On subgraphs of large girth

In Honor of the 50th Birthday of Robin Thomas

Vojtěch Rödl
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joint work with Domingos Dellamonica

May, 2012


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On the Genus of a Random Graph

Vojtěch Rödl*
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ABSTRACT

Let $p = p(n)$ be a function of n with $0 < p < 1$. We consider the random graph model $\mathcal{G}(n, p)$; that is, the probability space of simple graphs with vertex-set $\{1, 2, \dots, n\}$, where two distinct vertices are adjacent with probability p, and for distinct pairs these events are mutually independent. Archdeacon and Grable have shown that if $p^2(1 - p^2) \geq 8(\log n)^4/n$, then the (orientable) genus of a random graph in $\mathcal{G}(n, p)$ is $(1 + o(1))pn^2/12$. We prove that for every integer $i \geq 1$, if $n^{-i/(i+1)} \ll p \ll n^{-(i-1)/i}$, then the genus of a random graph in $\mathcal{G}(n, p)$ is $(1 + o(1))\frac{i}{4(i+2)} pn^2$. If $p = cn^{-(i-1)/i}$, where c is a constant, then the genus of a random graph in $\mathcal{G}(n, p)$ is $(1 + o(1))g(i, c, n)pn^2$ for some function $g(i, c, n)$ with $\frac{1}{12} \leq g(i, c, n) \leq 1$, but for $i > 1$ we were unable to compute this function.

ARRANGEABILITY AND CLIQUE SUBDIVISIONS

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ABSTRACT

Let k be an integer. A graph G is *k-arrangeable* (concept introduced by Chen and Schelp) if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every integer i with $1 \leq i \leq n$, at most k vertices among $\{v_1, v_2, \ldots, v_i\}$ have a neighbor $v \in \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ that is adjacent to v_i. We prove that for every integer $p \geq 1$, if a graph G is not p^8-arrangeable, then it contains a K_p-subdivision. By a result of Chen and Schelp this implies that graphs with no K_p-subdivision have "linearly bounded Ramsey numbers," and by a result of Kierstead and Trotter it implies that such graphs have bounded "game chromatic number."

An old question of Erdős and Hajnal

Is it true that for every k and g there exists $\chi = \chi(k, g)$ such that any graph G with $\chi(G) \geq \chi$ contains a subgraph $H \subset G$ with $\chi(H) \geq k$ and $\text{girth}(H) \geq g$?

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R (1977): True for $g = 4$, i.e., triangle-free subgraph H with large $\chi(H)$.

Thomassen's conjecture

Let $d(G) = \frac{2e(G)}{v(G)}$ denote the average degree of G.

Conjecture (1983)

For every k and g there exists $D = D(k, g)$ such that any graph G with $d(G) \geq D$ contains a subgraph $H \subset G$ with $d(H) \geq k$ and $\text{girth}(H) \geq g$.

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Note: trivial to get rid of odd cycles by taking a bipartite subgraph H.

Known results

- Pyber, R, and Szemerédi (1995): true for all graphs G satisfying $\Delta(G) \leq e^{\alpha_{k,g} d(G)}$.

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Alternative proof for $g \leq 6$

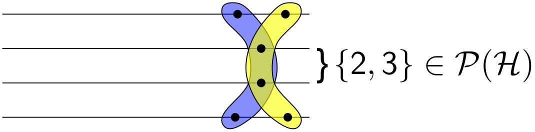
Definition (Intersection Pattern)

For k-partite k-graph $\mathcal{H}$ on $V = V_1 \cup \cdots \cup V_k$, the *intersection pattern* of distinct $e, f \in \mathcal{H}$ is

$$\{i \in [k] \, : \, e \cap V_i = f \cap V_i\}.$$

Let $\mathcal{P}(\mathcal{H})$ denote the set of intersection patterns of all $e \neq f \in \mathcal{H}$.

$J = \{2, 3\}$ is an intersection pattern



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Definition (Star, Kernel)

A hypergraph $\mathcal{S}$ is a *star* with *kernel* K if for all distinct $e, f \in \mathcal{S}$, $e \cap f = K$.

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Definition (Strong)

A k-partite k-graph $\mathcal{H}^*$ is called *t-strong* if for every $J \in \mathcal{P}(\mathcal{H}^*)$ and $e \in \mathcal{H}^*$ there is a star $\mathcal{S} \subset \mathcal{H}^*$, containing e, with kernel e_J and $|\mathcal{S}| = t$.

