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# Weak Alfvén-Wave Turbulence Revisited

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Weak Alfvénic turbulence in a periodic domain is considered as a mixed state of Alfvén waves interacting with the two-dimensional (2D) condensate. Unlike in standard treatments, no spectral continuity between the two is assumed and indeed none is found. If the 2D modes are not directly forced,  $k^{-2}$  and  $k^{-1}$  spectra are found for the Alfvén waves and the 2D modes, respectively, with the latter less energetic than the former. The wave number at which their energies become comparable marks the transition to strong turbulence. For imbalanced energy injection, the spectra are similar and the Elsasser ratio scales as the ratio of the energy fluxes in the counterpropagating Alfvén waves. If the 2D modes are forced, a 2D inverse cascade dominates the dynamics at the largest scales, but at small enough scales, the same weak and then strong regimes as described above are achieved.

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*Introduction.* It has been understood for many years that small-scale turbulence of a conducting fluid or plasma in a strong magnetic field consists of Alfvén-wave packets [1]. This is true in most astrophysical plasmas, including the weakly collisional ones, where Alfvénic fluctuations populate the scales above the ion Larmor scale [2]. On theoretical [3–5], numerical [6–9] and observational [10–15] grounds, it appears clear that Alfvénic turbulence is anisotropic with  $k_{\perp} \gg k_{\parallel}$ . The parallel scales are associated with the propagation of Alfvén waves, the perpendicular ones with the nonlinear interaction between them. The relative importance of these effects depends on the corresponding time scales,  $\tau_A \sim (k_{\parallel} v_A)^{-1}$  and  $\tau_{nl} \sim (k_{\perp} u_{\perp})^{-1}$ , where  $v_A$  is the Alfvén speed and  $u_{\perp}$  the perpendicular velocity perturbation. When  $\tau_A \gg \tau_{nl}$ , the nonlinearity dominates and the turbulence is effectively two-dimensional (2D); when  $\tau_A \ll \tau_{nl}$ , the wave propagation dominates and the turbulence is weak.

A causality argument suggests that a pure 2D regime cannot be sustained: for any given  $k_{\perp}$ , motions in two planes perpendicular to the mean field and separated by a distance  $\sim k_{\parallel}^{-1}$  can only remain correlated if the time it takes an Alfvén wave to propagate between the planes is  $\tau_A < \tau_{nl}$ . Thus, an initially 2D perturbation will naturally decay into a state of “critical balance,”  $\tau_A \sim \tau_{nl}$ . It has been argued [5, 16, 17] that weak turbulence, the limit case opposite to 2D, will also approach critical balance via a perpendicular cascade in which  $\tau_{nl}$  becomes ever smaller until  $\tau_{nl} \sim \tau_A$  at some sufficiently small scale. The critical balance thus appears to be the fundamental physical principle underpinning Alfvénic turbulence (and possibly in general turbulence in systems that support propagation of waves [18]). However, the structure of critically balanced turbulence remains contentious and poorly understood. Efforts to improve this understanding have often turned to various insights from the theory of weak turbulence for intuition and guid-

ance. Weak Alfvén-wave turbulence itself, while having for some time enjoyed the reputation of a solved problem [16], has nevertheless recently been subject of several new investigations that amended or disagreed with the established paradigm [19–22]. This paper is a contribution to this revisionist tendency, focusing on the structure of weak Alfvén-wave turbulence in finite periodic domains and on the transition to critical balance. While a periodic box may be an artificial setting, it is ubiquitous in numerical experiments. It is, therefore, important to understand turbulence in such domains and the extent to which it might belong to the same universality class as turbulence in natural systems.

