Largeness and SQ-universality of cyclically presented groups

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Abstract

Largeness, SQ-universality, and the existence of free subgroups of rank 2 are measures of the complexity of a finitely presented group. We obtain conditions under which a cyclically presented group possesses one or more of these properties. We apply our results to a class of groups introduced by Prishchepov which contain, amongst others, the various generalizations of Fibonacci groups introduced by Campbell and Robertson. Using the techniques developed we give a new, purely group-theoretic, proof of the (almost complete) classification of the finite Cavicchioli-Hegenbarth-Repovš groups.

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1 Introduction

Let $w = w(x_0, \ldots, x_{n-1})$ be a word in the free group F_n with generators $x_0, \ldots, x_{n-1}$ and let $\theta : F_n \to F_n$ be the automorphism of F_n given by $\theta(x_i) = x_{i+1}$ for each $0 \le i \le n-1$ (subscripts mod n). Define

$$G_n(w) = \langle\, x_0, \ldots, x_{n-1} \,|\, w, \theta(w), \ldots, \theta^{n-1}(w) \,\rangle.$$

Then $G_n(w)$ is said to be a *cyclically presented group* and the above presentation is said to be a *cyclic presentation*.

Cyclically presented groups may be trivial, finite and nontrivial, or infinite. Examples of cyclic presentations of the trivial group are of interest in connection with Andrews-Curtis conjecture [1] and have been researched in [15],[24] and elsewhere. In contrast, papers such as [2], [6], [9], [16], [28], [33], [43] give conditions for a cyclically presented group to be infinite, and in [32] for it to be SQ-universal. The classification of finite cyclically presented groups within certain families is a problem addressed in, for example, [14], [23], [42], [45].

In this paper we consider the "freeness" properties of largeness, SQ-universality, and the existence of free subgroups of rank 2. We investigate these properties both for arbitrary cyclically presented groups $G_n(w)$ and for the following family of groups, introduced and studied by Prishchepov in [33] and investigated further in [13],[40]. Let $n, r, s \ge 1$, $1 \le k \le n$, $0 \le q \le n-1$ and define the *Prischepov group* to be

$$\begin{aligned} P(r,n,k,s,q) &= G_n((x_0x_q \ldots x_{q(r-1)})(x_{(k-1)}x_{(k-1)+q} \ldots x_{(k-1)+q(s-1)})^{-1}) \\ &= \langle\, x_0, \ldots, x_{n-1} \,|\, x_ix_{i+q} \ldots x_{i+q(r-1)} = x_{i+(k-1)}x_{i+(k-1)+q} \ldots x_{i+(k-1)+q(s-1)} \ (0 \le i < n) \,\rangle. \end{aligned}$$

This family contains various other families of cyclically presented groups that have been considered in the literature, starting with *Conway's Fibonacci groups* $F(2,n) = P(2,n,3,1,1)$ of [14]. When $s = 1$

the Prischepov groups coincide with Campbell and Robertson's Fibonacci-type groups $R(r,n,k,h) = P(r,n,(r-1)h+k+1,1,h)$ of [5], which in turn contain the *Fibonacci groups* $F(r,n) = P(r,n,r+1,1,1)$ of [6], the *generalized Fibonacci groups* $F(r,n,k) = P(r,n,r+k,1,1)$ of [7]; the *Sieradski groups* $S(2,n) = P(2,n,2,1,2)$ of [39]; the *Gilbert-Howie* groups $H(n,t) = P(2,n,2,1,t)$ of [23]; the so-called *Cavicchioli-Hegenbarth-Repovš groups* $G_n(m,k) = P(2,n,k+1,1,m)$ which were introduced independently in [11] and [27]. For $s \geq 1$ we have the groups $F(r,n,k,s) = P(r,n,r+k,s,1)$ of [8] which contain the groups $H(r,n,s) = P(r,n,r+1,s,1)$ of [6]; and we have the *generalized Sieradski groups* $S(r,n) = P(r,n,2,r-1,2)$ $(r \geq 2)$ of [10]. We remark that there would be certain advantages in defining $P(r,n,k,s,q)$ to be the group $G_n((x_0x_q\ldots x_{q(r-1)})(x_kx_{k+q}\ldots x_{k+q(s-1)})^{-1})$ (and this was done in [41]) but in order to maintain consistency with [33], and also with [40], we use Prischepov's original definition.

We start by giving some definitions and background material in Section 2. In Section 3 we use free products, epimorphic images, amalgamated free products, and a Freiheitssatz to obtain conditions under which a cyclically presented group $G_n(w)$ is large, SQ-universal, or contains a free subgroup of rank 2. In corollaries we apply these results to the Prishchepov groups. In Section 4 we obtain other basic properties of these groups. In Section 5 we study $P(r,n,k,s,q)$ in greater depth by finding new large epimorphic images and by applying Freiheitssatz results of Shwartz [35].

Except for three groups the finite groups $H(n,t)$ were classified in [23],[30]; one of the outstanding cases was proved infinite in [12]. Except for the two remaining unresolved groups in the family $H(n,t)$ the finite groups $G_n(m,k)$ were classified in [45],[46]. In Section 6 we obtain a new proof of this (almost complete) classification; this proof avoids the algebraic number theory results of [30],[45] that were required in the first proof.

2 Preliminaries

A group G is *large* if it has a finite index subgroup that maps onto the free group of rank 2; G is *SQ-universal* if every countable group can be embedded in a quotient group of G. Any large group is SQ-universal and hence contains a free subgroup of rank 2. Not every SQ-universal group is large however, even within the class of cyclically presented groups: the Higman group $G_4(x_0x_1x_0^{-2}x_1^{-1})$ [25], which was proved to be SQ-universal in [34], has no proper subgroup of finite index and so cannot map onto the free group of rank 2. As is well known, not every group containing a free subgroup of rank 2 is SQ-universal and so we can consider three distinct levels of 'freeness': largeness, SQ-universality, and the existence of free subgroups of rank 2. Each of these properties is preserved when taking finite extensions or finite index subgroups; also, a group that maps onto a group with one of these freeness properties also satisfies that property.

A free product $H * K$ (where H, K are non-trivial) is large if and only if either H or K is large, or H, K have non-trivial finite homomorphic images $\bar{H}, \bar{K}$, not both of order 2 ([31, Theorem 3.7]). An amalgamated free product $H *_L K$ in which $[H:L] \geq 2$, $[K:L] \geq 2$ and $[H:L]+[K:L] \geq 5$ contains a free subgroup of rank 2 (this is well known but see, for example, [4, Lemma 1]); if additionally L is finite then the amalgamated free product is SQ-universal [29].

The automorphism θ of the introduction induces an action of the cyclic group $T = \langle\, t \,|\, t^n \,\rangle$ of order n on the presentation $G_n(w)$. Specifically, $t^{-1}x_it = x_{i+1}$ $(0 \leq i \leq n-1)$ and therefore $t^{-i}x_0t^i = x_i$. Writing $x = x_0$ we see that the split extension of $G_n(w)$ by T has a presentation $E_n(W) = \langle\, x, t \,|\, t^n = W(x,t) = 1 \,\rangle$ where $W(x,t) = x^{\alpha_1}t^{\beta_1}\ldots x^{\alpha_\ell}t^{\beta_\ell}$ (for some $\ell \geq 1$, $1 \leq \beta_i \leq n-1$, $\alpha_i \in \mathbb{Z}\backslash\{0\}$ $(1 \leq i \leq \ell)$) is a rewrite of $w = w(x_0,\ldots,x_{n-1})$. We remark that t has order n in $E_n(W)$ and that if

w is an mth power then $W(x,t)$ is also an mth power.

In the case of a Prischepov group $P(r,n,k,s,q)$ the relator

$$(x_0 x_q \dots x_{q(r-1)})(x_{(k-1)} x_{(k-1)+q} \dots x_{(k-1)+q(s-1)})^{-1}$$

rewrites to $(xt^{-q})^{r-1}xt^{-B}x^{-1}(xt^{-q})^{1-s}t^{A}$, where $A=(k-1)$, $B=(k-1)-q(r-s)$. Setting $y=t^q x^{-1}$ and eliminating x this becomes $y^{-r}t^{-B}y^s t^A$ and so the split extension of $P(r,n,k,s,q)$ by T has a presentation

$$M(r,n,k,s,q) = \langle\, y,t \,|\, t^n=1, y^s t^A = t^B y^r \,\rangle.$$

By the above comments $P(r,n,k,s,q)$ is large, SQ-universal, or contains a free subgroup of rank 2 if and only if $M(r,n,k,s,q)$ is large, SQ-universal, or contains a free subgroup of rank 2, respectively.

3 Free subgroups in cyclically presented groups

3.1 Free product of cyclically presented groups

The following theorem, formalizing a statement made in the introduction of [15], gives conditions under which a cyclically presented group $G_n(w)$ can be expressed as a free product; its corollary gives conditions for it to be large.