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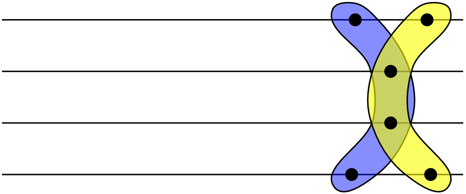

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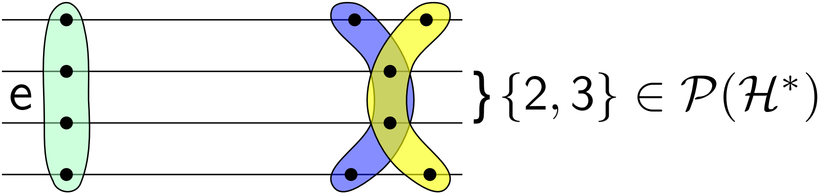

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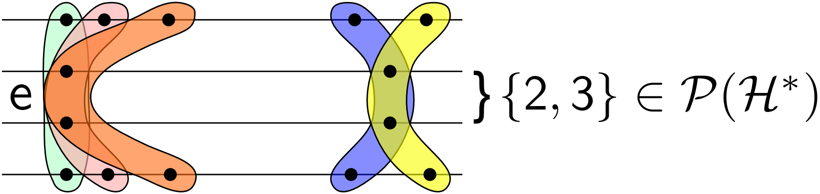
A t-star for $t = 3$



Alternative proof for $g \leq 6$

Lemma (Füredi, 1983)

For every k, t, there exists $c = c(k, t)$ such that any k-graph $\mathcal{H}$ contains a t-strong subhypergraph $\mathcal{H}^*$ with $|\mathcal{H}^*| \geq c\,|\mathcal{H}|$.

A common trick: take a bipartite subgraph

From arbitrary G_0 with $d(G_0) \geq D \gg k$, get $G \subset G_0$ with $\delta = \delta(G) \geq d(G_0)/2$.

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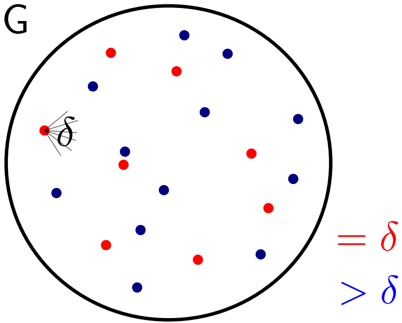

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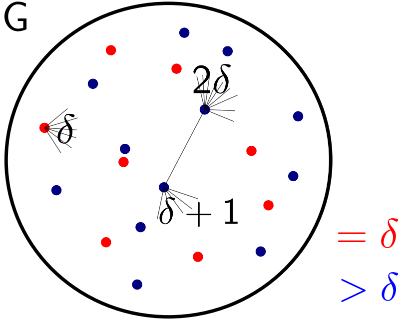

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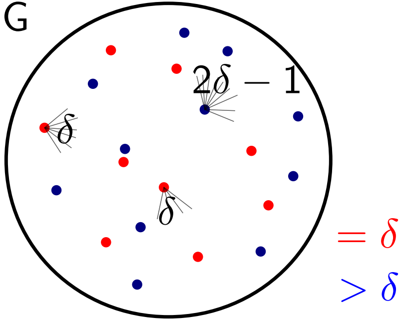

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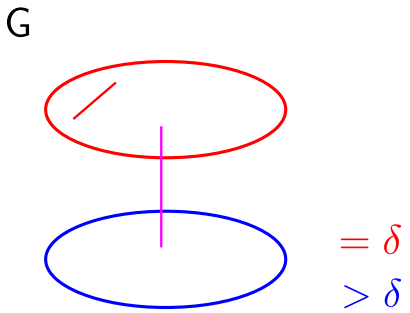
Repeat until no more edge between vertices of degree $> \delta$.



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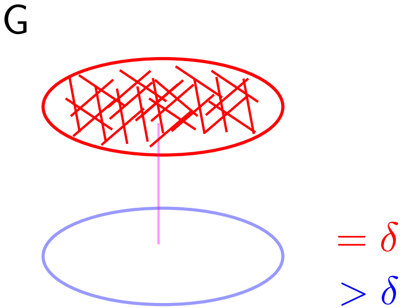

First case: at least half of the edges are red $\implies G_{\text{red}}$ is almost regular, i.e. $\Delta(G_{\text{red}}) \leq \delta = O(d(G_{\text{red}}))$.

