Guessing Axioms, Invariance and Suslin Trees

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Abstract

In this thesis we investigate the properties of a group of axioms known as 'Guessing Axioms,' which can be used to extend the standard axiomatisation of set theory, ZFC. In particular, we focus on the axioms called 'diamond' and 'club,' and ask to what extent properties of the former hold of the latter.

A question of I. Juhasz, of whether club implies the existence of a Suslin tree, remains unanswered at the time of writing and motivates a large part of our investigation into diamond and club. We give a positive partial answer to Juhasz's question by defining the principle Superclub and proving that it implies the existence of a Suslin tree, and that it is weaker than diamond and stronger than club (though these implications are not necessarily strict). Conversely, we specify some conditions that a forcing would have to meet if it were to be used to provide a negative answer, or partial answer, to Juhasz's question, and prove several results related to this.

We also investigate the extent to which club shares the *invariance* property of diamond: the property of being formally equivalent to many of its natural strengthenings and weakenings. We show that when certain cardinal arithmetic statements hold, we can always find different variations on club that will be provably equivalent. Some of these hold in ZFC. But, in the absence of the required cardinal arithmetic, we develop a general method, using forcing, for proving that most variants of club are pairwise inequivalent in ZFC.

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Chapter 1

Introduction: Guessing Axioms and Suslin Trees

In ordinary language the term 'guessing' means, roughly: 'Anticipating properties of something about which we do not have full knowledge.' This definition is our own, and its accuracy is perhaps debatable, but it seems to at least describe a phenomenon recognisable as an *instance* of guessing.

Combinatorial principles in set theory can sometimes be used in a manner that resembles this everyday notion of guessing. In this case, the epistemological emphasis of the above definition is replaced with a focus on cardinality: we wish to find a set of small cardinality that somehow captures non-trivial properties of the members of a larger set. This characterisation is entirely informal, of course, but it captures the intuition behind the usage and naming of a group of axioms known as 'Guessing Axioms.'

Throughout this thesis, the axiomatisation of set theory that we will use is ZFC, the Zermelo-Fraenkel axioms plus the Axiom of Choice. For a detailed account of these axioms, see [19, I] or [15, Chapter One]. A formal statement in the language of set theory, θ, is *independent* of ZFC if there is no formal derivation of θ or $\neg\theta$ from these axioms; Cohen's method of forcing (developed in [4] and [5]) can be used to show that a large number of statements are independent of ZFC. In this thesis we will mostly be interested in questions that ask whether ZFC $+\ \theta \rightarrow \varphi$, where both θ and φ will be statements independent of ZFC; we give a mixture of *combinatorial results* (positive answers to questions of this kind) and *consistency results* (negative answers to questions of this kind). We will frequently abuse notation by suppressing any reference to ZFC and simply asking whether $\theta \rightarrow \varphi$?

Formally, such θ and φ can be treated as *axioms*, without any issue, but because 'axiom' is something of a loaded term – often taken to imply that if θ is an axiom then we ought to have some intuitive reason to believe it to be true – we will more commonly refer to them as *statements* or *principles*.

1.1 Guessing Principles

We are interested in a group of combinatorial principles known as *Guessing Axioms* or *Guessing Principles*.

Throughout this thesis we will use these terms to describe several natural relatives of the axiom $\diamondsuit$. We do not give a formal definition of the term 'Guessing Principle,' but that need not concern us – there are many well-known statements to which it readily applies, including $\clubsuit$, *club guessing* and $\dagger$, and they are all relatives of $\diamondsuit$ – a recap of the definition of $\diamondsuit$ will remind us why the name 'Guessing Principle' is appropriate. In its simplest form, $\diamondsuit$ asserts the existence of a sequence, $\langle D_\delta : \delta < \omega_1$ and δ a is limit ordinal[1]$\rangle$, with $D_\delta \subseteq \delta$ for all $\delta \in Lim(\omega_1)$, such that for any $X \subseteq \omega_1$ the following set is stationary:

$$\{\delta \in Lim(\omega_1) : D_\delta = X \cap \delta\}.$$

Thus a sequence witnessing the truth of $\diamondsuit$, also called a $\diamondsuit$-*sequence*, manages to capture non-trivial properties of any arbitrary subset $X \subseteq \omega_1$, in the sense that the range of the $\diamondsuit$-sequence contains stationary many initial sections of X. There are (at least) $\aleph_2$-many such X, while the witness to $\diamondsuit$ is a sequence of length just ω_1. This fact makes $\diamondsuit$ particularly useful for inductive constructions of objects of size $\aleph_1$, and hence $\diamondsuit$ exemplifies the sense of 'guessing' that we attempted to describe in the opening paragraphs.

[1] In future we denote this by: $\delta \in Lim(\omega_1)$.

1.1.1 Jensen's $\diamondsuit$ and Ostaszewski's $\clubsuit$

The formulation of $\diamondsuit$ is due to the American logician R. B. Jensen (in [17]) and grew out of his close analysis of the set theoretic universe under Gödel's Axiom of Constructibility, $V = L$. He first proved that Suslin trees exist assuming $V = L$ and then extracted the definition of $\diamondsuit$ from this proof as a weaker, but still sufficient, assumption. Hence $\diamondsuit$ implies the existence of Suslin trees (we prove this fact in Chapter 3) and has many further applications as well, in various branches of mathematics. It has been used for example to establish the relative consistency (with ZFC) of a counterexample to Naimark's problem, a long-standing open question in operator algebras [1] and has applications to topology, see [20].

The following two facts come from Jensen [17]:

Fact 1.1.1. $V = L \to \diamondsuit$.

Fact 1.1.2. $\diamondsuit \to$ CH.

Proof For a proof of Fact 1.1.1, see [19, VI 5.2] or [15].

For 1.1.2, let $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ be a witness to $\diamondsuit$ and suppose x is an arbitrary subset of ω. Then the set $\{\delta \in Lim(\omega_1) : D_\delta = x \cap \delta\}$ is stationary, so in particular it is cofinal in ω_1. Let α be in this set and be greater than ω, then $x = x \cap \alpha = D_\alpha$. Hence the sequence witnessing $\diamondsuit$ contains a subsequence enumerating the continuum. This subsequence has length ω_1, so $2^\omega = \omega_1$.

$\square$

The above two facts combine to establish the independence of $\diamondsuit$ from ZFC, given that CH and $V = L$ are themselves independent of ZFC. Jensen was able to prove that CH is a strict weakening of $\diamondsuit$, using a complex forcing iteration to obtain a model of CH without Suslin trees (see [7]). This is a celebrated early result in the theory of forcing, which motivated many further developments in the field; Shelah later gave a considerably shorter proof of the same result, see [28, pp.228–236] for details. The extra power that $\diamondsuit$ has over CH is encapsulated in the principle $\clubsuit$, pronounced 'club,' which forms the focus of much of this thesis. In its simplest form, $\clubsuit$ asserts the following:

> There is a sequence $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ such that $A_\delta \subseteq \delta$ for all $\delta \in Lim(\omega_1)$, and $\sup(A_\delta) = \delta$, and if $X \subseteq \omega_1$ is uncountable then the set $\{\delta \in Lim(\omega_1) : A_\delta \subseteq X\}$ is stationary.

It is easy to see that $\clubsuit$ is a weakening of $\diamondsuit$: we need only note the fact that for any uncountable $X \subseteq \omega_1$, the set $\{\delta < \omega_1 : \sup(X \cap \delta) = \delta\}$ is always a closed unbounded subset of ω_1, so a witness to $\diamondsuit$ can easily be modified to produce a witness to $\clubsuit$. If $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ is a witness to $\diamondsuit$ then defining $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ by setting $A_\delta = D_\delta$ if $\sup(D_\delta) = \delta$ and to be an arbitrary cofinal subset of δ if $\sup(D_\delta) < \delta$, for all $\delta \in Lim(\omega_1)$, will give us a witness to $\clubsuit$. (We will frequently employ this trick to create witnesses to $\clubsuit$ from sequences that almost, but not quite, fulfil the definition of $\clubsuit$. We will not always describe it explicitly, and will usually just say that a given sequence can be 'easily modified' to give a witness to $\clubsuit$.)

In the presence of CH the two are in fact equivalent:

Theorem 1.1.3 (Devlin)**.** $(\clubsuit + \mathrm{CH}) \leftrightarrow \diamondsuit$.

The principle $\clubsuit$ was first formulated by Ostaszewski in [24], where it was used to establish the relative consistency of the existence of a non-compact, hereditarily separable, locally compact, perfectly normal, countably compact space. This came several years after the formulation of $\diamondsuit$. Theorem 1.1.3 is cited in Ostaszewski's original paper (and is attributed there to Devlin) and the construction in that paper uses CH as well as $\clubsuit$, so in fact uses the full power of $\diamondsuit$; a number of years passed before it was established that $\clubsuit$ is indeed not equivalent to $\diamondsuit$.

For completeness, we will give the full proof of Theorem 1.1.3:

Proof of Theorem 1.1.3 In light of the discussion preceding the statement of Theorem 1.1.3, it remains to prove that $(\clubsuit + \mathrm{CH}) \rightarrow \diamondsuit$. So choose an arbitrary witness to $\clubsuit$, $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$. Let $\langle c_\alpha : \alpha < \omega_1\rangle$ be an enumeration of the countable subsets of ω_1, such that each of these subsets appears uncountably often in the enumeration. We can do this because CH implies that $[\omega_1]^{\leq\omega} = \{Z \subseteq \omega_1 : |Z| \leq \omega\}$ has cardinality ω_1. We define the sequence $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ as follows: for all $\delta \in Lim(\omega_1)$ let $D_\delta = \delta \cap \bigcup_{\alpha \in A_\delta} c_\alpha$. We claim that $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ gives us a witness to $\diamondsuit$.

To see this, first let $X \subseteq \omega_1$ be a bounded subset. Then the set

$$Y = \{\alpha < \omega_1 : X = c_\alpha\}$$

will be unbounded in ω_1, and thus will contain stationary many A_δ as subsets. For any such δ, greater than $\sup(X)$, D_δ is equal to the set $\bigcup_{\alpha \in A_\delta} c_\alpha = \bigcup_{\alpha \in A_\delta} X$, which is equal to X, and hence is also equal to $X \cap \delta$. There are stationary many δ such that $A_\delta \subseteq Y$ and $\delta > \sup(X)$, so we obtain the stationary set required in the definition of $\diamondsuit$.

If $X \subseteq \omega_1$ is unbounded, then let $Y \subseteq \omega_1$ be such that for all $\alpha \in Y$, c_α is an initial section of X, and if $\alpha, \beta \in Y$ satisfy $\alpha < \beta$ then c_α is an initial section of c_β. (In other words, Y indexes an increasing chain in the ordering of $\{c_\alpha : \alpha < \omega_1\}$ by initial-sectionhood.) It is straightforward to define such a Y by induction, and to see that Y will be unbounded in ω_1. It is also clear that for a closed unbounded set, C, it will be the case that $\delta \in C$ implies $\bigcup_{\alpha \in \delta \cap Y} c_\alpha = X \cap \delta$, by a standard argument. So there will be a stationary set, S, such that $S \subseteq C$ and for $\delta \in S$ we get $A_\delta \subseteq Y$ and $\sup(A_\delta) = \delta$. For any δ in S we then have that $D_\delta = X \cap \delta$, which again gives us the stationary set required by the definition of $\diamondsuit$.

$\square$

From Theorem 1.1.3 and Fact 1.1.2 above, we conclude that to establish that $\clubsuit$ is a *strict* weakening of $\diamondsuit$ it is both necessary and sufficient to prove the relative consistency of $\clubsuit + \neg$CH. This was first done by Shelah in [27], via a proof that involved adding $\aleph_3$ many subsets of $\aleph_1$ to a model of GCH, through a countably closed forcing, and then collapsing $\aleph_1$ to $\aleph_0$. Shortly afterwards, Baumgartner proved the same result using a forcing that does not collapse cardinals, by adding $\aleph_2$-many Sacks reals by side-by-side product and showing that $\clubsuit$ is preserved if $\diamondsuit$ holds in the ground model (this proof was not published by Baumgartner himself, but see [14] for details). The simplest proof that Con(ZFC) $\rightarrow$ Con(ZFC + $\clubsuit$ + $\neg$CH) known to the author is that of Fuchino, Shelah and Soukup in [11]; we give a version of this proof in Chapter 5.

Thus it has been established that $\clubsuit$ is a strictly weaker axiom than $\diamondsuit$. The following informal question suggests itself as the natural thing to ask next: how *much* weaker is $\clubsuit$ than $\diamondsuit$?

1.1.2 How much weaker is $\clubsuit$ than $\diamondsuit$?

The relative consistency of ZFC + $\clubsuit$ + $\neg$CH, taken together with the fact that ($\clubsuit$ + CH) $\leftrightarrow \diamondsuit$, means that we can sensibly think of $\clubsuit$ as being '$\diamondsuit$ without the cardinal arithmetic assumptions.'[2] Due to the manifold applications of $\diamondsuit$, in many different areas of mathematics, we therefore consider the question 'which properties of $\diamondsuit$ are shared by $\clubsuit$?' to be important as a restricted version of the

[2]Several other axioms would also satisfy this description, by the same reasoning. This description of $\clubsuit$ is therefore arbitrary and is to be taken purely as an aid to intuition.

broader question: how crucial are cardinal arithmetic assumptions in determining the structure of the set-theoretic universe? The importance of this latter question is self-evident.

However, as it stands this is not a formal question. The simplest way to paraphrase it formally is to find statements ϕ such that $\diamondsuit \to \phi$ and to ask whether ϕ follows from $\clubsuit$ alone. We have seen already that when ϕ is the Continuum Hypothesis then the answer to this question is negative. This fact suggests a wealth of natural questions concerning weakenings of CH and their relation to $\clubsuit$; for instance, those concerned with *cardinal invariants of the continuum* (see [2]). When CH holds, all cardinal invariants are bounded by $\aleph_1$, trivially, so it is natural to ask: which cardinal invariants must necessarily have size $\aleph_1$ in models of $\clubsuit$? The answer to this question provides us with many non-trivial facts about $\clubsuit$. This thesis is not particularly concerned with cardinal invariants, except where they have relevance to Juhasz's question (see Chapter 3), but we mention two of the more notable known facts here:

Theorem 1.1.4 (J. Brendle, [3])**.**

$\clubsuit \to (\mathfrak{b} = \omega_1)$, where $\mathfrak{b}$ is the bounding number.

Theorem 1.1.5. *Con(ZFC) $\to$ Con(ZFC + $\clubsuit$ + $\mathfrak{d} = \omega_2$), where $\mathfrak{d}$ is the dominating number.*

Theorem 1.1.5 was first proved by I. Juhasz, though the proof was not published. See [22] for a proof due to H. Mildenberger; the model constructed in [11] also satisfies the conditions of the Theorem and was therefore the first published proof of this result.

Arguably the most prominent open question of the form: 'does $\clubsuit \to \phi$?', where ϕ is a consequence of $\diamondsuit$, was asked by the Hungarian set-theoretic topologist I. Juhasz ([23]). This is the question of whether $\clubsuit$ implies the existence of a Suslin tree. Juhasz's question forms the focus of Chapters 3 and 4 of this thesis; the question remains open (at the time of writing) but we consider some partial answers to it and prove some restrictions on potential techniques for forcing a negative answer to it. Chapters 6 to 8 of this thesis concern another property of $\diamondsuit$ and its relation to $\clubsuit$: following [9] we call this the *invariance property.* This is an informally defined notion roughly expressing the fact that $\diamondsuit$ is formally equivalent to many of its apparent weakenings and strengthenings. We look at the extent to which the same is true of $\clubsuit$. Both this and Juhasz's question fall broadly under the umbrella of the ubiquitous question: 'How much weaker is $\clubsuit$ than $\diamondsuit$?'

We give the background to the invariance property in Chapter 6. The background to Juhasz's question is given in the next section.

1.2 Suslin trees

Having introduced Guessing Axioms, and in particular $\clubsuit$, we turn now to the other combinatorial objects that dominate this thesis: *Suslin trees*.

