

Slice Segal-Bargmann transform

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Abstract

The Segal-Bargmann transform is a unitary map between the Schrödinger and the Fock space, which is e.g. used to show the integrability of quantum Rabi models. Slice monogenic functions provide the framework in which functional calculus for quaternionic quantum mechanics can be developed. In this paper, a generalisation of the Segal-Bargmann transform to the context of slice monogenic functions is constructed and studied in detail. It is shown to interact appropriately with the recently constructed slice Fourier transform. This leads furthermore to a construction of a slice Fock space, which is shown to be a reproducing kernel space.

Keywords: Clifford-Hermite functions, Segal-Bargmann transform, Fock space, Fourier transform, slice monogenic functions

1 Introduction

In quantum mechanics, the Segal-Bargmann transform [4, 5, 23] is a well-known unitary map between the Schrödinger space and the Fock space. The Schrödinger space (the space of wavefunctions) is denoted mathematically as the space $L^2(\mathbb{R})$ of square integrable functions. The Fock space, see e.g. [25], is a suitable closure of the space of holomorphic polynomials (where one can identify the monomial z^k with the ket $|k\rangle$). Recently the Segal-Bargmann transform gained renewed interest as it allowed for showing the integrability of quantum Rabi models, see the celebrated paper [6] as well as [22, 24]. It is moreover interesting to note that the Segal-Bargmann transform interacts nicely with the Fourier transform, see [15] or the subsequent Proposition 4.

In recent years, there has equally been a lot of interest in the theory of slice monogenic functions ([16, 12, 13]). The main object of study in this theory is a new class of functions, called slice monogenic or hyperholomorphic functions, which are Clifford algebra valued null-solutions of a generalised Cauchy-Riemann [9] or Dirac [8] operator. They are especially promising for the study of quaternionic quantum mechanics (see the book [1]). Indeed, a crucial problem in the study of quaternionic versions of quantum mechanics is to find a suitable definition for the spectrum of an operator on a quaternionic Hilbert space and to establish a mathematically sound framework of functional calculus. This was recently solved in an extended series of papers on functional calculus for slice functions (without claiming completeness, we refer to e.g. [10, 11, 13, 17, 3]).

Although several papers have recently dealt with Clifford algebra valued extensions of the Segal-Bargmann transform or of Fock spaces, see [19, 20, 21, 2, 14], such extensions have not yet been considered in full generality in the context of slice monogenic functions. Given the fact that a suitable slice Fourier transform was constructed in [7], a natural question is whether it has an accompanying Segal-Bargmann transform.

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The aim of our paper is therefore to introduce and study a slice Segal-Bargmann transform. It yields a generalisation of the Segal-Bargmann transform to the context of the slice Dirac operator of [8], that interacts appropriately with the slice Fourier transform of [7]. It turns out that this is possible because of the appearance of the $\mathfrak{osp}(1|2)$ Lie superalgebra, realised by the already mentioned slice Dirac operator. We expect that the slice Segal-Bargmann transform might play a role to study the integrability of quaternionic versions of certain quantum systems such as Rabi models.

The paper is organised as follows. Section 2 of this paper lists some preliminary results on the classical Segal-Bargmann transform and Fock space, as well as on the theory of slice monogenic functions in order to make this paper self-contained. Section 3 shows how these results can be used to construct a slice Fock space. Based on this reasoning, the slice Segal-Bargmann transform is defined in Section 4. Also its explicit expression is used to study the properties of the basis elements of the slice Fock space. In Section 5 this space is endowed with an inner product and an orthonormal basis is constructed. Section 6 further examines the slice Segal-Bargmann transform by constructing its inverse and studying its action on the slice Fourier transform. Finally, it is shown that the slice Fock space is a reproducing kernel space.

2 Preliminaries

This preliminary section contains some general background on the classical Segal-Bargmann transform and the slice approach in \mathbb{R}^{m+1} . Also it briefly summarises the definitions and properties of the Clifford-Hermite functions as defined and proved in [8].

2.1 Classical Segal-Bargmann transform and Fock space

This section gives an introduction to both the classical Segal-Bargmann transform and the corresponding Fock space (see [4, 5, 23, 25]).

Definition 1. *The Fock space F is the Hilbert space of entire functions f on \mathbb{C} for which*

$$\int_{\mathbb{C}} \overline{f(z)} f(z) \exp(-\bar{z}z) dz < +\infty,$$

where $dz = dx dy$ with $z = x + iy$. It is endowed with the inner product

$$\langle f, g \rangle_F = \int_{\mathbb{C}} \overline{f(z)} g(z) \exp(-\bar{z}z) dz \quad \forall f, g \in F.$$

Proposition 1. *An orthonormal basis for the Fock space is given by the monomials*

$$\left\{ e_m(z) = \frac{z^m}{\sqrt{m!}} \right\}.$$

Moreover we have the following proposition

Proposition 2. *The Fock space F is a reproducing kernel space of which the reproducing kernel is given by $\mathcal{K}^F = \exp(z\bar{w})$.*

Now the classical Fock space has been introduced, we can address the Segal-Bargmann transform, which will be shown to map the $L^2(\mathbb{R})$ -space onto it.

Definition 2. *The Segal-Bargmann transform $B : L^2(\mathbb{R}) \rightarrow F$ of a function f is given by*

$$B[f](z) = \frac{1}{\sqrt[4]{\pi}} \int_{\mathbb{R}} f(x) \exp\left(-\frac{z^2 + x^2 - 2\sqrt{2}zx}{2}\right) dx.$$

The Segal-Bargmann transform has the following properties:

Proposition 3. *One has*

$$\begin{aligned} \left(z + \frac{d}{dz}\right) B[f](z) &= \sqrt{2} B[xf(x)](z) \\ \sqrt{2}z B[f](z) &= B\left[\left(x - \frac{d}{dx}\right) f(x)\right](z) \end{aligned}$$

Corollary 1. *The Segal-Bargmann transform B maps the dimensionless raising and lowering operators*

$$a^\dagger = \left(x - \frac{d}{dx}\right) \quad \text{and} \quad a = \left(x + \frac{d}{dx}\right)$$

on $L^2(\mathbb{R})$ onto the respective raising and lowering operators

$$b^\dagger = \sqrt{2} z \quad \text{and} \quad b = \sqrt{2} \frac{d}{dz}$$

on the Fock space.

Therefore the Segal-Bargmann transform of the normalised Hermite functions

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{x^2}{2}\right)$$

is given by

$$B\left[\frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{x^2}{2}\right)\right](z) = \frac{z^n}{\sqrt{n! \sqrt{\pi}}} B\left[\exp\left(-\frac{x^2}{2}\right)\right] = \frac{z^n}{\sqrt{n!}}$$

because $B[\exp(-x^2/2)] = \pi^{1/4}$. These are exactly the normalised basis functions e_n of the Fock space. Given that the Segal-Bargmann transform maps the basis of $L^2(\mathbb{R})$ on the basis of F , it is a natural question to ask whether also an inverse transform exists.

Definition 3. *The inverse Segal-Bargmann transform $B^{-1} : F \rightarrow L^2(\mathbb{R})$ of a function g is given by*

$$B^{-1}[g](z) = \frac{1}{\pi \sqrt[4]{\pi}} \int_{\mathbb{C}} g(z) \exp\left(-\frac{\bar{z}^2 + x^2 - 2\sqrt{2}\bar{z}x}{2}\right) \exp(-\bar{z}z) dz,$$

where $dz = dx dy$ and $\bar{\cdot}$ denotes the complex conjugation.

We end this section on the Segal-Bargmann transform B by addressing its behaviour with respect to the Fourier transform \mathcal{F} .

Definition 4. *The Fourier transform \mathcal{F} of a function $f \in L^1(\mathbb{R})$ is defined as*

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x) dx.$$

It is well known that the Fourier transform can be extended to $L^2(\mathbb{R})$ and that it maps the Hermite functions ψ_n onto complex multiples of themselves, as to

$$\mathcal{F}\psi_n = (-i)^n \psi_n.$$

Performing the Segal-Bargmann transform on both sides of this equation, we obtain the following proposition.

Proposition 4. *The operator on the Fock space F corresponding to the Fourier transform on $L^2(\mathbb{R})$ is given by*

$$\mathcal{G} : F \rightarrow F : g(z) \mapsto g(-iz).$$

We thus have the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ B \downarrow & & \uparrow B^{-1} \\ F & \xrightarrow{\mathcal{G}} & F \end{array}$$

2.2 Slice approach in \mathbb{R}^{m+1}

The $(m + 1)$ -dimensional real Clifford algebra Cl_{m+1} has $m + 1$ basis vectors $e_i, i = 0, \dots, m$, which satisfy the anti-commutation relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 0, \dots, m.$$

A k -vector (where $k \leq m + 1$) is an element e_A of Cl_{m+1} such that $e_A = e_{i_1} \dots e_{i_k}$ where $i_j \in \{0, \dots, m\}$ for all $j \in \{1, \dots, k\}$ and with $i_1 < \dots < i_k$. The variable $\mathbf{x} \in Cl_{m+1}$ is defined as the 1-vector which corresponds to the $(m + 1)$ -tuple $(x_0, \dots, x_m) \in \mathbb{R}^{m+1}$ by

$$\mathbf{x} = x_0 e_0 + x_1 e_1 + \dots + x_m e_m.$$

Using spherical coordinates to describe the Cl_m -part \underline{x} of \mathbf{x} , one can also write

$$\begin{aligned} \mathbf{x} &= x_0 e_0 + \underline{x} \\ &= x_0 e_0 + r \underline{\omega}, \end{aligned}$$

where $r = \sqrt{x_1^2 + \dots + x_m^2}$ and $\underline{\omega} = \underline{x}/r$. A general element \mathbf{x} is thus defined by the triplet $(x_0, r, \underline{\omega}) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{m-1}$, where \mathbb{S}^{m-1} denotes the $(m - 1)$ -dimensional sphere in \mathbb{R}^m . The variable \mathbf{x} therefore lives in the subspace spanned by the fixed basis vector e_0 and the unit 1-vector $\underline{\omega}$. This subspace is called a slice, referring to the slice concept used in literature (see e.g. [12] and the book [13]).

