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Comparing Axiomatizations of Free Pseudospaces

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Abstract

Independently and pursuing different aims, Hrushovski and Srour [3] and Baudisch and Pillay [1] have introduced two free pseudospaces that generalize the well know concept of Lachlan's free pseudoplane. In this paper we investigate the relationship between these free pseudospaces, proving in particular, that the pseudospace of Baudisch and Pillay is a reduct of the pseudospace of Hrushovski and Srour.

Key words: free pseudoplane, free pseudospace, stable theories, equational theories
MSC (2000): 03C45

1 Introduction

Already back in 1974 Lachlan [8] introduced the free pseudoplane which is by now a well studied and well understood model-theoretic object. In particular, Hrushovski and Pillay [4] showed that 1-based or weakly normal theories do not contain a type-definable pseudoplane. Hence the free pseudoplane is the prototype of a stable and not 1-based theory. While the free pseudoplane is a 2-dimensional object in essence, two generalizations of the pseudoplane in form of 3-dimensional pseudospaces were independently introduced by Hrushovski and Srour [3] and Baudisch and Pillay [1]. The motivations for the construction of these pseudospaces differ, but the constructions itself share many common features.

The free pseudospace of Hrushovski and Srour is a coloured 3-dimensional pseudospace and was constructed as the first example of a stable and non-equational theory. Equational theories were introduced by Srour [10, 12–14] and further developed by Junker, Kraus, and Lascar [5–7]. A parameter-free formula $\varphi(\bar{x};\bar{y})$ with two sorts of variables $\bar{x}$ and $\bar{y}$ is called an equation, if every infinite conjunction $\bigwedge_{i\in I}\varphi(\bar{x};\bar{a}_i)$ of instances of φ is equivalent to a conjunction

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$\bigwedge_{i\in I_0} \varphi(\bar{x}, \bar{a}_i)$ with finite $I_0 \subseteq I$.[1] A theory is equational, if every formula is equivalent to a Boolean combination of equations.

By counting the number of types it is easy to see that equational theories are stable [10]. Thus Srour posed the question whether the class of equational theories is a proper subclass of the class of stable theories. This question was answered affirmatively by Hrushovski and Srour with the construction of their free pseudospace in the unfortunately unpublished manuscript [3]. The result from [4] mentioned above shows that Lachlan's pseudoplane is a typical example of a stable non-1-based theory. As equational theories provide a natural generalization of 1-based theories [10], this motivates the approach to search for a stable non-equational theory in form of a higher-dimensional version of the pseudoplane.

Independently of [3], Baudisch and Pillay [1] constructed another free pseudospace as an example of a non-CM-trivial stable theory in which no infinite field is interpretable. This shows that the hierarchy of n-ample theories, developed by Pillay [9], is strict up to its second level. The first level of this hierarchy is again formed by non-1-based theories, whereas 2-ample theories correspond to non-CM-trivial theories. It had already been conjectured in [1] that the pseudospace of Baudisch and Pillay is a reduct of the coloured pseudospace by Hrushovski and Srour, but the actual verification turned out to be far from obvious.

This paper is organized as follows. In Sect. 2 we review the pseudoplane of Lachlan [8]. We also introduce a coloured version of this pseudoplane which will serve as an essential ingredient for the analysis of the coloured pseudospace of Hrushovski and Srour.

In Sects. 3 and 4 we describe the free pseudospaces Σ of Baudisch and Pillay [1] and Γ of Hrushovski and Srour [3]. Using the standard model of Σ from [1] we construct a standard model of Γ.

The main results follow in Sect. 5 where we investigate the relationship between the axiom systems Σ and Γ. We prove that Σ is a reduct of Γ. The main technical difficulty for this result lies in deriving from Γ the axioms of Σ which expresses the freeness conditions. We achieve this by analyzing paths and cycles in models of Γ. As a byproduct we obtain a simplification of the axiom system Σ.

In the final section we explain the original purpose of Γ as a stable non-equational theory. In particular, we include a proof for the non-equationality of Γ which is based on the proof sketch given in the draft [3].

2 The Free Pseudoplane

First we will review the free pseudoplane of Lachlan [8], because it is of fundamental importance for the higher dimensional pseudospaces that are the topic

[1] This definition of equationality stems from [3]. Pillay and Srour [10] give a different definition where they only consider equations in *one* free variable x. Whether the two definitions are equivalent is open. In [5] Junker provided a detailed comparison of these different formalisations of equationality.

of this paper. The language contains unary predicates B and C for lines and points, respectively, and a binary incidence relation I between lines and points. The free pseudoplane is axiomatized by the following axiom set Δ:

Δ_1: Every element is a point or a line, but not both.

Δ_2: $I \subseteq (B \times C) \cup (C \times B)$ is a symmetric relation between lines and points.

Δ_3: Every point lies via I on infinitely many lines. Conversely, every line contains infinitely many points.

Δ_4: There are no cycles, i.e., there do not exist mutually distinct elements $x_0, \ldots, x_n$, $n \geq 2$, with $I(x_i, x_{i+1})$, $0 \leq i \leq n-1$, and $I(x_n, x_0)$.

The standard model N_0 of Δ has as its domain the set $\omega^{<\omega}$ of finite sequences of natural numbers. The lines of N_0 are the sequences of even length, whereas sequences of odd length are points. The incidence relation $I(x, y)$ holds between elements x and y, if x is either a direct predecessor or a direct successor of y. Thus N_0 is a countable model of Δ which, moreover, is connected. It is well known that Δ is a complete theory.

The Coloured Pseudoplane

Next we will describe a coloured modification of the free pseudoplane where lines and points are equipped with colours. This modification is not of independent interest, but it will serve as an important building block in subsequent sections. The language is enriched by unary relations C_r, C_w, B_r, and B_w for red and white points and red and white lines, respectively. In addition to the axioms Δ_1 to Δ_4, the axiom set Δ' contains the following three axioms regarding the colours:

Δ_5: Every line is either red or white, i.e., it fulfills exactly one of the predicates B_r or B_w. The analogous condition holds for points.

Δ_6: Every point lies on infinitely many white and on infinitely many red lines.

Δ_7: Every red (resp. white) line b contains exactly one white (resp. red) point, which is called the *exceptional point* of b.

Models of Δ' are called *free coloured pseudoplanes.* The standard model N_0' of the coloured pseudoplane is derived from the standard model N_0 of Δ by colouring lines and points. Lines are coloured according to

$$\begin{aligned} B_r(N_0') &= \{b \mid b \in B(N_0'),\ b_{\ell(b)} \text{ is even}\} \\ B_w(N_0') &= \{b \mid b \in B(N_0'),\ b_{\ell(b)} \text{ is odd}\}\ , \end{aligned}$$

where $\ell(b)$ denotes the length of the sequence b, and $b_{\ell(b)}$ is its last element. By this construction every point lies on infinitely many red and white lines.