While it might appear that weak turbulence is an analytically tractable and, therefore, easily understood limit, the weak turbulence of Alfvén waves is, in fact, difficult to treat in a rigorous fashion because of a special role played by the modes with  $k_{\parallel} = 0$ . Since Alfvén waves have frequencies  $\omega_{\mathbf{k}}^{\pm} = \pm k_{\parallel} v_A$  and only counter-propagating waves can interact, the resonance conditions  $\omega_{\mathbf{k}_1}^{+} + \omega_{\mathbf{k}_2}^{-} = \omega_{\mathbf{k}_3}^{\pm}$  and  $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$  imply that at least one mode in any interacting triad must have  $k_{\parallel} = 0$  [23, 24]. These modes are not Alfvén waves but rather 2D motions for which  $\tau_A = \infty$ , so they cannot be treated by the weak-turbulence approximation. The standard remedy for this complication has been to proceed with the weak-turbulence expansion anyway, assuming formally that the  $k_{\parallel}$  spectrum of the Alfvén waves is continuous across  $k_{\parallel} = 0$  [16, 17]. The resulting theory predicts a  $k_{\perp}^{-2}$  scaling of the energy spectrum. As this scaling is corroborated by direct numerical simulations [25], the continuity assumption might seem to be vindicated. In this paper, we propose a different way of treating the  $k_{\parallel} = 0$  modes, with no assumption of their spectral continuity with the Alfvén waves. It leads to a new phenomenological theory of the weak Alfvén-wave turbulence and to distinct scaling predictions for the energy spectra of the

Alfvén waves and of the  $k_{\parallel} = 0$  modes. We discuss various regimes: balanced and imbalanced, containing forced 2D motions or otherwise, and also describe the transition to critically balanced strong turbulence in a new way.

*Scaling theory.* Let us start with the equations of reduced magnetohydrodynamics (RMHD) [26, 27], which can be shown to describe correctly the anisotropic Alfvénic fluctuations both in an MHD fluid [28], and, above the ion Larmor scale, even in weakly collisional kinetic plasmas [2]. In RMHD, the velocity and magnetic field perturbations perpendicular to the mean field  $\mathbf{B}_0 = v_A \hat{z}$  (the Alfvénic perturbations) are two-dimensionally solenoidal, so they can be expressed in terms of stream and flux functions:  $\mathbf{u}_{\perp} = \hat{z} \times \nabla_{\perp} \Phi$  and  $\delta \mathbf{B}_{\perp} = \hat{z} \times \nabla_{\perp} \Psi$ . The RMHD equations can then be written in terms of the Elsasser potentials  $\zeta^{\pm} = \Phi \pm \Psi$  as follows [2]

$$\partial_t \zeta^{\pm} \mp v_A \partial_z \zeta^{\pm} = N[\zeta^{\mp}, \zeta^{\pm}] + F^{\pm}, \quad (1)$$

where  $F^{\pm}$  is the stream function of a body force representing energy injection and the nonlinear term is

$$N[\zeta^{\mp}, \zeta^{\pm}] = -\frac{1}{2} \nabla_{\perp}^{-2} (\{\zeta^{+}, \nabla_{\perp}^2 \zeta^{-}\} + \{\zeta^{-}, \nabla_{\perp}^2 \zeta^{+}\} \mp \nabla_{\perp}^2 \{\zeta^{+}, \zeta^{-}\}), \quad (2)$$

where  $\{A, B\} \equiv \hat{z} \cdot (\nabla_{\perp} A \times \nabla_{\perp} B)$ .

Let us now Fourier transform in  $z$ , factor out the oscillating in time part of the solution,  $\zeta^{\pm}(x, y, z) = \sum_{k_{\parallel}} \zeta_{k_{\parallel}}^{\pm}(x, y) e^{\pm i k_{\parallel} v_A t + i k_{\parallel} z}$ , and write separately the evolution equations for the Alfvén waves ( $k_{\parallel} \neq 0$ ),

$$\begin{aligned} \partial_t \zeta_{k_{\parallel}}^{\pm} &= N[\zeta_0^{\mp}, \zeta_{k_{\parallel}}^{\pm}] + N[\zeta_{k_{\parallel}}^{\mp}, \zeta_0^{\pm}] e^{\mp i 2 k_{\parallel} v_A t} \\ &+ \sum_{k'_{\parallel} \neq 0, k_{\parallel}} N[\zeta_{k'_{\parallel}}^{\mp}, \zeta_{k_{\parallel} - k'_{\parallel}}^{\pm}] e^{\mp i 2 k'_{\parallel} v_A t} + F_{k_{\parallel}}^{\pm} e^{\mp i k_{\parallel} v_A t}, \end{aligned} \quad (3)$$