We will see (Theorem 1.2.10) that $\diamondsuit$ implies the existence of a Suslin tree. It is not known whether $\clubsuit$ implies the existence of a Suslin tree. This question was asked in the 1980s[3] and has proved to be a remarkably persistent problem. We review its history here:

1.2.1 Suslin's Hypothesis

M. Y. Suslin (1894 – 1919) was a Russian mathematician, active in set theory at the start of the previous century. He is remembered for several developments in mathematics, and particularly for a paper that he contributed to the first issue of the journal *Fundamenta Mathematicae* ([31]), which was published in 1920. A question posed in that paper (on the properties needed to uniquely characterise the real number line) became widely known as 'Suslin's Problem.' The question persisted into the second half of the twentieth century, awaiting the arrival of Cohen's method of forcing, and later iterated forcing, which were used to conclusively attack it. By that time the problem was known in its modern formulation, concerning the existence of a certain type of tree. But first we shall state Suslin's

[3]Source: a personal conversation between I. Juhasz and the author, and see also [23].

Problem in its original form. Briefly put, Suslin asked whether the condition of *separability* can be weakened in the following well-known theorem:

Definition 1.2.1. Let $\langle L, \leq_L \rangle$ be a linearly ordered set (for convenience we will usually just denote it by L). Then L is *dense* if for all $a, b \in L$ with $a <_L b$ there is a $c \in L$ such that $a <_L c <_L b$. $A \subseteq L$ is a *dense subset* if A is dense and for all $a, b \in L$ with $a <_L b$ there is a $c \in A$ such that $a <_L c <_L b$. L is *complete* if for every set $A \subseteq L$ that has an upper bound in L, $\sup(A)$ exists in L. L is *separable* if L has a countable dense subset. L is *without end-points* if there is no greatest or least element in L.

Theorem 1.2.2 (Cantor, see [15])**.** Let $\langle L, \leq_L \rangle$ be a linearly ordered set. If L is:

(i) dense,

(ii) complete,

(iii) separable,

(iv) without end-points,

Then $\langle L, <_L \rangle$ is isomorphic to the real numbers, $\mathbb{R}$, with the usual ordering.

Proof This proof is well-known, so we only sketch it here. The result follows from both Dedekind's method of constructing the real numbers as sets of rationals and Cantor's back-and-forth argument establishing that any two countable dense linear orders without endpoints are isomorphic. The latter gives us an isomorphism between the countable dense subset of L (call it A) and $\mathbb{Q}$; identifying each $r \in \mathbb{R}$ with the set of rationals less than it, and each $l \in L$ with the set of $a \in A$ less than l, induces an isomorphism between L and $\mathbb{R}$. See [15] for details.

$\square$

Suslin asked whether condition (iii) in Theorem 1.2.2 could be weakened to the following:

Definition 1.2.3. Let $\langle L, \leq_L \rangle$ be a linearly ordered set. L has the *countable chain condition* (c.c.c.) if every set of pairwise disjoint open intervals from L is countable.

Suslin's Hypothesis (SH) states that any linearly ordered set satisfying conditions (i), (ii) and (iv) of Theorem 1.2.2, which also has the property of c.c.c., is isomorphic to the real numbers, $\mathbb{R}$. A linearly ordered set that satisfies these properties and which is not isomorphic to $\mathbb{R}$ is called a *Suslin line*. Thus, Suslin's Hypothesis states that there does not exist a Suslin line. This conjecture was shown to be independent of the axioms of ZFC in the 1960s and early 1970s, by the combined efforts of Jech, Solovay and Tennenbaum in [32], [16] and [30] (see [7]).

The modern formulation of SH uses the idea of a *Suslin tree*, which is a certain type of partially ordered set (that is, a set together with an ordering relation satisfying transitivity, anti-symmetry and reflexivity), as defined in Definition 1.2.5.

Definition 1.2.4. For a partial order $\langle P, \leq_P \rangle$:

- A *chain* is a set $Y \subseteq P$ such that for all $x, y \in Y$ with $x \neq y$, either $x <_P y$ or $y <_P x$.
- An *antichain* is a set $Y \subseteq P$ such that there is no $z \in P$ with $x \leq_P z$ and $y \leq_P z$ for any $x, y \in Y$.

Definition 1.2.5. A *tree*, $\langle T, \leq_T \rangle$, is a partial order such that for every $x \in T$, the set $\{y \in T : y \leq_T x\}$ is well-ordered by $\leq_T$. A *Suslin tree* is a tree of size $|T| = \aleph_1$, such that all chains and antichains in T are countable. (The fact that all antichains are countable is what will henceforth be meant when we say that a partial order is c.c.c.)

Definition 1.2.6. For a tree $\langle T, \leq_T \rangle$ and $x \in T$, $\mathrm{ht}(x)$ is the order type of the set $\{z : z <_T x\}$.

We now cite a useful theorem that allows us to forget about Suslin lines in favour of Suslin trees, which are easier to use in forcing arguments. For a mathematical account of this shift in emphasis, see [19, II], or [7]; we simply note here that this result was discovered independently by Kurepa, in 1935, and E. Miller in 1943:

Theorem 1.2.7 (Kurepa, Miller)**.** There exists a Suslin tree if and only if there exists a Suslin line.

From here onwards Suslin's Hypothesis (SH) is taken to be the assertion that there do not exist any Suslin trees. It is worth pointing out that Suslin's Hypothesis is also widely known as 'Souslin's Hypothesis.' Both are valid transliterations from the Cyrillic. We follow Kunen [19] and Jech [15] in using 'Suslin'. We will require some further notation:

Notation 1.2.8. Let $\langle T, \leq_T \rangle$ be a Suslin tree. Then $Lev_\alpha(T) = \{x \in T : \mathrm{ht}(x) = \alpha\}$.

$Lev_\alpha(T)$ will be referred to as the αth level of T. It is trivial that for all $\alpha < \omega_1$, $Lev_\alpha(T)$ is an antichain. Often one includes in the definition of a Suslin tree the fact that each level is countable, but the c.c.c. property makes this redundant. When, however, we talk about *Aronszajn* trees, it is to be understood that we are defining them by replacing the c.c.c. property in the definition of a Suslin tree with the requirement that $(|Lev_\alpha(T)| = \omega)$ for all $\alpha < \omega_1$.

Definition 1.2.9. An *Aronszajn tree* is a tree of size ω_1 such that all levels are countable and all chains are countable.

1.2.2 Juhasz's question

The definition of $\diamondsuit$ was extrapolated from Jensen's proof that Suslin trees exist assuming $V = L$. The following theorem is then immediate:

Theorem 1.2.10. $\diamondsuit$ implies that there is a Suslin tree (i.e. $\diamondsuit \to \neg\mathrm{SH}$).

Proof In Theorem 3.0.5 we prove a stronger statement. For a direct proof of Theorem 1.2.10 see [19, II].

$\square$

This leads us to the following natural question:

Question 1.2.11 (Juhasz)**.** Does $\clubsuit \to \neg\mathrm{SH}$?

Question 1.2.11 is commonly referred to as 'Juhasz's question'. Juhasz formulated a weak relative of the $\clubsuit$ principle in [18] and asked whether it implied $\neg$SH. He then observed that it wasn't known whether $\clubsuit$ itself implies $\neg$SH (though neither question actually appears in [18]); thirty years later both questions remain unanswered. It is also unknown whether $\clubsuit$ is relatively consistent with the assertion that all Suslin trees are isomorphic. $\diamondsuit$ implies that there are at least two non-isomorphic Suslin trees.

We ought to note here that a purported answer to Juhasz's question by Džamonja and Shelah was published in [8], but the authors later noticed a mistake in this paper rendering the proof incorrect [10]. The result they appeared to obtain there is stronger than Con($\clubsuit$ + SH), as their proof would in fact establish Con($\clubsuit$ + SH + $\mathrm{cov}(\mathcal{M}) = \omega_2$), if correct. But this contradicts a known (though at the time unpublished) theorem of Miyamoto (see Chapter 3).

We examine Juhasz's question in Chapters 3 and 4, and give some pertinent results there.

1.3 The structure of this thesis

The structure of this thesis is as follows:

- We begin in Chapter 2 by briefly reviewing some notation and preliminaries.
- Chapter 3 is concerned with partial answers to Juhasz's question. We first survey some of the existing partial answers, then we define the principle *Superclub* and prove that it implies the existence of a Suslin tree. We conjecture that it is strictly weaker than $\diamondsuit$ and give some related results to substantiate this conjecture.
- In Chapter 4 we discuss the possibility of forcing to obtain a model of Suslin's Hypothesis, and we establish some conditions that such a forcing would have to meet if it were to be used to give a negative answer to Juhasz's question. Specifically, we define several properties that a witness to $\clubsuit$ must not satisfy if it is to be preserved (as a witness to $\clubsuit$) over a forcing iteration giving us

a model of SH.

- In Chapter 5 we review some basic facts about $\clubsuit$ and its relation to cardinal arithmetic. We give a full proof that $\clubsuit$ is consistent with $\neg$CH and ask under what conditions can we force $\clubsuit$ to hold without collapsing cardinals. We show that there is a c.c.c. forcing that adds a $\clubsuit$-sequence (which, in particular, does not necessarily add a $\diamondsuit$-sequence) whenever a weak version of $\clubsuit$ holds.

- In Chapter 6 we prove some equivalences between different versions of $\clubsuit$. We show that a greater number of $\clubsuit$-like principles can be proved equivalent as increasingly stronger cardinal arithmetic statements are assumed to hold, though we also prove some equivalences in ZFC. Several known results on $\diamondsuit$ and club guessing follow from our results in this chapter as specific instances.

- Chapter 7 is a counterpart to Chapter 6. Here we extend work begun by Džamonja and Shelah in [9] and establish a general forcing technique to show that many of the equivalences in the previous chapter are not provable in ZFC alone. We show that several variants of $\clubsuit$, as defined on ω_1, can be proved to be pairwise inequivalent in ZFC.

- Chapter 8 generalises the results of Chapter 7 to successor cardinals greater than ω_1, and we discuss some limitations on the extent to which we can further generalise these results.

- Finally, in Chapter 9, we list some open questions relating to our results in the preceding chapters.

Chapter 2

Notation and Preliminaries

We assume the reader is familiar with the basics of set theory and logic. We take this to include everything implicit in the previous chapter, and in particular: the axioms of ZFC, the definitions of ordinals, cardinals, relations and functions, stationary sets, sequences, products, models of set theory, elementary submodels, Gödel's incompleteness theorems and the standard variations on the Löwenheim-Skolem theorem, which we will make frequent use of in the later chapters, and which we note in particular can be proved in ZFC.

We cite [15] as the standard reference for the basic facts and definitions listed above. We will now specify some of the notational conventions that are not necessarily universal but that are used frequently throughout this thesis:

- Let $f : A \to B$ be a function, and $C \subseteq A$. Then we write $f[C]$ to denote the set $\{b \in B : \exists c \in C(f(c) = b)\}$, and $f^{-1}(b)$ to denote $\{a \in A : f(a) = b\}$ if $b \in B$, and $f^{-1}(B') = \bigcup\{f^{-1}(b) : b \in B'\}$ if $B' \subseteq B$.

- A *partial function* $f : A \to B$ is a function $f : C \to B$ where $C \subseteq A$. The cardinality of a partial function refers to the cardinality of the set C.

- We write (a, b) to denote an ordered pair, unless we are defining a partial order (which, formally, is an ordered pair consisting of an underlying set and a relation), in which case we use angular brackets: $\langle A, \leq_A \rangle$. We write $\langle x_\alpha : \alpha < \lambda \rangle$ to denote a sequence of length λ and $\{x_\alpha : \alpha < \lambda\}$ to denote the unordered set of elements in the range of this sequence.

- When we refer to a *cardinal*, we allow for the possibility that the cardinal in question is finite, unless otherwise specified, but the word *countable* will be used exclusively to describe infinite sets of size $\aleph_0$.

- We introduced the principle $\clubsuit$ in the previous chapter. An uncountable sequence is called a $\clubsuit$-sequence if it witnesses the truth of $\clubsuit$. A $\diamondsuit$-sequence is defined analogously. We say that a forcing (that preserves ω_1) kills a $\clubsuit$-sequence, $\langle A_\delta : \delta \in Lim(\omega_1) \rangle$, if it adds an uncountable subset $X \subseteq \omega_1$ such that the set $\{\delta \in Lim(\omega_1) : A_\delta \subseteq X\}$ is empty. Likewise, we say that a (cardinal preserving) forcing kills a Suslin tree, T, if it adds an uncountable set that is an antichain in T.

- A *subtree* of a given tree $\langle T, \leq_T \rangle$ is a tree of the form $\langle T' \subseteq T, \leq_T \restriction T' \rangle$. We say a Suslin tree is *normal* if the following hold:

  (i) For every $x \in T$, and any $\alpha < \omega_1$, there is a $y \in Lev_\alpha(T)$ such that $x <_T y$.

  (ii) $Lev_0(T)$ has cardinality 1.

  (iii) For every $x \in Lev_\alpha(T)$, for any $\alpha < \omega_1$, there are $y_1, y_2 \in Lev_{\alpha+1}(T)$ with $x <_T y_1, y_2$.

  It is an easily provable fact that every Suslin tree has a normal Suslin subtree (see [19, Chapter Two]); therefore we will usually work with normal Suslin trees rather than with general Suslin trees.

Any other notation used in this thesis, where it is not in standard usage, will be introduced as and when it is needed.

2.1 Forcing notation

We assume some familiarity with the theory of forcing, but due to the wide variety of forcing notation that is used in the literature we will now briefly outline the development of forcing that we have chosen to adopt.

A *partial order*, $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$, consists of a set together with a relation that is transitive, reflexive and anti-symmetric. If $\mathbb{P}$ is infinite, has a maximal element (denoted $1_{\mathbb{P}}$), and is such that for any $p \in P$ there exists $p^1, p^2 \in P$ such that $\neg \exists q \in P(q \leq_{\mathbb{P}} p^1$ and $q \leq_{\mathbb{P}} p^2)$ then we call it a *forcing notion*, or just a *forcing*. In this case elements in the partial order will be called *conditions*. We will often abuse notation by writing $p \in \mathbb{P}$ rather than $p \in P$, and by dropping the subscript from $\leq_{\mathbb{P}}$ where $\mathbb{P}$ is clear from context.

When a partial order, $\mathbb{P}$, is a forcing, the definition of an antichain in $\mathbb{P}$ differs slightly from that given in the previous chapter. In this case an antichain is a set of conditions in $\mathbb{P}$ such that for any two of them, p and q, there is no condition r with $r \leq_{\mathbb{P}} p$ and $r \leq_{\mathbb{P}} q$.

A set $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is a $q \in D$ with $q \leq_{\mathbb{P}} p$. A filter G is a set such that if $p \in G$ and $p \leq_{\mathbb{P}} q$ then $q \in G$, and such that for any two $p^1, p^2 \in G$ there is a $q \in \mathbb{P}$ such that $q \leq_{\mathbb{P}} p^1$ and $q \leq_{\mathbb{P}} p^2$.

Let V be a model of ZFC. Then a filter G is $\mathbb{P}$-generic over V if G intersects every dense subset $D \subset \mathbb{P}$ that is in V. If V is a countable model of ZFC and $\mathbb{P} \in V$, then such a G can be shown to exist; we cannot, of course, prove in ZFC that a model of ZFC exists, so all our forcing proofs are in fact relative consistency proofs which begin by assuming the consistency of ZFC.

A set $\dot{\tau} \in V$ is a $\mathbb{P}$-*name* if and only if $\dot{\tau}$ is a set of ordered pairs and for all $(\dot{\sigma}, p) \in \dot{\tau}$, $\dot{\sigma}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. This is a recursive definition, trivially satisfied by $\emptyset$. For a $\mathbb{P}$-name $\dot{\tau}$ and a filter G, let $\dot{\tau}_G = \{\dot{\sigma}_G : \exists p \in G\,((\dot{\sigma}, p) \in \dot{\tau})\}$.

