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Stable Splitting of Bivariate Splines Spaces by Bernstein-Bézier Methods

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Abstract. We develop stable splitting of the minimal determining sets for the spaces of bivariate C^1 splines on triangulations, including a modified Argyris space, Clough-Tocher, Powell-Sabin and quadrilateral macro-element spaces. This leads to the stable splitting of the corresponding bases as required in Böhmer's method for solving fully nonlinear elliptic PDEs on polygonal domains.

Keywords: Fully nonlinear PDE, Monge-Ampère equation, multivariate splines, Bernstein-Bézier techniques

1 Introduction

Numerical solution of fully nonlinear elliptic partial differential equations is a topic of intensive research and great practical interest, see [2, 4]. Since no weak form formulation is available for the equations of this type in general, the standard Galerkin finite element method cannot be applied directly.

Recently, Böhmer [1, 2] introduced a general approach that solves the Dirichlet problem for fully nonlinear elliptic equations numerically with the help of a sequence of linear elliptic equations used within an appropriate Newton scheme. These linear elliptic equations can be solved by the finite element method, but the discretisation has to be done by appropriate spaces of C^1 finite elements (splines) that admit a stable splitting into a subspace satisfying zero boundary conditions, and its complement. Such a stable splitting has been developed in [6] for a modified space of the Argyris finite element.

In this paper we systematically study the problem of stable splitting for the spaces of bivariate C^1 splines on triangulations of low degree using the Bernstein-Bézier methods. It turns out that stable splitting can be easily formulated as splitting of the minimal determining sets (MDS). We revisit the modified Argyris space studied in [6] by a different technique, and show that its modification is necessary at least if the convenient MDS splitting approach is used. We also show that Clough-Tocher, Powell-Sabin and quadrilateral macro-element spaces admit the stable splitting and therefore can also be used in the Böhmer's numerical method.

The paper is organised as follows. Section 2 is devoted to an outline of Böhmer's method, whereas Section 3 introduces necessary definitions from the theory of Bernstein-Bézier methods [8], and defines the stable splitting of an MDS. In Section 4 we discuss the stable splitting for the Argyris space and its modification, and Section 5 is devoted to the C^1 macro-element spaces.

2 Böhmer's Method for Fully Nonlinear Elliptic PDEs

2.1 Fully Nonlinear Elliptic Operators

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $G : H^\gamma(\Omega) \to L^2(\Omega)$, $\gamma \geq 2$, be a second order differential operator of the form

$$G(u) = \widetilde{G}(\cdot, u, \nabla u, \nabla^2 u),$$

where $\widetilde{G}$ is a real valued function defined on a domain $\widetilde{\Omega} \times \Gamma$ such that

$$\overline{\Omega} \subset \widetilde{\Omega} \subset \mathbb{R}^n \quad \text{and} \quad \Gamma \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n},$$

and $\nabla u, \nabla^2 u$ denote the gradient and the Hessian of u, respectively. The points in $\widetilde{\Omega} \times \Gamma$ are denoted by $w = (x, z, p, r)$, with $x \in \widetilde{\Omega}$, $z \in \mathbb{R}$, $p = [p_i]_{i=1}^n \in \mathbb{R}^n$, $r = [r_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$, to indicate the product structure of this set.

The operator G is said to be *elliptic* in a subset $\widetilde{\Gamma} \subset \widetilde{\Omega} \times \Gamma$ if the matrix $[\frac{\partial \widetilde{G}}{\partial r_{ij}}(w)]_{i,j=1}^n$ is well defined and positive definite for all $w \in \widetilde{\Gamma}$ [2, 7]. If $\widetilde{G}$ is a linear function of (z, p, r) for each fixed x, then G is a linear differential operator. Under suitable restrictions on $\widetilde{G}$, classes of quasilinear and semilinear differential operators are obtained [2, p. 80], but in general G may be fully nonlinear.

In the neighborhood of a fixed function $\hat{u} \in H^\gamma(\Omega)$ the linear elliptic operator $G'(\hat{u})$ is defined by

$$G'(\hat{u})u = \frac{\partial \widetilde{G}}{\partial z}(\hat{w})u + \sum_{i=1}^n \frac{\partial \widetilde{G}}{\partial p_i}(\hat{w})\partial^i u + \sum_{i,j=1}^n \frac{\partial \widetilde{G}}{\partial r_{ij}}(\hat{w})\partial^i\partial^j u,$$

where $\hat{w} = (x, \hat{u}(x), \nabla\hat{u}(x), \nabla^2\hat{u}(x))$ is a function of $x \in \Omega$, and ∂^i denotes the partial derivative with respect to the i-th variable. If $G : H^\gamma(\Omega) \to L^2(\Omega)$ is Fréchet differentiable at $\hat{u}$, then $G'(\hat{u}) : H^\gamma(\Omega) \to L^2(\Omega)$ is its Fréchet derivative. If $G'(\hat{u})$ depends continuously on $\hat{u}$ with respect to the linear operator norm, then G is said to be *continuously differentiable* at $\hat{u}$.

Many nonlinear elliptic operators and corresponding equations $G(u) = 0$ are important for applications, for example the *Monge-Ampère equation* for $\Omega \subset \mathbb{R}^2$, given by

$$G_{\text{MA}}(u) := \det(\nabla^2 u) - f(x) = 0, \quad f(x) > 0 \text{ for } x \in \Omega.$$

The operator G_{MA} is fully nonlinear and $G_{\text{MA}}(u) \in L^2(\Omega)$ if u belongs to the Sobolev space $H^{5/2}(\Omega)$ and $f \in L^2(\Omega)$. Moreover, $G_{\text{MA}} : H^\gamma(\Omega) \to L^2(\Omega)$ is continuously differentiable if $\gamma \geq 5/2$.

