2, c_3) = (c_2, c_1)$. Thus $g_1 g_2^{-1} c_1 = c_2$ and $g_1 g_2^{-1} \in U_s(c_1)U_s(c_2) \subseteq H_1$.

The proofs $2 \Rightarrow 1$ and $4 \Rightarrow 2$ follow the strategy of [CM13, Lemma 3.11]. The idea is to prove that for $\alpha \in \Phi_T$ and $\varepsilon \in \{+, -\}$, the group $U_{\varepsilon\alpha}$ is contained in H_1 (or in a conjugate of H_1 in the second proof). The first key ingredient is the fact that $\mathrm{Aut}(\Delta)$ is strongly transitive and therefore $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ acts transitively on the set of apartments containing c_0. Hence we can consider $g \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ such that $gA_0 = A$ and work with standard parabolic subgroups. Moreover, by Lemma 3.3.5, there is $n \in \mathbb{Z}$ such that $U_{\varepsilon\alpha}$ is, in H_1 by transitivity of H_1 in the proof of $2 \Rightarrow 1$, or inside a conjugate of H_1 in the proof that $4 \Rightarrow 2$. $\square$

The reasoning of the proof of the next statement is analogous to the one in [CM13, Lemma 3.12]. We write it down for completeness.

Lemma 3.4.12. *Let $A \in \mathcal{A}_{\geq c_0}$. There exists $I_A \subseteq S$ such that $\overline{N_A}$ contains a parabolic subgroup P_{I_A} of W of type I_A as a finite index subgroup.*

Proof. We choose $h \in N_A$ as in Lemma 3.4.7. By construction $Pc(\overline{h})$ is generated by the reflections r_α with α a $\overline{h}$-essential root of A.

Let α be one of those roots. We want to prove that $r_\alpha \in \overline{N_A}$. Let $b \in \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(c_0)$ such that $\alpha = b\alpha_0$ with $\alpha_0 \in \Phi$, by strong transitivity of $\mathrm{Aut}(\Delta)$. In particular $bA_0 = A$.

Using Lemma 3.3.5 we obtain, for $\varepsilon \in \{+, -\}$ that $U_{\varepsilon\alpha_0} \subseteq b^{-1}H_1 b$. In particular the element $n_{\alpha_0} \in \langle U_{\alpha_0} \cup U_{-\alpha_0} \rangle \subseteq bH_1 b$. We have that r_{α_0} is the image in W of n_{α_0}. Moreover $r_\alpha = b r_{\alpha_0} b^{-1}$ so we obtain that $r_\alpha \in \overline{N_A}$ which implies that $Pc(\overline{h}) \subseteq \overline{N_A}$. Since $Pc(\overline{h})$ has finite index in $Pc(\overline{N_A})$ we conclude that $Pc(\overline{h})$ is the desired parabolic subgroup of W. $\square$

For each $A \in \mathcal{A}_{\geq c_0}$, we fix such an $I_A \subseteq S$ which, without loss of generality, we assume essential. We also consider the corresponding parabolic subgroup P_{I_A} contained in $\overline{N_A}$. Note then that $P_{I_{A_1}}$ has finite index in $Pc(\overline{N_{A_1}})$ by Lemma 1.4.10, where A_1 is the apartment fixed after Lemma 3.4.9. Therefore $I = I_{A_1}$.

The next task in the process of showing that H_1 contains L_J^+ is to prove that $J = I$, which is achieved by the following sequence of steps, each of which follows from the previous ones and which are analogues of results in [CM13]. Some of the proofs follow a similar reasoning as in the afore-mentioned paper. Therefore we present only the main keys steps and point to a precise reference in the paper.

Lemma 3.4.13. *Let $A \in \mathcal{A}_{\geq c_0}$ and retain the notation as before. Then the following hold.*

1. *H_1 contains L_I^+.*
2. *$I_A \subset I$.*
3. *$\overline{N_A}$ contains W_I as a subgroup of finite index.*
4. *$I = J$*

Proof.

1. Statement 1 follows from the fact that $I = I_{A_1}$ and $P_I = W_I$. So we just use the equivalence provided by Lemma 3.4.11.
2. To prove Statement 2 we use the same strategy as in [CM13, Lemma 3.14]. We consider $\mathcal{R}_1 = \mathcal{R}_{I,c_0} \cap A$ and $\mathcal{R}_2$ an I_A-residue in A on which N_A acts transitively and that is at minimal distance from R_1 amongst such residues. Using Lemma 3.4.11 we know that N_A is transitive on $\mathcal{R}_1$ as well. The idea of the proof is to consider a residue of type $I \cup I_A$ in A where N_A (or $\overline{N_A}$) acts transitively, which implies, by the maximality of I, that $I_A \subseteq I$.
3. We know by Statement 1 and Lemma 3.4.11 that $W_I \subseteq \overline{N_A}$. Furthermore, by Lemma 3.4.12, the group $\overline{N_A}$ also contains a finite index parabolic subgroup $P_{I_A} = wW_{I_A}w^{-1}$, for some $w \in W$. By Statement 2, we know that $I_A \subseteq I$ so we obtain that $W_{I_A} \subseteq \overline{N_A}$. Hence $P = W_{I_A} \cap wW_{I_A}w^{-1}$ is a parabolic subgroup with finite index in W_{I_A}. Since I_A was chosen essential, we have $P = W_{I_A}$ by [AB08, Proposition 2.43], because P has finite index in W_{I_A}.

   Therefore $W_{I_A} \subseteq wW_{I_A}w^{-1}$. As the chain $W_{I_A} \subseteq wW_{I_A}w^{-1} \subseteq w^2W_{I_A}w^{-2} \subseteq \cdots$ stabilizes we conclude that $W_{I_A} = P_{I_A}$ has finite index in $\overline{N_A}$. Thus W_I has finite index in $\overline{N_A}$.

4. To prove the last statement we use the reasoning of [CM13, Lemma 3.16] and we explain here the main lines of the proof. We denote by $\mathscr{R}$ the set of I-residues of Δ containing a chamber in the orbit $H_1.c_0$ and we denote $\mathcal{R} = \mathcal{R}_{I,c_0}$.

   The first step is to show that the distance from c_0 to the residues of $\mathscr{R}$ is bounded and hence $\mathscr{R}$ is finite. This is achieved by assuming the contrary and applying Lemma 3.4.8 and Statement 3.

   As the set $\mathscr{R}$ is finite and stabilized by H_1, the kernel H_2 of the induced action of H_1 on $\mathscr{R}$ is a finite index subgroup of H_1 stabilizing an I-residue.

   Up to conjugating by an element of H_1, we can assume that $H_2 \leq \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{I,c_0})$ and so H_2 has finite index in H_1 (and hence also in H). Thus the intersection $H_1 \cap \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{I,c_0})$ is open and contains H_2. Hence it has finite index in H_1 (and in H). This implies that $I = J$ by the minimality of J.

□

As a consequence of the previous lemma, and using Lemma 3.4.11, we obtain the following.

Lemma 3.4.14. H_1 *acts transitively on the chambers of* $\mathcal{R}_{J,c_0}$.

The next lemma is the first of the main goals of this section, and puts together all the construction of the sets I and J. We prove that gHg^{-1} is contained in the stabilizer of a residue that has $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ as finite index subgroup.

Lemma 3.4.15. *Every subgroup of* $\mathrm{Aut}(\Delta)$ *containing* H_1 *as a subgroup of finite index is contained in a stabilizer* $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})$ *with* J' *a spherical subset of* $J^{\perp}$.

Proof. To prove this result we can follow the arguments in [CM13, Lemma 3.19]. We highlight the main steps for the proof. Let O be a subgroup of $\mathrm{Aut}(\Delta)$ containing H_1 as finite index subgroup.

The group H_1 stabilizes the J-residue $\mathcal{R} = \mathcal{R}_{J,c_0}$ and acts transitively on its chambers by Lemma 3.4.11 and Corollary 3.4.14. Let $\mathscr{R}$ be the set of J-residues of Δ containing a chamber in the orbit $O.c_0$.

The set $\mathscr{R}$ is finite because O contains H_1 as a subgroup of finite index. The proof will be structured by the following set of claims.

Claim 1. *For any $\mathcal{R}' \in \mathscr{R}$ there is a constant M such that $\mathcal{R}$ is contained in an M-neighborhood of $\mathcal{R}'$. Since $\mathscr{R}$ is finite, we can assume that the constant is independent from $\mathcal{R}'$.*

This claim is proved in the same lines as in the mentioned lemma in [CM13].

Let $J' \in S \backslash J$ be minimal such that the right-angled building $\overline{\mathcal{R}} = \mathcal{R}_{J \cup J', c_0}$ contains the union of the residues in $\mathscr{R}$. This means that $O \leq \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J \cup J', c_0})$ with J' minimal for this property.

Claim 2. *$J' \subseteq J^{\perp}$.*

The idea is to show that O stabilizes $\mathcal{R}_{J \cup J', c_0}$ by using Claim 1.

Claim 3. *J' is spherical.*

By Claim 2, the building $\overline{\mathcal{R}}$ splits into a direct product of buildings $\overline{\mathcal{R}} = \mathcal{R}_{J,c_0} \times \mathcal{R}_{J',c_0}$. We get then a homomorphism $O \to \mathrm{Aut}(\mathcal{R}_{J,c_0}) \times \mathrm{Aut}(\mathcal{R}_{J',c_0})$. Since H_1 stabilizes $\mathcal{R}_{J,c_0}$ and has finite index in O, the image of O in $\mathrm{Aut}(\mathcal{R}_{J',c_0})$ has finite orbits in $\mathcal{R}_{J',c_0}$ because Δ is locally finite.

Then by the Bruhat-Tits Fixed Point Theorem we have that O fixes a point in the geometric realization of $\mathcal{R}_{J',c_0}$ and thus stabilizes a spherical residue of $\mathcal{R}_{J',c_0}$. By minimality of J', we obtain that O must stabilize $\mathcal{R}_{J',c_0}$ and thus J' is spherical. □

Since Δ is irreducible we observe that $J \cup J' = S$, for a subset $J' \subseteq J^{\perp}$ if and only if $J = S$.

Next we want to prove that the group H_1 has finite index in $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ in order to obtain that H has finite index in a conjugate of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J \cup J^{\perp}, c_0})$. We start by proving a general result, suggested by Pierre-Emmanuel Caprace, for locally compact groups acting on connected locally finite graphs.

Lemma 3.4.16. *Let G be a locally compact group of automorphisms of a connected locally finite graph Γ. Assume that G acts on Γ with finitely many orbits of vertices and with compact open vertex-stabilizers.*

Then any open subgroup $H \leq G$ acting on Γ with finitely many orbits of vertices has finite index in G.

Proof. Let F_G and F_H be sets of representatives for the G- and H-orbits of vertices of Γ, respectively. For each $v \in F_H$, we choose $g_v \in G$ such that $g_v(v)$ belongs to F_G and we denote $w(v) = g_v(v)$. Then we obtain that

$$V\Gamma = \bigcup_{v \in F_H} Hv = \bigcup_{v \in F_H} Hg_v^{-1}w(v).$$

Therefore $G = \bigcup_{v \in F_H} Hg_v^{-1} \operatorname{Stab}_G(w(v))$. Recall that by assumption the group $\operatorname{Stab}_G(w(v))$ is compact. Thus the finite union

$$\bigcup_{v \in F_H} g_v^{-1} \operatorname{Stab}_G(w(v))$$

is compact. So we have found a compact subset of G that maps onto to the coset space G/H. Thats means that G/H is compact. Since the group H is open, we obtain that the index of H on G is finite. $\square$

Now we apply the previous lemma to the setting of locally finite right-angled buildings, seen as chamber systems.

Lemma 3.4.17. *The group H_1 has finite index in* $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$.

Proof. Let $\Gamma = \mathcal{R}_{J,c_0}$ be the right-angled building (seen as a chamber system). The group $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ acts transitively on Γ since Δ is semi-regular and H_1 also acts transitively on Γ be Lemma 3.4.14. Thus the result follows directly from Lemma 3.4.16. $\square$

Lemma 3.4.18. *For every spherical subset J' of $J^\perp$, the index of* $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ *on* $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})$ *is finite.*

Proof. Let $G = \operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})$ and $H = \operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$. We have that G is transitive on $\mathcal{R}_{J\cup J',c_0}$ and H is transitive on $\mathcal{R}_{J,c_0}$, the latter being a subgraph of the former (seen as chambers systems). Since J' is spherical, every vertex (chamber) of $\mathcal{R}_{J\cup J',c_0}$ is at bounded distance from a vertex of $\mathcal{R}_{J,c_0}$. Moreover, as Δ is locally finite, all the balls in $\mathcal{R}_{J\cup J',c_0}$ are finite. Hence H acts on $\mathcal{R}_{J\cup J',c_0}$ with finite orbits of vertices. Thus we are in position to apply Lemma 3.4.16 to conclude that $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ has finite index on $\operatorname{Stab}_{\operatorname{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})$. $\square$

We are now ready to connect the dots of this section on open subgroups of the automorphism group of a locally finite thick irreducible semi-regular right-angled building.

Theorem 3.4.19. *Let Δ be a thick irreducible semi-regular locally finite right-angled building. Any proper open subgroup of* $\mathrm{Aut}(\Delta)$ *is contained with finite index in the stabilizer in* $\mathrm{Aut}(\Delta)$ *of a proper residue.*

Proof. Let H be a proper open subgroup of $\mathrm{Aut}(\Delta)$. If H is compact then it has a global fixed point in the geometric realization of the building. Hence it is contained in the stabilizer of a spherical residue, which is a compact group. Since H is open, this containment is with finite index.

If H is non-compact then it follows from Lemma 3.4.15 since, by definition of H_1, the group H contains a conjugate of H_1 as a subgroup of finite index. Hence H is contained in a conjugate of $\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})$. So we have

$$gH_1g^{-1} \overset{f.i.}{\leq} H \leq g\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})g^{-1}.$$

Using Lemmas 3.4.17 and 3.4.18 we obtain that

$$gH_1g^{-1} \overset{f.i.}{\leq} g\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g^{-1} \overset{f.i.}{\leq} g\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})g^{-1}.$$

Thus H is a subgroup of finite index of $g\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J\cup J',c_0})g^{-1}$.

Assume that $\mathcal{R}_{J\cup J',c_0}$ is the whole building Δ. Since $\mathrm{Aut}(\Delta)$ is simple and infinite, the only finite index subgroup of $\mathrm{Aut}(\Delta)$ is the whole group. Indeed, a finite index subgroup $G_1 \leq \mathrm{Aut}(\Delta)$ yields a homomorphism $\varphi : \mathrm{Aut}(\Delta) \to \mathrm{Aut}(\Delta)\backslash G_1$ whose kernel is non-trivial and contained in G_1. As $\ker\varphi$ is a normal subgroup of $\mathrm{Aut}(\Delta)$ we have that $\ker(\varphi) = \mathrm{Aut}(\Delta)$. Thus $H = G$ in this case, which finishes the proof of the theorem. $\square$

3.5 Consequences of the main theorem

In this last section we prove two results that are consequences from Theorem 3.4.19 regarding open subgroups of the automorphism group of a right-angled building. The first states that the automorphism group of a locally finite thick semi-regular right-angled building Δ is Noetherian (see Definition 3.5.1) and the second regards reduced envelopes in $\mathrm{Aut}(\Delta)$.

Definition 3.5.1. We call a topological group *Noetherian* if it satisfies the ascending chain condition on open subgroups.

We will prove that the group $\mathrm{Aut}(\Delta)$ is Noetherian by making use of the following characterization of the Noetherian property in locally compact groups.

Lemma 3.5.2 ([CM13, Lemma 3.22])**.** *Let G be a locally compact group. Then G is Noetherian if and only if every open subgroup of G is compactly generated.*

Proposition 3.5.3. *Let Δ be a locally finite thick semi-regular right-angled building. Then the group $\mathrm{Aut}(\Delta)$ is Noetherian.*

Proof. Using the previous Lemma, we have to show that any open subgroup of $\mathrm{Aut}(\Delta)$ is compactly generated. By Theorem 3.4.19, every open subgroup of $\mathrm{Aut}(\Delta)$ is contained with finite index in the stabilizer of a residue of Δ.

Stabilizers of residues are compactly generated since they act properly and cocompactly on the residue they stabilize. A cocompact subgroup of a group acting cocompactly on a set also acts cocompactly on that set. Therefore it implies that every open subgroup of $\mathrm{Aut}(\Delta)$ is compactly generated. Hence $\mathrm{Aut}(\Delta)$ is Noetherian. $\square$

Now we focus on reduced envelopes for sets of automorphisms of Δ, since this group is a totally disconnected and locally compact group.

Definition 3.5.4. Two subgroups H_1 and H_2 of a group G are called *commensurable* if $[H_i : H_1 \cap H_2] < \infty$ for $i \in \{1, 2\}$.

Definition 3.5.5. Let G be a totally disconnected locally compact (t.d.l.c.) group. Let $X \subseteq G$. An *envelope* of X in G is an open subgroup of G that contains X.

An envelope E of X is called *reduced* if given any open subgroup E_2 with $[X : X \cap E_2] < \infty$ we have $[E : E \cap E_2] < \infty$.

Reduced envelopes for t.d.l.c. groups have been studied by Reid (see [Rei16] and [Rei15]) and he proved the following.

Theorem 3.5.6 ([Rei16, Theorem B])**.** *Let G be a t.d.l.c. group and let H be a compactly generated subgroup of G. Then there exists a reduced envelope for H in G.*

The existence of reduced envelopes in the compactly generated case allow us to conclude the following.

Proposition 3.5.7. *Every open subgroup of* $\mathrm{Aut}(\Delta)$ *is commensurable with the reduced envelope of a cyclic subgroup.*

Proof. Let H be an open subgroup of $\mathrm{Aut}(\Delta)$. Using the notation of the previous section, consider $H_1 = gHg^{-1} \cap \mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})$ such that J is minimal amongst the subsets L of S such that there exists $g \in \mathrm{Aut}(\Delta)$ such that $H \cap g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{L,c_0})g$ has finite index in H.

Let h_{A_1} be a hyperbolic element of H_1 whose minimal displacement function gives rise to the type J, as described right before Remark 3.4.10. By Theorem 3.5.6, the group $\langle g^{-1}h_{A_1}g\rangle$ has a reduced envelope E in $\mathrm{Aut}(\Delta)$.

Then $[\langle g^{-1}h_{A_1}g\rangle : \langle g^{-1}h_{A_1}g\rangle \cap H] < \infty$ which implies, by definition of a reduced envelope, that $[E : H \cap E] < \infty$.

By definition of a reduced envelope, we also obtain that

$$[g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g : g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g \cap E] < \infty.$$

Since H has finite index in $g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g$ by Theorem 3.4.19, it follows that

$$\begin{aligned}[g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g \cap H : (g^{-1}\,\mathrm{Stab}_{\mathrm{Aut}(\Delta)}(\mathcal{R}_{J,c_0})g \cap H) \cap E]&\\ = [H : H \cap E] < \infty.&\end{aligned}$$

Thus H is commensurable with the reduced envelope of a cyclic group. ∎

CHAPTER 4

Universal group of a right-angled building

The goal of this chapter is to extend the concept of universal groups defined for regular trees by Burger and Mozes [BM00a] to the more general setting of right-angled buildings. This is motivated mainly by two facts. First, trees are instances of right-angled buildings so it is interesting to study in which sense those groups would be defined and have interesting properties in a more general setting. Secondly, Pierre-Emmanuel Caprace [Cap14] proved that the group of type-preserving automorphisms of a thick semi-regular right-angled building is an abstractly simple group. Therefore those automorphism groups fit in the class of simple totally disconnected locally compact groups discussed in Section 1.3. In the locally finite case they are furthermore compactly generated (see Proposition 4.2.5).

Hence we will define universal groups for semi-regular right-angled buildings (recall the definition in 2.3.1) and we will prove that those groups also fit in the class of topological groups mentioned above. We will start by presenting basic topological properties of these groups in Section 4.2 and then we will move towards the proof of simplicity of the universal groups.

In Section 4.3 we define a dense subgroup of the universal group constructed using actions on the tree-walls of the building and we use this group to prove that we can extend automorphisms of residues to automorphisms of the whole building. Then in Section 4.4 we present a property that resembles Tits's independence property in the setting of right-angled buildings. Of central importance will be to understand the action of the universal group on the tree-wall trees introduced in Definition 2.2.37, which is what we investigate in Section 4.5. We finish the chapter by proving simplicity in Section 4.6. Almost all the work of this chapter is presented in [DMSS16], with exception of some properties in Section 4.5 and of Section 4.7.

4.1 The definition

We will use the following notation throughout this chapter, unless otherwise stated. Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ with index-set S and set of generators $S = \{s_i\}_{i \in I}$. For each $s \in S$, let q_s be a cardinal number and Y_s be a set of size q_s, which we regard as the set of s-colors. Consider the right-angled building Δ of type (W, S) with parameters $(q_s)_{s \in S}$, which is unique up to isomorphism (see Theorem 2.3.2).

Definition 4.1.1. For each $s \in S$, let $G^s \leq \mathrm{Sym}(Y_s)$ be a transitive permutation group and $h_s \colon \mathrm{Ch}(\Delta) \to Y_s$ be a (weak) legal coloring of Δ (recall the definition in Section 2.3). We define the *universal group* of Δ *with respect to* the groups $(G^s)_{s \in S}$ as

$$\begin{aligned} U &= U((G^s)_{s \in S}) \\ &= \{g \in \mathrm{Aut}(\Delta) \mid (h_s|_{\mathcal{P}_{s,gc}}) \circ g \circ (h_s|_{\mathcal{P}_{s,c}})^{-1} \in G^s, \text{ for all } s \in S, \\ &\qquad\qquad \text{all } s\text{-panels } \mathcal{P}_s, \text{ and for all chambers } c \in \mathcal{P}_s\}, \end{aligned}$$

where $\mathcal{P}_{s,c}$ is the s-panel containing $c \in \mathrm{Ch}(\Delta)$.

If the group G^s equals $\mathrm{Sym}(Y_s)$, for all $s \in S$, then U is the group $\mathrm{Aut}(\Delta)$ of all type-preserving automorphisms of the right-angled building since we are assuming that the groups G^s are transitive.

Remark 4.1.2. The definition of a universal group for a right-angled building also makes sense when the groups G^s are not transitive. For instance, if the groups G^s are all trivial and Δ is locally finite, then U is a lattice in $\mathrm{Aut}(\Delta)$ since it acts freely and cocompactly on Δ. However, the universal group will be chamber-transitive only in the case when all groups G^s are transitive and we will require chamber-transitivity quite often.

In the next lemma we justify why in Definition 4.1.1 we consider h_s to be a (weak) legal coloring. To be precise, we justify the brackets on the word "weak".

Lemma 4.1.3. *For each $s \in S$, let $(h_s^1)_{s\in S}$ and $(h_s^2)_{s\in S}$ be two G^s-equivalent colorings. Then the universal groups constructed using $(h_s^1)_{s\in S}$ and $(h_s^2)_{s\in S}$ coincide.*

Proof. Let $s \in S$. Let $g \in U^{h_s^1}$ be an element of the universal group constructed with the coloring h_s^1. Let $c \in \mathrm{Ch}(\Delta)$ and $\mathcal{P}$ be the s-panel of c. By definition we know that $h_s^1 \circ g \circ (h_s^1)^{-1}|_{\mathcal{P}} \in G^s$. Further, since h_s^1 and h_s^2 are G^s-equivalent colors we know that $h_s^1|\mathcal{P} = g_1 \circ h_s^2|_{\mathcal{P}}$ and $h_s^1|\mathcal{P}_{s,gv} = g_2 \circ h_s^2|_{\mathcal{P}_{s,gv}}$ with $g_1, g_2 \in G^s$. Thus

$$h_s^1 \circ g \circ (h_s^1)^{-1}|_{\mathcal{P}} = g_2 \circ h_s^2 \circ g \circ (h_s^2)^{-1} \circ g_1^{-1}|_{\mathcal{P}} \in G^s,$$

which implies that $h_s^2 \circ g \circ (h_s^2)^{-1}|_{\mathcal{P}} \in G^s$. Hence g is an element of the universal group $U^{h_s^2}$ constructed using h_s^2. Exchanging the colorings in the reasoning, we obtain that the two universal groups coincide. □

This lemma implies that the definition of the universal group restricting to legal colorings, weak legal colorings or directed legal colorings with respect to a fixed chamber all yield the same universal group because such colorings are G^s-equivalent by Proposition 2.3.12 and Observation 2.3.15.

4.2 Basic properties

In this section we gather some basic properties concerning universal groups for right-angled buildings. We start by looking at the action of these groups on panels.

Definition 4.2.1. Let $H \leq \mathrm{Aut}(\Delta)$ and $\mathcal{P}$ be a panel of Δ. We define the *local action* of H at the panel $\mathcal{P}$, normally denoted as $H|_{\mathcal{P}}$, as the permutation group induced by $\mathrm{Stab}_H(\mathcal{P})$ on the panel $\mathcal{P}$.

This local action is isomorphic to $\mathrm{Stab}_H(\mathcal{P})/\operatorname{Fix}_H(\mathcal{P})$.

Lemma 4.2.2. *Let $s \in S$. The local action of the universal group U on any s-panel is isomorphic to the transitive group G^s.*

Proof. Consider some chamber c. The chambers in the s-panel $\mathcal{P} := \mathcal{P}_{s,c}$ containing c are parametrized by Y_s via the coloring. From the definition of the universal group it is immediate that the local action on the s-panel is a subgroup of G^s.

We will now show that this local action is indeed G^s. Let $g_s \in G^s$ and let c' be the chamber in $\mathcal{P}$ with color $g_s \circ h_s(c)$. Denote by $(h'_s)_{r\in S}$ the set of legal colorings obtained from $(h_r)_{r\in S}$ by replacing the coloring h_s by $g_s^{-1} \circ h_s$ and leaving the other colorings unchanged. Note that h'_s is again a legal coloring, as it will still satisfy the defining property (L) of legal colorings (see Definition 2.3.6). As $h_r(c)$ equals $h'_r(c')$ for every $r \in S$, we can apply Proposition 2.3.8 to find an automorphism $g \in \mathrm{Aut}(\Delta)$ mapping c to c'.

The automorphism g acts locally as the identity for r-panels where $r \in S \setminus \{s\}$, and as g_s on s-panels. Hence g is an element of the universal group and it has the desired action on the panel $\mathcal{P}_{s,c}$, whence the claim. ☐

Assume that the set of colors Y_s contains an element called 1, and let G_0^s be the stabilizer of this element in G^s. Recall that as G^s is assumed to be transitive, all its point-stabilizers are conjugate. Therefore the notation G_0^s is to stress that it is the stabilizer of a point in G^s.

Proposition 4.2.3. *Let Δ be the directed right-angled building with prescribed thickness $(q_r)_{r\in S}$ and base chamber c_0 (see Definition 2.4.5). Consider the universal group U with respect to the standard colorings $(f_s)_{s\in S}$ of Δ directed with respect to c_0. Let $\mathcal{T}$ be an s-tree-wall of Δ.*

For each $g \in G_0^s$, let $g_{\mathcal{T}}$ be a tree-wall automorphism as in Definition 2.4.8. Then $G_{\mathcal{T}} = \{g_{\mathcal{T}} \mid g \in G_0^s\}$ is a subgroup of U fixing the chambers of the s-wing with respect to $\mathcal{T}$ containing the chamber c_0. This subgroup acts locally as G_0^s on each s-panel of the tree-wall $\mathcal{T}$.

Proof. Let $g \in G_0^s$ and let $g_{\mathcal{T}} \in \mathrm{Aut}(\Delta)$ be as in Proposition 2.4.7. We will first prove that $g_{\mathcal{T}}$ is an element of the universal group U. Let $\mathcal{P}$ be a t-panel for some $t \in S$. It is clear that the automorphism $g_{\mathcal{T}}$ fixes the colorings of the chambers in $\mathcal{P}$ unless $t = s$.

Assume then that $t = s$. In the case that $\mathcal{P}$ is a panel in the s-tree-wall $\mathcal{T}$, we have $f_s|_{g_{\mathcal{T}}.\mathcal{P}} \circ g_{\mathcal{T}} \circ f_s|_{\mathcal{P}} = g \in G_0^s$. If $\mathcal{P} \notin \mathcal{T}$, then this permutation of the colors is the identity in G^s. Therefore $g_{\mathcal{T}} \in U$ as claimed.

Next, we prove that $g_{\mathcal{T}}$ fixes the wing $X_s(\mathrm{proj}_{\mathcal{T}}(c_0))$. Let

$$c = \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}$$

be a chamber in $X_s(\mathrm{proj}_{\mathcal{T}}(c_0))$, so that $\mathrm{proj}_{\mathcal{T}}(c) = \mathrm{proj}_{\mathcal{T}}(c_0)$. If there is an element $i \in \{1, \ldots, n\}$ such that $s_i = s$ and

$$\begin{pmatrix} s_1 & \cdots & s_{i-1} \\ \alpha_1 & \cdots & \alpha_{i-1} \end{pmatrix} \text{ and } \begin{pmatrix} s_1 & \cdots & s_i \\ \alpha_1 & \cdots & \alpha_i \end{pmatrix} \text{ are in } \mathcal{T},$$

then

$$\mathrm{proj}_{\mathcal{T}}(c) = \begin{pmatrix} s_1 & \cdots & s_{i-1} & s \\ \alpha_1 & \cdots & \alpha_{i-1} & \alpha_i \end{pmatrix} \neq \begin{pmatrix} s_1 & \cdots & s_{i-1} \\ \alpha_1 & \cdots & \alpha_{i-1} \end{pmatrix} = \mathrm{proj}_{\mathcal{T}}(c_0),$$

which is a contradiction. Hence there is no such i, and therefore $g_{\mathcal{T}}$ fixes c by definition.

Using the last statement of Proposition 2.4.7, we obtain that the set $G_{\mathcal{T}} = \{g_{\mathcal{T}} \mid g \in G_0^s\}$ forms a group. By construction, $G_{\mathcal{T}}$ acts locally as G_0^s on each s-panel of $\mathcal{T}$. Hence the proposition is proved. □

Definition 4.2.4. Let $s \in S$ and $\mathcal{T}$ be an s-tree-wall of Δ. The group $G_{\mathcal{T}}$ as in Proposition 4.2.3 is called a *tree-wall group.*

We will prove in the following proposition, among other properties, that different choices of legal colorings give rise to conjugate subgroups of $\mathrm{Aut}(\Delta)$. This will allow us to justify the omission of an explicit reference to the colorings in our notation for the universal groups.

Proposition 4.2.5 (Properties of U)**.** *Let Δ be a right-angled building with prescribed thickness $(q_s)_{s\in S}$. Let U be the universal group of Δ with respect to the finite transitive permutation groups $\{G^s \leq \mathrm{Sym}(Y_s)\}_{s\in S}$. Then the following hold.*

1. *The subgroup $U \leq \mathrm{Aut}(\Delta)$ is independent of the choice of the set of legal colorings up to conjugacy in $\mathrm{Aut}(\Delta)$.*
2. *U is a closed subgroup of $\mathrm{Aut}(\Delta)$.*
3. *U is a chamber-transitive subgroup of $\mathrm{Aut}(\Delta)$.*
4. *U is universal for $(G^s)_{s\in S}$. That is, if H is a closed chamber-transitive subgroup of $\mathrm{Aut}(\Delta)$ for which the local action in the s-panels is permutationally isomorphic to the group G^s, for all $s \in S$, then H is conjugate in $\mathrm{Aut}(\Delta)$ to a subgroup of U.*
5. *If Δ is locally finite, then U is compactly generated.*

Proof. 1. Let $(h_s^1)_{s\in S}$ and $(h_s^2)_{s\in S}$ be distinct sets of legal colorings. We want to show that the universal groups U_1 and U_2 defined using the legal colorings $(h_s^1)_{s\in S}$ and $(h_s^2)_{s\in S}$, respectively, are conjugate in $\mathrm{Aut}(\Delta)$. By Proposition 2.3.8 we know that there exists $g \in \mathrm{Aut}(\Delta)$ such that $h_s^2 = h_s^1 \circ g$ for all $s \in S$. If $u \in U_2$ then

$$h_s^2 \circ u \circ (h_s^2)^{-1} \in G^s \iff h_s^1 \circ g \circ u \circ g^{-1} \circ (h_s^1)^{-1} \in G^s,$$

for all $s \in S$. Hence $u^g = g \circ u \circ g^{-1}$ is an element of U_1. Therefore U_1 and U_2 are conjugate in $\mathrm{Aut}(\Delta)$.

2. To prove Statement 2, we will show that $\mathrm{Aut}(\Delta) \setminus U$ is open. Consider $u \in \mathrm{Aut}(\Delta) \setminus U$. By definition, there exists $s \in S$, an s-panel $\mathcal{P}_s$ and $v \in \mathcal{P}_s$ such that

$$h_s|_{\mathcal{P}_{s,uv}} \circ u \circ (h_s|_{\mathcal{P}_{s,v}})^{-1} \notin G^s.$$

But then the set $\{u' \in \mathrm{Aut}(\Delta) \mid u'|_{\mathcal{P}_{s,v}} = u|_{\mathcal{P}_{s,v}}\}$ is contained in $\mathrm{Aut}(\Delta) \setminus U$ and it is a coset of the stabilizer of $\mathcal{P}_{s,v}$. Hence $\mathrm{Aut}(\Delta) \setminus U$ is open by definition of the permutation topology.

3. Next we show that U is a chamber-transitive group. Since Δ is connected, it is enough to prove the result for two adjacent chambers. Let c_1 and c_2 be two adjacent chambers in the building Δ, i.e., there exists an $s \in S$ such that c_1 and c_2 are in the same s-panel $\mathcal{P}$. By Lemma 4.2.2, the induced action of $\mathrm{Stab}_U(\mathcal{P})$ on $\mathcal{P}$ is isomorphic to the group G^s. Since G^s is assumed to be transitive, there is an element in G^s mapping $h_s(c_1)$ to $h_s(c_2)$. Hence there is an element $g \in U$ such that $g(c_1) = c_2$. Thus U is a chamber-transitive subgroup of $\mathrm{Aut}(\Delta)$.

4. Now we prove that the group U is universal. Let H be a closed chamber-transitive subgroup of $\mathrm{Aut}(\Delta)$ for which the local action in the s-panels is permutationally isomorphic to the group G^s, for all $s \in S$. We will construct weak legal colorings $(h_s)_{s\in S}$ such that H is a subgroup of U^{h_s}, which denotes the universal group defined using the set of weak legal colorings $(h_s)_{s\in S}$.