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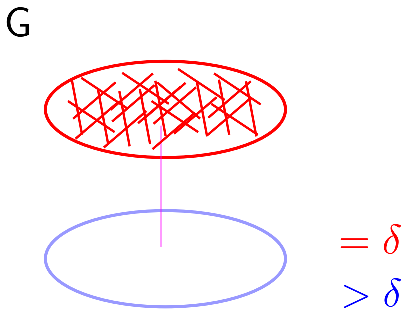

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Pyber-R-Szemerédi: G_{red} contains subgraph H with $d(H) \geq k$ and $\text{girth}(H) \geq g$.

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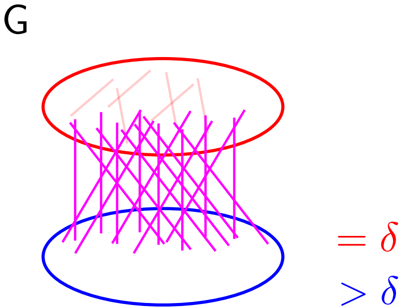

Second case: at least half of the edges are purple (crossing). Then set $A =$ red set, $B =$ blue set and $G = (A, B; E) = G_{\text{purple}}$.

Applying Füredi's Lemma

A little more work yields a bipartite graph $G = (A, B; E)$ where

- $\deg_G(v) = d \gg k$ for all $v \in A$ and,
- $N_G(v) \neq N_G(w)$ for all $v \neq w \in A$.

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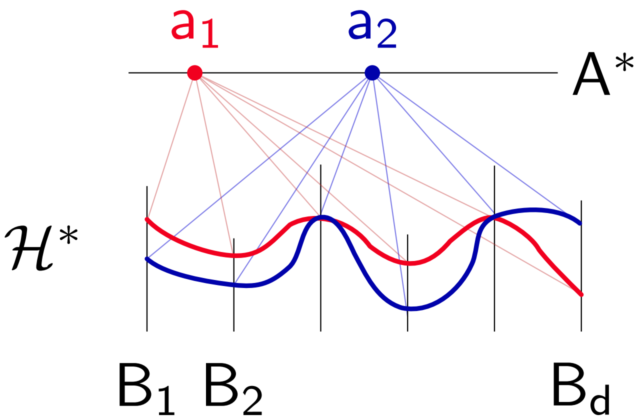
Apply Füredi's Lemma with some large t: $\mathcal{H}^* \subset \mathcal{H}$ is d-partite and t-strong. Note $\mathcal{H}^* \leftrightarrow A^* \subset A$.

The hypergraph from Füredi's Lemma



Dichotomy

Recall $\mathcal{P}(\mathcal{H}^*) \subset 2^{[d]}$ is the family of all intersection patterns of $\mathcal{H}^*$. Let Γ be the shadow graph of $\mathcal{P}(\mathcal{H}^*)$:

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1. Γ has a large independent set I, or
2. $\Delta(\Gamma)$, the max degree of Γ, is large.

Alternative Proof for $g = 6$

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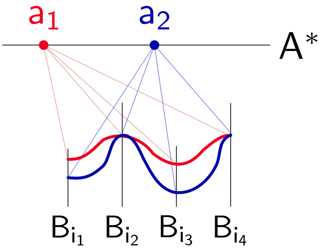

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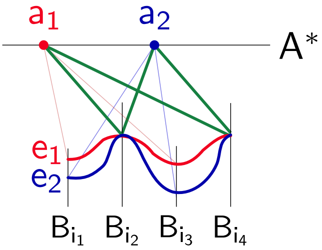
The intersection pattern J of $e_1 \cap e_2$ contains $\{i_2, i_4\} \subset I$. Hence $\{i_2, i_4\} \in \Gamma$, a contradiction!