We consider the Dirichlet problem for the operator G: Find u such that

$$G(u) = 0, \quad x \in \Omega \tag{1}$$
$$u = \phi, \quad x \in \partial\Omega \tag{2}$$

where ϕ is a continuous function defined on $\partial\Omega$. Under certain assumptions (including the exterior sphere condition for $\partial\Omega$ and sufficient smoothness of $\widetilde{G}$, satisfied in particular in the above mentioned examples if $f \in C^2(\Omega)$), this problem has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ [7, Theorem 17.17]. Note that the Monge-Ampère operator G_{MA} is elliptic in subsets $\widetilde{\Gamma}$ satisfying

$$\widetilde{\Gamma} \subset \widetilde{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \{r \in \mathbb{R}^{n\times n} : r \text{ is positive definite}\}.$$

Therefore there exists a unique convex solution of $G_{\mathrm{MA}}(u) = 0$, whereas it is known that the Monge-Ampère equation has another, concave solution [3, Chapter 4].

2.2 Spline Spaces and Stable Splitting

As usual in the finite element method, the discretisation of the Dirichlet problem is done with the help of spaces of piecewise polynomial functions (splines). Let $\triangle$ be a triangulation of a polyhedral domain $\Omega \subset \mathbb{R}^n$, that is a partition of Ω into simplices such that the intersection of every pair of simplices is either empty or a common face. The space of multivariate splines of degree d and smoothness r is defined by

$$S_d^r(\triangle) = \{s \in C^r(\Omega) : s|_T \in P_d \text{ for all simplices } T \text{ in } \triangle\}, \tag{3}$$

where $d > r \geq 0$ and P_d is the space of polynomials of total degree d in n variables. Recall that the *star* of a vertex v of $\triangle$, denoted by $\mathrm{star}(v) = \mathrm{star}^1(v)$, is the union of all triangles $T \in \triangle$ attached to v. We define $\mathrm{star}^j(v)$, $j \geq 2$, inductively as the union of the stars of all vertices of $\triangle$ contained in $\mathrm{star}^{j-1}(v)$.

Let $\{\triangle^h\}_{h\in H}$ be a family of triangulations of Ω, where h is the maximum edge length in $\triangle^h$. The triangulations in the family are said to be *quasi-uniform* if there is an absolute constant $c > 0$ such that $\rho_T \geq ch$ for all $T \in \triangle^h$, where ρ_T denotes the radius of the inscribed sphere of the simplex T.

Let $S^h \subset S_d^r(\triangle^h)$ be a linear subspace with basis $s_1, \ldots, s_N$ and dual functionals $\lambda_1, \ldots, \lambda_N$ such that $\lambda_i s_j = \delta_{ij}$. This basis is *stable* and *local* if there are three constants $m \in \mathbb{N}$ and $C_1, C_2 > 0$ independent of h such that (a) $\mathrm{supp}\, s_k$ is contained in $\mathrm{star}^m(v)$ for some vertex v of $\triangle^h$, (b) $\|s_k\|_{L^\infty(\Omega)} \leq C_1$, $k = 1, \ldots, N$, and (c) $|\lambda_k s| \leq C_2 \|s\|_{L^\infty(\mathrm{supp}\, s_k)}$, $k = 1, \ldots, N$, for all $s \in S^h$, see [5, 6] and [2, Section 4.2.6].

To handle the Dirichlet boundary conditions, the following subspace of S^h is important:

$$S_0^h := \left\{s \in S^h : s|_{\partial\Omega} = 0\right\}.$$

Moreover, the method of solving (1)–(2) proposed in [1, 2] requires a *stable splitting* of S^h into a direct sum

$$S^h = S_0^h + S_b^h,$$

such that a stable local basis $\{s_1, \ldots, s_N\}$ for S^h can be split into two parts

$$\{s_1, \ldots, s_N\} = \{s_1, \ldots, s_{N_0}\} \cup \{s_{N_0+1}, \ldots, s_N\},$$

where $\{s_1, \ldots, s_{N_0}\}$ and $\{s_{N_0+1}, \ldots, s_N\}$ are bases for S_0^h and S_b^h, respectively. Note that the space S_b^h is not uniquely defined by the pair S^h, S_0^h. It was shown in [6] (see also [2, Section 4.2.6]) how the stable splitting can be achieved for a modified space of Argyris finite element.

2.3 Böhmer's Method

Let $u = \hat{u}$ be the solution of (1)–(2). According to [1], its approximation $\hat{u}^h \approx \hat{u}$ is sought as a solution of the following problem: Find $\hat{u}^h \in S^h$ such that

$$(G(\hat{u}^h), v^h)_{L^2(\Omega)} = 0 \quad \forall v^h \in S_0^h, \quad \text{and} \tag{4}$$

$$(\hat{u}^h, v_b^h)_{L^2(\partial\Omega)} = (\phi, v_b^h)_{L^2(\partial\Omega)} \quad \forall v_b^h \in S_b^h, \tag{5}$$

where $(\cdot,\cdot)$ denotes the inner products in the respective Hilbert spaces. Since S_0^h and S_b^h are finite dimensional linear spaces, the problem (4)–(5) is equivalent to a system of algebraic equations with respect to the coefficients of $\hat{u}^h$ in a basis of S^h.