Theorem 3.1 *Let w be a word in $x_0,\dots,x_{n-1}$ involving only the m letters $x_{\lambda_0},\dots,x_{\lambda_{m-1}}$ $(0\le\lambda_1<\lambda_2<\dots<\lambda_{m-1}\le n-1)$ so that $w=v(x_{\lambda_0},\dots,x_{\lambda_{m-1}})$. Let $\Delta=(\lambda_0,\dots,\lambda_{m-1},n)$, $N=n/\Delta$ and $\mu_0=\lambda_0/\Delta,\dots,\mu_{m-1}=\lambda_{m-1}/\Delta$. Then $G_n(w)=G_n(v(x_{\lambda_0},\dots,x_{\lambda_{m-1}}))$ is isomorphic to the free product of Δ copies of $G_N(v(x_{\mu_0},\dots,x_{\mu_{m-1}}))$.*

Proof

The group $G_n(v(x_{\lambda_0},\dots,x_{\lambda_{m-1}}))$ has a presentation $\langle\, X \,|\, R\,\rangle$ where

$$\begin{aligned} X &= \{x_i \mid 0\le i\le n-1\},\\ R &= \{v(x_{\lambda_0+i},x_{\lambda_1+i},\dots,x_{\lambda_{m-1}+i}) \mid 0\le i\le n-1, \text{subscripts mod } n\}. \end{aligned}$$

For each $0\le\alpha\le\Delta-1$ set

$$\begin{aligned} X_\alpha &= \{x_i \mid i\equiv\alpha \bmod \Delta, 0\le i\le n-1\}\\ &= \{x_\alpha, x_{\alpha+\Delta},\dots,x_{\alpha+(N-1)\Delta}\},\\ R_\alpha &= \{v(x_{\lambda_0+i},x_{\lambda_1+i},\dots,x_{\lambda_{m-1}+i}) \mid i\equiv\alpha \bmod \Delta, 0\le i\le n-1\}\\ &= \{v(x_{\lambda_0+\alpha},x_{\lambda_1+\alpha},\dots,x_{\lambda_{m-1}+\alpha}), v(x_{\lambda_0+\alpha+\Delta},x_{\lambda_1+\alpha+\Delta},\dots,x_{\lambda_{m-1}+\alpha+\Delta}),\dots,\\ &\qquad v(x_{\lambda_0+\alpha+(N-1)\Delta},x_{\lambda_1+\alpha+(N-1)\Delta},\dots,x_{\lambda_{m-1}+\alpha+(N-1)\Delta})\}. \end{aligned}$$

Then R_α is a set of words involving only elements of X_α and the X_α form a partition of X and the R_α form a partition of R. Hence

$$\langle\, X \,|\, R\,\rangle \cong \langle\, X_0 \,|\, R_0\,\rangle * \dots * \langle\, X_{N-1} \,|\, R_{N-1}\,\rangle.$$

Fix a value of α $(0\le\alpha\le N-1)$ and set $y_0=x_\alpha, y_1=x_{\alpha+\Delta},\dots,y_{(N-1)}=x_{\alpha+(N-1)\Delta}$. Then $X_\alpha=\{y_0,\dots,y_{N-1}\}$ and $x_{\lambda_0+\alpha}=y_{\mu_0}, x_{\lambda_1+\alpha}=y_{\mu_1},\dots,x_{\lambda_{m-1}+\alpha}=y_{\mu_{m-1}}$ so R_α is the set

$$\{v(y_{\mu_0},y_{\mu_1},\dots,y_{\mu_{m-1}}), v(y_{\mu_0+1},y_{\mu_1+1},\dots,y_{\mu_{m-1}+1}),\dots,v(y_{\mu_0+(N-1)},y_{\mu_1+(N-1)},\dots,y_{\mu_{m-1}+(N-1)})\}.$$

Thus $\langle\, X_\alpha \,|\, R_\alpha\,\rangle \cong G_N(v(y_{\mu_0},y_{\mu_1},\dots,y_{\mu_{m-1}}))$ which (by relabeling) is $G_N(v(x_{\mu_0},x_{\mu_1},\dots,x_{\mu_{m-1}}))$ and the result follows. □

Corollary 3.2 *With the above notation let* $G = G_n(v(x_{\lambda_0}, \ldots, x_{\lambda_{m-1}}))$, $H = G_N(v(x_{\mu_0}, \ldots, x_{\mu_{m-1}}))$ *and suppose* $\Delta \geq 2$, $H \neq 1$. *Then* G *is large unless* $\Delta = 2$ *and* $H \cong \mathbb{Z}_2$, *in which case* $G \cong D_\infty$.

Now $|r - s|$ divides the determinant of the relation matrix of $P(r, n, k, s, q)$ and so it divides $|P(r, n, k, s, q)|$. Using this and applying Theorem 3.1 and Corollary 3.2 to Prischepov groups we have

Corollary 3.3 *Let* $\Delta = (n, k-1, q)$ *when* $r + s \geq 3$ *and let* $\Delta = (n, k-1)$ *when* $r = s = 1$. *Then* $P = P(r, n, k, s, q)$ *is isomorphic to the free product of* Δ *copies of* $H = P(r, N, K, s, Q)$ *where* $N = n/\Delta$, $Q = q/\Delta$, $K = (k-1)/\Delta + 1$. *If* $H \neq 1$, $\Delta \geq 2$ *then* P *is large unless* $H \cong \mathbb{Z}_2$ *and* $\Delta = 2$, *in which case* $P \cong D_\infty$. *In particular, if* $\Delta \geq 2$ *and* $|r - s| \neq 1$ *then* P *is large unless* $\Delta = 2$ *and* $|r - s| = 2$.

In particular we recover a result about the groups $R(r, n, k, h)$.

Corollary 3.4 ([5, Theorem 4]) *The group* $R(r, n, k, h)$ *is isomorphic to the free product of* (n, k, h) *copies of* $R(r, N, K, H)$ *where* $N = n/(n, k, h)$, $K = k/(n, k, h)$, $H = h/(n, k, h)$.

In particular, if $(n, k, h) > 1$ then $R(r, n, k, h)$ is large unless $R(r, N, K, H) = 1$ or $((n, k, h) = 2$, $r \leq 3$ and $R(r, N, K, H) \cong \mathbb{Z}_2)$. This corollary in turn contains a result about the groups $G_n(m, k)$.

Corollary 3.5 ([2, Lemma 1.2]) *The group* $G_n(m, k)$ *is isomorphic to the free product of* (n, m, k) *copies of* $G_N(M, K)$ *where* $N = n/(n, m, k)$, $M = m/(n, m, k)$, $K = k/(n, m, k)$.

In particular, if $(n, m, k) > 1$ then $G_n(m, k)$ is large unless $G_N(M, K) = 1$ or $((n, m, k) = 2$ and $G_N(M, K) \cong \mathbb{Z}_2)$.

3.2 Epimorphic images

If a group G maps homomorphically onto a large group, or onto a group that contains a free subgroup of rank 2 then G is large, or contains a free subgroup of rank 2, respectively. Our method of proof in this section is to find suitable epimorphic images of $E_n(W)$.

It was determined in [19] when the group $\langle\, x, t \,|\, t^n = W(x, t)^m = 1 \,\rangle$ $(m \geq 2)$ contains a free subgroup of rank 2 or is infinite and soluble. (Actually, it also gives conditions under which the group contains a Ree-Mendelsohn pair – see [19] for the definition – or is infinite and soluble.) Combining that theorem with [3] we can prove the following related result.

Theorem 3.6 *Let* $E_n(W) = \langle\, x, t \,|\, t^n = V(x, t)^m = 1 \,\rangle$ *where* $n, m \geq 2$ *and* $V(x, t) = x^{\alpha_1} t^{\beta_1} \ldots x^{\alpha_\ell} t^{\beta_\ell}$, $\ell \geq 1$, $1 \leq \beta_i \leq n - 1$, $\alpha_i \in \mathbb{Z} \backslash \{0\}$ $(1 \leq i \leq \ell)$.

- *(a) If* $n + m \geq 5$ *then* $E_n(W)$ *is large;*
- *(b) if* $n = m = 2$ *then* $E_n(W)$ *contains a free subgroup of rank 2 unless* $\ell = 1$ *and* $\alpha_1 \leq 2$, *in which case* $E_n(W)$ *is infinite and soluble.*

Proof

If $n + m \geq 5$ then choose $k \in \mathbb{N}$ with $k > \max\{6, |\alpha_1|, \ldots, |\alpha_\ell|\}$. Then $E_n(W)$ maps onto the group $\langle\, x, t \,|\, x^k = t^n = V(x, t)^m = 1 \,\rangle$ which is large by [3] since $1/k + 1/n + 1/m < 1$. If $n = m = 2$ then the result was proved in [19, Theorem 4] (see also [20, Theorem 8] or [21, Theorem 7.3.3.1]). □

Theorem 3.7 *Let $E_n(W) = \langle\, x,t \,|\, t^n = W(x,t) = 1 \,\rangle$ where $W(x,t) = x^{\alpha_1}t^{\beta_1} \ldots x^{\alpha_\ell}t^{\beta_\ell}$, $\ell \geq 1$, $1 \leq \beta_i \leq n-1$, $\alpha_i \in \mathbb{Z}\backslash\{0\}$ $(1 \leq i \leq \ell)$.*

(a) If $(\beta_1,\ldots,\beta_\ell,n) \geq 2$ and $|\sum_{i=1}^{\ell}\alpha_i| \neq 1$ then $E_n(W)$ is large except possibly when $(\beta_1,\ldots,\beta_\ell,n) = 2$ and $|\sum_{i=1}^{\ell}\alpha_i| = 2$, in which case $E_n(W)$ is infinite.