It remains to colour the points. If the predecessor point c of a line b in $B(N_0')$ has a different colour than b, then c is the exceptional point of b, and all successors of b are coloured with the colour of b. If, on the other hand, b and c

are of the same colour, then we can choose the exceptional point freely among the successors of b. Therefore Δ_7 is fulfilled, and hence N_0' is a model of Δ'.

It is not hard to directly construct an isomorphism between two countable connected free coloured pseudoplanes. Therefore Δ' has only one countable connected model up to isomorphism.

3 The Free Pseudospace of Baudisch and Pillay

In this section we describe a 3-dimensional analogue of the pseudoplane as developed by Baudisch and Pillay [1]. In addition to points and lines the pseudospace contains also planes. The language L of this pseudospace consists of unary predicates A, B, C for planes, lines and points, respectively, and binary predicates I and J for the incidence relations between planes and lines as well as between lines and points.

Before we describe the axioms of the pseudospace we need to introduce some terminology. By A, B, and C we also denote the set of planes, lines, and points, respectively. We will usually use letters $a, a', a_i \dots$ for planes, $b, b', b_i \dots$ for lines, and $c, c', c_i \dots$ for points, and we will often refrain from indicating explicitly the type of an element denoted in this way. For a plane a we define the sets $B(a) = \{b \in B \mid J(a, b)\}$ and $C(a) = \{c \in C \mid c \in a\}$. For a point c the sets $A(c)$ and $B(c)$ are defined analogously.

Elements $d_0, \dots, d_n$ form a *walk* if consecutive elements are incident to each other. If all elements are pairwise distinct except possibly for d_0 and d_n, the walk is called a *path*. If in addition $d_0 = d_n$, the path is also called a *cycle*. The length of a path is the number of distinct elements in it. If all elements are planes or lines, then we speak of an *AB-path*. *BC-paths* are defined analogously.

The free pseudospace of [1] is axiomatized by the following axioms:

Σ_0: Every element fulfills exactly one of the relations A, B, or C. The relations $J \subseteq (A \times B) \cup (B \times A)$ and $I \subseteq (B \times C) \cup (C \times B)$ are symmetric.

Σ_1: (A, B, J) is a free pseudoplane.

Σ_2: $(B(a), C(a), I)$ is a free pseudoplane for every plane a.

Σ_3: The intersection of two planes is either empty, or a point, or a line.

Σ_4: Let a be a plane and $X = (a, b, \dots, b', a)$ be a cycle of length n. Then there exists a BC-path between b and b' of length at most $n - 1$, which only contains points from X and lines from a.

For the axioms $\Sigma_1, \dots, \Sigma_4$ we also consider their dual versions $\Sigma_1^*, \dots, \Sigma_4^*$ which are formed by interchanging planes and points (e.g., Σ_2^* reads: for every point c, $(A(c), B(c), J)$ is a free pseudoplane). The axiom set Σ of the free pseudospace of Baudisch and Pillay [1] comprises of the axioms $\Sigma_0, \dots, \Sigma_4$ together with their dual versions $\Sigma_1^*, \dots, \Sigma_4^*$.

In [1] Baudisch and Pillay construct a particular countable connected model M_0 of Σ which is called the standard model. Further, it is shown that the theory Σ is complete, ω-stable, and not CM-trivial.

A Simplified Axiomatization

It is apparent from the axioms Σ that points and planes are completely dual to each other. Many arguments can therefore be simplified by establishing some property only for points and lines, which immediately implies this property for planes and lines as well. In this subsection we will simplify Σ and point out that it is in fact not necessary to include the point-plane duality in the axiomatization. It already follows from the first half $\Sigma_1, \ldots, \Sigma_4$ of the axioms of Σ.

First, let us introduce a convention which will help to ease the notation. From the axioms of Σ it is clear that planes and lines are uniquely determined by the set of its points. This allows us to relax our notation and occasionally identify planes and lines with the set of its points, i.e., $a = C(a)$ and $b = \{c \mid I(b,c)\}$. Hence we can use expressions like $c \in b$, $b \subset a$ or $a \cap b$, which are considered as abbreviations for the respective formulas involving the incidence relations I and J.

The next two lemmas are the first steps towards proving the point-plane duality in Σ.

Lemma 3.1 *Every model of* Σ_0*,* Σ_1*, and* Σ_2 *fulfills* Σ_2^**.*

Proof. Let c be a point. We have to show that $(A(c), B(c), J)$ is a free pseudoplane, i.e., we have to check the axioms Δ_1 to Δ_4. Axioms Δ_1 and Δ_2 follow immediately from Σ_0.

For Δ_3 let $a \in A(c)$. By Σ_2, $(B(a), C(a), I)$ is a free pseudoplane. Because $c \in C(a)$ there exist infinitely many lines in a that contain c. Therefore every plane in $(A(c), B(c), J)$ contains infinitely many lines. That every line b lies in infinitely many planes follows from Σ_1, because every plane $a \supset b$ also contains c. Finally, $(A(c), B(c), J)$ does not contain cycles as this is already true for (A, B, J) by Σ_1. Hence also Δ_4 is fulfilled. □

Lemma 3.2 *Every model of* Σ_0*,* Σ_1^**, and* Σ_3 *satisfies* Σ_3^**.*

Proof. Let c and c' be two distinct points. If there is none or exactly one plane containing c and c', then Σ_3^* is already fulfilled for c and c'.

Assume therefore that a and a' are two distinct planes that both contain c and c'. Then $\{c, c'\} \subseteq a \cap a'$, and by Σ_3 there exists a line b such that $c, c' \in b$ and $a \cap a' = b$. By Σ_1^* this line b is uniquely determined by c and c'. Hence the planes containing c and c' are exactly the planes that contain b. □

In the next lemma we notice that also Σ_3 follows from the other axioms.

Lemma 3.3 *Every model of* Σ_0*,* Σ_1*,* Σ_1^**,* Σ_2*, and* Σ_4 *satisfies* Σ_3*.*

Proof. Assume that a and a' are two distinct planes which both contain two distinct points c and c'. We will first show that both c and c' lie on a common line b. By Σ_2 we can choose appropriate lines such that $X = (a, b_1, c, b_2, a', b_3, c', b_4, a)$ forms a cycle. Now, by axiom Σ_4 there exists a BC-path Y connecting b_1 and

b_4 which contains only points from X and lines from a. There are three possibilities for such a path: either $Y = (b_1, c, b_4)$, yielding the desired line $b = b_4$, or $Y = (b_1, c', b_4)$, yielding $b = b_1$, or, finally, $Y = (a, b_1, c, b, c', b_4, a)$ with some line b fulfilling the claim. Thus we have obtained a line $b \subset a$ with $c, c' \in b$.

As the role of a and a' can be interchanged, we also get a line $b' \subset a'$ with $c, c' \in b'$. By Σ_1^* this line is already uniquely determined by c and c', i.e., $b = b'$ is a common line of a and a'. By Σ_1 the intersection of a and a' cannot contain more than one line, and hence Σ_3 is fulfilled. □

In the following proposition we provide a simplified axiomatization for Σ. As it seems more difficult to verify that also Σ_4^* is derivable from the remaining axioms, we have to postpone the proof until Sect. 5.