Theorem 1 ([1, Theorem 8.7] and [2, Theorem 5.2]). *Let Ω be a bounded convex polyhedral domain, and let $G : D(G) \to L^2(\Omega)$, with $D(G) \subset H^2(\Omega)$, satisfy Condition H of [2, Section 5.2.3]. Assume that G is continuously differentiable in the neighbourhood of an isolated solution $\hat{u}$ of (1)–(2), such that $\hat{u} \in H^\ell(\Omega)$, $\ell > 2$, and $G'(\hat{u}) : D(G) \cap H_0^1(\Omega) \to L^2(\Omega)$ is boundedly invertible. Furthermore, assume that the spline spaces $S^h \subset S_d^1(\triangle^h)$, $d \geq \ell - 1$, on quasi-uniform triangulations $\triangle^h$ possess stable local bases and stable splitting $S^h = S_0^h + S_b^h$, and include polynomials of degree $\ell - 1$. Then the problem (4)–(5) has a unique solution $\hat{u}^h \in S^h$ as soon as the maximum edge length h is sufficiently small. Moreover,*

$$\|\hat{u} - \hat{u}^h\|_{H^2(\Omega)} \leq Ch^{\ell-2}\|\hat{u}\|_{H^\ell(\Omega)}.$$

In particular, Condition H is satisfied by the Monge-Ampère operators on bounded convex polygonal domains in $\mathbb{R}^2$.

The nonlinear problem (4)–(5) can be solved iteratively by a Newton method [1], where the initial guess $u_0^h \in S^h$ satisfies the boundary condition

$$(u_0^h, v_b^h)_{L^2(\partial\Omega)} = (\phi, v_b^h)_{L^2(\partial\Omega)} \quad \forall v_b^h \in S_b^h,$$

and the sequence of approximations $\{u_k^h\}_{k\in\mathbb{N}}$ of $\hat{u}^h$ is generated by

$$u_{k+1}^h = u_k^h - w^h, \quad k = 0, 1, \ldots,$$

with $w^h \in S_0^h$ being the solution of the linear elliptic problem:

$$\text{Find } w^h \in S_0^h \text{ such that } (G'(u_k^h)w^h, v^h)_{L^2(\Omega)} = (G(u_k^h), v^h)_{L^2(\Omega)} \; \forall v^h \in S_0^h.$$

Clearly, w^h can be found by using the standard finite element method. Under some additional assumptions on G, it is proved in [1, Theorem 9.1] that u_i^h converges to $\hat{u}$ quadratically. Note that in the case when $G(u)$ is only conditionally elliptic (e.g. elliptic only for a convex u for Monge-Ampère equation) the ellipticity of the above linear problem is only guaranteed for u_k^h sufficiently close to the exact solution $\hat{u}$.

3 Bernstein-Bézier Techniques

Certain spaces of bivariate C^1 splines with stable local bases and stable splitting required in Böhmer's method have been investigated by nodal techniques in [6]. However, Bernstein-Bézier methods are often preferable. Let us recall some related key concepts here, see [8] for more details.

From now on we only consider the bivariate case. In particular, Ω is a polygonal domain in $\mathbb{R}^2$ and $\triangle$ is a triangulation of Ω.

Given $d \geq 1$, let $D_{d,\triangle} := \bigcup_{T\in\triangle} D_{d,T}$ be the set of *domain points*, where

$$D_{d,T} := \left\{ \xi_{ijk} = \frac{iv_1 + jv_2 + kv_3}{d} \right\}_{i+j+k=d}$$

for each triangle $T := \langle v_1, v_2, v_3 \rangle$ in $\triangle$. Also note that every $v \in \mathbb{R}^2$ can be uniquely represented in the form

$$v = \sum_{i=1}^{3} b_i v_i, \quad \sum_{i=1}^{3} b_i = 1.$$

The triplet (b_1, b_2, b_3) is called the *barycentric coordinates* of v relative to the triangle $T := \langle v_1, v_2, v_3 \rangle$, and

$$B_{ijk}^d(v) := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \quad i + j + k = d,$$

are the *Bernstein-Bézier basis polynomials* of degree d associated with triangle T. Every polynomial p of total degree d can be written uniquely as

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d,$$

where c_{ijk} are the *Bézier coefficients* of p. For each $s \in S_d^r(\triangle)$ and $\xi = \xi_{ijk} \in D_{d,\triangle}$ we denote by c_ξ the coefficient c_{ijk} of the restriction of s to any triangle

$T \in \triangle$ containing ξ. (Because of the continuity of s the coefficient c_ξ does not depend on the particular choice of such triangle.)

We now introduce two additional notations. We refer to the set

$$R_n(v_1) := \{\xi_{ijk} \in D_{d,\triangle} : i = d - n\}, \quad 0 \le n \le d,$$

of domain points as the *ring* of radius n arround the vertex v_1 and refer to the set

$$D_n(v_1) := \bigcup_{m=0}^{n} R_m(v_1)$$

as the *disk* of radius n arround the vertex v_1.

A key concept for dealing with spline spaces is that of a minimal determining set. Recall that the set $M \subset D_{d,\triangle}$ is a *determining set* for a linear space $S \subset S_d^r(\triangle)$ if

$$s \in S \text{ and } c_\xi = 0 \quad \forall \xi \in M \quad \Rightarrow \quad s = 0,$$

and M is a *minimal determining set* (MDS) for the space S if there is no smaller determining set. Then $\dim S$ equals the cardinality $\#\{M\}$ of M. Let

$$\Gamma_\eta := \{\xi \in M \;:\; c_\eta \text{ depends on } c_\xi\},$$

where we say that c_η depends on c_ξ, $\xi \in M$, if the value of c_η is changed when we change the value of c_ξ. A minimal determining set M for a space S is said to be *local* if there exists an absolute integer constant ℓ not depending on $\triangle$ such that

$$\Gamma_\eta \subset \text{star}^\ell(T_\eta) \quad \forall \eta \in D_{d,\triangle} \backslash M,$$

where T_η is a triangle containing η. And M is called *stable* if there exists a constant K which may depend only on d, ℓ and the smallest angle $\theta_\triangle$ in the triangulation $\triangle$ such that

$$|c_\eta| \le K \max_{\xi \in \Gamma_\eta} |c_\xi| \quad \forall \eta \in D_{d,\triangle} \backslash M.$$

Given a stable local minimal determining set M for $S \subset S_d^r(\triangle)$, a stable local basis $\{s_\xi\}_{\xi \in M}$ for S can be defined by requiring that the Bézier coefficients c_η, $\eta \in M$, of s_ξ satisfy $c_\xi = 1$ and $c_\eta = 0$ for all $\eta \in M \setminus \{\xi\}$, see [8, Section 5.8]. Clearly, a stable splitting of this basis is achieved by an appropriate splitting of the MDS, which leads to the following definition.