(b) If $(n,\sum_{i=1}^{\ell}\beta_i) \geq 2$ and $(\alpha_1,\ldots,\alpha_\ell) \geq 2$ then $E_n(W)$ is large except possibly when $(n,\sum_{i=1}^{\ell}\beta_i) = 2$ and $(\alpha_1,\ldots,\alpha_\ell) = 2$, in which case $E_n(W)$ is infinite.

Proof
For (a) observe that the group $E_n(W)$ maps onto $\langle\, x,t \,|\, t^{(\beta_1,\ldots,\beta_\ell,n)} = x^{|\sum_{i=1}^{\ell}\alpha_i|} = 1 \,\rangle \cong \mathbb{Z}_{(\beta_1,\ldots,\beta_\ell,n)} * \mathbb{Z}_{|\sum_{i=1}^{\ell}\alpha_i|}$ and for (b) that it maps onto $\langle\, x,t \,|\, t^{(n,\sum_{i=1}^{\ell}\beta_i)} = x^{(\alpha_1,\ldots,\alpha_\ell)} = 1 \,\rangle \cong \mathbb{Z}_{(n,\sum_{i=1}^{\ell}\beta_i)} * \mathbb{Z}_{(\alpha_1,\ldots,\alpha_\ell)}$. $\square$

Corollary 3.8 *(a) If $(n,A,B) \geq 2$ and $|r-s| \neq 1$ then $M(r,n,k,s,q)$ is large except possibly when $(n,A,B) = 2$ and $|r-s| = 2$, in which case it is infinite.*

(b) If $(n,A-B) \geq 2$ and $(r,s) \geq 2$ then $M(r,n,k,s,q)$ is large except possibly when $(n,A-B) = 2$ and $(r,s) = 2$, in which case it is infinite.

As an immediate corollary we get

Corollary 3.9 ([44, Theorems 1 and 2]) *If $(r,n) > 1$ then $F(r+1,n,0)$ is infinite.*

3.3 Amalgamated free products and the Freiheitssatz

The following theorem uses the fact that the group $E_n(W) = \langle\, x,t \,|\, t^n = W(x,t) = 1 \,\rangle$ can sometimes be expressed as an amalgamated free product, possibly with the amalgamation over a finite group, to prove SQ-universality of $E_n(W)$ or the existence of a free subgroup of rank 2. Since the split extension of any cyclically presented group $G_n(w)$ is of the form $E_n(W)$ the theorem can be used to prove SQ-universality of $G_n(w)$ or the existence of a free subgroup of rank 2.

Theorem 3.10 *Let $E_n(W) = \langle\, x,t \,|\, t^n = W(x,t) = 1 \,\rangle$ where $n \geq 2$ and $W(x,t) = x^{\alpha_1}t^{\beta_1} \ldots x^{\alpha_\ell}t^{\beta_\ell}$, $\ell \geq 1$, $1 \leq \beta_i \leq n-1$, $\alpha_i \in \mathbb{Z}\backslash\{0\}$ $(1 \leq i \leq \ell)$, and suppose x has infinite order and t has order n in $E_n(W)$.*

(a) If $(\alpha_1,\ldots,\alpha_\ell) \geq 2$, $n \geq 3$ then $E_n(W)$ contains a free subgroup of rank 2.

(b) If $(\beta_1,\ldots,\beta_\ell,n) \geq 2$ then $E_n(W)$ is SQ-universal.

In particular, if $n \geq 3$ and $E_n(W)$ does not contain a free subgroup of rank 2 then $(\alpha_1,\ldots,\alpha_\ell) = 1$ and $(\beta_1,\ldots,\beta_\ell,n) = 1$.

Proof
(a) Let $a = (\alpha_1,\ldots,\alpha_\ell)$, $\gamma_i = \alpha_i/a$ $(1 \leq i \leq \ell)$. Then $E_n(W) \cong H *_L K$ where

$$H = \langle\, x^a, t \,|\, t^n = (x^a)^{\gamma_1}t^{\beta_1} \ldots (x^a)^{\gamma_\ell}t^{\beta_\ell} = 1 \,\rangle,$$

$K = \langle\, x \,|\, \rangle$, $L = \langle\, x^a \,|\, \rangle$. Now $[K:L] = a \geq 2$. If $[H:L] = 1$ or 2 then $H \cong \mathbb{Z}$ or D_∞. But t has order $n \geq 3$ in H so $H \not\cong D_\infty$. Further, $H \not\cong \mathbb{Z}$ since t has order n. Thus $[H:L] \geq 3$.

(b) Let $b = (\beta_1, \ldots, \beta_\ell, n)$, $\delta_i = \beta_i/b$ $(1 \leq i \leq \ell)$, $N = n/b$. Then $E_n(W) \cong H *_L K$ where $H = \langle\, x, t^b \,|\, (t^b)^N = x^{\alpha_1}(t^b)^{\delta_1} \ldots x^{\alpha_\ell}(t^b)^{\delta_\ell} = 1 \,\rangle$, $K = \langle\, t \,|\, t^n \,\rangle$, $L = \langle\, t^b \,|\, (t^b)^N \,\rangle$. Now $[K : L] = b \geq 2$ and L has infinite index in H since L is finite and H is infinite. □

If $E_n(W)$ arises as a split extension of the group $G_n(w)$, as explained in Section 2, then t has order n in $E_n(W)$; x will not always have infinite order of course. When $\sum_{i=1}^{\ell} \alpha_i = 0$, however, there is an epimorphism $E \twoheadrightarrow \mathbb{Z}$ given by $t \mapsto 0, x \mapsto 1 \in \mathbb{Z}$ and so x has infinite order in $E_n(W)$.

We now consider the hypothesis "x has infinite order and t has order n in $E_n(W)$" in more detail. A one-relator product $G = (H * K)/ << R >>$ (where $<< R >>$ denotes the normal closure of R in $H * K$) is said to satisfy the *Freiheitssatz* if the natural homomorphisms $H \to G$, $K \to G$ are both embeddings. The Freiheitssatz for one-relator products has been considered in many papers – see [17],[18],[26],[38] and the references therein. Setting $H = \langle\, x \,|\, \rangle \cong \mathbb{Z}$, $K = \langle\, t \,|\, t^n \,\rangle \cong \mathbb{Z}_n$, $R = W(x,t)$ we see that $E_n(W) = (H * K)/ << R >>$. Clearly the Freiheitssatz holds here if and only if x has infinite order and t has order n in $E_n(W)$. Thus we can re-express Theorem 3.10 as

Theorem 3.10′ *Let $E_n(W) = (H * K)/ << R >>$ where $H = \langle\, x \,|\, \rangle \cong \mathbb{Z}$, $K = \langle\, t \,|\, t^n \,\rangle \cong \mathbb{Z}_n$, $R = W(x,t)$ where $n \geq 2$ and $W(x,t) = x^{\alpha_1}t^{\beta_1} \ldots x^{\alpha_\ell}t^{\beta_\ell}$, $\ell \geq 1$, $1 \leq \beta_i \leq n-1$, $\alpha_i \in \mathbb{Z}\backslash\{0\}$ $(1 \leq i \leq \ell)$, and suppose that the Freiheitssatz holds.*

(a) If $(\alpha_1, \ldots, \alpha_\ell) \geq 2$, $n \geq 3$ then $E_n(W)$ contains a free subgroup of rank 2.

(b) If $(\beta_1, \ldots, \beta_\ell, n) \geq 2$ then $E_n(W)$ is SQ-universal.

In particular, if $n \geq 3$ and $E_n(W)$ does not contain a free subgroup of rank 2 then $(\alpha_1, \ldots, \alpha_\ell) = 1$ and $(\beta_1, \ldots, \beta_\ell, n) = 1$.

Applying this to Prischepov groups we have

Corollary 3.11 *Let $M = M(r,n,k,s,q)$ where $n \geq 2$. Then M is the one-relator product $(H*K)/ << R >>$ where $\{H, K\} = \{\langle\, x \,|\, \rangle, \langle\, t \,|\, t^n \,\rangle\}$, $R = y^s t^A y^{-r} t^{-B}$ where $A = (k-1)$, $B = (k-1) - q(r-s)$. Suppose that the Freiheitssatz holds.*

(a) If $(r,s) \geq 2$, $n \geq 3$ then M contains a free subgroup of rank 2.

(b If $(A,B,n) \geq 2$ then M is SQ-universal.

In particular, if $n \geq 3$ and M does not contain a free subgroup of rank 2 then $(r,s) = 1$ and $(A,B,n) = 1$.

(We remark that alternative forms of the Freiheitssatz for cyclically presented groups and their extensions have been considered in [16],[28].)

4 Basic properties of Prishchepov groups

We first note some isomorphisms amongst the groups $P(r,n,k,s,q)$.