As a consequence, a general function f of \mathbf{x} will depend on x_0, r and $\underline{\omega}$. Throughout this article such functions f will be written both as $f(\mathbf{x})$ and as $f(x_0, r, \underline{\omega})$ because the former is more compact and the latter shows its dependencies explicitly. Based on considerations in [9], the following definition was proposed in [8].

Definition 5. *The slice Dirac operator D_0 is the partial differential operator defined as*

$$D_0 = e_0 \partial_{x_0} + \underline{\omega} \partial_r.$$

Given that $-e_0 D_0$ corresponds to the slice Cauchy-Riemann operator, null-solutions of D_0 correspond to slice monogenic functions as studied in e.g. [12, 16, 18]. Together with the multiplication operator \mathbf{x} and the slice Dirac operator D_0 , the Euler operator $\mathbb{E} = \sum_{i=0}^m x_i \partial_{x_i}$ establishes a realisation of the $\mathfrak{osp}(1|2)$ -superalgebra, see [8].

Definition 6. *The (complex) Clifford conjugation $\bar{\cdot}$ is defined as*

$$\begin{aligned} \bar{\lambda} &= \lambda^* & \lambda &\in \mathbb{C} \\ \bar{e}_i &= -e_i & i &= 0, \dots, m. \\ \overline{ab} &= \bar{b} \bar{a} & a, b &\in Cl_{m+1}. \end{aligned}$$

where $*$ denotes the standard complex conjugation.

Definition 7. *The vector space \mathcal{L}^2 is defined as*

$$\begin{aligned} \mathcal{L}^2 &= L^2(\mathbb{R}^{m+1}, r^{1-m} d\mathbf{x}) \otimes Cl_{m+1} \\ &= \left\{ f : \mathbb{R}^{m+1} \rightarrow Cl_{m+1} \mid \left[\int_{\mathbb{R}^{m+1}} \overline{f(\mathbf{x})} f(\mathbf{x}) r^{1-m} d\mathbf{x} \right]_0 < +\infty \right\} \end{aligned}$$

where $[\cdot]_0$ denotes the scalar part of the expression between the brackets.

On \mathcal{L}^2 an inner product was defined.

Proposition 5. *The vector space \mathcal{L}^2 is given the structure of a right Hilbert module by defining the inner product of two functions $f, g : \mathbb{R}^{m+1} \rightarrow Cl_{m+1}$ as*

$$\langle f, g \rangle = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}} \int_{\mathbb{R}^{m+1}} \overline{f(\mathbf{x})} g(\mathbf{x}) r^{1-m} d\mathbf{x} = \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}} \int_{\mathbb{R}^{m+1}} \overline{f(\mathbf{x})} g(\mathbf{x}) dx_0 dr d\sigma_{\underline{x}}$$

where $d\sigma_{\underline{x}}$ denotes the measure on the unit sphere \mathbb{S}^{m-1} corresponding to the \underline{x} -part of \mathbf{x} . This inner product obeys the relations

$$\begin{aligned} \langle D_0 f, g \rangle &= \langle f, D_0 g \rangle, \\ \langle \mathbf{x} f, g \rangle &= -\langle f, \mathbf{x} g \rangle \end{aligned}$$

on a dense subset of \mathcal{L}^2 .

2.3 The Clifford-Hermite functions

Based on the classical definitions, the Clifford-Hermite polynomials and functions are defined using the kernel of the differential operator D_0 and the $\mathfrak{osp}(1|2)$ -relations.

In [8] it was shown that the polynomial kernel of D_0 is a right Cl_{m+1} -module which is spanned by the homogeneous polynomials $m_k(\mathbf{x}) = (e_0 - 1)(x_0 + \underline{x})^k$ of degree $k \in \mathbb{N}$.

Definition 8. *The Clifford-Hermite polynomials $h_{j,k}$ of degree j and order k are defined as*

$$h_{j,k}(\mathbf{x}) m_k(\mathbf{x}) = (\mathbf{x} - cD_0)^j m_k(\mathbf{x})$$

where $c \in \mathbb{R}_0^+$ and $j \in \mathbb{N}$.

The parameter c adds some freedom to the definition. In order not to overload notation, however, its presence will not be denoted explicitly. We briefly summarise the most important properties of these polynomials (see [8]).

Theorem 1. *The polynomials $H_j(m_k)(\mathbf{x}) = h_{j,k}(\mathbf{x}) m_k(\mathbf{x})$ are solutions of the differential equation*

$$cD_0^2 H_j(m_k)(\mathbf{x}) - \mathbf{x} D_0 H_j(m_k)(\mathbf{x}) + C(j, k) H_j(m_k)(\mathbf{x}) = 0$$

with $C(j, k) = -2t$ if $j = 2t$ and $C(j, k) = -2(k + t + 1)$ if $j = 2t + 1$.

Based on explicit expressions of the Clifford-Hermite polynomials $h_{j,k}$ in terms of Laguerre polynomials, Clifford-Hermite functions $\psi_{j,k}$ are defined and their main properties are summarised as well.

Definition 9. *The Clifford-Hermite functions $\psi_{j,k}$ are defined as*

$$\begin{aligned} \psi_{2t,k}(\mathbf{x}) &= (2c)^t t! L_t^k \left(\frac{|\mathbf{x}|^2}{2c} \right) m_k(\mathbf{x}) \exp(-|\mathbf{x}|^2/4c) \\ \psi_{2t+1,k}(\mathbf{x}) &= (2c)^t t! \mathbf{x} L_t^{k+1} \left(\frac{|\mathbf{x}|^2}{2c} \right) m_k(\mathbf{x}) \exp(-|\mathbf{x}|^2/4c) \end{aligned}$$

where $c \in \mathbb{R}_0^+$ is the same parameter as in Definition 8 and L_t^k are the generalised Laguerre polynomials of degree t and order k on the real line.

Proposition 6. *The Clifford-Hermite functions $\psi_{j,k}$ satisfy the relations*

$$\begin{cases} \psi_{j,k} &= \tilde{D}_c \psi_{j-1,k} \\ \tilde{D}_c^\dagger \psi_{j,k} &= -c C(j, k) \psi_{j-1,k} \end{cases} \quad (1)$$

with $\tilde{D}_c = \frac{\mathbf{x}}{2} - cD_0$, $C(j, k)$ as in Theorem 1 and where \cdot^\dagger denotes the adjoint with respect to the inner product of Proposition 5.

Theorem 2. *The Clifford-Hermite functions $\psi_{j,k}$ are solutions of the scalar differential equation*

$$\left(cD_0^2 + \frac{|\mathbf{x}|^2}{4c}\right) \psi_{j,k}(\mathbf{x}) = (j+k+1) \psi_{j,k}(\mathbf{x}). \quad (2)$$

Definition 10. *The set of right finite linear combinations of Clifford-Hermite functions over Cl_{m+1} will be denoted as \mathcal{V} , so $\mathcal{V} = \text{span}_{Cl}\{\psi_{j,k}\}$ and $\mathcal{V} \subset \mathcal{L}^2$.*

In [7] the slice Fourier transform is defined as follows:

Definition 11. *The slice Fourier transform of a function $f \in \mathcal{V}$ is given by*

$$\mathcal{F}_S(f)(\mathbf{y}) = \frac{-i\Gamma\left(\frac{m}{2}\right)}{8c\pi^{m/2+1}} \int_{\mathbb{R}^{m+1}} \left[(1 + \underline{\eta}\underline{\omega})e^{-\frac{i}{2c}(x_0y_0 - rg)} + (1 - \underline{\eta}\underline{\omega})e^{-\frac{i}{2c}(x_0y_0 + rg)}\right] f(\mathbf{x}) \, dx_0 dr d\sigma_{\underline{x}}$$

with $\mathbf{x} = x_0e_0 + r\underline{\omega}$ and $\mathbf{y} = y_0e_0 + g\underline{\eta}$.

3 Towards a slice Fock space

The classical one-dimensional Segal-Bargmann transform B maps square integrable functions $f \in L^2(\mathbb{R})$ into the classical Fock space $\mathcal{F}(\mathbb{C})$ (see [4, 5]). Requiring the slice Segal-Bargmann transform to exhibit analogous behaviour with respect to the Clifford-Hermite functions and a slice analogue of the Fock space, we should first obtain an appropriate basis for the latter. To do so, we will write the Clifford-Hermite functions in terms of raising operators and assume the slice Segal-Bargmann transform to treat these operators as in the classical case.

The purpose of this section is thus to express the Clifford-Hermite functions $\psi_{j,k}$ in terms of raising operators only. According to Proposition 6 the operator $\tilde{D}_c = \frac{\underline{x}}{2} - cD_0^{\underline{x}}$ is a raising operator for the first index j of the Clifford-Hermite functions $\psi_{j,k}$: its action raises the first index by one. Therefore, only two issues remain to be solved in order to build an appropriate basis for the slice Fock space:

- first, we lack a raising operator for k , the second index of $\psi_{j,k}$,
- second, the operator \tilde{D}_c depends on both x_0 and r whereas the behaviour of the transform as described above is one-dimensional.

The first problem asks for which operator $\tilde{D}_{c,k}$ the equality $\psi_{0,k+1} = \tilde{D}_{c,k}\psi_{0,k}$ with $k = 0, 1, 2, \dots$ is fulfilled and will be addressed in the section at hand. To solve the second problem, we will introduce an alternative basis $\{\phi_{n_1, n_2}\}$ for the \mathcal{L}^2 -space which will allow us to construct a basis for the slice Fock space in Section 5.