   Let us fix $c_0 \in \mathrm{Ch}(\Delta)$ and $s \in S$. We choose a bijection $h_s^0 \colon \mathcal{P}_{s,c_0} \to Y_s$ such that

   $$h_s^0 \circ H|_{\mathcal{P}_{s,c_0}} \circ (h_s^0)^{-1}|_{\mathcal{P}_{s,c_0}} = G^s.$$

   Let $\mathcal{T}_0 = \mathcal{T}_{s,c_0}$. For each s-panel $\mathcal{P}$ in the s-tree-wall $\mathcal{T}_0$ we define

   $$h_s(c) = h_s^0(\mathrm{proj}_{\mathcal{P}_{s,c_0}}(c)), \text{ for every } c \in \mathcal{P}.$$

   With this procedure we have colored with a color from Y_s all the chambers in $\mathcal{T}_0$. Moreover, since the chambers of parallel panels are in bijection through the projection map, we have $h_s \circ H|_{\mathcal{P}} \circ (h_s|_{\mathcal{P}})^{-1} = G^s$ for each panel $\mathcal{P}$ of $\mathcal{T}_0$.

   Assume by induction hypothesis that the coloring h_s is defined in the s-tree-walls of Δ at tree-wall distance smaller or equal to $n-1$ from $\mathcal{T}_0$ (see Definition 2.2.40).

   Let $\mathcal{T}$ be an s-tree-wall at tree-wall distance n from $\mathcal{T}_0$. Fix $c \in \mathrm{Ch}(\mathcal{T})$. Define a bijection $h_s^n \colon \mathcal{P}_{s,c} \to Y_s$ such that $h_s^n \circ H|_{\mathcal{P}_{s,c}} \circ (h_s^n)^{-1} = G^s$, and for all chambers $v \in \mathrm{Ch}(\mathcal{T})$ we define

   $$h_s(v) = h_s^n(\mathrm{proj}_{\mathcal{P}_{s,c}}(v)).$$

In this fashion we color all the chambers of the building with the colorings $(h_s)_{s\in S}$. Now we have to prove that h_s is indeed a weak legal coloring, for each $s \in S$. It is clear that h_s is a coloring since it was defined as a bijection in a panel of each s-tree-wall and parallel panels in $\mathcal{T}$ are in bijection through the projection map.

To prove that h_s is a weak legal coloring, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be s-panels in a common s-tree-wall $\mathcal{T}$. Let $c_1 \in \mathcal{P}_1$ and $c_2 = \mathrm{proj}_{\mathcal{P}_2}(c_1)$. Let $c = \mathrm{proj}_{\mathcal{P}}(c_1)$, where $\mathcal{P}$ is the s-panel of $\mathcal{T}$ that was used to define h_s in the recursive process. Then, by Proposition 2.2.32, $\mathrm{proj}_{\mathcal{P}}(c_2) = c$ and hence $h_s(c_1) = h_s(c) = h_s(c_2)$. Thus h_s satisfies (W) in Definition 2.3.9 and hence it is a weak legal coloring.

Finally we show that H is a subgroup of U^{h_s}. If $g \in H$ then $h_s \circ g \circ (h_s|_{\mathcal{P}})^{-1} \in G^s$ for every s-panel $\mathcal{P}$ and every $s \in S$, by the construction of the weak legal colorings $(h_s)_{s\in S}$. Therefore H is a subgroup of U^{h_s}. Hence U is the largest vertex-transitive closed subgroup of $\mathrm{Aut}(\Delta)$ which acts locally as the groups $(G^s)_{s\in S}$.

5. To conclude, we prove that U is compactly generated when the building Δ is locally finite. Since U is chamber-transitive, its action on the chambers of Δ has only one orbit. Let $c \in \mathrm{Ch}(\Delta)$. We know that the stabilizer U_c is a compact open subgroup of U by the definition of the permutation topology on U. Let $\{c_1, \dots, c_n\}$ be the set of chambers of Δ adjacent to c. For each c_i, there exists an element $g_i \in U$ such that $g_i(c_i) = c$. Let $C = \{g_1, \dots, g_n\}$.

   Let $g \in U$. We claim that there exists some $g' \in \langle C\rangle$ such that $g'gc = c$. This is proved by induction on the discrete distance from c to gc. If $\mathrm{dist}(c, gc) = 1$ it follows from the definition of the set C. Assume that the claim holds if $\mathrm{dist}(c, gc) \leq n$. If $\mathrm{dist}(c, gc) = n + 1$, let $\gamma = (c, c', \dots, gc)$ be a minimal gallery from c to gc. We have $\mathrm{dist}(c, c') = 1$ therefore there is $\overline{g} \in C$ such that $\overline{g}c' = c$. As $\mathrm{dist}(c', gc) = n$ we have $\mathrm{dist}(\overline{g}c', \overline{g}gc) = \mathrm{dist}(c, \overline{g}gc) = n$. Hence, by induction hypothesis, there exists $g^* \in \langle C\rangle$ such that $g^*\overline{g}gc = c$ and $g^*\overline{g} \in \langle C\rangle$. But then $g^*\overline{g}g \in$

U_c. Therefore we conclude that the compact subset $U_c \cup C$ generates the group U. $\square$

4.3 Extending elements of the universal group

Every residue $\mathcal{R}$ of a right-angled building Δ is on its own a right-angled building by Theorem 1.4.35. In this section we prove that given a residue $\mathcal{R}$ of type $J \subseteq S$ and a set of transitive groups $\{G^t \leq \mathrm{Sym}(q_s)\}_{t\in J}$, we can extend any automorphism of $\mathcal{R}$ acting locally like G^t on its t-panels to an element of the universal group $U(\{G^s\}_{s\in S})$, where $G^s = G^t$ for all $s \in J$. In this way we extend elements of the universal group of a residue $\mathcal{R}$ to elements of universal group of any building containing $\mathcal{R}$.

We start by defining a dense subgroup of the universal group using its action on tree-walls. We keep the notation of the previous sections.

Definition 4.3.1. Fix a chamber $c_0 \in \mathrm{Ch}(\Delta)$ and let $s \in S$.

1. Let B be a connected subset of the building Δ containing c_0 and $\mathcal{T}$ be an s-tree-wall of Δ. If B is not entirely contained in one wing of $\mathcal{T}$ (see Definition 2.2.29), we say that B *crosses* $\mathcal{T}$.

2. Let c be the projection of c_0 on an s-tree-wall $\mathcal{T}$. We call the Weyl distance $\delta(c, c_0)$ the *distance between* c_0 *and* $\mathcal{T}$.

   Let us now consider, for a fixed $w \in W$, the collection $T(w, s)$ of s-tree-walls at distance w from c_0.

Observe that the set $T(w, s)$ might be empty. For instance if $w \sim w't$ with $|ts| = 2$ in the Coxeter diagram then that is the case. Indeed, let c be a chamber at Weyl distance w from the fixed c_0 and let c' be the chamber at Weyl distance w' from c_0 that is at distance t from c. Then c and c' are in the same s-tree-wall $\mathcal{T}$ and $\mathsf{d}_W(c_0, c') < \mathsf{d}_W(c_0, c)$ (the gallery distance). Therefore $\mathrm{proj}_{\mathcal{T}}(c_0) \neq c$. Thus in this case there are no s-tree-walls at distance w from c_0.

Lemma 4.3.2. *Let c_0 be a fixed chamber. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be distinct s-tree-walls in $T(w, s)$, for some $s \in S$ and $w \in W$. Let $G_{\mathcal{T}_1}$ and $G_{\mathcal{T}_2}$ be the respective tree-wall subgroups of U (see Definition 4.2.4).*

Then $G_{\mathcal{T}_1}$ and $G_{\mathcal{T}_2}$ have disjoint supports.

Proof. Consider two different tree-walls $\mathcal{T}_1$ and $\mathcal{T}_2$ in $T(w,s)$. This property can be observed in the tree-wall tree Γ_s. Indeed the s-wing with respect to $\mathcal{T}_1$ containing c_0 contains every s-wing with respect to $\mathcal{T}_2$, except the one containing c_0. Therefore the supports of the two respective tree-wall groups are disjoint. $\square$

Remark 4.3.3. The previous lemma considers a semi-regular right-angled building in a directed description. By Proposition 2.4.4, we can fix a chamber $c \in \Delta$ and, given a set of directed legal colorings, we can consider an isomorphism from Δ with that set of colorings to a directed right-angled building with its set of standard legal colorings (*cf.* Definition 2.4.3). Moreover, by Lemma 4.1.3, universal groups considering directed legal colorings and equivalent legal colorings coincide. Therefore, without loss of generality, we will consider directed right-angled buildings when it suits better our purposes.

Before proceeding to the definition of a dense subgroup of the universal group, we mention the following technical lemma.

Lemma 4.3.4 ([Cap14, Lemma 5.3])**.** *Let $n > 0$ be an integer, let C, W be sets and $\delta\colon C^n \to W$ be a map. Let G denote the group of all permutations $g \in \mathrm{Sym}(C)$ such that $\delta(gx_1, \dots, gx_n) = \delta(x_1, \dots, x_n)$ for all $(x_1, \dots, x_n) \in C^n$. Let moreover $(H_s)_{s\in S}$ be a collection of groups indexed by a set S, and for all $s \in S$, let $\varphi_s\colon H_s \to G$ be an injective homomorphism, such that for all $s \neq r$, the subgroups $\varphi(H_s)$ and $\varphi(H_r)$ have disjoint supports. Then there is a unique homomorphism*

$$\varphi\colon \prod_{r\in S} H_r \to G$$

such that $\varphi \circ \iota_s = \varphi_s$ for all $s \in S$, where $\iota_s\colon H_s \to \prod_{r\in S} H_r$ is the canonical inclusion.

For each tree-wall $\mathcal{T} \in T(w,s)$ we have the tree-wall group $G_{\mathcal{T}}$ of the universal group U, acting faithfully and locally like G_0^s on each s-panel of $\mathcal{T}$, and fixing the chambers in the wing of $\mathcal{T}$ containing c_0.

By Lemma 4.3.2, we know that these groups have disjoint supports for any distinct tree-walls in $T(w,s)$. Hence we can apply Lemma 4.3.4 and consider the action of the product of the $G_{\mathcal{T}}$'s (one for each tree-wall $\mathcal{T}$ in $T(w,s)$), which we denote by $\prod_{\mathcal{T}\in T(w,s)} G_{\mathcal{T}}$.

Lemma 4.3.5. *Let Δ be a semi-regular right-angled building of type (W, S) with parameters $(q_s)_{s\in S}$. Fix a chamber c_0 of Δ. Let K be a connected subset of W (as a Coxeter complex) containing the identity. Let B be the set of chambers c in $\mathrm{Ch}(\Delta)$ such that $\delta(c_0, c) \in K$.*

Then

$$\left\langle \left\{ \prod_{\mathcal{T}\in T(w,s)} G_{\mathcal{T}} \right\} \middle| \begin{array}{r} s \in S, w \in K \text{ such that } B \text{ crosses} \\ \text{the tree-walls in } T(w,s) \end{array} \right\rangle$$

is a dense subgroup of $U_{c_0}|_B$.

Proof. Note that the subgroups $\prod_{\mathcal{T}\in T(w,s)} G_{\mathcal{T}}$ fix the chamber c_0 and stabilize the set B, so we may consider them to be subgroups of $U_{c_0}|_B$.

We first prove the result for finite subsets K by induction on the size of K. If $|K| = 1$ then $K = \{\mathrm{id}\}$ and $B = \{c_0\}$, hence $U_{c_0}|_B$ is trivial.

Assume now by induction hypothesis that the result holds for every set K of size $n \leq N$. Let $K \subset W$ be a connected subset of $N + 1$ elements containing the identity. Let $w \in K$ be such that $K \setminus \{w\}$ is still connected. (This is always possible, for instance by picking a vertex of valency one in a spanning tree of K). Let

$$K_1 = K \setminus \{w\} \ \text{ and } \ B_1 = \{c \in \mathrm{Ch}(\Delta) \mid \delta(c_0, c) \in K_1\}.$$

Let $g \in U_{c_0}|_B$ fixing B_1 and let $c \in B \setminus B_1$. Note that $B \setminus B_1$ consists exactly of those chambers at Weyl distance w from c_0.

As K is connected, we know that there exists a $w_1 \in K_1$ which is s-adjacent to w for some $s \in S$. Hence c is s-adjacent to some $c_1 \in B_1$. Note that g stabilizes the s-panel $\mathcal{P}$ of c and c_1. Let $\mathcal{T}$ be the s-tree-wall containing $\mathcal{P}$, and let w_2 be its Weyl distance to c_0.

If $\mathcal{T}$ is already crossed in B_1 then B_1 contains chambers in the same s-wing of the tree-wall $\mathcal{T}$ as c. So this wing is stabilized, and as the s-panel containing c is also stabilized, we can conclude that g fixes the chamber c.

If $\mathcal{T}$ was not crossed by B_1 then, as g fixes c_1, it acts on the s-panel $\mathcal{P}$ as an element of G_0^s (see Lemma 4.2.2). Repeating this reasoning for each possible chamber c at Weyl distance w from c_0, we conclude that that g is contained in $\prod_{T_{w_2,s}} G_{\mathcal{T}}$ considered as subgroup of $U_{c_0}|_B$. By the induction hypothesis, any element $g_1 \in U_{c_0}|_{B_1}$ is in the desired conditions, and every element $h \in U_{c_0}|_B$ can be written as a product $g \circ g_1$ with g and g_1 as before. Hence we conclude that the statement of the lemma holds for K, and hence for every finite K.

If the set K is infinite then we can approximate K by a sequence $(K_1, K_2, \dots)$ of finite connected subsets of K containing the identity, and such that for every n, there is an N such that K_i agrees with K on the ball of radius n around the identity in W for every $i > N$. Let $(B_1, B_2, \dots)$ be the corresponding sets of chambers $c \in \Delta$ such that $\delta(c_0, c) \in K_i$. The group $U_{c_0}|_B$ is then the inverse limit of the finite groups $U_{c_0}|_{B_i}$. Moreover, each $U_{c_0}|_{B_i}$ is a quotient of $U_{c_0}|_B$. Since the statement holds for each of these finite quotients and because we take the closure, we can conclude that the statement holds for every K, finite or infinite. $\square$

Now we are in conditions of extending automorphisms of a residue to automorphisms of the whole building.

Definition 4.3.6. Let $J \subseteq S$ and $\mathcal{R}$ be a residue of type J in Δ. We define

$$U(\mathcal{R}) := \{g \in \mathrm{Aut}(\mathcal{R}) \mid (h_s|_{\mathcal{P}_{s,gc}}) \circ g \circ (h_s|_{\mathcal{P}_{s,c}})^{-1} \in G^s, \text{ for all } s \in J \text{ and for all chambers } c \in \mathrm{Ch}(\mathcal{R})\}.$$

In particular, $U(\Delta) = U$.

Lemma 4.3.7. *Let $\mathcal{R}$ be a J-residue containing c_0 for some $J \subseteq S$. With the same notation as above, $U_{c_0}|_{\mathrm{Ch}(\mathcal{R})}$ and $U_{c_0}(\mathcal{R})$ acting on* $\mathrm{Ch}(\mathcal{R})$ *are permutationally isomorphic.*

Proof. We consider both groups as subgroups of the symmetric group acting on $\mathrm{Ch}(\mathcal{R})$.

We observe that $\mathrm{Ch}(\mathcal{R})$ corresponds to the set of chambers at distance w from c_0, for some $w \in W_J$, the parabolic subgroup of W with set of generators $J \subseteq S$. As W_J is a connected subset of W (as a Coxeter chamber system), we can apply Lemma 4.3.5 and obtain that $U_{c_0}(\Delta)|_{\mathrm{Ch}(\mathcal{R})}$ is the closure of its subgroup generated by the groups $\prod_{\mathcal{T} \in T(w,s)} G_{\mathcal{T}}$ (regarded as subgroups of $\mathrm{Sym}(\mathrm{Ch}(\mathcal{R}))$ such that $w \in W_J$ and $s \in J$.

On the other hand, the residue $\mathcal{R}$ is, in its own right, a right-angled building of type (W_J, J), to which we may apply the same Lemma 4.3.5, yielding that $U_{c_0}(\mathcal{R})$ is the closure of the subgroup generated by the same $\prod_{\mathcal{T}\in T(w,s)} G_{\mathcal{T}}$ where $w \in W_J$ and $s \in J$.

We therefore conclude that $U_{c_0}|_{\mathrm{Ch}(\mathcal{R})}$ and $U_{c_0}(\mathcal{R})$ have the same action on $\mathrm{Ch}(\mathcal{R})$. $\square$

Lemma 4.3.8. *Let $\mathcal{R}$ be a J-residue of Δ. The groups $U(\Delta)|_{\mathrm{Ch}(\mathcal{R})}$ and $U(\mathcal{R})$ acting on $\mathrm{Ch}(\mathcal{R})$ are permutationally isomorphic.*

Proof. It is clear that $U(\Delta)|_{\mathrm{Ch}(\mathcal{R})} \subseteq U(\mathcal{R})$. Let g be an arbitrary element of $U(\mathcal{R})$ and let c_0 a chamber in $\mathcal{R}$. We want to prove that there exists a g' in the stabilizer of $\mathcal{R}$ in $U(\Delta)$ with the same action on $\mathrm{Ch}(\mathcal{R})$ as g.

Since $U(\Delta)$ is chamber-transitive there exists an $h \in U(\Delta)$ such that $h(c_0) = g(c_0)$. Note that h necessarily stabilizes the residue $\mathcal{R}$ because g does. The automorphism $h^{-1} \circ g$ fixes the chamber c_0, hence we may apply Lemma 4.3.7 and conclude that there exists an element g_0 with the same action as $h^{-1} \circ g$ on $\mathrm{Ch}(\mathcal{R})$. This yields that $g' := h \circ g_0$ has the same action on $\mathrm{Ch}(\mathcal{R})$ as g, proving the lemma. $\square$

Another way to state Lemma 4.3.8, and the way that we will normally refer to this result is stated in the next proposition.

Proposition 4.3.9. *Let $J \subseteq S$ and $\mathcal{R}$ be a residue of type J in Δ. Let $g \in U(\mathcal{R})$. Then g extends to an element $\widetilde{g} \in U$.*

Proof. This follows directly from Lemma 4.3.8. $\square$

4.4 Subgroups of U with support on wings

In this section we define subgroups of the universal group with support on a wing with respect to a tree-wall (*cf.* Definition 2.2.29) and we state a property for right-angled buildings that generalizes Tits independence property for groups acting on trees (see Definition 1.5.3). This property will be a key step to prove simplicity of the universal group.

We keep the notation from the previous sections. For $s \in S$ and $c \in \mathrm{Ch}(\Delta)$, let

$$\begin{aligned} V_s(c) &= \{g \in U \mid g(v) = v \text{ for all } v \notin X_s(c)\}, \\ U_s(c) &= \{g \in U \mid g(v) = v \text{ for all } v \in X_s(c)\}. \end{aligned} \tag{4.4.1}$$

Both the subgroups $V_s(c)$ and $U_s(c)$ fix the chamber c and stabilizes its s-panel. Since their supports are disjoint, $V_s(c)$ and $U_s(c)$ commute and have trivial intersection. In [Cap14, Section 5] similar subgroups are defined in the whole group of type-preserving automorphisms of a right-angled building.

The next few results demonstrate the importance of these subgroups of the universal group. In particular, the following proposition generalizes the *independence property* (see Definition 1.5.3) for groups acting on trees.

Proposition 4.4.1. *Let $c \in \mathrm{Ch}(\Delta)$ and $s \in S$. Let $\mathcal{T}$ be the s-tree-wall of c. Then*

$$\mathrm{Fix}_U(\mathcal{T}) = \prod_{d \in \mathcal{P}_{s,c}} V_s(d).$$

To be precise, Proposition 4.4.1 is a slight variation of the independence property, since an s-tree-wall in a tree is a single panel and therefore it corresponds to a star around a vertex in the tree.

We remark that when we consider wings with respect to an s-tree-wall $\mathcal{T}$ we can choose one of the s-panels of $\mathcal{T}$ and consider the s-wings with respect to that panel. The corresponding partition of the building in wings is independent of the choice of the panel the tree-wall.

Proof of Proposition 4.4.1. We start by showing that $\prod_{d \in \mathcal{P}_{s,c}} V_s(d)$ is a subgroup of the fixator $\mathrm{Fix}_U(\mathcal{T})$. Let $d \in \mathcal{P}_{s,c}$. Given $x \in \mathrm{Ch}(\mathcal{T})$, we deduce from Lemma 2.2.31 that $V_s(d)$ fixes all chambers of the s-panel $\mathcal{P}_{s,x}$ different from the projection of d to that panel. Hence $V_s(d)$ fixes $\mathcal{P}_{s,x}$. This proves that $V_s(d)$ is contained in $\mathrm{Fix}_U(\mathcal{T})$. As the supports of each of the subgroups $V_s(d)$ are disjoint, we can apply Lemma 4.3.4 and consider $\prod_{d \in \mathcal{P}_{s,c}} V_s(d)$ as a subgroup of the universal group and, in particular, as a subgroup of $\mathrm{Fix}_U(\mathcal{T})$.

In order to show the other inclusion, pick an element $g \in \mathrm{Fix}_U(\mathcal{T})$. Let d be an arbitrary chamber in the s-panel $\mathcal{P}_{s,c}$. Consider the permutation g_d of $\mathrm{Ch}(\Delta)$ defined as

$$g_d \colon \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) \colon x \mapsto \begin{cases} g(x) & \text{if } x \in X_s(d), \\ x & \text{otherwise.} \end{cases}$$

Proving that g_d is a type-preserving automorphism of Δ uses the same arguments of the proof of Proposition 5.2 in [Cap14]. Hence we refer to that for this part of the proof.

Clearly g_d fixes the tree-wall $\mathcal{T}$, therefore it preserves projections to the s-panels in that tree-wall. Hence it also preserves the s-wings with respect to the s-panel of c. Now we have to show that it is an element of the universal group, i.e., we have to prove that

$$h_t \circ g_d \circ (h_t|_{\mathcal{P}_{t,x}})^{-1} \in G^t, \text{ for all } t \in S \text{ and for all } x \in \mathrm{Ch}(\Delta).$$

We observe that any panel $\mathcal{P}$ not in the tree-wall $\mathcal{T}$ is not parallel to $\mathcal{P}_{s,c}$. Hence by Lemma 2.2.18 we conclude that $\mathrm{proj}_{\mathcal{P}_{s,c}}(\mathcal{P})$ is a chamber. Therefore either $\mathrm{Ch}(\mathcal{P}) \subseteq X_s(d)$ or $\mathrm{Ch}(\mathcal{P}) \subseteq \mathrm{Ch}(\Delta) \setminus X_s(d)$.

Let $\mathcal{P}$ be a t-panel for some $t \in S$. If $\mathcal{P}$ is in the tree-wall $\mathcal{T}$ or in one of its wings different from $X_s(d)$, then $\mathcal{P}$ is fixed by g_d so the permutation $h_t \circ g_d \circ (h_t|_{\mathcal{P}})^{-1}$ is the identity, therefore it is in G^t. If $\mathcal{P} \in X_s(d)$ then $g_d(x) = g(x)$ for all $v \in \mathrm{Ch}(\mathcal{P})$. Thus we obtain that $g_d \in U$ because g is also an element of the universal group.

We conclude that g_d is an element of $\mathrm{Fix}_U(\mathcal{T})$ and by construction it is also an element of $V_s(d)$. Moreover, the tuple $(g_d)_{d \in \mathcal{P}_{s,c}}$, which is an element of $\prod_{d \in \mathcal{P}_{s,c}} V_s(d)$, coincides with g. Therefore $g \in \prod_{d \in \mathcal{P}_{s,c}} V_s(d)$. □

In the same spirit we can exhibit the fixator of an $s^{\perp}$ residue using the groups in Equation (4.4.1) and its induced action on the respective tree-wall. Observe that all the chambers of an $s^{\perp}$-residue have the same s-color by definition of legal colorings (or weak legal colorings).

Lemma 4.4.2. *Let $s \in S$ and let Q be an $s^{\perp}$-residue of Δ. Let $c \in \mathrm{Ch}(Q)$ and $\mathcal{T}$ be the s-tree-wall containing Q. Then the following hold.*

1. $\mathrm{Fix}_U(Q) = V_s(c)U_s(c) = U_s(c)V_s(c)$.
2. *The induced action of* $\mathrm{Fix}_U(Q)$ *in the* s*-panels of* $\mathcal{T}$ *is permutationally isomorphic to* $\mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$*, where* α *is the* s*-color of the chambers of* Q*.*

Proof. Let us prove Statement 1. It is clear that $U_s(c)$ fixes Q so $U_s(c) \subseteq \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q)$. By definition $V_s(c)$ stabilizes Q. We claim that $V_s(c)$ fixes Q.

Assume that there exist $g \in V_s(c)$ and chambers $c_1, c_2 \in \mathrm{Ch}(Q)$ such that $gc_1 = c_2$. The other chambers in the s-panel of c_1 do not belong to $X_s(c)$ and therefore they must be fixed by g. Hence $\mathcal{P}_{s,c_1}$ coincides with $\mathcal{P}_{s,c_2}$ which implies that $c_1 = c_2$ since they are in the same s-wing. Therefore $V_s(c)$ fixes Q which implies the containment $V_s(c)U_s(c) \subseteq \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q)$.

Now let $g \in \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q)$ and let $c \in \mathrm{Ch}(Q)$. Consider the permutation g_c of the chambers

$$g_c \colon \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) \colon x \mapsto \begin{cases} g.x & \text{if } x \in X_s(c), \\ x & \text{otherwise.} \end{cases}$$

The proof that g_c is an element of the universal group is analogous as the reasoning carried out in Proposition 4.4.1. On the other hand, $g_c \in \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q)$ and by construction it is also an element of $V_s(c)$. Now define u_c as

$$u_c \colon \mathrm{Ch}(\Delta) \to \mathrm{Ch}(\Delta) \colon x \mapsto \begin{cases} x & \text{if } x \in X_s(c), \\ g.x & \text{otherwise.} \end{cases}$$

We want to show that also u_c is a type-preserving automorphism, that is, that for $x, y \in \mathrm{Ch}(\Delta)$ we have $\delta(x,y) = \delta(u_c x, u_c y)$. If x, y are both in $X_s(c)$ then u_g fixes these chambers. If both x and y are not in $X_s(c)$ then u_c acts on these chambers as g, which is a type-preserving automorphism.

So it remains to show the case where $x \in X_s(c)$ and $y \notin X_s(c)$. Let $x' = \mathrm{proj}_{\mathcal{T}}(x)$ and $y' = \mathrm{proj}_{\mathcal{T}}(y)$. By Lemma 3.3 in [Cap14] we have that

$$\delta(u_c x, u_c x')\delta(u_c x', u_c y')\delta(u_c y', u_c y)$$

is a minimal gallery. Observe that u_c stabilizes each s-panel of $\mathcal{T}$ since it fixes the chambers in $X_s(c)$. Therefore $u_c y' \overset{s}{\sim} y'$ and it follows that $u_c y' \notin X_s(c)$ as it is a permutation of the chambers. Hence $\mathrm{proj}_{\mathcal{P}_{s,y'}}(x') \notin \{y', u_c y'\}$ which implies that $\delta(x', y') = \delta(x', u_c y')$. We then obtain that

$$\begin{aligned}
\delta(u_c x, u_c y) &= \delta(u_c x, u_c x')\delta(u_c x', u_c y')\delta(u_c y', u_c y)\\
&= \delta(x, x')\delta(x' u_c y')\delta(gy, gy')\\
&= \delta(x, x')\delta(x', y')\delta(y, y')\\
&= \delta(x, y),
\end{aligned}$$

where the last step is obtained by applying [Cap14, Lemma 3.3] to $\delta(x, y)$. Therefore u_c is a type-preserving automorphism of Δ. To show that u_c is an element on the universal group we observe that if $\mathcal{P}$ is a panel in $\mathcal{T}$ then the induced action of u_c on $\mathcal{P}$ is the same as g since $g \in \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q)$. Otherwise we know, using the reasoning of proof of the last proposition, that $\mathcal{P}$ is completely contained in an s-wing with respect to $\mathcal{T}$. If $\mathcal{P}$ is a panel in $X_s(c)$ then $\mathcal{P}$ is fixed and therefore the permutation induced is the identity. If $\mathcal{P}$ is not in $X_s(c)$ then $u_c\mathcal{P} = g\mathcal{P}$ which is an element of the universal group. Thus $u_c \in U$ and, by construction, it belongs to $U_s(c)$.

Therefore the product $g_c u_c$, which is an element of $V_s(c)U_s(c)$, coincides with g. Thus $g \in V_s(c)U_s(c)$ and the first statement is proved.

Let us now prove Statement 2. Let $\mathcal{P}$ be an s-panel of $\mathcal{T}$. Let α be the s-color of $c = \mathrm{Ch}(Q) \cap \mathrm{Ch}(\mathcal{P})$. Since $\mathrm{Fix}_U(Q)$ fixes c, its induced action on $\mathcal{P}$ is permutationally isomorphic to a subgroup of $\mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$.

Conversely, take $g \in \mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$. It induces a permutation, also denoted g, of the chambers of $\mathrm{Ch}(\mathcal{P})$ fixing the chamber c. By Proposition 4.2.3, we can extend g to a tree-wall automorphism $\widetilde{g} \in U$ fixing the s-wing of c. Since $Q \subseteq X_s(c)$, we have that $\widetilde{g} \in \mathrm{Fix}_U(Q)$ and the local action of $\widetilde{g}$ on $\mathcal{P}$ is the initial element g. Thus the induced action of $\mathrm{Fix}_U(Q)$ on $\mathcal{P}$ is permutationally isomorphic to $\mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$. □

The next technical lemma will be used in the next section to prove that fixators of tree-walls are contained in any normal subgroup of the universal group.

Lemma 4.4.3. *Let $s \in S$ and let c_1 and c_2 be two s-adjacent chambers in an s-panel $\mathcal{P}$. Let $g \in U$, and let $s_1 \cdots s_n$ be a reduced representation of $\delta(c_2, gc_1)$. Assume moreover that*

(1) *there exists an $i \in \{1, \ldots, n\}$ such that $|s_i s| = \infty$, and*

(2) $\operatorname{proj}_{\mathcal{P}}(gc_1) = c_2$ *and* $\operatorname{proj}_{\mathcal{P}_{s,gc_1}}(c_2) = gc_1$.

Then for each $h \in \prod_{d \in \mathrm{Ch}(\mathcal{P}) \setminus \{c_1, c_2\}} V_s(d)$, there exists an element $u \in U$ such that $h = [u, g] = ugu^{-1}g^{-1}$.

Proof. Let $V_0 = \prod_{d \in \mathrm{Ch}(\mathcal{P}) \setminus \{c_1, c_2\}} V_s(d)$. We know that V_0 is subgroup of U. For each $n > 0$ let

$$\mathcal{P}_n = g^n(\mathcal{P}), \; c_1^n = g^n(c_1), \; c_2^n = g^n(c_2) \; \text{ and } \; V_n = g^n V_0 g^{-n}.$$

For each $n \geq 0$ the support of the group V_n is contained in the set $\bigcup_{d \in \mathrm{Ch}(\mathcal{P}_n) \setminus \{c_1^n, c_2^n\}} X_s(d)$. Since $\operatorname{proj}_{\mathcal{P}}(g.c_1) = c_2$ and $\operatorname{proj}_{\mathcal{P}_{s,g.c_1}}(c_2) = g.c_1$, given a chamber $d \in \mathrm{Ch}(\mathcal{P}_n) \setminus \{c_1^n, c_2^n\}$ and $m > n$ we have

$$d \in X_s(c_1^m) \text{ and } c_1^m \notin X_s(d).$$

Thus $X_s(d) \subset X_s(c_1^m)$ by [Cap14, Lemma 3.4]. Similarly we have $X_s(c) \subset X_s(c_2^n)$ for any $c \in \mathrm{Ch}(\mathcal{P}_m) \setminus \{c_1^m, c_2^m\}$. This implies that the sets

$$\bigcup_{d \in \mathrm{Ch}(\mathcal{P}_n) \setminus \{c_1^n, c_2^n\}} X_s(d) \quad \text{and} \quad \bigcup_{d \in \mathrm{Ch}(\mathcal{P}_m) \setminus \{c_1^m, c_2^m\}} X_s(d)$$

are disjoint. In other words, this means that, for $m > n \geq 0$, the products V_m and V_n have disjoint support. Using Lemma 4.3.4 we know that the direct product $V = \prod_{n \geq 0} V_n$ is a subgroup of U. Moreover $gV_n g^{-1} = V_{n+1}$.

Let $h \in V_0$. For each $n \in \mathbb{N}$, let $u_n = g^n h g^{-n}$. Then the tuple $u = (u_n)_{n \geq 0}$ is an element of the product $V \leq U$ and so is the commutator $[u, g]$. We observe that the commutator $[u, g]$ fixes c_1^n and c_2^n for all $n \geq 0$.

Furthermore, denoting by y_n the n-th component of an element $y \in V$ according to the decomposition $V = \prod_{n \geq 0} V_n$ we obtain that $[u, g]_n = u_n (gu^{-1}g^{-1})_n$ for all $n \geq 0$. Hence $[u, g]_0 = h$ and

$$[u, g]_n = u_n g u_{n-1}^{-1} g^{-1} = u_n u_n^{-1} = 1.$$

Therefore $[u, g] = h$, which proves the lemma. □

4.5 Action of U on the tree-wall trees

A key point in proving simplicity of the universal group will be to prove that a normal subgroup of U contains the fixators of tree-walls. Therefore, having that as a motivation, it is suitable to have a closer look at the action of the universal group on the tree-wall trees presented in Definition 2.2.37.

That is the goal of this section. We will prove that any normal subgroup of U induces a translation axis in any tree-wall tree and we will also present some results regarding the action of open subgroups of the universal group on these trees.

We recall the notation that we are using. Let (W, S) be a right-angled Coxeter system and $(q_s)_{s\in S}$ be a set of cardinal numbers. Consider Δ the unique semi-regular right-angled building of type (W, S) and prescribed thickness $(q_s)_{s\in S}$. For each $s \in S$, let $G^s \leq \mathrm{Sym}(q_s)$ be a transitive permutation group and let U denote the universal group with respect to the groups $\{G^s\}_{s\in S}$.