Definition 1. *Assume that the space $S \subset S_d^r(\triangle)$ has a stable local MDS M and let*

$$S_0 := \{s \in S \;:\; s|_{\partial\Omega} = 0\}. \tag{6}$$

The MDS M is said to admit a stable splitting *if M is the disjoint union of two subsets $M_0, M_b \subset M$ such that*

$$S_0 = \{s \in S : c_\xi = 0 \; \forall \xi \in M_b\} \tag{7}$$

and M_0 and M_b are stable local MDS for the spaces S_0 and S_b, respectively, where

$$S_b := \{s \in S : c_\xi = 0 \; \forall \xi \in M_0\}. \tag{8}$$

Note that if M is a stable local MDS, and $M = M_0 \cup M_b$ is a disjoint union, then it is a stable splitting as soon as (7) holds. Indeed, assume (7) is correct. If $s \in S_0$, then its coefficients related to M_b are zero, and similarly if $s \in S_b$ then its coefficient related to M_0 are zero. Hence computing s from coefficient corresponding to points in M_0 (respectively, M_b) is equivalent to computing from M, and so M_0 and M_b are determining sets for S_0 and S_b, respectively. They are minimal determining sets because otherwise M would not be minimal. Clearly, stability and locality properties of M_0 and M_b are also inherited from M.

If M admits a stable splitting, then $S = S_0 + S_b$ and it is easy to see that

$$\{s_\xi\}_{\xi\in M} = \{s_\xi\}_{\xi\in M_0} \cup \{s_\xi\}_{\xi\in M_b}$$

is a stable splitting of the stable local basis $\{s_\xi\}_{\xi\in M}$.

4 Stable Splitting for Argyris Finite Element

Recall that the superspline subspaces $S_d^{r,\rho}(\triangle)$, $r \le \rho \le d$, of $S_d^r(\triangle)$ are defined as

$$S_d^{r,\rho}(\triangle) = \{\, s \in S_d^r(\triangle) : s \in C^\rho(v)\ \forall v \in V\}\,, \tag{9}$$

where V is the set of all vertices of $\triangle$.

Consider the Argyris finite element space obtained with $d = 5$, $r = 1$ and $\rho = 2$ in (9). Now for each $v \in V$, let T_v be any one of the triangles sharing the vertex v and let $M_v := D_2(v) \cap T_v$. For each edge e of the triangulation $\triangle$, let $T_e := \langle v_1, v_2, v_3\rangle$ be one of the triangles sharing the edge $e := \langle v_2, v_3\rangle$ and let $M_e := \left\{\xi_{122}^{T_e}\right\}$. Then from [8, Theorem 6.1] we have

Theorem 2. $\dim S_5^{1,2}(\triangle) = 6\#\{V\} + \#\{E\}$ *and*

$$M = \bigcup_{v\in V} M_v \cup \bigcup_{e\in E} M_e \tag{10}$$

is a stable local minimal determining set for $S_5^{1,2}(\triangle)$.

An example is given in Figure 1 (left).

4.1 Modified Argyris Space

We now modify the Argyris space to achieve the stable splitting. This construction is discussed in term of nodal basis functions in [6]. We will explain in Section 4.3 why this modification is required. Let us denote the modified Argyris space by $\tilde S$, where

$$\tilde S := \left\{s \in S_5^1(\triangle) : s \in C^2(v),\ \text{for all } \textit{interior} \text{ vertices } v \text{ of } \triangle\right\}. \tag{11}$$

Let us now differentiate between boundary vertices and interior vertices by using V_I and V_B for the sets of interior and boundary vertices respectively. And let E_I and E_B denote interior and boundary edges respectively, such that

$$V = V_I \cup V_B, \quad E = E_I \cup E_B.$$

We describe a minimal determining set $\tilde{M}$ for this modified space $\tilde{S}$. Since we have modified the space only at the boundary vertices, so the points in M related to interior vertices and related to all edges, will belong to $\tilde{M}$. That is,

$$\left(\bigcup_{v \in V_I} M_v \cup \bigcup_{e \in E} M_e \right) \subset \tilde{M}.$$

However, we will have to modify the sets corresponding to the boundary vertices $v \in V_B$. First of all, we require that each T_v, $v \in V_B$, is a triangle sharing an edge with the boundary of Ω (we call it a *boundary triangle*). Furthermore, we add some more points to M_v, $v \in V_B$, as follows. Let us denote all edges of $\triangle$ emanating from a vertex $v \in V_B$, in counterclockwise order, by

$$E_v = \{e_1, e_2, \cdots, e_n\}.$$

Then clearly $e_1, e_n \in E_B$, and the triangle T_v is formed by either e_1, e_2 or e_{n-1}, e_n. For each e_i, let ξ_i be the (unique) domain point in $R_2(v) \cap e_i$, $i = 1, \dots, n$. We set

$$\tilde{M}_v := M_v \cup \{\xi_1, \xi_2, \cdots, \xi_n\}.$$

Theorem 3. $\dim \tilde{S} = 6\#\{V_I\} + \#\{E\} + \sum_{v \in V_B} (4 + \#E_v)$ *and*

$$\tilde{M} := \bigcup_{v \in V_I} M_v \cup \bigcup_{e \in E} M_e \cup \bigcup_{v \in V_B} \tilde{M}_v. \tag{12}$$

is stable local MDS for modified Argyris space $\tilde{S}$.

