

# On the invertibility of the XOR of rotations of a binary word

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November 10, 2009

## Abstract

We prove the following result regarding operations on a binary word whose length is a power of two: computing the exclusive-or of a number of rotated versions of the word is an invertible (one-to-one) operation if and only if the number of versions combined is odd.

(This result is not new; there is at least one earlier proof, due to Thomsen in his PhD thesis [12]. Our proof may be new.)

**Keywords:** invertibility, exclusive-or, rotation, binary words, circulant matrix.

## 1 Introduction and proof of main result

This short note considers some simple operations on binary words.

We only consider binary words whose length is a power of two, as this is typically the case for actual computer operations (e.g., with 32-bit or 64-bit words).

We focus on operations based on rotations and exclusive-ors, as these are typically standard built-in operations.

Simple invertible operations such as these are used in many applications, such pseudo-random number generation [7, 9], encryption [4], and cryptographic hash function design [10].

We state and prove the main result, and then provide some related discussion afterwards.

**Theorem 1** *If  $n$  is a power of two,  $v$  is an  $n$ -bit word, and  $r_1, r_2, \dots, r_k$  are distinct fixed integers modulo  $n$ , then the function*

$$R(v) = R(v; r_1, r_2, \dots, r_k) = (v \lll r_1) \oplus (v \lll r_2) \oplus \dots \oplus (v \lll r_k) \quad (1)$$

*is invertible if and only if  $k$  is odd, where  $(v \lll r)$  denotes the  $n$ -bit word  $v$  rotated left by  $r$  positions, and where “ $\oplus$ ” denotes the bit-wise “exclusive-or” of  $n$ -bit words.*

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**Proof:** Let  $V = \{0, 1\}$ , and let  $V^n$  denote the set of all  $n$ -bit words. We identify  $V^n$  with  $GF(2)^n$ , the set of  $n$ -element vectors over the finite field  $GF(2)$ .

With this identification,  $R$  is a linear operation over  $V^n$ ;  $R(v)$  may be obtained by multiplying  $v$  by an  $n \times n$  circulant matrix over  $GF(2)$  having  $k$  ones per row and per column. (An equivalent statement of our theorem is that when  $n$  is a power of two, an  $n \times n$  circulant matrix over  $GF(2)$  is invertible if and only if the number  $k$  of ones in each row is odd.)

We define the Hamming weight (or weight) of an  $n$ -bit word  $v$  to be the number of ones in  $v$ .

Our proof identifies words in  $V^n$  with polynomials in  $GF(2)[x]$  of degree less than  $n$ .

For each  $n$ -bit word  $v$  we define an associated polynomial  $v(x)$  in  $GF(2)[x]$  in the natural way: if

$$v = (v_{n-1}, v_{n-2}, \dots, v_1, v_0)$$

then the associated polynomial  $v(x)$  is

$$v(x) = \sum_{i=0}^{n-1} v_i x^i.$$

For example, the unit-weight word  $u_i$  having a one in position  $i$  is associated with the polynomial  $u_i(x) = x^i$ . This association between words and polynomials is one-to-one.

Let  $f_n(x) = x^n + 1$ , a polynomial in  $GF(2)[x]$ . We now work with polynomials modulo  $f_n(x)$ , so that rotation can be effected by polynomial multiplication modulo  $f_n(x)$ , as is typically done when working with cyclic error-correcting codes (see [6, Section 9.2]) or circulant matrices (see [1]).

Now the word

$$(v \lll r)$$

is associated with the polynomial

$$v(x) * u_r(x) \pmod{f_n(x)} ;$$

reducing modulo  $f_n$  captures the effects of the rotation. In other words, multiplying by  $u_r(x)$  modulo  $f_n(x)$  represents a left-rotation by  $r$  positions.

Computing  $R(v)$  combines the effect of several rotations, so the word  $R(v)$  is associated with the polynomial

$$v(x) * r(x) \pmod{f_n(x)}$$

where

$$r(x) = x^{r_1} + x^{r_2} + \dots + x^{r_k} .$$

Note that  $R$  is an invertible operation if and only if  $r(x)$  is relatively prime to  $f_n(x)$ ; (This result is due to Guan et al. [5, Theorem 2.4]; see also Bini et al. [1, Theorem 2.2].) If  $\gcd(r(x), f_n(x)) = 1$ , then an inverse to  $r(x)$  modulo  $f_n(x)$  can be found by the extended version of Euclid's algorithm, otherwise no inverse exists. These propositions hold whether or not  $n$  is a power of two.