and the  $k_{\parallel} = 0$  modes,

$$\partial_t \zeta_0^{\pm} = N[\zeta_0^{\mp}, \zeta_0^{\pm}] + \sum_{k_{\parallel} \neq 0} N[\zeta_{k_{\parallel}}^{\mp}, \zeta_{-k_{\parallel}}^{\pm}] e^{\mp i 2 k_{\parallel} v_A t}. \quad (4)$$

The standard approximation of the weak-turbulence theory is, roughly speaking, to neglect the nonlinear terms in Eq. (3) that have oscillatory factors, so the dominant effect is the Alfvén-wave “scattering” off the  $k_{\parallel} = 0$  modes (the first term on the right-hand side). This gives rise to a cascade of energy to small perpendicular scales (large  $k_{\perp}$ ), while the transfer of energy from the directly forced  $k_{\parallel}$  to other  $k_{\parallel}$  is small. For simplicity, let us assume that only one  $k_{\parallel} = k_{\parallel f}$  is forced. If we denote  $\zeta_n^{\pm}(k_{\perp})$  the characteristic amplitudes corresponding to  $k_{\parallel} = n k_{\parallel f}$  and the perpendicular wave number  $k_{\perp}$  and take the interactions to be local in  $k_{\perp}$ , we may estimate

$$(\nabla_{\perp}^2 \zeta_1^{\pm}) N[\zeta_0^{\mp}, \zeta_1] \sim k_{\perp}^4 \zeta_0^{\mp}(k_{\perp}) [\zeta_1^{\pm}(k_{\perp})]^2 \sim \varepsilon^{\pm}, \quad (5)$$

where  $\varepsilon^{\pm} = \langle (\nabla_{\perp} \zeta^{\pm}) \cdot (\nabla_{\perp} F^{\pm}) \rangle$  is the mean power injected by the forcing into the “+” and “−” modes. We first consider the *balanced* case:  $\varepsilon^{+} = \varepsilon^{-} = \varepsilon$  [43].

Eq. (5) shows that in order to make scaling predictions for the Alfvén waves, we must know the scaling of the amplitudes of the  $k_{\parallel} = 0$  modes — these modes determine the cascade rate  $\sim k_{\perp}^2 \zeta_0^{\mp}(k_{\perp})$  for the Alfvén waves. The  $k_{\parallel} = 0$  modes are very different from the Alfvén waves: they are described by 2D magnetohydrodynamics with an oscillatory nonlinear source term representing the coupling of  $k_{\parallel}$  and  $-k_{\parallel}$  Alfvén waves [Eq. (4)]. We shall first consider the situation in which the  $k_{\parallel} = 0$  modes are not forced externally, so this nonlinear source is the only source of energy in the  $k_{\parallel} = 0$  modes. Since the nonlinear source in Eq. (4) has an oscillatory factor, there is a strong cancellation effect and the amplitude of the  $k_{\parallel} = 0$  modes can be estimated as [44]

$$\zeta_0^{\pm}(k_{\perp}) \sim \omega_A^{-1} k_{\perp}^2 \zeta_1^{+}(k_{\perp}) \zeta_1^{-}(k_{\perp}), \quad (6)$$

where  $\omega_A = k_{\parallel f} v_A$ . Combining Eqs. (5) and (6), we find

$$\zeta_1^{\pm}(k_{\perp}) \sim (\varepsilon \omega_A)^{1/4} k_{\perp}^{-3/2} \Rightarrow E_1^{\pm}(k_{\perp}) \sim (\varepsilon \omega_A)^{1/2} k_{\perp}^{-2}, \quad (7)$$