Proposition 3.4 *Every model of $\Sigma_0, \Sigma_1, \Sigma_1^*, \Sigma_2$, and Σ_4 fulfills all axioms of Σ.*

4 The Coloured Pseudospace of Hrushovski and Srour

This section is devoted to another free pseudospace, introduced by Hrushovski and Srour [3]. Although this pseudospace is very similar to the pseudospace of Baudisch and Pillay [1], it also contains a number of additional features. Before giving the full axiomatization we will provide an informal description.

As in the pseudospace of [1] models consist of points, lines, and planes. As before there are incidence relations I between points and lines and J between lines and planes, but additionally there are two direct incidence relations I_r and I_w between points and planes. Lines are either red or white, indicated by unary relations B_r and B_w. Points have on a given plane also a colour red or white, indicated by the relations I_r and I_w. The colour of one point can be different on different planes. Via I_r and I_w planes split into a red and a white section. A red line b of a plane a contains only points from the red section of a, except for one white point, the exceptional point of b in a. The same holds for white lines. Lines and points of a plane therefore form a free coloured pseudoplane. Finally, there are axioms stating that models are maximally free of cycles.

The language L' consists of unary relation symbols A, B, B_r, B_w, and C for planes, lines (red and white) and points, and binary relation symbols I, J, I_r, and I_w for the incidence relations. Therefore L' extends the language L from the previous section. The axiom set Γ from [3] contains the following axioms:

Γ_0: Every element fulfills exactly one of the relations A, B, or C.
$J \subset (A \times B) \cup (B \times A)$ is a symmetric incidence between planes and lines.
$I \subset (B \times C) \cup (C \times B)$ is a symmetric incidence between lines and points.

Γ_1: The intersection of two lines is either empty or a single point. Every line contains infinitely many points. The set of lines is nonempty.

Γ_2: For every plane a and every point $c \in a$ there exist infinitely many lines in a that contain c.

Γ_3: Every line lies in infinitely many planes.

Γ_4: If $b_1, \ldots, b_n$, $n \geq 2$, are pairwise different lines with $b_i \cap b_{i+1} \neq \emptyset$, $1 \leq i \leq n-1$, then $b_1 \cap b_n = \emptyset$, or there exists a point c with $c \in b_i$ for all $i = 1, \ldots, n$.

Γ_5: Planes are nonempty. The intersection of two planes is either empty or a point or a line.

Γ_6: If $a_1, \ldots, a_n$, $n \geq 2$, are pairwise distinct planes such that $a_i \cap a_{i+1}$ is a line for $i = 1, \ldots, n-1$, then $a_1 \cap a_n = \emptyset$, or $a_1 \cap a_n$ is a point, or $a_1, \ldots, a_n$ contain a common line.

Γ_7: If a_1, a_2, a_3 are three distinct planes such that $a_i \cap a_j \neq \emptyset$, $1 \leq i, j \leq 3$, then a_1, a_2, a_3 contain a common point.

Γ_8: If $a_1, \ldots, a_n$, $n \geq 3$, are pairwise distinct planes such that $a_i \cap a_{i+1} \neq \emptyset$, $1 \leq i \leq n-1$, and $a_i \cap a_{i+2} = \emptyset$, $1 \leq i \leq n-2$, then $a_1 \cap a_n = \emptyset$.

Γ_9: Lines are either red or white, i.e., every line fulfills exactly one of the relations B_r or B_w.

Γ_{10}: $I_r, I_w \subset (A \times C) \cup (C \times A)$ are the symmetric incidence relations between planes and their red and white points. $I_r \cap I_w = \emptyset$. The red and white sections of a plane a are defined as $a_r = \{x \mid I_r(x,a)\}$ and $a_w = \{x \mid I_w(x,a)\}$, respectively.

Γ_{11}: $J(a,b) \leftrightarrow \forall x(I(x,b) \rightarrow I_r(x,a) \vee I_w(x,a))$ holds for all $a \in A, b \in B$.

Γ_{12}: For every plane a and every point $c \in a$ there are infinitely many red and infinitely many white lines in a containing c.

Γ_{13}: For every red (resp. white) line b in a plane a there exists exactly one *exceptional point* $c \in a_w \cap b$ (resp. $c \in a_r \cap b$).

Γ_{14}: For every line b and every point $c \in b$ there exist infinitely many planes a with $b \subset a$, such that c is the exceptional point of b in a.

To obtain consistent notation we slightly modified the description of Γ from [3] (where other symbols than A, B, B_r, B_w, C are used and incidence relations are not symmetric). We also grouped the axioms in such a way that the first axioms Γ_0 to Γ_8 do not refer to the colour relations B_r, B_w and I_r, I_w. We denote the axioms Γ_0 to Γ_8 by Γ'. Together with the colour axioms Γ_9 to Γ_{14} we obtain the full set Γ. Clearly, every model of Γ is also a model of Γ'.

The notions of walks, paths, and cycles are easily adapted to the language L'. Because in Γ we also have direct point-plane incidence relations, walks in models of Γ are not necessarily walks in the sense of Σ.

In contrast to the pseudospace of Baudisch and Pillay the duality between points and planes is not so apparent from the axioms of Γ. Because of the colours (points are red or white, and planes do not have colours) full duality is not even possible. We will, however, show in the next section that the role of points and planes can be interchanged if colours are omitted.

First we will show the consistency of Γ by constructing a coloured version M_0' of the standard model M_0 of Σ from [1]. The planes and lines form the free pseudoplane $\omega^{<\omega}$, where the set of planes is $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is even}\}$ and lines correspond to $\{\eta \in \omega^{<\omega} \mid \ell(\eta) \text{ is odd}\}$. The incidence $J(\eta, \tau)$ holds, if η is a direct predecessor or successor of τ. In analogy to Sect. 2, lines are coloured according to $B_r(M_0') = \{b \in B(M_0') \mid b_{\ell(b)} \text{ is even}\}$ and $B_w(M_0') = \{b \in B(M_0') \mid b_{\ell(b)} \text{ is odd}\}$. Hence every plane contains infinitely many red and infinitely many white lines. Planes and lines therefore form a free coloured pseudoplane, where the colour of planes is neglected.

Now we inductively add points for the planes and define the relations I, I_r, and I_w. The induction is carried out on the length of a plane a as an element of $\omega^{<\omega}$. The induction hypothesis consists of the following two assertions:

1. For all planes a of length $\leq 2n$, points $C(a)$ have been added such that

$$P(a) = (B(a), C(a), I_a, B_{ra}, B_{wa}, I_{ra}, I_{wa})$$

   is a connected free coloured pseudoplane. Here I_a, B_{ra}, B_{wa} are the restrictions of I, B_r, B_w to lines and points from a, and I_{ra}, I_{wa} are the unary relations $I_r(\cdot, a)$ and $I_w(\cdot, a)$, respectively, which indicate the colour of a point in a.