Again this is a recursive definition. We also set $V[G] = \{\dot{\tau}_G : \dot{\tau} \in V \text{ is a } \mathbb{P}\text{-name}\}$. We make numerous uses of the following crucial theorem:

Theorem 2.1.1. Let $V \vDash \mathrm{ZFC}$ and $\mathbb{P}$ be a notion of forcing. If G is a $\mathbb{P}$-generic filter over V, then $V[G] \vDash \mathrm{ZFC}$ and $G \in V[G]$.

The forcing relation $\Vdash$ is defined as follows:

Definition 2.1.2. Let $\dot{\tau}_1, \ ..., \dot{\tau}_n$ be $\mathbb{P}$-names and $\phi(\dot{\tau}_1, \ ..., \dot{\tau}_n)$ be a sentence in the language of set theory. Then for a condition $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}}$ "$\phi(\dot{\tau}_1, \ ..., \dot{\tau}_n)$" if and only if for any generic filter G such that $p \in G$, we have $V[G] \vDash \phi((\dot{\tau}_1)_G, \ ..., (\dot{\tau}_n)_G)$. We usually drop the subscript from $\Vdash_{\mathbb{P}}$ when $\mathbb{P}$ is clear from context.

We use dotted Greek letters to denote $\mathbb{P}$-names, usually $\dot{\tau}$. If we are dealing with a name for a function (or a name forced to be a function by a particular p under consideration), then we will sometimes use $\dot{f}$ to denote it; the dot is intended to make it clear that this is a name and not a function in V. When $x \in V$ there is a canonical $\mathbb{P}$-name for x, $\check{x} = \{(\check{y}, 1_{\mathbb{P}}) : y \in x\}$, such that $\check{x}_G = x$ for any filter G. Hence $V \subseteq V[G]$. In practice we will normally use x instead of $\check{x}$ when writing statements of the form $p \Vdash$ "$\phi(\check{x})$"; the quotation marks surrounding the $\phi(\check{x})$ are for the purposes of clarity, as it is infeasible to write $\phi(\check{x})$ as a fully formal statement in the language of set theory. Note that we have developed our notation for forcing so that for $p, q \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ means q is a stronger condition that p. That is, if $p \Vdash$ "ϕ" then $q \Vdash$ "ϕ".

If V is a transitive model of ZFC then $V[G]$ is also transitive; it is to be implicitly understood that this will always be the case. It is straightforward to check that ω is absolute for transitive models of ZFC. We say that ω_1^V is collapsed by G if ω_1^V is countable in $V[G]$. Similarly, for λ^V, an arbitrary cardinal in V, we say that λ is collapsed to κ if there is a bijective map from λ^V to $\kappa^{V[G]}$ in $V[G]$. If a forcing $\mathbb{P}$ has the λ-c.c. (i.e. all antichains in $\mathbb{P}$ have size $< \lambda$) then no cardinal greater than or equal to λ is collapsed by $\mathbb{P}$.

When defining a forcing we will often use the phrase "let χ be a sufficiently large cardinal..." Specifically, we want χ to be large enough such that $(\mathcal{H}(\chi), \in)$ encompasses enough of V to reflect certain statements in which we are interested. These will always be clear from context. In all cases where this phrase is used, the forcing being defined will be formed from a set of partial functions $f : \kappa \to 2$ for some cardinal κ, and we will use a chain of elementary submodels of $(\mathcal{H}(\chi), \in)$ to define this set. Hence, setting χ to be strictly greater than 2^κ will be sufficient, so e.g. letting $\chi = 2^{2^\kappa}$ works for this. We won't explicitly state this each time the phrase is used, but it is always possible to find a relevant χ.

Chapter 3

Partial Answers to Juhasz's Question

Informally, we can state that the expected answer to Juhasz's question is negative (this view was expressed to the author in conversation by M. Džamonja, co-author of [9] and [8], and by I. Juhasz himself), though the following principles are two of the strongest weakenings of $\clubsuit$ that have been shown to be consistent with SH. Both are much weaker than $\clubsuit$, in the sense that even in the presence of CH they do not imply $\diamondsuit$, unlike those weakenings we consider later in this thesis:

(⫯) There is a set $\mathcal{S}$ with $|\mathcal{S}| = \omega_1$ and $|s| = \omega$ for all $s \in \mathcal{S}$, such that if $X \in [\omega_1]^{\omega_1}$ then for some $s \in \mathcal{S}$ we have $s \subseteq X$.

($\clubsuit_{W^2}$) There is a sequence $\langle A_\delta : \delta \in Lim(\omega_1) \rangle$, with $A_\delta \subseteq \delta$ and

$\sup(A_\delta) = \delta$ for all $\delta \in \mathit{Lim}(\omega_1)$, such that if $X \subseteq \omega_1$ is unbounded then the following set is stationary:

$$\{\delta \in \mathit{Lim}(\omega_1) : \text{ either } A_\delta \setminus X \text{ or } A_\delta \setminus (\omega_1 \setminus X) \text{ is finite } \}.$$

Both of these principles are implied by $\clubsuit$. The principle $\stick$ is also implied by CH; in this case the set of all countably infinite subsets of ω_1 forms a suitable $\mathcal{S}$. So the relative consistency of ($\stick$ + SH) follows from Jensen's proof of Con(CH + SH) assuming Con(ZFC). The principle $\clubsuit_{W^2}$ was shown to be consistent with SH by H. Mildenberger (in [21]).

There is a notable lack of *positive* partial answers to Juhasz's question. The most prominent result that could be so described is due to Miyamoto:

Theorem 3.0.3 (Miyamoto)**.** If $\mathrm{cov}(\mathcal{M}) \geq \omega_2$ and $\stick$ holds, then there is a Suslin tree.

But there are no known[1] guessing principles φ such that $\varphi \to \neg\mathrm{SH}$ and $\diamondsuit \to \varphi \to \clubsuit$, where these implications are not reversible.

In this chapter we present a candidate for such a φ. We prove that it can be used to construct a Suslin tree, and that it implies $\clubsuit$. We conjecture that it is strictly weaker than $\diamondsuit$.

Definition 3.0.4. The principle *Superclub* states that there is a sequence $\langle B_\delta :$

[1]Known to the author, at least. We are using here our informal characterisation of guessing principles, as discussed in Chapter One.

$\delta \in Lim(\omega_1)\rangle$ such that for any $X \in [\omega_1]^{\omega_1}$, there is a $Y \in [\omega_1]^{\omega_1}$ such that $Y \subseteq X$ and the set $\{\delta \in Lim(\omega_1) : Y \cap \delta = B_\delta\}$ is stationary.

So a witness to Superclub (a *Superclub sequence*) acts like $\diamondsuit$, but on a cofinal subset of every unbounded $X \subseteq \omega_1$ rather than on X itself. It therefore follows immediately that $\diamondsuit \to$ Superclub. It is also easy to see that Superclub $\to \clubsuit$.

Superclub is notable mainly for the following theorem:

Theorem 3.0.5. Superclub $\to \neg$SH.

Proof Let $\langle B_\delta : \delta \in Lim(\omega_1)\rangle$ be a witness to Superclub; we construct $\langle \omega_1, \leq_T\rangle$ to be a Suslin tree, by inductively specifying the behaviour of $\leq_T$ restricted to initial sections of ω_1. Our induction will ensure that if $x \ <_T \ y$ then $x < y$ as an ordinal and that $Lev_\beta(T) = [\omega.\beta,\, \omega.\beta + \omega)$ for each β less than ω_1, except where $\beta = 0$ or 1. Throughout the proof we frequently abuse notation by writing $Lev_\beta(T)$ as shorthand for $[\omega.\beta, \omega.\beta + \omega)$ when $2 \leq \beta < \omega_1$, $\{0\}$ when $\beta = 0$, and $[1, \omega + \omega)$ when $\beta = 1$. We also write $\leq_T\restriction A$ to denote $\{(a, b) \in A \times A : a \leq_T b\}$, and similarly for $<_T\restriction A$. The induction is on the levels $Lev_\alpha(T)$, for $\alpha < \omega_1$, and proceeds as follows:

1. We set $<_T\restriction \{0\}$ to be empty and $<_T\restriction [0, \omega + \omega)$ to be the set of all ordered pairs $(0, y)$ such that y is in the interval $[1,\, \omega + \omega)$. Hence $Lev_0(T) = \{0\}$ and $Lev_1(T) = [1,\, \omega + \omega)$ as desired. Choose an enumeration $\langle i_n : n < \omega\rangle$ of the set $[1,\, \omega + \omega)$ and let $<_T\restriction (\omega.2 + \omega)$ be the set $<_T\restriction [0, \omega + \omega)$ together

with all those ordered pairs of the form $(i_n, \omega.2+2n)$ or $(i_n, \omega.2+2n+1)$ for $n<\omega$, and then take the transitive closure of this. This ensures that $Lev_2(T) = [\omega.2, \omega.2+\omega)$.

2. If α is a successor ordinal greater than 2, assume $\alpha = \beta+1$. Then we assume that $Lev_\beta(T) = [\omega.\beta, \omega.\beta+\omega)$ and that $<_T \restriction (\omega.\beta+\omega)$ is already defined. We extend the ordering $<_T$ to include $[\omega.\alpha, \omega.\alpha+\omega)$ as follows. Let $y \in Lev_\beta(T)$, then $y = \omega.\beta+n$ for some $n<\omega$. The ordering $<_T$ is extended by setting $y <_T x_1$ and $y <_T x_2$ where $x_1 = \omega.\alpha+2n$ and $x_2 = \omega.\alpha+2n+1$, and also setting $z <_T x_1, x_2$ for all $z <_T y$. Each element in $Lev_\beta(T)$ has exactly two successors at the level $Lev_\alpha(T)$.

3. If α is a countable limit ordinal then we assume that $<_T\restriction \bigcup_{\beta<\alpha} Lev_\beta(T)$ is already defined. Let $\langle x_i : i<\omega\rangle$ enumerate $\bigcup_{\beta<\alpha} Lev_\beta(T)$. If $\bigcup_{\beta<\alpha} Lev_\beta(T) = \alpha$ and B_α is an antichain in the tree:

$$\langle \alpha, \leq_T\restriction \bigcup_{\beta<\alpha} Lev_\beta(T)\rangle$$

Then for each $i<\omega$ we choose a branch $\mathrm{br}_\alpha(x_i)$ such that $x_i \in \mathrm{br}_\alpha(x_i)$, $\sup(\mathrm{br}_\alpha(x_i)) = \alpha$ and *if* there is some γ in B_α with $\gamma \leq_T x_i$ or $x_i <_T \gamma$ then the least such γ is in $\mathrm{br}_\alpha(x_i)$, and we also insist that if $j<i<\omega$, then $\mathrm{br}_\alpha(x_i) \neq \mathrm{br}_\alpha(x_j)$.

If $\bigcup_{\beta<\alpha} Lev_\beta(T) \neq \alpha$ or B_α is not an antichain, then we choose a branch for each $i < \omega$ such that $x_i \in \mathrm{br}_\alpha(x_i)$ and $\mathrm{br}_\alpha(x_i)$ intersects $[\omega.\beta, \omega.\beta + \omega)$ for every $\beta < \alpha$. Again, we also insist that if $j < i < \omega$, then $\mathrm{br}_\alpha(x_i) \neq \mathrm{br}_\alpha(x_j)$. Having defined $\{\mathrm{br}_\alpha(x_i) : i < \omega\}$, let $y \in [\omega.\alpha, \omega.\alpha + \omega)$. Then $y = \omega.\alpha + n$ for some $n < \omega$. We then set $z <_T y$ if and only if $z \in \mathrm{br}_\alpha(x_n)$. This extends the ordering $<_T$ to $\bigcup_{\beta \leq \alpha}[\omega.\beta, \omega.\beta + \omega)$.

This is identical to Jensen's construction of a Suslin tree from $\diamondsuit$ (see [19, II]) except that at those limit stages α where B_α is an antichain, when we choose a branch that passes through a given x and goes cofinal in the initial section of the tree already defined, we only insist that it intersects B_α if it is possible for it to do so (regardless of whether or not B_α is maximal in that initial section of the tree).

T is clearly a tree. We show that T is Suslin. Since every element of T has (at least) two immediate successors, it is enough to show that T has no uncountable antichains. So assume for a contradiction that $X \subseteq \omega_1$ is a maximal uncountable antichain in T. Then there is a cofinal subset $Y \subseteq X$ such that $\delta \cap Y = B_\delta$ for stationary many δ. Let

$$T' = \{x \in \omega_1 : \exists y \in Y(y \leq_T x \text{ or } x \leq_T y)\}$$

and $\leq_{T'} = \leq_T \restriction T'$. Clearly $\langle T', \leq_{T'}\rangle$ is a tree of size ω_1, and Y is a maximal antichain in T'. So there will be stationary many δ where $Y \cap \delta$ is a maximal antichain in $\langle \delta, \leq_{T'} \restriction \bigcup_{\alpha<\delta} Lev_\alpha(T')\rangle$ and $B_\delta = Y \cap \delta$ and $\bigcup_{\alpha<\delta} Lev_\alpha(T') = \delta$. Take such a δ. We show that for every $x \in Lev_\delta(T')$ there is some $y \in Y \cap \delta$ with

$y <_{T'} x$, meaning $Y \cap \delta$ is already a maximal antichain in T', a contradiction. So assume that for some $x \in Lev_\delta(T')$ there is no such y. By the construction of the level $Lev_\delta(T')$ there is some $x' <_{T'} x$ such that x is an upper bound to all the elements in $\mathrm{br}_\delta(x')$. But since $x' \in T'$ and hence is clearly comparable with some $y \in B_\delta$, there must be a $y \in B_\delta$ such that $y \in \mathrm{br}_\delta(x')$, giving $y <_T x$. This is a contradiction.

So $\delta \cap Y$ is a maximal antichain in T', which is also contradictory. Hence T cannot have an uncountable antichain. This means that T is a Suslin tree.

$\square$

It is not known if Superclub is consistent with $\neg$CH; it may in fact be equivalent to $\diamondsuit$. But it is worth pointing out that the restriction of Superclub to closed unbounded sets is demonstrably weaker than the restriction of $\diamondsuit$ to closed unbounded sets, which is equivalent to $\diamondsuit$. We will briefly develop this argument here:

Definition 3.0.6. The principle $\text{Superclub}_{\text{CLUB}}$ asserts the existence of a sequence $\langle B_\delta : \delta \in Lim(\omega_1)\rangle$ such that for any closed unbounded $C \subseteq \omega_1$, there is an unbounded $D \subseteq \omega_1$ such that $D \subseteq C$ and the set $\{\alpha \in Lim(\omega_1) : D \cap \alpha = B_\alpha\}$ is stationary.

Equivalently, we can insist that the $D \subseteq \omega_1$ in the above definition is closed unbounded (simply replace each B_δ with the following set: $B'_\delta = B_\delta \cup \{\alpha < \delta :$

$\sup(B_\delta \cap \alpha) = \alpha\}$ and then $\langle B'_\delta : \delta \in Lim(\omega_1)\rangle$ will give us a witness to this seemingly stronger statement). The following is well-known:

Lemma 3.0.7. If $\mathbb{P}$ is a c.c.c. forcing in V and G is a $\mathbb{P}$-generic filter over V, then if $E \in V[G]$ is a closed unbounded subset of ω_1 there exists a closed unbounded set $E' \subseteq \omega_1$ in V such that $E' \subseteq E$.

This gives us the following result:

Theorem 3.0.8. Con(ZFC) $\rightarrow$ Con(ZFC + $\text{Superclub}_{\text{CLUB}}$ + $\neg$CH).

Proof Start with a model of $\diamondsuit$. Use the forcing consisting of finite partial functions from ω_2 to 2. This is a c.c.c. forcing and gives a generic extension in which $2^\omega = \omega_2$. It is easy to see that any witness to $\diamondsuit$ in the ground model will be a witness to $\text{Superclub}_{\text{CLUB}}$ in the generic extension, by Lemma 3.0.7.