Lemma 1. *One has*

$$\psi_{0,k+1} = \tilde{D}_{c,k}\psi_{0,k}$$

with

$$\tilde{D}_{c,k} = -e_0\tilde{D}_c.$$

Consequently, $\tilde{D}_{c,k}$ does not depend on the integer k .

Proof. Given that

$$\psi_{0,k} = (e_0 - 1)(x_0 + r\underline{\omega})^k \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right)$$

the raising operator $\tilde{D}_{c,k}$ for the index k of $\psi_{0,k}$ has to be such that

$$\psi_{0,k+1} = \tilde{D}_{c,k}\psi_{0,k} = -\frac{1}{2}(e_0 - 1)(x_0 + r\underline{\omega})(e_0 + 1)\psi_{0,k} = -e_0\mathbf{x}\psi_{0,k}.$$

With respect to the raising operator for j we have

$$\begin{aligned}\tilde{D}_c \left[m_k(\mathbf{x}) \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right) \right] &= \left(\frac{\mathbf{x}}{2} \psi_{0,k}(\mathbf{x}) - cD_0^{\mathbf{x}} [m_k(\mathbf{x})] \exp \left(\frac{|\mathbf{x}|^2}{4c} \right) + \frac{\mathbf{x}}{2} \psi_{0,k}(\mathbf{x}) \right) \\ &= \mathbf{x} \psi_{0,k}(\mathbf{x})\end{aligned}$$

because the polynomials m_k span the kernel of D_0 . Hence $\tilde{D}_{c,k} = -e_0 \tilde{D}_c$, and the proof is complete. \square

Remark 1. *Some caution has to be taken when using the expression ‘raising operator with respect to k ’ because this operator only raises the second index when it is applied to $\psi_{0,k}$. When the first index is not zero, Proposition 6 yields $-e_0 \tilde{D}_c \psi_{j,k} = -e_0 \psi_{j+1,k}$.*

Corollary 2. *One has*

$$\psi_{j,k}(\mathbf{x}) = \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}} \right)^j \left[-e_0 \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}} \right) \right]^k (e_0 - 1) \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right).$$

Proof. The successive application of Proposition 6 and the above Lemma 1 prove this corollary. \square

Theorem 3. *The basis elements $\psi_{j,k}$ of the \mathcal{L}^2 -space can be written as specific, Clifford-valued combinations of the real-valued functions*

$$\phi_{n_1, n_2}(x_0, r) = \left[\left(\frac{x_0}{2} - c\partial_{x_0} \right)^{n_1} \left(\frac{r}{2} - c\partial_r \right)^{n_2} \right] \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right).$$

Proof. Based on Corollary 2 and the fact that

$$\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}} = e_0 \left(\frac{x_0}{2} - c\partial_{x_0} \right) + \underline{\omega} \left(\frac{r}{2} - c\partial_r \right)$$

the binomial theorem yields

$$\psi_{j,k}(x_0, r) = \sum_{\substack{n_1, n_2 \\ n_1 + n_2 = j+k}} \mathbf{a}_{n_1, n_2}(\underline{\omega}) \left[\left(\frac{x_0}{2} - c\partial_{x_0} \right)^{n_1} \left(\frac{r}{2} - c\partial_r \right)^{n_2} \right] \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right).$$

where the coefficients \mathbf{a}_{n_1, n_2} are specific, Clifford-valued combinations of $1, e_0, \underline{\omega}$ and $e_0 \underline{\omega}$. This proves the theorem. \square

4 Slice Segal-Bargmann transform

As was demonstrated in Theorem 3 of the previous section, the Clifford-Hermite functions can be expressed in terms of classical two-dimensional Hermite functions in the variables x_0 and r .

For the classical Hermite functions, the operators $\frac{x_0}{2} - c\partial_{x_0}$ and $\frac{r}{2} - c\partial_r$ in Theorem 3 act as raising operators for the indices n_1 and n_2 of ϕ_{n_1, n_2} , respectively. A logical requirement for the slice Segal-Bargmann transform SB would thus be to transform these operators to respective complex variables z_1 and z_2 , acting as multiplication operators on the slice Fock space.

This already suggests the following notation:

Definition 12. *In the complexified setting the following notations will be used:*

$$\mathbf{z} = z_1 e_0 + z_2 \underline{\zeta} = (x_1 e_0 + x_2 \underline{\zeta}) + i(y_1 e_0 + y_2 \underline{\zeta})$$

and

$$D_0^{\mathbf{z}} = e_0 \partial_{z_1} + \underline{\zeta} \partial_{z_2},$$

where $\underline{\zeta} \in \mathbb{S}^{m-1}$ and $\partial_{z_\ell}, \ell \in \{1, 2\}$ denotes the classical Cauchy-Riemann operator with respect to z_ℓ , so

$$\partial_{z_\ell} = \frac{1}{2} (\partial_{x_\ell} - i\partial_{y_\ell}).$$

Finally, $|\mathbf{z}|^2$ is a shorthand notation for $|z_1|^2 + |z_2|^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2$ so $|\mathbf{z}|^2 \in \mathbb{R}$. Mind that $|\mathbf{z}|^2 \neq \mathbf{z}\bar{\mathbf{z}} = |z_1|^2 + |z_2|^2 + (z_2\bar{z}_1 - z_1\bar{z}_2)e_0\underline{\zeta} \notin \mathbb{R}$.

Based on Corollary 2, Definition 12, and the above reasoning, we propose the following expression for the elements $\varphi_{j,k}$ spanning the slice Fock space.

Definition 13. The monomials $\varphi_{j,k}$ spanning the slice Fock space F^S are defined as

$$\begin{aligned} \varphi_{j,k}(\mathbf{z}) &= \mathbf{z}^j (-e_0\mathbf{z})^k (e_0 - 1) \\ &= (z_1e_0 + z_2\underline{\zeta})^j [-e_0(z_1e_0 + z_2\underline{\zeta})]^k (e_0 - 1) \end{aligned} \quad (3)$$

with $z_\ell = x_\ell + iy_\ell$, where $\ell \in \{1, 2\}$, x_ℓ and y_ℓ are real and i denotes the classical complex unit commuting with all basis elements $e_j, j \in \{0, \dots, m\}$.

In the classical case the exponential factor in the Hermite functions is mapped onto a constant function so this behaviour has been assumed here as well.

Corollary 3. One has

$$\begin{aligned} \varphi_{j+1,k}(\mathbf{z}) &= \mathbf{z}\varphi_{j,k}(\mathbf{z}) \\ \varphi_{0,k+1}(\mathbf{z}) &= -e_0\mathbf{z}\varphi_{0,k}(\mathbf{z}). \end{aligned}$$

Proof. This follows from the definition of $\varphi_{j,k}$. □

The operator \mathbf{z} thus acts as a raising operator with respect to the first index of $\varphi_{j,k}$.

Lemma 2. The differential operator $D_0^{\mathbf{z}} = e_0\partial_{z_1} + \underline{\zeta}\partial_{z_2}$ acts as a lowering operator with respect to the first index of the monomials $\varphi_{j,k}$ and one has

$$D_0^{\mathbf{z}}\varphi_{j,k}(\mathbf{z}) = C(j, k)\varphi_{j-1,k}$$

with $C(j, k)$ as in Theorem 1.

Proof. Using the definition of the monomials $\varphi_{j,k}$ for $j = 2t$, one obtains

$$\begin{aligned} D_0^{\mathbf{z}}\varphi_{2t,k}(\mathbf{z}) &= (e_0\partial_{z_1} + \underline{\zeta}\partial_{z_2}) \left[(-z_1^2 - z_2^2)^t [-e_0(z_1e_0 + z_2\underline{\zeta})]^k (e_0 - 1) \right] \\ &= e_0(-2tz_1) \left[(-z_1^2 - z_2^2)^{t-1} [(z_1 - z_2e_0\underline{\zeta})]^k (e_0 - 1) \right] \\ &\quad + ke_0 \left[(-z_1^2 - z_2^2)^t [(z_1 - z_2e_0\underline{\zeta})]^{k-1} (e_0 - 1) \right] \\ &\quad + \underline{\zeta}(-2tz_2) \left[(-z_1^2 - z_2^2)^{t-1} [(z_1 - z_2e_0\underline{\zeta})]^k (e_0 - 1) \right] \\ &\quad - k\underline{\zeta}e_0\underline{\zeta} \left[(-z_1^2 - z_2^2)^t [(z_1 - z_2e_0\underline{\zeta})]^{k-1} (e_0 - 1) \right] \\ &= -2t(z_1e_0 + z_2\underline{\zeta}) \left[(-z_1^2 - z_2^2)^{t-1} [(z_1 - z_2e_0\underline{\zeta})]^k (e_0 - 1) \right] \\ &= -2t\varphi_{2t-1,k}(\mathbf{z}), \end{aligned}$$

which proves the lemma for even j . An analogous calculation proves the statement for $j = 2t + 1$. □

4.1 Explicit expression

As we are searching for an integral transform which maps the Clifford-Hermite functions $\psi_{j,k}$ onto the polynomials $\varphi_{j,k}$, we can establish a system of partial differential equations the kernel function \mathcal{K}^{SB} has to obey. Writing

$$SB(\psi_{j,k})(\mathbf{z}) = \int_{\mathbb{R}_{m+1}} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \psi_{j,k}(\mathbf{x}) dx_0 dr d\sigma_{\underline{x}},$$

where \mathbf{z} denotes $z_1 e_0 + z_2 \zeta$, the kernel function must be such that the raising and lowering operators for the Clifford-Hermite functions are mapped on the raising and lowering operators for the monomial basis of the Fock space as given in Corollary 3 and Lemma 2.