2. Axiom Γ_{14} is fulfilled for all lines of length at most $2n-1$, i.e., for every line $b \in \omega^{<2n}$ and every point $c \in b$ there exist infinitely many planes $a \in \omega^{\leq 2n}$ with $b \subset a$, such that c is the exceptional point of b in a.

At the end of this construction the set of all points is formed by the union $\bigcup\{C(a) \mid \ell(a) \text{ even}\}$. Let us already remark here that in the construction we have many choices concerning the distribution of the colours. All these choices, however, will provide elementarily equivalent models.

In the initial step of the construction we choose $C(<>)$ as a countable set of points. Colours of $B(<>)$ are already determined by the colouring of (A, B, J) in such a way that $B(<>)$ contains infinitely many red and white lines. On $B(<>) \cup C(<>)$ we define the relations $I_{<>}, I_{r<>}, I_{w<>}$ such that $P(<>)$ is a countable connected free coloured pseudoplane. In the initial step, the second part of the induction claim does not apply. Colours of $C(<>)$ in planes of length two are chosen in the next step of the construction.

For the induction step, assume that the induction hypothesis holds for all planes of length at most $2n$. Let b be a line of length $2n+1$. We will simultaneously define $P(a)$ for all planes a of length $2n+2$ which are incident to b. Let a be one of these planes, i.e., $b = a_{2n+1}$ is the predecessor line of a. Let further C^0 be the set of points of the line b and let C^1 be a countable set of new elements. We define the points of a as $C(a) = C^0 \cup C^1$. As in the initial step of the induction, the colours of C^0 in planes of length $2n+2$ have not been determined yet. This will be done below, observing the second part of the induction hypothesis regarding axiom Γ_{14}.

Now we define I_a, I_{ra}, I_{wa} on $B(a) \cup C(a)$ such that $P(a)$ becomes a connected countable free coloured pseudoplane. We do not introduce any new points on the line b, i.e., $I_a(b, c)$ holds if and only if $c \in C^0$. Colours of points can be chosen independently in each plane a of length $2n+2$. Additionally, the exceptional point of b is chosen such that for each $c \in C^0$ there are infinitely many planes a of length $2n + 2$ such that $a_{2n+1} = b$ and c is the exceptional point of b on a. This is possible because C^0 is countable and also b contains countably many successor planes a on which the exceptional point can be chosen arbitrarily. Hence also the second part of the induction claim is fulfilled.

Finally, the set of all points of M_0' is the union of all sets $C(a)$, and the relations I, I_r, I_w are the unions of the respective relations I_a, I_{ra}, I_{wa} for all planes a.

The next theorem is our main result in this section.

Theorem 4.1 *M_0' is a model of Γ, and hence Γ is consistent.*

The theorem is proven by the following remarks and lemmas.

Let M_0 be the L-reduct of M_0'. Then M_0 is exactly the standard model of Σ constructed in [1]. Therefore Σ is valid in M_0'. It remains to show that also Γ is fulfilled in M_0'. To show this we will first derive all axioms of Γ' from Σ. As M_0' is a model of Σ this implies the validity of Γ' in M_0'.

Lemma 4.2 *Every model of Σ satisfies Γ_0 to Γ_6.*

Proof. The axioms Γ_0 and Σ_0 are identical. Axiom Σ_1^* implies Γ_1 and Γ_4. Axiom Γ_2 follows from Σ_2, and Γ_3 follows from Σ_1. Finally, Γ_5 and Γ_6 follow from Σ_1 and Σ_3. □

Deriving Γ_7 and Γ_8 from Σ requires some extra arguments.

Lemma 4.3 *Every model of Σ satisfies Γ_7.*

Proof. Let a_1, a_2, a_3 be distinct planes and let c_1, c_2, c_3 be points such that $c_1 \in a_2 \cap a_3$, $c_2 \in a_1 \cap a_3$, $c_3 \in a_1 \cap a_2$. We have to show the existence of a point $c \in a_1 \cap a_2 \cap a_3$. If $c_1 \in a_1$, then $c = c_1$ is such a point. Likewise, if $c_2 \in a_2$ or $c_3 \in a_3$. Assume now that

$$c_i \notin a_i \text{ for } i = 1, \ldots, 3. \tag{1}$$

We will derive a contradiction. By assumption c_1, c_2, and c_3 are pairwise distinct. Hence there exists a cycle

$$(a_1, b_3, c_3, b_3', a_2, b_1, c_1, b_1', a_3, b_2, c_2, b_2', a_1)$$

with pairwise distinct lines $b_1, b_2, b_3, b_1', b_2', b_3'$. Choosing such lines is possible by (1) and because $(B(a_i), C(a_i), I)$ is a free pseudoplane. By Σ_4 there exists a BC-path X between b_3 and b_2', containing only lines from a_1 and points from $\{c_1, c_2, c_3\}$. The point c_1 cannot occur in X because $c_1 \notin a_1$. Hence we have either

(a) $X = (b_3, c_3, b_2')$	or	
(b) $X = (b_3, c_2, b_2')$	or	
(c) $X = (b_3, c_3, b', c_2, b_2')$	with $b' \subset a_1$.	

In every case there exists a line $b_1'' \subset a_1$ containing both c_2 and c_3, namely in (a) $b_1'' = b_2'$, in (b) $b_1'' = b_3$ and in (c) $b_1'' = b'$. Analogously, using $c_2 \notin a_2$ and $c_3 \notin a_3$ we get lines $b_2'' \subset a_2$ and $b_3'' \subset a_3$ with $c_1, c_3 \in b_2''$ and $c_1, c_2 \in b_3''$. By (1) the lines b_1'', b_2'', and b_3'' are pairwise distinct. Hence there exists a cycle of lines and points

$$(b_1'', c_3, b_2'', c_1, b_3'', c_2, b_1'') \ ,$$

contradicting Σ_1^*. □

Lemma 4.4 *Every model of* Σ *satisfies* Γ_8.