Proof. We set the coefficients $\{c_\xi\}_{\xi \in \tilde{M}}$ for any spline $s \in \tilde{S}$ to arbitrary values and show that all other coefficients, i.e. $\{c_\xi\}_{\xi \in D_{5,\triangle} \setminus \tilde{M}}$, of s can be determined consistently.

Now first note that for each $v \in V_I$ and for each $e \in E$ the points in M_v and M_e are the same as for Argyris space. So we only need to prove that for each $v \in V_B$ the set $\tilde{M}_v$ is an MDS on $D_2(v)$. To this end, for each $v \in V_B$, we set the coefficients of s corresponding to points in $\tilde{M}_v$ and see that, in view of C^1 smoothness conditions, all coefficients corresponding to domain points in $D_2(v)$ can be determined consistently. Thus by [8, Theorem 5.15] $\tilde{M}$ is minimal determining set for the space $\tilde{S}$. Observe that $\tilde{M}$ is a stable MDS. Indeed, for each $v \in V_I$ and all edges $e \in E$ the stability follows from [8, Lemma 2.29]. And for each $v \in V_B$ the set $\tilde{M}_v$ is a stable MDS for S_5^1 on $D_2(v)$ by [8, Theorem 11.7]. Standard arguments show that $\tilde{M}$ is local. □

The minimal determining sets for Argyris space and for modified Argyris space over a small triangulation with nine triangles are illustrated in Figure 1.

4.2 Stable Splitting

Now we show how to determine a stable splitting $\tilde{M} = \tilde{M}_0 \cup \tilde{M}_b$ of the MDS $\tilde{M}$ for modified Argyris space $\tilde{S}$.

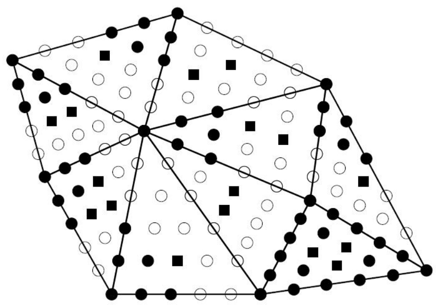

Fig. 1. Minimal determining sets for the Argyris space (*left*) and for the modified modified Argyris space (*right*). The points in the sets M_v, $\tilde{M}_v$ are marked by black dots, and those in M_e by black squares.

It is already understood that all those points of $\tilde{M}$ which are on the boundary will be in $\tilde{M}_b$ and those points lying in M_v, $v \in V_I$, and M_e along with the points in $R_2(v)$, $v \in V_B$, but not on either e_1 or e_n, will be in $\tilde{M}_0$. Consider, for each $v \in V_B$, the remaining point which lies in $R_1(v)$, $v \in V_B$, but not on the boundary edges. We denote this point by ξ_v. Whether ξ_v belongs to $\tilde{M}_0$ or $\tilde{M}_b = \tilde{M} \setminus \tilde{M}_0$ depends on the geometry of the boundary edges e_1 and e_n, as follows.

- If e_1 and e_n are non-collinear, then $\xi_v \in \tilde{M}_b$.
- If e_1 and e_n are collinear, then $\xi_v \in \tilde{M}_0$.

Indeed, in the non-collinear case the coefficient corresponding to ξ_v is zero for all $s \in \tilde{S}_0$, wheras in the collinear case it can be chosen freely. The Figures 2 and 3 show points in $\tilde{M}_0$ and $\tilde{M}_b$ for the boundary vertex with collinear and non-collinear edges respectively.

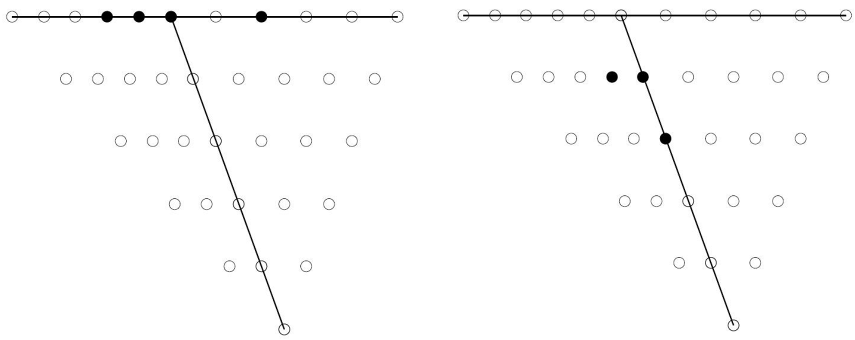

Fig. 2. Splitting of points in $\tilde{M}_v$, $v \in V_B$ for modified Argyris space with collinear boundary edges. *Left*: $\tilde{M}_v \cap \tilde{M}_b$, *right*: $\tilde{M}_v \cap \tilde{M}_0$.

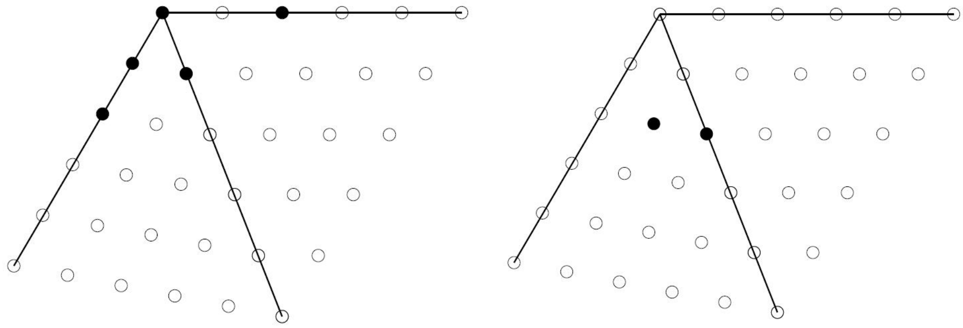

Fig. 3. Splitting of points in $\tilde{M}_v$, $v \in V_B$ for modified Argyris space with noncollinear boundary edges. *Left*: $\tilde{M}_v \cap \tilde{M}_b$, *right*: $\tilde{M}_v \cap \tilde{M}_0$.