Lemma 4.1 $P(r,n,k,s,q) \cong P(r',n,k',s',q)$, *where $r' = s, s' = r, k' = n - k + 2$.*

Proof

Let r', s', k' be as stated and for each $0 \leq i \leq n-1$, set $j = i + (k-1) \bmod n$. Then the relators of $P(r,n,k,s,q)$, namely $(x_i x_{i+q} \ldots x_{i+q(r-1)})(x_{i+(k-1)} x_{i+(k-1)+q} \ldots x_{i+(k-1)+q(s-1)})^{-1}$ become $(x_{j+(k'-1)} x_{j+(k'-1)+q} \ldots x_{j+(k'-1)+q(r-1)})(x_j x_{j+q} \ldots x_{j+q(s-1)})^{-1}$. Inverting these we get the relators $(x_j x_{j+q} \ldots x_{j+q(s-1)})(x_{j+(k'-1)} x_{j+(k'-1)+q} \ldots x_{j+(k'-1)+q(r-1)})^{-1}$ which are the relators of $P(r', n, k', s', q)$. □

Thus the roles of r, s may be interchanged. For $P(r,n,k,s,q)$ we have $A = (k-1)$, $B = (k-1) - q(r-s)$; the corresponding values for $P(r', n, k', s', q)$ are $A' = (k'-1) \equiv -A \bmod n$, $B' = (k'-1) - q(r'-s') \equiv -B \bmod n$ – that is, A and B are negated (mod n).

Lemma 4.2 *(i)* $P(r,n,k,s,q) \cong P(r,n,k-q(r-s),s,n-q)$;

(ii) $P(r,n,k,s,q) \cong P(s,n,k-q(r-s),r,q)$.

Proof

(i) Setting $y_i = x_i^{-1}$ ($0 \leq i \leq n-1$, subscripts mod n) the relators

$$(x_i x_{i+q} \ldots x_{i+q(r-2)} x_{i+q(r-1)})(x_{i+(k-1)} x_{i+(k-1)+q} \ldots x_{i+(k-1)+q(s-2)} x_{i+(k-1)+q(s-1)})^{-1}$$

of $P(r,n,k,s,q)$ become

$$(y_i^{-1} y_{i+q}^{-1} \ldots y_{i+q(r-2)}^{-1} y_{i+q(r-1)}^{-1})(y_{i+(k-1)}^{-1} y_{i+(k-1)+q}^{-1} \ldots y_{i+(k-1)+q(s-2)}^{-1} y_{i+(k-1)+q(s-1)}^{-1})^{-1}$$

which is a cyclic permutation of

$$(y_{i+(k-1)+q(s-1)} y_{i+(k-1)+q(s-2)} \ldots y_{i+(k-1)+q} y_{i+(k-1)})(y_{i+q(r-1)} y_{i+q(r-2)} \ldots y_{i+q} y_i)^{-1}. \qquad (1)$$

Inverting gives

$$(y_{i+q(r-1)} y_{i+q(r-2)} \ldots y_{i+q} y_i)(y_{i+(k-1)+q(s-1)} y_{i+(k-1)+q(s-2)} \ldots y_{i+(k-1)+q} y_{i+(k-1)})^{-1}$$

and then setting $j = i + q(r-1) \bmod n$ (for each $0 \leq i \leq n-1$) these become

$$(y_j y_{j+(n-q)} \ldots y_{j+(r-2)(n-q)} y_{j+(r-1)(n-q)})$$
$$(y_{j+(k-1)-q(r-s)} y_{j+(k-1)-q(r-s)+(n-q)} \ldots y_{j+(k-1)-q(r-s)+(s-2)(n-q)} y_{j+(k-1)-q(r-s)+(s-1)(n-q)})^{-1}$$

which are the relators of $P(r,n,k-q(r-s),s,n-q)$.

(ii) Setting $z_i = y_{-i}$ ($0 \leq i \leq n-1$, subscripts mod n) in (1) we get

$$(z_{-i-(k-1)-q(s-1)} z_{-i-(k-1)-q(s-2)} \ldots z_{-i-(k-1)-q} z_{-i-(k-1)})(z_{-i-q(r-1)} z_{-i-q(r-2)} \ldots z_{-i-q} z_{-i})^{-1}.$$

Letting $j = -i - (k-1) - q(s-1) \bmod n$ (for each $0 \leq i \leq n-1$) these become

$$(z_j z_{j+q} \ldots z_{j+q(s-2)} z_{j+q(s-1)})$$
$$(z_{j+(k-1)+q(s-r)} z_{j+(k-1)+q(s-r)+q} \ldots z_{j+(k-1)+q(s-r)+q(r-2)} z_{j+(k-1)+q(s-r)+q(r-1)})^{-1}.$$

These are the relators of $P(s,n,k-q(r-s),r,q)$ so the proof is complete. □

For $P(r,n,k,s,q)$ we have $A = (k-1)$, $B = (k-1) - q(r-s)$; the corresponding values for the isomorphic copy of $P(r,n,k,s,q)$ (in either (i) or (ii)) are $A' = B$, $B' = A$ so the roles of A, B may also be interchanged. Thus while part (ii) interchanges the roles of r, s we now have a different effect on A, B than that obtained when we use Lemma 4.1.

Corollary 4.3 ([2, Lemma 1.1(3)]) $G_n(m,k) \cong G_n(n-m, n+(k-m))$.

Applying the technique used in [5, Lemma 2] more generally we have

Theorem 4.4 *Let* $(\alpha, n) = 1$. *Then* $G_n(w(x_0, x_1, \ldots, x_{n-1})) \cong G_n(w(x_0, x_\alpha, \ldots, x_{\alpha(n-1)}))$. *In particular if* $(q,n) = 1$ *then* $P(r,n,k,s,q) \cong P(r,n,(k-1)Q+1,s,1)$ *where* $qQ \equiv 1$ *mod* n.

Proof
Let a satisfy $a\alpha \cong 1$ mod n and for each $0 \le j \le n-1$ set $i = aj$ mod n, so $j \equiv \alpha i$ mod n and define $y_i = x_{\alpha i}$ $(0 \le i \le n-1$, subscripts mod $n)$. Then the set of generators of $G_n(w(x_0, x_\alpha, \ldots, x_{\alpha(n-1)}))$ $\{x_0, x_1, \ldots, x_{n-1}\} = \{y_0, y_1, \ldots, y_{n-1}\}$ and the set of relators

$$\{w(x_j, x_{j+\alpha}, \ldots, x_{j+(n-1)\alpha}) \mid 0 \le j \le n-1\} = \{w(y_i, y_{i+1}, \ldots, y_{i+(n-1)}) \mid 0 \le i \le n-1\}$$

and the result follows. □

As a corollary we of course recover [5, Lemma 2] which states that $R(r,n,k,h) \cong R(r,n,\alpha k,\alpha h)$ for any $(\alpha, n) = 1$. This in turn implies the following, which we record for later use.

Corollary 4.5 ([2, Lemma 1.3]) *(i) If* $(n,k) = 1$ *then* $G_n(m,k) \cong G_n(t,1) = H(n,t)$ *where* $tk = m$ *mod* n.

(ii) If $(n, k-m) = 1$ *then* $G_n(m,k) \cong G_n(t,1) = H(n,t)$ *where* $t(k-m) = n-m$ *mod* n.

Let $P = P(r,n,k,s,q)$. If $A \equiv 0$ mod n then $k \equiv 1$ mod n so $P = P(r,n,1,s,q)$; if $B \equiv 0$ mod n then $k - q(r-s) \equiv 1$ mod n and using the equivalent presentation $P(s,n,k-q(r-s),r,q)$ (of Lemma 4.2(ii)) we see that $P \cong P(s,n,1,r,q)$. Furthermore, a direct consideration of the cyclic presentation shows that $P(r,n,1,s,q) = P(|r-s|+1,n,1,1,q)$. We can classify when these groups are large:

Theorem 4.6 *Let* $P = P(r,n,1,1,q)$ *with* $r \ge 1$ *and let* $d = (n,(r-1)q)$.

(a) If $r = 1$ *then* $P \cong \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{n}$;

(b) if $r = 2$ *then* $P = 1$;

(c) if $d = 1$ *then* $P \cong \mathbb{Z}_{r-1}$;

(d) if $r = 3$ *and* $d = 2$ *then* $P \cong \begin{cases} D_\infty & \text{if } (n,q) = 2 \text{ and } n = 2 \text{ mod } 4, \\ \mathbb{Z} & \text{if } (n,q) = 1; \end{cases}$

(e) if $r \ge 4$ *or* $d \ge 3$ *then* P *is large.*

Proof
The cases $r = 1$ and $r = 2$ (parts (a) and (b)) are immediate by considering the cyclic presentation.

Suppose $d = 1$. Then $(n,q) = 1$ so, by Theorem 4.4, $P \cong P(r,n,1,1,1) = G_n(x_0x_1\ldots x_{r-2})$. Moreover $(n, r-1) = 1$ so by [44, Theorem 3] $G_n(x_0x_1\ldots x_{r-2}) \cong \mathbb{Z}_{r-1}$, proving part (c). Suppose then that $r \ge 3$ and $d \ge 2$.