$$\zeta_0^{\pm}(k_{\perp}) \sim \left( \frac{\varepsilon}{\omega_A} \right)^{1/2} k_{\perp}^{-1} \Rightarrow E_0^{\pm}(k_{\perp}) \sim \frac{\varepsilon}{\omega_A} k_{\perp}^{-1}, \quad (8)$$

where  $E_n^{\pm}(k_{\perp})$  is the one-dimensional energy spectrum, related to the characteristic amplitudes via  $k_{\perp} E_n^{\pm}(k_{\perp}) \sim k_{\perp}^2 [\zeta_n^{\pm}(k_{\perp})]^2$ . The energy injection is balanced, so the the “+” and “−” spectra have the same scaling.

The phenomenological argument presented above has led to a prediction of the Alfvén-wave spectrum [Eq. (7)] that is formally the same as the prediction of the standard weak-turbulence theory (this is only true for the balanced case; see below). However, the physical origin of this spectrum is different and the assumption of continuity across  $k_{\parallel} = 0$  is certainly not satisfied:

$$\frac{E_0^{\pm}(k_{\perp})}{E_1^{\pm}(k_{\perp})} \sim \frac{\tau_A}{\tau_{nl,0}} \sim \left( \frac{\tau_A}{\tau_{nl,1}} \right)^2 \sim \frac{k_{\perp}}{k_c} \ll 1, \quad (9)$$

where  $\tau_A = \omega_A^{-1}$ ,  $\tau_{nl,0} \sim [k_{\perp}^2 \zeta_0^{\pm}(k_{\perp})]^{-1}$ ,  $\tau_{nl,1} \sim [k_{\perp}^2 \zeta_1^{\pm}(k_{\perp})]^{-1}$ , and  $k_c = (\omega_A^3 / \varepsilon)^{1/2}$ . The amplitudes of the  $k_{\parallel} = 0$  modes are smaller than those of the Alfvén waves as long as  $\tau_A \ll \tau_{nl}$ , i.e., as long as the weak-turbulence limit holds. However, their spectrum is shallower than that of the Alfvén waves, so the ratio between the amplitudes increases with  $k_{\perp}$  until they become comparable at  $k_{\perp} \sim k_c$ . At this point the turbulence becomes critically balanced and is no longer weak, developing a  $k_{\perp}^{-5/3}$  [4, 29] or perhaps  $k_{\perp}^{-3/2}$  [30] spectrum.

*Higher- $k_{\parallel}$  modes.* In the same way that a small leakage of energy from the forced modes ( $k_{\parallel} = k_{\parallel f}$ ) via oscillatory nonlinear couplings gives rise to a spectrum of  $k_{\parallel} = 0$  modes, similar leakages arise from  $k_{\parallel} = k_{\parallel f}$  to  $k_{\parallel} = 2k_{\parallel f}$  and from there onwards to  $k_{\parallel} = 3k_{\parallel f}$ ,  $4k_{\parallel f}$ , ...,  $n k_{\parallel f}$  (the third term on the right-hand of Eq. (3)).

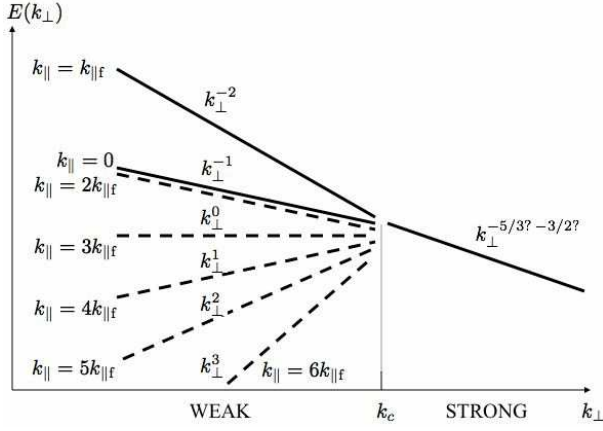


FIG. 1: A schematic illustration of the weak-turbulence spectra and transition to strong, critically balanced turbulence.