Definition 4.5.1. For every $s \in S$, any automorphism of the right-angled building Δ also induces an automorphism of the s-tree-wall tree Γ_s. As the universal group U acts chamber-transitively on Δ, it has a natural edge-transitive type-preserving action on this tree Γ_s.

We say that an element $g \in U$ is *s-hyperbolic*, for $s \in S$, if it induces a hyperbolic action on the s-tree-wall tree Γ_s.

Lemma 4.5.2. *If the right-angled building Δ is irreducible, then the universal group U acts faithfully on the s-tree-wall tree Γ_s, for all $s \in S$.*

Proof. Assume by way of contradiction that some non-trivial group element $g \in U$ acts trivially on Γ_s for some $s \in S$. This implies that g stabilizes every s-tree-wall and every residue of type $S \setminus \{s\}$ in the building Δ. The residues of types $s^\perp$, which are the non-trivial intersections of these two, are hence also stabilized. As g is non-trivial, there exist distinct chambers c_1 and c_2 of Δ such that $gc_1 = c_2$. These chambers have to be contained in a common residue $\mathcal{R}$ of type $s^\perp$. Let $s_1 \cdots s_n$ be a reduced word representing the Weyl distance between c_1 and c_2.

For each $i \in \{1, \ldots, n\}$, let γ_i be a shortest path in the Coxeter diagram Σ between s_i and an element $t \in S$ such that $|st| = \infty$. Both the elements t and the paths exist because we are assuming Σ to be connected, i.e., Δ to be irreducible.

Let γ_j be such a path of minimal length between s_i and t (minimized over all possible $i \in \{1, \ldots, n\}$ and elements t). Denote $\gamma_j = (r_1, \ldots, r_k)$, with $r_1 = s_j$ and $r_k = t$ with $|ts| = \infty$. We observe that $|r_i r_{i+1}| = \infty$ for all i by definition of a path in Σ. Moreover, $r_i \in s^\perp \setminus \{s_1, \ldots, s_n\}$ for all $i \in \{2, \ldots k-1\}$ since γ_j was chosen as a shortest path of minimal length. Therefore

$$\begin{aligned} w_1 &= r_{k-1} \cdots r_2 s_1 \cdots s_n r_2 \cdots r_{k-1} \text{ and} \\ w_2 &= t r_{k-1} \cdots r_2 s_1 \cdots s_n r_2 \cdots r_{k-1} t \end{aligned}$$

are reduced words in M_S with respect to Σ.

Pick a chamber d_1 at Weyl distance $r_2 \cdots r_{k-1}$ from c_1 and let $d_2 = g d_1$. Observe that the word w_1 represents the Weyl distance from d_1 to d_2. Hence d_1 and d_2 are also in the $s^\perp$-residue $\mathcal{R}$ and therefore in the same s-tree-wall $\mathcal{T}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the t-panels of d_1 and d_2, respectively. These panels are not contained in $\mathcal{R}$ as $|ts| = \infty$. So, if $x_1 \in \mathrm{Ch}(\mathcal{P}_1) \setminus \{d_1\}$, then the s-tree-wall $\mathcal{T}_1$ of x_1 is distinct from $\mathcal{T}$.

Let $x_2 = g x_1$. Then $x_2 \in \mathrm{Ch}(\mathcal{P}_2) \setminus \{d_2\}$. Analogously, the s-tree-wall $\mathcal{T}_2$ of x_2 does not coincide with $\mathcal{T}$. Moreover w_2 is a reduced representation of the Weyl distance between x_1 and x_2. This implies that $\mathcal{T}_1$ and $\mathcal{T}_2$ are distinct because $|r_i r_{i+1}| = \infty$ for every element $i \in \{2, \ldots, k-1\}$ and $r_k = t$. As g maps x_1 to x_2, it maps $\mathcal{T}_1$ to $\mathcal{T}_2$. We have hence arrived at a contradiction as g should stabilize $\mathcal{T}_1$, since it acts trivially on the s-tree-wall tree Γ_s. Thus we conclude that the action is faithful. □

Lemma 4.5.3. *Assume that Δ is thick and irreducible. Then every non-trivial normal subgroup of U contains an s-hyperbolic element, for every $s \in S$.*

Proof. Let $s \in S$ and N be a non-trivial normal subgroup of U. Assume that N does not contain any s-hyperbolic element. By [Tit70, Proposition 3.4] we know that either N fixes some vertex or it fixes an end of the tree Γ_s, as all its elements are elliptic.

First assume that N fixes some vertex. This implies, since N is normal in the type-preserving edge-transitive group U, that N fixes all vertices of the same type of this vertex. Therefore it fixes every vertex of the tree Γ_s and hence, by Lemma 4.5.2, it contradicts the non-triviality of N.

Next assume that N fixes an end of the tree. Again using normality and edge-transitivity (and the fact that the tree Γ_s contains vertices of valency at least three because Δ is thick), we obtain that N fixes at least three ends of Γ_s. Since three ends of a tree determine a unique vertex we get that N has a global fixed point, a possibility already handled in the previous paragraph.

We conclude then that for every $s \in S$, the normal subgroup N contains an s-hyperbolic element. □

Next we investigate the action of open subgroups of U on the tree-wall trees. Since we want a faithful action on the tree-wall trees, we assume henceforth that the building Δ is irreducible.

Lemma 4.5.4. *Let $s \in S$. Let $H \leq U$ be an open subgroup with an s-hyperbolic element h in its action on the s-tree-wall tree Γ_s. Then* $\mathrm{Fix}_U(Q) \subseteq H$ *for any $s^\perp$-residue Q of Δ corresponding to an edge in the axis A.*

Proof. Let Q be an $s^\perp$-residue corresponding to an edge e in the axis A.

Since H is open, there exists, by definition, a finite set $C \subseteq \mathrm{Ch}(\Delta)$ such that the fixator $\mathrm{Fix}_U(C)$ is contained in H. This finite set of chambers corresponds to a finite set of vertices (of s-tree-wall type for instance) in Γ_s. Therefore we can apply enough positive and negative powers of the hyperbolic element h so that

$$C_1 = h^n(C) \subseteq X_s(c) \text{ and } C_2 = h^{-m}(C) \cap X_s(c) = \emptyset.$$

Then we have that $\mathrm{Fix}_U(C_i) = \langle \mathrm{Fix}_U(C), h\rangle \subset H$, for $i \in \{1, 2\}$.

Moreover $V_s(c) \subseteq \mathrm{Fix}_U(C_2)$ and $U_s(c) \subseteq \mathrm{Fix}_U(C_1)$. Since we have $\mathrm{Fix}_U(Q) = \langle V_s(c), U_s(c)\rangle$ by Lemma 4.4.2(1), we obtain that $\mathrm{Fix}_U(Q) \subset H$, as desired. □

Corollary 4.5.5. *Fix $s \in S$. Let $H \leq \mathrm{Aut}(\Delta)$ be an open subgroup.*

1. $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathcal{T}) \subseteq H$ *for all s-tree-walls in an s-axis of an s-hyperbolic element of H.*

2. $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathcal{R}) \subseteq H$ *for all $S\backslash\{s\}$-residues in an s-axis of an s-hyperbolic element of H.*

Proof. This follows from Lemma 4.5.4. Since any $s^{\perp}$-residue Q corresponding to an edge e in an s-axis in Γ_s, we have that $Q \subseteq \mathcal{T}$ and $Q \subseteq \mathcal{R}$, where $\mathcal{T}$ and $\mathcal{R}$ are the s-tree-wall and the $S\backslash\{s\}$-residue, respectively, corresponding to the vertices of e. Then $\mathrm{Fix}_{\mathrm{Aut}(\Delta)}(\mathcal{T}) \subseteq \mathrm{Fix}_{\mathrm{Aut}(\Delta)}(Q) \subseteq H$ and the same holds for $\mathcal{R}$. $\square$

Now we prove, if the s-local action is primitive and generated by point stabilizers, that we can conclude further results on the action of open subgroups of U with s-hyperbolic elements.

We first present a lemma of group theoretical nature.

Lemma 4.5.6. *If $G \leq \mathrm{Sym}(Y)$ is primitive and generated by point stabilizers then, for all $\alpha \in Y$, the stabilizer $\mathrm{Stab}_G(\alpha)$ only fixes α. In particular $G = \langle \mathrm{Stab}_G(\alpha), \mathrm{Stab}_G(\beta)\rangle$, for any $\alpha \neq \beta$ in Y.*

Proof. Let us define an equivalence relation on Y by $\alpha \sim \beta \Leftrightarrow \mathrm{Stab}_G(\alpha) = \mathrm{Stab}_G(\beta)$. The relation $\sim$ is a congruence relation on G since if $g \in G$ then, for $\alpha \sim \beta$ we have

$$\mathrm{Stab}_G(g\alpha) = g\,\mathrm{Stab}_G(\alpha)g^{-1} = g\,\mathrm{Stab}_G(\beta)g^{-1} = \mathrm{Stab}_G(g\beta)$$

that is, $g\alpha \sim g\beta$.

As G is primitive either the equivalence classes are singletons or there is only one equivalence class. If there is only one equivalence class then all the point stabilizers are equal, which is a contradiction to the fact that G is generated by point stabilizers. Therefore the equivalence classes of $\sim$ are singletons. Hence if $\mathrm{Stab}_G(\beta) \neq \mathrm{Stab}_G(\alpha)$ for some $\alpha, \beta \in Y$, since these stabilizers are conjugate, we obtain that $\mathrm{Stab}_G(\beta) = \mathrm{Stab}_G(\alpha)$ which implies that $\alpha = \beta$ by the assumption on $\sim$. Thus $\mathrm{Stab}_G(\alpha)$ only fixes α. $\square$

Lemma 4.5.7. *Let $s \in S$ and assume that G^s is primitive and generated by point stabilizers. Let H be an open subgroup of U with an s-hyperbolic element with axis A in the s-tree-wall tree Γ_s. Let $\mathcal{T}$ be an s-tree-wall of Δ.*

1. *The local action of H on the s-panels of $\mathcal{T}$ is permutationally isomorphic to* $\mathrm{Sym}(q_s)$*.*
2. $\mathrm{Fix}_U(Q) \subseteq H$ *for all $s^\perp$-residues Q of $\mathcal{T}$.*

Proof. Let $\mathcal{P}$ be an s-panel of $\mathcal{T}$. Let e_1 and e_2 be the two edges in the star of $\mathcal{T}$ (regarded as a vertex in Γ_s) that are in the axis and let Q_1 and Q_2 be the corresponding $s^\perp$-residues of Δ. We know by Lemma 4.5.4 that $\mathrm{Fix}_U(Q_1)$ and $\mathrm{Fix}_U(Q_2)$ are contained in H.

Moreover, by Lemma 4.4.2(2), the induced action of $\mathrm{Fix}_U(Q_i)$ in $\mathrm{Ch}(\mathcal{P})$ is permutationally isomorphic to $\mathrm{Stab}_{G^s}(\alpha_i)$, where α_i is the s-color of all the chambers in Q_i, for $i \in \{1, 2\}$. Furthermore $\alpha_1 \neq \alpha_2$.

On the other hand, as by hypothesis $G^s = \langle \mathrm{Stab}_{G^s}(\alpha), \mathrm{Stab}_{G^s}(\beta)\rangle$, for any $\alpha \neq \beta \in Y_s$, the induced action of H on $\mathrm{Ch}(\mathcal{P})$ contains (up to permutational isomorphism) $\langle \mathrm{Stab}_{G^s}(\alpha_1), \mathrm{Stab}_{G^s}(\alpha_2)\rangle = G^s$. Therefore Statement 1 follows.

Now let Q be an $s^\perp$-residue of $\mathcal{T}$. Let $\mathcal{P}$ be an s-panel of $\mathcal{T}$. We know by the previous statement that the induced action of H on $\mathcal{P}$ is permutationally isomorphic to G^s. Let $c = \mathrm{Ch}(Q) \cap \mathrm{Ch}(\mathcal{P})$ and let $c' = \mathrm{Ch}(Q') \cap \mathrm{Ch}(\mathcal{P})$, where Q' is an $s^\perp$-residue of $\mathcal{T}$ corresponding to an edge in the axis. In particular $\mathrm{Fix}_U(Q') \subseteq H$ by Lemma 4.5.4.

As G^s is transitive, there exists $g \in G^s$ such that $gh_s(c') = h_s(c)$. We can extend this permutation g to an element $\widetilde{g} \in H$ and then we have that $\mathrm{Fix}_U(Q) = \widetilde{g}\,\mathrm{Fix}_U(Q')\widetilde{g}^{-1}$. Therefore $\mathrm{Fix}_U(Q) \subseteq H$. □

We finish the section by proving that any s-tree-wall can be considered to be in an s-axis of an element of the universal group.

Lemma 4.5.8. *Let $s \in S$ and assume that G^s is primitive and generated by point stabilizers. Then every s-tree-wall is contained in the axis of an s-hyperbolic element of U.*

Proof. It is clear that U contains s-hyperbolic elements as Δ is irreducible and U is chamber-transitive. Let h be an s-hyperbolic element of U with axis A.

Let $\mathcal{T}$ be an s-tree-wall of Δ not in A. Let $c = \mathrm{proj}_{\mathcal{T}}(A)$ and let $\mathcal{P}$ be its s-panel. Observe that $A \subseteq X_s(c)$.

Let $g \in G^s$ such that $gh_s(c) \neq h_s(c)$. Such an element exists by Lemma 4.5.6. We can extend this permutation of the chambers of $\mathcal{P}$ to an element $\widetilde{g} \in U$ stabilizing $\mathcal{P}$ which does not stabilize A.

Let Q_1 be an $s^\perp$-residue in A such that $X_s(Q_1) \cap \mathcal{T} = \emptyset$ and let Q_2 be the $s^\perp$-residue in the same s-tree-wall such that $T \subseteq X_s(Q_2)$. Let $g_2 \in \mathrm{Fix}_U(Q_1)$ not stabilizing Q_2. This element exists because the induced action of $\mathrm{Fix}_U(Q_1)$ on the s-panels of this tree-wall is permutationally isomorphic to $\mathrm{Stab}_{G^s}(h_s(Q_1))$, which only fix $h_s(Q_1)$ by Lemma 4.5.6. Then g_2 also does not stabilize the initial s-tree-wall $\mathcal{T}$.

Both elements $\widetilde{g}$ and g_2 are s-elliptic on their action on the s-tree-wall tree since they stabilize $\mathcal{T}$ and Q_1, respectively. Moreover, their fixed vertex sets in Γ_s are disjoint. Therefore $\widetilde{g}g_2$ is an s-hyperbolic element of U by [Ser80, Proposition 26] and by construction $\mathcal{T}$ is in the s-axis of $\widetilde{g}g_2$. □

4.6 Simplicity of the universal group

In this section we prove the simplicity of the universal group provided that the following two conditions are satisfied.

(IR) The right-angled building is thick, irreducible, semi-regular and has rank ≥ 2.

(ST) For each $s \in S$, the group G^s is transitive and generated by its point stabilizers.

Remark 4.6.1. We observe that if we make the stronger assumption that the local action of the universal group is given by 2-transitive groups, i.e. if each local group $G^s \leq \mathrm{Sym}(Y_s)$ is assumed to be 2-transitive, then we can use similar arguments to the ones in [Cap14, Proposition 6.1] to show that the action of U on $\mathrm{Ch}(\Delta)$ is strongly transitive and to prove simplicity in that manner.

The second condition (ST) is necessary, as is clear from the following proposition.

Proposition 4.6.2. *Let U^+ be the subgroup of U generated by chamber stabilizers. Then $U = U^+$ if and only if for every $s \in S$ the group G^s is transitive and generated by point stabilizers.*

Proof. Notice that, in the definition of U, we already assume the groups G^s to be transitive. Assume first that $U = U^+$. Fix an element

$s \in S$. We will show that G^s is generated by its point stabilizers. Let $(G^s)^+$ denote the subgroup of G^s generated by the point stabilizers $(G^s_\alpha)_{\alpha \in Y_s}$.

For any s-panel $\mathcal{P}$ and any $g \in U$, we define

$$\sigma(g, \mathcal{P}) := h_s|_{g.\mathcal{P}} \circ g \circ (h_s|_{\mathcal{P}})^{-1}, \tag{4.6.1}$$

which is an element of G^s by the very definition of the universal group U. Observe that

$$\sigma(g_1 g_2, \mathcal{P}) = \sigma(g_1, g_2\mathcal{P}) \circ \sigma(g_2, \mathcal{P}) \tag{4.6.2}$$

for all $g_1, g_2 \in U$.

We first claim that if $g \in \mathrm{Stab}_U(c)$ for some chamber $c \in \mathrm{Ch}(\Delta)$, then

$$\sigma(g, \mathcal{P}) \in (G^s)^+ \tag{4.6.3}$$

for all s-panels $\mathcal{P}$ of Δ. We will prove this claim by induction on the distance

$$\mathrm{dist}(c, \mathcal{P}) := \min\{\mathrm{dist}(c, d) \mid d \in \mathrm{Ch}(\mathcal{P})\}.$$

If $\mathrm{dist}(c, \mathcal{P}) = 0$, then g stabilizes $\mathcal{P}$ since g fixes c. Hence $\sigma(g, \mathcal{P}) \in G^s_{h_s(c)} \leq (G^s)^+$.

Assume now that the claim is true whenever $\mathrm{dist}(c, \mathcal{P}) \leq n$ and let $\mathcal{P}$ be a panel at distance $n+1$ from c. Then there exists an s-panel $\mathcal{P}_1$ at distance n from c such that there are chambers $d \in \mathrm{Ch}(\mathcal{P})$ and $d_1 \in \mathrm{Ch}(\mathcal{P}_1)$ that are t-adjacent for some $t \neq s$. Then $h_s(d) = h_s(d_1)$; denote this value by $\alpha \in Y_s$. Similarly, $g.d$ and $g.d_1$ are t-adjacent, hence $h_s(g.d) = h_s(g.d_1)$. By (4.6.1), this implies $\sigma(g, \mathcal{P}_1)(\alpha) = \sigma(g, \mathcal{P})(\alpha)$. We conclude that $\sigma(g, \mathcal{P}) = \sigma(g, \mathcal{P}_1) g_\alpha$ for some $g_\alpha \in G^s_\alpha$. By our induction hypothesis, $\sigma(g, \mathcal{P}_1) \in (G^s)^+$, and therefore $\sigma(g, \mathcal{P}) \in (G^s)^+$, which proves the claim (4.6.3).

Now let $g \in G^s$ be arbitrary. Choose an arbitrary s-panel $\mathcal{P}$. By Lemma 4.2.2, the action of $U|_{\{\mathcal{P}\}}$ on $\mathcal{P}$ is permutationally isomorphic to G^s, so in particular, we can find an element $u \in U|_{\{\mathcal{P}\}}$ such that its local action on $\mathcal{P}$ is given by g, that is, $g = \sigma(u, \mathcal{P})$. Since $U = U^+$, we can write u as a product $u_{c_1} \cdots u_{c_n}$, where each u_{c_i} fixes a chamber c_i. Then $g = \sigma(u, \mathcal{P}) = \sigma(u_{c_1} \cdots u_{c_n}, \mathcal{P})$. It now follows from (4.6.2) and (4.6.3) that $g \in (G^s)^+$, so G^s is indeed generated by its point stabilizers.

Conversely, assume that for each $s \in S$, the group G^s is generated by point stabilizers. Let $s \in S$ and $\mathcal{P}$ be an s-panel. Then, as $U|_{\{\mathcal{P}\}}$ is permutationally isomorphic to G^s, if $u \in U|_{\{\mathcal{P}\}}$ for some s-panel $\mathcal{P}$, then $h_s \circ u \circ (h_s)^{-1}|_{\mathcal{P}} = g_{\alpha_1} \cdots g_{\alpha_n}$, where g_{α_i} fixes the color α_i. Lift arbitrarily the elements $g_{\alpha_1}, \dots, g_{\alpha_{n-1}}$ to elements $u_{c_1}, \dots, u_{c_{n-1}} \in U$, and then define $u_{c_n} \in U$ such that $u = u_{c_1} \cdots u_{c_n}$. Then each u_{c_i} fixes the corresponding chamber $c_i \in \mathcal{P}$ of color α_i, and hence $u \in U^+$. As G^s is transitive, U^+ contains elements of U mapping one chamber to any other s-adjacent chamber. Since this is true for any $s \in S$, the subgroup U^+ is chamber-transitive. As in addition U^+ contains the full chamber stabilizers in U, we conclude that U^+ is indeed all of U. □

Next we prove that fixators of tree-walls are contained in any normal subgroup of the universal group.

Proposition 4.6.3. *Assume that Δ is thick and of irreducible type. Let $s \in S$ and $\mathcal{T}$ be an s-tree-wall. Then any non-trivial normal subgroup N of U contains $\mathrm{Fix}_U(\mathcal{T})$.*

Proof. Let $g \in N$ be an s-hyperbolic element with axis $A(g)$, which exists by Lemma 4.5.3. Let $\mathcal{T}$ be an s-tree-wall for which the corresponding vertex of Γ_s lies on the axis $A(g)$. Let $\mathcal{P}$ be an s-panel of $\mathcal{T}$, $\mathcal{P}_1 = g(\mathcal{P})$ and $\mathcal{T}_1 = g(\mathcal{T})$ (so $\mathcal{P}_1 \in \mathcal{T}_1$). We note that $\mathcal{T}$ and $\mathcal{T}_1$ are distinct since g is s-hyperbolic.

Let $c = \mathrm{proj}_{\mathcal{P}}(\mathcal{P}_1)$ and $c_1 = \mathrm{proj}_{\mathcal{P}_1}(\mathcal{P})$ (the projections are unique chambers since the panels are not parallel). We claim that $g(c) \neq c_1$. Assume by way of contradiction that $g(c) = c_1$. Then the residue $\mathcal{R} = \mathcal{R}_{S\setminus\{s\},c}$ is mapped to $\mathcal{R}_1 = \mathcal{R}_{S\setminus\{s\},c_1}$, both corresponding to vertices of Γ_s. Both of these residues belong to the wings $X_s(c)$ and $X_s(c_1)$ implying that $\mathrm{dist}(\mathcal{R}, \mathcal{R}_1)$ (as vertices of Γ_s) is strictly smaller than $\mathrm{dist}(\mathcal{T}, \mathcal{T}_1)$. This a contradiction to the fact that $\mathcal{T} \in A(g)$. Hence $g(c) \neq c_1$.

Let $c_2 = g^{-1}c_1$. Applying Lemma 4.4.3 to c and c_2 combined with the fact that N is a normal subgroup of U yields that

$$\prod_{d \in \mathrm{Ch}(\mathcal{P})\setminus\{c,c_2\}} V_s(d) \subseteq N.$$

As $\mathcal{P}$ is thick, we can pick a chamber c_3 in $\mathcal{P} \setminus \{c, c_2\}$. As G^s is transitive, Lemma 4.2.2 implies that there exists a $u \in U$ such that $uc_3 = c$. We observe that

$$u^{-1} \circ V_s(c_3) \circ u = V_s(uc_3) = V_s(c),$$

hence $V_s(c) \subseteq N$. Analogously one proves that $V_s(c_2) \subseteq N$. We therefore obtain that

$$\prod_{d \in \mathrm{Ch}(\mathcal{P})} V_s(d) \subseteq N$$

and, by Proposition 4.4.1, that $\mathrm{Fix}_U(\mathcal{T}) \subseteq N$.

Since U is chamber-transitive, there exists, for each tree-wall $\mathcal{T}'$, an element $h \in U$ mapping $\mathcal{T}$ to $\mathcal{T}'$. Hence $h^{-1} \circ \mathrm{Fix}_U(\mathcal{T}) \circ h = \mathrm{Fix}_U(\mathcal{T}') \subseteq N$. □

Proposition 4.6.4. *Let Δ be a right-angled building satisfying (IR) and (ST). Let N be a non-trivial normal subgroup of U. Then for each panel $\mathcal{P}$ of Δ, we have that* $\mathrm{Fix}_N(\mathcal{P}) \neq \mathrm{Stab}_N(\mathcal{P})$*.*

Proof. Let $s \in S$ and $\mathcal{P}$ be an s-panel of Δ. Let $g \in G_0^s$ be a non-trivial element. Then g induces a non-trivial permutation of $\mathrm{Ch}(\mathcal{P})$ fixing some $c_0 \in \mathrm{Ch}(\mathcal{P})$. By Proposition 4.2.3, we can find a corresponding tree-wall automorphism $\widetilde{g} \in U$ stabilizing $\mathcal{P}$, acting locally as g on $\mathcal{P}$, and fixing the chambers in the s-wing $X_s(c_0)$ (considering Δ to be directed with respect to a chamber in the wing $X_s(c_0)$).

Let $t \in S$ be such that $|ts| = \infty$ (such an element always exists because Δ is irreducible). Let $\mathcal{T}$ be the t-tree-wall of c_0. Then $\mathrm{Ch}(\mathcal{T}) \subseteq X_s(c_0)$. Therefore $\widetilde{g}$ fixes $\mathcal{T}$. Since $\widetilde{g} \in \mathrm{Fix}_U(\mathcal{T})$ we have $\widetilde{g} \in N$ by Proposition 4.6.3. We conclude hence that $\widetilde{g} \in N \cap \mathrm{Stab}_U(\mathcal{P})$ and thus $\widetilde{g} \in \mathrm{Stab}_N(\mathcal{P}) \setminus \mathrm{Fix}_N(\mathcal{P})$. □

With the previous proposition at hand we can show that a normal subgroup of the universal group is transitive on the chambers of the building. That is achieved in the next two results.

Lemma 4.6.5. *Let Δ be a right-angled building satisfying (IR) and (ST). Let N be a normal subgroup of U and $\mathcal{P}$ be an s-panel from some $s \in S$. Then* $\mathrm{Stab}_N(\mathcal{P}) / \mathrm{Fix}_N(\mathcal{P})$ *is permutationally isomorphic to G^s.*

Proof. Let T denote $\mathrm{Stab}_N(\mathcal{P})/\mathrm{Fix}_N(\mathcal{P})$. By Proposition 4.6.4 we have that $G_0^s \leq T$, up to permutational isomorphism, where G_0^s fixes the color 1 of Y_s. Let $c_0 \in \mathrm{Ch}(\mathcal{P})$ with s-color 1.

Let $c \in \mathrm{Ch}(\mathcal{P})\backslash\{c_0\}$ and $g \in U$ such that $gc = c_0$. Then $g' = g|_{\mathcal{P}}$ is such that $G^s_{h(c)} = g' G_0^s g'^{-1}$.

Let $g_1 \in G^s_{h(c)}$. Then there is $g_2 \in G_0^s$ such that $g_1 = g' g_2 g'^{-1}$. By Lemma 4.6.4, there exists $\widetilde{g_2} \in N$ such that $\widetilde{g_2}|_{\mathcal{P}} = g_2$. Thus $g\widetilde{g_2}g^{-1} \in N$ and $g\widetilde{g_2}g^{-1}|_{\mathcal{P}} = g' g_2 g'^{-1} = g_1$. Therefore $G^s_{h(c)} \leq T$. Since $G^s = \langle G^s_\alpha, G^s_\beta\rangle$ for any $\alpha \neq \beta$, the results follows. □

The next corollary is a direct consequence of the previous lemma.

Corollary 4.6.6. *Let Δ be a right-angled building satisfying (IR) and (ST). The group U is quasi-primitive in* $\mathrm{Ch}(\Delta)$. *Therefore any normal subgroup of U acts transitively on the chambers of Δ.*

We are now ready to prove the main result of this section.

Theorem 4.6.7. *Let Δ be a thick semi-regular right-angled building of irreducible type (W, S) with prescribed thickness $(q_s)_{s\in S}$ and rank at least 2. For each $s \in S$, let $h_s\colon \mathrm{Ch}(\Delta) \to Y_s$ be a legal s-coloring and $G^s \leq \mathrm{Sym}(Y_s)$ be a transitive group generated by point stabilizers.*

Then the universal group U of Δ with respect to the groups $(G^s)_{s\in S}$ is simple.

Proof. We first observe that by Proposition 4.6.2, U coincides with U^+. We will prove the simplicity by induction on the rank of Δ.

If Δ has rank 2 then the simplicity of $U^+ = U$ follows from [Tit70, Theorem 4.5] since by definition U is the universal group of a biregular tree and it has Tits independence property.

Now assume that the rank of Δ is at least three, and that we have proven simplicity for lower rank. Let N be a non-trivial normal subgroup of the universal group U. It suffices to show that N contains the chamber stabilizer $\mathrm{Stab}_U(c)$ of c in U, for each chamber $c \in \mathrm{Ch}(\Delta)$. Let then $c \in \mathrm{Ch}(\Delta)$. We will show that the stabilizers $\mathrm{Stab}_N(c)$ and $\mathrm{Stab}_U(c)$ coincide, showing what we want.

Pick a generator $s \in S$ such that $S \setminus \{s\}$ is irreducible. (Note that this is always possible, by picking s to be a leaf of a spanning tree of the unlabeled Coxeter diagram.) Let $\mathcal{R}$ be the $S \setminus \{s\}$-residue of

Δ containing c. This residue is a right-angled building on its own of irreducible type and of rank at least two.

Claim 1. *N contains the fixator $\mathrm{Fix}_U(\mathcal{R})$ of $\mathcal{R}$ in U.*

Let r be an element of S not commuting with s (which is always possible by the irreducibility of the Coxeter system). The set $\{r\} \cup r^{\perp}$ is a subset of $S \setminus \{s\}$ and therefore the residue $\mathcal{R}$ contains a residue of type $\{r\} \cup r^{\perp}$, which forms an r-tree-wall $\mathcal{T}$. The normal subgroup N contains the fixator $\mathrm{Fix}_U(\mathcal{T})$ by Proposition 4.6.3, hence it also contains its subgroup $\mathrm{Fix}_U(\mathcal{R})$.

Claim 2. *The stabilizer $\mathrm{Stab}_N(\mathcal{R})$ maps surjectively onto $U(\mathcal{R})$.*

We first observe that the image of $N \cap \mathrm{Stab}_U(\mathcal{R})$ in $U(\mathcal{R})$ (which is permutationally isomorphic to $\mathrm{Stab}_U(\mathcal{R})/\,\mathrm{Fix}_U(\mathcal{R})$) is non-trivial since $\mathrm{Stab}_U(\mathcal{P}) \subseteq \mathrm{Stab}_U(\mathcal{R})$ for any panel $\mathcal{P}$ in $\mathcal{R}$ and $\mathrm{Fix}_N(\mathcal{P}) \neq \mathrm{Stab}_N(\mathcal{P})$ by Proposition 4.6.4.

By induction on the rank we know that $U(\mathcal{R})$ is simple. Since moreover the natural homomorphism from $\mathrm{Stab}_U(\mathcal{R})$ to $U(\mathcal{R})$ is surjective by Proposition 4.3.9, it follows that it remains surjective in restriction to $N \cap \mathrm{Stab}_U(\mathcal{R})$. We conclude then that $\mathrm{Stab}_N(\mathcal{R})$ maps surjectively to $U(\mathcal{R})$, by simplicity of $U(\mathcal{R})$, proving Claim 2.

The chamber stabilizers $\mathrm{Stab}_N(c)$ and $\mathrm{Stab}_U(c)$ also stabilize the residue $\mathcal{R}$, hence we may consider their image in the universal group $U(\mathcal{R})$. By Claim 2, the images of $\mathrm{Stab}_N(c)$ and $\mathrm{Stab}_U(c)$ in $U(\mathcal{R})$ are both equal to the entire group $\mathrm{Stab}_{U(\mathcal{R})}(c)$. The kernels of the maps from $\mathrm{Stab}_N(c)$ and $\mathrm{Stab}_U(c)$ to $U(\mathcal{R})$ are the fixators $\mathrm{Fix}_N(\mathcal{R})$ and $\mathrm{Fix}_U(\mathcal{R})$, respectively, which also coincide by Claim 1. We conclude that $\mathrm{Stab}_U(c)$ and $\mathrm{Stab}_N(c)$ are equal for all c, hence N contains all chamber stabilizers U_c and thus coincides with U by Proposition 4.6.2, proving the simplicity. $\square$

4.7 Open subgroups of the universal group

Let Δ be a thick irreducible semi-regular right-angled building with prescribed thickness $(q_s)_{s \in S}$. In Chapter 3 we proved that proper open subgroups of $\mathrm{Aut}(\Delta)$ are contained with finite index in stabilizers of proper residues, considering the locally finite case.

As observed right after the definition of the universal group, the whole automorphism group of Δ can be visualized as a universal group, where the local s-action is prescribed by the symmetric group on q_s elements, for all $s \in S$. In this section we make explicit how much of the work done in Chapter 3 can be applied for a general universal group for a right-angled building.

We recall the notation that we are using. Let (W, S) be a irreducible right-angled Coxeter system and $(q_s)_{s\in S}$ be a set of cardinal numbers. Consider Δ the unique right-angled building of type (W, S) and prescribed thickness $(q_s)_{s\in S}$. For each $s \in S$, let $G^s \leq \mathrm{Sym}(q_s)$ be a transitive permutation group and let U denote the universal group with respect to the groups $\{G^s\}_{s\in S}$.

We start by looking at the fixed-point set of the fixator of a ball in the universal group. The proof of Proposition 3.2.6, other than the geometry of the right-angled building, only uses two facts. The first is that $\mathrm{Stab}_{\mathrm{Sym}(q_s)}(\alpha)$ only fixes α. The second is an extension result corresponding to Proposition 2.2.13.

For universal groups, we also have an extension result given by Proposition 4.2.3, namely the tree-wall automorphisms. Moreover, the condition on the local action can be obtained by weaker requirements, as shown in Lemma 4.5.6. Therefore, using the same arguments as in Section 3.2, we can prove the following.

Proposition 4.7.1. *Let Δ be a thick irreducible semi-regular right-angled building of type (W, S). For each $s \in S$, let q_s be a cardinal number and $G^s \leq \mathrm{Sym}(q_s)$ be primitive and generated by point stabilizers. Consider the universal group U for Δ with respect to the groups $\{G^s\}_{s\in S}$. Let $n \in \mathbb{N}$ and c_0 be a fixed chamber in $\mathrm{Ch}(\Delta)$.*

Denote $K = \mathrm{Fix}_U(\mathsf{B}(c_0, n))$. Then Δ^K is bounded.