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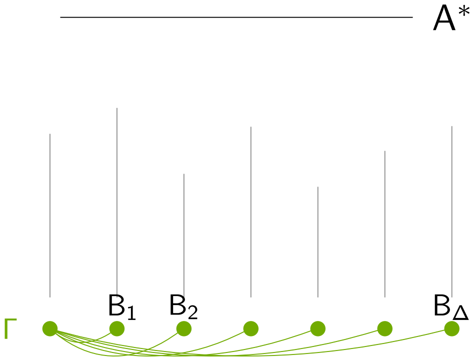

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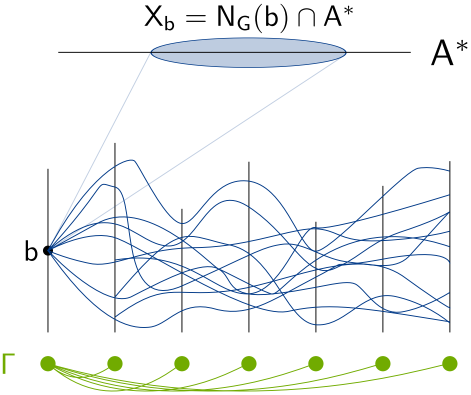

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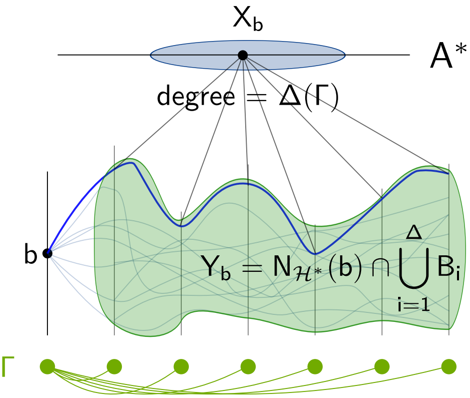

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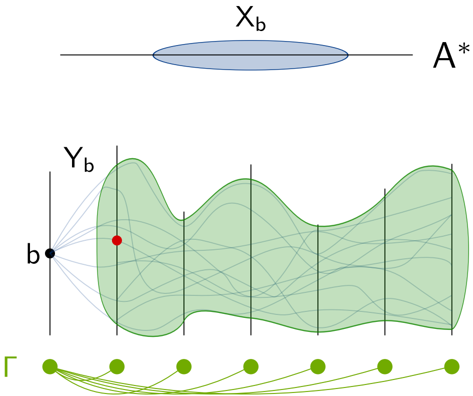

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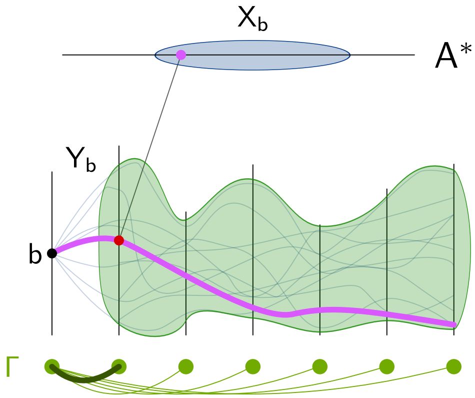

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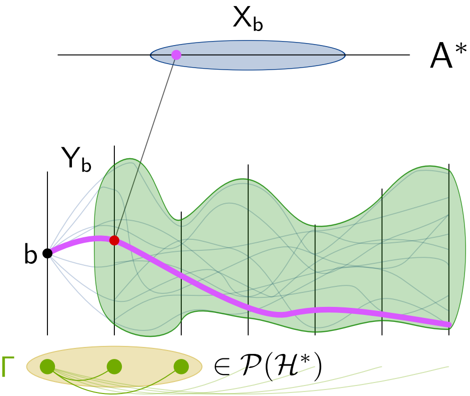

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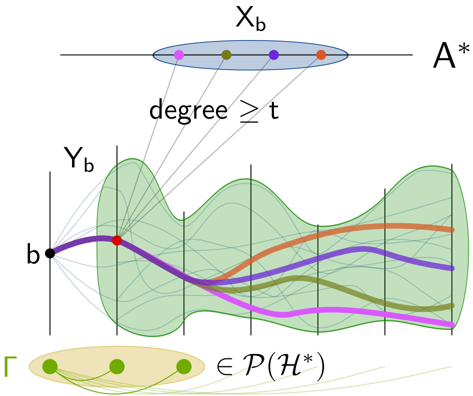

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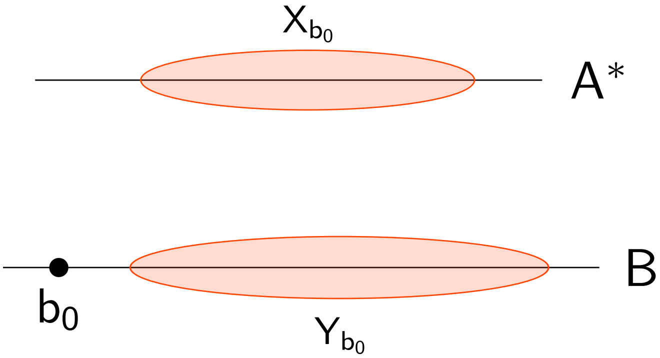