If $r \ge 4$ or $d \ge 3$ then Corollary 3.8 implies that P is large so assume that $r = 3$ and $d = 2$ (i.e. $(n,2q) = 2$). Now Corollary 3.3 implies that P is isomorphic to (n,q) copies of $G =$

$P(3,N,1,1,Q)$ where $N=n/(n,q)$, $Q=q/(n,q)$, and since $(Q,N)=1$ Theorem 4.4 implies that $G\cong P(3,N,1,1,1)=G_N(x_0x_1)$. Eliminating generators x_N,x_{N-1} shows that $G_N(x_0x_1)\cong G_{N-2}(x_0x_1)$ and it is clear that $G_3(x_0x_1)\cong\mathbb{Z}_2$ and $G_2(x_0x_1)\cong\mathbb{Z}$ so $G_N(x_0x_1)\cong\mathbb{Z}$ when N is even and $G_N(x_0x_1)\cong\mathbb{Z}_2$ when N is odd. If $(n,q)=2$ then $n\equiv 2 \bmod 4$, since $(n,2q)=2$, so N is odd and hence $P\cong\mathbb{Z}_2*\mathbb{Z}_2\cong D_\infty$. If $(n,q)=1$ then $n=N$ so $P\cong\mathbb{Z}$ since n is even. □

5 Free subgroups in Prishchepov groups

5.1 Largeness

In this section we extend ideas that were first used in [9]. As in Section 3 we prove largeness by finding a large epimorphic image. When we consider (the split extension of) Prishchepov groups $P(r,n,k,s,q)$, rather than arbitrary cyclically presented groups, there is a new epimorphic image that we can use. Let $d=(n,A+B)$; then by killing t^d we see that $M(r,n,k,s,q)$ maps onto

$$N=\langle\, y,t \,|\, t^d=1, (y^st^A)^2=y^{r+s}\,\rangle.$$

(Note that $N=M(r,d,(r-s)q/2+1,s,q)$.) The group N in turn maps onto the generalized triangle group $\langle\, y,t \,|\, y^{r+s}=t^d=(y^st^A)^2=1\,\rangle$.

Let $G(l,m,n)=\langle\, a,b \,|\, a^l=b^m=(a^\alpha b^\beta)^n=1\,\rangle$. In [19, Theorem 6] (see also [20, Theorem 2],[21, Theorem 7.3.2.2]) it was determined when $G(l,m,n)$ contains a free subgroup of rank 2, is infinite and soluble, or is finite; independently in [9, Theorem 2.5] the finite groups $G(l,m,n)$ were classified. Refining these results slightly we can classify when $G(l,m,n)$ is large, infinite and soluble, or finite.

Theorem 5.1 *Let $G=\langle\, a,b \,|\, a^l=b^m=(a^\alpha b^\beta)^n=1\,\rangle$ where $1\le\alpha\le l-1$, $1\le\beta\le m-1$ and let $\kappa=1/l+1/m+1/n-1$.*

(a) If $(\alpha,l)=1$ and $(\beta,m)=1$ then G is large if $\kappa<0$, infinite and soluble if $\kappa=0$, finite if $\kappa>1$.

(b) If $(\alpha,l)>1$ or $(\beta,m)>1$ then G is large unless either:

(i) $l=2,n=2$ and $(\beta,m)=2$; or

(ii) $m=2,n=2$ and $(\alpha,l)=2$;

in which case G is infinite and soluble.

Proof
(a) If $(\alpha,l)=1$ and $(\beta,m)=1$ then we may assume $\alpha=\beta=1$, in which case G is an ordinary triangle group and the result is well known. (b) If $\kappa<1$ then G is large by [3, Theorem B]. If $\{l,m\}=\{2,2\},\{2,3\},\{2,5\},\{3,3\}$, or $\{3,5\}$ then $(\alpha,l)=(\beta,m)=1$. Thus we only need to consider the cases $(\{l,m\},n)=(\{2,k\},2)$ $(k\ge 4)$, $(\{3,4\},2),(\{3,6\},2),(\{2,4\},3),(\{2,6\},3),(\{2,4\},4)$, $(\{4,4\},2)$ where $(\alpha,l)>1$ or $(\beta,m)>1$. If $(l,m,n)=(2,m,2)$ then G maps onto $\langle\, a,b \,|\, a^2=b^{(\beta,m)}=1\,\rangle$ which is large unless $(\beta,m)=2$ and in this case the cyclic subgroup $H=\langle\, b^\beta \,|\, b^m\,\rangle$ is normal in G and $G/H\cong D_\infty$ so G is infinite and soluble. Similarly, if $(l,m,n)=(l,2,2)$ then G is large unless $(\alpha,l)=2$ in which case G is infinite and soluble. By passing to another generating pair if necessary we may assume $\alpha|l,\beta|m$ which means that for the remaining triples there are nine groups to consider. In each case we can use GAP [22] to find a subgroup (of index at most 6) that maps onto a free product of two cyclic groups

(other than $\mathbb{Z}_2 * \mathbb{Z}_2$) and hence G is large. $\square$

We now classify the large 2-generator Prischepov groups.

Theorem 5.2 *Let $M = M(r, 2, k, s, q)$, $P = P(r, 2, k, s, q)$, $A = (k-1)$, $B = (k-1) - q(r-s)$. If $(r, s) = 1$ let $\alpha, \beta \in \mathbb{Z}$ be such that $\alpha r + \beta s = 1$ and set $g = |s^2 - r^2|(\alpha, \beta)$. Then M is large unless one of the following holds:*

(a) A,B are both even and either

  (i) $|r - s| = 2$ in which case $M \cong D_\infty$,

    1. if q is even then $P \cong D_\infty$;

    2. if q is odd then $P \cong \mathbb{Z}$;

  (ii) $|r - s| = 1$ in which case $M \cong \mathbb{Z}_2$ and $P = 1$;

(b) A,B are of opposite parity, in which case $M \cong \mathbb{Z}_{2|r-s|}$, $P \cong \mathbb{Z}_{|r-s|}$;

(c) A,B are both odd and one of the following holds:

  (i) $(r, s) = 2$, in which case M and P are infinite and soluble;

  (ii) $r = s = 1$, in which case $M \cong \mathbb{Z}_2 \times \mathbb{Z}$, $P \cong \mathbb{Z}$;

  (iii) $(r, s) = 1$ and $r + s \geq 3$, in which case M soluble and finite of order $2g$,

    1. if q is even then $P \cong \mathbb{Z}_g$;

    2. if q is odd then P is non-abelian and soluble of order g.

Proof
If A,B are both even then $M \cong \mathbb{Z}_2 * \mathbb{Z}_{|r-s|}$ which is large unless $|r - s| = 2$ or 1. If $|r - s| = 2$ then $M \cong D_\infty$ and $P \cong D_\infty$ when q is even and $P \cong \mathbb{Z}$ when q is odd. If $|r - s| = 1$ then $M \cong \mathbb{Z}_2$ so $P = 1$. If A,B are of opposite parity then $M \cong \mathbb{Z}_{2|r-s|}$ and hence $P \cong \mathbb{Z}_{|r-s|}$. Suppose then that A, B are both odd.

Now $M = \langle\, y, t \,|\, t^2 = 1, y^s t = t y^r \,\rangle$ maps onto $\langle\, y, t \,|\, t^2 = y^{(r,s)} = 1 \,\rangle$ which is large when $(r, s) \geq 3$ so assume $(r, s) = 1$ or 2. If $r = s = 1$ then $M \cong \mathbb{Z}_2 \times \mathbb{Z}$ and $P \cong \mathbb{Z}$ so assume $r + s \geq 3$. Let $G = \langle\, y, t \,|\, y^{r+s} = t^2 = (y^s t)^2 = 1 \,\rangle$. If $(r, s) = 1$ then $G \cong D_{2(r+s)}$, which is soluble; if $(r, s) = 2$ then Theorem 5.1 implies that G is infinite and soluble. The cyclic subgroup $H = \langle\, (y^s t)^2 \,\rangle$ is normal in M and $M/H \cong G$, which is soluble, so M is soluble. Since M maps onto G we have that M is infinite when $(r, s) = 2$. Assume then that $(r, s) = 1$.

Suppose q is even, so $P = \langle\, x_0, x_1 \,|\, x_0^r = x_1^s, x_1^r = x_0^s \,\rangle$, and let α, β, g be as defined in the statement. Then $x_0^{\beta r} = x_1^{\beta s} = x_1^{1-\alpha r} = x_1 x_0^{-\alpha s}$ and hence $x_1 = x_0^{\alpha s + \beta r}$ and so

$$\begin{aligned} P &= \langle\, x_0 \,|\, x_0^{\alpha s^2 + \beta r s - r} = x_0^{\beta r^2 + \alpha r s - s} = 1 \,\rangle \\ &= \langle\, x_0 \,|\, x_0^{\alpha(s^2 - r^2)} = x_0^{\beta(s^2 - r^2)} = 1 \,\rangle \cong \mathbb{Z}_g. \end{aligned}$$

Now $[M : P] = 2$ so $|M| = 2g$ and is soluble (regardless of the parity of q).