Keeping in mind the properties of the inner product on \mathcal{L}^2 (see Proposition 5), one obtains the following partial differential system:

$$\begin{cases} [\mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) (\frac{x}{2} + cD_0^x)] &= \mathbf{z} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \\ [\mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) (\frac{x}{2} - cD_0^x)] &= cD_0^z \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \end{cases} \quad (4)$$

or, equivalently,

$$\begin{cases} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \mathbf{x} &= (\mathbf{z} + cD_0^z) \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \\ 2c [\mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) D_0^x] &= (\mathbf{z} - cD_0^z) \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \end{cases}.$$

Mind that the square brackets denote the differential operator is acting from the right, as the Clifford multiplication is non-commutative. Based on the latter partial differential system and the particular structure of the kernel function \mathcal{K}^M of the slice Fourier transform \mathcal{F}_S as defined in Definition 11, the following expression for \mathcal{K}^{SB} is proposed:

$$\begin{aligned} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) &= A \frac{\Gamma(\frac{m}{2})}{2\pi^{m/2}} \exp\left(-\frac{|\mathbf{x}|^2 - 4x_0 z_1 + 2(z_1^2 + z_2^2)}{4c}\right) \\ &\quad \times \left[(1 - \underline{\zeta}\omega) \exp\left(\frac{rz_2}{c}\right) + (1 + \underline{\zeta}\omega) \exp\left(-\frac{rz_2}{c}\right) \right], \end{aligned}$$

where the appropriate value for the constant A can be obtained by having a closer look at the transform of $\psi_{0,0}$. One gets

$$\begin{aligned} SB(\psi_{0,0})(z) &= \int_{Cl_{m+1}} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) (e_0 - 1) \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right) dx_0 dr d\sigma_{\underline{x}} \\ &= A 2c \pi (e_0 - 1). \end{aligned} \quad (5)$$

Requiring $SB(\psi_{0,0})$ to be equal to the Clifford-valued constant $e_0 - 1$, we put $A = 1/2c\pi$.

Definition 14. *The slice Segal-Bargmann transform of a function $f \in \mathcal{V}$ is given by*

$$\begin{aligned} SB(f)(\mathbf{z}) &= \frac{\Gamma(\frac{m}{2})}{4c\pi^{m/2+1}} \exp\left(-\frac{z_1^2 + z_2^2}{2c}\right) \times \\ &\quad \int_{\mathbb{R}^{m+1}} \exp\left(-\frac{|\mathbf{x}|^2 - 4x_0 z_1}{4c}\right) \left[(1 - \underline{\zeta}\omega) \exp\left(\frac{rz_2}{c}\right) + (1 + \underline{\zeta}\omega) \exp\left(-\frac{rz_2}{c}\right) \right] f(\mathbf{x}) dx_0 dr d\sigma_{\underline{x}} \end{aligned}$$

with $\mathbf{x} = x_0 e_0 + r\underline{\omega}$, $\mathbf{z} = z_1 e_0 + z_2 \zeta$ and $z_1, z_2 \in \mathbb{C}$.

Theorem 4. *The slice Segal-Bargmann transform is a linear integral transform which obeys*

$$\begin{aligned} SB(\mathbf{x}\psi_{j,k}) &= (cD_0^z + \mathbf{z}) \varphi_{j,k}(\mathbf{z}) \\ SB(2cD_0^x \psi_{j,k}) &= (cD_0^z - \mathbf{z}) \varphi_{j,k}(\mathbf{z}). \end{aligned}$$

Proof. The linearity of the slice Segal-Bargmann transform follows directly from the definition. In order to prove these relations, we will write the above kernel function as a product of two commuting functions \mathcal{K}_1^{SB} and \mathcal{K}_2^{SB} defined as

$$\begin{aligned}\mathcal{K}_1^{SB}(x_0, z_1) &= \exp\left(\frac{-x_0^2 + 4x_0z_1 - 2z_1^2}{4c}\right) \\ \mathcal{K}_2^{SB}(x, z_2) &= \exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) \left[(1 - \underline{\zeta}\underline{\omega}) \exp\left(\frac{rz_2}{c}\right) + (1 + \underline{\zeta}\underline{\omega}) \exp\left(-\frac{rz_2}{c}\right) \right].\end{aligned}$$

In this proof the prefactor $\frac{\Gamma(\frac{m}{2})}{2\pi^{m/2}}$ is omitted since it is not affected by the statement of the lemma. With respect to the first expression, we observe that

$$\mathcal{K}_1^{SB}(x_0, z_1)x_0 = (c\partial_{z_1} + z_1) \exp\left(\frac{-x_0^2 + 4x_0z_1 - 2z_1^2}{4c}\right)$$

and analogously $\mathcal{K}_2^{SB}(\underline{x}, z_2)r\underline{\omega}$ equals

$$\begin{aligned}(c\partial_{z_2} + z_2) \exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) &\left[(\underline{\omega} + \underline{\zeta}) \exp\left(\frac{rz_2}{c}\right) - (\underline{\omega} - \underline{\zeta}) \exp\left(-\frac{rz_2}{c}\right) \right] \\ &= \underline{\zeta} (c\partial_{z_2} + z_2) \exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) \left[(1 - \underline{\zeta}\underline{\omega}) \exp\left(\frac{rz_2}{c}\right) + (1 + \underline{\zeta}\underline{\omega}) \exp\left(-\frac{rz_2}{c}\right) \right].\end{aligned}$$

Combining these results with the fact that $\mathcal{K}_2^{SB}e_0 = e_0\mathcal{K}_2^{SB}$ yields

$$\begin{aligned}\int_{\mathbb{R}^{m+1}} \mathcal{K}_1^{SB}(x_0, z_1)\mathcal{K}_2^{SB}(\underline{x}, z_2)(x_0e_0 + r\underline{\omega})\psi_{j,k}(\mathbf{x})dx_0drd\sigma_{\underline{x}} \\ = [\mathbf{z} + cD_0^z] \int_{\mathbb{R}^{m+1}} \mathcal{K}_1^{SB}(x_0, z_1)\mathcal{K}_2^{SB}(\underline{x}, z_2)\psi_{j,k}(\mathbf{x})dx_0drd\sigma_{\underline{x}},\end{aligned}$$

which proves the first part of the lemma.

In an analogous reasoning we now investigate the action of the partial derivatives on \mathcal{K}_1^{SB} and \mathcal{K}_2^{SB} . The former yields

$$\begin{aligned}\partial_{x_0}\mathcal{K}_1^{SB}(x_0, z_1) &= \left(\frac{z_1}{c} - \frac{x_0}{2c}\right) \exp\left(\frac{-x_0^2 + 4x_0z_1 - 2z_1^2}{4c}\right) \\ &= \left(\frac{z_1}{c} - \frac{1}{2c}(c\partial_{z_1} + z_1)\right) \exp\left(\frac{-x_0^2 + 4x_0z_1 - 2z_1^2}{4c}\right) \\ &= \left(\frac{z_1}{2c} - \frac{1}{2}\partial_{z_1}\right) \mathcal{K}_1^{SB}\end{aligned}$$

and $\partial_r\mathcal{K}_2^{SB}(\underline{x}, z_2)\underline{\omega}$ equals

$$\begin{aligned}\exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) &\left[(\underline{\omega} + \underline{\zeta}) \left(\frac{z_2}{c} - \frac{r}{2c}\right) \exp\left(\frac{rz_2}{c}\right) + (\underline{\omega} - \underline{\zeta}) \left(-\frac{z_2}{c} - \frac{r}{2c}\right) \exp\left(-\frac{rz_2}{c}\right) \right] \\ &= \underline{\zeta} \exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) \left[(1 - \underline{\zeta}\underline{\omega}) \left(\frac{z_2}{c} - \frac{1}{2}\partial_{z_2}\right) \exp\left(\frac{rz_2}{c}\right) \right. \\ &\quad \left. + (1 + \underline{\zeta}\underline{\omega}) \left(\frac{z_2}{c} - \frac{r}{2}\partial_{z_2}\right) \exp\left(-\frac{rz_2}{c}\right) \right] \\ &= \underline{\zeta} \left(\frac{z_2}{c} - \frac{1}{2}(\partial_{z_2} + \frac{z_2}{c})\right) \exp\left(-\frac{r^2 + 2z_2^2}{4c}\right) \left[(1 - \underline{\zeta}\underline{\omega}) \exp\left(\frac{rz_2}{c}\right) + (1 + \underline{\zeta}\underline{\omega}) \exp\left(-\frac{rz_2}{c}\right) \right] \\ &= \underline{\zeta} \left(\frac{z_2}{2c} - \frac{1}{2}\partial_{z_2}\right) \mathcal{K}_2^{SB}.\end{aligned}$$

When performing partial integration on the full integral expression of $SB(D_0^{\mathbf{x}}\psi_{j,k})$, two m -dimensional integrals show up of which the limiting values have to be calculated for $x_0 \rightarrow \pm\infty$ and for r going to 0 and $+\infty$. However all four of the corresponding terms will disappear because, on the one hand,

$$\lim_{x_0 \rightarrow \infty} \exp(-|\mathbf{x}|^2/4c) = \lim_{r \rightarrow \infty} \exp(-|\mathbf{x}|^2/4c) = 0 \quad (6)$$

and, on the other hand,

$$\int_{\mathbb{S}^{m-1}} \underline{\omega} \psi_{j,k}(x_0, 0, \underline{\omega}) \, d\sigma_{\underline{x}} = 0 \quad \forall j, k \in \mathbb{N} \quad (7)$$

because the functions

$$\psi_{j,k}(x_0, 0, \underline{\omega}) = h_{j,k}(\mathbf{x}) m_k(\mathbf{x}) \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right) \Big|_{r=0}$$

do not depend on $\underline{\omega}$ (see [8]). All remaining terms correspond to Equations (6) and (7) and we finally get