$\square$

We contrast this with the following theorem:

Definition 3.0.9. The principle $\diamondsuit_{\text{CLUB}}$ states that there is a sequence $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ with $D_\delta \subseteq \delta$ for all δ, such that if $C \subseteq \omega_1$ is a closed unbounded set then the set $\{\alpha \in Lim(\omega_1) : C \cap \alpha = D_\alpha\}$ is stationary in ω_1.

Theorem 3.0.10. $\diamondsuit_{\text{CLUB}} \rightarrow \diamondsuit$.

Proof Let $\langle D_\delta : \delta \in Lim(\omega_1)\rangle$ be a witness to $\diamondsuit_{\text{CLUB}}$. We use Devlin's result (in [24]) that ($\clubsuit$ + CH) $\rightarrow \diamondsuit$ and thus split the proof into two stages.

To see that $\diamondsuit_{\mathrm{CLUB}} \to \mathrm{CH}$, observe that if $x \subseteq \omega$ and $C \subseteq \omega_1$ is a closed unbounded set then $C' = (C \setminus \omega) \cup x \cup \{\omega\}$ is also closed unbounded. Hence there is some $\delta \geq \omega$ such that $D_\delta = C' \cap \delta$, giving $D_\delta \cap \omega = x$.

So $\langle \omega \cap D_\delta : \delta \in \mathit{Lim}(\omega_1)\rangle$ contains a subsequence enumerating $\mathcal{P}(\omega)$, implying the continuum has size ω_1.

To see that $\diamondsuit_{\mathrm{CLUB}} \to \clubsuit$, let $X \subseteq \omega_1$ be unbounded. So $X' = X \cup \{\alpha < \omega_1 : \sup(X \cap \alpha) = \alpha\}$ is a closed unbounded set. And $X'' = X' \setminus \{\alpha < \omega_1 : \sup(X \cap \alpha) = \alpha\}$ is unbounded and is a subset of X. Whenever $D_\delta = \delta \cap X'$ we will get $D'_\delta = D_\delta \setminus \{\beta < \delta : \sup(D_\delta \cap \beta) = \beta\} \subseteq X'' \cap \delta \subseteq X$, which will be cofinal in δ if and only if X'' is cofinal in δ. So there will be stationary many δ such that $D'_\delta \subseteq X$ and $\sup(D'_\delta) = \delta$, hence $\langle D'_\delta : \delta \in \mathit{Lim}(\omega_1)\rangle$ can be easily modified to give a witness to $\clubsuit$.

$\square$

Hence $\diamondsuit_{\mathrm{CLUB}} \leftrightarrow \diamondsuit$.

The following definition also seems to be pertinent:

Definition 3.0.11. *Superstick* asserts that there is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ with $|\mathcal{S}| = \omega_1$ such that for any $X \in [\omega_1]^{\omega_1}$ the set $\{x \in \mathcal{S} : x \subseteq X\}$ when ordered by strict inclusion contains a chain of length ω_1.

Superstick implies $\dagger$ and is a consequence of CH, so it does not imply $\clubsuit$ or Superclub. It stands in a similar relation to Superclub as CH does to $\diamondsuit$, so by analogy with the proof that $(\clubsuit + \mathrm{CH}) \to \diamondsuit$ we can prove:

Theorem 3.0.12. $(\clubsuit + \text{Superstick}) \to \text{Superclub}$.

Proof Let $\mathcal{S}$ witness Superstick and $\langle s_\alpha : \alpha < \omega_1 \rangle$ enumerate $\mathcal{S}$. If $X \in [\omega_1]^{\omega_1}$ then there is an uncountable set $S' \subseteq \omega_1$ indexing the chain asserted to exist by Superstick, with $i, j \in S'$ and $i < j$ implying s_i is a subset of s_j. Let $Y = \bigcup\{s_\alpha : \alpha \in S'\}$, then $Y \subseteq X$ and $|Y| = \omega_1$. Let $\langle A_\delta : \delta \in Lim(\omega_1) \rangle$ witness $\clubsuit$ and set $B_\delta = \bigcup_{\alpha \in A_\delta} s_\alpha$, unless the latter is not a cofinal subset of δ in which case we choose it to be an arbitrary cofinal subset of δ. There is a closed unbounded set $C \subseteq \omega_1$ for which $\delta \in C$ implies that $\sup \left(\bigcup_{\alpha \in \delta \cap S'} s_\alpha \right) = \delta$ and $\bigcup_{\alpha \in \delta \cap S'} s_\alpha = Y \cap \delta$, hence for $\delta \in C$ where $A_\delta \subseteq S'$ also holds we will have $Y \cap \delta = B_\delta$ and $\sup(B_\delta) = \delta$. There is a stationary set of such δ so $\langle B_\delta : \delta \in Lim(\omega_1) \rangle$ is a witness to Superclub.

$\square$

It is not known whether Superstick $\to$ CH. However, based on the above results (notably Theorems 3.0.8 and 3.0.10) we form the following conjecture:

Conjecture 3.0.13. We believe the following to be true:

(i) Con(ZFC) $\to$ Con(ZFC + Superstick + $\neg$CH),

(ii) Con(ZFC) $\to$ Con(ZFC + Superclub + $\neg\diamondsuit$).

It is not clear how we could prove either of these using existing forcing techniques. But it is clear that if 3.0.13 (ii) is true then Superclub gives a strong positive partial answer to Juhasz's question, as discussed at the start of this chapter.

Chapter 4

$\clubsuit$, Forcing and Suslin Trees

We mentioned in the previous chapter that the expected answer to Juhasz's question is negative. If this is indeed the case, then giving a proof of this would require us to find a model of ($\clubsuit + \neg$CH) in which there are no Suslin trees. The usual method for finding a model of ($\clubsuit + \neg$CH), that doesn't involve collapsing cardinals, is to preserve a witness to $\clubsuit$ from an initial model while using forcing to add reals (see the discussion of this in Chapter 5), though it is also possible for such a forcing to introduce a new witness to $\clubsuit$, not present in the ground model (see for example [11]). In this chapter we present several conditions that such a witness to $\clubsuit$ would have to satisfy. In particular, we prove that any forcing that adds an uncountable antichain to a single Suslin tree cannot preserve *every* ground model witness to $\clubsuit$. We also show that Juhasz's question could potentially be answered (negatively) by preserving a certain kind of $\clubsuit$-sequence while killing off another.

Unfortunately, a method for carrying out this line of attack is not known to us; we merely prove that it would be sufficient.

4.1 T-preserving ♣-sequences

It is of interest to us to examine different types of ♣-sequences that exist in the ground model and consider their relation to Suslin trees. The following observation highlights a link between ♣-sequences and Suslin trees that is otherwise hidden by their seemingly unrelated definitions:

Observation 4.1.1. Let $R \subseteq [\omega_1]^2$ be a set of unordered pairs of countable ordinals. We will call this a *pre-relation*. We define the ordering $\leq_R$ from R as follows: $x \leq_R y$ iff $x = y$ or $\{x, y\} \in R$ and x is less than y as an ordinal. Then $\langle \omega_1, \leq_R \rangle$ is a Suslin tree if and only if $\langle \omega_1, \leq_R \rangle$ is an Aronszajn tree and for any uncountable $X \subseteq \omega_1$ there is a $z \in R$ such that $z \subseteq X$.

Proof We know that $\langle \omega_1 \ \leq_R \rangle$ is a Suslin tree if and only if it is an Aronszajn tree and does not cannot contain any uncountable antichains. The latter condition is equivalent to saying that any uncountable subset of ω_1, X, must contain two ordinals that are compatible with respect to the tree ordering, $\leq_R$. Let x and y be two such ordinals. Then $\{x, y\} \in R$ and $\{x, y\} \subseteq X$.

□

This gives us another characterisation of a Suslin tree. And note that the last line of this characterisation bears a strong resemblance to the definition of ⫯. Hence we can say that a Suslin tree has a certain (albeit very weak) *guessing property* for unbounded subsets of ω_1. Furthermore, it is precisely this guessing property that distinguishes it from an Aronszajn tree.

Observation 4.1.1 motivates the following definition:

Definition 4.1.2. Let $\bar{A} = \langle A_\delta : \delta \in Lim(\omega_1)\rangle$ be a witness to $\clubsuit$ and T be a Suslin tree such that both are in V, a model of ZFC + $\clubsuit$ + $\neg$SH. If $\bar{A}$ is such that for any forcing, $\mathbb{P} \in V$, and any filter G that is $\mathbb{P}$-generic over V, if $\bar{A}$ remains a witness to $\clubsuit$ in $V[G]$ then T remains a Suslin tree in $V[G]$, then we say that $\bar{A}$ is *T-preserving over V*.

Normally we will just write that $\bar{A}$ is T-preserving, when V is clear from context. The existence of T-preserving $\clubsuit$-sequences for normal Suslin trees (see Chapter 2 for the definition of *normal*) is easy to establish:

Theorem 4.1.3. Let $\bar{A} = \langle A_\delta : \delta \in Lim(\omega_1)\rangle$ be a $\clubsuit$-sequence and $T = \langle \omega_1, \leq_T\rangle$ be a normal Suslin tree, both in V. Then we can define a further $\clubsuit$-sequence $\bar{A}^T = \langle A^T_\delta : \delta \in Lim(\omega_1)\rangle$ in V, such that $\bar{A}^T$ is T-preserving over V.

Proof We assume without loss of generality that for all $0 < \alpha < \omega_1$, $Lev_\alpha(T) = [\omega.\alpha, \omega.\alpha + \omega)$, and we give the construction of $\bar{A}^T$. Choose $\langle e_\epsilon : \epsilon < \omega_1\rangle$ to be an enumeration of $[\omega_1]^2$, and let $Z \subseteq \omega_1$ be such that $\{e_\epsilon : \epsilon \in Z\} = \{\{\alpha, \beta\} : \alpha <_T$

$\beta\}$. Let $\langle z_i : i < \omega_1 \rangle$ be an enumeration of Z, and set $A^T_\delta = \delta \cap \bigcup_{i \in A_\delta} e_{z_i}$ for all $\delta \in Lim(\omega_1)$, unless this gives us a set that is bounded in δ or is an antichain in T, in which case set A^T_δ to be an arbitrary cofinal subset of δ containing two ordinals that are compatible in T. Then because T is Suslin, if $X \subseteq \omega_1$ is unbounded we can find an uncountable set $Y \subseteq Z$ such that $\gamma \in Y \Rightarrow e_\gamma \subseteq X$, and for $\gamma, \xi \in Y$ with $\gamma < \xi$ we have $\max(e_\gamma) < \min(e_\xi)$. The fact that we can find such a Y follows from the fact that X cannot be (or contain) an uncountable antichain. We will make use of the following standard definition:

Definition 4.1.4. For an unbounded set $E \subseteq \omega_1$, we write $\mathrm{acc}(E)$ to denote the set: $\{\zeta < \omega_1 : \sup(\zeta \cap E) = \zeta\}$.

Continuation of the Proof of Theorem 4.1.3. Let δ be such that $\delta \in \mathrm{acc}\,(Y) \cap \mathrm{acc}\left(\bigcup_{\gamma \in Y} e_\gamma\right)$ and $A_\delta \subseteq Y$. This is possible because the set $\mathrm{acc}(E)$ is always closed and unbounded in ω_1 for an unbounded set E, and by the definition of $\clubsuit$. Then $\delta \cap \bigcup_{i \in A_\delta} e_{z_i}$ is a subset of X and has supremum δ. Hence $\bar{A}^T$ is also a $\clubsuit$-sequence.

We finish the proof of the theorem by establishing that $\bar{A}^T$ is T-preserving. So observe that the set A^T_δ, for any $\delta \in Lim(\omega_1)$, contains a γ and ξ with $\gamma <_T \xi$. Hence if X is an uncountable antichain for T, in $V[G]$, then we cannot have $A^T_\delta \subseteq X$ for any $\delta \in Lim(\omega_1)$. So if $\bar{A}^T$ remains a $\clubsuit$-sequence in the generic extension then T must have no uncountable antichains in $V[G]$, and since T is normal this is sufficient to prove that T remains a Suslin tree.

$\square$

There are several ways we could have constructed such an $\bar{A}^T$, but we will retain the definition used in the proof of Theorem 4.1.3 (when we write $\bar{A}^T$ we take it to be assumed that T is normal). So henceforth, given $\bar{A}$, we set:

$$\bar{A}^T = \langle \delta \cap \bigcup_{i \in A_\delta} e_{z_i} : \delta \in Lim(\omega_1) \rangle,$$

except where this gives us an A^T_δ that is not a cofinal subset of δ or that is an antichain in T, in which case we choose such a set arbitrarily as in the above proof.

Corollary 4.1.5. If we preserve every witness to $\clubsuit$ in a given forcing extension, then every ground model normal Suslin tree remains Suslin in the generic extension.

The contrapositive to this is as follows:

Corollary 4.1.6. If $V \models \clubsuit + \neg$SH then any forcing $\mathbb{P} \in V$ that kills normal Suslin trees must also kill some $\clubsuit$-sequences.

Proofs: Both by Theorem 4.1.3.

$\square$

We could easily alter the definition of a T-preserving $\clubsuit$-sequence so as to apply to Suslin trees T that are not normal, but since every Suslin tree contains a normal subtree (see Chapter 2), and the existence of Suslin trees is therefore equivalent to the existence of normal Suslin trees, we feel justified in restricting our attention to trees that are normal as well as Suslin.

The next fact shows that Juhasz's question can be reduced to a question about separating $\clubsuit$-sequences.

Fact 4.1.7. In V, let $\bar{A}$ be a $\clubsuit$-sequence and T be a normal Suslin tree. Let $\mathbb{P}$ be a cardinal preserving forcing notion and G a $\mathbb{P}$-generic filter over V. Then:

> If $V[G] \models$ "$\bar{A}$ is a $\clubsuit$-sequence" and $V[G] \models$ "$\bar{A}^T$ is not a $\clubsuit$-sequence,"
>
> then $V[G] \models$ "T is not a Suslin tree".

Proof Assume T is Suslin in the generic extension. Let $X \in [\omega_1]^{\omega_1} \cap V[G]$. Then because X cannot be an uncountable antichain in T there must be some $\{x, y\} \subseteq X$ with $x <_T y$. By the uncountability of X there must be uncountably many such pairs $\{x, y\}$. Let Z and $Y \subseteq X$ be as in the proof of Theorem 4.1.3. Then $\bar{A}^T$ witnesses $\clubsuit$, as before. This is a contradiction.

$\square$

4.2 Directly T-preserving $\clubsuit$-sequences

With the following definition we can isolate the property of $\bar{A}^T$ that causes it to be T-preserving. Any $\clubsuit$-sequence that we hope to preserve over an iteration killing all ground model Suslin trees must *not* have this property.

Definition 4.2.1. Let $\bar{A}$ be a witness to $\clubsuit$ and T be a normal Suslin tree. We say that $\bar{A}$ is *directly T-preserving* if there exists a club set $C \subseteq \omega_1$ such that for $\delta \in C \cap Lim(\omega_1)$ there exist $x, y \in A_\delta$ with $x <_T y$.

Clearly $\bar{A}^T$ is directly T-preserving, so the existence of directly T-preserving $\clubsuit$-sequences (assuming $\clubsuit + \neg$SH) is immediate. But the following question is unresolved:

Question 4.2.2. Can there exist a $\clubsuit$-sequence, $\bar{A}$, such that $\bar{A}$ is T-preserving but not directly T-preserving, for a normal Suslin tree T?

Assuming $\diamondsuit$ we can construct a $\clubsuit$-sequence that is not directly T-preserving for *any* normal Suslin tree T.

Theorem 4.2.3. $\diamondsuit$ implies the existence of a $\clubsuit$-sequence, $\bar{A}$, such that if T is a normal Suslin tree then $\bar{A}$ is not directly T-preserving.

We prove this theorem by a series of lemmas. Let T be a Suslin tree and x, y be elements in the tree, then we write $y \perp_T x$ to denote the following: $(x \not\leq_T y \wedge y \not\leq_T x)$. In this case we say that x and y are *incomparable.*

Lemma 4.2.4. Let $T = \langle \omega_1, \leq_T \rangle$ be a Suslin tree. If $A \subseteq \omega_1$ is uncountable then there is an $x \in A$ such that $\{y \in A : y \perp_T x\}$ is uncountable.