Suppose then that q is odd and so r, s are both odd. Then

$$P = \langle\, x_0, x_1 \,|\, (x_0 x_1)^{(r-1)/2} x_0 = (x_1 x_0)^{(s-1)/2} x_1, (x_1 x_0)^{(r-1)/2} x_1 = (x_0 x_1)^{(s-1)/2} x_0 \,\rangle,$$

being an index 2 subgroup of M is soluble of order g. The determinant of the relation matrix of P gives $|P^{\mathrm{ab}}| = 2|r - s|$. But $g = |r - s|(r + s)(\alpha, \beta) \geq 3|r - s|$ so $|P| \neq |P^{\mathrm{ab}}|$ so P is non-abelian. $\square$

Theorem 5.3 *Let* $N = \langle y, t \,|\, t^d = 1, y^s t^A = t^{-A} y^r \rangle$ $(d \geq 2)$. *Then* N *is large unless one of the following holds:*

(a) $A \equiv 0 \bmod d$ *and either*

- *(i)* $d = 2$ *and* $|r - s| = 2$, *in which case* $N \cong D_\infty$; *or*
- *(ii)* $|r - s| = 1$ *in which case* $N \cong \mathbb{Z}_d$.

(b) $(A, d) = 1$ *and one of the following holds:*

- *(i)* $d = 2$ *and* $(r, s) = 2$ *in which* N *is infinite and soluble; or*
- *(ii)* $r = s = 1$, *in which case* $N \cong \mathbb{Z} \rtimes \mathbb{Z}_d$, *which is infinite and soluble; or*
- *(iii)* $r + s \geq 3$, $(r, s) = 1$ *and one of the following holds:*
  1. $(d, \{r, s\}) = (3, \{1, 5\})$,
  2. $(d, \{r, s\}) = (4, \{1, 3\})$,
  3. $(d, \{r, s\}) = (6, \{1, 2\})$,

  in which case N *is infinite and soluble; or*
- *(iv)* $r + s \geq 3$, $(r, s) = 1$, *and one of the following holds:*
  1. $d = 2$, *in which case* N *is soluble and finite of order* $2|s^2 - r^2|(\alpha, \beta)$, *where* $\alpha r + \beta s = 1$,
  2. $(d, \{r, s\}) = (3, \{1, 2\})$, *in which case* N *is soluble and finite of order 24,*
  3. $(d, \{r, s\}) = (3, \{1, 3\})$, *in which case* N *is soluble and finite of order 144,*
  4. $(d, \{r, s\}) = (3, \{1, 4\})$, *in which case* N *is insoluble and finite of order 1080,*
  5. $(d, \{r, s\}) = (3, \{2, 3\})$, *in which case* N *is insoluble and finite of order 360,*
  6. $(d, \{r, s\}) = (4, \{1, 2\})$, *in which case* N *is soluble and finite of order 96,*
  7. $(d, \{r, s\}) = (5, \{1, 2\})$, *in which case* N *is insoluble and finite of order 600.*

Proof

If $d = 2$ then the result follows from Theorem 5.2 so assume $d \geq 3$.
(a) If $A \equiv 0 \bmod d$ then $N \cong \mathbb{Z}_d * \mathbb{Z}_{|r-s|}$ which is large unless either $d = 2$ and $|s - r| = 2$, in which case $N \cong D_\infty$, or $|s - r| = 1$, in which case $N \cong \mathbb{Z}_d$.
(b) The group N maps onto $G = \langle y, t \,|\, y^{r+s} = t^d = (y^s t^A)^2 = 1 \rangle$. If $(A, d) > 1$ then Theorem 5.1 implies that G, and hence N, is large unless $r = s = 1$, $(A, d) = 2$, in which case N maps onto $\langle y, t \,|\, t^2 \rangle \cong \mathbb{Z} * \mathbb{Z}_2$, which is large. Suppose then that $(A, d) = 1$. By applying an automorphism of $\langle t \,|\, t^d \rangle$ we may assume $A = 1$. If $r = s = 1$ then $N \cong \mathbb{Z} \rtimes \mathbb{Z}_d$ so assume $r + s \geq 3$. By Theorem 5.1 G, and hence N is large unless $(r, s) = 1$ and $1/(r + s) + 1/d \geq 1/2$. When we have equality the conditions are equivalent to (b)(iii) and G is infinite and soluble. The cyclic subgroup $H = \langle (y^s t)^2 \rangle$ is normal in N and $N/H \cong G$, which is soluble, so N is soluble. When the inequality is strict the conditions are equivalent to (b)(iv) and computations using GAP show that N is finite of the given order and soluble or insoluble as indicated. □

This yields

Corollary 5.4 *Let* $d = (n, A + B) = (n, 2(k - 1) - q(r - s))$ *and assume* $d \geq 2$.

(a) Suppose none of the conditions in Theorem 5.3(a),(b) hold. Then $M(r, n, k, s, q)$ *is large.*

(b) *Suppose none of the conditions in Theorem 5.3(a)(ii) or (b)(iv) hold. Then* $M(r,n,k,s,q)$ *is infinite.*

Combining the results of this section with those of Section 3 we have

Corollary 5.5 *Suppose* $(n,A,B)\geq 2$, $|s-r|\geq 2$, $(A-B,n)\geq 2$ *and* $(r,s)\geq 2$. *Then* $M(r,n,k,s,q)$ *is large unless* $r=2R$, $s=2(R+\epsilon)$, $k=2K-1$, $n=2N$, *for some* $R,N,K\geq 1$, $\epsilon=\pm 1$, *where* $(q,N)=1$ *and* $(N,(2K-1)+q\epsilon)=1$.

Proof
By Corollary 3.8 we may assume $(n,A,B)=|s-r|=(A-B,n)=(r,s)=2$. The conditions $|s-r|=2$, $(r,s)=2$ are equivalent to $r=2R$, $s=2(R+\epsilon)$, for some $R\geq 1$, $\epsilon=\pm 1$. The condition $(A-B,n)=2$ is then equivalent to $n=2N$ and $(q,N)=1$ for some $N\geq 1$. The condition $(n,A,B)=2$ implies that $k=2K-1$ for some $K\geq 1$. Applying Corollary 5.4 we see that if $M(r,n,k,s,q)$ is not large then only case of Theorem 5.3 that can hold is (a)(i), and this is equivalent to $(N,2(K-1)+q\epsilon)=1$. □

5.2 Freiheitssatz methods for Prischepov groups

In this section we will regard $M(r,n,k,s,q)$ as a one-relator product $(H*K)/<<R>>$ where $\{H,K\}=\{\langle x\,|\,\rangle,\langle t\,|\,t^n\rangle\}$, $R=y^st^Ay^{-r}t^{-B}$ where $A=(k-1)$, $B=(k-1)-q(r-s)$. In view of Corollary 3.11 we now investigate when the Freiheitssatz holds in this situation. The following result is contained in [33, Theorem C].

Theorem 5.6 ([33]) *Suppose* $A\not\equiv 0$ *mod* n, $B\not\equiv 0$ *mod* n, $2A\not\equiv 0$ *mod* n, $2B\not\equiv 0$ *mod* n, $A\not\equiv \pm B$ *mod* n, *and* $r>2s$ *or* $s>2r$. *Then the Freiheitssatz holds for* $M(r,n,k,s,q)$ *if any of the following hold.*

(a) $3A,4A,5A\not\equiv 0$ *mod* n, $B\not\equiv\pm 2A$ *mod* n, $B\not\equiv -3A$ *mod* n, $A\not\equiv -2B$ *mod* n*;*

(b) $3B,4B,5B\not\equiv 0$ *mod* n, $A\not\equiv\pm 2B$ *mod* n, $A\not\equiv -3B$ *mod* n, $B\not\equiv -2A$ *mod* n*;*

(c) $3A,3B\not\equiv 0$ *mod* n, $B\not\equiv -2A$ *mod* n, $A\not\equiv -2B$ *mod* n.

In [35, 36, 37, 38] Shwartz considered the Freiheitssatz for one-relator product $(H*K)/<<R>>$ where $R=abcd\in H*K$ with $a,c\in H$, $b,d\in K$ and these results can be applied to our situation to obtain other conditions under which the Freiheitssatz holds. We now review Shwartz's results. Let H_1 be the subgroup of H generated by $\{a,c\}$ and let K_1 be the subgroup of K generated by $\{b,d\}$. We assume that there are no relations of length 1 or 2 among $\{a,c\}$ in H_1 or among $\{b,d\}$ in K_1. By interchanging the roles of H,K, cyclically permuting the relator R, and replacing a,b,c,d by their inverses we can reduce to the following four cases:

0. $c=_H a^{\pm 2}, d=_K b^{\pm 2}$;

1. $c=_H a^2$, $b\neq_K d^{\pm 2}$, $d\neq_K b^{\pm 2}$;

2. $c=_H a^{-2}$, $b\neq_K d^{\pm 2}$, $d\neq_K b^{\pm 2}$;

3. $c\neq_H a^{\pm 2}$, $a\neq_H c^{\pm 2}$, $b\neq_K d^{\pm 2}$, $d\neq_K b^{\pm 2}$;

where the subscripts indicate the group in which equality or inequality is considered. Freiheitssatz theorems were obtained by Shwartz for Case 1 in [36], for Case 2 in [37], and for Case 3 in [38]; all of these results are contained in [35]. Case 0 was considered in [17],[18] and our arguments below may be applied to this case; however, since these results are more intricate we limit ourselves to applying the results of Cases 1–3. We summarize Shwartz's results in the following theorem. (In this theorem A_4, S_4, A_5 denote alternating and symmetric groups and Q_{12} denotes the quarternionic group of order 12.)