Theorem 4. *$\tilde{M} = \tilde{M}_0 \cup \tilde{M}_b$ is stable splitting of MDS $\tilde{M}$.*

Proof. If $s \in \tilde{S}_0$, then all its Bézier coefficient on the boundary are zero since $s|_{\partial\Omega} = 0$. For those $v \in V_B$ where the boundary edges are non-collinear, the C^1 smoothness implies that the gradient at v is also zero, and hence the coefficient of s at ξ_v is also zero. This shows that $\tilde{S}_0 \subset \{s \in \tilde{S} : c_\xi = 0 \; \forall \xi \in \tilde{M}_b\}$. Conversely, assume $s \in \tilde{S}$ and $c_\xi = 0$ for all $\xi \in \tilde{M}_b$. Let $v \in V_B$ and $E_v = \{e_1, e_2, \cdots, e_n\}$ as before. Without loss of generality assume that $D_2(v) \cap e_1 \subset \tilde{M}_v$ and $R_2(v) \cap e_n \subset \tilde{M}_v$. Therefore $c_\xi = 0$ at all these points. However, due to the C^1 smoothness $c_\xi = 0$ also for the domain point in $R_1(v) \cap e_n$, both in the collinear and non-collinear case. This shows that $c_\xi = 0$ for all domain points on the boundary of Ω and hence $s|_{\partial\Omega} = 0$. Thus, $\tilde{S}_0 = \{s \in \tilde{S} : c_\xi = 0 \; \forall \xi \in \tilde{M}_b\}$, which completes the proof, see the discussion following Definition 1. □

4.3 Why Modification in Argyris Space is Required

We now prove that modification is needed in Argyris space at the boundary vertices to achieve a stable splitting.

We first consider the Argyris space $S_5^{1,2}(\triangle)$ with M in Theorem 2 being its MDS, and show that no splitting $M = M_0 \cup M_b$ is possible in this case if there is a boundary vertex v with two triangles attached, and the boundary edges are non-collinear. On contrary, assume that such a splitting has been found. Let $T := \langle v_1, v_2, v_3 \rangle$ and $\tilde{T} := \langle v_4, v_3, v_2 \rangle$ be two triangles in $\triangle$ with v_3 as boundary vertex and assume that the edges $\langle v_3, v_4 \rangle$ and $\langle v_3, v_1 \rangle$ are boundary edges. Consider the set

$$M_{v_3} := D_2(v_3) \cap T = \{\xi_{005}, \xi_{014}, \xi_{023}, \xi_{104}, \xi_{113}, \xi_{203}\} \subset M,$$

see the Figure 4, and let

$$s|_T = \sum_{i+j+k=5} c_{ijk} B^5_{ijk}, \quad s|_{\tilde{T}} = \sum_{i+j+k=5} \tilde{c}_{ijk} \tilde{B}^5_{ijk},$$

where B^5_{ijk} and $\tilde{B}^5_{ijk}$ are Bernstein basis polynomials associated with T and $\tilde{T}$ respectively. In the case that the edges $\langle v_3, v_4\rangle$ and $\langle v_3, v_1\rangle$ are non-collinear, the points $\{\xi_{005}, \xi_{014}, \xi_{104}, \xi_{203}\}$ must be in M_b, because $s \in S$ has zero coefficients at these points. We show that $\{\xi_{113}, \xi_{023}\} \not\subset M_0$. Let (b_1, b_2, b_3) be barycentric coordinates of v_4 relative to T. Then by a C^2 smoothness condition, see [8, Theorem 2.28], across the edge $e := \langle v_3, v_2\rangle$ we can write

$$\tilde{c}_{230} = b_1^2 c_{203} + 2b_1 b_2 c_{113} + 2b_2 b_3 c_{014} + b_2^2 c_{023} + 2b_1 b_3 c_{104} + b_3^2 c_{005},$$

and because $\tilde{c}_{230} = c_{203} = c_{014} = c_{104} = c_{005} = 0$,

$$0 = 2b_1 c_{113} + b_2 c_{023},$$

which shows that c_{113} and c_{023} are linearly dependent so that ξ_{113}, ξ_{023} cannot be both in M_0. Moreover, we cannot shift one of these points to M_b because there is a spline $s \in S_0$ such that

$$c_{113}, c_{023} \neq 0,$$

e.g. s with $c_{113} = b_2$ and $c_{023} = -2b_1$. Note that $b_2 \neq 0$ if the boundary edges are non-collinear.

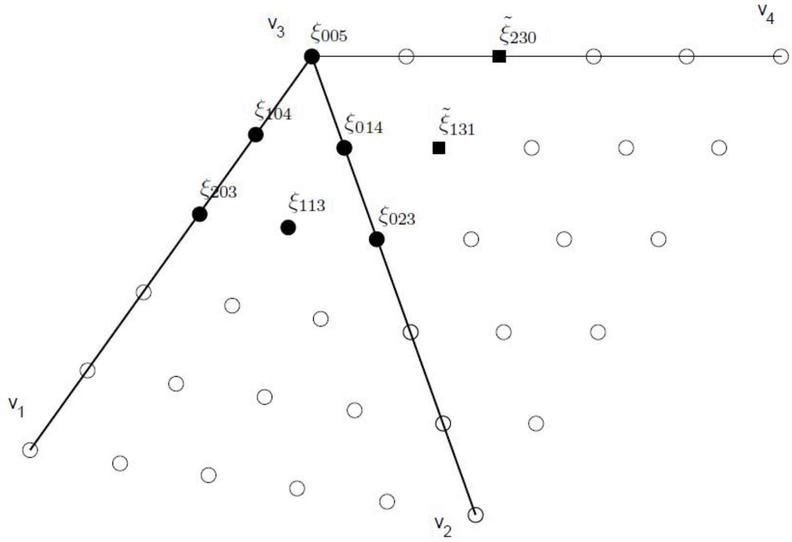

Fig. 4. The black dots are MDS points in $M_{v_3}, v_3 \in V_B$, for Argyris space. The two domain points marked by black squares are involved in the smoothness conditions discussed in the proof of Theorem 5.