These modes do not contribute to the scattering of the driven modes and are just induced by the latter. Analogously to Eq. (6), it is not hard to see that for  $n \geq 2$ ,

$$\zeta_n^\pm(k_\perp) \sim \frac{k_\perp^{2(n-1)} [\zeta_1^\pm(k_\perp)]^n}{\omega_A^{n-1}}, \quad (10)$$

whence follow the spectra of these modes [using Eq. (7)]

$$E_n(k_\perp) \sim \frac{k_\perp^{3(n-1)}}{\omega_A^{2(n-1)}} E_1^n(k_\perp) \sim \frac{\varepsilon^{n/2}}{\omega_A^{(3n-4)/2}} k_\perp^{n-3}. \quad (11)$$

Similarly to Eq. (9), we have  $E_n(k_\perp)/E_1(k_\perp) \sim (k_\perp/k_c)^{n-1}$ , so the amplitudes of the  $k_\parallel = nk_{\parallel f}$  modes with  $n \geq 2$  become comparable to the amplitude of the forced mode ( $n = 1$ ) at the same wave number as does the amplitude of the  $k_\parallel = 0$  mode. This situation is illustrated schematically in Fig. 1.

*Imbalanced turbulence.* If the energy injection into the “+” and “−” Elsasser modes is not the same, say,  $\varepsilon^+ > \varepsilon^-$ , the resulting turbulence is known as *imbalanced* — this is, in fact, a generic situation in the solar wind [31, 32] and also in numerical simulations if one considers local subdomains of the simulation box [33].

Our arguments are easily adapted to this case. Eqs. (5) and (6) hold unchanged (note that the latter formula implies  $\zeta_0^+ \sim \zeta_0^-$ ). Eqs. (7) and (8) generalize to

$$\zeta_1^\pm(k_\perp) \sim \frac{(\varepsilon^\pm)^{3/8} \omega_A^{1/4}}{(\varepsilon^\mp)^{1/8} k_\perp^{3/2}} \Rightarrow E_1^\pm(k_\perp) \sim \omega_A^{1/2} \frac{(\varepsilon^\pm)^{3/4}}{(\varepsilon^\mp)^{1/4}} k_\perp^{-2}, \quad (12)$$

$$\zeta_0^\pm(k_\perp) \sim \frac{(\varepsilon^+ \varepsilon^-)^{1/4}}{\omega_A^{1/2} k_\perp} \Rightarrow E_0^\pm(k_\perp) \sim \frac{(\varepsilon^+ \varepsilon^-)^{1/2}}{\omega_A} k_\perp^{-1}, \quad (13)$$

These imply that the cross-helicity and the Elsasser ratio in weak imbalanced turbulence are independent of  $k_\perp$ :

$$E_0^+(k_\perp) \sim E_0^-(k_\perp), \quad \frac{E_1^+(k_\perp)}{E_1^-(k_\perp)} \sim \frac{\varepsilon^+}{\varepsilon^-}, \quad (14)$$

Note that the standard weak-turbulence treatment [5] of the imbalanced case was insufficient to fix the individual scalings of the “+” and “−” spectra [17]. Our argument does not have this problem and is able predict the relationship between amplitudes and fluxes [Eq. (14)], for which the standard weak-turbulence theory had to make recourse to solving the kinetic equation, formally invalid for the  $k_\parallel = 0$  modes [16].