Also the work developed in Section 3.4.1 goes through in the setting of universal groups. That yields the following.

Lemma 4.7.2. *Retain the notation from Proposition 4.7.1 for Δ and U. An open subgroup of U is compact if and only if it is X-locally elliptic on the Davis realization of the building.*

In the locally finite case, to prove that open subgroups of $\mathrm{Aut}(\Delta)$ are contained with finite index in stabilizers of residues, we used, as main ingredient, the fact that the action of $\mathrm{Aut}(\Delta)$ on Δ is strongly transitive. By Remark 4.6.1, we can obtain strong transitivity of the universal group on Δ if we require the local action to be 2-transitive. Hence, using the reasoning of the proof of Theorem 3.4.19 (and all the constructions associated) we can conclude:

Proposition 4.7.3. *Let Δ be a locally finite thick irreducible semi-regular building of type (W, S). For each $s \in S$, let $q_s \geq 3$ be a natural number and let $G^s \leq \mathrm{Sym}(q_s)$ be a 2-transitive permutation group. Consider the universal group U for Δ with respect to the groups $\{G^s\}_{s \in S}$.*

Then any proper open subgroup of U is contained with finite index in the stabilizer in U of a proper residue of Δ.

In particular, any proper open subgroup of U is commensurable with the stabilizer in U of a proper residue of Δ.

CHAPTER 5

Compact open subgroups of the universal group

The universal group of a locally finite semi-regular right-angled building is a locally compact group. Crucial in understanding such a group is to investigate its compact open subgroups. This is the aim of this chapter.

We start by describing the maximal compact open subgroups of U and then we focus in the structure of chamber stabilizers in the universal group, which are finite index subgroups of the maximal compact open subgroups of U (see Proposition 5.1.2).

The interest in the study of such chamber stabilizers is that one can use distinct group theoretical constructions that, in the case of right-angled buildings, give rise to permutationally isomorphic groups. For any chamber c, the group $\mathrm{Stab}_U(c)$ is profinite so we will present in Section 5.2 an explicit description of the projective limit of finite groups as an iteration of semidirect products. As the name suggests, this description will provide a way of constructing the finite groups in a recursive way.

Those finite groups appearing in the projective limit correspond to the induced action of a chamber stabilizer in balls (or spheres) around the chamber and these induced actions also deserve some investigation, which we will do using different points of view.

We will regard the induced action on an n-sphere as a subdirect product of the induced actions on w-spheres, for reduced words w of length n. Then we prove in Section 5.3.1 that the induced actions of chamber stabilizers on w-spheres are permutationally isomorphic to a generalized wreath product constructed using a partial order on the letters of w (see Proposition 5.3.3).

In Section 5.3.2 we connect these generalized wreath products with intersections of complete wreath product in imprimitive action and show that we can also describe the structure of the induced action on a w-sphere using those intersections.

We finish the chapter by defining a new partial order $\prec_n$ on the tree-walls of the right-angled building and by describing the induced action on the whole n-sphere directly as a generalized wreath product with respect to $\prec_n$, without looking at w-spheres (*cf.* Theorem 5.3.13).

In this chapter we will use the description of semi-regular right-angled buildings as directed objects and we assume that the legal colorings of the chambers are always directed with respect to a fixed chamber c_0. We recall that such a description is explained in Section 2.4.

5.1 Maximal compact open subgroups of U

We start by describing the maximal compact open subgroups of the universal group for a right-angled building. We will use the following notation throughout the chapter.

Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ. Let $(q_s)_{s\in S}$ be a set of natural numbers with $q_s \geq 3$ and, for each $s \in S$, let $Y_s = \{1, \dots, q_s\}$ be the set of colors. Consider the locally finite thick semi-regular right-angled building Δ of type (W, S) with prescribed thickness $(q_s)_{s\in S}$. Fix a base chamber $c_0 \in \mathrm{Ch}(\Delta)$. For each $s \in S$, let $h_s \colon \mathrm{Ch}(\Delta) \to Y_s$ be a directed legal s-coloring with respect to c_0 and let $G^s \leq \mathrm{Sym}(Y_s)$ be a transitive permutation group. Consider the universal group U of Δ with respect to the groups $(G^s)_{s\in S}$, as in Definition 4.1.1.

Lemma 5.1.1. *Let $\mathcal{R}$ be a spherical residue of Δ. Then the stabilizer* $\mathrm{Stab}_U(\mathcal{R})$ *of $\mathcal{R}$ in U is compact.*

Proof. The result follows the argument of Lemma 1.3.9 where it is proved that each chamber stabilizer is compact. Indeed, since $\mathcal{R}$ is a spherical residue, the set of chambers of $\mathcal{R}$ is finite. □

Proposition 5.1.2. *The maximal compact open subgroups of U are exactly the stabilizers of maximal spherical residues of Δ.*

Proof. Let H be a compact open subgroup of U. By Lemma 5.1.1 it is sufficient to show that H is contained in the stabilizer of a spherical residue of Δ.

Since H is compact, the orbits of H on $\mathrm{Ch}(\Delta)$ are finite. In particular, H acts (type-preservingly) on any s-tree-wall tree with finite orbits. Therefore, it fixes a vertex in each s-tree-wall tree. Such a vertex corresponds to a residue $\mathcal{R}$ of Δ, which is, on its own, a right-angled building. Therefore $H \subseteq \mathrm{Stab}_U(\mathcal{R})$.

As the orbits of action of H on $\mathrm{Ch}(\mathcal{R})$ are still finite, we can repeat the above procedure and obtain a residue $\mathcal{R}'$ of smaller rank than $\mathcal{R}$ such that $H \subseteq \mathrm{Stab}_U(\mathcal{R}')$. We can continue this procedure until there are no non-trivial tree-wall trees left, which happens exactly when the right-angled building is spherical (*cf.* Remark 2.2.39).

We conclude that H is indeed contained in the stabilizer of a spherical residue. □

5.2 The structure of a chamber stabilizer

The chamber stabilizers (as well as the maximal compact subgroups of which they are finite index subgroups) are totally disconnected compact groups, and are therefore profinite (see [Wil98, Corollary 1.2.4]), i.e., they are a projective limit of finite groups.

As Burger and Mozes did in [BM00a] the goal of this section is to make this inverse limit explicit, by means of describing the finite groups taking part in the limit. The commutation relations between the generators of the Coxeter group W (which in the case of trees are inexistent) play an important role in this description, as they make possible more than one reduced representation of an element of W.

Let $U_{c_0}|_{\mathsf{B}(c_0,n)}$ be the induced action of U_{c_0} on the n-ball $\mathsf{B}(c_0, n)$, that is, the group $U_{c_0}|_{\mathsf{B}(c_0,n)} \cong U_{c_0}/\operatorname{Fix}_{U_{c_0}}(\mathsf{B}(c_0, n))$. (Recall that $\mathsf{B}(c_0, n)$ is the set of chambers of Δ at gallery distance smaller than or equal to n, as in Definition 1.4.48).

The restriction of the action of $U_{c_0}|_{\mathsf{B}(c_0,n+1)}$ to $\mathsf{B}(c_0, n)$ maps $U_{c_0}|_{\mathsf{B}(c_0,n+1)}$ to $U_{c_0}|_{\mathsf{B}(c_0,n)}$ and this restriction is onto. Moreover we have

Lemma 5.2.1. *Let $n \in \mathbb{N}$. Then $U_{c_0}|_{\mathsf{B}(c_0,n)} = U_{c_0}|_{\mathsf{S}(c_0,n)}$.*

Proof. It is clear that $\operatorname{Fix}_{U_{c_0}}(\mathsf{S}(c_0, n)) \subseteq \operatorname{Fix}_{U_{c_0}}(\mathsf{B}(c_0, n))$. On the other hand, since there is a unique path of each type between any two chambers of Δ and since we are considering type-preserving automorphisms only, if an element fixes every element in $\mathsf{S}(c_0, n)$ then it has to fix every chamber in $\mathsf{B}(c_0, n)$. Therefore the two induced actions coincide. $\square$

Therefore we obtain

$$U_{c_0} = \varprojlim_n U_{c_0}|_{\mathsf{B}(c_0,n)} = \varprojlim_n U_{c_0}|_{\mathsf{S}(c_0,n)}. \tag{5.2.1}$$

We will make the extension of the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ to $\mathsf{B}(c_0, n+1)$ explicit following the strategy described in [Gib14, Section 9], that is, by splitting each sphere according to the commutator relations of the generators of the Coxeter group. This partition of the sphere was already considered in Chapter 3 (see Definition 3.1.1) and we quickly recall it here.

Definition 5.2.2. For any $w \in W$, let

$$L(w) = \{s \in S \mid l(ws) < l(w)\}. \tag{5.2.2}$$

Let $W(n)$ denote the set of elements $w \in W$ of length n and define

$$W_1(n) = \{w \in W(n) \mid |L(w)| = 1\},$$
$$W_2(n) = \{w \in W(n) \mid |L(w)| \geq 2\}.$$

Observe that if $w \in W_1(n)$, then the last letter of w is independent of the choice of the reduced representation for w since otherwise there would be two generators s_i and s_j such that $l(ws_i) < l(w)$ and

$l(ws_j) < l(w)$. We will write this last letter as r_w. Therefore, if $w \in W_1(n)$ then $L(w) = \{r_w\}$.

We consider a partition of $\mathsf{S}(c_0, n)$ by defining, for each $i \in \{1, 2\}$,

$$A_i(n) = \{c \in \mathsf{S}(c_0, n) \mid \delta(c_0, c) \in W_i(n)\}. \tag{5.2.3}$$

(a graphical visualization of these sets is presented in Figure 3.1).

We observe that if two chambers c_1 and c_2 in the sphere $\mathsf{S}(c_0, n)$ are s-adjacent for some element $s \in S$, then there is a unique chamber c in their s-panel that is in $\mathsf{S}(c_0, n-1)$. Therefore it follows that $\delta(c_0, c_1) = \delta(c_0, c)s = \delta(c_0, c_2)$ and hence c_1 and c_2 are in the same part $A_i(n)$. Thus $A_1(n)$ and $A_2(n)$ are mutually disconnected parts of $S(n)$.

We will extend the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ to $\mathsf{B}(c_0, n+1)$ by considering the extension to $A_1(n+1)$ and to $A_2(n+1)$ separately.

Definition 5.2.3. We define the set

$$C_n = \begin{array}{r} \{c \in \mathsf{B}(c_0, n) \mid c \stackrel{s}{\sim} c' \text{ for some } c' \in A_1(n+1) \\ \text{and some } s \in S\} \end{array}, \tag{5.2.4}$$

and for each $c \in C_n$, let

$$S_c = \{s \in S \mid c \stackrel{s}{\sim} c' \text{ for some } c' \in A_1(n+1)\}. \tag{5.2.5}$$

Now consider the set of pairs

$$\begin{aligned} Z_n &= \{(c, s) \mid c \in C_n \text{ and } s \in S_c\}, \text{ or equivalently,} \\ &= \{(c, s) \in \mathsf{S}(c_0, n) \times S \mid \delta(c_0, c)s \in W_1(n+1)\}. \end{aligned} \tag{5.2.6}$$

For each element $z = (c, s) \in Z_n$, let $\mathcal{P}_z$ be the s-panel containing c and let $G_z = G_0^s$, where G_0^s is the stabilizer of the element $1 \in Y_s$ in G^s. Observe that $h_s(c) = 1$ because we are considering directed legal colorings with respect to c_0.

The first step in the extension of this action is to extend the action of U_{c_0} to the set $\mathsf{B}(c_0, n) \cup A_1(n+1)$.

Lemma 5.2.4. *Let $E := \mathsf{B}(c_0, n) \cup A_1(n+1)$. For each element $z = (c, s) \in Z_n$, there is a subgroup $U_z \leq U_{c_0}|_E$ acting trivially on $E \setminus \mathcal{P}_z$ and acting as G_z on $\mathcal{P}_z$. In particular, U_z fixes $\mathsf{B}(c_0, n)$.*

Proof. Let $z = (c, s) \in Z_n$, and consider the s-panel $\mathcal{P} = \mathcal{P}_z$. Notice that $c = \text{proj}_{\mathcal{P}}(c_0)$. Since h_s is a directed legal coloring, we know that $h_s(c) = 1$ and that, for any $u \in U_{c_0}|_{\mathsf{B}(c_0,n)}$, also $h_s(u(c)) = 1$ because u is a type-preserving automorphism.

For each $g \in G_0^s$, we consider the induced automorphism of $\mathcal{P}$ given by

$$g_{\mathcal{P}} = (h_s|_{g\mathcal{P}})^{-1} \circ g \circ h_s|_{\mathcal{P}} \in \text{Aut}(\mathcal{P}).$$

By Lemma 4.2.3 we know that there exists a tree-wall automorphism $\widetilde{g_{\mathcal{P}}} \in U$ acting locally as $g_{\mathcal{P}}$ on the panel $\mathcal{P}$ and fixing the s-wing $X_s(c)$. In particular, $\widetilde{g_{\mathcal{P}}} \in U_{c_0}$ because $c_0 \in X_s(c)$.

We claim that $\widetilde{g_{\mathcal{P}}}$ fixes each chamber $d \in E \setminus \text{Ch}(\mathcal{P})$. We start by showing that $\text{proj}_{\mathcal{P}}(d) = c$. Indeed, if $\text{proj}_{\mathcal{P}}(d) = c_2 \in \mathsf{S}(c_0, n+1)$, then by Lemma 2.2.12 there would exist $c_3 \in \mathsf{S}(c_0, n)$ such that c_2 is t-adjacent to c_3 with $st = ts$, contradicting the fact that $c_2 \in A_1(n+1)$. Therefore $\text{proj}_{\mathcal{P}}(d) = c$. It follows that $d \in X_s(c)$, and hence $\widetilde{g_{\mathcal{P}}}$ fixes d, proving our claim. In particular, the restriction of $\widetilde{g_{\mathcal{P}}}$ to E is independent of the choice of the extension $\widetilde{g_{\mathcal{P}}}$. Let us denote this element of $U_{c_0}|_E$ by $\widetilde{g_{\mathcal{P}}}|_E$. Then

$$U_z := \{\widetilde{g_{\mathcal{P}}}|_E \mid g \in G_0^s\}$$

is a subgroup of $U_{c_0}|_E$ isomorphic to G_0^s, fixing every chamber outside $\text{Ch}(\mathcal{P}) \cap A_1(n+1)$. We also know that it fixes c, therefore U_z fixes pointwise $\mathsf{B}(c_0, n)$. □

Theorem 5.2.5. *Let us keep the notation in Definition 5.2.3. The group* $U_{c_0}|_{\mathsf{B}(c_0,n)\cup A_1(n+1)}$ *is isomorphic to*

$$U_{c_0}|_{\mathsf{B}(c_0,n)} \ltimes \Big(\prod_{z\in Z_n} G_z\Big),$$

where the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ *on* $\prod_{z\in Z_n} G_z$ *is given by permuting the entries of the direct product according to the action of* $U_{c_0}|_{\mathsf{B}(c_0,n)}$ *on* $C_n \subseteq \mathsf{B}(c_0, n)$.

Proof. Let $E = \mathsf{B}(c_0, n) \cup A_1(n+1)$ as in Lemma 5.2.4 and consider, for each $z \in Z_n$, the subgroup $U_z \leq U_{c_0}|_E$. Then the subgroups U_z

for different $z \in Z_n$ all have disjoint support, and hence form a direct product

$$D := \prod_{z \in Z_n} U_z \leq U_{c_0}|_E.$$

Moreover, D fixes $\mathsf{B}(c_0, n)$ pointwise.

On the other hand, it is clear that each element $g \in U_{c_0}|_{\mathsf{B}(c_0,n)}$ can be uniquely extended to an element $\tilde{g} \in U_{c_0}|_E$ in such a way that for each $z \in Z_n$, the induced map from $\mathcal{P}_z$ to $\mathcal{P}_{g.z}$ preserves the s-coloring, i.e., it induces the identity of G^s. Let

$$T := \{\tilde{g} \mid g \in U_{c_0}|_{\mathsf{B}(c_0,n)}\} \leq U_{c_0}|_E,$$

with $\widetilde{g}$ as described above. Then indeed $U_{c_0}|_E = T \ltimes D$, and the conjugation action of T on D is given by permuting the entries of the direct product according to the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ on the set $C_n \subseteq \mathsf{B}(c_0, n)$. □

We observe that the group $U_{c_0}|_{\mathsf{B}(c_0,n)} \ltimes (\prod_{z \in Z_n} G_z)$ resembles a wreath product as defined in Section 1.1.1. Moreover, the group T in the proof above plays a similar role as the top group plays in the complete wreath product.

Next we see what happens in the extension of the action of the group $U_{c_0}|_{\mathsf{B}(c_0,n)}$ to the set $A_2(n+1)$.

Proposition 5.2.6. *Each element of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ extends uniquely to $\mathsf{B}(c_0, n) \cup A_2(n+1)$. In particular, $U_{c_0}|_{\mathsf{B}(c_0,n) \cup A_2(n+1)} \cong U_{c_0}|_{\mathsf{B}(c_0,n)}$.*

Proof. Let $c \in A_2(n+1)$ and let w be a reduced representation of $\delta(c_0, c)$. Since $|L(w)| \geq 2$, we can choose $s, t \in L(w)$ with $s \neq t$. Consider the chambers $c_s = \mathrm{proj}_{\mathcal{P}_{s,c}}(c_0)$ and $c_t = \mathrm{proj}_{\mathcal{P}_{t,c}}(c_0)$ in $\mathsf{B}(c_0, n)$. Observe that c is the unique chamber of $A_2(n+1)$ that is s-adjacent to c_s and t-adjacent to c_t. Indeed, if d were another such chamber, then c and d would be both s-adjacent and t-adjacent, which is impossible. In particular, for any $u \in U_{c_0}$, the restriction of u to $\mathsf{B}(c_0, n) \cup A_2(n+1)$ is already determined by u restricted to $\mathsf{B}(c_0, n)$, and the result follows. □

As the sets $A_1(n+1)$ and $A_2(n+1)$ are mutually disconnected, we can combine the two previous results to obtain an extension of the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ to $\mathsf{B}(c_0, n+1)$ and a description of the groups $U_{c_0}|_{\mathsf{B}(c_0,n)}$.

Theorem 5.2.7. *For each n, we have*

$$U_{c_0}|_{\mathsf{B}(c_0,n+1)} \cong U_{c_0}|_{\mathsf{B}(c_0,n)} \ltimes \big(\prod_{z\in Z_n} G_z\big), \tag{5.2.7}$$

where the conjugation action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ on $\prod_{z\in Z_n} G_z$ is given by permuting the entries of the direct product according to the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ on the set $C_n \subseteq \mathsf{B}(c_0, n)$. Moreover,

$$U_{c_0} \cong \varprojlim_n U_{c_0}|_{\mathsf{B}(c_0,n)}.$$

Proof. This is now an immediate consequence of Theorem 5.2.5 and Proposition 5.2.6. □

As observed before Proposition 5.2.6, the semidirect product occurring in Theorem 5.2.7 is almost a (complete) wreath product. The only difference is that the groups G_z might be distinct for each orbit of the action of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ on Z_n.

Remark 5.2.8. In the case of regular trees T_q, all the groups G_z correspond to the same group G_0 acting on $\{2, \dots, q\}$. Moreover, $U_{c_0}|_{\mathsf{B}(c_0,1)} \cong G$ and $A_2(n)$ is empty for all $n \in \mathbb{N}$. Therefore we get exactly the same description as provided by Burger and Mozes [BM00a] since also $|Z_n| = |\{1, \dots, q\}| \times |\{2, \dots, q\}|^{n-1}$ in this case.

Theorem 5.2.7 gives a procedure to *construct* the group U_{c_0} recursively, together with its action on the directed right-angled building Δ. Indeed, given the group $U_n = U_{c_0}|_{\mathsf{B}(c_0,n)}$ acting on $\mathsf{B}(c_0, n)$, we can define a group U_{n+1} as in (5.2.7), and endow it with a faithful action on $\mathsf{B}(c_0, n+1)$ precisely by extending the action of $\mathsf{B}(c_0, n)$ following the description given in the proofs of Theorem 5.2.5 and Proposition 5.2.6. In particular, the group $\prod_{z\in Z_n} G_z$ acts trivially on $\mathsf{B}(c_0, n)$ and on $A_2(n+1)$, and acts naturally on $A_1(n+1)$ via the isomorphisms $G_z \cong U_z$, where U_z is as in Lemma 5.2.4.

5.3 The structure of the n-sphere stabilizers

We saw in the previous section that we can describe the chamber stabilizers in the universal group as an inverse limit of its induced actions on the n-spheres or n-balls around the fixed chamber.

It might be handy however to have a more direct description of $U_{c_0}|_{\mathsf{S}(c_0,n)}$. That is the goal of this part of the thesis. We retain the notation of the previous sections.

Definition 5.3.1. Let $w \in W$. The *w-sphere* of Δ around c_0 is the set

$$\mathsf{S}(c_0, w) = \{c \in \mathrm{Ch}(\Delta) \mid \delta(c_0, c) = w\},$$

where δ is the Weyl distance.

The group $U_{c_0}|_{\mathsf{S}(c_0,n)}$ leaves invariant each w-sphere, with $l(w) = n$, because it is a group of type-preserving automorphisms of Δ. Hence each of these spheres is a union of orbits for the action of $U_{c_0}|_{\mathsf{S}(c_0,n)}$ on $\mathsf{S}(c_0, n)$. Therefore, regarding $\mathsf{S}(c_0, n) = \sqcup_{l(w)=n}\mathsf{S}(c_0, w)$, we have that the group $U_{c_0}|_{\mathsf{S}(c_0,n)}$ is a subdirect product of the groups $U_{c_0}|_{\mathsf{S}(c_0,w)}$ running over all $w \in W$ with $l(w) = n$ as in Definition 1.1.18. This subdirect product gives another point of view in the structure of U_{c_0} and shows the interest in studying the groups $U_{c_0}|_{\mathsf{S}(c_0,w)}$. Moreover, it turns out that these groups have a very interesting structure that can be described only by looking at the Coxeter diagram of the building and at the commutation relations between the generators. Therefore one does not have to look at the building since the description will rely only on the geometry of the associated Coxeter group.

We will describe the groups $U_{c_0}|_{\mathsf{S}(c_0,w)}$ as generalized wreath products and show, for the partial orders coming from right-angled Coxeter groups, that there is a connection between generalized wreath products and intersections of complete wreath products in imprimitive action.

We finish the section by describing the group $U_{c_0}|_{\mathsf{S}(c_0,n)}$ itself as a generalized wreath product. For that we will introduce a new partial order on the tree-walls of the building (see Definition 5.3.9).

5.3.1 The w-sphere stabilizers as generalized wreath products

We will prove that $U_{c_0}|_{\mathsf{S}(c_0,w)}$ can be realized as a generalized wreath product as in Section 1.1.2. For that we will use the partial order on reduced words of a right-angled Coxeter system defined in Chapter 2.

Definition 5.3.2. Let $w = s_1 \cdots s_{n-1}s_n$ be a reduced word in M_S and write $\hat{w} = s_1 \cdots s_{n-1}$. Consider the associated partially ordered set $(I_w, \prec_w)$ as in Definition 2.1.9. For each $i \in \{1, \dots, n\}$, denote $X_i = \{2, \dots, q_{s_i}\}$. We define

$$X_w = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i \in X_i\}.$$

By the definition of a directed right-angled building (described in Definition 2.4.1), the chambers in the w-sphere $\mathsf{S}(c_0, w)$ are indexed by the elements of X_w. Moreover, every element of $U_{c_0}|_{\mathsf{S}(c_0,w)}$ induces an element of $U_{c_0}|_{\mathsf{S}(c_0,\hat{w})}$ by restricting the elements of X_w to their first $n-1$ coordinates. Let

$$D_w = \{g \in U_{c_0}|_{\mathsf{S}(c_0,w)} \mid g \text{ fixes } \mathsf{S}(c_0, \hat{w})\}.$$

Notice that D_w depends on the word w in M_S, that is, different reduced representations w_1, w_2 for a given element of W might have different associated groups D_{w_1}, D_{w_2}.

Proposition 5.3.3. *Let $w = s_1 \cdots s_{n-1}s_n$ be a reduced word in M_S.*

1. *The group $U_{c_0}|_{\mathsf{S}(c_0,w)}$ is permutationally isomorphic to the generalized wreath product $G = X_w\text{–}\mathrm{WR}_{i\in I_w} G_0^{s_i}$ with respect to the partial order $\prec_w$.*

2. *Under this isomorphism, the subgroup $D_w \leq U_{c_0}|_{\mathsf{S}(c_0,w)}$ corresponds to the subgroup $D(n) \leq G$ from Definition 1.1.12.*

3. *If we write $\widehat{w} = s_1 \cdots s_{n-1}$, then $U_{c_0}|_{\mathsf{S}(c_0,w)} \cong U_{c_0}|_{\mathsf{S}(c_0,\widehat{w})} \ltimes D_w$.*

Proof. We will identify $\mathsf{S}(c_0, w)$ and X_w through the direct description of the chambers, i.e., we will view both $U_{c_0}|_{\mathsf{S}(c_0,w)}$ and G as subgroups of $\mathrm{Sym}(X_w)$, and we will show by induction on $n = l(w)$ that they are equal. Notice that if $n = 1$, then these two groups coincide by definition. Hence we may assume that $n \geq 2$.

Consider $w_1, w_2, \ldots, w_n$ to be the sequence of elements in W represented by the words $s_1, s_1s_2, \ldots, s_1s_2\cdots s_n$, respectively, and let $K := \{1_W, w_1, w_2, \ldots, w_n\} \subseteq W$. As K is a (finite) connected subset of W containing the identity (regarded as a Coxeter chamber system), we can apply Lemma 4.3.5 and obtain a set of generators for $U_{c_0}|_{\mathsf{S}(c_0,w)}$, with an explicit description as in Propositions 2.4.7 and 4.2.3. Each of these generators $G_{\mathcal{T}}$ (acting on X_w) belongs to the generalized wreath product G because each of their elements induces a permutation of G_0^s on the s-panels of one s-tree-wall and is the identity on the other panels. Hence $U_{c_0}|_{\mathsf{S}(c_0,w)} \leq G$.

Now consider $I_{\widehat{w}} = \{1, \ldots, n-1\}$ with the partial order $\prec_{\widehat{w}}$. Let

$$X_{\widehat{w}} = \{(\alpha_1, \ldots, \alpha_{n-1}) \mid \alpha_i \in X_i\}$$

and let $\widehat{G} = X_{\widehat{w}}\text{–WR}_{i\in I_{\widehat{w}}} G_0^{s_i}$. By the induction hypothesis, we have $\widehat{G} \cong U_{c_0}|_{\mathsf{S}(c_0,\widehat{w})}$ as subgroups of $\mathrm{Sym}(X_{\widehat{w}})$.

Observe that $\{n\}$ is an ideal of I_w. Hence Lemma 1.1.15 now implies that

$$G = H \ltimes D(n), \tag{5.3.1}$$

where H is a subgroup of G isomorphic to $\widehat{G}$. Comparing Definition 1.1.12 and Definition 5.3.2, we see that $D(n) = D_w$, and then $D_w \leq U_{c_0}|_{\mathsf{S}(c_0,w)}$. On the other hand, the embedding of $\widehat{G}$ into G as in Lemma 1.1.15. corresponds precisely to the embedding of $U_{c_0}|_{\mathsf{S}(c_0,\widehat{w})}$ into $U_{c_0}|_{\mathsf{S}(c_0,w)}$ obtained by extending each element trivially on the last coordinate of X_w. Hence $H \leq U_{c_0}|_{\mathsf{S}(c_0,w)}$. We conclude that $G \leq U_{c_0}|_{\mathsf{S}(c_0,w)}$ and thus we have equality. The last statement now follows from (5.3.1). □

Remark 5.3.4. When Δ is a tree, for each reduced word w, the partial order considered in the set I_w is a chain. Therefore, by Remark 1.1.10, the generalized wreath products considered in this construction are iterated wreath products in their imprimitive action. Hence Proposition 5.3.3 translates, for the case of trees, to the description provided by Burger and Mozes in [BM00a, Section 3.2]. Notice that the chamber c_0 corresponds to an edge of the tree, and for each of the two reduced words w of length n, the corresponding w-sphere stabilizer corresponds to the stabilizer of a ball in the tree around one of the two vertices of that edge.

This description using generalized wreath products allow us furthermore to compute the order of the groups $U_{c_0}|_{\mathsf{S}(c_0,w)}$ and $U_{c_0}|_{\mathsf{S}(c_0,n)}$.

Definition 5.3.5. Let $w = s_1 \cdots s_n$ be a reduced word in M_S. Let $(I_w, \prec_w)$ be the poset as before. We define

$$d_w = \prod_{j \succ_w n} (q_{s_j} - 1)$$

(where $d_w = 1$ if there is no $j \succ_w n$). Notice that if $w \in W_1(n)$, then each element $j \in \{1, \ldots, n-1\}$ satisfies the condition $j \succ_w n$, so in that case, $d_w = \prod_{j=1}^{n-1}(q_{s_j} - 1)$.

Proposition 5.3.6. *Let $w = s_1 \cdots s_n$ be a reduced word. Then*

$$\left|U_{c_0}|_{\mathsf{S}(c_0,w)}\right| = \prod_{i=1}^{n} |G_0^{s_i}|^{d_i},$$

where $d_i = d_{s_1 \cdots s_i} = \prod_{j \succ_w i} (q_{s_j} - 1)$, for each $i \in \{1, \ldots, n\}$.

Proof. This follows from Proposition 5.3.3 and Proposition 1.1.14 using induction on n. Notice that for each initial subword $w_i = s_1 \cdots s_i$, the poset $\prec_{w_i}$ is a sub-poset of $\prec_w$, so that we can indeed express all d_i using $\prec_w$ only. $\square$

Proposition 5.3.7. *For each generator $s \in S$, let*

$$d(s,n) = \sum_{\substack{w \in W_1(n) \\ s.t.\ r_w = s}} d_w \quad \textit{and} \quad t(s,n) = \sum_{i=1}^{n} d(s,i).$$

Then

$$|U_{c_0}|_{\mathsf{B}(c_0,n)}| = \prod_{s \in S} |G_0^s|^{t(s,n)}.$$

Proof. Recall from Definition 5.2.3 that

$$Z_{n-1} = \{(c,s) \in \mathsf{S}(c_0, n-1) \times S \mid \delta(c_0,c)s \in W_1(n)\},$$

and observe that this set can be partitioned as

$$Z_{n-1} = \bigsqcup_{w \in W_1(n)} \big(\mathsf{S}(c_0, wr_w) \times \{r_w\}\big).$$

(Notice that w is the unique initial subword of wr_w of length $n-1$). By the remark in Definition 5.3.5, the sphere $\mathsf{S}(c_0, wr_w)$ contains precisely d_w chambers. It follows that

$$\prod_{z \in Z_{n-1}} G_z = \prod_{s \in S} \prod_{\substack{w \in W_1(n) \\ \text{s.t. } r_w = s}} G_0^s = \prod_{s \in S} (G_0^s)^{d(s,n)}.$$

The result now follows from Theorem 5.2.7 by induction on n. $\square$

5.3.2 Connection with intersections of complete wreath products

In this section we prove, for the case of right-angled Coxeter groups and respective partial orders obtained from it, that some generalized wreath products correspond to intersection of complete wreath products in imprimitive action.

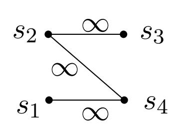
As a consequence we show that the induced action of U_{c_0} on a w-sphere, which in the last section was described as a generalized wreath product, is permutationally isomorphic to an intersection of complete iterated wreath products.

We start with an example that triggered the interest on this connection.

A first motivating example

Consider the Coxeter group of Example 2.1.4.

$$\begin{aligned} W = \langle s_1, s_2, s_3, s_4 \,|& (s_1)^2 = (s_2)^2 = (s_3)^2 = (s_4)^2 = 1 \\ & (s_1 s_2)^2 = (s_1 s_3)^2 = (s_3 s_4)^2 = 1 \rangle \end{aligned}$$



Let $\{q_{s_i}\}_{i \in \{1,\dots,4\}}$ be a set of finite parameters with $q_{s_i} \geq 3$ for all i, and consider Δ the unique locally finite semi-regular right-angled building whose s_i-panels have size q_{s_i} for all i.

For each $i \in \{1, \dots, 4\}$, let G^{s_i} be a transitive permutation group acting on $\{1, \dots, q_{s_i}\}$ and let h_{s_i} be a directed legal coloring with respect to a fixed chamber $c_0 \in \mathrm{Ch}(\Delta)$. Consider the universal group U of Δ with respect to the local groups G^{s_i}.

Let $w = s_1s_2s_3s_4$ and consider the poset $(I_w, \prec_w)$ as described in Definition 2.1.9. Let $X_i = \{2, \dots, q_{s_i}\}$, for $i \in \{1, \dots, 4\}$ and let $X_w = \{(\alpha_1, \dots, \alpha_4) \mid \alpha_i \in X_i\}$. Then by Proposition 5.3.3 the group $U_{c_0}|_{\mathsf{S}(c_0,w)}$ is permutationally isomorphic to the generalized wreath product $G = X_w\text{–}\mathrm{WR}_{i\in I_w} G_0^{s_i}$.

Let us have a closer look at the poset $(I_w, \prec_w)$. We have the relations

$$3 \prec_w 2,\ 4 \prec_w 2 \text{ and } 4 \prec_w 3. \tag{5.3.2}$$

Therefore the poset $(I_w, \prec_w)$ corresponds to the poset $(S, \prec)$ that we studied in detail in Example 1.1.11 while introducing generalized wreath products. Hence, considering

$$A_{s_i} = X_i, H_{s_i} = G_0^{s_i} \text{ and } X = X_w \text{ for } i \in \{1, \dots, 4\}, \tag{5.3.3}$$

we have that $U_{c_0}|_{\mathsf{S}(c_0,w)}$ is permutationally isomorphic to the generalized wreath product G of the afore-mentioned example.