Proof. Let $a_1, \ldots, a_n$, $n \geq 3$, be distinct planes with $c_i \in a_i \cap a_{i+1}$, $1 \leq i < n$, and $a_i \cap a_{i+2} = \emptyset$, $1 \leq i < n-1$. We have to prove $a_1 \cap a_n = \emptyset$. We will show this by induction on n. The base case $n = 3$ is clear. Let $n > 3$ and assume that Γ_8 is valid for all $3 \leq k < n$. By hypothesis we have

$$a_i \cap a_j = \emptyset \text{ for } 1 \leq i < j \leq n \text{ with } i+1 \neq j \text{ and } (i,j) \neq (1,n). \qquad (2)$$

Assume now, that there exists a point $c_n \in a_1 \cap a_n$. We will construct a contradiction, similarly as in the previous lemma. By (2) and the assumption there exists a cycle

$$(a_1, b, c_1, b', a_2, \ldots, a_n, b'', c_n, b''', a_1) \ .$$

Applying Σ_4 yields a path X between b and b''', containing only lines from a_1 and points from $\{c_1, \ldots, c_n\}$. By (2), the points $c_2, \ldots, c_{n-1}$ cannot appear in X. Hence we have $X = (b, c_1, b''')$, or $X = (b, c_n, b''')$, or $X = (b, c_1, b_1, c_n, b''')$ with $b_1 \subset a_1$. In each case there exists a line b_1, containing the points c_1 and c_n. Analogously, (2) yields lines $b_2, \ldots, b_n$ with $c_{i-1} \in b_i$ and $c_i \in b_i$, $2 \leq i \leq n$. By (2) all lines $b_1, \ldots, b_n$ are distinct. Hence there is a cycle

$$(b_1, c_1, b_2, c_2, \ldots, b_n, c_n, b_1) \ ,$$

contradicting axiom Σ_1^*. □

Finally, the colour axioms in Γ follow directly from the construction of M_0':

Lemma 4.5 M_0' *satisfies the axioms* Γ_9 *to* Γ_{14}.

5 The Relationship between the Two Pseudospaces

The aim of this section is to prove Theorem 5.5: Σ and Γ' are equivalent. Already in the last section we have shown that the axioms Γ' are derivable from Σ. Now we will prove that the axioms of Γ' also imply all axioms from Σ. We will first derive Σ_4 from Γ', which requires the following lemma.

Lemma 5.1 *Let* $M \models \Gamma'$ *and let* $X = (a, b, c_0, a_1, c_1, \ldots, c_{n-1}, a_n, c_n, b', a)$ *be a cycle in* M *consisting of planes* $a = a_0, a_1, \ldots, a_n$, *lines* b, b', *and points* $c_0, \ldots, c_n$. *Then there exists a* BC*-path* $Y = (b, c_0', b_1', c_1', \ldots, c_{m-1}', b_m', c_m', b')$ *such that* $\{c_0', \ldots, c_m'\} \subseteq \{c_0, \ldots, c_n\}$ *and* $c_i' \in a$, $0 \leq i \leq m$. *Additionally, we have* $b_i' \subset a$, $1 \leq i \leq m$.

Proof. The last sentence follows from the first part of the lemma. Namely, if $a'_i \neq a$ is a plane with $b'_i \subset a'_i$, then $c'_{i-1}, c'_i \in a'_i \cap a$, and hence $b'_i = a'_i \cap a$. The first part of the lemma is shown by induction on n.

Base cases. For $n = 0$ we have $X = (a, b, c_0, b', a)$, and the claim is true.

For $n = 1$ we have $X = (a, b, c_0, a_1, c_1, b', a)$, i.e., $c_0, c_1 \in a \cap a_1$. Then there exists a line $b'' = a \cap a_1$, and hence there is the walk (b, c_0, b'', c_1, b'). If $b = b''$ or $b'' = b'$, then the walk can be shortened. In the following we will not explicitly mention, if a walk can be shortened in such a way.

For $n = 2$ we have $X = (a, b, c_0, a_1, c_1, a_2, c_2, b', a)$. By Γ_7 there exists a point $c \in a \cap a_1 \cap a_2$. We will distinguish four cases.

Case 1. $c = c_0$, i.e., in particular $c \neq c_2$. Then there exists $b'' = a \cap a_2$ such that $c_0, c_2 \in b''$. Therefore the desired path Y is obtained from the walk (b, c_0, b'', c_2, b').

Case 2. $c = c_1$, hence $c \neq c_0$ and $c \neq c_2$. Then there exist lines $b'' = a \cap a_1$ and $b''' = a \cap a_2$ such that $c_0, c_1 \in b''$ and $c_1, c_2 \in b'''$. Therefore we have the walk $(b, c_0, b'', c_1, b''', c_2, b')$.

Case 3. $c = c_2$. Like case 1.

Case 4. $c \neq c_0$, $c \neq c_1$, and $c \neq c_2$. Then we have lines $b_0 = a \cap a_1$, $b_1 = a_1 \cap a_2$, and $b_2 = a_2 \cap a$, i.e., there exists the walk $(a, b_0, a_1, b_1, a_2, b_2, a)$. Because a, a_1, a_2 are pairwise distinct, they contain a common line b'' by Γ_6, which is identical with b_0, b_1, and b_2 by Γ_5, i.e., $b'' = b_0 = b_1 = b_2$. Therefore we get the walk (b, c_0, b_0, c_2, b').

Induction step. Let the claim be true for $k < n$, $n \geq 3$. If there exists an index i, $1 \leq i \leq n-1$ such that $c_i \in a$, then we choose $b'' \subset a$ with $c_i \in b''$. Hence we have the paths $(a, b, c_0, \ldots, a_i, c_i, b'', a)$ and $(a, b'', c_i, a_{i+1}, \ldots, a_n, c_n, b', a)$. By induction hypothesis there exist BC-paths connecting b and b'' as well as b'' and b', that only use points from $\{c_0, \ldots, c_n\}$ lying in a. We obtain the desired BC-path between b and b' by concatenation. We can therefore make the following

Assumption 1 $c_i \notin a$ *for* $1 \leq i \leq n-1$.

If there exist i and j such that $0 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j, j-1$ and $c_i \in a_j$, then we can shorten the path X to $X' = (a, b, c_0, \ldots, a_i, c_i, a_j, c_j, \ldots, b', a)$ if $i < j-1$, and to $X' = (a, b, c_0, \ldots, c_{j-1}, a_j, c_i, a_{i+1}, \ldots, b', a)$ if $i > j$. In this case the induction hypothesis for X' yields the claim. In addition to Assumption 1 we therefore make

Assumption 2 $c_i \notin a_j$ *for* $0 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j, j-1$.

Finally, if c_0 and c_n are on a common line b'', then we directly get the path Y from (b, c_0, b'', c_n, b'). We therefore also assume

Assumption 3 c_0 *and* c_n *do not lie on a common line.*

From Assumptions 1 to 3 we will derive a contradiction, hence for any given X at least one of these assumptions does not hold, and thus the claim is proved. By Γ_8 there exists some j, $0 \leq j \leq n-2$ such that $a_j \cap a_{j+2} \neq \emptyset$. Let $c \in a_j \cap a_{j+2}$. For this situation we will prove the following claim.

Claim 1 *If there exists a point* $c \in a_j \cap a_{j+2}$ *with* $0 \le j \le n-2$*, then there exists a* BC*-path* Y' *which connects* b *and* b' *and does not use any points except* c_0*,* c_n*, and* c*. Further,* c *appears in* Y'*, and we have* $c \neq c_0$*,* $c \neq c_n$*, and* $c \in a$*.*

Proof of Claim 1. To prove the first sentence we will distinguish two cases.