The  $k_\parallel = 0$  modes are still small compared to the Alfvénic modes:

$$\frac{E_0^\pm(k_\perp)}{E_1^\pm(k_\perp)} \sim \left( \frac{\tau_A}{\tau_{nl,1}} \right)^2 \sim \frac{k_\perp}{k_c^\pm} \ll 1, \quad (15)$$

where the nonlinear times  $\tau_{nl,1}^\pm \sim [k_\perp^2 \zeta_1^\mp(k_\perp)]^{-1}$  are now different for the “+” and “−” modes and so are the wave numbers  $k_c^\pm = \omega_A^{3/2} (\varepsilon^\pm)^{1/4} (\varepsilon^\mp)^{-3/4}$  at which the weak-turbulence approximation breaks down. It must of course break down already at the smaller of the two, viz.,  $k_c^-$  (we have assumed  $\varepsilon^+ > \varepsilon^-$ ). This opens the possibility of a “twilight range” of  $k_\perp$  between  $k_c^-$  and  $k_c^+$  (or another threshold dependent on the intermediate scalings) — an extended transition from weak to strong regime, where the “−” modes are strongly nonlinear, while the “+” modes are not. It is indeed possible to construct such mixed theories, featuring steeper spectra for the “+” modes and shallower ones for the “−” modes (cf. [34]). We are tempted to speculate that this might help explain the origin of apparently non-universal (and different) slopes of the “+” and “−” spectra found in simulations of strongly imbalanced Alfvénic turbulence [35, 36]. However, the more or less arbitrary assumptions necessary to fix scalings in this regime and the consequent uncertainties in the outcome are so numerous that we prefer not to treat this subject here. The precise scalings of strong turbulence in the imbalanced regime also remain theoretically uncertain [33–35, 37, 38].

*Case of hydrodynamically forced  $k_\parallel = 0$  modes.* We have so far considered a special case in which the  $k_\parallel = 0$  modes were not forced. Let us now allow comparable power to be injected into  $k_\parallel = 0$  as into  $k_\parallel = k_{\parallel f}$ . The situation is now radically different because the amplitude of the  $k_\parallel = 0$  modes is no longer determined by a small leakage of the Alfvén-wave energy via the oscillatory term in Eq. (4), but by a direct forcing. If we ignore the oscillatory term altogether (assuming it averages out to lowest order in  $\tau_A/\tau_{nl}$ ), the  $k_\parallel = 0$  modes decouple and form an independent 2D turbulent condensate. We write the equations for this condensate in terms of its velocities, given by the stream function  $\Phi_0 = (\zeta_0^+ + \zeta_0^-)/2$ , and magnetic fields, given by the flux function  $\Psi_0 = (\zeta_0^+ - \zeta_0^-)/2$ :

$$\partial_t \nabla_\perp^2 \Phi_0 + \{\Phi_0, \nabla_\perp^2 \Phi_0\} = \{\Psi_0, \nabla_\perp^2 \Psi_0\} + \nabla_\perp^2 F_0, \quad (16)$$

$$\partial_t \Psi_0 + \{\Phi_0, \Psi_0\} = \text{Re} \sum_{k_\parallel \neq 0} \{\zeta_{k_\parallel}^+, \zeta_{-k_\parallel}^-\} e^{-i2k_\parallel v_A t}, \quad (17)$$



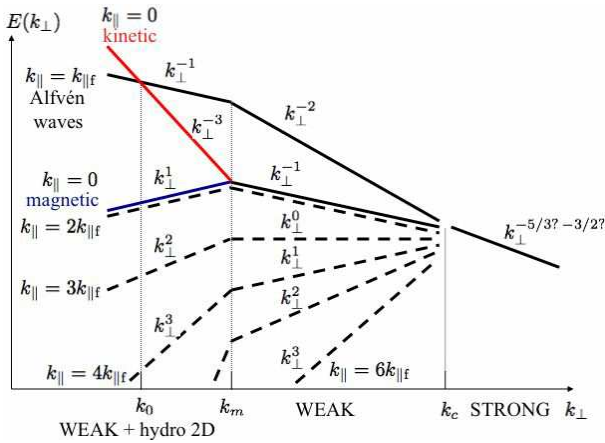


FIG. 2: A schematic illustration of the weak-turbulence spectra for the case of velocity-forced  $k_{\parallel} = 0$  modes.