We proved that G was isomorphic to a specific intersection I of iterated complete wreath products from Example 1.1.8. We will present these groups here to make the connection clear, taking in account the identification of notation as in Equation (5.3.3). The group I was taken to be the intersection of the groups

$$\begin{aligned}
G_1 &= G_0^{s_1} \wr G_0^{s_2} \wr G_0^{s_3} \wr G_0^{s_4}\\
G_2 &= G_0^{s_1} \wr G_0^{s_2} \wr G_0^{s_4} \wr G_0^{s_3}\\
G_3 &= G_0^{s_2} \wr G_0^{s_1} \wr G_0^{s_3} \wr G_0^{s_4}\\
G_4 &= G_0^{s_2} \wr G_0^{s_1} \wr G_0^{s_4} \wr G_0^{s_3}\\
G_5 &= G_0^{s_2} \wr G_0^{s_3} \wr G_0^{s_1} \wr G_0^{s_4}
\end{aligned}$$

acting on $X_w = \{1, \dots, |X_1 \times X_2 \times X_3 \times X_4|\}$.

On the other hand, the other reduced representations of w, as an element of the right-angled Coxeter group W, are

$$w_2 = s_1s_2s_4s_3,\ w_3 = s_2s_1s_3s_4,\ w_4 = s_2s_1s_4s_3 \text{ and } w_5 = s_2s_3s_1s_4.$$

So considering $w = w_1$, we have that $U_{c_0}|_{\mathsf{S}(c_0,w)}$ is permutationally isomorphic to the intersection of the iterated complete wreath products corresponding to the distinct reduced representations of the word w.

The previous example was not a one time coincidence. In fact, we can derive a general result.

Let (W, S) be a right-angled Coxeter system and, for each $s \in S$, let $q_s \geq 3$ be a natural number and $G^s \leq \mathrm{Sym}(q_s)$ be a transitive permutation group. For each reduced word $w = s_1 \cdots s_n$ in the free monoid M_S with respect to Σ, let

$$G_w = G_0^{s_1} \wr \cdots \wr G_0^{s_n}$$

denote the iterated wreath product acting on the set X_w of tuples $(\alpha_1, \ldots, \alpha_n)$, where each α_i is an element in $\{2, \ldots, q_{s_i}\}$, for $i \in \{1, \ldots, n\}$, with its imprimitive action as defined in Section 1.1.1. We can view the iterated wreath product as a subgroup of $\mathrm{Sym}(X_w)$.

For each $\sigma \in \mathrm{Rep}(w)$ (recall this notion in Definition 2.1.7), we have

$$G_{\sigma.w} = G_0^{s_{\sigma(1)}} \wr \cdots \wr G_0^{s_{\sigma(n)}}.$$

We can view this as a group acting on the same set X_w (but of course taking into account that the entries have been permuted by σ), so in particular, it makes sense to consider the intersection

$$I = \bigcap_{\sigma \in \mathrm{Rep}(w)} G_{\sigma.w}$$

as a subgroup of $\mathrm{Sym}(X_w)$.

Let Δ be the locally finite thick semi-regular right-angled building of type (W, S) with prescribed thickness $(q_s)_{s \in S}$. Let U be the universal group for Δ with respect to the groups $\{G^s\}_{s \in S}$.

Proposition 5.3.8. *Fix a chamber $c_0 \in \mathrm{Ch}(\Delta)$. Let $w = s_1 \cdots s_n$ be a reduced word in M_S. For each $\sigma \in \mathrm{Rep}(w)$ let*

$$G_{\sigma.w} = G_0^{s_{\sigma(1)}} \wr \cdots \wr G_0^{s_{\sigma(n)}}$$

be the iterated wreath product in imprimitive action on the set X_w of tuples $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in X_i = \{2, \ldots, q_{s_i}\}$, for $i \in \{1, \ldots, n\}$.

Then $U_{c_0}|_{\mathsf{S}(c_0, w)}$ is permutationally isomorphic to the intersection $I = \bigcap_{\sigma \in \mathrm{Rep}(w)} G_{\sigma.w}$.

Proof. We know by Proposition 5.3.3 that $U_{c_0}|_{\mathsf{S}(c_0,w)}$ is permutationally isomorphic to $G = X_w\text{–}\mathrm{WR}_{i\in I_w} G_0^{s_i}$. Therefore we will prove that the groups I and G are isomorphic.

We first observe that for each $\sigma \in \mathrm{Rep}(w)$, the group $G_{\sigma.w}$ can be regarded as acting on the set X_w, taking in account that the entries have been permuted by σ. Hence it makes sense to consider this intersection I as a subgroup of $\mathrm{Sym}(X_w)$.

Moreover, each of the groups $G_{\sigma.w}$ can be viewed as a generalized wreath product as follows. Consider the set $I_\sigma = \{1, \ldots, n\}$ with the chain $\{\sigma(1) > \sigma(2) > \cdots > \sigma(n)\}$ as a partial order $\prec_\sigma$. Define equivalence relations on X_w, for $i \in I_\sigma$, as

$$x \sim_{\sigma(i)} y \Leftrightarrow x_{\sigma(j)} = y_{\sigma(j)} \text{ for all } \sigma(j) > \sigma(i)$$

and $\cong_{\sigma(i)}$ as in Equation (1.1.9). Let $E_\sigma = \{\sim_{\sigma(i)} |\ i \in I_\sigma\} \cup \{\simeq_{\sigma(i)} |\ i \in I_\sigma\}$ be the set of those equivalence relations. Then

$$\begin{aligned} G_{\sigma.w} &= X_w\text{–}\mathrm{WR}_{i\in I_\sigma} G_0^{s_i} \\ &= \left\{ g \in \mathrm{Aut}(X_w, E_\sigma) \;\middle|\; \begin{array}{c} \text{for each } x \in X_w \text{ and } \sigma(i) \in I_\sigma \\ \text{there is } g_{\sigma(i),x} \in G_0^{s_{\sigma(i)}} \text{ such that} \\ (g.y)_{\sigma(i)} = g_{\sigma(i),x} . y_{\sigma(i)} \\ \text{for all } y \in X_w \text{ with } y \sim_{\sigma(i)} x \end{array} \right\}. \end{aligned}$$

For all $\sigma \in \mathrm{Rep}(w)$, the sets I_σ coincide. Therefore we can consider the partial order $\prec$ defined by $\bigcap_{\sigma\in\mathrm{Rep}(w)} \prec_\sigma$ and the poset $(I_\sigma, \prec)$, where

$$i \prec j \Leftrightarrow \sigma(i) \prec_\sigma \sigma(j) \Leftrightarrow \sigma(i) > \sigma(j) \text{ for all } \sigma \in \mathrm{Rep}(w). \tag{5.3.4}$$

An element $g \in I$ has to preserve, for all $i \in I_\sigma$, the intersection of the equivalence relations $\sim_{\sigma(i)}$ (and of $\simeq_{\sigma(i)}$) for all $\sigma \in \mathrm{Rep}(w)$. These equivalence relations are defined using the chain partial orders. Hence, for $i \in I_\sigma$, the intersection $\sim_i$ of the $\sim_{\sigma(i)}$'s corresponds, for $x, y \in X_w$, to

$$x \sim_i y \Leftrightarrow x_j = x_i \text{ for all } i \prec j,$$

where $\prec$ is defined in Equation (5.3.4). Now we just observe that $\prec$ coincides with $\prec_w$ and $I_\sigma = I_w$. Therefore $I \subseteq G$.

Conversely, let $g \in G$ and $i \in I_\sigma = I_w$. For each $j \in I_w$ with $i \prec_w j$, one can define a function

$$f_i : X_{j_1} \times \cdots \times X_{j_k} \to G_0^{s_i} \text{ by } f_i(\alpha_{j_1}, \ldots, \alpha_{j_k}) = g_{i,x},$$

where $x \in X_w$ has the j_ℓ-th coordinate α_{j_ℓ} for every $\ell \in \{1, \ldots, k\}$ and $g_{i,x}$ is the element of $G_0^{s_i}$ such that g acts on x_i as $g_{i,x}$. An element $y \sim_i x$ has also α_{j_ℓ} in its j_ℓ-th coordinate, so the element $g_{i,x}$ is well defined and independent of the choice of x. We define f_i for every element $i \in I_w$ and we can view $g = (f_1, f_2, \ldots, f_n)$ acting on X_w has an element of an iterated wreath product. As observed before, $\prec$ and $\prec_w$ coincide. Therefore the functions f_i are defined in the same way as the functions of elements in the intersection I. Hence we can regard g as an element of I. Thus G and I are permutationally isomorphic, which proves the proposition. □

5.3.3 The n-sphere stabilizers as generalized wreath products

The goal now is to realize the induced action of chamber stabilizers in the universal group on the whole n-spheres also as generalized wreath products. For that we will need to define a new partial order. We recall the notation that we are considering.

Let (W, S) be a right-angled Coxeter system with Coxeter diagram Σ. Let $(q_s)_{s \in S}$ be a set of natural numbers with $q_s \geq 3$ and let $Y_s = \{1, \ldots, q_s\}$ be the set of colors, for each $s \in S$. Consider the locally finite thick semi-regular right-angled building Δ of type (W, S) with prescribed thickness $(q_s)_{s \in S}$. Fix a base chamber $c_0 \in \mathrm{Ch}(\Delta)$.

For each $s \in S$, let $h_s \colon \mathrm{Ch}(\Delta) \to Y_s$ be a directed legal s-coloring with respect to c_0 and let $G^s \leq \mathrm{Sym}(Y_s)$ be a transitive permutation group. Consider the universal group U of Δ with respect to the groups $(G^s)_{s \in S}$.

Definition 5.3.9. Let I_n be the set of all tree-walls in Δ that cross the ball $\mathsf{B}(c_0, n)$, i.e., the set of tree-walls $\mathcal{T}$ such that $\mathsf{B}(c_0, n)$ is not completely contained in one wing of $\mathcal{T}$ (as in Definition 4.3.1). We define a partial order $\prec_n$ on I_n as follows. For $\mathcal{T}_1, \mathcal{T}_2 \in I_n$,

$$\mathcal{T}_1 \prec_n \mathcal{T}_2 \;\Leftrightarrow\; \begin{array}{l} \mathcal{T}_2 \text{ is contained in the wing of } \mathcal{T}_1 \text{ containing } c_0 \text{ and} \\ \mathcal{T}_1 \text{ is contained in a wing of } \mathcal{T}_2 \text{ not containing } c_0. \end{array}$$

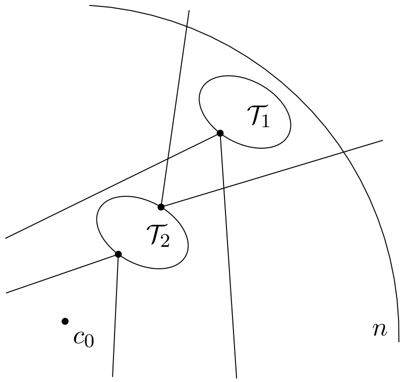


Figure 5.2: $\mathcal{T}_1 \prec_n \mathcal{T}_2$.

Lemma 5.3.10. *For each $n \in \mathbb{N}$, the relation $\prec_n$ is a partial order on I_n.*

Proof. We prove the transitivity of this relation. Let $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$ be distinct tree-walls in I_n such that

$$\mathcal{T}_3 \prec_n \mathcal{T}_2 \text{ and } \mathcal{T}_2 \prec_n \mathcal{T}_1.$$

Let s_1, s_2, s_3 be the types of the three tree-walls, respectively. Consider

$$d_2 = \operatorname{proj}_{\mathcal{T}_2}(c_0),\ d_3 = \operatorname{proj}_{\mathcal{T}_3}(c_0),\ e_1 \in \operatorname{proj}_{\mathcal{T}_1}(\mathcal{T}_2) \text{ and } e_2 \in \operatorname{proj}_{\mathcal{T}_2}(\mathcal{T}_3).$$

We know, by definition of $\prec_n$, that $\mathrm{Ch}(\mathcal{T}_1) \subseteq X_{s_2}(d_2)$ and that $\mathrm{Ch}(\mathcal{T}_2) \subseteq X_{s_3}(d_3)$. Moreover, also using the definition, we have $\mathrm{Ch}(\mathcal{T}_2) \subseteq X_{s_1}(e_1)$ and $\mathrm{Ch}(\mathcal{T}_3) \subseteq X_{s_2}(e_2)$.

Let $c_1 \in \mathrm{Ch}(\mathcal{T}_1)$. We want to prove that $c_1 \in X_s(d_3)$. Let

$$w_1 \sim \delta(c_1, e_1),\ w_2 \sim \delta(e_1, d_2),\ w_3 \sim \delta(d_2, e_2) \text{ and } w_4 \sim \delta(e_2, d_3).$$

We have that $w_1w_2w_3w_4$ is a representation of $\delta(c_1, d_3)$. Furthermore $l(w_1w_2w_3w_4s_3) > l(w_1w_2w_3w_4)$ by the way the words were considered and because the tree-walls are distinct. Hence $\operatorname{proj}_{\mathcal{P}_{s,d_3}}(c_1) = d_3$ and therefore $\mathrm{Ch}(\mathcal{T}_1) \subseteq X_{s_3}(d_3)$.

Now let $c_3 \in \mathrm{Ch}(\mathcal{T}_3)$ and let $w_5 \sim \delta(c_3, d_3)$. With an analogous reasoning we obtain that $l(s_1w_1w_2w_3w_4w_5) > l(w_1w_2w_3w_4w_5)$, which means that $c_3 \in X_{s_1}(e_1)$. Hence $\mathrm{Ch}(\mathcal{T}_3) \subseteq X_{s_1}(e_1)$. Thus, by definition of $\prec_n$, it follows that $\mathcal{T}_3 \prec_n \mathcal{T}_1$. □

We present a couple of observations regarding this new partial order.

Remark 5.3.11. 1. We first observe that if $\mathcal{T}$ crosses $\mathsf{B}(c_0, n)$ then $\mathrm{Ch}(\mathcal{T}) \cap \mathsf{B}(c_0, n) \neq \emptyset$ by making use of a closing squares argument (see Lemmas 2.2.6 and 2.2.7).

2. We are considering the directed representation of the buildings Δ with respect to the chamber c_0 as described in Section 2.4. This means that, for $c \in \mathrm{Ch}(\Delta)$ and $\delta(c_0, c) \sim s_1 \cdots s_n$, we know that

$$c \sim \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix} \text{ with } \alpha_i \in Y_{s_i} \backslash \{1\}$$

Let $c_i \sim \begin{pmatrix} s_1 & \cdots & s_i \\ \alpha_1 & \cdots & \alpha_i \end{pmatrix}$ be the chamber in the minimal gallery of type $s_1 \cdots s_n$ between c_0 and c in Δ that is at distance $s_1 \cdots s_i$ from c_0 and at distance $s_n \cdots s_{i+1}$ from c. Then the s_{i+1}-tree-wall of c_i crosses $\mathsf{B}(c_0, n)$.

3. If we consider another reduced representation of $\delta(c_0, c)$ then, for every $i \in \{1, \ldots n\}$ the s_i-tree-wall that crosses $\mathsf{B}(c_0, n)$ corresponding to the generator s_i is the same, as the elementary operations performed only interchange generators that commute with s_i and such tree-wall is a residue of type $s_i \cup \{s_i^{\perp}\}$.

   Thus if $\mathcal{T}_i$ crosses $\mathsf{B}(c_0, k)$ but not $\mathsf{B}(c_0, k-1)$ then $\mathcal{T}_i \prec_n \mathcal{T}_j$ for all $j \in \{1, \ldots, k-1\}$.

Lemma 5.3.12. *For $n < m$, we have that $(I_n, \prec_n)$ is a subposet of $(I_m, \prec_m)$.*

Proof. It is clear that $I_n \subseteq I_m$ since if $\mathsf{B}(c_0, n)$ is not contained in one wing of a tree-wall $\mathcal{T}$, then $\mathsf{B}(c_0, m)$ is also not contained in a wing of $\mathcal{T}$. The partial order is defined in the building independently of the balls, so that follows as well. □

With this partial new order at hand, we can visualize the induced action on an n-sphere (or equivalently, on an n-ball) as a generalized wreath product.

Theorem 5.3.13. *For each s-tree-wall $\mathcal{T} \in I_n$, we set $s_{\mathcal{T}} = s$ and $X_{\mathcal{T}} = Y_s = \{1, \ldots, q_s\}$. Let X_n be the Cartesian product of all the sets $X_{\mathcal{T}}$, for every tree-wall $\mathcal{T}$ in I_n.*

Then the group $U_{c_0}|_{\mathsf{B}(c_0,n)}$ is permutationally isomorphic to the generalized wreath product $G_n = X_n\text{–}\mathrm{WR}_{\mathcal{T}\in I_n} G_0^{s_{\mathcal{T}}}$ with respect to the partial order $\prec_n$.

Proof. We start by regarding the chambers in $\mathsf{B}(c_0, n)$ as elements of the set X_n. Let $c \sim \begin{pmatrix} s_1 & \cdots & s_n \\ \alpha_1 & \cdots & \alpha_n \end{pmatrix}$ be a chamber in Δ, where $\delta(c_0, c) \sim s_1 \cdots s_n$. By Remark 5.3.11, for each s_i, there is an s_i-tree-wall $\mathcal{T}_i$ in I_n. Hence we can regard c as an element of the product $X_{\mathcal{T}_1} \times \cdots \times X_{\mathcal{T}_n}$. This is exactly the identification of $X_{\mathcal{T}_1} \times \cdots \times X_{\mathcal{T}_n}$ with $\mathsf{S}(c_0, s_1 \cdots s_n)$. Now we embed this direct product on X_n by adding the element $1 \in X_{\mathcal{T}}$ for every tree-wall not intersecting the chambers in the minimal gallery of type $s_1 \cdots s_n$ from c_0 to c.

Since $U_{c_0}|_{\mathsf{B}(c_0,n)}$ is type-preserving, the coordinates 1 of c as an element of X_n remain 1 in the chamber gc, for any $g \in U_{c_0}|_{\mathsf{B}(c_0,n)}$. Moreover, the generalized wreath product G_n preserves this characterization as $G_{\mathcal{T}}$ fixes the element $1 \in X_{\mathcal{T}}$. Like this we see c as an element of X_n and we can consider the group $U_{c_0}|_{\mathsf{B}(c_0,n)}$ as a subgroup of $\mathrm{Sym}(X_n)$ since it preserves the set X_n.

We will prove the result by induction on n. If $n = 1$ then we have that $\mathsf{S}(c_0, 1) = \sqcup_{s\in S} \mathsf{S}(c_0, s)$. The group $X_1\text{–}\mathrm{WR}_{\mathcal{T}\in I_1} G_0^{s_{\mathcal{T}}}$ is isomorphic to $\prod_{s\in S} G_0^s$ because the partial order $\prec_1$ is empty and the group $U_{c_0}|_{\mathsf{B}(c_0,1)}$ is also isomorphic to that direct product.

Assume by induction hypothesis that if $N \leq n$ then the result holds. Observe that the set $K = \{\delta(c_0, c) \in W \mid c \in \mathsf{B}(c_0, n+1)\}$ is a connected subset of W (regarded as a thin building) that contains the identity since $c_0 \in \mathsf{B}(c_0, n+1)$. Hence we can apply Lemma 4.3.5 and obtain a set of generators for $U_{c_0}|_{\mathsf{B}(c_0,n+1)}$ given by the tree-wall groups (*cf.* Definition 4.2.4). Each of these generators $G_{\mathcal{T}}$ (acting on X_{n+1}) induces a permutation of G_0^s (for some s) on the s-panels of an s-tree-wall and it is the identity on the other panels. Hence $U_{c_0}|_{\mathsf{B}(c_0,n+1)} \leq G_{n+1}$.

Since $(I_n, \prec_n)$ is a subposet of $(I_{n+1}, \prec_{n+1})$ by Lemma 5.3.12, we can regard G_n as a subgroup of G_{n+1} by extending each element trivially in the coordinates corresponding to $D_{n+1} = I_{n+1} \backslash I_n$. By induction hypothesis, G_n is permutationally isomorphic to $U_{c_0}|_{\mathsf{B}(c_0,n)}$.

The set D_{n+1} corresponds to the set of tree-walls crossing the ball $\mathsf{B}(c_0, n+1)$ but not $\mathsf{B}(c_0, n)$ and it its an ideal of I_{n+1} (see Definition 1.1.12). We can define, for $x, y \in X_n$,

$$x \sim_{D_{(n+1)}} y \Leftrightarrow x_{\mathcal{T}} = y_{\mathcal{T}} \text{ for all } \mathcal{T} \notin D_{n+1},$$

and consider the subgroup

$$G(D_{n+1}) = \{g \in G_{n+1} \mid x \sim_{D_{n+1}} gx \text{ for all } x \in X_n\}$$

of G_{n+1}. Then, by construction, $G(D_{n+1})$ fixes $\mathsf{B}(c_0, n)$ and moreover it is a normal subgroup of G_{n+1} by Proposition 1.1.13.

Using the same reasoning as Lemma 1.1.15, and considering

$$X_{n+1} = X_n \times \prod_{\mathcal{T} \in D_{n+1}} X_{\mathcal{T}},$$

we can write G_{n+1} as $H \ltimes G(D_{n+1})$, where H is a subgroup of G_{n+1} isomorphic to G_n by extending each element trivially on the coordinates corresponding to tree-walls in D_{n+1}. This is exactly the same embedding of $U_{c_0}|_{\mathsf{B}(c_0,n)}$ on $U_{c_0}|_{\mathsf{B}(c_0,n+1)}$ by acting trivially on the $(n+1)$-th coordinate of the chambers described in a directed way. The coordinates 1 remain 1 as observed in the first paragraph of this proof.

Let us now have a closer look at the group $G(D_{n+1})$. If $\mathcal{T} \in D_{n+1}$ is an s-tree-wall then $\mathcal{T}$ crosses $\mathsf{B}(c_0, n+1)$ but not $\mathsf{B}(c_0, n)$. This implies that the ball $\mathsf{B}(c_0, n)$ is completely contained in one s-wing of $\mathcal{T}$. Since $\mathcal{T}$ crosses $\mathsf{B}(c_0, n+1)$, it means that $\mathrm{Ch}(\mathcal{T}) \cap \mathsf{B}(c_0, n) = c_{\mathcal{T}}$ with $c_{\mathcal{T}} \in \mathsf{S}(c_0, n)$ and, furthermore, that $c_{\mathcal{T}} \overset{s}{\sim} c'$ for some chamber $c' \in \mathsf{S}(c_0, n+1)$.

We claim that $c' \in A_1(n+1)$. If $c' \in A_2(n+1)$ then there is $c_2 \in \mathsf{S}(c_0, n)$ and $t \in S$ such that $c' \overset{t}{\sim} c_2$. Using closing squares (Lemma 2.2.6), we obtain $c_3 \in \mathsf{S}(c_0, n-1)$ such that $c_3 \overset{t}{\sim} c_{\mathcal{T}}$ and $c_3 \overset{s}{\sim} c_2$ with $|st| = 2$. Thus $c_3 \in \mathrm{Ch}(\mathcal{T})$ which is a contradiction to the fact that $\mathcal{T}$ does not cross $\mathsf{B}(c_0, n)$. Hence $c' \in A_1(n+1)$.

Therefore the chamber $c_{\mathcal{T}}$ is in the set C_n as in Equation (5.2.4) and $s \in S_{c_{\mathcal{T}}}$ as in Equation (5.2.5). This means that the set D_{n+1}

is in bijection with the set $Z_n = \{(c,s) \mid c \in C_n \text{ and } s \in S_c\}$ from Equation (5.2.6).

As the partial order $\prec_n$ restricted to D_{n+1} is empty, we obtain that $G(D_{n+1})$ is isomorphic to $\prod_{(c,s)\in Z_n} G_0^{s\mathcal{T}_{s,c}}$ by [Beh90, Proposition 7.2]. Using Theorem 5.2.7 we know that

$$U_{c_0}|_{\mathsf{B}(c_0,n+1)} \cong U_{c_0}|_{\mathsf{B}(c_0,n)} \ltimes \Big(\prod_{z\in Z_n} G_z\Big).$$

Thus we obtain that $G_{n+1} \leq U_{c_0}|_{\mathsf{B}(c_0,n+1)}$ and therefore we have a permutational isomorphism. $\square$

CHAPTER 6

Universal groups for polygonal complexes

This chapter is devoted to the study of the concept of universal groups for other geometrical structures, rather than right-angled buildings, that are also CAT(0) spaces with some local regularity.

We will define these groups on polygonal complexes and show that the idea of Burger and Mozes in [BM00a] of prescribing a local action on the geometrical object and studying the group using local-to-global arguments also makes sense in the universe of polygonal complexes.

Bourdon's buildings studied in [Bou97] are examples of right-angled buildings that also belong to the class of polygonal complexes that we are interested in studying but these are one of the only examples in the intersection of these two categories of geometric objects.

Therefore this chapter presents a different flavor of defining universal groups acting on completely distinct geometric objects which, in the case of locally finite polygonal complexes, also belong to the class of totally disconnected locally compact groups.

The chapter begins with the definition of the polygonal complexes that are the object of our study, mostly following [GLST15]. We describe when these objects are CAT(0) spaces and the topology that we will consider on their automorphism group.

Then we present a construction from Ballmann and Brin [BB94] of regular polygonal complexes with a prescribed link. They construct these polygonal complexes in a inductive way as a limit of subcomplexes, that are obtained from each other by gluing polygons according to the prescribed link. Moreover, they prove that any such polygonal complex can be obtained with their procedure. We will use this construction for our arguments to extend automorphisms, for instance in Proposition 6.3.2.

As in the case of right-angled buildings, under some conditions, the polygonal complexes that we will be interested are, up to isomorphism, unique objects. These results are discussed in Section 6.2.2 and follow the approach taken by Nir Lazarovich in [Laz14].

Then in Section 6.3 we define colorings in polygonal complexes that are necessary for the definition of the universal group and we show, as in the case of trees and right-angled buildings, that, up to isomorphism, these colorings are unique (see Proposition 6.3.2).

In Section 6.4 we are finally in a condition to define the universal group for a polygonal complex and we prove basic properties of these groups. The work presented in this chapter is joint work with Anne Thomas and it is still work in progress, therefore we add a section in the end of the chapter with open questions regarding our construction and future developments.

6.1 Polygonal Complexes

Polygonal complexes play an important role in combinatorial and geometric group theory. The Cayley 2-complex of a group presentation is a polygonal complex, and many groups are investigated by considering their action on an associated polygonal complex of non-positive or negative curvature.

Moreover, if X is a simply-connected, locally finite polygonal complex, then the automorphism group of X, endowed with the permutation topology defined in Section 1.3.2, is a totally disconnected locally compact group, which is the class we aim to contribute to in the work of this thesis.

We will start the section with the definition of polygonal complexes in general. For the construction of a particular class of polygonal complexes, called (Γ, k)-complexes, we will follow [BB94].

6.1.1 2-dimensional CW-complexes

We present a constructive definition of a 2-dimensional CW-complex, also called 2-dimensional cell complex. We will be faithful to the process described in [GLST15] which refers to [Hag02].

Definition 6.1.1. Denote by D^1 the closed interval $[-1,1]$ with boundary ∂D^1 the points $S^0 = \{-1,1\}$. Let D^2 denote the closed unit disk in the Euclidean plane with boundary ∂D^2 the unit circle S^1.

A space X is called a *2-dimensional CW-complex* if it is constructed as follows:

1. Begin with a (possible infinite) set $X^{(0)}$, called the *0-skeleton*, whose points are the *0-cells* and equip $X^{(0)}$ with the discrete topology.

2. The *1-skeleton* $X^{(1)}$ is the quotient space that is obtained from the disjoint union $X^{(0)} \sqcup_\alpha D^1_\alpha$ of $X^{(0)}$, with a possible infinite collection of closed intervals D^1_α, by identifying each boundary point $x \in S^0 = \partial D^1_\alpha$ with a closed 0-cell $\varphi_\alpha(x) \in X^{(0)}$.

   In other words, each function φ_α is a function from $S^0 = \{-1,1\}$ to $X^{(0)}$, that is necessarily continuous. We equip $X^{(1)}$ with the quotient topology. The images of the D^1_α in $X^{(1)}$ are called the *1-cells.*

3. The *2-skeleton* $X^{(2)}$ is the quotient space obtained from the disjoint union $X^{(1)} \sqcup_\beta D^2_\beta$ of $X^{(1)}$ with a possible infinite collection of closed disks D^2_β, by identifying each boundary point $x \in \partial D^2_\beta$ with a point $\varphi_\beta(x) \in X^{(1)}$, where each φ_β is a continuous function from the circle $\partial D^2_\beta = S^1$ to $X^{(1)}$.

   We equip $X^{(2)}$ with the quotient topology. The images of the D^2_β in $X^{(2)}$ are called the (closed) *2-cells.*

4. Since X is 2-dimensional, it is equal to its 2-skeleton $X^{(2)}$.

The maps φ_α and φ_β are called the *attaching maps*. The (closed) *cells* of X are its (closed) 0-, 1- and 2-cells. The 1-skeleton of X may be thought as a graph (not necessarily simple), with vertex-set the 0-skeleton and edges the 1-cells.

6.1.2 (Γ, k)-complexes

Roughly speaking, a polygonal complex is obtained by gluing together polygons, for some constant curvature space, by isometries along faces. More formally,

Definition 6.1.2. A *polygonal complex* Y is a 2-dimensional CW-complex such that

1. the attaching maps of Y are homeomorphisms; and
2. the intersection of any two closed cells of Y is either empty or exactly one closed cell.

The two conditions of Definition 6.1.2 imply that the 1-skeleton of Y is simple, i.e., it has no loops nor multiple edges. We will refer to the closed 0-cells as vertices, to the closed 1-cells as edges and to the closed 2-cells as faces, polygons or k-gons (since the boundary of any face is a cycle of k-edges).

Definition 6.1.3. Let v be a vertex in a polygonal complex Y. The *link* of v in Y, denoted by $\mathrm{Lk}(v, Y)$, is the simple graph whose vertices are the edges of Y containing v, whose edges are the faces of Y containing v and two vertices of $\mathrm{Lk}(v, Y)$ are connected by an edge if and only if the corresponding edges in Y are contained in a common face.

We present now a couple of examples of polygonal complexes, some that we already introduced before, so the connection with the world of buildings is understood.

Example 6.1.4. The product of trees is an example of a polygonal complex. Let T_1 and T_2 be two trees. The product space $T_1 \times T_2$ is a polygonal complex, where each 2-cell is a square (edge $\times$ edge) and the link of each vertex is a complete bipartite graph.

Example 6.1.5. Consider the Bourdon's building $I_{p,q}$ as introduced in Example 2.2.5 in the chapter regarding right-angled buildings.

Through a geometric realization point of view, each chamber of $I_{p,q}$ is isomorphic to a right-angled hyperbolic p-gon P. The panels are the edges of those polygons and each panel is contained in q chambers. The apartments of $I_{p,q}$ are isomorphic to the hyperbolic plane tessellated by P. The link of each vertex is the complete bipartite graph $K_{q,q}$.

The Bourdon's building $I_{5,3}$ is partially depicted in Figure 6.1, taking in account our limitations in drawing right-angled pentagons.

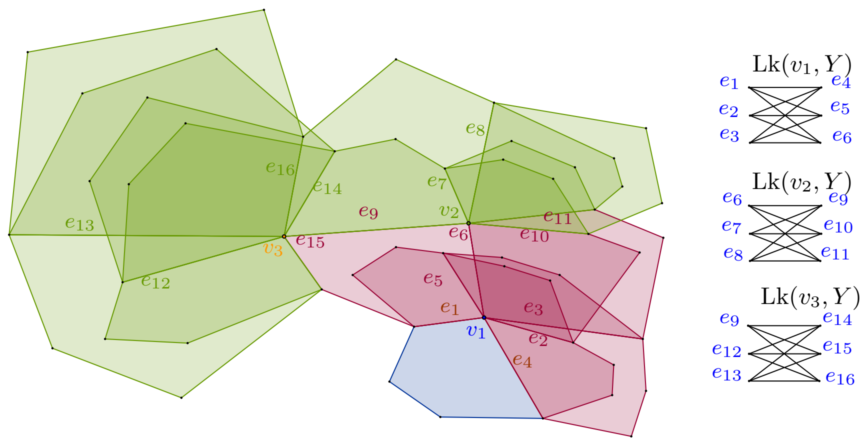


Figure 6.1: The Bourdon's building $I_{5,3}$.

Next we define (Γ, k)-complexes, which is the class of polygonal complexes where we will define universal groups.

Definition 6.1.6. Given a finite connected simple graph Γ and an integer $k \geq 3$, we define a (Γ, k)-*complex* Y to be a polygonal complex Y such that

1. each 2-cell of Y is a regular k-gon, and
2. the link of each vertex of Y is isomorphic to Γ.

Example 6.1.7. 2-dimensional Euclidean or hyperbolic buildings (see the definitions in Examples 1.4.39(4) and 2.2.5, respectively) with all the links isomorphic, are examples of (Γ, k)-complexes, where Γ is an one-dimensional spherical building.

In particular Bourdon's $I_{p,q}$ buildings are examples of $(K_{q,q}, k)$-complexes.

Example 6.1.8. The Davis-Moussong complexes defined in [Dav83] and [Mou88b] are barycentric subdivisions of (Γ, k)-complexes, with $k \geq 4$ even and Γ a connected finite simplicial graph with girth at least 4. We will not work with these complexes specifically, but we present this example without definition, together with the respective references, for general culture purposes.

Example 6.1.9. Let Γ be the Petersen graph and $k \geq 4$. Then we can consider the polygonal complex X whose 2-cells are k-gons and the link of each vertex is isomorphic to the Petersen graph. Then X is a (Γ, k)-complex. We will study more these complexes in the following sections. In Figure 6.2, there is a partial representation of such a complex with $k = 6$. If k is odd, then a (Γ, k)-complex, with

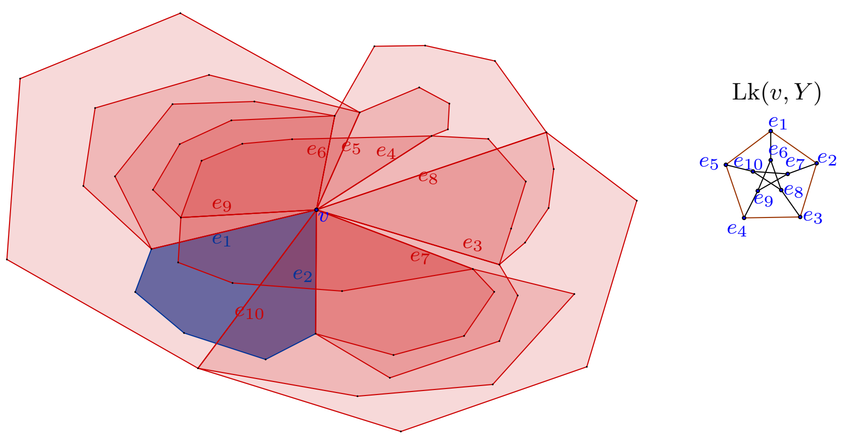


Figure 6.2: A (Petersen graph, 6)-complex.