Case 1. $j = 0$, i.e., $c \in a \cap a_2$. Then there exists $b'' \subset a$ such that $c \in b''$. Hence we get cycles $(a, b, c_0, a_1, c_1, a_2, c, b'', a)$ and $(a, b'', c, a_2, \dots, a_n, c_n, b', a)$. Applying the induction hypothesis twice yields BC-paths between b and b'' as well as between b'' and b'. By Assumption 1 these paths only contain points from $\{c_0, c_n, c\}$. By concatenation we get a BC-path connecting b and b'.

Case 2. $1 \le j \le n-2$. Then we have a cycle $(a, b, \dots, a_j, c, a_{j+2}, \dots, b', a)$, and by induction hypothesis and Assumption 1 we get a BC-path that contains only the points c_0, c_n, c.

Thus the path Y' only uses the points c_0, c_n, and c. The point c is included in Y' because all choices for Y' omitting c, i.e., (b, c_0, b'), (b, c_n, b'), and (b, c_0, b'', c_n, b') contradict Assumption 3. Thus the only possible configurations for Y' are (b, c, b'), (b, c, b'', c_n, b'), and $(b, c_0, b'', c, b''', c_n, b')$, in which case $c \neq c_0$ and $c \neq c_n$ follow by Assumption 3 and because Y' is a path. Hence c appears in Y', and therefore we get $c \in a$ by induction hypothesis. □

As argued before, axiom Γ_8 assures the existence of an index j, $0 \le j \le n-2$ such that a_j and a_{j+2} contain a common point c. Applying Claim 1 to this point we obtain a BC-path between b and b', using the point c and possibly also c_0 and c_n. We will call this path Y_1. In the next claim we prove that c is contained in all planes a_i.

Claim 2 *For all* $1 \le i \le n$ *we have* $c \in a_i$*.*

Proof of Claim 2. We will prove inductively the following claim: if $c \in a_i$ and $c \in a_j$, $0 \le i < j \le n+1$ (with $a_0 = a_{n+1} = a$), then $c \in a_k$ for all $i \le k \le j$. The proof proceeds by induction on $l = j - i$.

Base case. For $l = 1$ there is nothing to show. Let $l = 2$, i.e., $c \in a_i \cap a_{i+2}$. By Γ_7 we have $a_i \cap a_{i+1} \cap a_{i+2} \neq \emptyset$. Let $c' \in a_i \cap a_{i+1} \cap a_{i+2}$. Claim 1 for c' yields a BC-path Y_2 between b and b', that contains c' and possibly also c_0 and c_n. As (B, C, I) is a free pseudoplane, BC-paths are unique, and therefore $Y_1 = Y_2$ and in particular $c = c'$. Hence $c \in a_{i+1}$.

Induction step. Let $l \ge 3$ and let the claim be true for all $k < l$. Then we have the situation $(a_i, c_i, \dots, c_{i+l-1}, a_{i+l}, c, a_i)$. By Γ_8 there exists an index m such that $i \le m \le i + l - 2$ and $a_m \cap a_{m+2} \neq \emptyset$. Let $c' \in a_m \cap a_{m+2}$. As before, applying Claim 1 to c' we get a BC-path Y_2 between b and b', using only the points c_0, c_n, and c'. Then we have $Y_1 = Y_2$ and hence $c = c'$. Therefore $c \in a_i \cap a_{m+1} \cap a_{i+l}$ and the induction hypothesis yields $c \in a_{i+k}$, $0 \le k \le l$. □

By Claim 1 we have $c \in a$ and by Assumption 1 $c \neq c_i$, $1 \le i \le n-1$. Together with Claim 1 we get $c \neq c_i$, $0 \le i \le n$. By Claim 2 this means $c, c_i \in a_i \cap a_{i+1}$, $0 \le i \le n$. Thus by Γ_5 there exist lines $b_i = a_i \cap a_{i+1}$ such that $c, c_i \in b_i$, $0 \le i \le n$. The lines $b_0, \dots, b_{n-1}$ are pairwise distinct, because if $b_i = b_j$, $0 \le i < j \le n-1$, then $c_i \in b_i = b_j = a_j \cap a_{j+1}$ in contradiction to Assumption 2. By the same argument the lines $b_1, \dots, b_n$ are pairwise distinct.

Additionally, Assumption 3 yields $b_0 \neq b_n$, hence all of $b_0, \dots, b_n$ are pairwise distinct. Therefore we get an AB-cycle $(a, b_0, a_1, b_1, \dots, b_{n-1}, a_n, b_n, a)$ in contradiction to Γ_6. Hence Assumptions 1 to 3 cannot hold simultaneously, and the proof is complete. $\square$

This enables us to prove the validity of Σ_4 in Γ.

Lemma 5.2 *Every model of Γ' satisfies Σ_4.*

Proof. Let $X = (a, b, \dots, b', a)$ be an ABC-cycle. We have to construct a BC-path Y connecting b and b' and consisting only of points from X which are in a. To apply the previous lemma we transform X to a cycle X' that contains no lines except b and b'. To achieve this we apply the following steps a) to c) to the inner part $b, \dots, b'$ of X:

a) Every walk of the form a_1, b_1, c_1 is replaced by a_1, c_1. Similarly, every walk c_1, b_1, a_1 is shortened to c_1, a_1.

b) Every walk c_1, b_1, c_2 is substituted by c_1, a_1, c_2, where the plane a_1 contains the line b_1 and does not occur in X.

c) Finally, every walk of the form a_1, b_1, a_2 is changed to a_1, c_1, a_2 with an arbitrary point c_1 from b_1 that does not occur in X.

After these steps have been performed on X we apply the following rule:

d) If the cycle X obtained after the steps a) to c) starts with a, b, a_1, then we choose some point c_0 from b, not contained in X, and replace a, b, a_1 by a, b, c_0, a_1. Similarly, if X ends with a_n, b', a, then we insert a new point $c_n \in b'$, obtaining a_n, c_n, b', a.

The cycle X' thus obtained has the form $X' = (a, b, c_0, a_1, c_1, \dots, a_n, c_n, b', a)$ and contains only planes, the lines b and b', and all points from X as well as new points inserted by the rules c) and d). Applying Lemma 5.1 to X' yields a BC-path Y between b and b' with points from X' and lines from a. The new points introduced by the rules c) and d) can be chosen arbitrarily from infinitely many possibilities. Therefore, as BC-paths are unique, these new points cannot appear in Y. Hence the path Y is in accordance with the requirements from axiom Σ_4. $\square$

It remains to derive axiom Σ_4^* from Γ. This requires a similar result as Lemma 5.1, but with a considerably simpler proof.

Lemma 5.3 *In a model of Γ' let $X = (c, b, a_0, c_0, \dots, c_{n-1}, a_n, b', c)$ be a cycle consisting of planes $a_0, \dots, a_n$, lines b, b', and points $c_0, \dots, c_{n-1}$. Then there exists an AB-path $Y = (b, a'_0, b'_0, \dots, b'_{m-1}, a'_m, b')$ with $\{a'_0, \dots, a'_m\} \subseteq \{a_0, \dots, a_n\}$ and $c \in a'_i$, $0 \leq i \leq m$. Additionally, we have $c \in b'_i$, $0 \leq i \leq m-1$.*

Proof. The last assertion $c \in b'_i$ follows from $c \in a'_i \cap a'_{i+1} = b'_i$. We will show the first part of the claim by induction on n.