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{m-1}} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) (\partial_{x_0} e_0 + \partial_r \underline{\omega}) \psi_{j,k}(\mathbf{x}) \, dx_0 \, dr \, d\sigma_{\underline{x}} \\ &= \left[\frac{1}{2} D_0^{\mathbf{z}} - \frac{\mathbf{z}}{2c} \right] \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{m-1}} \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) \psi_{j,k}(\mathbf{x}) \, dx_0 \, dr \, d\sigma_{\underline{x}}, \end{aligned}$$

which proves the second part of the lemma. \square

Corollary 4. *One has*

$$SB(\psi_{j,k})(\mathbf{z}) = \mathbf{z}^j (-e_0 \mathbf{z})^k (e_0 - 1) = \varphi_{j,k}(\mathbf{z}).$$

Proof. As was shown in the previous section, the Clifford-Hermite functions can be written as

$$\psi_{j,k}(\mathbf{x}) = \left(\frac{\mathbf{x}}{2} - cD_0\right)^j \left[-e_0 \left(\frac{\mathbf{x}}{2} - cD_0\right)\right]^k (e_0 - 1) \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right).$$

By Theorem 4 and because $\mathcal{K}^{SB}(\mathbf{x}, \mathbf{z}) e_0 = e_0 \mathcal{K}^{SB}(\mathbf{x}, \mathbf{z})$ it is straightforward to verify that

$$SB(\psi_{j,k})(\mathbf{z}) = \mathbf{z}^j (-e_0 \mathbf{z})^k SB(\psi_{0,0})(\mathbf{z}).$$

Putting $A = 1/2c\pi$ in (5) it follows that $SB(\psi_{0,0})(\mathbf{z}) = (e_0 - 1)$, which proves the corollary. \square

Theorem 4 allows to transform several properties of the Clifford-Hermite functions to the slice Fock space. In particular, the analogue of the scalar differential equation turns out to be a very intuitive equality for the monomials $\varphi_{j,k}$

Corollary 5. *The scalar differential equation $(4c^2 D_0^2 + |\mathbf{x}|^2) \psi_{j,k}(\mathbf{x}) = 4c(j+k+1) \psi_{j,k}(\mathbf{x})$ is transformed into*

$$(\mathbf{z} D_0^{\mathbf{z}} + D_0^{\mathbf{z}} \mathbf{z}) \varphi_{j,k}(\mathbf{z}) = -2(j+k+1) \varphi_{j,k}(\mathbf{z}).$$

Proof. Theorem 4 allows to write the action of the slice Segal-Bargmann transform on $4c^2 D_0^2 + |\mathbf{x}|^2$ as

$$\left[(cD_0^{\mathbf{z}} - \mathbf{z})^2 - (cD_0^{\mathbf{z}} + \mathbf{z})^2 \right] = -2c(D_0^{\mathbf{z}} \mathbf{z} + \mathbf{z} D_0^{\mathbf{z}}),$$

which proves the corollary. \square

Remark 2. *Note that the scalar differential equation for the Clifford-Hermite functions reduces to a more intuitive form on the slice Fock space. Indeed, given that $\mathbf{z} D_0^{\mathbf{z}} + D_0^{\mathbf{z}} \mathbf{z} = -2(z_1 \partial_{z_1} + z_2 \partial_{z_2} + 1)$, the above expression can be rewritten as $\mathbb{E} \varphi_{j,k} = (j+k) \varphi_{j,k}$, where \mathbb{E} denotes the Euler operator $\mathbb{E} = z_1 \partial_{z_1} + z_2 \partial_{z_2}$ which measures the degree of a homogeneous polynomial in z_1 and z_2 .*

Corollary 6. *The polynomials $\varphi_{j,k}$ satisfy the following relations:*

$$\begin{aligned}\mathbf{z}D_0^{\mathbf{z}} \varphi_{j,k} &= C(j, k) \varphi_{j,k} \\ D_0^{\mathbf{z}} \varphi_{j,k} &= C(j+1, k) \varphi_{j,k}.\end{aligned}$$

Proof. These relations follow from consecutive application of Corollary 3 and Lemma 2. \square

Remark 3. *The scalar differential equation of Corollary 5 can also be retrieved by taking the sum of the identities of Corollary 6.*

5 Monomial basis of the slice Fock space

The purpose of this section is to prove the orthogonality of the monomials $\varphi_{j,k}$ with respect to a well-defined inner product. To do so we will thoroughly use the results of the previous sections.

5.1 Inner product

Inspired by the particular behaviour of the inner product $\langle \psi_{j_1, k_1}, \psi_{j_2, k_2} \rangle_{\mathcal{L}^2}$ of two Clifford-Hermite functions, as to

$$\begin{aligned}\langle \psi_{j_1, k_1}, \psi_{j_2, k_2} \rangle_{\mathcal{L}^2} &= \langle \tilde{D}_c \psi_{j_1-1, k_1}, \psi_{j_2, k_2} \rangle_{\mathcal{L}^2} \\ &= \langle \psi_{j_1-1, k_1}, \tilde{D}_c^\dagger \psi_{j_2, k_2} \rangle_{\mathcal{L}^2} \\ &= -c C(j_2, k_2) \langle \psi_{j_1-1, k_1}, \psi_{j_2-1, k_2} \rangle_{\mathcal{L}^2},\end{aligned}$$

where $\tilde{D}_c = \frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}$ and $\tilde{D}_c^\dagger = -\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}$ denotes its adjoint with respect to the inner product, we require the inner product $\langle \varphi_{j_1, k_1}, \varphi_{j_2, k_2} \rangle_{FS}$ on the Fock space to establish an analogous property. For the time being, we write this inner product as

$$\langle \varphi_{j_1, k_1}, \varphi_{j_2, k_2} \rangle_{FS} = \int_{\mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1}} \overline{\varphi_{j_1, k_1}(\mathbf{z})} \varphi_{j_2, k_2}(\mathbf{z}) h(\mathbf{z}) dz_1 dz_2 d\sigma_\zeta,$$

where $dz_j = dx_j dy_j$ and $h(\mathbf{z})$ denotes the weight function $h(z_1, z_2, \zeta)$ that has to be determined. From Lemma 2 we know that $D_0^{\mathbf{z}}$ acts as a lowering operator with respect to the index j of $\varphi_{j,k}$. The above requirement therefore translates to the following proportionality

$$\langle \mathbf{z} \varphi_{j_1-1, k_1}, \varphi_{j_2, k_2} \rangle_{FS} \sim \langle \varphi_{j_1-1, k_1}, D_0^{\mathbf{z}} \varphi_{j_2, k_2} \rangle_{FS}.$$

Given that a well-defined inner product has to be symmetric as well, the same should hold with respect to the second argument:

$$\langle \varphi_{j_1, k_1}, \mathbf{z} \varphi_{j_2-1, k_2} \rangle_{FS} \sim \langle D_0^{\mathbf{z}} \varphi_{j_1, k_1}, \varphi_{j_2-1, k_2} \rangle_{FS}.$$

Expressing both conditions with respect to the weight function h one obtains the following partial differential systems:

$$\begin{cases} \overline{z_1} h(\mathbf{z}) &= B \partial_{z_1} h(\mathbf{z}) \\ \overline{z_2} h(\mathbf{z}) &= B \partial_{z_2} h(\mathbf{z}) \end{cases} \quad \text{and} \quad \begin{cases} B \partial_{\overline{z_1}} h(\mathbf{z}) &= z_1 h(\mathbf{z}) \\ B \partial_{\overline{z_2}} h(\mathbf{z}) &= z_2 h(\mathbf{z}), \end{cases}$$

where the proportionality factor $B \in \mathbb{R}$ remains to be fixed. Observing that these expressions are invariant under complex conjugation, h should depend on $|z_1|$ and $|z_2|$ only. Inspired by the inner product on the classical Fock space (see Definition 1), we propose the following weight function:

$$h(\mathbf{z}) = A \exp\left(\frac{|z_1|^2 + |z_2|^2}{2B}\right),$$

where the constants A and B remain to be fixed. A straightforward calculation shows that this expression satisfies all of the above partial differential equations. The constants can be fixed by requiring the functions $\psi_{j,k}$ with low indices j and k to be orthogonal. This end result yields the following definition.

Definition 15. The inner product $\langle f, g \rangle_{FS}$ of two polynomials $f, g : \mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1} \rightarrow \mathbb{C}_{m+1}$ in F^S is defined as

$$\langle f, g \rangle_{FS} = \frac{1}{c\pi} \frac{\Gamma(m/2)}{2\pi^{m/2}} \int_{\mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1}} \overline{f(\mathbf{z})} g(\mathbf{z}) e^{-|\mathbf{z}|^2/c} dz_1 dz_2 d\sigma_{\underline{\zeta}},$$

where $|\mathbf{z}|^2 = |z_1|^2 + |z_2|^2$ and $\overline{\cdot}$ denotes the complex Clifford conjugation.

Corollary 7. One has $\langle e_0 f, g \rangle_{FS} = -\langle f, e_0 g \rangle_{FS}$ for all $f, g \in F^S$.

Proof. This follows from the above definition of the inner product and the fact that $\overline{e_0 f} = \overline{f} \overline{e_0} = -\overline{f} e_0$. \square

Corollary 8. One has $\langle f, g \rangle_{FS} = \overline{\langle g, f \rangle_{FS}}$ for all $f, g \in F^S$.