Theorem 5.7 ([35, 36, 37, 38]) *Let $G = (H * K)/ << R >>$ where $R = abcd$, $a, c \in H, b, d \in K$; let H_1 be the subgroup of H generated by $\{a, c\}$, K_1 be the subgroup of K generated by $\{b, d\}$. Suppose $a \neq_H 1$, $c \neq_H 1$, $a^2 \neq_H 1$, $c^2 \neq_H 1$, $a \neq_H c^{\pm 1}, b \neq_K 1$, $d \neq_K 1$, $b^2 \neq_K 1$, $d^2 \neq_K 1$, $b \neq_K d^{\pm 1}$, and that $b \neq_K d^{\pm 2}$, $d \neq_K b^{\pm 2}$. The Freiheitssatz holds in each of the following cases.*

1. *$c = a^2$ and either*
   - *(i) $|H_1| \geq 12$, $|K_1| \geq 10$ and $K_1 \not\in \{A_4, S_4, A_5\}$; or*
   - *(ii) $|H_1| \in \{7, 9, 10, 11\}$, $|K_1| \geq 11$ and $K_1 \not\in \{A_4, S_4, A_5\}$.*
2. *$c = a^{-2}$ and either*
   - *(i) $|H_1| \geq 9$, $|K_1| \geq 11$ and $K_1 \not\in \{\mathbb{Z}_{12}, A_4, S_4, A_5\}$; or*
   - *(ii) $|H_1| = 7$, $|K_1| \geq 11$ and $K_1 \not\in \{\mathbb{Z}_{12}, A_4, S_4, A_5\}$.*
3. *$c \neq_H a^{\pm 2}$, $a \neq_H c^{\pm 2}$, and*
   - *(i) $H_1 \not\in \{A_4, \mathbb{Z}_3 \oplus \mathbb{Z}_3\}$ and $K_1 \not\in \{A_4, S_4, A_5, \mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9, \mathbb{Z}_{12}, \mathbb{Z}_{15}, Q_{12}\}$; or*
   - *(ii) $H_1 \not\in \{A_4, S_4, A_5, \mathbb{Z}_3 \oplus \mathbb{Z}_3, \mathbb{Z}_9, \mathbb{Z}_{12}, \mathbb{Z}_{15}, Q_{12}\}$ and $K_1 \not\in \{A_4, \mathbb{Z}_3 \oplus \mathbb{Z}_3\}$.*

We may regard $M = M(r, n, k, s, q)$ as a one-relator product $(H * K)/ << R >>$ where $R = abcd$ in two ways:

(a) $H = \langle\, t \,|\, t^n \,\rangle$, $K = \langle\, y \,|\, \rangle$, $\{a, c\} = \{t^A, t^{-B}\}$, $\{b, d\} = \{y^s, y^{-r}\}$, and so $H_1 \cong \mathbb{Z}_N$, $K_1 \cong \mathbb{Z}$, where $N = n/(n, A, B)$; or

(b) $H = \langle\, y \,|\, \rangle$, $K = \langle\, t \,|\, t^n \,\rangle$, $\{a, c\} = \{y^s, y^{-r}\}$, $\{b, d\} = \{t^A, t^{-B}\}$, and so $H_1 \cong \mathbb{Z}$, $K_1 \cong \mathbb{Z}_N$, where $N = n/(n, A, B)$.

By replacing R by R^{-1}, inverting generators of H and K, and cyclically permuting R we may interchange the roles of A, B and interchange the roles of r, s. Therefore in (a) we may take (without loss of generality) $a = t^A, b = y^s, c = t^{-B}, d = y^{-r}$, and in (b) we may take $a = y^s, b = t^A, c = y^{-r}, d = t^{-B}$. Applying Theorem 5.7 and then including the cases obtained by interchanging r, s and interchanging A, B we obtain the following theorem. Note that by definition $r \geq 1, s \geq 1$ so many hypotheses are automatic, and note that in (b) Case 1 does not occur. To make clear where each case comes from we keep the numbering here consistent with that of Theorem 5.7.

Theorem 5.8 *Suppose $A \not\equiv 0 \bmod n$, $B \not\equiv 0 \bmod n$, $2A \not\equiv 0 \bmod n$, $2B \not\equiv 0 \bmod n$, $A \not\equiv \pm B \bmod n$, $r \neq s$ and let $N = n/(n, A, B)$. Then the Freiheitssatz holds for $M(r, n, k, s, q)$ if any of the following hold.*

(a) $r \neq 2s$, $s \neq 2r$ and one of the following holds:

1. $(A \equiv -2B \bmod n$ or $B \equiv -2A \bmod n)$ and $N \geq 7$, $N \neq 8$;

2. $(A \equiv 2B \bmod n$ or $B \equiv 2A \bmod n)$ and $N \geq 7$, $N \neq 8$;

3. $A \not\equiv \pm 2B \bmod n$, $B \not\equiv \pm 2A \bmod n$.

(b) $A \not\equiv \pm 2B \bmod n$, $B \not\equiv \pm 2A \bmod n$ *and one of the following holds:*

2. $(r = 2s$ or $s = 2r)$ and $N \geq 11$, $N \neq 12$;

3. $r \neq 2s$, $s \neq 2r$.

Observe that the conditions in b)(3) are the same as a)(3) and that the hypothesis $r \neq s$ can be removed (as in that case killing t shows that y has infinite order). Further, the condition a)(3) does not hold when $N < 7$ so we obtain the following tidier formulation.

Theorem 5.8′ *Let* $N = n/(n, A, B)$ *and suppose* $N \geq 7$, $A \not\equiv 0 \bmod n$, $B \not\equiv 0 \bmod n$, $2A \not\equiv 0 \bmod n$, $2B \not\equiv 0 \bmod n$, $A \not\equiv \pm B \bmod n$ *and that either*

(i) $(A \equiv \pm 2B \bmod n$ *or* $B \equiv \pm 2A \bmod n)$ *and* $r \neq 2s, s \neq 2r$, $N \neq 8$; *or*

(ii) $A \not\equiv \pm 2B \bmod n$, $B \not\equiv \pm 2A \bmod n$ *and if* $(r = 2s$ *or* $s = 2r)$ *then* $N \geq 11$, $N \neq 12$.

Then the Freiheitssatz holds for $M(r, n, k, s, q)$.

Note that for any of the conditions of Theorem 5.6 to hold we require $N \geq 7$ (where $N = n/(n, A, B)$). However, Theorem 5.8′ does not generalize Theorem 5.6 since, for example, the group $M(2, 16, 3, 4, 1)$ satisfies the hypotheses of Theorem 5.6 but not of Theorem 5.8′.

By Corollary 3.11 we now have

Corollary 5.9 *Suppose that the hypotheses of Theorem 5.8′ or Theorem 5.6 hold. Then* $M = M(r, n, k, s, q)$ *is infinite; moreover,*

(a) *if* $(r, s) \geq 2$ *then* M *contains a free subgroup of rank 2;*

(b) *if* $(A, B, n) \geq 2$ *then* M *is SQ-universal.*

Note that for the cases $A \equiv 0 \bmod n$ or $B \equiv 0 \bmod n$ largeness of $P(r, n, k, s, q)$ was completely dealt with in Theorem 4.6 and for the case $A \equiv -B \bmod n$ it was dealt with in Theorem 5.3.

Using Theorem 5.8′(ii) we can obtain a result about Cavicchioli-Hegenbarth-Repovš groups $G_n(m, k)$.

Corollary 5.10 *Let* $N = n/(n, m, k)$ *and suppose* $N \geq 11$, $N \neq 12$ *and* $k \not\equiv 0$, $k \not\equiv m$, $2k \not\equiv 0$, $2(k-m) \not\equiv 0$, $m \not\equiv 0$, $m \not\equiv 2k$, $k \not\equiv 2m$, $k + m \not\equiv 0$, $3k \not\equiv 2m$, $m \not\equiv 3k$ *(all mod n) then* $G_n(m, k)$ *is infinite.*

It also follows from Theorem 5.8′ that the group $G_7(x_0^{-1}x_1^{-1}x_2^{-1}x_4x_3x_2x_1) \cong P(4, 7, 3, 3, 1)$ is infinite. This is the one group in [15] that could not be dealt with by computational techniques and required detailed curvature analysis. (Though, of course, Theorem 5.8′ relies on the detailed curvature analysis of Shwartz.) In fact, Theorem B of [15], which states that $\langle\, y, t \mid t^n, ty^{-3}t^{-2}y^4 \,\rangle \cong M(4, n, 3, 3, 1)$ is infinite for all $n \geq 6$ can be recovered as a corollary of Theorem 5.8′ apart from in the cases $n = 6, 8$.

6 The finite Cavicchioli-Hegenbarth-Repovš groups

Recall that the Cavicchioli-Hegenbarth-Repovš groups $G_n(m,k)$ are the groups $P(2,n,k+1,1,m) = G_n(x_0x_mx_k^{-1})$. Bardakov and Vesnin [2, Question 1] have asked for a classification of the finite groups $G_n(m,k)$. With the exception of two unresolved groups the classification has now been obtained. The existing proof of the classification relies on techniques from algebraic number theory [30],[45]. In this section we first review that proof and then build on the results of Section 3 to obtain a new proof that is purely group theoretic.