Proof Assume not. So for every $x \in A$ there are only countably many elements of A that are incomparable with x. We will inductively define an uncountable chain

$\{x_\alpha : \alpha < \omega_1\}$ in T, thus obtaining a contradiction. Let x_0 be $\min(A)$, which is well-defined because A is a set of ordinals. Now, assume $\{x_\alpha : \alpha < \beta\}$ is already defined and is a chain in T. We will define x_β. If $\beta = \alpha + 1$ for some α, then by assumption there are only countably many members of A that are incomparable with x_α, so $\{y \in A : x_\alpha <_T y\}$ is uncountable. Let $x_\beta = \min\{y \in A : x_\alpha <_T y\}$. Then $\{x_\alpha : \alpha \leq \beta\}$ is a chain in T.

Now assume that β is a limit ordinal. Let $Y = \bigcup_{\alpha<\beta}\{y \in A : y \perp_T x_\alpha\}$. So Y is a countable union of countable sets, and hence is countable, which implies that $A \setminus Y$ is uncountable. For all $z \in A \setminus Y$ and all $\alpha < \beta$ we have either $z \leq_T x_\alpha$ or $x_\alpha <_T z$. By the fact that $A \setminus Y$ is uncountable we can find a z such that for all $\alpha < \beta$, $x_\alpha <_T z$. Let x_β be the least ordinal such that $x_\beta \in A \setminus Y$ and $x_\alpha <_T x_\beta$ for all $\alpha < \beta$, then $\{x_\alpha : \alpha \leq \beta\}$ is a chain in T.

So $\{x_\alpha : \alpha < \omega_1\}$ is an uncountable chain in T, giving us a contradiction.

$\square$

We henceforth assume without loss of generality that all Suslin trees with underlying set ω_1 that we consider are such that $Lev_\beta(T) = [\omega.\beta, \omega.\beta + \omega)$, when $2 \leq \beta < \omega_1$.

Lemma 4.2.5. Let $T = \langle \omega_1, \leq_T \rangle$ be a Suslin tree. If $A \subseteq \omega_1$ is uncountable then the set of $\delta < \omega_1$ such that there is a countably infinite antichain $X \subseteq A \cap \delta$ with $\sup(X) = \delta$, is unbounded in ω_1.

Proof Let $\gamma < \omega_1$ be arbitrary. We define the antichain $X = \{x_n : n < \omega\}$ by induction. First, observe that we can assume $A \subseteq \omega_1 \setminus \gamma$, without loss of generality. Let x_0 be the least ordinal in A satisfying the claim of Lemma 4.2.4. Given $X_n = \{x_m : m \leq n\}$, let us assume that X_n satisfies the following statement:

$$(A'_n = \{z \in A : \forall x \in X_n (z \perp_T x)\} \text{ is uncountable}) \qquad (*)_n$$

Clearly X_0 satisfies $(*)_0$, and our induction will be such that if X_n satisfies $(*)_n$ then X_{n+1} satisfies $(*)_{n+1}$. By the previous lemma there is a $z \in A'_n$ such that $\{y \in A'_n : y \perp z\}$ is uncountable. Let x_{n+1} be the least ordinal in A'_n having this property. Clearly the set $X_{n+1} = \{x_m : m \leq n+1\}$ satisfies $(*)_{n+1}$, and is an antichain.

The set $\{x_n : n < \omega\}$ is therefore a countably infinite antichain, contained within $A = A \setminus \gamma$. Let $\delta = \sup\{x_n : n < \omega\}$, which will be a limit ordinal because $\langle x_n : n < \omega \rangle$ is an increasing sequence under the usual ordering of ordinals, so $\delta > \gamma$ and the lemma is proved.

□

The next lemma tells us that the set of such δ is not only unbounded, it is closed too.

Lemma 4.2.6. Let $T = \langle \omega_1, \leq_T \rangle$ be a Suslin tree, and $\{\beta_n : n < \omega\}$ be such that for all $n < \omega$, β_n is a limit ordinal and $\beta_n < \beta_{n+1} < \omega_1$, and there is an antichain

$B_n \subseteq [\beta_n, \beta_{n+1})$ with $\sup(B_n) = \beta_{n+1}$. Let $\gamma = \sup\{\beta_n : n < \omega\}$. Then there is an antichain $B \subseteq \bigcup_{n<\omega} B_n$ such that $\sup(B) = \gamma$.

Proof First, we observe that for any $n \in \omega$, and any infinite $B'_n \subseteq B_n$, there is a $b \in B'_n$ such that the set $\{m < \omega : |\{y \in B_m : y \perp b\}| = \omega\}$ is infinite. In other words, there is a $b \in B'_n$ such that there is a cofinal subsequence $\{B_{m_i} : i < \omega\}$ of sets that contain infinitely many elements incomparable with b. To see that this is true, assume it is not. Given some n and $B'_n \subseteq B_n$, every $x \in B'_n$ fails to have such a cofinal subsequence. Fix such an $x \in B'_n$. Then for some finite m we have that for all $p \in \omega \setminus m$ and all but finitely many $y \in B_p$, $x <_T y$. Then let $x' \in B'_n$ be distinct from x. Clearly $x' \perp_T x$, because B'_n is an antichain, so for all $p \in \omega \setminus m$ and all but finitely many $y \in B_p$ we have $x <_T y$ and consequently $x' \perp_T y$. This contradicts our assumption that no such $x' \in B'_n$ exists.

We will use this fact to define an antichain $X = \{x_n : n < \omega\}$ by induction. Let x_0 be the least ordinal in B_0 that satisfies the claim in the previous paragraph. Assume $X_n = \{x_m : m \leq n\}$ is defined and satisfies $(*)_n$:

$$(|\{l < \omega : B_l \setminus \{y \in B_l : \exists m(m \leq n \text{ and } x_m <_T y)\} \text{ is countable}\}| = \omega)$$

Clearly $X_0 = \{x_0\}$ satisfies $(*)_0$. Now let n' be the least finite ordinal greater than n such that $B'_{n'} = B_{n'} \setminus \{y : \exists m(m \leq n \text{ and } x_m <_T y)\}$ is countable, and let x_{n+1} be the smallest ordinal $b \in B'_{n'}$ that satisfies the claim in the first paragraph.

Clearly, since $b \in B'_{n'}$, $X_{n+1} = \{x_m : m \leq n+1\}$ is an antichain satisfying $(*)_{n+1}$, and with $\sup(X_{n+1}) \geq \beta_{n'}$.

So $\{x_n : n < \omega\} \subseteq \bigcup_{n<\omega} B_n$ is an antichain with supremum γ.

$\square$

So combining the previous two lemmas we get: for any Suslin tree T and unbounded $A \subseteq \omega_1$, there is a closed unbounded set of $\delta < \omega_1$ such that we can find an infinite antichain $X \subseteq A \cap \delta$ which (considered as a set of ordinals) is unbounded in δ. We now use this fact to prove our initial theorem.

Proof of Theorem 4.2.3: Let $\langle B_\delta : \delta \in Lim(\omega_1)\rangle$ be a witness to $\Diamond$. Choose disjoint uncountable sets, A_1 and A_2, such that $\omega_1 = A_1 \cup A_2$. Fix bijections $\tau_1 : A_1 \to [\omega_1]^2$ and $\tau_2 : A_2 \to \omega_1$. We define $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ as follows. If $\tau_1[B_\delta \cap A_1]$ is the pre-relation for a tree ordering on the ordinal δ, and $\tau_2[B_\delta \cap A_2]$ is an unbounded subset of δ that is a superset of some B with order-type ω such that B is unbounded in δ and also forms an antichain in the tree given by $\langle \delta, \leq_{\tau_1[B_\delta \cap A_1]}\rangle$, then set $A_\delta = B$ (choose such a B arbitrarily). Otherwise, let A_δ be an arbitrary sequence cofinal in δ, of order-type ω.

We will show that $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ is a $\clubsuit$-sequence that is not directly T-preserving for any Suslin tree T. Assume that this is not the case, and that in fact either there is such a T (with underlying set ω_1) or there is an uncountable set $X \subseteq \omega_1$ contradicting $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ being a witness to $\clubsuit$. Thus we can either find a closed unbounded set $E \subseteq \omega_1$ such that if $\delta \in E$ then A_δ is not an antichain in T or a closed unbounded set E_1 such that if $\delta \in E$ then $A_\delta \not\subseteq X$. The following set is also closed unbounded:

$$E' = \{\delta < \omega_1 : \langle T \cap \delta, \leq_T \restriction \delta\rangle \text{ is a tree}\} \cap \{\delta < \omega_1 : \sup(X \cap \delta) = \delta\} \cap$$
$$\{\delta < \omega_1 : \delta \cap X \text{ contains an antichain in } T\text{, cofinal in } \delta\}.$$

This follows from the previous two lemmas, as well as basic facts about closed unbounded sets. Let $Y = \tau_1^{-1}[\leq_T] \cup \tau_2^{-1}[X]$, and $S = \{\alpha \in Lim(\omega_1) : B_\alpha = Y \cap \alpha\}$. The latter is stationary, so $S \cap E'$ is also stationary. If $\delta \in S \cap E'$, then by our definition of $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$, A_δ must be an antichain in $\langle \delta, \leq_T \restriction \delta\rangle$ and we must also have $A_\delta \subseteq X$. This contradicts the fact that either E or E_1 is closed unbounded. So the theorem is proved.

$\square$

If $\bar{A}$ is a witness to $\clubsuit$ in V and if we hope to prove Con(ZFC) $\rightarrow$ Con(ZFC + SH + $\clubsuit$) by preserving $\bar{A}$ as a witness to $\clubsuit$ over a forcing iteration, then $\bar{A}$ must *not* be directly T-preserving for any normal Suslin tree T in V. This is perhaps not a sufficient condition for the existence of an appropriate forcing, but it is certainly a necessary condition. We have shown that under $\diamondsuit$ there is a $\clubsuit$-sequence, $\bar{A}$, satisfying this necessary condition.

The following is unknown:

Question 4.2.7. Can there be a model of $\neg$CH + $\neg$SH + $\clubsuit$ in which for any witness to $\clubsuit$, $\bar{A}$, there is a normal Suslin tree T such that $\bar{A}$ is directly T-preserving?

Chapter 5

Cardinal Arithmetic and $\clubsuit$

We have seen that $(\clubsuit + \mathrm{CH}) \to \diamondsuit$, and that $\diamondsuit$ implies the Continuum Hypothesis (see Chapter 1); intuitively, we think of $\clubsuit$ as being $\diamondsuit$ with this cardinal arithmetic assumption removed. In this chapter we give this intuition some further justification, by proving that $\clubsuit$ is consistent with the negation of the Continuum Hypothesis. This result is originally due to Shelah [27]. The proof we give is due to Fuchino, Shelah and Soukup [11] and uses forcing; it proceeds by starting from a model of $\diamondsuit$ + GCH and adding Cohen reals to it while simultaneously ensuring that a witness to $\clubsuit$ in the ground model remains a witness to $\clubsuit$ in the generic extension. Most of the known proofs of the relative consistency of $\clubsuit + \neg\mathrm{CH}$, that do not involve collapsing cardinals, proceed in this manner. In Section 5.2 we ask whether the same result can be established in a different manner: by starting with a model of $\neg$CH (and possibly some other assumptions) and forcing $\clubsuit$ to hold without collapsing the continuum. It is not known whether this can be done in general. If it were indeed possible to find such a forcing then it could potentially be used to obtain results on Juhasz's question and other related matters. We give a partial result here, showing that $\clubsuit$ can always be forced when a weaker version of $\clubsuit$ holds, without collapsing 2^ω.

5.1 The consistency of $\clubsuit$ with $\neg$CH

There are many proofs of the following theorem (see Section 1.1.1). The one we give here, which we believe to be the shortest, is due to Fuchino, Shelah and Soukup [11]:

Theorem 5.1.1. Con(ZFC) $\rightarrow$ Con(ZFC + $\clubsuit$ + $\neg$CH).

We start with a model of ZFC satisfying $\diamondsuit$ + GCH. It is straightforward to prove that the consistency of ZFC implies the existence of such a model (see for example [19, VI]). The forcing we use is defined as follows:

Definition 5.1.2. We define a partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ as follows:

- Let P be the set of all countable partial functions, f, from ω_2 to 2 such that for any ordinal $\alpha \in Lim(\omega_2)$, $\mathrm{dom}(f) \cap [\alpha, \alpha + \omega)$ is finite.
- Let $p, q \in P$. Then $q \leq_{\mathbb{P}} p$ (q is stronger than p) if and only if both of the following hold:

(i) q extends p as a function, i.e. $q \supseteq p$.

(ii) The set of $\alpha \in Lim(\omega_2)$, with $\mathrm{dom}(p) \cap [\alpha, \alpha+\omega) \neq \mathrm{dom}(q) \cap [\alpha, \alpha+\omega)$ and $\mathrm{dom}(p) \cap [\alpha, \alpha+\omega) \neq \emptyset$, is finite.

It is easy to check that $\mathbb{P}$ is a notion of forcing. We will henceforth abuse notation by writing $p \in \mathbb{P}$ rather than $p \in P$ when p is a condition in this forcing.

Theorem 5.1.3. Let V be a model of ZFC such that $V \vDash \diamondsuit + \mathrm{GCH}$, and let G be a $\mathbb{P}$-generic filter over V. Then the generic extension $V[G]$ satisfies the following:

(i) $\omega_1^V = \omega_1^{V[G]}$ and $\omega_2^V = \omega_2^{V[G]}$

(ii) $\clubsuit$

(iii) $\neg\mathrm{CH}$.

For the rest of this section we fix G to be a specific $\mathbb{P}$-generic filter over V, as above; we split the proof of the theorem into a series of lemmas and a proposition:

Proposition 5.1.4. Let $\dot{f}$ be a $\mathbb{P}$-name for a function and $p \in \mathbb{P}$ be a condition such that $p \Vdash$ "$\dot{f} : \omega_1^V \to \omega_1^V$". Then there is an unbounded set $A^{p,\dot{f}} \subseteq \omega_1$ in V and a function $g^{p,\dot{f}} : A^{p,\dot{f}} \to \omega_1$ also in V such that for every ordinal $\delta < \omega_1^V$ there exists a $q^\delta \leq_{\mathbb{P}} p$ in $\mathbb{P}$ for which $q^\delta \Vdash$ "$g^{p,\dot{f}} \restriction (A^{p,\dot{f}} \cap \delta) = \dot{f} \restriction (A^{p,\dot{f}} \cap \delta)$".

Proof We will make use of the following Δ-system Lemma: if $\kappa^{<\kappa} = \kappa$, and W is a collection of sets of cardinality less than κ, with $|W| = \kappa^+$, then there is a

$U \subseteq W$ with $|U| = \kappa^+$ and a set v such that for any distinct $x, y \in U$ we have $x \cap y = v$. (For a proof of this Lemma see [19, II].)