Base case. For $n = 0$ we have $X = (c, b, a_0, b', c)$, and the claim holds.

For $n = 1$ we have $X = (c, b, a_0, c_0, a_1, b', c)$. Because $c, c_0 \in a_0 \cap a_1$ there is a line $b_0 = a_0 \cap a_1$, and hence we get the walk (b, a_0, b_0, a_1, b').

For $n = 2$ we have $X = (c, b, a_0, c_0, a_1, c_1, a_2, b', c)$. By Γ_7 there exists a point $c' \in a_0 \cap a_1 \cap a_2$. We will distinguish four cases.

Case 1. $c' = c_0$. Then $c_0 \in a_2$, and there exists the cycle $(c, b, a_0, c_0, a_2, b', c)$. We can therefore continue as in the case $n = 1$.

Case 2. The case $c' = c_1$ is analogous to case 1.

Case 3. $c' = c$, and therefore in particular $c' \neq c_0$ and $c' \neq c_1$. Then there exist lines $b_0 = a_0 \cap a_1$ and $b_1 = a_1 \cap a_2$, yielding the path $(b, a_0, b_0, a_1, b_1, a_2, b')$.

Case 4. $c' \neq c$, $c' \neq c_0$, and $c' \neq c_1$. Then there exist lines $b_0 = a_0 \cap a_1$, $b_1 = a_1 \cap a_2$, and $b_2 = a_2 \cap a_0$, i.e., we have the walk $(a_0, b_0, a_1, b_1, a_2, b_2, a_0)$. From this we infer $b_0 = b_1 = b_2$ (cf. the resp. part of the proof of Lemma 5.1), and thus we obtain the path (b, a_0, b_0, a_2, b').

Induction step. Let the claim be true for $k < n$, $n \geq 3$. By Γ_8 there exists an index i, $0 \leq i \leq n-2$ such that $a_i \cap a_{i+2} \neq \emptyset$. Let $c' \in a_i \cap a_{i+2}$. Applying the induction hypothesis to $(c, b, a_0, \ldots, c_{i-1}, a_i, c', a_{i+2}, c_{i+2}, \ldots, a_n, b', c)$ yields the desired AB-path Y. □

Lemma 5.4 *Every model of* Γ' *satisfies* Σ_4^*.

Proof. Let $X = (c, b, \ldots, b', c)$ be an ABC-cycle. We search for an AB-path between b and b' with planes from X that contain c. We transform X to a circle X' that does not contain lines others than b, b' by applying the dual procedure of the proof of Lemma 5.2. Applying Lemma 5.3 to X' yields an AB-path Y between b and b' with planes from X' and lines containing c. The planes inserted in the construction of X' cannot appear in Y, because AB-paths are unique, and for the new planes there exist infinitely many different choices. Therefore Y meets the requirements of Σ_4^*. □

These preparations enable us to characterize the relationship between Σ and Γ as follows:

Theorem 5.5 Σ *and* Γ' *are equivalent.*

Proof. In the last section we have already shown $\Sigma \models \Gamma'$.

Concerning the converse $\Gamma' \models \Sigma$, axiom Σ_0 is identical to Γ_0. To derive Σ_1, Σ_1^*, and Σ_2 from Γ' we have to check the axioms Δ for the respective pseudoplanes. Axioms Δ_1 and Δ_2 defining the incidence relations are apparently fulfilled. Δ_3 is easily checked to follow from Γ_1, Γ_2, and Γ_3. For Δ_4 we have to verify the absence of cycles. AB-cycles do not exist by Γ_5 and Γ_6. By Γ_1 and Γ_4 there are also no BC-cycles. Clearly, then there are also no cycles in the pseudoplanes mentioned in Σ_2.

Σ_3 is equivalent to Γ_5. By Lemmas 3.1 and 3.2 the axioms Σ_2^* and Σ_3^* hold in Γ'. Finally, Σ_4 and Σ_4^* were proved in Lemmas 5.2 and 5.4. □

Corollary 5.6 Σ *is the* L*-reduct of* Γ*, i.e., if* M *is an* L'*-structure such that* $M \models \Gamma$*, then* $M|L$ *is a model of* Σ.

This corollary also clarifies the duality between points and lines in models of Γ, namely, if the colours are removed, then points and planes can be interchanged. In fact, this duality is a very natural concept, that does not even have to be required axiomatically. We have already formulated this observation before as Proposition 3.4, but now we can give the full proof.

Proof of Proposition 3.4. Let M be an L-structure satisfying $\Sigma_0, \Sigma_1, \Sigma_1^*, \Sigma_2$, and Σ_4. By Lemmas 3.1, 3.2, and 3.3, M also satisfies Σ_2^*, Σ_3, and Σ_3^*. In the proof of $\Sigma \models \Gamma'$ we only used the above mentioned axioms of Σ. In particular, the proofs of Lemmas 4.3 and 4.4 do not involve Σ_4^*. Therefore $M \models \Gamma'$, and with Theorem 5.5 we get $M \models \Sigma$. □

6 On the Non-Equationality of Γ

Baudisch and Pillay proved in [1] that the pseudospace Σ is a complete and ω-stable theory. Once we know that Σ is a reduct of Γ, the same line of arguments can be also used to show the completeness and ω-stability of Γ. This involves in particular exploring the fine structure of sufficiently saturated models of Γ and a detailed type analysis together with the computation of Morley ranks. In comparison to [1], however, the details are somewhat more tedious due to the richer language of Γ (a complete analysis was carried out in [2]). We will omit this altogether and proceed to explain the original purpose of Γ as an example of a stable and non-equational theory.

Computing Morley ranks in Γ it turns out that, as in Σ, the Morley rank of a plane a is ω. However, in contrast to Σ, where we have $MD(a) = 1$, the Morley degree of a increases to 2 in Γ, owing to the fact that a splits into a white and a red section. For these we get $MR(a_r) = MR(a_w) = \omega$ and $MD(a_r) = MD(a_w) = 1$. We collect all these facts in the following list. For the proof we refer to [3] and [2]. For the rest of this section we work in a big saturated model of Γ.

Fact 6.1

1. *Γ is complete and ω-stable.*
2. *The set of points of a line b is indiscernible over b. Similarly, for every plane a, every two points from the red section a_r of a are conjugate over a.*
3. *For each plane a, $MR(a_r) = \omega$ and $MD(a_r) = 1$.*

Building on this analysis, the next result from [3] is the key lemma for showing the non-equationality of Γ. In fact, it is the only place in the whole argument where equations come into play.