Moreover, we prove that no other MDS admits a stable splitting, either.

Theorem 5. *No MDS for the Argyris space can be stably split on arbitrary triangulations.*

Proof. Assume that the triangulation $\triangle$ is such that there is a boundary vertex v with two triangles T and $\tilde{T}$ attached, and the boundary edges are non-collinear at v, as in the above proof. Let M be some MDS for Argyris space.

From the dimension argument we know that there must be exactly six points in $M \cap D_2(v)$. For the non-collinear boundary edges, no points on boundary edges or in $R_1(v)$ can be in M_0 because, all the corresponding coefficients of splines in S_0 are zero. So the only candidates for M_0 are the points in $R_2(v)$ not on boundary edges. Now we discuss the relation between the coefficients $\tilde{c}_{131}, c_{113}, c_{023}$ of $s \in S_0$ at these points. By using C^1 and C^2 condition across the common edge of T and $\tilde{T}$ we get

$$\begin{aligned} \tilde{c}_{131} &= b_1 c_{113} + b_2 c_{023} \\ 0 &= 2b_1 c_{113} + b_2 c_{023} \end{aligned}$$

By subtracting these equations we can write

$$\tilde{c}_{131} = -b_1 c_{113}$$

Hence the three coefficients cannot be set arbitrarily. Only one of them can be chosen freely, which cannot be either $\tilde{c}_{131}$ or c_{113}. Indeed, let us choose e.g. c_{113} arbitrarily, then from the above equations we obtain

$$c_{023} = \frac{-2b_1 c_{113}}{b_2}$$

and hence $c_{023} \to \infty$ for $b_2 \to 0$ as the boundary edges get collinear. This would be unstable as the minimum angles in $T, \tilde{T}$ do not degenerate.

Thus ξ_{023} is the only point to be in M_0. It is easy to see that M_b must contain $\xi_{203}, \tilde{\xi}_{230}$ and three points in $D_1(v)$. Consider the basis spline s in S_b corresponding to $\tilde{\xi}_{230}$. Then its coefficient satisfy

$$\tilde{c}_{230} = 1, \quad c_{203} = c_{023} = 0, \quad c_\xi = 0, \quad \xi \in D_1(v)$$

Now again using C^1 and C^2 conditions we find

$$\tilde{c}_{230} = 2b_1 b_2 c_{113} \;\text{ or }\; c_{113} = \frac{1}{2b_1 b_2},$$

which is unbounded for $b_2 \to 0$ as the boundary gets flat. □

Remark 1. If a boundary vertex v has exactly two triangles attached and the boundary edges are not collinear at v, then stable splitting of an MDS is impossible for any spline space S where each spline is C^2 continuous at v. Indeed, this follows by the arguments in the proof of Theorem 5. In fact, it is easy to see that the set $D_2(v) \cap T$ as MDS for S on $D_2(v)$ cannot be split stably for a boundary vertex with any number of triangles attached.

5 C^1 Macro-element Spaces

Now we discuss the possibility of stable splitting of minimal determining sets of some of the C^1 macro-element spaces.

5.1 Stable Splitting of Clough-Tocher Macro-element Space

Given a triangulation $\triangle$ of a domain Ω, let $\triangle_{CT}$ be corresponding Clough-Tocher refinement of $\triangle$, where each triangle is split into three subtriangles, see Figure 5.

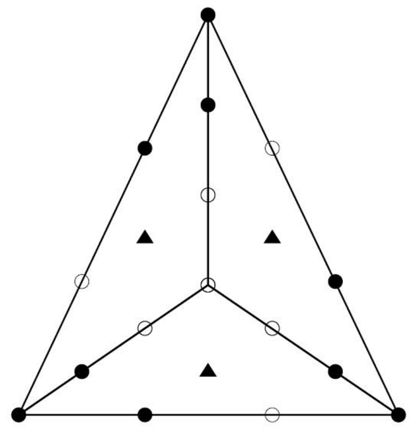

Fig. 5. A typical Clough-Tocher refinement of one triangle with points in M_v marked as black dots and points in M_e marked as black triangles.

Consider the stable local MDS M given in [8, Theorem 6.5] for C^1 Clough-Tocher Macro-element space $S_3^1(\triangle_{CT})$ as

$$M = \bigcup_{v \in V} M_v \cup \bigcup_{e \in E} M_e, \tag{13}$$

where $M_v := D_1(v) \cap T_v$ and $M_e := \left\{ \xi_{111}^{T_e} \right\}$, and T_v and T_e are triangles in $\triangle_{CT}$. Denote by V and E the sets of vertices and edges in $\triangle$, respectively. Let

$$S_0 := \left\{ s \in S_3^1(\triangle_{CT}) \ : \ s|_{\partial\Omega} = 0 \right\}.$$

Let V_I and V_B be the sets of interior and boundary vertices of $\triangle$, respectively. We assume that T_v is a boundary triangle for each M_v, $v \in V_B$. Then stable splitting for M is possible as follows. Clearly,

$$\left(\bigcup_{v \in V_I} M_v \cup \bigcup_{e \in E} M_e \right) \subset M_0. \tag{14}$$

However, M_0 may contain some more points from M_v, $v \in V_B$. Note that, for boundary vertices v, two points in M_v are always on the boundary and one is not. These two boundary points are in M_b but the point in M_v, which is not on the boundary, belongs to either M_0 or M_b depending on the geometry of boundary edges attached to v in the same way as the point ξ_v in Section 4.2. This point will be in M_0 for those boundary vertices where boundary edges are collinear. Otherwise it will be in M_b. Stability and locality follows as M is a stable local MDS for $S_3^1(\triangle_{CT})$.