Γ the Petersen graph, is an example that is neither a building nor a Davis-Moussong complex (see [GLST15, Section 3.7]).

Remark 6.1.10. We observe that we are considering polygonal complexes with unoriented edges. Nevertheless, we keep in mind that each unoriented edge $e = \{v_1, v_2\}$ has two oriented edges (v_1, v_2) and (v_2, v_1) associated to it, since for instance the transfer map in Definition 6.1.16 requires a direction to pass from a vertex to the other in the edge.

When we consider automorphisms of Y it is enough to look at the 1-skeleton of Y, that is, to look at graph automorphisms. Therefore we consider the permutation topology on the group of automorphisms $\mathrm{Aut}(Y)$ of Y as defined in Section 1.3.2. In the case that the link of each vertex is isomorphic to a finite graph the group $\mathrm{Aut}(Y)$ is a totally disconnected and locally compact group (see Proposition 1.3.11).

We finish this section by defining the concepts that resemble balls and spheres around a vertex of Y. We will use these notions for inductive arguments, for instance in the prove of uniqueness, up to isomorphism, of colorings of the polygons of a $(3, 6)$-complex.

Definition 6.1.11. Let $v \in VY$. For a polygon P of Y, we define

$$\mathrm{dist}(v, P) = \min\{\mathrm{dist}(v, x) \mid x \in VP\},$$

where $\mathrm{dist}(v, x)$ is the discrete distance, i.e., the length of a shortest path, between the vertices v and x in the 1-skeleton of Y. Then we consider the sets

$$\begin{aligned} &\mathsf{B}(v, n) = \{P \in Y \mid \mathrm{dist}(v, P) \leq n\} \quad \text{and} \\ &\mathsf{S}(v, n) = \mathsf{B}(v, n) \backslash \mathsf{B}(v, n-1) = \{P \in Y \mid \mathrm{dist}(v, P) = n\}. \end{aligned}$$

6.1.3 CAT(0) (Γ, k)-complexes

Now we show in which way a (Γ, k)-complex Y is a CAT(0)-space. Assume that each face of Y is metrized as regular Euclidean k-gon, or that each face of Y is metrized as a regular hyperbolic k-gon. Then by Theorem I.7.50 of [BH99], Y is a complete geodesic metric space when equipped with the “taut string” metric, in which each geodesic is a concatenation of a finite number of geodesics contained in faces.

Definition 6.1.12 (Gromov Link Condition)**.** Let Y be a (Γ, k)-complex. We say that the pair (Γ, k) satisfies the *Gromov Link Condition* if $k \geq n$ and the $girth(\Gamma) \geq m$ where $(m, n) \in \{(3, 6), (4, 4), (6, 3)\}$.

If the faces of a (Γ, k)-complex Y are regular k-gons and (Γ, k) satisfies Gromov Link Condition then Y is locally CAT(0) (*cf.* [BH99, I.5.24]). Hence the Cartan-Hadamard Theorem (see [BH99, II.4.1]) gives that the universal cover of Y is a CAT(0)-space. Thus to see whether a simply-connected space X has a global metric of non-positive curvature, we only need to check locally a neighborhood of each point $x \in X$, that is, the link of x.

Similarly, assume that either $k > n$ and that $girth(\Gamma) \geq m$ or $k \geq n$ and $girth(\Gamma) > m$, for $(m, n) \in \{(3, 6), (4, 4), (6, 3)\}$. Then the faces of Y can be metrized as regular hyperbolic k-gons with vertex angles $2\pi/girth(\Gamma)$ and the complex Y is locally CAT(-1). Hence its universal cover is a CAT(-1)-space.

6.1.4 Ballmann and Brin's construction of polygonal complexes

Let Γ be a simple connected graph and let $k \geq 6$. Ballmann and Brin in [BB94] developed a procedure to construct CAT(0) (Γ, k)-complexes inductively.

Remark 6.1.13. We make a remark on the notation that we will use from now on, since there will be vertices and edges in polygonal complexes and vertices and edges of the links (graphs) of vertices of polygonal complexes. These objects will be in a clear correspondence that we will try to make less confusing.

Therefore we will use Roman letters to denote the vertices $(v, \dots)$ and edges $(e, \dots)$ in the polygonal complex and we will use Greek letters for the vertices $(\xi, ...)$ and the edges $(\varepsilon, \dots)$ of the links of the vertices of the polygonal complex (this is also the notation adopted in [BB94]).

Definition 6.1.14. Let X be a polygonal complex. Given a subset Z of X we define the *star* $\mathrm{St}(Z, X)$ as the subcomplex of X consisting of all closed cells of X intersecting Z.

Remark 6.1.15. We observe that when we consider the star of a closed 0-cell in the definition above, that is, when we consider $\mathrm{St}(v, X)$ for some $v \in VX$, then there is a clear identification between the closed 1-cells in $\mathrm{St}(v, X)$ and the vertices of $\mathrm{Lk}(v, X)$ and between the closed 2-cells of $\mathrm{St}(v, X)$ and the edges of $\mathrm{Lk}(v, X)$. We will use this identification interchangeably.

Definition 6.1.16. Let X be a polygonal complex and $e = (v_1, v_2)$ be an oriented edge of X. Let ξ_1 and ξ_2 be vertices of $\mathrm{Lk}(v_1, X)$ and $\mathrm{Lk}(v_2, X)$, respectively, which correspond to the edge e. The map

$$\tau_e : \mathrm{St}(\xi_1, \mathrm{Lk}(v_1, X)) \to \mathrm{St}(\xi_2, \mathrm{Lk}(v_2, X))$$

which sends adjacent edges of ξ_1, which correspond to polygons in X adjacent to e, to the corresponding adjacent edges of ξ_2 is called the *transfer map along e.*

The construction of Ballmann and Brin produces polygonal complexes with a prescribed link on the vertices as a limit of an increasing sequence of subcomplexes

$$X(1) \subset X(2) \subset \cdots \subset \cdots X(n) \subset \cdots ,$$

such that the subcomplex $X(n+1)$ is obtained from $X(n)$ by attaching edges and polygons in such a way that the links of $X(n-1)$ do not change. In each step, the vertices of $X(n-1)$ are considered as "interior" vertices of $X(n)$. We now make more precise the definition of interior and boundary cells of a polygonal complex.

Definition 6.1.17. Let X be a polygonal complex. We chose a subset of vertices $V^i \subset VX$ which are called the *interior vertices.* The remaining vertices $V^\partial = VX \backslash V^i$ are called the *boundary vertices.* An edge of X is called a *boundary edge* if both its end points are boundary vertices. Otherwise it is called an *interior edge.* A polygon of X is called a *boundary polygon* if is is adjacent to a boundary edge.

As referred before, Ballmann and Brin construct CAT(0) polygonal complexes. We now describe when a general polygonal complex is a CAT(0) space.

Definition 6.1.18. Let X be a polygonal complex. We call X an *(m,n)-complex* if the following hold:

1. Each 2-cell of X is a polygon with at least n sides,

2. For all $v \in VX$, the girth of the link $\mathrm{Lk}(v, X)$, i.e., the length of a shortest cycle, is at least m.

By definition, any polygonal complex is a $(3,3)$-complex. If X is a (Γ, k)-complex (see Definition 6.1.6), then X is an (m, n)-complex if $k \geq n$ and the girth of Γ is at least m.

As for (Γ, k)-complexes, we can say when an (m, n)-complex is a CAT(0) and CAT(-1) space.

Lemma 6.1.19 ([BH99, Proposition 5.25])**.** *Let Y be an (m, n)-complex.*

1. *If $(m, n) \in \{(3,6), (4,4), (6,3)\}$ then Y can be metrized as a piecewise Euclidean complex of non-positive curvature.*

2. *If $(m, n) \in \{(3,6), (4,5), (5,4), (7,3)\}$ then Y can be metrized as a piecewise hyperbolic complex of negative curvature.*

The Euclidean plane tessellated by triangles, squares and hexagons, respectively, provides examples of such complexes in the non-positive curvature case.

From now on we will be interested in $(3,6)$-complexes. Although in [BB94], the authors treat all the cases in Lemma 6.1.19(1), we will only describe Balmann and Brin's construction for this particular case.

Definition 6.1.20. Let X be a $(3,6)$-complex. A choice (X, V^i) of interior vertices of X is called *(3,6)-admissible* if each boundary vertex v of X is either:

1. *Free*, i.e., the vertex v is in the boundary of a unique polygon P and the two edges adjacent to v are boundary edges. In this case the link $\mathrm{Lk}(v, X)$ consists of only one edge, the one corresponding to the polygon P.

2. *partly free*, i.e., there is exactly one interior edge e adjacent to v. Any other edge adjacent to v connects v to a free vertex and bounds a boundary polygon which is adjacent to e as well. In this situation the link $\mathrm{Lk}(v, X)$ is the star of a vertex ξ which corresponds to e.

Note that no two partly free vertices are connected by an edge because we are now only considering $(3,6)$ complexes. Moreover, any boundary edge of X is adjacent to precisely one (boundary) polygon and any simple loop of boundary edges contains at least 6 edges.

We are now in a position to describe the data needed to do the inductive step in the construction of Ballmann and Brin, that is, how to obtain a $(3,6)$-complex $Z = X(n+1)$ from $X = X(n)$.

Definition 6.1.21. Assume that we have a (3,6)-admissible choice of interior vertices (X, V^i) on a $(3,6)$-complex X. Let Γ be a simple connected graph with girth at least 3.

1. For each boundary vertex $v \in V^\partial$, let
$$i_v : \mathrm{Lk}(v, X) \to \Gamma$$
be a simplicial embedding such that if v is partly free and ξ is the vertex in $\mathrm{Lk}(v, X)$ corresponding to the unique interior edge e adjacent to v then the map $i_v : \mathrm{Lk}(v, X) \to \mathrm{St}(i_v(\xi), \Gamma)$ is an isomorphism.

2. For each boundary edge $\{v_1, v_2\}$ in X, we can consider the two oriented edges (v_1, v_2) and (v_2, v_1). Let e be one of those (oriented) edges (and then e^{-1} will be the other one). Let $\xi_1 \in i_{v_1}(\mathrm{Lk}(v_1, X)) \subset \Gamma$ and $\xi_2 \in i_{v_2}(\mathrm{Lk}(v_2, X)) \subset \Gamma$ be the vertices corresponding to e. Fix an isomorphism
$$\tau_e : \mathrm{St}(\xi_1, \Gamma) \to \mathrm{St}(\xi_2, \Gamma)$$
such that the edge in $\mathrm{St}(\xi_1, \Gamma)$ corresponding to the unique polygon of X adjacent to e is mapped onto the edge in $\mathrm{St}(\xi_2, \Gamma)$ corresponding to that polygon and such that $\tau_{e^{-1}} = (\tau_e)^{-1}$.

A choice of maps i_v, τ_e as above will be called an *admissible choice of data.*

Remark 6.1.22. We observe that the maps τ_e exist if and only if the stars involved are isomorphic. This is an implicit restriction on the data. Therefore, it can happen, for a certain $(3,6)$-admissible choice (X, V^i), that there does not exist an admissible choice of data.

That is the case for instance if X is a k-gon with k odd, $\Gamma = K_{p,q}$ (the complete bipartite graph with $p+q$ vertices) with $p \neq q$ and $V^i = \emptyset$.

In [Laz14], Lazarovich proved that, for k even, under certain conditions, there is always an admissible choice of data for certain (Γ, k)-complexes (see Continuation Lemma in the afore-mentioned paper).

Theorem 6.1.23 (Theorem 1.4 in [BB94])**.** *Let X be a $(3,6)$-complex and let (V^i, X) be a (3,6)-admissible choice of interior vertices of X. Moreover, assume that $\mathrm{Lk}(v,X) \cong \Gamma$ for every interior vertex v. Then for any admissible choice of data (i_v, τ_e) there is, up to isomorphism, a unique $(3,6)$-complex Z containing X as a subcomplex and such that*

1. *The pair (Z, VX) is a $(3,6)$-admissible choice of interior vertices for Z,*
2. $\mathrm{Lk}(v,Z) \cong \Gamma$ *for all $v \in VX$,*
3. *For any boundary edge e of X, the transfer map along e with respect to Z is τ_e,*
4. *The interior edges of any boundary polygon P of Z form a connected part of ∂P of length 2, 3 or 4.*
5. *The complex X is a deformation retract of Z.*

The next result states that every $(3,6)$-complex can be constructed using the procedure of Theorem 6.1.23.

Theorem 6.1.24 (Theorem 1.5 in [BB94])**.** *Let X be a simply-connected (3,6)-complex and $v_0 \in VX$. Define a nested sequence of subcomplexes $X(n)$ of X by*

$$X(v_0, 0) = \{v_0\} \text{ and } X(v_0, n) = \mathrm{St}(X(v_0, n-1), X).$$

Then $(X(v_0,n), VX(v_0,n-1))$ is a (3,6)-admissible choice of interior vertices, for every $n \geq 1$, and $X(v_0, n+1)$ is obtained from $X(v_0,n)$ by the construction of Theorem 6.1.23.

Observe that, denoting $X(v,n)$ as in Theorem 6.1.24, we have that

$$\mathsf{B}(v,0) = X(v,1) = \mathrm{St}(v,X) \text{ and } \mathsf{B}(v,n) = X(v,n+1),$$

where $\mathsf{B}(v,n)$ is introduced in Definition 6.1.11.

6.2 When Γ is a finite cover of the Petersen graph

In this section, and henceforth, we will only consider CAT(0) (Γ,k)-complexes for which Γ is a regular connected finite cover of the Petersen graph for which all automorphisms lift. We will use the graph theoretical properties of odd graphs (for which the Petersen graph is an example) since for these class of graphs, under some conditions, we have that (Γ,k)-complexes are unique, up to isomorphism of polygonal complexes.

We will define an edge-star coloring for a regular graph and we will show, for the case of a finite cover of the Petersen graph Γ, that there is not a lot of freedom to define such a coloring (see Lemma 6.2.11).

These colorings on Γ will be used to color the polygons of a (Γ,k)-complex and the uniqueness of edge-star colorings for Γ will be the main key to prove that legal colorings in (Γ,k)-complexes Y are unique up to automorphisms of Y and, later on, to define the universal group independently of the coloring.

6.2.1 Odd graphs and their covers

Definition 6.2.1. A graph Γ is called *odd* if there is a d such that the vertices of Γ can be labeled by the $(d-1)$-subsets of $\{1,\dots,2d-1\}$ in a way that two vertices are connected by an edge if and only if the two respective subsets are disjoint.

The odd graphs are regular of degree d, so sometimes we will denote $\Gamma = O_d$ if Γ is the odd graph of degree d.

Definition 6.2.2. Let Γ be s simple graph.

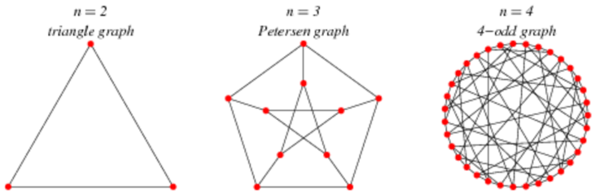


Figure 6.3: Odd graphs of small degrees. **Source:** Weisstein, Eric W. "Odd Graph" From MathWorld-A Wolfram Web Resource. http://mathworld.wolfram.com/OddGraph.html

1. We say that Γ is *vertex-star-transitive* if for every two vertices v_1, v_2 of Γ and for every bijection $\alpha : \mathrm{St}(v_1) \to \mathrm{St}(v_2)$ such that $\alpha(v_1) = v_2$ there exists an automorphism $\widetilde{\alpha} \in \mathrm{Aut}(\Gamma)$ such that $\widetilde{\alpha}|_{\mathrm{St}(v_1)} = \alpha$.

2. Let $e \in E\Gamma$. Considering $e = \{v_1, v_2\}$, the *star of the edge* e in Γ is defined as

$$\mathrm{St}(e, \Gamma) = \mathrm{St}(v_1, \Gamma) \cup \{e\} \cup \mathrm{St}(v_2, \Gamma),$$

that is, the set of edges of Γ which are adjacent to at least one of the vertices v_1 and v_2. If the graph Γ is clear from the context, we will denote such stars simply as $\mathrm{St}(e)$.

3. Given two edges $e_1, e_2 \in E\Gamma$, an *edge-star isomorphism* is a bijection $\alpha : \mathrm{St}(e_1) \to \mathrm{St}(e_2)$ such that $\alpha(e_1) = e_2$ (that is, the vertices of e_1 are mapped in some order to the vertices of e_2) and moreover α is incidence-preserving, meaning that, if $e_1 = \{v_1, v_2\}$ then $e \in E\Gamma \cap \mathrm{St}(e_1)$ is incident to v_1 if and only if $\alpha(e)$ is incident to $\alpha(v_1)$ and $f \in E\Gamma \cap \mathrm{St}(e_1)$ is incident to v_2 if and only if $\alpha(f)$ is incident to $\alpha(v_2)$.

4. We say that Γ is *edge-star-transitive* if for every two edges e_1, e_2 of Γ and for every edge-star isomorphism $\alpha : \mathrm{St}(e_1) \to \mathrm{St}(e_2)$ mapping e_1 to e_2 there is $\widetilde{\alpha} \in \mathrm{Aut}(\Gamma)$ such that $\widetilde{\alpha}|_{\mathrm{St}(e_1)} = \alpha$.

5. If $F \leq \mathrm{Aut}(\Gamma)$ is such that for every two edges e_1, e_2 of Γ and for every edge-star isomorphism $\alpha : \mathrm{St}(e_1) \to \mathrm{St}(e_2)$ mapping e_1 to e_2 there is $\widetilde{\alpha} \in F$ such that $\widetilde{\alpha}|_{\mathrm{St}(e_1)} = \alpha$ then we call Γ

an *F-edge-star-transitive group.* A similar definition holds for vertex-star-transitive subgroups of Γ.

We remark that if $girth(\Gamma) \leq 4$ then the definition of edge-star isomorphism does not require that it preserves the possible adjacency relations among the neighbors of vertices in the star of an edge. However, the graph automorphisms extending edge-star isomorphisms must preserve such adjacencies.

We collect some properties of the odd graphs which will be useful later on.

Proposition 6.2.3 (Properties of odd graphs)**.** *Let $\Gamma = O_d$ be the odd graph of degree d. Then the following hold.*

1. $|E\mathrm{St}(e)| = 2d - 1$ *for any edge $e \in E\Gamma$.*
2. *The diameter of Γ is $d - 1$.*
3. *The girth of Γ is 3 if $d = 2$, 5 if $d = 3$ and 6 if $n \geq 4$.*
4. $\mathrm{Aut}(\Gamma) = \mathrm{Sym}(2d - 1)$.
5. *Γ is vertex-star and edge-star-transitive.*

Proof. The first 4 statements can be found in [HS93]. Statement 4 is Theorem 7.1 in the afore-mentioned book and Statement 5 is a consequence of the fact that $\mathrm{Sym}(n)$ is n-transitive. □

Definition 6.2.4. Let $f : \Gamma_1 \to \Gamma_2$ be a covering map. An automorphism $g \in \mathrm{Aut}(\Gamma_2)$ is said to have a *lift* if there is an automorphism $\widetilde{g} \in \mathrm{Aut}(\Gamma_1)$ of the cover such that $g \circ f = f \circ \widetilde{g}$.

Lemma 6.2.5. *Let Γ_1 be the odd graph of degree d, with $d \geq 3$. Let Γ_2 be a regular cover of Γ_1 for which all automorphisms lift, with covering map f. Then Γ_2 is also vertex-star- and edge-star-transitive.*

Proof. We prove edge-star-transitivity. Vertex-star-transitivity can be proved in an analogous way. Let e_1 and e_2 be two edges of Γ_2 and let $\widetilde{\alpha} : \mathrm{St}(e_1) \to \mathrm{St}(e_2)$ be an edge-star isomorphism. Then we get, since $d \geq 3$, that $\mathrm{St}(e_i) \cong \mathrm{St}(f(e_i))$, for $i \in \{1, 2\}$, which implies that we get an induced map $\alpha : \mathrm{St}(f(e_1)) \to \mathrm{St}(f(e_2))$ such that

$$f \circ \widetilde{\alpha}(e) = \alpha \circ f(e) \text{ for all } e \in \mathrm{St}(e_1).$$

Since Γ_1 is edge-star-transitive, the map α extends to an automorphism $g \in \mathrm{Aut}(\Gamma_1)$. By assumption, the automorphism g lifts to an automorphism $\widetilde{g} \in \mathrm{Aut}(\Gamma_2)$. We have that $\widetilde{g}$ maps e_1 to an edge e_2' in the same fibre as e_2.

Using the regularity of the cover Γ_2, we know that there exists $\widetilde{g_2} \in \mathrm{Aut}(\Gamma_2)$ mapping e_2' to e_2 and that acts as the identity when pushed down to Γ_1. Then $\widetilde{g_2}\widetilde{g}$ is an automorphism of Γ_2 which extends $\widetilde{\alpha}$. Thus Γ_2 is edge-star-transitive. □

6.2.2 Uniqueness of polygonal complexes

Nir Lazarovich in [Laz14] studied combinatorial conditions on the link of a CAT(0) (k, Γ)-complex Y in order that Y is unique, up to isomorphism of polygonal complexes. The main result of his paper the we will use is the following.

Theorem 6.2.6 ([Laz14, Uniqueness Theorem])**.** *Let Γ be a finite connected graph and $k \geq 4$ such that (Γ, k) satisfies Gromov's Link Condition. If Γ is vertex-star- and edge-star-transitive then the number of isomorphism classes of CAT(0) (Γ, k)-complexes is at most 1.*

In particular, considering polygons with an even number of sides, Lazarovich provides a full characterization of unique (Γ, k)-complexes, up to isomorphism.

Theorem 6.2.7 ([Laz14, Theorem A])**.** *Let Γ be a finite connected graph. Let $k \in \mathbb{N}$ be greater than or equal to 4 and even such that (Γ, k) satisfies Gromov's Link Condition. Then there is, up to isomorphism of polygonal complexes, a unique CAT(0) (Γ, k)-complex if and only if the graph Γ is vertex-star-transitive and edge-star-transitive.*

The last theorem shows the interest behind considering (Γ, k)-complexes where Γ is a finite regular cover of an odd graph for which all the automorphisms lift, as in this case Γ is vertex-star- and edge-star-transitive. We state it in the next corollary for future references.

Corollary 6.2.8. *Let Γ be a finite cover of an odd graph for which all automorphisms lift. Let $k \in \mathbb{N}$ be greater than or equal to 4 and even . Then, up to isomorphism of polygonal complexes, there is a unique (Γ, k)-complex.*

Hence, as in the case of right-angled buildings, we are in a situation where our geometrical objects are unique. So it seems that we are "in business" to define the universal group for an interesting class of geometric objects.

6.2.3 Edge-star colorings of finite covers of the Petersen graph

We will define the universal group for CAT(0) (Γ, k)-complexes Y where Γ is a regular connected finite cover of the Petersen graph for which all automorphisms lift. For that we will introduce colorings on the polygons of the complex in the next section.

Since the link of each vertex v of Y is isomorphic to Γ, the polygons of Y incident to v correspond to edges of $\mathrm{Lk}(v, Y) = \Gamma$. Hence in this section we are interested in considering colorings of the edges of a finite cover of the Petersen graph Γ.

We define edge-star colorings for regular graphs and we show, for the particular case of finite covers of the Petersen graph, that these colorings are unique up to automorphism (see Lemma 6.2.15).

Definition 6.2.9. Let Γ be a regular graph of degree m. We define an *edge-star coloring* in Γ as a map $\iota : E\Gamma \to \{1, \dots, 2m-1\}$ such that, for all $e \in E\Gamma$,

$$\iota|_{\mathrm{St}(e)} : E\mathrm{St}(e) \to \{1, \dots, 2m-1\} \text{ is a bijection.}$$

Example 6.2.10. The *Petersen graph* O_3 is the odd graph of degree 3. Therefore its edge-star colorings are defined using 5 colors.

An example of such a coloring is presented in Figure 6.4, by considering the vertices as 2-subsets of $\{1, \dots, 5\}$ and two vertices being connected by an edge if the respective subsets are disjoint. Then the color of the respective edge is the element of the set $\{1, \dots, 5\}$ that is not in the subsets corresponding to the vertices incident to that edge. We will call this coloring the *standard edge-coloring* of the Petersen graph.

As the next lemma shows, there is not a lot of freedom in the choice of the colors for the edges of the Petersen graph if we want to construct an edge-star coloring.

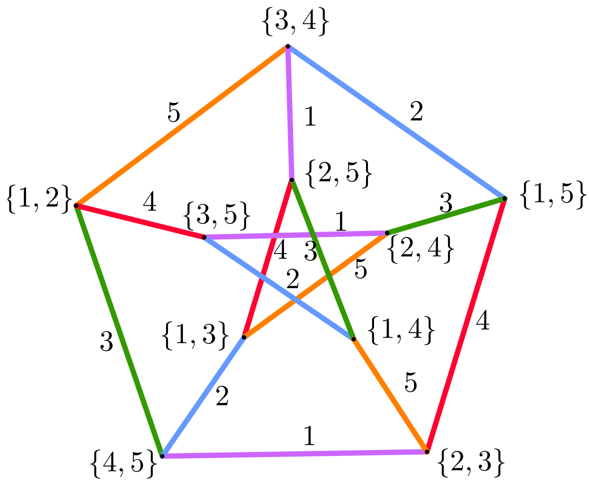


Figure 6.4: The standard edge-coloring of the Petersen graph.

Lemma 6.2.11. *Let O_3 be the Petersen graph and let ι_1 and ι_2 be two edge-star colorings of the edges of O_3. If ι_1 and ι_2 coincide in the star of an edge of O_3 then ι_1 and ι_2 coincide in the whole graph.*

Proof. The Petersen graph has a partition $\mathcal{P}$ of EO_3 into 5 sets of size 3, with edges at distance 3 from each other (that is, the minimum number of vertices that one has to cross in a minimal path between two such edges is 3). Therefore, two edges in distinct parts of $\mathcal{P}$ are at distance at most 2 and hence cannot have the same edge-star color, because they belong to the star of a common edge.

Thus the partition $\mathcal{P}$ corresponds to the edges with the same color from an edge-star coloring of O_3. Therefore, as soon as we define an edge-star coloring in the star of an edge in O_3, we attribute one color to each part of $\mathcal{P}$ and hence the coloring is completely determined in the whole graph.

Thus, if ι_1 and ι_2 coincide in the star of an edge then they are equal. $\square$

The previous lemma implies in particular that all the edge-star colorings are the same up to an automorphism of the Petersen graph.

Corollary 6.2.12. *Let O_3 be the Petersen graph. Then the following hold.*

1. *The choice of the colors of ι in the star of an edge determines ι in the whole graph.*

2. *If ι_1 and ι_2 are two edge-star colorings of O_3 then there is $g \in \operatorname{Aut}(O_3)$ such that $\iota_1 = \iota_2 \circ g$.*

Proof. We prove Statement 2. Without loss of generality we can assume that ι_2 is the edge-star coloring described in Example 6.4. Fix an edge $e = \{v_1, v_2\} \in EO_3$.

Let $g \in \operatorname{Sym}(5)$ be such that $\iota_1(e') = g \circ \iota_2(e')$ for all $e' \in \operatorname{St}(e)$. So be Lemma 6.2.11 ι_1 and $g \circ \iota_2$ coincide in O_3 as they coincide in the star of one edge.

The element g induces an automorphism $\widetilde{g} \in \operatorname{Aut}(O_3)$ given by $\widetilde{g}(\{a,b\}) = \{ga, gb\}$ for any vertex $v = \{a,b\} \in VO_3$ with $\{a,b\}$ a 2-subset of $\{1,\ldots,5\}$. Moreover, each edge of O_3 is determined by the two-subsets corresponding to the vertices incident to it and the ι_2-color of the edge is also determined. So, in particular, $\iota_1(e') = \iota_2 \circ \widetilde{g}(e')$ for all $e' \in EO_3$. Thus $\iota_1 = \iota_2 \circ \widetilde{g}$. □

Definition 6.2.13. Let $f : \widetilde{\Gamma} \to \Gamma$ be a finite cover of a m-regular graph Γ. We define an *edge-star coloring* in the edges of $\widetilde{\Gamma}$ as a map $\widetilde{\iota_f} : E\widetilde{\Gamma} \to \{1,\ldots,2m-1\}$ satisfying

1. $\widetilde{\iota_f}|_{\operatorname{St}(e)} : E\operatorname{St}(e) \to \{1,\ldots,2m-1\}$ is a bijection for all $e \in E\widetilde{\Gamma}$,

2. any two edges of $\widetilde{\Gamma}$ in the same fibre have the same color.

Lemma 6.2.14. *Let Γ be a m-regular graph and let $\widetilde{\Gamma}$ be a finite cover of Γ with covering map f. An edge-star coloring of Γ induces an edge-star coloring of $\widetilde{\Gamma}$ and vice versa.*

Proof. Let ι be an edge-star coloring of Γ. We consider a coloring $\widetilde{\iota_f}$ of the edges of $\widetilde{\Gamma}$ by

$$\widetilde{\iota_f}(e) = \iota(f(e)), \text{ for all } e \in E\widetilde{\Gamma}.$$

It is clear that $\widetilde{\iota_f}$ is an edge-star coloring of $\widetilde{\Gamma}$.

Conversely, if $\widetilde{\iota_f}$ is an edge-star-coloring of the finite cover $\widetilde{\Gamma}$ then we define a coloring $\iota : E\Gamma \to \{1, \ldots, 2m-1\}$ by $\iota(e) = \widetilde{\iota_f}(f^{-1}(e))$, for $e \in E\Gamma$. The coloring ι is well-defined because $\widetilde{\iota_f}$ satisfies Property 2 of Definition 6.2.13 and by Property 1 it follows that ι is an edge-star coloring of Γ. □

The next result shows that edge-star colorings of finite covers of the Petersen graph are also uniquely determined by the colors in the star of an edge.

Lemma 6.2.15. *Let $f : \widetilde{\Gamma} \to O_3$ be a connected finite regular cover of the Petersen graph. Let $\widetilde{\iota_1}$ and $\widetilde{\iota_2}$ be two edge-star colorings of $\widetilde{\Gamma}$ that coincide on* $\mathrm{St}(e)$ *for some $e \in E\widetilde{\Gamma}$. Then $\widetilde{\iota_1} = \widetilde{\iota_2}$.*

Proof. This result follows directly from Lemmas 6.2.11 and 6.2.14. □

Using an argument similar to the one of Corollary 6.2.12, we also conclude that edge-star colorings of finite covers of the Petersen graph are unique up to automorphism.

6.3 Colorings of (Γ, k)-complexes

Let Γ be a regular connected finite cover of the Petersen graph with covering map f for which all the automorphisms lift and let $k \geq 6$ be even. Then a (Γ, k)-complex Y is unique up to isomorphism (see Corollary 6.2.8) and Y is a CAT(0) $(3, 6)$-complex. Therefore it can be obtained using the construction of Ballmann and Brin as described in Section 6.1.4.

Let $\mathcal{P}(Y)$ denote the set of all k-gons of Y. We will define legal colorings in the set $\mathcal{P}(Y)$ that will later on be used to prescribe the local action of the universal group in the links of the vertices of Y.

Definition 6.3.1. A *legal coloring* of the (Γ, k)-complex Y is a map $c : \mathcal{P}(Y) \to \{1, \ldots, 5\}$ such that

1. For all $x \in VY$, $c|_{\mathrm{Lk}(x,Y)} : E\Gamma \to \{1, \ldots, 5\}$ is an edge-star coloring of the covering Γ (see Definition 6.2.13).

2. Let $e = \{v_1, v_2\}$ be an edge in Y and ξ_1 and ξ_2 be the vertices in $\mathrm{Lk}(v_1, Y)$ and $\mathrm{Lk}(v_2, Y)$, respectively, corresponding to e. Consider the two oriented edges $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_1)$ associated to e. Then

$$c(\varepsilon) = c(\tau_{e_1}(\varepsilon)) \text{ and } c(\varepsilon') = c(\tau_{e_2}(\varepsilon'),$$

for all $\varepsilon \in \mathrm{St}(\xi_1, \mathrm{Lk}(v_1, Y))$ and for all $\varepsilon' \in \mathrm{St}(\xi_2, \mathrm{Lk}(v_2, Y))$, where τ_{e_1} and τ_{e_2} are the transfer maps introduced in Definition 6.1.21.

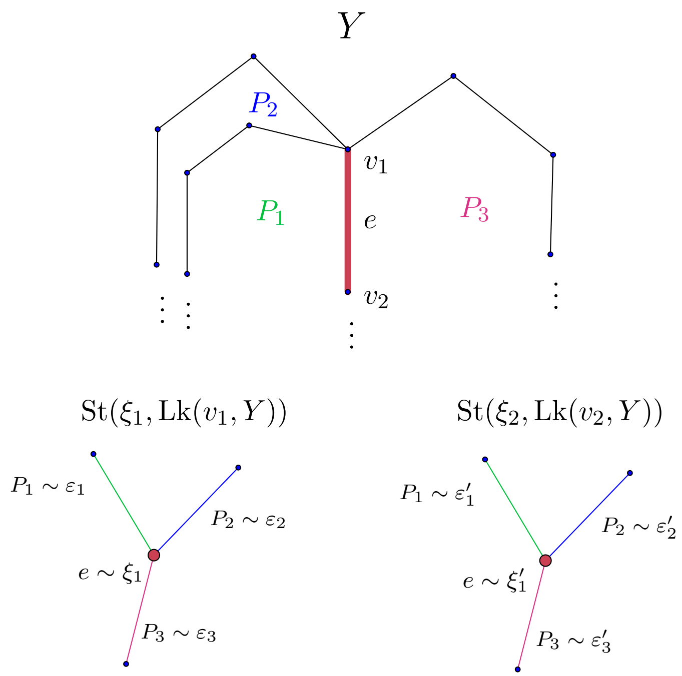


Figure 6.5: Second condition for a legal coloring on Y.