Now fix p and $\dot{f}$ to be as in the statement of the Proposition. We define two sequences $\langle p_\alpha : \alpha < \omega_1 \rangle$ and $\langle q_\alpha : \alpha < \omega_1 \rangle$ of conditions in $\mathbb{P}$, and a sequence $\langle u_\alpha : \alpha < \omega_1 \rangle$ of finite subsets of ω_2, by induction. Let $p_0 = p = q_0$ and $u_0 = \emptyset$. When $\alpha = \beta + 1$ and p_β, q_β and u_β are defined, we choose q_α to be a condition such that $q_\alpha \leq_{\mathbb{P}} p_\beta$ and $q_\alpha \Vdash$ "$\dot{f}(\alpha) = \gamma_\alpha$" for some countable ordinal γ_α. Let p_α be equal to:

$$p_\beta \cup \left(q_\alpha \upharpoonright \bigcup \{ [\zeta, \zeta + \omega) : \zeta \in Lim(\omega_2) \text{ and } \mathrm{dom}(p_\beta) \cap [\zeta, \zeta + \omega) = \emptyset \} \right).$$

This gives us $p_\alpha \leq_{\mathbb{P}} p_\beta$. Set u_α to be:

$$\{ \zeta \in Lim(\omega_2) : q_\alpha \upharpoonright [\zeta, \zeta + \omega) \neq p_\beta \upharpoonright [\zeta, \zeta + \omega) \text{ and } p_\beta \upharpoonright [\zeta, \zeta + \omega) \neq \emptyset \}.$$

When α is a limit ordinal, let $p'_\alpha = \bigcup_{\beta < \alpha} p_\beta$, which will be a condition in $\mathbb{P}$ due to the way we are constructing $\langle p_\alpha : \alpha < \omega_1 \rangle$, and choose a condition q_α such that $q_\alpha \leq_{\mathbb{P}} p'_\alpha$ and $q_\alpha \Vdash$ "$\dot{f}(\alpha) = \gamma_\alpha$" for some $\gamma_\alpha < \omega_1$. Let p_α be equal to:

$$p'_\alpha \cup \left(q_\alpha \upharpoonright \bigcup \{ [\zeta, \zeta + \omega) : \zeta \in Lim(\omega_2) \text{ and } \mathrm{dom}(p'_\alpha) \cap [\zeta, \zeta + \omega) = \emptyset \} \right).$$

Set u_α to be:

$$\{ \zeta \in Lim(\omega_2) : q_\alpha \upharpoonright [\zeta, \zeta + \omega) \neq p'_\alpha \upharpoonright [\zeta, \zeta + \omega) \text{ and } p'_\alpha \upharpoonright [\zeta, \zeta + \omega) \neq \emptyset \}.$$

The collection $\{ u_\alpha : \alpha < \omega_1 \}$ is an uncountable set of finite sets, so by the Δ-system Lemma there is a cofinal subsequence $\langle u_{\alpha_\epsilon} : \epsilon < \omega_1 \rangle$ and a finite set $u \subseteq \omega_2$

such that for all $i, j < \omega_1$, $u_{\alpha_i} \cap u_{\alpha_j} = u$ and because there are only countably many possibilities for $q_{\alpha_i} \restriction \bigcup_{\beta \in u}[\beta, \beta + \omega)$ we can choose this cofinal subsequence to have the further property that $q_{\alpha_i} \restriction \bigcup_{\beta \in u}[\beta, \beta + \omega) = q_{\alpha_j} \restriction \bigcup_{\beta \in u}[\beta, \beta + \omega)$ for all $i, j < \omega_1$, and to be such that if $i < j < \omega_1$ then

$$\{\beta \in Lim(\omega_2) : q_{\alpha_j} \restriction [\beta, \beta + \omega) \neq q_{\alpha_i} \restriction [\beta, \beta + \omega) \text{ and } q_{\alpha_i} \restriction [\beta, \beta + \omega) \neq \emptyset\}$$

is a subset of u.

This latter requirement is possible because if we are given an $i < \omega_1$ then $\mathrm{dom}(q_{\alpha_i})$ is always countable and so, by the fact that these sets form a Δ-system, we can find a countable ordinal α' for which all α_j with $\alpha' < \alpha_j < \omega_1$ are such that q_{α_j} meets this requirement.

Once this is done, the sequence $\langle q_{\alpha_\epsilon} : \epsilon < \omega_1 \rangle$ will be a decreasing sequence of conditions such that any countable initial subsequence $\langle q_{\alpha_\epsilon} : \epsilon < \gamma < \omega_1 \rangle$ has a lower bound in $\mathbb{P}$. We define the lower bound to be: $q^{\alpha_\gamma} = \bigcup_{\epsilon < \gamma} q_{\alpha_\epsilon}$.

To see that this is the case, let $i < j$ be less than γ. The only $\beta \in Lim(\omega_2)$ for which q_{α_i} and q_{α_j} both differ from p_{α_j} on the interval $[\beta, \beta + \omega)$ are those $\beta \in u$, in which case we have chosen q_{α_i} and q_{α_j} to be identical on this interval, or those where $q_{\alpha_i} \restriction [\beta, \beta + \omega)$ is empty. This means that $q^{\alpha_\gamma} = \bigcup_{\epsilon < \gamma} q_{\alpha_\epsilon}$ is a condition in $\mathbb{P}$, and is a lower bound to all q_{α_ϵ} for $\epsilon < \gamma$, and is also less than p.

So any countable initial subsequence $\langle q_{\alpha_\epsilon} : \epsilon < \gamma < \omega_1 \rangle$ has a lower bound, q^{α_γ}. Let $A^{p,\dot{f}} = \{\alpha_\epsilon : \epsilon < \omega_1\}$. Then the function $g^{p,\dot{f}} : A^{p,\dot{f}} \to \omega_1$, given by setting

$g^{p,\dot{f}}(\alpha_\epsilon) = \gamma_{\alpha_\epsilon}$, can be defined in V, by the definability of the forcing relation, and is such that $q^{\alpha_\gamma} \Vdash$ "$g^{p,\dot{f}} \restriction (A^{p,\dot{f}} \cap \alpha_\gamma) = \dot{f} \restriction (A^{p,\dot{f}} \cap \alpha_\gamma)$". The sequence $\langle \alpha_\epsilon : \epsilon < \omega_1 \rangle$ goes cofinal in ω_1, so for an arbitrary $\delta < \omega_1$ we can find a q^δ as required.

$\square$

Lemma 5.1.5. Let V and G be as in Theorem 5.1.3, then $\omega_1^V = \omega_1^{V[G]}$.

Proof Assume this is not the case. Let p be a condition and $\dot{f}$ be a $\mathbb{P}$-name for a function such that $p \Vdash$ "$\dot{f} : \omega_1^V \to \omega$ and $\dot{f}$ is injective". Applying Proposition 5.1.4 gets us a function $g^{p,\dot{f}}$ and an uncountable set $A^{p,\dot{f}} \subseteq \omega_1$, both in V, such that $g^{p,\dot{f}} : A^{p,\dot{f}} \to \omega_1$ and which witnesses the Proposition. But $g^{p,\dot{f}} \in V$ so cannot both be injective and have $\mathrm{ran}(g^{p,\dot{f}}) \subseteq \omega$. Let $\delta < \omega_1$ be such that $g^{p,\dot{f}} \restriction (A^{p,\dot{f}} \cap \delta)$ is either not injective or its range is not a subset of ω. Then we can find a $q^\delta \leq_{\mathbb{P}} p$ as in the conclusion of Proposition 5.1.4, in which case we have $q^\delta \Vdash$ "$\dot{f}$ is both injective and not injective," or $q^\delta \Vdash$ "$\mathrm{ran}(\dot{f}) \subseteq \omega$ and $\mathrm{ran}(\dot{f}) \not\subseteq \omega$", which either way is a contradiction.

$\square$

Lemma 5.1.6. Let V and G be as in Theorem 5.1.3, then $\omega_2^V = \omega_2^{V[G]}$.

Proof The result follows from the fact that $\mathbb{P}$ has the $\aleph_2$-c.c. To see this, assume otherwise and let $\langle p_\alpha : \alpha < \omega_2 \rangle$ be an antichain of size $\aleph_2$. Then the set $\{\{\beta \in Lim(\omega_2) : \mathrm{dom}(p_\alpha) \cap [\beta, \beta + \omega) \neq \emptyset\} : \alpha < \omega_2\}$ is a collection of countable sets. $V \models$ GCH, so applying the Δ-system Lemma (as stated in the proof of

Proposition 5.1.4) gives us a subsequence $\langle p_{\alpha_\epsilon} : \epsilon < \omega_2 \rangle$ such that for all $i, j < \omega_2$ we have some fixed u for which $\{\gamma \in Lim(\omega_2) : \mathrm{dom}(p_i) \cap [\gamma, \gamma + \omega) \neq \emptyset\} \cap \{\gamma \in Lim(\omega_2) : \mathrm{dom}(p_j) \cap [\gamma, \gamma + \omega) \neq \emptyset\} = u$. If any two such p_i and p_j are equal when restricted to $\bigcup_{\gamma \in u} [\gamma, \gamma + \omega)$ then they will be compatible elements, by the definition of the forcing. Furthermore, u is a countable set. But this means there can only be ω_1 many functions $f : \bigcup_{\gamma \in u} [\gamma, \gamma + \omega) \to 2$, because $V \vDash 2^\omega = \omega_1$. So by the pigeonhole principle we can find a cofinal subsequence of our original antichain, $\langle p_{\alpha_\epsilon} : \epsilon < \omega_2 \rangle$, consisting of pairwise compatible conditions, which contradicts its being an antichain.

□

Lemma 5.1.7. Let V and G be as in Theorem 5.1.3, then $V[G] \vDash \neg$CH.

Proof The generic function $G' = \bigcup G$ is a total function from ω_2 to 2 because for each $\alpha < \omega_2$ the set $D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\}$ is a dense subset of $\mathbb{P}$ in V. For each $\alpha \in Lim(\omega_2)$, the set $N_\alpha = \{n < \omega : G'(\alpha + n) = 1\}$ is a subset of ω in $V[G]$. Let $\alpha < \beta$ both be in $Lim(\omega_2)$, then $D_{(\alpha,\beta)} = \{p \in \mathbb{P} : \langle p(\alpha + n) : n < \omega\rangle \neq \langle p(\beta + n) : n < \omega \rangle\}$ is a dense subset of $\mathbb{P}$, because $p \restriction [\alpha, \alpha + \omega)$ is finite for any $\alpha \in Lim(\omega_2)$. So for any $\alpha < \beta$ in $Lim(\omega_2)$ we get $N_\alpha \neq N_\beta$, giving us a set $\{N_\alpha : \alpha < \omega_2\}$ of $\aleph_2$ distinct subsets of ω in $V[G]$, by Lemma 5.1.6.

□

Lemma 5.1.8. Let V and G be as in Theorem 5.1.3, then $V[G] \vDash \clubsuit$.

Proof Let $\langle A_\delta : \delta < \omega_1 \rangle$ be a witness to $\clubsuit$ in V. Let $\dot{f}, \dot{\tau}$ be $\mathbb{P}$-names and $p \in \mathbb{P}$ be such that $p \Vdash$ "$\dot{f} : \omega_1 \to \omega_1$ is injective and $\dot{\tau} = \mathrm{ran}(\dot{f})$". Then apply Proposition 5.1.4 to find in V a $g^{p,\dot{f}}$ and $A^{p,\dot{f}}$ with the properties stated there. Let $B = \mathrm{ran}(g^{p,\dot{f}})$, which will be an uncountable set in V. Let $\delta < \omega_1$ be such that $A_\delta \subseteq B$, then there is a $\delta' < \omega_1$ such that A_δ is contained in $\mathrm{ran}(g^{p,\dot{f}} \restriction (A^{p,\dot{f}} \cap \delta'))$. Then let $q^{\delta'}$ be as defined in Proposition 5.1.4, giving $q^{\delta'} \leq p$ and $q^{\delta'} \Vdash$ "$A_\delta \subseteq \dot{\tau}$". Since p was arbitrary, except for the properties stated above, this establishes that if r and $\dot{\tau}$ are such that $r \Vdash$ "$\dot{\tau} \in [\omega_1]^{\omega_1}$" then the set of conditions forcing $A_\delta \subseteq \dot{\tau}$ for some $\delta < \omega_1$ is dense below r. Hence $\langle A_\delta : \delta < \omega_1 \rangle$ remains a witness to $\clubsuit$ in $V[G]$. (Strictly speaking we need to show there are stationary many such δ, but in fact it is sufficient to just show that there is at least one. Here we are implicitly using Theorem 6.1.2, which is proved in the next chapter. A direct proof without using this theorem is possible, but involves a slightly longer argument.)

$\square$

This completes the proof of Theorem 5.1.3 and consequently of Theorem 5.1.1. We have in fact proved that every witness to $\clubsuit$ in the ground model remains a witness to $\clubsuit$ in the generic extension. This fact gives us the following:

Remark 5.1.9. Let T be a normal Suslin tree in V. Then T is a normal Suslin tree in $V[G]$.

Proof By Corollary 4.1.5, see Chapter 4.

$\square$

In fact, every Suslin tree in V, whether normal or not, remains Suslin in $V[G]$, as discussed in Chapter 4.

5.2 A different approach to Con($\clubsuit + \neg$CH)

Now we consider whether the same result can be obtained by starting with a model of $\neg$CH and $\neg\clubsuit$ (for example, a model of Martin's Axiom, MA(ω_1)) and forcing to get a $\clubsuit$-sequence without also forcing CH to hold. Specifically, we ask: when can there consistently exist a forcing $\mathbb{Q}$ in a model $V \vDash$ ZFC + $\neg$CH + $\neg\clubsuit$ such that forcing with $\mathbb{Q}$ causes $\clubsuit$ to hold in the generic extension and doesn't collapse cardinals?

We will show here that such a $\mathbb{Q}$ exists when we assume that a weak version of $\clubsuit$ holds in V (one that in particular is not compatible with Martin's Axiom), but we would conjecture that in general such a $\mathbb{Q}$ need not exist. If such a forcing *could* be constructed (in ZFC or from weaker assumptions than those in Theorem 5.2.2, such as ⫯) then questions such as Juhasz's question could potentially be approached by, for example, starting from a model of $\neg$CH with no Suslin trees and forcing $\clubsuit$ to hold via a forcing that doesn't collapse cardinals or add Suslin trees. We do not know if this is possible.

Definition 5.2.1. $\clubsuit^{\omega}$ denotes the following statement: there exists a sequence

$\langle \mathcal{A}_\delta : \delta \in Lim(\omega_1)\rangle$ such that for all $\delta \in Lim(\omega_1)$, $|\mathcal{A}_\delta| = \omega$ and for all $A \in \mathcal{A}_\delta$ we have $\sup(A) = \delta$ and $\mathrm{otp}(A) = \omega$, and if $X \subseteq \omega_1$ is uncountable then the set $\{\delta \in Lim(\omega_1) : \exists A \in \mathcal{A}_\delta (A \subseteq X)\}$ is stationary.

Theorem 5.2.2. Let $V \models \mathrm{ZFC} + \clubsuit^\omega$. Then there is a c.c.c. forcing $\mathbb{Q}$ in V such that if G is a $\mathbb{Q}$-generic filter over V then $V[G] \models \clubsuit$.

Proof Begin by fixing $\langle \mathcal{A}_\delta : \delta \in Lim(\omega_1)\rangle$, a witness to $\clubsuit^\omega$ in V. We also fix χ to be a 'sufficiently large' cardinal (see the discussion of this term in Chapter 2); taking $\chi = 2^{2^{2^\omega}}$ will suffice.

We will define $\mathbb{Q} = \langle Q, \leq_\mathbb{Q}\rangle$ by inductively defining two uncountable sequences $\langle N_\alpha : \alpha < \omega_1\rangle$ and $\langle Q_\alpha : \alpha < \omega_1\rangle$. As we go along we will also define an uncountable sequence of functions $\langle f_\alpha : \alpha < \omega_1\rangle$, though we ought to note that we do not in general define f_α at the α^{th} stage of the induction.

We first choose N_0, a countable elementary submodel of $(\mathcal{H}(\chi), \in)$ containing ω_1, in V, and let Q_0 be the set:

$$\{f \in N_0 : f \text{ is a partial function from } \omega_1 \text{ to } 2 \text{ and } \mathrm{otp}(\mathrm{dom}(f)) < \omega^\omega\}.$$

Now let $\alpha = \beta + 1$ and assume that Q_β and N_β are already defined. If β is a limit ordinal then assume we have also defined a sequence $\langle f_\gamma : \gamma < \beta\rangle$. We describe the construction of Q_α. First we choose N_α, a countable elementary submodel of $(\mathcal{H}(\chi), \in)$ such that $N_\beta \subseteq N_\alpha$, $\beta \subseteq N_\alpha$ and if β is a limit ordinal then

$\langle f_\gamma : \gamma < \beta \rangle \in N_\alpha$ and for each $A \in \mathcal{A}_\beta$ the set $\bigcup_{\gamma \in A} f_\gamma$ is in N_α. We can always find a suitable N_α by the Löwenheim-Skolem Theorem.