Proof. Taking the complex Clifford conjugation of the definition of the inner product proves the corollary. \square

Theorem 5. The inner product $\langle f, g \rangle_{FS}$ of two polynomials $f, g \in F^S$ satisfies

$$\begin{aligned} \langle f, -cD_0^{\mathbf{z}} g \rangle_{FS} &= \langle \mathbf{z} f, g \rangle_{FS} \\ \langle -cD_0^{\mathbf{z}} f, g \rangle_{FS} &= \langle f, \mathbf{z} g \rangle_{FS} \end{aligned}$$

Proof. Putting $z_\ell = x_\ell + iy_\ell$ for $\ell = 1, 2$, one has

$$\begin{aligned} D_0^{\mathbf{z}} \exp\left(-\frac{|\mathbf{z}|^2}{c}\right) &= \frac{1}{2} \left[e_0 \left(-\frac{2\overline{z_1}}{c}\right) + \underline{\zeta} \left(-\frac{2\overline{z_2}}{c}\right) \right] \exp\left(-\frac{|\mathbf{z}|^2}{c}\right) \\ &= -\frac{\overline{\mathbf{z}}}{c} \exp\left(-\frac{|\mathbf{z}|^2}{c}\right). \end{aligned}$$

Performing partial integration on $D_0^{\mathbf{z}}$ one gets

$$\langle f, D_0^{\mathbf{z}} g \rangle_{FS} = -\frac{1}{c\pi} \frac{\Gamma(m/2)}{2\pi^{m/2}} \int_{\mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1}} \overline{f(\mathbf{z})} D_0^{\mathbf{z}} \left[\exp\left(-\frac{|\mathbf{z}|^2}{c}\right) \right] g(\mathbf{z}) dz_1 dz_2 d\sigma_{\underline{\zeta}}$$

where all the additional terms vanished because

$$\lim_{x_i \rightarrow \pm\infty} \overline{f(\mathbf{z})} g(\mathbf{z}) \exp\left(-\frac{|\mathbf{z}|^2}{c}\right) = \lim_{y_i \rightarrow \pm\infty} \overline{f(\mathbf{z})} g(\mathbf{z}) \exp\left(-\frac{|\mathbf{z}|^2}{c}\right) = 0$$

for $i = 1, 2$ and because $\overline{f(\mathbf{z})}$ only depends on $\overline{z_1}$ and $\overline{z_2}$. Putting things together, we finally obtain $\langle f, D_0^{\mathbf{z}} g \rangle_{FS} = -\frac{1}{c} \langle \mathbf{z} f, g \rangle_{FS}$. The proof of the second equation is achieved by taking the Clifford conjugation of the above expression and performing partial integration on $\overline{D_0^{\mathbf{z}}}$ in the expression for $\langle D_0^{\mathbf{z}} f, g \rangle_{FS}$. \square

5.2 Orthogonality of the basis functions

Before proving the orthogonality of the full set of polynomials $\{\varphi_{j,k}\}$, the following lemma addresses the specific case where $j = 0$.

Lemma 3. One has

$$\langle \varphi_{0,k_1}, \varphi_{0,k_2} \rangle_{FS} = 2c\pi (2c)^{k_1} k_1! \delta_{k_1 k_2}.$$

Proof. Combining Corollary 3, Corollary 7 and Corollary 6, one obtains

$$\begin{aligned} \langle \varphi_{0,k}, \varphi_{0,k} \rangle_{FS} &= \langle -e_0 \mathbf{z} \varphi_{0,k-1}, \varphi_{0,k} \rangle_{FS} = \langle \mathbf{z} \varphi_{0,k-1}, e_0 \varphi_{0,k} \rangle_{FS} \\ &= \langle \mathbf{z} \varphi_{0,k-1}, \mathbf{z} \varphi_{0,k-1} \rangle_{FS} = -c \langle \varphi_{0,k-1}, D_0^{\mathbf{z}} \mathbf{z} \varphi_{0,k-1} \rangle_{FS} \\ &= -c C(1, k-1) \langle \varphi_{0,k-1}, \varphi_{0,k-1} \rangle_{FS} = 2ck \langle \varphi_{0,k-1}, \varphi_{0,k-1} \rangle_{FS} \\ &= \dots \\ &= (2c)^k k! \langle \varphi_{0,0}, \varphi_{0,0} \rangle = 2(2c)^k k! (c\pi)^2 \end{aligned}$$

because a straightforward calculation yields $\langle \varphi_{0,0}, \varphi_{0,0} \rangle = 2(c\pi)^2$. \square

Now we can determine the inner product of two monomials φ_{j_1, k_1} and φ_{j_2, k_2} in the slice Fock space. Different cases will be distinguished corresponding to the parity of j_1 and j_2 .

Theorem 6. *Let $\varphi_{j_i, k_i}(\mathbf{z}) = \mathbf{z}^{j_i} [-e_0 \mathbf{z}]^{k_i} (e_0 - 1)$ for $i = 1, 2$. The inner product of these two monomials φ_{j_1, k_1} and φ_{j_2, k_2} is given by*

$$\langle \varphi_{j_1, k_1}, \varphi_{j_2, k_2} \rangle_{FS} = B(j_1, k_1) \delta_{j_1 j_2} \delta_{k_1 k_2}$$

with

$$B(j_1, k_1) = \begin{cases} (2c)^{2t_1+k_1+1} \pi t_1! (t_1 + k_1)! & j_1 = 2t_1, \\ (2c)^{2t_1+k_1+2} \pi t_1! (t_1 + k_1 + 1)! & j_1 = 2t_1 + 1. \end{cases}$$

Proof. By Corollary 8 we can assume $j_1 \geq j_2$. Using Corollary 3, Lemma 2 and Theorem 5, one has

$$\begin{aligned} \langle \varphi_{j_1, k_1}, \varphi_{j_2, k_2} \rangle_{FS} &= \langle \mathbf{z} \varphi_{j_1-1, k_1}, \varphi_{j_2, k_2} \rangle_{FS} \\ &= -c \langle \varphi_{j_1-1, k_1}, D_0^{\mathbf{z}} \varphi_{j_2, k_2} \rangle_{FS} \\ &= -c C(j_2, k_2) \langle \varphi_{j_1-1, k_1}, \varphi_{j_2-1, k_2} \rangle_{FS} \\ &= \dots \\ &= (-c)^{j_2} C(j_2, k_2) \dots C(1, k_2) \langle \varphi_{j_1-j_2, k_1}, \varphi_{0, k_2} \rangle_{FS} \end{aligned}$$

where the third equality is due to Lemma 2.

If now $j_1 - j_2$ would be bigger than 0, this procedure could be repeated at least one more time, yielding a factor $C(0, k_2)$ to show up. Given that $C(j, k) = -j$ for even j , this additional factor would make the inner product vanish.

The inner product of two monomials φ_{j_1, k_1} and φ_{j_2, k_2} can thus only be different from zero if $j_1 = j_2$. In this case the inner product in the last expression reads $\langle \varphi_{0, k_1}, \varphi_{0, k_2} \rangle_{FS}$ and the previous lemma can be used. Simplifying the final expressions yields the theorem. \square

5.3 Normalised basis functions

Denoting the normalised Clifford-Hermite functions $\psi_{j, k}$ as $\psi_{j, k}^\circ$, one has

$$\begin{cases} \psi_{2t, k}^\circ(\mathbf{x}) & := \frac{1}{2c\pi} \frac{\sqrt{t!}}{\sqrt{(k+t)!}} \frac{(e_0 - 1)(x_0 + \underline{x})^k}{(\sqrt{2c})^k} L_t^k \left(\frac{|\mathbf{x}|^2}{2c} \right) \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right) \\ \psi_{2t+1, k}^\circ(\mathbf{x}) & := \frac{1}{2c\pi} \frac{\sqrt{t!}}{\sqrt{(k+t+1)!}} \mathbf{x} \frac{(e_0 - 1)(x_0 + \underline{x})^k}{(\sqrt{2c})^{k+1}} L_t^{k+1} \left(\frac{|\mathbf{x}|^2}{2c} \right) \exp \left(-\frac{|\mathbf{x}|^2}{4c} \right), \end{cases}$$

where the original polynomials $\psi_{j, k}$ have been divided by the square root of their respective norms, given by (see [7])

$$\begin{cases} \sqrt{\langle \psi_{2t, k}, \psi_{2t, k} \rangle_{\mathcal{L}^2}} & = \sqrt{2c\pi} (\sqrt{2c})^{2t+k} \sqrt{t!(k+t)!} \\ \sqrt{\langle \psi_{2t+1, k}, \psi_{2t+1, k} \rangle_{\mathcal{L}^2}} & = \sqrt{2c\pi} (\sqrt{2c})^{2t+k+1} \sqrt{t!(k+t+1)!}. \end{cases}$$

With respect to the above defined inner product $\langle \cdot, \cdot \rangle_{FS}$, the monomials $\varphi_{j, k}$ can as well be normalised, which yields

$$\begin{cases} \varphi_{2t, k}^\circ(\mathbf{z}) & = \frac{1}{\sqrt{2c\pi}} \frac{(-(z_1^2 + z_2^2))^t}{(2c)^t \sqrt{t!}} \frac{(z_1 - e_0 \zeta z_2)^k (e_0 - 1)}{(\sqrt{2c})^k \sqrt{(t+k)!}} \\ \varphi_{2t+1, k}^\circ(\mathbf{z}) & = \frac{1}{\sqrt{2c\pi}} \mathbf{z} \frac{(-(z_1^2 + z_2^2))^t}{(2c)^t \sqrt{t!}} \frac{(z_1 - e_0 \zeta z_2)^k (e_0 - 1)}{(\sqrt{2c})^{k+1} \sqrt{(t+k+1)!}}, \end{cases}$$

because the square roots of their respective norms are given by (see Theorem 6)

$$\begin{cases} \sqrt{\langle \varphi_{2t,k}, \varphi_{2t,k} \rangle_{F^S}} &= \sqrt{2c\pi}(\sqrt{2c})^{2t+k} \sqrt{t!(t+k)!} \\ \sqrt{\langle \varphi_{2t+1,k}, \varphi_{2t+1,k} \rangle_{F^S}} &= \sqrt{2c\pi}(\sqrt{2c})^{2t+k+1} \sqrt{t!(t+k+1)!} \end{cases}.$$

Given the linearity of the slice Segal-Bargmann transform, we can end this section with the following theorem.