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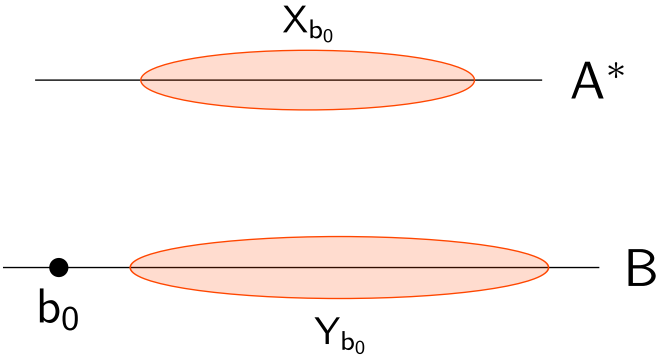

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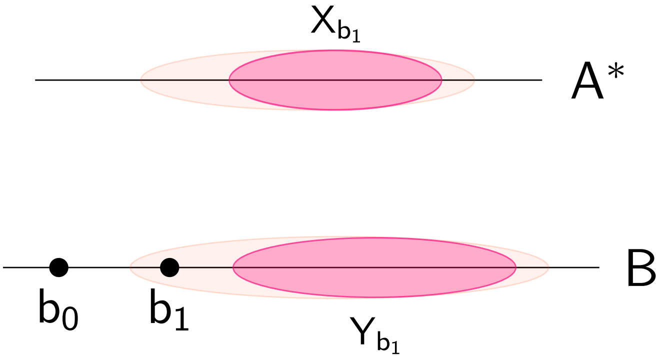

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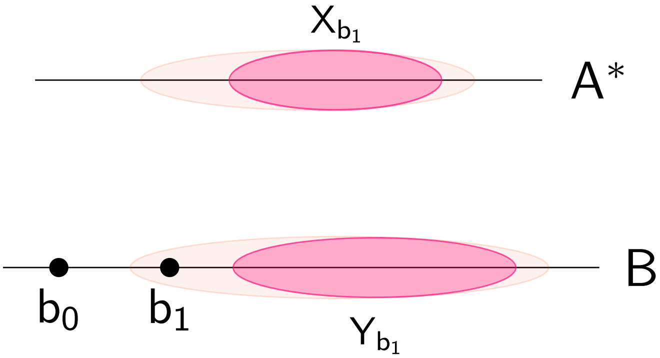

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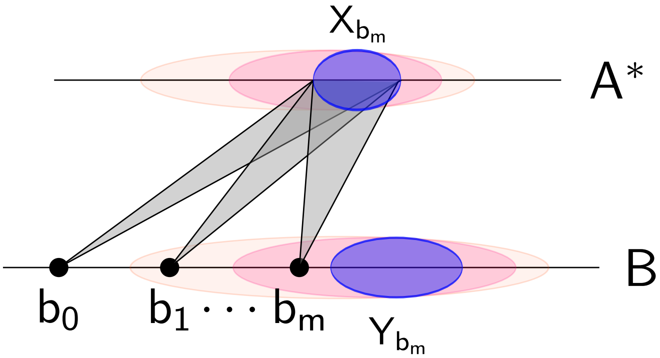

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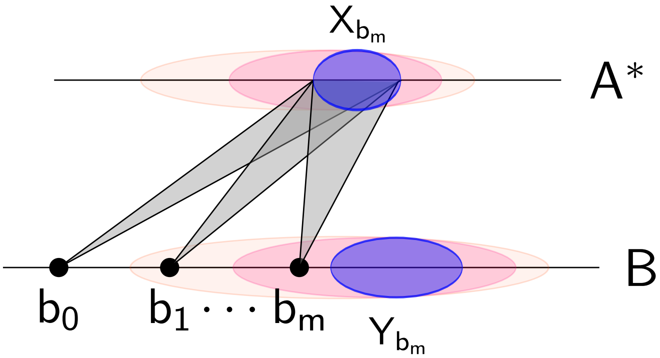

Found a complete bipartite subgraph $G[\{b_0, \ldots, b_m\}, X_{b_m}]$. □

Question

Is there a hypergraph lemma similar to Füredi's that could be used to prove the general case of the conjecture?