Then let Q_α be the set:

$$\{f \in N_\alpha : f \text{ is a partial function from } \omega_1 \text{ to } 2$$

$$\text{and } \mathrm{otp}(\mathrm{dom}(f)) < \omega^\omega \text{ and if } i < \beta \text{ then } f \restriction i \in N_{i+1}\}.$$

When α is a limit ordinal and Q_β is defined for all β less than α, then let $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, which is also an elementary submodel of $(\mathcal{H}(\chi), \in)$. Let $Q_\alpha = \bigcup_{\beta<\alpha} Q_\beta$. We may also need to extend our sequence of functions to be of length α: specifically, if α is not a limit of limits, so is of the form $\alpha'+\omega$ for some $\alpha' < \alpha$, then extend the existing sequence of functions, $\langle f_\gamma : \gamma < \alpha' \rangle$, to a sequence $\langle f_\gamma : \gamma < \alpha \rangle$ that enumerates all of Q_α without any repetitions. When α is a limit of limits then $\langle f_\gamma : \gamma < \alpha \rangle$ will already be defined.

Finally, let $Q = \bigcup_{\alpha<\omega_1} Q_\alpha$ and let $q \leq_{\mathbb{Q}} p$ for $p, q \in Q$ if and only if $q \supseteq p$. We now prove the following:

Claim 5.2.3. Given $W = \{r_\alpha : \alpha < \omega_1\}$, an uncountable set of conditions in $\mathbb{Q}$, we can find an uncountable $U \subseteq \omega_1$ such that $\{r_\alpha : \alpha \in U\}$ is a set of pairwise compatible conditions and for stationary many $\delta < \omega_1$ there is a countable set $x \subseteq U$ with $\mathrm{otp}(x) = \omega$, $\sup(x) = \delta$ and such that $\{r_\alpha : \alpha \in x\}$ has a lower bound (i.e. there is a condition $q \in \mathbb{Q}$ such that for any $\alpha \in x$, $q \leq_{\mathbb{Q}} r_\alpha$).

Proof We will need to make use of the following result of Fodor: if $S \subseteq \omega_1$ is stationary and $h : S \to \omega_1$ is such that $h(\alpha) < \alpha$ for all $\alpha \in S$ (in this case we say that h is *regressive*), then there is an $\epsilon < \omega_1$ such that the set $\{\alpha \in S : h(\alpha) = \epsilon\}$ is stationary.

Now, let W be an uncountable set of conditions in $\mathbb{Q}$, as in the statement of the claim. There is a closed unbounded set of limit ordinals $C \subseteq \omega_1$ such that for $\delta \in C$ there are uncountably many conditions $p \in W$ with $\sup(\mathrm{dom}(p) \cap \delta) < \delta$. To see this, assume not and let $T \subseteq \omega_1$ be a stationary set of limit ordinals such that for $\delta \in T$ there are at most countably many $p \in W$ with $\sup(\mathrm{dom}(p) \cap \delta) < \delta$. Let γ be such that $\mathrm{otp}(T \cap \gamma) = \omega^\omega$. $T \cap \gamma$ is countable, so by assumption the following set must be countable: $W' = \{p \in W : \sup(\mathrm{dom}(p) \cap \delta) < \delta \text{ for some } \delta \in T \cap \gamma\}$. Let $q \in W \setminus W'$. Then $\sup(\mathrm{dom}(q) \cap \delta) = \delta$ for all $\delta \in T \cap \gamma$, but this means $\mathrm{dom}(q)$ must have order type greater than or equal to ω^ω, which contradicts the definition of $\mathbb{Q}$.

We will now define a sequence of conditions in W, $\langle p_\alpha : \alpha \in C \rangle$ where C is as above, by induction. (Formally, this will be a cofinal subsequence of the enumeration of W, $\langle r_\alpha : \alpha < \omega_1 \rangle$, but to avoid an excessive use of subscripts we write e.g. p_β rather than r_{α_β}.) So let $p_{\min(C)}$ be an arbitrary member of W. Now assume that $\alpha < \omega_1$ and for all $i < \alpha$, p_i has been defined. Choose p_α to be any condition in W not already equal to p_i for any $i < \alpha$, such that

$\sup(\mathrm{dom}(p_\alpha) \cap \alpha) < \alpha$. C was defined so as to make this possible. For each $\alpha \in C$ we let $h(\alpha) = \sup(\mathrm{dom}(p_\alpha) \cap \alpha)$, giving us $h : C \to \omega_1$, a regressive function. By Fodor's Lemma we get a stationary set S and some $\epsilon < \omega_1$ such that $\alpha \in S$ implies $\sup(\mathrm{dom}(p_\alpha) \cap \alpha) = \epsilon$, hence $\mathrm{dom}(p_\alpha) \cap [\epsilon + 1, \alpha) = \emptyset$.

Let $S' \subseteq S$ be given by:

$S' = \{\alpha \in S : \text{ for all } \beta < \alpha,\ \sup(\mathrm{dom}(p_\beta)) < \alpha\}$, which is a stationary set because the conditions in $\mathbb{Q}$ have countable domains. Then for any $\alpha, \beta \in S'$, $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) \subseteq \epsilon$. But for any $\alpha \in S'$, $p_\alpha \restriction \epsilon \in N_{\epsilon+1}$ by the definition of $\mathbb{Q}$, and $N_{\epsilon+1}$ is countable, so there are only countably many possibilities for $p_\alpha \restriction \epsilon$. There are also only countably many possibilities for the order type of p_α.

Thus, because S' is uncountable, we can find a $\rho < \omega^\omega$ and a function f such that there is an uncountable $S'' \subseteq S'$ for which $\alpha \in S''$ implies $p_\alpha \restriction \epsilon = f$ and the order type of p_α is ρ. We define a sequence enumerating a subset of $\{p_\alpha : \alpha \in S''\}$ as follows:

Recall the sequence $\langle f_\gamma : \gamma < \omega_1 \rangle$ we defined in the definition of $\mathbb{Q}$. This sequence enumerates all conditions in $\mathbb{Q}$ with no repetitions. Let $\alpha_0 = \min(S'')$. Now assume α_i is defined for all $i < j < \omega_1$. Let $\alpha_j \in S'' \setminus (\sup\{\alpha_i : i < j\} + 1)$ be such that p_{α_j} is equal to f_γ for some γ greater than $\sup\{\beta < \omega_1 : \exists i < j (f_\beta = p_{\alpha_i})\}$. This sequence, $\langle p_{\alpha_i} : i < \omega_1 \rangle$, thins out the set $\{p_\alpha : \alpha \in S''\}$ so as to ensure that an increasing subsequence of $\langle p_{\alpha_i} : i < \omega_1 \rangle$ will correspond to an increasing subsequence of $\langle f_\gamma : \gamma < \omega_1 \rangle$.

Let $U = \{\alpha_i : i < \omega_1\}$, which is uncountable. Then the sequence $\langle p_{\alpha_i} : i < \omega_1 \rangle$ is not only a subsequence of the enumeration of W but also a cofinal increasing subsequence of $\langle f_\gamma : \gamma < \omega_1 \rangle$, which enumerates all of $\mathbb{Q}$. For every $i < \omega_1$, let γ_i be the unique ordinal such that $p_{\alpha_i} = f_{\gamma_i}$. Then $U' = \{\gamma_i : i < \omega_1\}$ is also an uncountable set. So by the definition of $\clubsuit^\omega$ there are stationary many $\delta < \omega_1$ having an $A \in \mathcal{A}_\delta$ with $\mathrm{otp}(A) = \omega$ and $A \subseteq U$ and $\sup(A) = \delta$. Fix such a δ. To prove the claim we need to show that the set $\bigcup_{\gamma_i \in A} p_{\alpha_i}$ is a lower bound to $\{p_{\alpha_i} : \gamma_i \in A\}$. First, note that it is a function because p_α, p_β agree on their common domain, for $\alpha, \beta \in S''$, and its domain has order type at most $\rho.\omega$, by the construction of S''. This is less than ω^ω because ρ is less than ω^ω. Also $\bigcup_{\gamma_i \in A} p_{\alpha_i} \in N_{\delta+1}$ by the definition of $\mathbb{Q}$ (our forcing was cooked up specifically for this purpose; the fact that $\left(\bigcup_{\gamma_i \in A} p_{\alpha_i}\right) \restriction j \in N_{j+1}$ for $i < \delta$ follows from the fact that this is a union of only finitely many functions in N_{j+1}), so it is a condition in $\mathbb{Q}$. Setting $x = \{\alpha_i : \gamma_i \in A\}$ gives us a countable set of the kind stated in the Claim, and it is clearly the case that $\mathrm{otp}(x) = \mathrm{otp}(A) = \omega$ and $x \subseteq U$. And $\sup(x)$ is equal to δ for at least stationary many of the δ under consideration. So the claim is proved.

$\square$

Continuation of the proof of Theorem 5.2.2: Let G be a $\mathbb{Q}$-generic filter over V and $f_G = \bigcup G$ be the generic function. Let $G' = f_G^{-1}(1)$, an unbounded subset of ω_1. Fix a series of functions $\langle h_\alpha : \alpha \in Lim(\omega_1)\rangle$ in V such that $h_\alpha : [\alpha, \alpha+\omega) \to \alpha$ and h_α is a bijection. Then we claim that $\langle h_\alpha\,[([\alpha, \alpha+\omega) \cap G')] : \alpha \in Lim(\omega_1)\rangle$ is a $\clubsuit$-sequence in $V[G]$.

To see this, let p, $\dot{f}$ and $\dot{\tau}$ be such that $p \Vdash$ "$\dot{\tau} \in [\omega_1]^{\omega_1}$ and $\dot{f} : \omega_1 \to \omega_1$ is its increasing enumeration". Then let $\langle q_\alpha : \alpha < \omega_1\rangle$ be a sequence of conditions such that for each $\alpha < \omega_1$, $q_\alpha \Vdash$ "$\dot{f}(\alpha) = \gamma_\alpha$" for some $\gamma_\alpha < \omega_1$. There is a closed unbounded set $E \subseteq \omega_1$ such that for all $\delta \in E$, $\sup\{\gamma_i : i < \delta\} = \delta$. So by applying Claim 5.2.3 we can find stationary many $\delta \in E$ and for each one a countable set $x^\delta \subseteq \omega_1$ such that $q^\delta = \bigcup\{q_\alpha : \alpha \in x^\delta\}$ is a condition in $\mathbb{Q}$ with $\sup(\mathrm{dom}(q^\delta)) = \sup\{\gamma : q^\delta \Vdash \text{“}\gamma \in \dot{\tau}\text{”}\} = \delta$. Let $Y = h_\delta^{-1}[\{\gamma : q^\delta \Vdash \text{“}\gamma \in \dot{\tau}\text{”}\}]$, a subset of $[\delta, \delta+\omega)$. Then $q^\delta_+ = q^\delta \cup h^Y_\delta$, where h^Y_δ is the function with domain $[\delta, \delta+\omega)$ such that $h^Y_\delta[Y] = \{1\}$ and $h^Y_\delta[[\delta, \delta+\omega) \setminus Y] = \{0\}$, is a condition in $\mathbb{Q}$ and clearly it is the case that $q^\delta_+ \Vdash$ "$\mathrm{ran}(h_\delta \restriction ([\delta, \delta+\omega) \cap G')) \subseteq \dot{\tau}$". So we have shown that the set of $\delta \in Lim(\omega_1)$ for which there exists a dense (below p) set of conditions forcing "$h_\delta\,[([\delta, \delta+\omega) \cap G')] \subseteq \dot{\tau}$" is stationary. Hence the sequence $\langle h_\alpha\,[([\alpha, \alpha+\omega) \cap G')] : \alpha \in Lim(\omega_1)\rangle$ is a $\clubsuit$-sequence in $V[G]$.

It remains to check that $\mathbb{Q}$ does not collapse cardinals. But this is an immediate Corollary to Claim 5.2.3, which actually establishes that the forcing has a very strong form of the countable chain condition (stronger even than the Knaster property).

$\square$

Chapter 6

Sometimes the Same: ♣ and the Invariance Property

In this chapter, and in Chapters 7 and 8, we define several variations on the axiom ♣ and ask whether they are formally equivalent (either in ZFC or with the assumption of extra axioms); we present both positive and negative results. Consistency results are dealt with in the latter two chapters, while the present chapter is devoted to giving combinatorial results.

We begin by observing that $\diamondsuit$ is formally equivalent to many of its apparent weakenings and strengthenings: this phenomenon is widely documented (see [19, II] or Section 6.1, below) and leads us to say, following [9], that $\diamondsuit$ has an *invariance property*. The extent to which ♣ shares this invariance property is not as widely known, although the paper [9] has answered several key questions in this area. Our purpose here and in the following two chapters is to extend the known results on this and to refine the techniques that can be used to attain them.

In this chapter, for the first time, we will work with slightly different definitions of both $\diamondsuit$ and $\clubsuit$, which allow us to take a stationary set as a parameter and which can be immediately generalised to uncountable regular cardinals other than ω_1. So let S be a stationary subset of a regular cardinal λ, consisting only of limit ordinals. We generalise $\diamondsuit$ and $\clubsuit$ as follows:

> ($\diamondsuit(S)$) There exists a sequence $\langle B_\delta : \delta \in S \rangle$ such that $B_\delta \subseteq \delta$ for all $\delta \in S$ and if $X \subseteq \lambda$, where $\lambda = \sup(S)$, then the set $\{\delta \in S : X \cap \delta = B_\delta\}$ is a stationary subset of λ.

> ($\clubsuit(S)$) There exists a sequence $\langle A_\delta : \delta \in S \rangle$ such that $A_\delta \subseteq \delta$ with $\sup(A_\delta) = \delta$, for all $\delta \in S$, and if $X \in [\lambda]^\lambda$, where $\lambda = \sup(S)$, then the set $\{\delta \in S : A_\delta \subseteq X\}$ is a stationary subset of λ.

The specific axioms $\clubsuit$ and $\diamondsuit$ that we have been working with up to now are therefore $\clubsuit(Lim(\omega_1))$ and $\diamondsuit(Lim(\omega_1))$ respectively, though in future we will denote them $\clubsuit(\omega_1)$ and $\diamondsuit(\omega_1)$ for the sake of convenience. Our notation for $\clubsuit(S)$ and $\diamondsuit(S)$ does not make explicit reference to λ, but it will always be clear from context.

6.1 The invariance property of $\diamondsuit$

Perhaps the most well-known result exemplifying the invariance property of $\diamondsuit$ is due to K. Kunen, who considered the following axiom:

> ($\diamondsuit^-(S)$) Let $\lambda = \kappa^+$ and $S \subseteq \lambda$ be stationary. Then there exists a sequence $\langle \mathcal{B}_\delta : \delta \in S \rangle$, for which $\mathcal{B}_\delta = \{B^i_\delta : i < \kappa\}$ and $B^i_\delta \subseteq \delta$ for each $\delta \in S$ and $i < \kappa$, such that if $X \subseteq \lambda$ then the set $\{\delta \in S : X \cap \delta \in \mathcal{B}_\delta\}$ is a stationary subset of λ.

Kunen proved that this apparent weakening of $\diamondsuit$ is not in fact a weakening at all. The proof generalises to any uncountable successor ordinal λ, and any stationary S:

Theorem 6.1.1 (Kunen)**.** $\diamondsuit^-(S) \leftrightarrow \diamondsuit(S)$.

Proof See [19, II].

$\square$

Many other equivalent versions of $\diamondsuit$ have since been found, all of which seem at first sight to be substantially different statements; see for example [6]. (We should point out, however, that there are also many variants of $\diamondsuit$ that are known to be *strictly* weaker or stronger that $\diamondsuit$: for instance $\diamondsuit^*$ and $\diamondsuit^+$ are both stronger. See [19, II].) Equivalences between different versions of $\clubsuit$ have not been explored to the same extent. The equivalence of the following two statements is perhaps the most widely known positive result in this vein. Let S be a stationary subset of a regular uncountable cardinal λ:

($\clubsuit^1(S)$) There exists a sequence $\langle A_\delta : \delta \in S\rangle$ such that $A_\delta \subseteq \delta$ with $\sup(A_\delta) = \delta$, for all $\delta \in S$, and if $X \in [\lambda]^\lambda$ then the set $\{\delta \in S : A_\delta \subseteq X\}$ is a stationary subset of λ.