where we have returned to the assumption that all forcing is in the velocity field. Therefore, the magnetic flux function  $\Psi_0$  is a passive scalar and its only source of energy is the oscillatory coupling to the Alfvén waves, which has consequently been retained in Eq. (17). Since this energy source is small,  $\Psi_0 \ll \Phi_0$  and the Lorentz force can be neglected in Eq. (16), leaving a 2D Euler equation. For the forced Alfvén waves with  $k_{\parallel} = k_{\parallel f}$ , we have from from Eq. (3), neglecting the oscillatory terms and  $\Psi_0$ ,

$$\partial_t \zeta_1^{\pm} = N[\Phi_0, \zeta_1^{\pm}] + F_1 e^{\mp i k_{\parallel} v_A t}. \quad (18)$$

The unforced Alfvén waves with  $k_{\parallel} \neq k_{\parallel f}$  will have small amplitudes due to the oscillatory terms in Eq. (3).

Thus, the turbulence has split into the following distinct components: a 2D ( $k_{\parallel} = 0$ ) forced *hydrodynamical* condensate, an ensemble of forced Alfvén waves passively advected by this 2D condensate [Eq. (18)], small 2D magnetic fluctuations feeding off the Alfvén waves and also passively advected by the 2D hydrodynamic condensate [Eq. (17)], and small unforced Alfvén waves again passively advected by the 2D condensate and feeding off the forced modes and each other (in a way similar to that described above for the case of no forcing of the  $k_{\parallel} = 0$  modes). This situation is somewhat similar to the “slaved” regime proposed in [19], where everything is unilaterally controlled by the 2D condensate.

Since the condensate is hydrodynamical and 2D, one expects an inverse energy cascade to perpendicular scales larger than the forcing scale (in a finite system leading to energy accumulation at the system scale) [45]. Below the forcing scale, the direct enstrophy cascade produces a well-known kinetic-energy spectrum:

$$E_0(k_{\perp}) \sim k_{\perp} [\Phi_0(k_{\perp})]^2 \sim \gamma_0^2 k_{\perp}^{-3}, \quad (19)$$

where  $\gamma_0 = (k_{\perp f}^2 \varepsilon_0)^{1/3}$  is the rate of strain (the same for all modes),  $\varepsilon_0 = \langle (\nabla_{\perp} \Phi_0) \cdot (\nabla_{\perp} F_0) \rangle$  the mean power injected into the  $k_{\parallel} = 0$  modes and  $k_{\perp f}$  the perpendicular

wave number of the forcing [46]. The spectra of the forced Alfvén waves and the  $k_{\parallel} = 0$  magnetic fluctuations follow from Eq. (5) with  $\zeta_0^{\pm} = \Phi_0$  and Eq. (6) with  $\zeta_0^{\pm} = \Psi_0$ , respectively:

$$E_1^{\pm}(k_{\perp}) \sim k_{\perp} [\zeta_1^{\pm}(k_{\perp})]^2 \sim \frac{\varepsilon_1}{\gamma_0} k_{\perp}^{-1}, \quad (20)$$

$$M_0(k_{\perp}) \sim k_{\perp} [\Psi_0(k_{\perp})]^2 \sim \left( \frac{\varepsilon_1}{\omega_A \gamma_0} \right)^2 k_{\perp}, \quad (21)$$

where  $\varepsilon_1$  is the mean power injected into the  $k_{\parallel} = k_{\parallel f}$  Alfvén waves (assumed balanced). This situation is illustrated in Fig. 2. The intersection wave number between  $E_0$  and  $E_1$  is  $k_0 = (\gamma_0^3 / \varepsilon_1)^{1/2} \sim k_{\perp f}$  if  $\varepsilon_0 \sim \varepsilon_1$ . The intersection between  $E_0$  and  $M_0$  is at  $k_m = (\gamma_0^2 \omega_A / \varepsilon_1)^{1/2}$ . For  $k_{\perp} > k_m$ , the rate of strain  $\gamma_0$  of the hydrodynamic condensate is smaller than the rate of strain associated with the  $k_{\parallel} = 0$  modes induced by the oscillatory coupling to the Alfvén waves ( $\gamma_0 \tau_{nl,0} < 1$ ) and so the turbulence reverts to the regime described earlier for the case of no forcing of the  $k_{\parallel} = 0$  modes.