More modestly, is there a hypergraph lemma that would allow us to prove the $g = 8$ case?

Graphs with large degree gap

Lemma (Dellamonica–R, 2011)

For all k and g there exist c and d_0 such that the following holds. In any C_4-free bipartite graph $G = (A, B; E)$ satisfying

$$d \geq \max\{c \log\log \Delta(G), d_0\},$$

there exists $H \subset G$ such that $d(H) \geq k$ and $\mathrm{girth}(H) \geq g$.

Graphs with large degree gap

Using the result of Kühn–Osthus (the numerical bounds in DKMR are inferior), yields:

Theorem

For all k and g there exist α, β, and d_0 such that for any graph G with

$$d(G) \geq \max\{\alpha(\log \log \Delta(G))^{\beta}, d_0\}$$

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Recall: Pyber-R-Szemerédi yields same conclusion under stronger assumption

$$d \geq c\log\Delta(G).$$

Proof of our lemma

Assume wlog that $G = (A, B; E)$ is such $\deg_G(v) = d$ for all $v \in A$ and $|B| \leq |A|$.

Take $\varrho = e^{\alpha_{k,g} d}$, $\varepsilon > 0$, and partition $B = B_0 \cup B_1 \cup \cdots \cup B_t$, where

$$B_0 = \left\{ v \in B \;:\; \deg_G(v) \leq \varrho \right\}, \text{ and}$$
$$B_j = \left\{ v \in B \;:\; \varrho^{(1+\varepsilon)^{j-1}} < \deg_G(v) \leq \varrho^{(1+\varepsilon)^j} \right\}, \text{ for } j = 1, \ldots, t.$$

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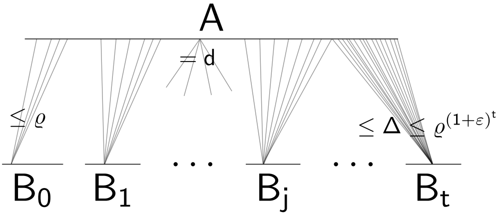
$t = O\big(\log\log \Delta(G)\big)$.
For appropriate choice of $c = c(k, g)$, $\Delta(G) \leq e^{e^{cd}} \implies t < \frac{d}{8k}$.

Proof of our lemma



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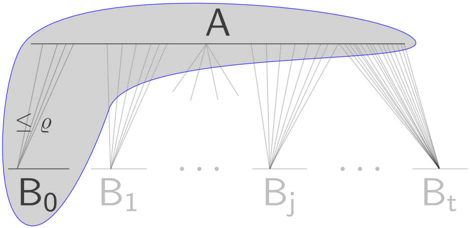

Easy case: $e_G(A, B_0) \geq \frac{e(G)}{2}$. Take $G^* = G[A \cup B_0]$ and note that $\Delta(G^*) \leq \varrho \leq e^{c^* d(G^*)}$.

Pyber–R–Szemerédi's result implies that G^* contains the graph we are looking for.

Proof of our lemma

Otherwise: by averaging, there exists $j \in [t]$ such that

$$e_G(A, B_j) \geq \frac{\sum_{i=1}^{t} e_G(A, B_i)}{t} = \frac{e(G) - e_G(A, B_0)}{t}$$
$$\geq \frac{e(G)}{2t} \geq \frac{d\,|A|}{2\frac{d}{8k}} = 4k\,|A|.$$

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1. Delete all vertices in $B \setminus B_j$.
2. Sequentially delete vertices from A with degree $< k$ and vertices of B_j with degree $< D/4$, where $D = \varrho^{(1+\varepsilon)^{j-1}}$.

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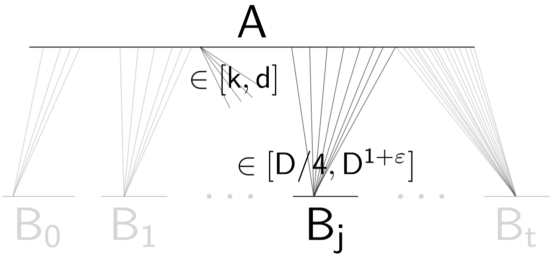
Note: deleted at most $k\,|A| + \frac{D}{4}|B_j| < \frac{e_G(A, B_j)}{2}$ edges.