Observe that the second requirement in the definition of a legal coloring assures that each k-gon of Y has the same color, no matter in which link it is considered an edge.

The next proposition shows that legal colorings of CAT(0) $(3, 6)$ (Γ, k)-complexes are unique up to automorphism of polygonal complexes.

Proposition 6.3.2. *Let $k \geq 6$ and let Γ be a regular connected finite cover of the Petersen graph for which all the automorphisms lift. Let Y be the unique (Γ, k)-complex. Let c_1 and c_2 be colorings of Y and P_1 and P_2 be two polygons of Y such that $c_1(P_1) = c_2(P_2)$.*

Then there is $g \in \mathrm{Aut}(Y)$ such that $g(P_2) = P_1$ and $c_2 = c_1 \circ g$.

Proof. Consider two polygons such that $c_1(P_1) = c_2(P_2)$. Fix a vertex $v_2 \in VP_2$. Let

$$A_n = \{g \in \mathrm{Aut}(Y) \mid P_1 = g(P_2) \text{ and } c_2|_{\mathsf{B}(v_2,n)} = c_1 \circ g|_{\mathsf{B}(v_2,n)}\},$$

where $\mathsf{B}(v, n)$ was introduced in Definition 6.1.11. The strategy of the proof will be to recursively construct a sequence of elements $g_k \in A_k$ such that g_k and g_t agree on the ball of radius $\min\{k, t\}$ around v_2.

Lets us deal first with the case $n = 0$. Let $g \in \mathrm{Aut}(Y)$ be such that $g(P_2) = P_1$ and let $v_1 = g(v_2)$. Let $\varepsilon_1 \in \mathrm{Lk}(v_1, Y)$ and $\varepsilon_2 \in \mathrm{Lk}(v_2, Y)$ be the edges corresponding, respectively, to the polygons P_1 and P_2. The automorphism g induces an isomorphism, by abuse of notation also denoted by g, between $\mathrm{Lk}(v_2, Y)$ and $\mathrm{Lk}(v_1, Y)$ mapping $\mathrm{St}(\varepsilon_2, \mathrm{Lk}(v_2, Y))$ to $\mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1, Y))$. Consider a map

$$\varphi : \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1, Y)) \to \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1, Y)) \text{ s.t. } c_2(g^{-1}(\varepsilon)) = c_1(\varphi(\varepsilon)),$$

for every edge $\varepsilon \in \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1, Y))$. Observe that φ is well defined since we have $c_1(\varepsilon_1) = c_2(\varepsilon_2)$ by assumption and therefore $\varphi(\varepsilon_1) = \varepsilon_1$. Moreover, since both c_1 and c_2 are edge-star colorings of Γ, the map φ is actually an automorphism of $\mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1, Y))$. Since the graph Γ is edge-star-transitive by Lemma 6.2.5, we can extend φ to an automorphism g_{v_1} of $\mathrm{Lk}(v_1, Y)$ fixing ε_1. Observe that $c_2|_{\mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1,Y))} = c_1 \circ g_{v_1}|_{\mathrm{St}(\varepsilon_1, \mathrm{Lk}(v_1,Y))}$ by construction of φ. Therefore using Lemma 6.2.15, we also conclude that $c_2|_{\mathrm{Lk}(v_1,Y)} = c_1 \circ g_{v_1}|_{\mathrm{Lk}(v_1,Y)}$.

The automorphism g_{v_1} of $\mathrm{Lk}(v_1, Y)$ induces an automorphism of Y, also denoted by g_{v_1}, which fixes v_1 and stabilizes P_1. Thus by construction we have that $g_{v_1} \circ g \in A_0$.

Let $g_0 \in A_0$. Let us assume that we already constructed an automorphism $g_i \in A_i$ for every $i \leq n$ as described above. In particular $g_n(P_2) = P_1$ and $c_2(P) = c_1 \circ g_n(P)$ for all $P \in \mathsf{B}(v_2, n)$. Therefore, without loss of generality, we can assume that $P_1 = P_2$ and that

$c_1(P) = c_2(P)$ for every $P \in \mathsf{B}(v_2, n)$. We will construct an element $g_{n+1} \in A_{n+1}$ by extending g_n (which now acts trivially on $\mathsf{B}(v_2, n)$).

We can consider $\mathsf{B}(v_2, n)$ as $Y(v_2, n+1)$ in Ballmann and Brin's construction (see Theorem 6.1.24). Hence, using the data characterization of Definition 6.1.20, the boundary vertices of $Y(v_2, n+1)$ are either free or partly free (in which case they have a unique interior edge adjacent to it). Furthermore, since $k \geq 6$, two partly free vertices are not joined by an edge. We will extend the element g_n by considering separately the local action on partly free and free vertices. Let us denote $\mathsf{B}_n = \mathsf{B}(v_2, n) = Y(v_2, n+1)$ for sake of simplicity in the notation.

Step 1 - Extend g_n to the link of a partly free boundary vertex.

Let v be a partly free boundary vertex. Let e_1 be its unique interior edge in B_n and let e_2 be some other (boundary) edge incident to v in B_n.

Observe that in B_n the link of v is isomorphic to $\mathrm{St}(\xi, \Gamma)$, where ξ is the vertex corresponding to e_1. However, in B_{n+1} the link of v is isomorphic to Γ, since by the construction, v will be an interior vertex of B_{n+1}.

Let ξ_1 and ξ_2 be the vertices in $\mathrm{Lk}(v, \mathsf{B}_{n+1})$ corresponding to e_1 and e_2, respectively. Since v is partly free, this two vertices are connected by an edge ε_1 as demonstrated in Figure 6.6.

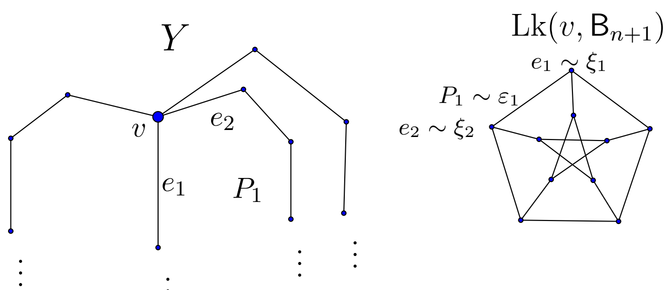


Figure 6.6: Step 1.

Consider a map

$$\alpha_v : \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v, \mathsf{B}_{n+1})) \to \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v, \mathsf{B}_{n+1}))$$

defined by $c_2(\varepsilon) = c_1 \circ \alpha_v(\varepsilon)$ for all $\varepsilon \in \mathrm{St}(\varepsilon_1, \mathrm{Lk}(v, \mathsf{B}_{n+1}))$. We have that c_1 and c_2 are edge-star colorings of Γ which agree on P_1 and coincide in every edge corresponding to a poygon in B_n. Hence the map α_v is an automorphism of $\mathrm{St}(varepsilon_1, \mathrm{Lk}(v, \mathsf{B}_{n+1}))$.

Using again the edge-star-transitivity of Γ we can extend α_v to an automorphism g_v of $\mathrm{Lk}(v, \mathsf{B}_{n+1})$.

For every partly free boundary vertex v we perform Step 1 and get an automorphism g_v of $\mathrm{Lk}(v, \mathsf{B}_{n+1})$.

Step 2 - Extend g_n to the link of boundary free vertices.

Observe that a boundary free vertex is in a path in $\partial\mathsf{B}_n$ bounded by two partly free vertices, i.e., there is a path

$$(x_0, x_1, \ldots, x_{m-2}, x_{m-1}, x_m),$$

in the 1-skeleton of a boundary polygon P in which x_0 and x_m are the only partly free vertices as depicted in Figure 6.7. For every $i \in \{1, \ldots m\}$, let $e_i = (x_{i-1}, x_i)$ and ε denote the edge in $\mathrm{Lk}(x_i, \mathsf{B}_{n+1})$ corresponding to P.

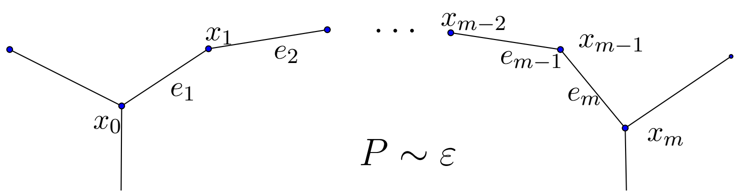


Figure 6.7: Step 2.

We will start by defining an automorphism of $\mathrm{Lk}(x_1, \mathsf{B}_{n+1})$ that is consistent with g_{x_0}, which was already defined in Step 1. Assume that the edge ε in $\mathrm{Lk}(x_1, \mathsf{B}_{n+1})$ is incident to two vertices ξ_1 and ξ_2, that correspond, respectively, to the edges e_1 and e_2. Define

$$\alpha_{x_1} : \mathrm{St}(\varepsilon, \mathrm{Lk}(x_1, \mathsf{B}_{n+1})) \to \mathrm{St}(\varepsilon, \mathrm{Lk}(x_1, \mathsf{B}_{n+1}))$$

by

$$\alpha_{x_1}|_{\mathrm{St}(\xi_1, \mathrm{Lk}(x_1, \mathsf{B}_{n+1}))} = \tau_{e_1} \circ g_{x_0} \circ \tau_{e_1}^{-1},$$

and $\alpha_{x_1}|_{\mathrm{St}(\xi_2, \mathrm{Lk}(x_1, \mathsf{B}_{n+1}))}$ is such that

$$c_2(\varepsilon') = c_1 \circ \alpha_{x_1}(\varepsilon') \text{ for all } \varepsilon' \in \mathrm{St}(\xi_2, \mathsf{B}_{n+1}).$$

(τ_{e_1} is the transfer map as in Definition 6.1.16). Observe that α_{x_1} is well defined since c_1 and c_2 coincide in P and therefore in ε. Moreover, by construction, it is an automorphism.

Using edge-star-transitivity of Γ we obtain an automorphism g_{x_1} of the link of x_1 in B_{n+1} extending α_{x_1} and such that $c_2 = c_1 \circ g_{x_1}$ in $\mathrm{Lk}(x_1, \mathsf{B}_{n+1})$ (by making use of Lemma 6.2.15).

We do this procedure up to $m-2$ and we obtain elements $g_{x_1}, \dots g_{x_{m-2}}$. Now we have to construct $g_{x_{m-1}}$ (observe that g_{x_m} is already defined using Step 1). We consider

$$\alpha_{x_{m-1}} : \mathrm{St}(\varepsilon, \mathrm{Lk}(x_{m-1}, \mathsf{B}_{n+1})) \to \mathrm{St}(\varepsilon, \mathrm{Lk}(x_{m-1}, \mathsf{B}_{n+1}))$$

defined by

$$\alpha_{x_{m-1}}|_{\mathrm{St}(\xi_{m-1}, \mathrm{Lk}(x_{m-1}, \mathsf{B}_{n+1}))} = \tau_{e_{m-1}} \circ g_{x_{m-2}} \circ \tau_{e_{m-1}}^{-1},$$
$$\alpha_{x_{m-1}}|_{\mathrm{St}(\xi_m, \mathrm{Lk}(x_{m-1}, \mathsf{B}_{n+1}))} = \tau_{e_m}^{-1} \circ g_{x_m} \circ \tau_{e_m}.$$

Once more, combining the edge-star-transitivity of Γ with Lemma 6.2.15 we get an automorphism g_{m-1} of the link of x_{m-1} where the two colorings coincide.

Observe that by construction the elements $g_{x_0}, g_{x_1}, \dots, g_{x_{x-1}}$ and g_{x_m} are consistent with each other.

We perform Step 2 in every path in $\partial \mathsf{B}_n$ bounded by two partly free vertices and hence we define g_v for every boundary vertex v. Thus we have extended g_n to an element $g_{n+1} \in A_{n+1}$ that agrees with g_n on B_n.

The sequence $g_0, g_1, \ldots$ obtained by repeating this procedure by levels converges (with respect to the permutation topology on $\mathrm{Aut}(Y)$) to an automorphism $g \in \mathrm{Aut}(Y)$. By construction, $g(P_2) = P_1$ and $c_2 = c_1 \circ g$, so the conditions of the proposition are satisfied. □

6.4 The universal group for a (Γ, k)-complex

In this section we are ready to consider the idea of Burger and Mozes and to define the universal group for a polygonal complex.

We will then prove basic properties of these groups and since this is still work in progress, we will end the chapter with comments and open questions about possible developments for these groups.

6.4.1 The definition

Let $k \geq 6$ and let Γ be a regular connected finite cover of the Petersen graph O_3 with covering map f for which all the automorphisms lift. Let Y be the unique CAT(0) (Γ, k)-complex.

Lemma 6.4.1. *Let $g \in \mathrm{Aut}(Y)$ and c be a legal coloring for Y. Then, for any $v \in VY$, the map $\sigma(g, v) = c|_{\mathrm{Lk}(g(v),Y)} \circ g \circ (c|_{\mathrm{Lk}(v,Y)})^{-1}$ is well defined as a permutation on the set $\{1, \ldots, 5\}$.*

Proof. Let $v \in VY$ and let P be a polygon in $\mathrm{St}(v, Y)$. Let ε be the edge in $\mathrm{Lk}(v, Y)$ corresponding to P and ε' be the edge in $\mathrm{Lk}(g(v), Y)$ corresponding to $g(P)$. Observe that the element g maps $\mathrm{St}(\varepsilon, \mathrm{Lk}(v, Y))$ to $\mathrm{St}(\varepsilon', \mathrm{Lk}(g(v), Y))$.

Let $\sigma \in \mathrm{Sym}(5)$ be a permutation such that $\sigma \circ c(\mu) = c \circ g(\mu)$ for every edge $\mu \in \mathrm{St}(\varepsilon, \mathrm{Lk}(v, Y))$.

Since c is an edge-star coloring, such a permutation σ in the star of ε determines a permutation of the colors in the whole graph $\mathrm{Lk}(v, Y)$, that is, we have $\sigma \circ c(\mu) = c \circ g(\mu)$ for all $\mu \in E(\mathrm{Lk}(v, Y))$.

Now assume that ε_1 and ε_2 are two edges in $\mathrm{Lk}(v, Y)$ with the same color. By slight abuse of notation, let $g(\varepsilon_1)$ and $g(\varepsilon_2)$ denote the edges in $\mathrm{Lk}(g(v), Y)$ corresponding to the image under g of the polygons corresponding to ε_1 and ε_2. Then we have

$$c(g(\varepsilon_1)) = \sigma \circ c(\varepsilon_1) = \sigma \circ c(\varepsilon_2) = c(g(\varepsilon_2)).$$

This means that if two polygons in the star of a vertex in Y have the same color a (and hence belong to $(c|_{\mathrm{Lk}(v,Y)})^{-1}(a)$, for some vertex v), then they are mapped through g to two polygons whose colors also coincide. Therefore $\sigma(g, v) = c|_{\mathrm{Lk}(g(v),Y)} \circ g \circ (c|_{\mathrm{Lk}(v,Y)})^{-1}$ is a permutation. □

Now we are ready to define the universal group for a CAT(0) (Γ, k)-complex Y.

Definition 6.4.2. Let c be a legal coloring of the k-gons of Y and $F \leq \mathrm{Sym}(5)$ be a permutation group.

We define the *universal group* for Y with respect to F as

$$U(F) = \{g \in \mathrm{Aut}(Y) \mid c|_{\mathrm{Lk}(g(v),Y)} \circ g \circ (c|_{\mathrm{Lk}(v,Y)})^{-1} \in F,\ \forall v \in VY\}.$$

We remark that in this definition we are making use of the clear identification between the sets $\mathrm{Lk}(v, Y)$ and $\mathrm{St}(v, Y)$, for a vertex $v \in VY$ (see Remark 6.1.15). Indeed, by definition, a legal coloring on the polygons in $\mathrm{St}(v, Y)$ induces an edge-star coloring on the graph $\mathrm{Lk}(v, Y)$. By Lemma 6.4.1, the set $U(F)$ is well defined. It is a group as showed by the next lemma whose proof uses only the definition of $\sigma(g, v)$ as in Lemma 6.4.1.

Lemma 6.4.3. *Let $v \in VY$ and $g, h \in \mathrm{Aut}(Y)$.*

Then $\sigma(gh, v) = \sigma(g, hv)\sigma(h, v)$.

6.4.2 Properties

We retain the notation as before. Observe that any star-edge coloring c of the covering Γ induces an edge-star coloring c_{O_3} of the Petersen graph O_3 as described in the proof of Lemma 6.2.14. Concretely, $c_{O_3}(e) = c(f^{-1}(e))$ for any $e \in EO_3$.

Moreover, by Lemma 6.2.15, we can assume without loss of generality that the coloring c induces the standard legal coloring c_{O_3} in O_3, as in Example 6.4. We will use this assumption from now on to make explicit in which way we can see directly an element $g \in \mathrm{Sym}(5)$ as an automorphism of the Petersen graph, that preserves the standard legal edge-star coloring.

Definition 6.4.4. Let $G \leq \mathrm{Aut}(Y)$ and let v be a vertex of the complex Y. We define the *local action* of G at v as the permutation group induced by $\mathrm{Stab}_G(v)$ on $\mathrm{St}(v, Y)$, which is isomorphic to $\mathrm{Stab}_G(v)/\,\mathrm{Fix}_{\mathrm{Stab}_G(v)}(\mathrm{St}(v, Y))$.

Observe that the subcomplex $\mathrm{St}(v, Y)$ of Y introduced in Definition 6.1.14 is in direct correspondence with the graph $\mathrm{Lk}(v, Y)$ and each element $g \in \mathrm{Stab}_G(v)$ induces an automorphism of $\mathrm{Lk}(v, Y)$. Hence we can consider the local action of G at v as the induced action of $\mathrm{Stab}_G(v)$ on $\mathrm{Lk}(v, Y)$.

Definition 6.4.5. Let $F \leq \mathrm{Sym}(5)$. We consider the group $\mathrm{Aut}(\Gamma)(F)$ to be the subgroup of automorphisms $g \in \mathrm{Aut}(\Gamma)$ such that $f \circ g = h \circ f$, for some $h \in F$.

Observe that, for $h \in F$, there might be more than one element $g \in \mathrm{Aut}(\Gamma)$ such that $f \circ g = f \circ h$, by considering, for instance, distinct permutations of the fibres of Γ, whose edges of each fibre all have the same color.

Lemma 6.4.6. *The local action of the universal group $U(F)$ on each vertex of the (Γ, k)-complex Y is permutationally isomorphic to the group $\mathrm{Aut}(\Gamma)(F)$.*

Proof. Let $v \in VY$. The edges of $\mathrm{Lk}(v, Y) \cong \Gamma$ are parametrized by the elements of the set $\{1, \dots, 5\}$ using the legal coloring c.

Let $g \in \mathrm{Stab}_{U(F)}(v)$. We know that it induces an automorphism of $\Gamma \cong \mathrm{Lk}(v, Y)$, also denoted by g. Moreover, by definition of the universal group, we have $c|_{\mathrm{Lk}(v,Y)} \circ g \circ (c|_{\mathrm{Lk}(v,Y)})^{-1} = h \in F$.

By the assumption that the edge-star coloring $c_{O_3} = c \circ f^{-1}$ of O_3 induced from c is standard, we know that the element h induces an automorphism $\widetilde{h} \in F \leq \mathrm{Aut}(O_3)$ such that, for a vertex $\{a, b\} \in VO_3$, $\widetilde{h}(\{a, b\}) = \{ha, hb\}$. Furthermore $h \circ c_{O_3} = c_{O_3} \circ \widetilde{h}$.

By construction we have that $f \circ g = \widetilde{h} \circ f$, as shown in Figure 6.8. Thus $g \in \mathrm{Aut}(\Gamma)(F)$.

Now we have to show that this local action is actually the whole group $\mathrm{Aut}(\Gamma)(F)$. Let $g \in \mathrm{Aut}(\Gamma)(F)$ and let $h \in F$ such that $f \circ g = h \circ f$.

Consider $c' = h \circ c$. Observe that $c(\varepsilon) = h \circ c(g(\varepsilon))$ for all edge $\varepsilon \in \mathrm{St}(v, Y)$, as the edges in the same fibre all have the same color, by definition of an edge-star coloring of the cover Γ. Furthermore c'

$$\begin{tikzcd}
\Gamma \arrow[r, "c"] \arrow[d, "g"'] \arrow[rr, bend left, "f"] & \{1,\dots,5\} \arrow[d, "h"'] & O_3 \arrow[l, "c_{O_3}"'] \arrow[d, "\widetilde{h}"'] \\
\Gamma \arrow[r, "c"] \arrow[rr, bend right, "f"'] & \{1,\dots,5\} & O_3 \arrow[l, "c_{O_3}"']
\end{tikzcd}$$

Figure 6.8: Proof of Lemma 6.4.6

is a legal coloring of the polygons of Y as it is obtained from c by only applying a permutation of the set $\{1, \dots, 5\}$.

Let $P \in \mathrm{St}(v, Y)$ and let $\varepsilon \in \mathrm{Lk}(v, Y)$ be the corresponding edge in the link. Then $c(\varepsilon) = c'(g(\varepsilon))$. Using a similar construction to the one of the proof of Proposition 6.3.2, we can obtain an automorphism $\widetilde{g} \in \mathrm{Aut}(Y)$ such that $c \circ \widetilde{g} = c'$ and $\widetilde{g}|_{\mathrm{St}(v,Y)} = g$.

The element $\widetilde{g}$ acts locally on $\mathrm{Lk}(v, Y)$ as g and, for any $v' \in VY$, we have

$$(c \circ g \circ (c)^{-1})|_{\mathrm{Lk}(v',Y)} = (c' \circ c^{-1})|_{\mathrm{Lk}(v',Y)} = (h \circ c \circ c^{-1})|_{\mathrm{Lk}(v',Y)} = h \in F.$$

Thus $\widetilde{g} \in U(F)$. □

Proposition 6.4.7. *1. The group $U(F)$ is independent of the legal coloring up to conjugacy in* $\mathrm{Aut}(Y)$.

2. $U(F)$ is a closed subgroup of $\mathrm{Aut}(Y)$.

3. F is an edge-transitive subgroup of $\mathrm{Aut}(O_3)$ *if and only if $U(F)$ is face-transitive, that is, $U(F)$ is transitive on the polygons of Y.*

Proof. 1. Let c_1 and c_2 be two legal colorings of the polygons of Y and let $U^1(F)$ and $U^2(F)$ be the respective universal groups constructed using those colorings. We want to show that these two groups are conjugate in $\mathrm{Aut}(Y)$.

By Lemma 6.3.2 we know that there exists $g \in \mathrm{Aut}(Y)$ such that $c_1 = c_2 \circ g$. Let $u \in U^1(F)$. Then we know that on any link

$$c_1 \circ u \circ c_1^{-1} \in F \Leftrightarrow c_2 \circ g \circ u \circ g^{-1} \circ c_2^{-1} \in F, \text{ for all } v \in VY.$$

This means that $u^g \in U^2(F)$ and thus $U^1(F)$ and $U^2(F)$ are conjugate in $\mathrm{Aut}(Y)$.

2. We will show that $\mathrm{Aut}(Y)\backslash U(F)$ is open. Let $g \in \mathrm{Aut}(Y)\backslash U(F)$. Then there is $v \in VY$ such that

$$c|_{\mathrm{Lk}(g(v),Y)} \circ g \circ (c|_{\mathrm{Lk}(v,Y)})^{-1} \notin F.$$

Then it follows that the set $\{g' \in \mathrm{Aut}(Y) \mid g|_{\mathrm{St}(v,Y)} = g'|_{\mathrm{St}(v,Y)}\}$ is also contained in $\mathrm{Aut}(Y)\backslash U(F)$ and this set is a coset of the stabilizer in $\mathrm{Aut}(Y)$ of v (which is open). Hence $U(F)$ is closed.

3. Assume that F acts edge-transitively on O_3 As Y is a connected complex, it is enough to prove the result for two polygons P_1 and P_2 sharing an edge e incident to a vertex v, and then proceed inductively.

   Let ε_1 and ε_2 be the edges corresponding, respectively, to P_1 and P_2 in $\mathrm{Lk}(v, Y)$. Let $\varepsilon_1' = f(\varepsilon_1)$ and $\varepsilon_2' = f(\varepsilon_2)$ be edges in O_3, where $f : \Gamma \to O_3$ is the covering map. Since F is edge-transitive, there exists $h \in F$ such that $h\varepsilon_1' = \varepsilon_2'$.

   Extend this element h to an element $\widetilde{h_1} \in \mathrm{Aut}(\Gamma)(F)$. Then $\widetilde{h_1}$ maps ε_1 to an element $\overline{\varepsilon_2}$ in the same fibre as ε_2.

   Due to the regularity of the cover Γ, we know that there is $\widetilde{h_2} \in \mathrm{Aut}(\Gamma)$ such that $\widetilde{h_2}\overline{\varepsilon_2} = \varepsilon_2$ and $f \circ \widetilde{h_2} = \mathrm{id}_{O_3} \circ f$. Hence $\widetilde{h} = \widetilde{h_2}\widetilde{h_1} \in \mathrm{Aut}(\Gamma)(F)$ and $\widetilde{h}\varepsilon_1 = \varepsilon_2$.

   As the local action of $U(F)$ on the vertices of Y is permutationally isomorphic to $\mathrm{Aut}(\Gamma)(F)$, we obtain that $\widetilde{h}$ extends to an element $g \in U(F)$ and by construction $gP_1 = P_2$.

   Conversely, assume that $U(F)$ is face-transitive. Let e_1 and e_2 be two edges of the Petersen graph O_3. Let $\varepsilon_1 \in f^{-1}(e_1)$ and $\varepsilon_2 \in f^{-1}(e_2)$ be edges of the cover Γ.

   Then there is $v \in VY$ and $P_1, P_2 \in \mathcal{P}(Y)$ such that ε_1 and ε_2 are the edges in $\mathrm{Lk}(v, Y)$ corresponding to P_1 and P_2, respectively. Since $U(F)$ is face-transitive, there is $g \in U(F)$ such that

$gP_1 = P_2$. The local action of g on $\mathrm{Lk}(v, Y)$ is permutationally isomorphic to an element $\widetilde{g} \in \mathrm{Aut}(\Gamma)(F)$ by Lemma 6.4.6. Moreover we have $\widetilde{g}\varepsilon_1 = \varepsilon_2$.

By definition of $\mathrm{Aut}(\Gamma)(F)$, there is $h \in F$ such that $f \circ \widetilde{g} = h \circ f$. Then $h(e_1) = h \circ f(\varepsilon_1) = f \circ \widetilde{g}(\varepsilon_1) = f(\varepsilon_2) = e_2$. Therefore F is an edge-transitive subgroup of $\mathrm{Aut}(O_3)$.

$\square$

Definition 6.4.8. A *flag* in Y is a triple (v, e, P) where v is a vertex incident to an edge e that is incident to a polygon P. We say that a group $G \leq \mathrm{Aut}(Y)$ is *flag-transitive* if G is transitive on the set of flags of Y.

We can give (necessary and sufficient) conditions on the local action in order to obtain flag-transitivity of the universal group of a polygonal complex.

Proposition 6.4.9. *$U(F)$ is flag-transitive if and only if F is edge- and vertex-transitive as an automorphism of the Petersen graph O_3.*

Proof. Assume that $U(F)$ is flag-transitive. In particular it is transitive in the set of polygons and therefore we obtain that F is an edge-transitive subgroup of $\mathrm{Aut}(O_3)$ by Proposition 6.4.7(3).

Now we show that F is vertex-transitive as a subgroup of $\mathrm{Aut}(O_3)$. Let $v_1, v_2 \in VO_3$ and let $\xi_i \in f^{-1}(v_i) \in V\Gamma$ for $i \in \{1, 2\}$. The vertices ξ_1 and ξ_2 correspond to edges in the complex Y incident to a vertex $v \in VY$. As $U(F)$ is flag-transitive, there is $g \in U(F)$ such that $g\xi_1 = \xi_2$.

The local action of g on $\mathrm{Lk}(v, Y)$ is then permutationally isomorphic to $g \in \mathrm{Aut}(\Gamma)(F)$. This means that there exists $h \in F$ such that $f \circ g = h \circ f$, with f the covering map. Therefore we have $h(v_1) = h \circ f(\xi_1) = f \circ g(\xi_1) = f(\xi_2) = v_2$. Thus F is vertex-transitive as an automorphism group of O_3.

Conversely, assume that F is edge- and vextex-transitively as a subgroup of $\mathrm{Aut}(O_3)$. Let (v_1, e_1, P_1) and (v_2, e_2, P_2) be two flags in Y. Since F is edge-transitive, we know that $U(F)$ is face-transitive by Proposition 6.4.7(3). Therefore there is $g \in U(F)$ such that $gP_1 = P_2$.

Consider a minimal path

$$(gv_1, t_1, u_2, t_2, u_3, \ldots, t_n, v_2) \text{ with } u_i \in VY \text{ and } t_i \in EY,$$

between gv_1 and v_2. Regarding t_1 and t_2 as vertices of $\mathrm{Lk}(u_2, Y)$ ($\cong \Gamma$), let $t_1' = f(t_1)$ and $t_2' = f(t_2)$ be vertices of O_3. As F is vertex-transitive on O_3, there is $h_1 \in F$ such that $h_1 t_1' = t_2'$. Extend h_1 to $\widetilde{g_1} \in \mathrm{Aut}(\Gamma)(F)$. Then $\widetilde{g_1}(t_1) = \overline{t_2}$, with $\overline{t_2}$ in the same fibre as t_2 with respect to the covering map f.

Since Γ is a regular cover, let $\widetilde{\alpha_1} \in \mathrm{Aut}(\Gamma)$ such that $\widetilde{\alpha_1}(\overline{t_2}) = t_2$ and $f \circ \widetilde{\alpha_1} = \mathrm{id}_{O_3} \circ f$. Then $g_1 = \widetilde{\alpha_1}\widetilde{g_1}$ is an element of $\mathrm{Aut}(\Gamma)(F)$ such that $g_1(t_1) = t_2$. By Lemma 6.4.6 g_1 extends to an element of $U(F)$, also denoted g_1, which stabilizes u_2 and maps t_1 to t_2 as edges of Y. In particular $g_1 g v_1 = u_3$ and $g_1 g P_1 = P_2$.

Consider $g_i \in U(F)$ in an analogous way for $i \in \{1, \dots, n\}$. Then we obtain that $\widetilde{g} = g_n \cdots g_1 g \in U(F)$ is such that $\widetilde{g} v_1 = v_2$ and $\widetilde{g} P_1 = P_2$. If $\widetilde{g} e_1 = e_2$ then we are done.

If not, consider e_2 and $\widetilde{g} e_1$ as vertices of $\mathrm{Lk}(v_2, Y)$. Let $e_2' = f(e_2)$ and $\widetilde{g} e_1' = f(\widetilde{g} e_1)$ as vertices of O_3. Using vertex-transitivity of F, we obtain an element $h \in F$ such that $h(\widetilde{g} e_1') = e_2'$. By assumption on the cover, we can extend h to an element $\widetilde{h_1} \in \mathrm{Aut}(\Gamma)(F)$ mapping e_1 to a vertex $\overline{e_2}$ in the same fibre as e_2.

As before, we make use of the regularity of Γ and we consider and element $\widetilde{h_2} \in \mathrm{Aut}(\Gamma)(F)$ such that $\widetilde{h_2}(\overline{e_2}) = e_2$ and $f \circ \widetilde{h_2} = \mathrm{id}_{O_3} \circ f$. Then $\widetilde{h_2}\widetilde{h_1} \in \mathrm{Aut}(\Gamma)(F)$ and using Lemma 6.4.6 we can extend it to an element $\widetilde{h} \in U(F)$ such that $\widetilde{h} v_2 = v_2, \widetilde{h} P_2 = P_2$ and $\widetilde{h}\widetilde{g} e_1 = e_2$. Thus we obtain that $\widetilde{h}\widetilde{g} \in U(F)$ and $\widetilde{h}\widetilde{g}(v_1, e_1, P_1) = (v_2, e_2, P_2)$. Hence $U(F)$ is flag-transitive. □

6.4.3 Universality of $U(F)$

In this section we assume that Γ is the Petersen graph O_3 and we prove that the groups of automorphisms of CAT(0) (Γ, k)-complexes that we defined are actually universal, by assuming some extra conditions on the local action, namely 3-transitivity.

First we show that this extra assumption on the local action covers the local-to-global properties stated in the previous section.

Lemma 6.4.10. *If $F \leq \mathrm{Sym}(5)$ is a 3-transitive group then F is edge-transitive and vertex-transitive as a subgroup of* $\mathrm{Aut}(O_3)$.

Proof. Consider the Petersen graph O_3 with vertex-set the 2-subsets of $\{1, \dots, 5\}$ and with the standard edge-star coloring c_{O_3} on its edges.

Each vertex of O_3 is determined by a 2-subset of $\{1, \dots, 5\}$ so if F is 3-transitive then it is 2-transitive and vertex-transitivity follows.

Now let $e_1 = \{v_1, v_2\}$ and $e_2 = \{u_1, u_2\}$ be edges of O_3, with v_i and u_i 2-subsets of $\{1, \dots, 5\}$ for $i \in \{1, 2\}$. Assume $v_i = \{a_1, b_i\}$ for $i = \{1, 2\}$. Since we are asuming that c_{O_3} is the standard edge-star coloring of the Petersen graph, the edge e_1 is determined by a_1, b_1 and the color $c_{O_3}(e_1)$. Similarly, if $u_i = \{c_i, d_i\}$ for $i = \{1, 2\}$ then e_2 is determined by c_1, d_1 and $c_{O_3}(e_2)$. As the group F is 3-transitive, there exists $g \in F$ such that $g(a_1, b_1, c_{O_3}(e_1)) = (c_1, d_1, c_{O_3}(e_2))$. This element, as an automorphism of the Petersen graph (defined by $h(v) = h(\{a, b\}) = \{ha, hb\})$ maps e_1 to e_2. Hence F is edge-transitive. $\square$

Observe that the converse of the previous lemma is not true. If we assume that F is vertex- and edge-transitive as a subgroup of $\mathrm{Aut}(O_3)$ then we obtain only that F is transitive on the triples $(a, b, c) \in (\{1, \dots, 5\})^3$ such that $\{a, b, c\}$ is a 3-subset of $\{1, \dots, 5\}$.