Theorem 7. *The slice Segal-Bargmann transform maps the \mathcal{L}^2 basis of orthonormal Clifford-Hermite functions $\psi_{j,k}^\circ$ onto the orthonormal basis $\varphi_{j,k}^\circ$ of the slice Fock space F^S :*

$$SB(\psi_{j,k}^\circ)(\mathbf{z}) = \varphi_{j,k}^\circ(\mathbf{z}).$$

6 Properties of the slice Segal-Bargmann transform

6.1 Inverse slice Segal-Bargmann transform

The integral expression for the inverse slice Segal-Bargmann transform has to show analogous behaviour as the forward slice Segal-Bargmann transform, apart from some signs and conjugations. Moreover, integration will be performed over complex variables now. In order to write the inverse slice Segal-Bargmann transform as an integral transform

$$SB^{-1}(\varphi_{j,k})(\mathbf{x}) = \int_{\mathbb{C} \times \mathbb{C} \times S^{m-1}} \mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x}) \varphi_{j,k}(\mathbf{z}) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{c}\right) dz_1 dz_2 d\sigma_{\underline{z}},$$

where dz_j denotes $dx_j dy_j$ for $j = 1, 2$, its kernel function $\mathcal{K}^{SB,-1}$ thus has to satisfy a partial differential system, as was the case for the forward slice Segal-Bargmann transform. In this case we want SB^{-1} to behave as follows:

$$\begin{aligned} SB^{-1}(\mathbf{z}g)(\mathbf{x}) &= \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}\right) SB^{-1}(g)(\mathbf{x}) \\ SB^{-1}(cD_0^{\mathbf{z}}g)(\mathbf{x}) &= \left(\frac{\mathbf{x}}{2} + cD_0^{\mathbf{x}}\right) SB^{-1}(g)(\mathbf{x}) \end{aligned} \quad (8)$$

where $g \in F^S$. This system, however, has to be translated into partial differential equations with respect to the kernel function $\mathcal{K}^{SB,-1}$. This is why the above integral expression has been written in a rather suggestive way: by including a factor $\exp(-(|z_1|^2 + |z_2|^2)/c)$ in the integrand, the expression already refers to the inner product on F^S as defined in Definition 15.

Though the integral expression can not be an inner product (because $\mathcal{K}^{SB,-1} \notin F^S$), the relations of Theorem 5 remain valid and can be used to transform the above partial differential system. Indeed, denoting the integral formally as $\langle \overline{\mathcal{K}^{SB,-1}}, \varphi \rangle_{F^S}$, one has

$$\begin{aligned} \langle \overline{\mathcal{K}^{SB,-1}}, D_0^{\mathbf{z}}\varphi \rangle_{F^S} &= -\frac{1}{c} \langle \mathbf{z}\overline{\mathcal{K}^{SB,-1}}, \varphi \rangle_{F^S} \\ \langle \overline{\mathcal{K}^{SB,-1}}, \mathbf{z}\varphi \rangle_{F^S} &= -c \langle D_0^{\mathbf{z}}\overline{\mathcal{K}^{SB,-1}}, \varphi \rangle_{F^S}. \end{aligned}$$

Therefore the above conditions on the inverse slice Segal-Bargmann transform SB^{-1} yield the following conditions on $\mathcal{K}^{SB,-1}$:

$$\begin{cases} \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}\right) \mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x}) &= -c [\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x}) \overline{D_0^{\mathbf{z}}}] \\ \left(\frac{\mathbf{x}}{2} + cD_0^{\mathbf{x}}\right) \mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x}) &= -\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x}) \bar{\mathbf{z}}. \end{cases}$$

Now taking the Clifford conjugate of these expressions and keeping in mind that x_0 and r are real variables, we obtain the following equations:

$$\begin{cases} \left[\overline{\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x})} \left(\frac{\mathbf{x}}{2} + cD_0^{\mathbf{x}} \right) \right] = \mathbf{z} \overline{\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x})} \\ \left[\overline{\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x})} \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}} \right) \right] = cD_0^{\mathbf{z}} \overline{\mathcal{K}^{SB,-1}(\mathbf{z}, \mathbf{x})}, \end{cases}$$

which are identical to the equations that were proposed for the kernel function \mathcal{K}^{SB} of the forward slice Segal-Bargmann transform, if \mathcal{K}^{SB} is substituted by $\overline{\mathcal{K}^{SB,-1}}$. We thus obtain the kernel function for the inverse transform immediately by taking the full conjugation of \mathcal{K}^{SB} .

This leads to the following definition of the inverse slice Segal-Bargmann transform SB^{-1} :

Definition 16. *The inverse slice Segal-Bargmann transform of a function $g \in F^S$ is given by*

$$\begin{aligned} SB^{-1}(g)(\mathbf{x}) &= \frac{1}{c^2 \pi^2} \frac{\Gamma\left(\frac{m}{2}\right)}{2\pi^{m/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right) \int_{\mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1}} \exp\left(-\frac{\bar{z}_1^2 + \bar{z}_2^2 - 2x_0 \bar{z}_1}{2c}\right) \\ &\times \left[(1 - \underline{\omega} \underline{\zeta}) \exp\left(\frac{r \bar{z}_2}{c}\right) + (1 + \underline{\omega} \underline{\zeta}) \exp\left(-\frac{r \bar{z}_2}{c}\right) \right] g(\mathbf{z}) \exp\left(-\frac{|z_1|^2 + |z_2|^2}{c}\right) dz_1 dz_2 d\sigma_{\underline{\zeta}}, \end{aligned}$$

where $dz_j = dx_j dy_j$ for $j = 1, 2$.

Remark 4. *Note that in this definition the prefactor has already been adapted in such a way that $SB^{-1}(\varphi_{0,0}) = \psi_{0,0}$.*

Theorem 8. *The integral transform SB^{-1} on F^S is the inverse of the slice Segal-Bargmann transform SB on \mathcal{L}^2 .*

Proof. The functions $\varphi_{j,k}$ span the slice Fock space F^S so it suffices to check the statement for these monomials in order to prove the theorem. Writing $\varphi_{j,k}(\mathbf{z}) = \mathbf{z}^j (-e_0 \mathbf{z})^k (e_0 - 1)$ and using (8), one gets

$$\begin{aligned} SB^{-1}(\varphi_{j,k})(\mathbf{x}) &= SB^{-1}(\mathbf{z}^j (-e_0 \mathbf{z})^k (e_0 - 1))(\mathbf{x}) \\ &= \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}\right)^j \left[-e_0 \left(\frac{\mathbf{x}}{2} - cD_0^{\mathbf{x}}\right)\right]^k SB^{-1}(\varphi_{0,0})(\mathbf{x}). \end{aligned}$$

Given that

$$SB^{-1}(\varphi_{0,0})(\mathbf{x}) = \exp\left(-\frac{|\mathbf{x}|^2}{4c}\right) (e_0 - 1) = \psi_{0,0}(\mathbf{x}),$$

one has $SB^{-1}(\varphi_{j,k}) = \psi_{j,k}$ and the lemma has been proven. \square

6.2 Slice Fourier transform on the slice Fock space

Now we have explicit integral expressions for the forward and the inverse slice Segal-Bargmann transform, we can investigate which operator on the slice Fock space corresponds to taking the slice Fourier transform on \mathcal{L}^2 . In other words, we want to find the operator \mathcal{G}_S on F^S that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{L}^2 & \xrightarrow{\mathcal{F}_S} & \mathcal{L}^2 \\ SB \downarrow & & \uparrow SB^{-1} \\ F^S & \xrightarrow{\mathcal{G}_S} & F^S \end{array}$$

We obtain the following theorem.

Theorem 9. *The operator on the slice Fock space F^S corresponding to the slice Fourier transform \mathcal{F}_S on \mathcal{L}^2 is given by*

$$\mathcal{G}_S : F^S \rightarrow F^S : f(\mathbf{z}) \mapsto -i f(-i\mathbf{z}).$$

Proof. Taking the slice Segal-Bargmann transform of the basis functions $\psi_{j,k}$ and their slice Fourier transforms $(-i)^{j+k+1}\psi_{j,k}$, the operator on F^S that maps the functions $\varphi_{j,k}(\mathbf{z})$ onto $-i\varphi_{j,k}(-i\mathbf{z})$ is given by the above operator \mathcal{G}_S . \square

Otherwise stated, the action of the slice Fourier transform in the slice Fock space is the combination of a multiplication with $(-i)$ and a substitution of the argument by $(-i)$ times the argument.

Corollary 9. *The basis functions $\varphi_{j,k}$ of F^S are eigenfunctions of \mathcal{G}_S with respective eigenvalues $(-i)^{j+k+1}$.*

Proof. Performing \mathcal{G}_S on $\varphi_{j,k}$ yields

$$\mathcal{G}_S \varphi_{j,k}(\mathbf{z}) = -i \varphi_{j,k}(-i\mathbf{z}) = (-i)^{j+k+1} \varphi_{j,k}(\mathbf{z}),$$

because $\varphi_{j,k}$ is a homogeneous monomial of degree $j+k$. \square

6.3 Reproducing kernel space

Our final aim is to show that the slice Fock space F^S is a reproducing kernel space. It is to say, there exists a reproducing kernel \mathcal{K}_{F^S} such that

$$\begin{aligned} \varphi_{j,k}(\mathbf{z}) &= \left\langle \overline{\mathcal{K}_{F^S}(\mathbf{u}, \mathbf{z})}, \varphi_{j,k}(\mathbf{u}) \right\rangle_{F^S} \\ &= \frac{1}{c\pi} \frac{\Gamma(m/2)}{2\pi^{m/2}} \int_{\mathbb{C} \times \mathbb{C} \times \mathbb{S}^{m-1}} \mathcal{K}_{F^S}(\mathbf{u}, \mathbf{z}) \varphi(\mathbf{u}) e^{-|\mathbf{u}|^2/c} du_1 du_2 d\sigma_{\underline{\nu}}, \end{aligned} \quad (9)$$

for all basis functions $\varphi_{j,k}$.