5.2 Powell-Sabin Macro-element Space

Now let for a given triangulation $\triangle$ of a domain Ω, $\triangle_{PS}$ be the corresponding Powell-Sabin refinement [8, Definition 4.18], see the Figure 6. For each $v \in V$, let T_v be some triangle of $\triangle_{PS}$ attached to v, and $M_v := D_1(v) \cap T_v$. Then

$$M = \bigcup_{v \in V} M_v \tag{15}$$

is a stable local minimal determining set for Powell-Sabin space $S_2^1(\triangle_{PS})$ [8, Theorem 6.9]. Now similarly if

$$S_0 := \left\{ s \in S_2^1(\triangle_{PS}) \ : \ s|_{\partial\Omega} = 0 \right\}$$

and if we take T_v to be a boundary triangle for M_v, $v \in V_B$, then M given in (15) for $S_2^1(\triangle_{PS})$ can be split stably in the same way as discussed above for the Clough-Tocher macro-element space.

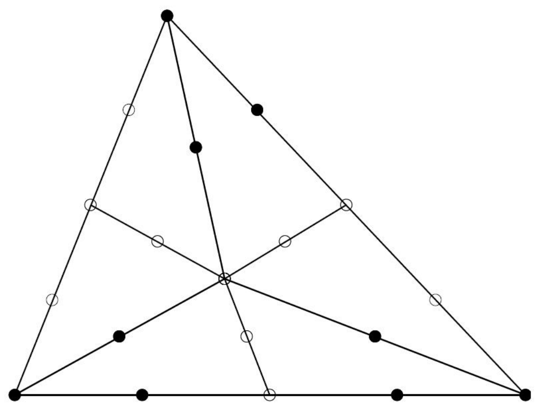

Fig. 6. Powell-Sabin refinement of one triangle with points in M_v marked as black dots.

5.3 Powell-Sabin-12 Macro-element Space

Let $\triangle_{PS12}$ be the Powell-Sabin-12 refinement [8, Definition 4.21] of a given triangulation $\triangle$ of a domain Ω, see Figure 7. For each e of $\triangle$, let u_e be the midpoint of e and let v_T be the incenter of a triangle T in $\triangle$ attached to e. Let $\xi_e := \frac{v_T + u_e}{2}$ and $M_e := \{\xi_e\}$. For each vertex $v \in V$, let T_v be a triangle of $\triangle_{PS12}$ attached to v, and let $M_v := D_1(v) \cap T_v$. Then the set

$$M = \bigcup_{v \in V} M_v \cup \bigcup_{e \in E} M_e \tag{16}$$

is a stable local MDS for the space $S_2^1(\triangle_{PS12})$ [8, Theorem 6.13]. Now let

$$S_0 := \left\{ s \in S_2^1(\triangle_{PS12}) \ : \ s|_{\partial\Omega} = 0 \right\}.$$

Again, assuming that T_v is a boundary triangle of $\triangle_{PS12}$ for any bondary vertex v, we can split M into M_0 and M_b by the same method as for the Clough-Tocher elements. Then $M = M_0 \cup M_b$ is a stable splitting for $S_2^1(\triangle_{PS12})$.

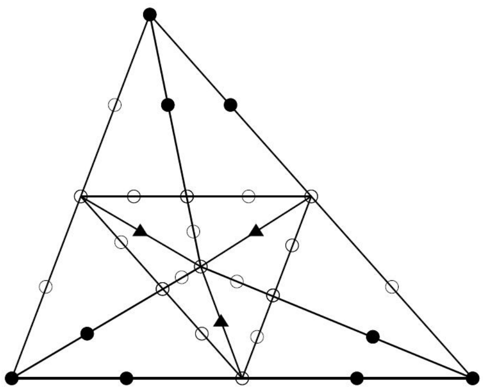

Fig. 7. A Powell-Sabin-12 refinement of one triangle with points in M_v marked as black dots and points in M_e marked as black triangles.

5.4 Quadrilateral Macro-element Space

Let $\diamondsuit$ be a strictly convex quadrangulation of a polygonal domain Ω and let $\triangle_Q$ be triangulation obtained by drawing in the diagonals of each quadriletral of $\diamondsuit$. Let V and E be the sets of vertices and edges of $\diamondsuit$. Here we will discuss the cubic spline space $S_3^1(\triangle_Q)$. Again let $M_v := D_1(v) \cap T_v$, for each $v \in V$, where T_v is a triangle in $\triangle_Q$ attached to v, and T_v is a boundary triangle in case of a boundary vertex v. For each $e \in E$, let T_e be some triangle in $\triangle_Q$ containing e and let $M_e := \left\{\xi_{111}^{T_e}\right\}$. Then

$$M = \bigcup_{v\in V} M_v \cup \bigcup_{e\in E} M_e \tag{17}$$

is a stable local MDS for the space $S_3^1(\triangle_Q)$ [8, Theorem 6.17]. Again the stable splitting of M for $S_3^1(\triangle_Q)$ is possible by the argument discussed above for other C^1 macro-elements.

Note that in [8, Section 6.5] the above triangle T_v is chosen such that it has the largest shape ratio $\operatorname{diam}(T)/\rho(T)$ among all triangles attached to v. This allows stable MDS even in the presence of small angles in $\triangle_Q$ if the smallest angle in $\diamondsuit$ is separated from zero. However, this choice of T_v might be unsuitable for stable splitting if v is a boundary vertex because we need T_v to be a boundary triangle whereas the shape ratio might be larger for some interior triangle attached to v. Therefore, our construction of stable splitting is valid only if $\triangle_Q$ satisfies the minimum angle condition.

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