Proposition 6.4.11. *Let Γ be the Petersen graph. Assume that F is a 3-transitive subgroup of* $\mathrm{Sym}(5)$. *Let H be a face-transitive closed subgroup of* $\mathrm{Aut}(Y)$ *such that the local action on the links of Y is permutationally isomorphic to F.*

Then there is a legal coloring on the polygons of Y that turns H into a subgroup of $U(F)$.

Proof. We will construct such a legal coloring c.

Let $v_0 \in VY$ and $P \in \mathrm{St}(v_0, Y)$, the star of v_0 in the complex Y. Let ε be the edge corresponding to P in $\mathrm{Lk}(v_0, Y)$. We choose a bijection $c_0 : E\mathrm{St}(\varepsilon, \mathrm{Lk}(v_0, Y)) \to \{1, \dots, 5\}$ such that

$$\left(c_{v_0} \circ H|_{\mathrm{St}(v_0,Y)} \circ (c_0)^{-1}\right)|_{\mathrm{Lk}(v_0,Y)} = F.$$

Observe that once we choose this bijection in the star of an edge, we know by Lemma 6.2.15 that the choice of an edge-star coloring in the edges of $\mathrm{Lk}(v_0, Y)$ is determined.

With this procedure we color the polygons in the complex $\mathsf{B}(v_0, 0) = Y(v_0, 1) = \mathrm{St}(v_0, Y)$ by defining $c(P) = c_0(\varepsilon)$, where ε is the edge in $\mathrm{Lk}(v_0, Y)$ corresponding to P. Assume now that we have defined a legal coloring c in $Y(v_0, n) = \mathsf{B}(v_0, n - 1)$.

Let $v \in \partial Y(v_0, n)$ be a partly free vertex. Observe that c is already defined in the polygons in $D = \mathrm{St}(v, Y) \cap Y(v_0, n)$ (see Figure 6.6). Let $P \in \mathcal{P}(D)$ and let ε be the corresponding edge in $\mathrm{Lk}(v, Y)$. Let $e = \{v_1, v\}$ be the unique interior edge of $Y(v_0, n)$ incident to v and let ξ be the corresponding vertex in $\mathrm{Lk}(v, Y)$.

Choose a bijection $\overline{c_v} : E\mathrm{St}(\varepsilon, \mathrm{Lk}(v, Y)) \to \{1, \dots 5\}$ such that

$$\left(\overline{c_v} \circ H|_{\mathrm{St}(v,Y)} \circ (\overline{c_v})^{-1}\right)|_{\mathrm{Lk}(v,Y)} = F.$$

By Lemma 6.2.11 this bijection defines an edge-star coloring of the Petersen graph. Let $\varepsilon' \in E\mathrm{St}(\varepsilon, \mathrm{Lk}(v, Y))$ such that $\overline{c_v}(\varepsilon') = c(\varepsilon)$ (which coincides with $c(P)$). Let ε_1 and ε_2 be the remaining edges in $\mathrm{St}(\xi, \mathrm{Lk}(v, Y))$ corresponding to polygons P_1 and P_2, respectively, that are in D.

Let $\varepsilon_1', \varepsilon_2' \in \mathrm{St}(\varepsilon, \mathrm{Lk}(v, Y))$ be edges such that

$$(c(P_1), c(P_2)) = (\overline{c_v}(\varepsilon_1'), \overline{c_v}(\varepsilon_2')).$$

Since F is 3-transitive (on the set of colors), we know that there is an element $g \in F$ such that

$$g(\overline{c_v}(\varepsilon), \overline{c_v}(\varepsilon_1), \overline{c_v}(\varepsilon_2)) = (\overline{c_v}(\varepsilon'), \overline{c_v}(\varepsilon_1'), \overline{c_v}(\varepsilon_2')).$$

Define $c_v = g \circ \overline{c_v}$, which is also an edge-star coloring of Γ. By construction we have

$$\left((c_v \circ H|_{\mathrm{St}(v,Y)} \circ (c_v)^{-1}\right)|_{\mathrm{Lk}(v,Y)} = F.$$

and moreover the colors of the polygons in D coincide in c and $c_v = g \circ \overline{c_v}$. Then, for $P \in \mathrm{St}(v, Y)$, we set $c(P) = c_v(\mu)$, where μ is the edge in $\mathrm{Lk}(v, Y)$ corresponding to P.

We do this procedure for all partly free vertices in $\partial Y(v_0, n)$.

Now we will construct the legal coloring on the link of free vertices in $\partial Y(v_0, n)$. Such vertices are in paths

$$\gamma = (x_0, x_1, \dots, x_{m-2}, x_{m-1}, x_m)$$

in $\partial Y(v_0, n)$ such that x_0 and x_m are the only partly free vertices (see Figure 6.7). Take one of these paths γ, which is along the boundary of a polygon P. Let ε be the edge corresponding to P in the links of all the vertices x_i.

We will define edge-star colorings on $\mathrm{Lk}(x_i, Y)$ that are consistent with each other and that give rise to a legal coloring of the polygons of $Y(v_0, n+1)$. Observe that c_{x_0} and c_{x_m} are already defined by the previous paragraph and they coincide in P by the definition of a legal coloring of the polygons of Y(see condition 2 in Definition 6.3.1).

Let $e_i = (x_{i-1}, x_i)$ and let ξ_i be the vertex in the links of x_{i-1} and x_i corresponding to e_i. Hence $\varepsilon = \{\xi_i, \xi_{i+1}\}$ in $\mathrm{Lk}(x_i, Y)$ for all $i \in \{1, \ldots, m-1\}$ (see Figure 6.7). The strategy will be similar to the case of partly free vertices, but now we have to match the colorings c_{x_i} not in the star of an interior edge, but in the star of the edge incident to the vertex x_{i-1} where the coloring was already defined.

Choose a bijection $\overline{c_{x_1}} : E\mathrm{St}(\varepsilon, \mathrm{Lk}(x_1, Y)) \to \{1, \ldots 5\}$ such that

$$\left(\overline{c_{x_1}} \circ H|_{\mathrm{St}(x_1,Y)} \circ (\overline{c_{x_1}})^{-1}\right)|_{\mathrm{Lk}(x_1,Y)} = F.$$

and therefore $\overline{c_{x_1}}$ defines an edge-star coloring of $\mathrm{Lk}(x_1, Y)$, making use of Lemma 6.2.11. Let $\varepsilon' \in E\mathrm{St}(\varepsilon, \mathrm{Lk}(x_1, Y))$ such that $\overline{c_{x_1}}(\varepsilon') = c(\varepsilon)(= c(P))$. Let ε_1 and ε_2 be the remaining edges in $\mathrm{St}(\xi_1, \mathrm{Lk}(v, Y))$ corresponding to polygons P_1 and P_2, in $\mathrm{St}(x_0, Y)$.

Let $\varepsilon_1', \varepsilon_2' \in \mathrm{St}(\varepsilon, \mathrm{Lk}(x_1, Y))$ be such that

$$(c(P_1), c(P_2)) = (c_{x_0}(P_1), c_{x_0}(P_2)) = (\overline{c_{x_1}}(\varepsilon_1'), \overline{c_{x_1}}(\varepsilon_2')).$$

Since F is 3-transitive, we know that there is $g \in F$ such that $g(\overline{c_{x_1}}(\varepsilon), \overline{c_{x_1}}(\varepsilon_1), \overline{c_{x_1}}(\varepsilon_2) = (\overline{c_{x_1}}(\varepsilon'), \overline{c_{x_1}}(\varepsilon_1'), \overline{c_{x_1}}(\varepsilon_2'))$.

Define $c_{x_1} = g \circ \overline{c_{x_1}}$, which is also an edge-star coloring of Γ. By construction, we have

$$\left(c_{x_1} \circ H|_{\mathrm{St}(x_1,Y)} \circ c_{x_1}^{-1}\right)|_{\mathrm{Lk}(x_1,Y)} = F.$$

and the colors of the polygons in $\mathrm{St}(x_0, Y) \cap \mathrm{St}(x_1, Y)$ coincide in c_{x_0} and c_{x_1}. Thus, for $P \in \mathrm{St}(x_1, Y)$ we define $c(P) = c_{x_1}(\mu)$, where μ is the edge in $\mathrm{Lk}(x_1, Y)$ corresponding to P.

Choose these bijections (and therefore we define edge-star colorings of $\mathrm{Lk}(x_i, Y)$) until $m-2$. Now we have to make sure to define the bijection $c_{x_{m-1}}$ in a consistent way, that is,

$$c_{x_{m-1}} : E\mathrm{St}(\varepsilon, \mathrm{Lk}(x_{m-1}, Y)) \to \{1, \ldots 5\}$$

is defined so that the following conditions hold

1) $c_{x_{m-1}}(\beta) = c_{x_{m-1}}(\tau_{e_{m-1}^{-1}}(\beta))$ for all $\beta \in \mathrm{St}(\xi_{m-2}, \mathrm{Lk}(x_{m-1}, Y))$,
2) $c_{x_{m-1}}(\alpha) = c_{x_m}(\tau_{e_m}(\alpha))$ for all $\alpha \in \mathrm{St}(\xi_m, \mathrm{Lk}(x_{m-1}, Y))$,

where τ_{e_i} is the transfer map as in Definition 6.1.16. Observe that $c_{x_{m-1}}$ is completely determined by the edge-star colorings c_{x_m-2} and c_{x_m} and it is an edge-star coloring of $\mathrm{Lk}(x_{m-1}, Y)$. Moreover, the induced action of H on $\mathrm{St}(x_{m-1}, Y)$ is isomorphic to F because c_{m-2} and c_m were constructed like that.

Performing a similar argument for each of the boundary paths $\gamma = (x_0, x_1, \ldots, x_{m-2}, x_{m-1}, x_m)$ in $\partial Y(v_0, n)$ we define a legal coloring on $Y(v_0, n+1)$ and by construction we have that H is a subgroup of the universal group $U(F)$ with this legal coloring. □

Remark 6.4.12. We remark that the conditions on the group F imply that either $F = \mathrm{Sym}(5)$ or $F = \mathrm{Alt5}$.

6.5 Open questions

In this section we present a couple of open questions that, at the time of writing of this thesis, we did not have chance to consider or to give satisfactory answers. This new class of groups is very promising and after an initial look done in this chapter, there are many more questions than answers. Some of these questions are stated here with some informal comments about a possible way to solve them.

In which other instances are the groups $U(F)$ universal?

In Proposition 6.4.11 we proved, if we consider the links of our polygonal complexes to be isomorphic to the Petersen graph, that we get obtain groups $U(F)$ that are universal, by assuming some conditions in the local action F. Namely, if F is 3-transitive then $U(F)$ is universal on its action on the CAT(0) (Γ, k)-complex Y. That is, if H is any closed face-transitive subgroup of $\mathrm{Aut}(Y)$ whose local action in the links of vertices is isomorphic to F, then H embeds in $U(F)$. We can consider F to be $\mathrm{Sym}(5)$ or Alt5 which are both 3-transitive groups.

The first immediate question that we could not solve at the time of the end of this thesis, is under which conditions on the local actions can we also obtain universality by considering the links of vertices to be isomorphic to a finite regular connected cover Γ of the Petersen graph for which all automorphisms lift.

Another interesting question is then whether in the same setting, that is, if the links of vertices of a (3,6)-complex are all isomorphic to Γ as above, one can get other conditions on the local action, weaker or different than the ones assumed now for the Petersen graph for instance, and still obtain groups with the universal property.

Can we get conditions on the local action F so that $U(F)$ is generated by face stabilizers?

Or, more generally, can we describe a nice set of generators of $U(F)$ prescribing some conditions on F?

The CAT(0) polygonal complexes that we are considering to define the universal group are not buildings. However, after writing this thesis and doing these 4 years of research, it is almost impossible to take buildings from the mind-set or to avoid trying some analogy with the buildings case.

If one considers Bourdon's buildings $I_{p,q}$, with $p \geq 6$ even and $q \geq 3$ then they are right-angled buildings and also a $(3,6)$-complexes, with the links of vertices being isomorphic to the complete bipartite graph $K_{q,q}$. The chambers of $I_{p,q}$ are the polygons and, considering the universal group $U(I_{p,q})$ defined in Chapter 4, we know that $U(I_{p,q})$ is generated by chamber stabilizers if and only if the local actions used to define the universal group are transitive and generated by point stabilizers (see Proposition 4.6.2). So, from a geometric point of view, the chamber stabilizers correspond to stabilizers of polygons.

That brings the question, can we adapt this for our setting? Can we generate $U(F)$ through stabilizers of k-gons prescribing some conditions on F? Or, if polygons wouldn't do the job, can we get any other set of nice enough generators for the group $U(F)$?

And since we are in the mood of questioning, could we describe those group generators in some nice way (as an analogy of what we did in Chapter 5)?

Simplicity

One of the interesting features in defining the universal group in the setting of right-angled buildings is that one gets examples of simple totally disconnected locally compact groups. Therefore we provide new building blocks for the understanding of this class of topological groups.

In the case of buildings, we are equipped with all the machinery coming from the geometry of the building and from the combinatorial properties of the associated Coxeter groups. We study and developed some of those properties in the right-angled case in Chapter 2 and we considered a distance between tree-walls in the building, constructing a tree out of the partition of the chambers into wings.

So the first question to be asked is whether we can consider convex subcomplexes that separate a CAT(0) (Γ, k)-complex. In the case of CAT(0) square complexes, one has the notion of hyperplane that separates the complex in half-spaces (see for instance Lecture 1 by Michah Sageev in [BSV14] for a detailed description of these notions).

We can consider the square subdivision X of any of our CAT(0) (Γ, k)-complexes where each k-gon is subdivided in k squares, each square having one vertex at the center of the k-gon and its opposite at the vertex of the k-gon, as illustrated in Figure 6.9 for an hexagon. Then X has the structure of a cube complex. Would hyperplanes play

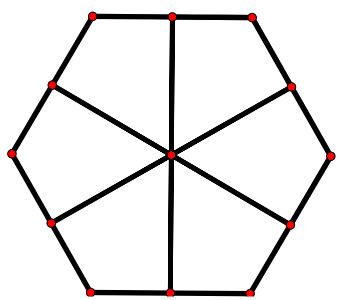


Figure 6.9: Square subdivision of a (Γ, k)-complex

the role of walls in the polygonal complex and the half-spaces replace the wings? Moreover, can we describe the fixator of a hyperplane in the group $U(F)$?

Then, to move towards simplicity, we would like to describe some properties of normal subgroups of $U(F)$. For instance whether is true that a normal subgroup of $U(F)$ contains a hyperbolic isometry of the polygonal complex. Or, using another approach, relate normal subgroups with hyperplanes.

Giving satisfactory answers to the questions above would be almost a finish line to prove simplicity of these groups under some additional conditions on the local actions. Of course taking into account that when one runs a marathon, almost a finish line might mean that we still have to run more than 10 kilometers.

What happens if Γ is a general odd graph?

This question regards the curiosity of knowing how much of this chapter goes through if we consider the links of the vertices of our (Γ, k)-complexes to be isomorphic to a general odd graph of degree bigger than 3 (the Petersen graph is the case of degree 3) or to one of its finite regular covers for which all the automorphisms lift. In this setting we still get polygonal complexes which are unique up to isomorphism because these graphs are vertex-star and edge-star-transitive.

The main motivation to consider CAT(0) (Γ, k)-complexes where Γ is a finite cover of the Petersen graph is the fact that edge-star colorings of these graphs are unique up to automorphism. Therefore one can prove that legal colorings on the polygons of the complex are also unique and define the universal group independently of the coloring.

The initial point would be to define edge-star colorings in odd graphs. The first natural choice is to define those colorings in an analogous way as in Example 6.4, which can easily be generalized to any odd graph, regarding the vertices of the odd graph Γ_d, for $d \geq 3$, as $(d-1)$-subsets of $\{1, \ldots, 2d-1\}$. Call those colorings *standard edge-star colorings* (those are used to study strong chromatic indexes in [WZ14]). Would one obtain that any other edge-star coloring of an odd graph is, up to automorphism, a standard edge-star coloring?

Moreover, as in the case of the Petersen graph, would these colors be determined (or almost determined) by making a choice in the star of an edge? If that would be the case, would then the work carried out in this chapter be generalized for odd graphs Γ_d by considering the local action to be a subgroup of $\mathrm{Aut}(\Gamma_d) = \mathrm{Sym}(2d-1)$?

We believe the the above questions have an affirmative answer at the time of writing this thesis.

How about considering Γ to be not even odd?

A natural question to ask is how can we generalize these groups to other/any CAT(0) (Γ, k)-complexes? Or, in other words, is there a general construction of universal groups for CAT(0) polygonal complexes whose links are all isomorphic? By this we mean that we would like to construct these groups not relying so heavily in the geometry of the Peterson graph, or of odd graphs if the previous question has a satisfatory answer.

We are interested to consider geometric objects that are unique up to automorphism. Moreover, it is handy to consider the CAT(0) polygonal complexes in [BB94] so that we have a constructive way of considering these complexes. Observe these complexes can also be (4,4)-complexes and (6,3)-complexes and not only the (3,6)-complexes that we described in this chapter. Therefore, using Lemma 6.2.7, we would start by considering CAT(0) (Γ, k)-complexes with $k \in \mathbb{N}$ greater than or equal to 4 and even, and Γ vertex-star- and edge-star-transitive.

Even if one does not get uniqueness of an edge-star coloring of the links, considering a fixed edge-star coloring of Γ as part of the data, maybe one can still define interesting groups. A good first example to start would be to consider CAT(0) (Γ, k)-complexes with Γ the complete bipartite graph $K_{m,m}$. That, in particular, would include a rather large class of Bourdon's buildings in the study.

APPENDIX A

Nederlandstalige samenvatting

In deze doctoraatsthesis onderzoeken we groepen die werken op rechthoekige gebouwen, en veralgemenen we het werk dat Burger en Mozes hebben verricht voor reguliere bomen [BM00a]. Wanneer we deze groepen uitrusten met de permutatietopologie (zie Sectie 1.3.2), behoren deze tot de klasse van de totaal onsamenhangende lokaal compacte groepen.

In deze Nederlandstalige samenvatting zullen we de context aangeven waarin het werk van deze thesis kan gesitueerd worden, en we presenteren de voornaamste resultaten van ons onderzoek.

Totaal onsamenhangende lokaal compacte groepen

De studie van lokaal compacte groepen kan op natuurlijke wijze worden opgesplitst in het samenhangende en het totaal onsamenhangende geval. De reden hiervoor is dat de samenhangscomponent van de identiteit van een lokaal compacte groep G een gesloten normaaldeler G_0 van G is, waarbij G/G_0 dan totaal onsamenhangend is.

Het samenhangende geval heeft een bevredigend antwoord omwille van de oplossing van het vijfde probleem van Hilbert.

Stelling (Gleason [Gle52], Montgomery and Zippin [MZ52], Yamabe [Yam53b], [Yam53a])**.** *Zij G een samenhangende lokaal compacte groep en zij $\mathcal{O}$ een omgeving van de identiteit. Dan is er een compacte normaaldeler $K \trianglelefteq G$ met $K \subseteq \mathcal{O}$ zodanig dat G/K een Lie groep is.*

Dit betekent ruwweg dat een samenhangende lokaal compacte groep kan benaderd worden door Lie groepen. Deze Lie groepen op hun beurt worden afzonderlijk bestudeerd in het oplosbare geval en in het enkelvoudige geval. De enkelvoudige Lie groepen zijn geklasseerd, eerst door werk van Killing [Kil88, Kil89], en vervolgens vervolledigd door Cartan [Car84].

Omwille hiervan vindt een groot deel van het huidige onderzoek in de lokaal compacte groepen plaats in het gebied van de totaal onsamenhangende lokaal compacte groepen. Gedurende een lange tijd was het enige gekende algemene structurele resultaat de stelling van van Dantzig uit 1936:

Stelling (van Dantzig [VD36])**.** *Elke totaal onsamenhangende lokaal compacte groep bevat een compacte open deelgroep.*

Pas een hele tijd later initieerde George Willis in [Wil94] de studie van de totaal onsamenhangende lokaal compacte groepen, ondermeer door het invoeren van het concept van de schaalfunctie; dit werd dan ondermeer verdergezet in [CRW13] en [CRW14].

Er zijn verscheidene stellingen die een verband leggen tussen de globale structuur van totaal onsamenhangende lokaal compacte groepen en die van hun compacte open deelgroepen; zie bijvoorbeeld [BEW11] en [Wil07]. Het idee om eigenschappen over de globale structuur af te leiden uit lokale eigenschappen wordt vaak omschreven als “lokaal-naar-globaal argumenten”; een belangrijk werk in dit opzicht was de studie van (specifieke) automorfismengroepen van bomen in het werk van Marc Burger en Shahar Mozes [BM00a].

Universele groepen voor reguliere bomen

Het startpunt en de motivatie voor deze thesis was het bovenvermelde baanbrekende werk van Burger en Mozes [BM00a] in verband met groepen die werken op bomen, met een vooropgegeven lokale actie; zie Secti 1.5 voor een precieze beschrijving van deze groepen.

Meer bepaald introduceerden ze in dit werk de universele groepen $U(F)$ als automorfismengroepen van een lokaal eindige boom, waarbij de lokale actie rond elke top van de boom wordt vooropgegeven door een eindige permutatiegroep F.

Universele groepen vormen een grote klasse van deelgroepen van de volledige automorfismengroep $\mathrm{Aut}(T)$ van een lokaal eindige boom T. De reden hiervoor is het feit dat elke gesloten top-transitieve deelgroep van $\mathrm{Aut}(T)$ waarvan de lokale actie rond elke top precies gelijk is aan F, kan ingebed worden in de corresponderende universele groep $U(F)$. Bovendien leveren deze universele groepen voorbeelden van compact voortgebrachte totaal onsamenhangende lokaal compacte groepen, en onder bepaalde milde restricties op F zijn ze niet-discreet. Ze voldoen aan de onafhankelijkheidseigenschap van Tits, en in het bijzonder hebben ze, opnieuw onder bepaalde milde condities op F, een enkelvoudige deelgroep van index 2.

Universele groepen zijn ook van fundamenteel belang in de studie van roosters in de automorfismengroep van het product van twee bomen, en bleken van cruciaal belang in het bewijzen van de normaaldelerstelling hiervoor, een resultaat dat analoog is aan de bekende normaaldelerstelling van Margulis voor halfenkelvoudige Lie groepen; zie [BM00b].

Verscheidene lokaal-naar-globaal resultaten voor deze universele groepen werden bewezen, bijvoorbeeld in het geval waarin de lokale actie tweevoudig transitief is [BM00a], of primitief is [CD11] (zie Sectie 1.5); dit soort resultaten bevestigen de schoonheid van de universele groepen.

Rechthoekige gebouwen

Gebouwen werden ingevoerd door Tits [Tit74] als een methode om halfenkelvoudige Lie groepen te begrijpen als automorfismengroepen van meetkundige structuren. Hierbij definieerde hij gebouwen in eerste instantie als simpliciale complexen die uitgerust zijn met welbepaalde deelcomplexen, appartementen genoemd, en waar de groep op werkt met een hoge vorm van regulariteit. Door de appartementen samen te lijmen volgens bepaalde axioma's bekwam Tits dan de definitie van een gebouw.

Enkele jaren later herschreef hij de definitie van een gebouw als een zogenaamd kamersysteem [Tit81], en het is deze beschrijving die we in deze thesis zullen gebruiken. Hoewel dit misschien niet onmiddellijk duidelijk is, zijn deze twee definities van gebouwen equivalent, zoals bijvoorbeeld wordt toegelicht in [AB08].

Ondanks deze oorspronkelijke motivatie van Tits, die ondermeer heeft geleid tot de classificatie van de sferische gebouwen [Tit74], zijn deze meetkundige objecten hun eigen leven gaan leiden, en hebben aanleiding gegeven tot talrijke onderzoeksaspecten. De classificatie van affiene gebouwen ([Tit86]), de studie van Moufang veelhoeken [TW02] en Moufang verzamelingen (zie bijvoorbeeld [DMW06]) die Moufang gebouwen zijn van respectievelijk rang 2 en rang 1, of the studie van Kac-Moody groepen die werken op tweelinggebouwen [Tit92] zijn slechts enkele voorbeelden uit een lange lijst.

Er is nog een andere wijze om naar gebouwen te kijken waar we af en toe gebruik van maken in deze thesis, met name als metrische ruimten. Dit gebeurt door de zogenaamde metrische realisatie van het gebouw te beschouwen; zie Sectie 1.4.4 voor meer details. In het sferische en in het Euclidische geval is dit vrij duidelijk, omdat appartementen van dergelijke gebouwen kunnen gezien worden als reflectiegroepen van een sfeer (in het sferische geval) en als tesselaties van de Euclidische ruimte (in het Euclidische geval). Moussong heeft in zijn thesis [Mou88a] een algemene constructie beschreven die een dergelijke constructie mogelijk maakt voor elk gebouw. Deze constructie werd mede ontwikkeld door Davis, die ondermeer een expliciet bewijs heeft gegeven dat deze metrische ruimte een zogenaamde CAT(0)-ruimte is (zie [Dav98]); daarom wordt deze metrische ruimte meestal de Davis realisatie van het gebouw genoemd.

De hoofdrolspelers in deze thesis zijn de rechthoekige gebouwen. De eenvoudigste voorbeelden hiervan zijn bomen, maar ook sommige hyperbolische gebouwen, zoals de Bourdon gebouwen (zie de definitie in Example 2.2.5) en sommige Euclidische gebouwen, zoals het direct product van bomen, behoren tot deze klasse.

Er zijn verschillende onderzoeksrichtingen in de wereld van de rechthoekige gebouwen. Zo heeft Anne Thomas een theorie ontwikkeld van roosters in rechthoekige gebouwen [TW11, Tho06], hebben Dymara, Osajda [DO07] en Clais [Cla16] de rand van dergelijke gebouwen bestudeerd, en is er een constructie van Rémi en Ronan [RR06] van tweelinggebouwen van rechthoekig Coxeter type waar Kac-Moody groepen op werken.

De klasse van semireguliere rechthoekige gebouwen, d.w.z. gebouwen waarvan alle panelen van een gegeven type even groot zijn, zullen voor ons een centrale rol spelen, aangezien dit precies de klasse zal zijn waarvoor we de ideeën van Burger en Mozes zullen veralgemenen. Haglund and Paulin [HP03] hebben aangetoond dat voor een gegeven Coxeter groep W en een verzameling parameters Q er op isomorfisme na een uniek rechthoekig gebouw bestaat van type W zodat de panelen van elk type precies de kardinaliteit hebben die gegeven is door de respectieve parameter in Q (zie Theorem 2.3.2 voor een precieze formulering). Bovendien heeft Caprace in [Cap14] bewezen dat indien het gebouw dik is en W irreducibel is, de volledige automorfismengroep van een dergelijk gebouw een enkelvoudige groep is.

Belangrijkste resultaten en methodologie

Na een uitgebreide studie van rechthoekige gebouwen in Hoofdstuk 2 bestuderen we open deelgroepen van de volledige automorfismengroep van een rechthoekig gebouw in Hoofdstuk 3. Het was eerder geweten dat elke open deelgroep van de automorfismengroep van een boom compact is (*cf.* [CD11, Theorem A]). We veralgemenen dit resultaat naar rechthoekige gebouwen, en we verkrijgen in dit geval het volgend resultaat.

Stelling 1. *Zij Δ een lokaal eindig semiregulier dik rechthoekig gebouw, en zij $G = \mathrm{Aut}(\Delta)$. Dan is elke echte open deelgroep van G bevat in de stabilisator in G van een echt residu van Δ.*

Dit resultaat is te vinden in de thesis als Theorem 3.4.19. Het bewijs ervan maakt gebruik van groepen die gelijken op wortelgroepen, namelijk de wortelvleugelgroepen die we invoeren in Sectie 3.3, waardoor het, verrassend genoeg, mogelijk is om een strategie toe te passen uit [CM13] voor Kac-Moody groepen die werken op tweelinggebouwen.

Nog in Hoofdstuk 3 tonen we aan dat de fixator van een bal in de automorfismengroep op het gebouw werkt met een begrensde fixpuntverzameling (Proposition 3.2.6). In het bijzonder impliceert dit dat een open deelgroep van de automorfismengroep van een dik semiregulier rechthoekig gebouw compact is als en slechts als hij lokaal elliptisch werkt op de Davis realisatie van het gebouw (Corollary 3.4.3).

In Hoofdstuk 4 definiëren we de universele groep voor een semiregulier rechthoekig gebouw, en na eerst enkele basiseigenschappen te bewijzen in Sectie 4.2 voeren we een gedetailleerde studie uit van deze groepen, om uiteindelijk te komen tot volgende stelling.

Stelling 2. *Een universele groep voor een dik semiregulier rechthoekig gebouw is enkelvoudig als en slechts als elk van de lokale acties is vooropgegeven door eindige groepen die transitief zijn en voorgebracht worden door hun puntstabilisatoren.*

Het bewijs van deze stelling, die terug te vinden is in de thesis als Stelling 4.6.7, steunt op de ontwikkeling van een aantal nieuwe concepten. Het eerste hiervan is een veralgemening (of beter gezegd, een aanpassing) van de onafhankelijkheidseigenschap van Tits in de setting van rechthoekige gebouwen. Deze eigenschap werd bewezen in [Cap14] voor de volledige automorfismengroep, en bewijzen we hier voor de universele groepen in Proposition 4.4.1.

Het tweede nieuwe concept is dat van een boommuur-boom gedefinieerd in Sectie 2.2.4, en we onderzoeken de actie van de universele groepen op deze bomen in Sectie 4.5.

In Hoofdstuk 5 bestuderen we de structuur van de compacte open deelgroepen van de universele groepen van lokaal eindige dikke rechthoekige gebouwen; deze universele groepen zijn compact voortgebrachte totaal onsamenhangende lokaal compacte groepen. De maximale compacte open deelgroepen zijn precies de stabilisatoren van sferische residus, zoals we aantonen in Proposition 5.1.2, en de kamerstabilisatoren zijn deelgroepen hiervan van eindige index. Deze groepen zijn pro-eindig, en in Hoofdstuk 5 beschrijven we elk van de eindige groepen die in deze projectieve limiet optreedt vanuit verschillende standpunten. (Deze eindige groepen zijn precies de groepen geïnduceerd door de actie van een kamerstabilisator op de eindige ballen rond die kamer.)

De eerste beschrijving van deze groepen (Theorem 5.2.7 in de thesis) is via een iteratief proces, gelijkaardig aan wat Burger en Mozes deden voor bomen in [BM00a, Section 3.2]. In Sectie 5.3 beschrijven we de geïnduceerde actie op de n-sferen op een directere manier:

Stelling 3. *De geïnduceerde actie van de kamerstabilisatoren in de universele groep op de w-sferen is permutatie-isomorf met veralgemeende kransproducten die beschreven kunnen worden gebruik makend van de eindige groepen die de lokale actie vooropgeven.*

Een preciezere versie van deze stelling is terug te vinden in Hoofdstuk 5 als Proposition 5.3.3. Het bewijs van dit resultaat beschouwt een "gerichte parametrisatie" van de kamers van een rechthoekig gebouw. We ontwikkelen dit concept in Sectie 2.4.

De veralgemeende kransproducten, die we bespreken in Sectie 1.1.2, hangen af van een zekere partiële orderelatie. We definiëren een dergelijke partiële orderelatie op de verzameling van gereduceerde woorden van de geassocieerde Coxeter groep (zie Definition 2.1.9). We stellen vast dat dit verband houdt met de complete kransproducten, en we onderzoeken dit verder in Sectie 5.3.2, waar we volgend resultaat bewijzen.

Stelling 4. *De doorsnede van geïtereerde complete kransproducten in hun imprimitieve actie, overeenkomend met de verschillende gereduceerde representaties van een element w van een rechthoekige Coxeter groep, is permutatie-isomorf met een veralgemeend kransproduct verkregen uit de partiële orderelatie op de letters van w.*

Dit resultaat is terug te vinden in de thesis als Proposition 5.3.8, en het mooie eraan is dat de beschrijving van de geïnduceerde actie van een kamerstabilisator op het gebouw enkel afhangt van het Coxeter diagram van het gebouw.

Een open hoofdstuk

In Hoofdstuk 6, dat gebaseerd is op samenwerking met Anne Thomas, zetten we het idee van universele groepen over naar een klasse van polygonale complexen.

We introduceren het concept van een universele groep voor (Γ, k)-complexen (zie Definition 6.1.6). De automorfismengroepen van deze complexen zijn opnieuw totaal onsamenhangende lokaal compacte groepen. De Bourdon gebouwen, die we reeds hebben vermeld als voorbeelden van rechthoekige gebouwen, vormen ook voorbeelden van (Γ, k)-complexen, maar het is een van de weinige voorbeelden in de doorsnede van deze twee klassen van meetkundige objecten.

In [BB94] hebben Ballmann en Brin een proces beschreven om CAT(0) (Γ, k)-complexen op inductieve wijze te construeren, en ze toonden aan dat elk (Γ, k)-complex op die wijze kan geconstrueerd worden. Onder zekere graaftheoretische condities op de link van deze complexen (zie Theorem 6.2.6) heeft Nir Lazarovich bewezen dat (Γ, k)-complexen uniek zijn op isomorfisme na [Laz14].

In deze thesis focussen we ons op het geval waarin de links van de toppen van het complex isomorf zijn met een eindige overdekking van het Petersen graaf waarvoor alle automorfismen liften; in dit geval is het eerder vermelde uniciteitsresultaat van toepassing.

We definiëren legale kleuringen op de veelhoeken van een dergelijk (Γ, k)-complex, en we bewijzen dat deze legale kleuringen uniek zijn op isomorfisme na (Proposition 6.3.2). In Sectie 6.4 definiëren we dan de universele groep $U(F)$ voor een (Γ, k)-complex met vooropgegeven lokale actie F als deelgroep van het Petersen graaf. In Sectie 6.4.2 bewijzen we enkele basiseigenschappen van deze groepen, en we beschrijven lokaal-naar-globaal resultaten voor de universele groep van een (Γ, k)-complex. Onder zekere restricties op de lokale actie F kunnen we bewijzen dat deze groepen inderaad universeel zijn (Proposition 6.4.11).

Op het moment van het voltooien van dit doctoraatswerk zijn er nog tal van open vragen in verband met deze groepen, en in feite zijn er meer vragen dan antwoorden. Sommige van deze vragen worden in de laatste sectie van deze thesis vermeld, voorzien van enige commentaar.

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