Lemma 6.2 (Hrushovski, Srour [3]) *Let $\varphi(x, \bar{y})$ be an equation and let $\bar{d}$ be parameters corresponding to the variables $\bar{y}$. Let further D be the realization set of the instance $\varphi(x, \bar{d})$. Then for every line b and every plane a the following holds:*

1. *If b is almost in D, i.e., all points of b except finitely many are in D, then already all points of b are in D.*
2. *If $MR(a_r \setminus D) < \omega$, then $a \subseteq D$.*

Proof. For the first item let us assume that there exists a point $c \in b \setminus D$, and let c' be an arbitrary point from b. By Fact 6.1 points are indiscernible over lines, i.e., there exists an automorphism f mapping c to c' and fixing b. We will denote $f(D)$ by $D_{c'}$. Because $c \notin D$ we also have $c' \notin D_{c'}$. As b is almost in D and is fixed by f, the line b is also almost in $D_{c'}$. Varying the point c' we get $\bigcap_{c' \in b} D_{c'} = \emptyset$, because $c' \notin D_{c'}$. The sets $D_{c'}$ are all defined by instances of the equation φ, hence there exist points $c_1, \ldots, c_n$ from b such that

$$\bigcap_{i=1}^{n} D_{c_i} = \bigcap_{c' \in b} D_{c'} = \emptyset \ .$$

But by assumption b is almost in D_{c_i} for $1 \leq i \leq n$ and therefore also almost in $\bigcap_{i=1}^{n} D_{c_i}$, which gives a contradiction.

For part 2 we first prove $a_r \subseteq D$ by a similar argument as in part 1. Assume that there exists a point $c \in a_r \setminus D$, and let $c' \in a_r$ be arbitrary. By Fact 6.1 there exists an automorphism f such that $f(c) = c'$ and $f(a) = a$. Let again $D_{c'}$ denote $f(D)$. By $c \notin D$ we get $c' \notin D_{c'}$. As f also fixes the red section a_r we have $f(a_r \setminus D) = a_r \setminus D_{c'}$. Morley ranks are preserved by automorphisms, hence $MR(a_r \setminus D_{c'})$ is finite. As all the sets $D_{c'}$ are defined by instances of the equation φ, there exist points $c_1, \ldots, c_n$ such that $\bigcap_{i=1}^{n} D_{c_i} = \bigcap_{c' \in a_r} D_{c'} = \emptyset$. Therefore $\bigcup_{i=1}^{n} a_r \setminus D_{c_i} = a_r$. But $MR(a_r \setminus D_{c_i}) < \omega$, contradicting $MR(a_r) = \omega$. This shows $a_r \subseteq D$.

It remains to show $a_w \subseteq D$. For this let $c \in a_w$. There exists a red line b in a that contains c, i.e., c is the exceptional point of b in a. Then b is contained almost in a_r, hence also almost in D. By part 1 we conclude that the whole line b lies in D, hence in particular $c \in D$. □

This lemma enables us to give the proof of the main theorem of [3] stating the non-equationality of the pseudospace Γ. More concretely, the theorem also exhibits a non-equational formula: $I_r(x; y)$ defining the red section of the plane specified by the parameter y. Before giving the precise argument, let us provide a more intuitive explanation why $I_r(x; y)$ is no equation. Let a_i, $i \in \omega$, be pairwise distinct planes which intersect in a common line b. Moreover, assume that b is red, but the exceptional point c_i of b is different on all planes a_i. Then $\bigcap_{i \in \omega} (a_i)_r = b \setminus \{c_i \mid i \in \omega\}$. It is clear that the intersection $\bigcap_{i \in \omega} (a_i)_r$ is not equal to a finite sub-intersection. Hence $I_r(x; y)$ is no equation.

Of course, to prove non-equationality of Γ we need to show that $I_r(x; y)$ is not even equivalent to a Boolean combination of equations. This is done in the proof of the following theorem.

Theorem 6.3 (Hrushovski, Srour [3]) Γ *is not equational. More precisely, the formula $I_r(x; y)$, defining the red section of a plane, is not equivalent to a Boolean combination of equations.*

Proof. We fix a big saturated model M of Γ and denote by φ^M the realization set of a formula φ in M. Let now a be a plane and assume that its red section a_r can be defined by a Boolean combination of equations

$$a_r = \left(\bigvee_{i=1}^{n} \left(\bigwedge_{j=1}^{n_i} \psi_{ij} \wedge \bigwedge_{j=1}^{n'_i} \neg \psi'_{ij} \right) \right)^M ,$$

where ψ_{ij} and ψ'_{ij} are instances of equations φ_{ij} and φ'_{ij}. Finite conjunctions and finite disjunctions of equations are again equations (cf. [5]). Using the abbreviations $\varphi_i = \bigwedge_{j=1}^{n_i} \varphi_{ij}$ and $\varphi'_i = \bigvee_{j=1}^{n'_i} \varphi'_{ij}$ we can therefore write a_r as

$$a_r = \left(\bigvee_{i=1}^{n} \psi_i \wedge \neg \psi'_i \right)^M$$

where ψ_i and ψ'_i are instances of the equations φ_i and φ'_i.

Because $MR(a_r) = \omega$, there exists an index j, $1 \leq j \leq n$, such that $MR((\psi_j \wedge \neg\psi'_j)^M) = \omega$. Let $Y = (\psi_j \wedge \neg\psi'_j)^M$. From $MD(a_r) = 1$ and $MR(a_r) = MR(Y) = \omega$ we conclude $MR(a_r \setminus Y) < \omega$.

Because $a_r \setminus \psi_j^M \subseteq a_r \setminus Y$, we get $MR(a_r \setminus \psi_j^M) \leq MR(a_r \setminus Y)$, hence in particular $MR(a_r \setminus \psi_j^M)$ is finite. Part 2 of Lemma 6.2 then yields $a \subseteq \psi_j^M$. As $Y \subseteq a_r$, this implies $a_w \subseteq {\psi'_j}^M$. As in the proof of part 2 of Lemma 6.2 this extends to $a \subseteq {\psi'_j}^M$. Namely, if $c \in a_r$, then there exists a white line b in a such that c is the exceptional point of b in a. As b is almost in ${\psi'_j}^M$ we get by part 1 of Lemma 6.2 $b \subseteq {\psi'_j}^M$ and hence $c \in {\psi'_j}^M$. Now we have $Y \subseteq a_r$ and $a \subseteq {\psi'_j}^M$ which implies $Y = \emptyset$. But this means $MR(a_r) = MR(a_r \setminus Y) < \omega$ in contradiction to $MR(a_r) = \omega$. □

The free pseudospace Γ was constructed as a first example of a stable and non-equational theory. Recently, Sela [11] has shown that also non-abelian free groups are stable non-equational. Already Hrushovski and Srour remark in [3] that, although Γ is not equational, it is almost equational, a weakening of equationality where the forking relation is controlled by equations (cf. [6]). It remains as an open problem to construct a theory that is simple (or even stable) but not almost equational.

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