If  $n$  is a power of two, then

$$f_n(x) = x^n + 1 = (x + 1)^n ,$$

since we are working in  $GF(2)$  (see [6, Thm. 1.46]). In this case,  $r(x)$  is relatively prime to  $f_n(x)$  if and only if  $r(x)$  is relatively prime to the polynomial  $x + 1$ .

Polynomials that are *not* relatively prime to  $x + 1$  must be multiples of  $x + 1$ , since  $x + 1$  is irreducible. A polynomial in  $GF(2)[x]$  is a multiple of  $x + 1$  if and only if its value at  $x = 1$  is 0. But  $r(1) = 0$  if and only if  $r(x)$  has an even number of non-zero coefficients. Therefore  $r(x)$  is relatively prime to  $f_n(x)$  if and only if  $k$  is odd.

Thus, when  $n$  is a power of two,  $R$  is an invertible operation on  $GF(2)^n$  if and only if  $k$  is odd. ■

## 2 Discussion

The inverse operation to  $R$  can be found using Euclid’s extended algorithm on input polynomials  $r(x)$  and  $f_n(x)$ , to find polynomials  $s(x)$  and  $t(x)$  such that

$$s(x) \cdot r(x) + t(x) \cdot f_n(x) = 1 .$$

The inverse operation  $S$  to  $R$  corresponds to the polynomial  $s(x)$ , representing another function of the same form as  $R$  (that is, an xor of rotations). In matrix terms, the inverse of a circulant matrix is another circulant matrix.

In terms of computational complexity,  $R(v)$  is easy to compute when  $k$  is small, requiring not more than  $k$  rotations and  $k - 1$  xors. Although the inverse  $S$  has the same form as  $R$ , it may require considerably more work to compute. For example, if  $r(x)$  has degree  $d$ , then  $s(x)$  must have degree at least  $n/d$  and at least  $n/d$  terms, so that evaluating  $S(v)$  requires at least  $\log_2(n/d)$  additions, since each addition in a computation chain can at most double the number of terms. Here multiplication by  $x^f$  (rotations) are “free” and we are only counting exclusive-ors. The exact complexity, in terms of rotations and xors, of evaluating  $R(v)$  or  $S(v)$  may be non-trivial to determine precisely, and we leave these questions as open problems. Thus, when  $k$  and  $d$  are small  $R$  may be considered to be in some sense “very modestly one-way”—easier to compute in one direction than another. Stephen Boyack [3] has interesting related results on the complexity of matrix operations over  $GF(2)$  and their inverses.

Efficient invertible operations are useful in many applications. A linear operation somewhat similar to the one studied here is the “xorshift” operation:

$$v = v \oplus (v \lll r)$$

where “ $\lll$ ” is the “left-shift” operator; xorshift has been used in pseudo-random number generation [7, 9] and hash-function design [10]. Schnorr and Vaudenay [11, Lemma 5] study the related operation

$$(v \wedge d) \oplus (v \lll r)$$

where “ $\wedge$ ” denote bitwise “and” and where  $d$  is a constant  $n$ -bit word; they show that this operation is invertible if and only if the iterates ( $d \lll (r \cdot i)$ ) take for each bit position the value 0 for some  $i$ .

The result of this paper may be useful to those working on similar applications. For example, we began our study of  $R$  when thinking about possible improvements to the MD6 hash function [10]. We also note that the  $k = 3$  version of the operation discussed here is used in the C2 cipher [2] (although not in manner that required its invertibility (it is part of the feedback function in a Feistel block-cipher)), and in the SHA hash function standard message expansion computation [8] (as the  $\Sigma$  function; invertibility of  $\Sigma$  is not claimed or proven).

When  $n$  is not a power of 2, we don’t know of any comparably simple characterization of when  $R(v)$  is invertible, other than the requirement that  $\gcd(f_n(x), r(x)) = 1$ ; perhaps simpler characterizations can be found for some cases, such as when  $n = 3 \cdot 2^k$ .

### 3 Related Work

Lars Knudsen points out that a different proof for the same result is available in the the Ph.D. thesis [12, Theorem 3.3, pages 86–87] of Søren Thomsen. Thomsen’s cute proof considers powers  $R^{2^i}$  of the original operation, notes that

$$R^2(v; r_1, r_2, \dots, r_k) = R(v; 2r_1, 2r_2, \dots, 2r_k)$$

from which it follows that  $R$  is invertible since  $R^n$  will be the identity function (if and only if  $k$  is odd).

### 4 Conclusions

This note provides an alternate proof of a characterization as to when an easily computed operation, based on the exclusive-or of rotated versions of a word, is invertible.

### Acknowledgments

I’d like to thank Lars Knudsen, Niels Ferguson, Louay M. J. Bazzi, and Sanjoy K. Mitter for helpful comments and suggestions.

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