Proof of our lemma

Result: non-empty graph G with classes A and $B = B_j$ such that

- degrees in A are between k and d;
- degrees in B are between $D/4$ and $D^{(1+\varepsilon)} = \varrho^{(1+\varepsilon)^j}$.



Back to hypergraphs

Let $\mathcal{H}$ be a hypergraph with vertex set B and edges

$$\{N_G(v) \,:\, v \in A\}.$$

Obs. 1: G is C_4-free $\implies |e \cap f| \leq 1$ for all $e, f \in \mathcal{H}$ (i.e., $\mathcal{H}$ is a linear hypergraph).

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Obs. 2: a cycle of length 2ℓ in G corresponds to a cycle of length ℓ in $\mathcal{H}$: i.e., cycle $(v_0, v_1, \ldots, v_{2\ell-1})$ corresponds to $(e_0, e_1, \ldots, e_{\ell-1})$, where $e_i = N_G(v_{2i}) \in \mathcal{H}$.

Cycles in linear hypergraphs

Recall: $\delta(\mathcal{H}) \geq D/4$ and $\Delta(\mathcal{H}) \leq D^{1+\varepsilon}$. Let's count cycles of length ℓ.

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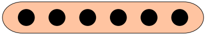

$\# \leq |\mathcal{H}| \times$

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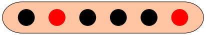

$\# \leq |\mathcal{H}| \times d^2 \times$

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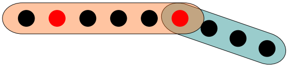

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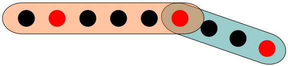

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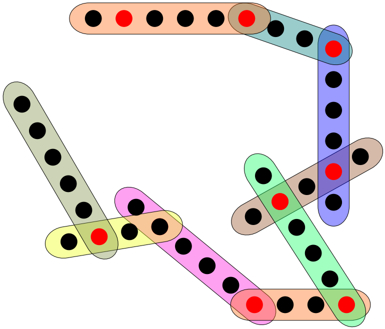

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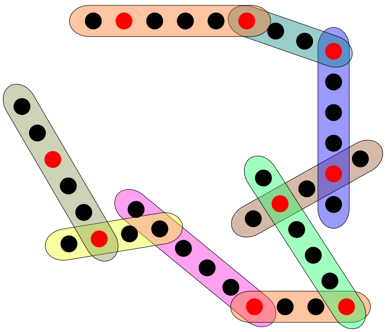

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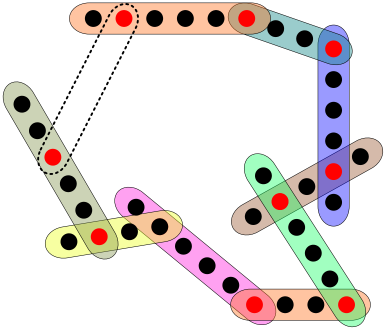
7. at most one possible edge $e_{\ell-1} \ni v_0, v_{2\ell-1}$.



$\# \leq |\mathcal{H}| \times d^{\ell} \times D^{(1+\varepsilon)(\ell-2)} \times 1.$

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Randomly select edges of $\mathcal{H}$ with probability p. How many cycles survive? In expectation, at most

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Delete an edge for each cycle and destroy them all!
Resulting hypergraph $\mathcal{H}^*$ has $\Omega\big(D^\varepsilon\,|V(\mathcal{H})|\big)$ edges and no cycles of length $< g/2$.

Back to graphs

Recall: edges of $\mathcal{H}^*$ correspond to vertices in $A^* \subset A$ $(e \in \mathcal{H}^* \leftrightarrow v \in A^*, e = N_G(v))$.

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Hence, H is the desired subgraph! □

Open questions

- The conjectures of Thomassen and Erdős–Hajnal remain open.
- Is there a hypergraph lemma similar to Füredi's that can be used to prove Thomassen's conjecture for, say, $g = 8$?
- Can one extend the gap between $d(G)$ and $\Delta(G)$ for which we can establish that the conjecture is true?
- Hypergraph version of Thomassen's conjecture: any "degree-gap" result in hypergraphs extends to graphs from our proof. Here the degree gap is polynomial $\Delta(\mathcal{H}) \leq \delta(\mathcal{H})^{1+\varepsilon}$.

Happy Birthday, Robin!!