*Other regimes.* Possibilities proliferate if we allow imbalanced energy injection and/or let the  $k_{\parallel} = 0$  modes be forced magnetically as well as hydrodynamically. The majority of the resulting regimes are probably not physically realizable, so we will not pursue this line of inquiry further, except for the following observation. If  $\Psi_0$  is forced, this direct injection of energy into 2D magnetic fluctuations will in general trump the oscillatory coupling to Alfvén waves (the right-hand side of Eq. (17)). Thus, the 2D condensate decouples fully from the latter. Magnetic field of the  $k_{\parallel} = 0$  modes is now dynamically significant, so there will be a 2D inverse cascade of the variance of the magnetic flux function,  $|\Psi_0|^2$ , and a direct cascade of the two Elsasser energies  $|\nabla_{\perp} \zeta_0^{\pm}|^2$ . The inverse cascade of  $|\Psi_0|^2$  to  $k_{\perp} < k_{\perp f}$  produces a  $k_{\perp}^{-1/3}$  spectrum of magnetic energy and possibly a shallower spectrum of the kinetic energy [39]. This may well be the physical mechanism responsible for the relative preponderance of magnetic energy over kinetic at large scales in weak MHD turbulence [21]. The direct cascade is numerically known to give rise to  $k_{\perp}^{-3/2}$  spectra [40–42][47]. The Alfvén waves ( $k_{\parallel} = k_{\parallel f}$ ) will be passively mixed by this strong 2D MHD turbulence. However, by the causality argument given in the Introduction, the latter is, in fact, likely to be unstable, develop parallel decorrelations and become strong, 3D and critically balanced.

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- [43] Physically speaking, this is the only case we are allowed to model by Eq. (1) because we must in fact set  $F^+ = F^-$  in order to ensure that only velocity field is directly forced. Indeed, if  $F^+ - F^- \neq 0$ , we would be introducing an inhomogeneous forcing term in the induction equation, leading to unphysical breaking of the magnetic flux conservation at the forcing scale. This said,  $F^+ \neq F^-$  is a popular modeling choice for simulating imbalanced MHD turbulence and we will consider this case later on.
- [44] A reader who does not find this estimate obvious on dimensional grounds, may be convinced by the following argument. Consider a model stochastic equation for the  $k_{\parallel} = 0$  modes:  $\partial_t \zeta_0 = -\zeta_0/\tau_c + \chi(t)e^{-i\omega_A t}$ , where  $\tau_c \sim \tau_{nl,0}$  is the decorrelation time due to nonlinear mixing of  $\zeta_0$  by and the  $\chi$  term models the oscillatory coupling to the Alfvén waves [see Eq. (4)]. The forcing amplitude  $\chi(t) \sim k_{\perp}^2 \zeta_1^+ \zeta_1^-$  also has decorrelation time  $\tau_c$  because  $\zeta_1^{\pm}$  are also mixed by  $\zeta_0$  [see Eq. (3)]. Taking  $\langle \chi(t)\chi^*(t') \rangle = \langle |\chi|^2 \rangle e^{-|t-t'|/\tau_c}$  and solving our model equation, we obtain  $\langle |\zeta_0|^2 \rangle \sim \langle |\chi|^2 \rangle / \omega_A^2$ , q.e.d.
- [45] An inverse cascade in weak MHD turbulence has recently been reported [22]. The regime with an inverse cascade and spectra similar to Fig. 2 was also found by T. A. Yousef in unpublished numerical work (2008).
- [46] In a finite box, the energy will accumulate at the box wave number  $k_{\perp \text{box}}$  and so  $\gamma_0 = k_{\perp \text{box}}(\varepsilon_0 t)^{1/2}$ .
- [47] A possible explanation for this in the form of an adaptation of the dynamical alignment argument [30] to 2D is found at the end of Sec. 4 of [18].