In order to get some grip on the structure of \mathcal{K}_{F^S} , we first approach this problem using a Mehler formula. Indeed, a formal series expression for the reproducing kernel reads

$$\begin{aligned} \mathcal{K}_{F^S}(\mathbf{u}, \mathbf{z}) &= \sum_{j,k=0}^{\infty} \frac{\varphi_{j,k}(\mathbf{z}) \overline{\varphi_{j,k}(\mathbf{u})}}{\langle \varphi_{j,k}, \varphi_{j,k} \rangle} \\ &= \frac{1}{c\pi} \sum_{t=0}^{\infty} \frac{(z_1^2 + z_2^2)^t (\bar{u}_1^2 + \bar{u}_2^2)^t}{(2c)^{2t} t!} \sum_{k=0}^{\infty} \frac{(z_1 - z_2 e_0 \zeta)^k (\bar{u}_1 + \bar{u}_2 e_0 \underline{\nu})^k}{(2c)^k (t+k)!} \\ &\quad + \frac{1}{c\pi} \sum_{t=0}^{\infty} \frac{(z_1^2 + z_2^2)^t (\bar{u}_1^2 + \bar{u}_2^2)^t}{(2c)^{2t} t!} \sum_{k=0}^{\infty} \frac{(z_1 + z_2 e_0 \zeta)^{k+1} (\bar{u}_1 - \bar{u}_2 e_0 \underline{\nu})^{k+1}}{(2c)^{k+1} (t+k+1)!}, \end{aligned}$$

where the series has been split with respect to even and odd values of j . Changing the summation index $k \rightarrow k-1$ in the second series, we can recombine both series expressions and obtain

$$\begin{aligned} &\frac{1}{c\pi} \sum_{t=0}^{\infty} \frac{(z_1^2 + z_2^2)^t (\bar{u}_1^2 + \bar{u}_2^2)^t}{(2c)^{2t} t!} \\ &\quad \times \left[\sum_{k=1}^{\infty} \frac{(z_1 - z_2 e_0 \zeta)^k (\bar{u}_1 + \bar{u}_2 e_0 \underline{\nu})^k + (z_1 + z_2 e_0 \zeta)^k (\bar{u}_1 - \bar{u}_2 e_0 \underline{\nu})^k}{(2c)^k (t+k)!} + \frac{1}{t!} \right]. \end{aligned} \quad (10)$$

Instead of summing this series directly, we proceed in a different way.

Proposition 7. *The reproducing kernel for the slice Fock space F^S only consists of a scalar part and a two-vector part which does not contain e_0 . In other words, one has*

$$\mathcal{K}_{F^S} = f + g \zeta \underline{\nu}$$

where $f, g : \mathbb{C}^4 \rightarrow \mathbb{C}$.

Proof. Formula (10) is Clifford-valued because $e_0\underline{\nu}$ and $e_0\underline{\zeta}$ appear in its numerator. Given that these expressions both square to -1 , the only possible Clifford-valued parts are $e_0\underline{\nu}$, $e_0\underline{\zeta}$ and $\underline{\zeta}\underline{\nu}$. At the same time, one observes that $e_0\mathcal{K}_{FS}e_0 = -\mathcal{K}_{FS}$. Indeed, pulling e_0 from the left through the above series expression (10) and multiplying it with the e_0 at the right, only yields a minus sign. Therefore the closed form of \mathcal{K}_{FS} can only consist of a scalar part and a $\underline{\zeta}\underline{\nu}$ -part. \square

Equation (9) should hold for all basis functions $\varphi_{j,k}$, so in particular for $\varphi_{j+1,k}$. According to Corollary 3 one has $\varphi_{j+1,k}(\mathbf{z}) = \mathbf{z}\varphi_{j,k}(\mathbf{z})$ and therefore

$$\mathbf{z}\varphi_{j,k}(\mathbf{z}) = \left\langle \overline{\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})}, \mathbf{u}\varphi_{j,k}(\mathbf{u}) \right\rangle_{FS}. \quad (11)$$

The corresponding requirement for $\varphi_{j-1,k}$ reads

$$D_0^{\mathbf{z}}\varphi_{j,k}(\mathbf{z}) = \left\langle \overline{\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})}, D_0^{\mathbf{u}}\varphi_{j,k}(\mathbf{u}) \right\rangle_{FS}. \quad (12)$$

Proposition 8. *Writing the reproducing kernel as*

$$\mathcal{K}_{FS} = f + g \underline{\zeta}\underline{\nu},$$

where $f, g : \mathbb{C}^4 \rightarrow \mathbb{C}$, the following relations for f and g have to be met:

$$\begin{cases} c\partial_{\bar{u}_1} f = z_1 f \\ c\partial_{\bar{u}_1} g = z_1 g \\ c\partial_{z_1} f = \bar{u}_1 f \\ c\partial_{z_1} g = \bar{u}_1 g, \end{cases} \quad \text{and} \quad \begin{cases} c\partial_{\bar{u}_2} f = -z_2 g \\ c\partial_{\bar{u}_2} g = -z_2 f \\ c\partial_{z_2} f = -\bar{u}_2 g \\ c\partial_{z_2} g = -\bar{u}_2 f, \end{cases}$$

where $\bar{\cdot}$ denotes the complex conjugation.

Proof. Using Theorem 5, equations (11) and (12) can be written as

$$\begin{aligned} \mathbf{z}\varphi_{j,k}(\mathbf{z}) &= -c \left\langle D_0^{\mathbf{u}} \overline{\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})}, \varphi_{j,k}(\mathbf{u}) \right\rangle_{FS} \\ -cD_0^{\mathbf{z}}\varphi_{j,k}(\mathbf{z}) &= \left\langle \mathbf{u} \overline{\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})}, \varphi_{j,k}(\mathbf{u}) \right\rangle_{FS}. \end{aligned}$$

Because these identities must hold for all $j, k \in \mathbb{N}$, one has

$$\begin{aligned} \mathbf{z}\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z}) &= -c[\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})\overline{D_0^{\mathbf{u}}}] \\ -cD_0^{\mathbf{z}}\mathcal{K}_{FS}(\mathbf{u}, \mathbf{z}) &= \mathcal{K}_{FS}(\mathbf{u}, \mathbf{z})\bar{\mathbf{u}}, \end{aligned}$$

which translate to the above conditions on f and g . \square

Proposition 9. *The function*

$$A \exp\left(\frac{z_1 \bar{u}_1}{c}\right) \left[A_1 \cosh\left(\frac{z_2 \bar{u}_2}{c}\right) - \underline{\zeta}\underline{\nu} A_2 \sinh\left(\frac{z_2 \bar{u}_2}{c}\right) \right]$$

with $A, A_1, A_2 \in \mathbb{R}$ solves the systems of differential equations of Proposition 8.

Proof. In the above systems of differential equations, one observes that f and g have to obey the same conditions with respect to z_1 and \bar{u}_1 . Up to a multiplicative constant, these conditions yield a common factor $\exp(z_1 \bar{u}_1 / c)$.

The other system implies that, with respect to z_2 and \bar{u}_2 , both f and g are a linear combination of the hyperbolic functions $\cosh(z_2 \bar{u}_2 / c)$ and $\sinh(z_2 \bar{u}_2 / c)$. \square

The particular choice to write f as a cosh function and g as a sinh function is motivated in the following theorem.

Theorem 10. *The reproducing kernel of the slice Fock space F^S reads*

$$\mathcal{K}_{F^S}(\mathbf{u}, \mathbf{z}) = \frac{1}{c\pi} \exp\left(\frac{z_1 \bar{u}_1}{c}\right) \left[\cosh\left(\frac{z_2 \bar{u}_2}{c}\right) - \underline{\zeta} \underline{\nu} \sinh\left(\frac{z_2 \bar{u}_2}{c}\right) \right].$$

Proof. To prove this theorem, we use the fact that the classical Fock space F is a reproducing kernel space (see [25]). To be more precise, on F one has

$$\frac{1}{\pi} \int_{\mathbb{C}} e^{z\bar{u}} u^k e^{-|u|^2} du = z^k$$

for all $k \in \mathbb{N}$ and where u and z are complex variables. Summing this expression with the same expression where z is substituted by $-z$ and dividing the resulting integral by 2, one gets

$$\frac{1}{\pi} \int_{\mathbb{C}} \cosh(z\bar{u}) u^k e^{-|u|^2} du = \begin{cases} z^k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

Analogously, dividing the difference of these integrals by two yields

$$\frac{1}{\pi} \int_{\mathbb{C}} \sinh(z\bar{u}) u^k e^{-|u|^2} du = \begin{cases} 0 & k \text{ even} \\ z^k & k \text{ odd} \end{cases}.$$

For \mathcal{K}_{F^S} to be the reproducing kernel, equation (9) should be satisfied for all $j, k \in \mathbb{N}$. Now we have that

$$\varphi_{j,k}(\mathbf{u}) = (u_1 e_0 + u_2 \underline{\nu})^j (u_1 - u_2 e_0 \underline{\nu})^k (e_0 - 1).$$

With respect to the Clifford-valued parts of the functions $\varphi_{j,k}$, one observes that \mathcal{K}_{F^S} commutes with e_0 and that the spherical integral transforms $\underline{\nu}$ into $\underline{\zeta}$. With respect to u_1 and u_2 , combining the above three identities yields the desired behaviour.

Finally, the constants A, A_1 and A_2 have been fixed by expressing equation (9) for $\varphi_{0,0}$ and $\varphi_{1,0}$. \square

7 Conclusion

In this paper we have introduced the Segal-Bargmann transform in the context of the slice Dirac operator. Based on results obtained in the articles [8] and [7], an appropriate slice Fock space could be defined. Next the slice Segal-Bargmann transform was constructed such that it mapped the Clifford-Hermite basis functions onto a monomial basis of the slice Fock space. The same approach could be used to find the kernel of the inverse slice Segal-Bargmann transform as well. Putting all of this together, we showed that under the slice Segal-Bargmann transform the action of the slice Fourier transform is mapped onto multiplying the variable on the slice Fock space with $-i$, where i denotes the complex unit. Finally we showed that the slice Fock space is also a reproducing kernel space.

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