If $k \equiv 0 \bmod n$ or $(k-m) \equiv 0 \bmod n$ then $G_n(m,k) = 1$. If $m \equiv 0 \bmod n$ then Corollary 3.5 implies that $G_n(m,k)$ is isomorphic to the free product of (k,n) copies of $G_N(M,K)$ where $N = n/(n,k)$, $K = k/(n,k)$. By [2, Lemma 1.1(1)] if $K \not\equiv 0 \bmod N$ then $G_N(0,K) \cong \mathbb{Z}_{2^N-1}$ so $G_n(0,k) \cong \mathbb{Z}_{2^N-1}$ if $(n,k) = 1$ and is infinite otherwise. Thus we may assume $1 \leq m,k \leq n-1$, $m \neq k$.

Theorem 6.1 ([46]) *Let $(n,m,k) = 1$, $(n,k) > 1$, $(n,k-m) > 1$, $1 \leq k,m \leq n-1$, $k \neq m$ and suppose $G_n(m,k) \neq 1$. Then $G_n(m,k)$ is finite if and only if $(m,k) = 1$ and ($n = 2k$ or $n = 2(k-m)$), in which case $G \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

The following theorem was proved (in number theoretic terms) in [30] for the case $k = 1$ and in [45] for the general case.

Theorem 6.2 ([30],[45]) *The group $G_n(m,k)$ is perfect if and only if either ($m = 2k$ mod n and $(n/(n,m,k),6) = 1$) or $k = 0$ or m mod n.*

Thus we have

Corollary 6.3 ([45],[46]) *Suppose $(n,k) > 1$ and $(n,m-k) > 1$, $1 \leq k,m \leq n-1$, $k \neq m$. Then $G_n(m,k)$ is finite if and only if $(m,k) = 1$ and ($n = 2k$ or $n = 2(k-m)$), in which case $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

Proof
Suppose first that $(n,m,k) = 1$. Then $m \neq 2k$ mod n (for otherwise $(n,m,k) = (n,k) > 1$) so by Theorem 6.2 $G_n(m,k)$ is not perfect, and hence is not trivial, so the result follows from Theorem 6.1. Suppose then that $d = (n,m,k) > 1$. Then by Corollary 3.5 $G_n(m,k)$ is isomorphic to the free product of d copies of $G_N(M,K)$ where $N = n/d, M = m/d, K = k/d$. Since $(N,M,K) = 1$ each of these is non-trivial by the above argument, so $G_n(m,k)$ is infinite. □

By Corollary 4.5 if $(n,k) = 1$ or $(n,m-k) = 1$ then $G_n(m,k)$ is isomorphic to some Gilbert-Howie group $H(n,t)$.

Theorem 6.4 ([23]) *Suppose $n \geq 2$, $t \geq 0$, $(n,t) \neq (8,3),(9,3),(9,4),(9,6),(9,7)$ and suppose $H(n,t) \neq 1$. Then $H(n,t)$ is finite if and only if $t = 0,1$ or $(n,t) = (2k,k+1)$ where $k \geq 1$ (in which case $H(n,t) \cong \mathbb{Z}_{2^k+1}$), or $(n,t) = (3,2)$, $(4,2)$, $(5,2)$, $(5,3)$, $(5,4)$, $(6,3)$, $(7,4)$, $(7,6)$.*

We have that $H(n,t)$ is non-trivial by [43, Theorem B] when $t = 2$ and by Theorem 6.2 for the case $k = 1$ ([30]) otherwise. Moreover the group $H(9,3) \cong H(9,6)$ was proved to be infinite in [12, Lemma 15]. (We remark that the extension of this group also appears in [17, page 228] as $G(-,9)$.) A calculation in GAP shows that $H(8,3)$ is soluble and of order $3^{10} \cdot 5$. Thus there is the following almost complete classification of the finite groups $H(n,t)$:

Corollary 6.5 *Suppose $n \geq 2$, $t \geq 0$, $(n,t) \neq (9,4),(9,7)$. Then $H(n,t)$ is finite if and only if $t = 0,1$ or $(n,t) = (2k, k+1)$ where $k \geq 1$ (in which case $H(n,t) \cong \mathbb{Z}_{2^k+1}$), or $(n,t) = (3,2)$, $(4,2)$, $(5,2)$, $(5,3)$, $(5,4)$, $(6,3)$, $(7,4)$, $(7,6)$, $(8,3)$.*

In particular, for $n \geq 10$ there are only finite groups in the families $t = 0$, $t = 1$, or $(n,t) = (2k, k+1)$. Combining Corollary 6.3 and Corollary 6.5 and restricting to the cases $n \geq 10$ we have a classification of the finite groups $G_n(m,k)$:

Corollary 6.6 *Suppose $n \geq 10$, $1 \leq m,k \leq n-1$, $m \neq k$. Then $G_n(m,k)$ is finite if and only if $(n,m,k) = 1$ and ($2k \equiv 0$ mod n or $2(k-m) \equiv 0$ mod n) in which case $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

Applying Corollary 3.5 we have that $G_n(m,k)$ is large whenever $(n,m,k) > 1$.

In the next theorem we give a proof of Corollary 6.6 for $n/(n,m,k) \geq 11$, $n/(n,m,k) \neq 12$ that is purely group theoretic. Since, by Corollary 3.5, $G_n(m,k)$ is perfect if and only if $G_N(M,K)$ is perfect (where $N = n/(n,m,k), M = m/(n,m,k), K = k/(n,m,k)$), to verify Theorem 6.2 for $n/(n,m,k) \leq 10$ and $n/(n,m,k) = 12$ we may assume $(n,m,k) = 1$ and so it suffices to verify it for $n \leq 10$ and $n = 12$. This can easily be done by group theoretic methods (for example using GAP). Therefore the proof described above gives a purely group theoretic proof of the classification of the finite groups $G_n(m,k)$ for $n/(n,m,k) \leq 10$ and $n/(n,m,k) = 12$. This, together with the proof of Theorem 6.7 provides a proof of the (almost complete) classification of the finite groups $G_n(m,k)$ that does not involve the algebraic number theory used to prove Theorem 6.2.

Theorem 6.7 *Suppose $n/(n,m,k) \geq 11$, $n/(n,m,k) \neq 12$, $1 \leq m,k \leq n-1$, $m \neq k$. Then $G_n(m,k)$ is finite if and only if $(n,m,k) = 1$ and ($2k \equiv 0$ mod n or $2(k-m) \equiv 0$ mod n) in which case $G_n(m,k) \cong \mathbb{Z}_s$ where $s = 2^{n/2} - (-1)^{m+n/2}$.*

Proof

As in Corollary 6.3 it suffices to prove the result for $(n,m,k) = 1$. By Corollary 5.10 we need to consider the cases $2k \equiv 0$, $2(k-m) \equiv 0$, $m \equiv 2k$, $k \equiv 2m$, $k+m \equiv 0$, $3k \equiv 2m$, $m \equiv 3k$ (all mod n).

If $2k \equiv 0$ mod n or $2(k-m) \equiv 0$ mod n then $G_n(m,k) \cong \mathbb{Z}_s$ by [45, Lemma 3]. If $m = 2k$ then $(k,n) = 1$ so by Theorem 4.4 we may assume $k = 1$ so $m = 2$. Then $G_n(m,k) = G_n(2,1) = S(2,n)$, the Sieradski group. By [43, Theorem B] this is infinite for all $n \geq 6$. If $k = 2m$ then $(m,n) = 1$ so by Theorem 4.4 we may assume $m = 1$ so $k = 2$. Then $G_n(m,k) \cong G_n(1,2) = F(2,n)$, the Fibonacci group. If $k+m \equiv 0$ mod n then $(k,n) = 1$ so we may assume $k = 1, m = n-1$ so $G_n(m,k) = G_n(n-1,1) \cong G_n(1,2) = F(2,n)$ by Corollary 4.3. The Fibonacci group $F(2,n)$ is infinite for all $n \geq 9$ (see [42] for a survey of such results).

This leaves the cases $3k = 2m$ and $m = 3k$. The split extension of $G_n(m,k) = P(2,n,k+1,1,m)$ is

$$M = M(2,n,k+1,1,n) = \langle\, y,t \,|\, t^n = 1, yt^A = t^B y^2 \,\rangle$$

where $A = k$, $B = k-m$ mod n, and so $(A,B,n) = (n,m,k) = 1$. The condition $3k = 2m$ is equivalent to $A \equiv -2B$ mod n and the condition $3k = m$ mod n is equivalent to $B = -2A$ mod n. In the first case we have $(B,n) = 1$ so we may assume $B = 1$, $A = -2$; in the second case we have $(A,n) = 1$ so we may assume $A = 1$, $B = -2$. Either way we get (by replacing y with y^{-1}, if necessary) that $M = \langle\, y,t \,|\, t^n, y^2t^2y^{-1}t \,\rangle$. This maps onto $L = \langle\, y,t \,|\, y^l, t^n, y^2t^2y^{-1}t \,\rangle$ for any $l \geq 1$. By [18, Theorem 3] if $l \geq 36$ then y has order l in L, so l divides $|M|$. Thus $|M| \geq l$ for any $l \geq 36$, so M is infinite. $\square$

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