($\clubsuit^2(S)$) There exists a sequence $\langle A_\delta : \delta \in S\rangle$ such that $A_\delta \subseteq \delta$ with $\sup(A_\delta) = \delta$, for all $\delta \in S$, and if $X \in [\lambda]^\lambda$ then the set $\{\delta \in S : A_\delta \subseteq X\}$ is non-empty.

Theorem 6.1.2. $\clubsuit^1(S) \leftrightarrow \clubsuit^2(S)$.

Proof $\clubsuit^1(S) \rightarrow \clubsuit^2(S)$ is trivial, so we prove $\clubsuit^2(S) \rightarrow \clubsuit^1(S)$.

In fact, we show that a witness to $\clubsuit^2(S)$ is also a witness to $\clubsuit^1(S)$. Let $\langle A_\delta : \delta \in S\rangle$ witness $\clubsuit^2(S)$, and X be in $[\lambda]^\lambda$. Assume that $C \subseteq \lambda$ is a closed unbounded set having empty intersection with $\{\delta \in S : A_\delta \subseteq X\}$. Choose an increasing sequence of ordinals less than λ, denoted $\langle \gamma_\alpha : \alpha < \lambda\rangle$ such that if α is a successor ordinal then $\gamma_\alpha \in C$ and if α is a limit ordinal then $\gamma_\alpha \in X$. Both C and X are cofinal in λ so this can be easily done. Then $\{\gamma_\alpha : \alpha \in Lim(\lambda)\} \subseteq X$ is unbounded so there is a $\delta \in S$ with $A_\delta \subseteq \{\gamma_\alpha : \alpha \in Lim(\lambda)\} \subseteq X$. From our construction of $\langle \gamma_\alpha : \alpha < \lambda\rangle$ we can find a set of ordinals in C with supremum δ, hence $\delta \in C$ because it is closed. This is a contradiction.

$\square$

There is a good explanation as to why Theorem 6.1.2 is perhaps the only well-known example of an equivalence between two variants of $\clubsuit$: it is one of only very few such statements that are actually true. We give a further example in Corollary 6.2.4, but most of the variants of $\clubsuit$ that we consider in this thesis can be shown to be pairwise inequivalent (see Chapters 7 and 8).

However, we have seen in Chapter 1 that ($\clubsuit$ + CH) is equivalent to $\diamondsuit$; in fact, many weaker variants of $\clubsuit$ are also equivalent to $\diamondsuit$ in the presence of CH. So if the Continuum Hypothesis holds, the equivalence of two different variations on $\clubsuit$ can often be inferred from the fact that they are both equivalent to $\diamondsuit$. We conclude from this that the invariance property of $\clubsuit$ is dependent on the cardinal arithmetic statements that are assumed to hold in the set-theoretic universe. In this chapter we show that, even with seemingly weak cardinal arithmetic assumptions (in particular, those that allow $\diamondsuit$ to fail), we can find non-trivial variants of $\clubsuit$ that are formally equivalent. Our technique derives from the proof of a recent theorem of Shelah that improved on a classical result of Gregory:

Definition 6.1.3. Let λ and κ be infinite regular cardinals with $\kappa < \lambda$. Then S^λ_κ denotes the set $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$, which will always be stationary. And $S^\lambda_{\neq\kappa} = \{\alpha < \lambda : \mathrm{cf}(\alpha) \neq \kappa\}$, which will be stationary when $\lambda > \omega_1$.

Theorem 6.1.4 (Gregory, [13])**.** If κ is regular and λ is such that $\lambda^\kappa = \lambda$ and

$2^\lambda = \lambda^+$, then $\diamondsuit(S^{\lambda^+}_\kappa)$ holds.

Shelah's result removes one of the conditions from this theorem, giving us full equivalence between $\diamondsuit(\lambda^+)$ and $2^\lambda = \lambda^+$:

Theorem 6.1.5 (Shelah, [29])**.** Let λ be uncountable and $S \subseteq S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ be a stationary subset of λ^+. If $2^\lambda = \lambda^+$ then there exists a sequence witnessing $\diamondsuit(S)$.

Notation 6.1.6. We will write CH_λ to denote the statement that $2^\lambda = \lambda^+$.

Shelah's result established the equivalence between CH_λ and $\diamondsuit(\lambda^+)$ for all uncountable cardinals λ, but there remain open questions concerning the stationary sets S that can be taken as parameters. For example, when λ is singular it is not known whether $\diamondsuit(S^{\lambda^+}_{\mathrm{cf}(\lambda)})$ follows from CH_λ. M. Zeman proved that the answer is positive assuming the weak square, $\square^*_\lambda$ (see [33]). A. Rinot isolated the use of $\square^*_\lambda$ in this proof and was able to replace it with a weaker assumption called the Stationary Approachability Property (SAP_λ), see [26]. The common methods used in all of these proofs are foreshadowed in at least two classical results of combinatorial set theory: Shelah's theorems on club guessing and Kunen's result in Theorem 6.1.1. Both Theorem 6.1.5 and 6.1.1, as well as several club guessing theorems, follow from our results in this chapter as specific instances.

6.2 $\clubsuit$ with multiple guesses

We begin by defining a weakening of $\clubsuit(S)$ that generalises the axiom $\clubsuit^\omega$ we encountered in the previous chapter. Again this axiom asserts the existence of a sequence indexed by a stationary set of limit ordinals, but rather than presenting us with a cofinal subset of δ, for each δ in the indexing set, the sequence now presents us with a *set* of cofinal subsets of δ. We signify this by writing $\mathcal{A}_\delta$ instead of A_δ, and we specify some bound on the size of $\mathcal{A}_\delta$ to avoid trivialities. The axiom $\clubsuit^\omega$ was first introduced by M. Rajagopalan in [25].

Definition 6.2.1. For λ a regular cardinal, $\kappa < \lambda$ a cardinal, and $S \subseteq \lambda$ a stationary set, the axiom $\clubsuit^\kappa(S)$ is the statement that there exists a sequence $\langle \mathcal{A}_\delta : \delta \in S \rangle$ such that $|\mathcal{A}_\delta| = \kappa$ for all $\delta \in S$, and for every unbounded subset $X \subseteq \lambda$ there exists a $\delta \in S$ and an $A^i_\delta \in \mathcal{A}_\delta$, such that $A^i_\delta \subseteq X$ and $\sup(A^i_\delta) = \delta$.

Notation 6.2.2. Let κ and λ be ordinals and $X \subseteq \kappa \times \lambda$. Then for $i < \kappa$, let $(X)_i = \{\beta < \lambda : (i, \beta) \in X\}$.

Theorem 6.2.3. Let $\kappa < \lambda$ be cardinals, with λ regular. If $\lambda^\kappa = \lambda$ then $\clubsuit^\kappa(S) \leftrightarrow \clubsuit(S)$, for any stationary $S \subseteq \lambda$.

Proof Let $\langle \mathcal{A}_\delta : \delta \in S \rangle$ be a witness to $\clubsuit^\kappa(S)$ and let $\langle A^i_\delta : i < \kappa \rangle$ enumerate $\mathcal{A}_\delta$ for each $\delta \in S$. Let $\langle D_\alpha : \alpha < \lambda \rangle$ be an enumeration of $[\kappa \times \lambda]^{\leq \kappa}$, which is possible because $\lambda^\kappa = \lambda$ and $|\kappa \times \lambda| = \lambda$. Then for some $i < \kappa$ the sequence $\langle B^i_\delta : \delta \in S \rangle$, given by setting $B^i_\delta = \delta \cap \bigcup_{\alpha \in A^i_\delta} (D_\alpha)_i$ for all $\delta \in S$ (unless this gives us a bounded subset of δ, in which case choose an appropriate B^i_δ arbitrarily) is a witness to $\clubsuit(S)$.

To prove this, assume it is not the case. Then for each $i < \kappa$ there is an unbounded set $X_i \subseteq \lambda$ and a closed unbounded set $E_i \subseteq \lambda$ such that X_i is not a superset of B^i_δ for any $\delta \in E_i \cap S$. Let $E = \bigcap_{i<\kappa} E_i$, and for each $i < \kappa$, let $\langle x^i_\epsilon : \epsilon < \lambda \rangle$ be an increasing enumeration of X_i. Then set $Z_\epsilon = \bigcup_{i<\kappa} (\{i\} \times \{x^i_\epsilon\})$. Clearly each Z_ϵ has size κ, and is a subset of $\kappa \times \lambda$.

We define two sequences of ordinals $\langle \alpha_\rho : \rho < \lambda \rangle$ and $\langle \beta_\rho : \rho < \lambda \rangle$ by induction. Let α_0 be the least ordinal such that $Z_0 = D_{\alpha_0}$, and let $\beta_0 = 0$. Assume α_μ and β_μ are defined for all $\mu < \rho$. Let α_ρ be the least ordinal greater than $\sup\{x^i_{\beta_\mu} : i < \kappa, \mu < \rho\}$ so that if β_ρ is such that $Z_{\beta_\rho} = D_{\alpha_\rho}$ then $\min(\{x^i_{\beta_\rho} : i < \kappa\})$ is greater than $\sup(\{\alpha_\mu : \mu < \rho\})$. Fix β_ρ to be as specified. This completes the definitions of $\langle \alpha_\rho : \rho < \lambda \rangle$ and $\langle \beta_\rho : \rho < \lambda \rangle$.

The set $\{\alpha_\rho : \rho < \lambda\}$ is an unbounded subset of λ, so there will be some $j < \kappa$ and a stationary $S' \subseteq S$ such that for $\delta \in S'$ we have $A^j_\delta \subseteq \{\alpha_\epsilon : \epsilon < \lambda\}$. (The existence of such a j follows from the fact that the union of κ many non-stationary subsets of λ cannot be stationary, so assuming there is no such j gives an immediate contradiction. Of course, j depends on the set $\{\alpha_\rho : \rho < \lambda\}$, otherwise the theorem would be trivial.)

Let $\delta \in S' \cap E$; there are stationary many such δ. We have set $B^j_\delta = \bigcup_{\alpha \in A^j_\delta} (D_\alpha)_j$ and A^j_δ is a subset of $\{\alpha_\rho : \rho < \lambda\}$, so for all $\beta \in A^j_\delta$ there is an ϵ with $D_\beta = Z_\epsilon = \bigcup_{i<\kappa} (\{i\} \times \{x^i_\epsilon\})$. Hence $B^j_\delta \subseteq X_j$. The fact that $\sup(B^j_\delta) = \delta$ follows from the construction of the sequence $\langle \alpha_\rho : \rho < \lambda \rangle$. This contradicts our choice of X_j and the statement is proved. The reverse direction of the theorem is trivial.

$\square$

It is worth noting that Kunen's result in Theorem 6.1.1 is a specific instance of the above theorem, telling us that $\clubsuit(\omega_1) \leftrightarrow \clubsuit^\omega(\omega_1)$ if CH holds (though to obtain this fact from Kunen's proof we would have to reason via the chain of equivalences: $(\mathrm{CH} + \clubsuit^\omega(\omega_1)) \leftrightarrow \diamondsuit^\omega(\omega_1) \leftrightarrow \diamondsuit(\omega_1) \leftrightarrow (\mathrm{CH} + \clubsuit(\omega_1))$).

We also obtain the following ZFC result:

Corollary 6.2.4. For $n < \omega$, λ regular and $S \subseteq \lambda$ stationary, $\clubsuit^n(S)$ is equivalent to $\clubsuit(S)$.

This answers a question asked by Rajagopalan in [25].

6.3 Another weak $\clubsuit$ principle

We now prove a similar result for a variation on $\clubsuit(S)$ where the guessing property is weakened from subsethood to cofinal intersection. This holds trivially if we don't put some kind of bound on the size of each A^i_δ (otherwise we could set $A^i_\delta = \delta$).

Even with such a bound, a version of this principle holds in ZFC for successor cardinals greater than ω_1.

Definition 6.3.1. For λ a regular cardinal, $\eta < \lambda$ a cardinal, and $S \subseteq \lambda$ a stationary set, the axiom $\clubsuit^{\sim\eta,\kappa}(S)$ is the statement that there exists a sequence $\langle \mathcal{A}_\delta : \delta \in S \rangle$ with $\mathcal{A}_\delta = \{A^i_\delta : i < \kappa\}$ and $|A^i_\delta| < \eta$ for all $\delta \in S$ and $i < \kappa$, such that for any cofinal subset $X \subseteq \lambda$ the following set is stationary: $\{\delta \in S : \exists i < \kappa$ $(\sup(A^i_\delta \cap X) = \delta)\}$.

When $\lambda^\eta = \lambda$ this apparent weakening is equivalent to $\clubsuit^\kappa(S)$. We prove this by using a sequence of possible counterexamples to filter out those $x \in A^i_\delta$ that prevent $\langle \mathcal{A}_\delta : \delta \in S \rangle$ from having the required guessing property.

Assuming $\lambda^\eta = \lambda$, once we have fixed a $\clubsuit^{\sim\eta,\kappa}(S)$-sequence $\langle \mathcal{A}_\delta : \delta \in S \rangle$ and an enumeration $\langle d_\alpha : \alpha < \lambda \rangle$ of $[\eta \times \lambda]^{\leq\eta}$ then we can make the following definition:

Definition 6.3.2. For a sequence of sets $\langle X_\alpha : \alpha < \gamma \leq \eta \rangle$ with $X_\alpha \in [\lambda]^\lambda$ for each $\alpha < \gamma$, we define $\bar{V}^{\delta,i}$ (for $\delta \in S$ and $i < \kappa$) to be the sequence $\langle V^{\delta,i}_\alpha : \alpha \leq \gamma \rangle$ where $V^{\delta,i}_\alpha = \{\epsilon \in A^i_\delta :$ for all $\beta < \alpha, (d_\epsilon)_\beta \subseteq X_\beta\}$.

Lemma 6.3.3. If $\langle \mathcal{A}_\delta : \delta \in S \rangle$ is a witness to $\clubsuit^{\sim\eta,\kappa}(S)$ and $\langle X_\alpha : \alpha < \gamma \rangle$ is such that for each $\alpha < \gamma$ there exists a club set E_α with $\delta \in E_\alpha \cap S$ implying that either $V^{\delta,i}_{\alpha+1} \subsetneq V^{\delta,i}_\alpha$ or $\sup(V^{\delta,i}_\alpha) < \delta$ for all $i < \kappa$, then we must have $\gamma < \eta$.

Proof Assume not. Then let $\langle X_\alpha : \alpha < \eta \rangle$ be a sequence contradicting the lemma, and $\langle E_\alpha : \alpha < \eta \rangle$ the associated club sets. Let $E' = \bigcap_{\alpha<\eta} E_\alpha$, and let $\langle \xi^\alpha_\mu : \mu < \lambda \rangle$

be the increasing enumeration of X_α. Then the family of sets $\{e_\mu : \mu < \lambda\}$, defined by setting $e_\mu = \langle \xi^\alpha_\mu : \alpha < \eta \rangle$ for all $\mu < \lambda$, is a subset of $[\eta \times \lambda]^{\leq \eta}$, so there is a sequence $\langle \epsilon_\mu : \mu < \lambda \rangle$ of ordinals less than λ such that $d_{\epsilon_\mu} = e_\mu$ for each $\mu < \lambda$. Then if $\beta < \eta$ we have $(e_\mu)_\beta = \{\xi^\beta_\mu\}$, so $(e_\mu)_\beta \subseteq X_\beta$. Let $\delta \in S$ and $i < \kappa$ be such that $\sup(A^i_\delta \cap Y) = \delta$, where $Y = \{\epsilon_\mu : \mu < \lambda\}$. Hence $A^i_\delta \cap Y \subseteq V^{\delta,i}_\alpha$ for all $\alpha < \eta$, so $\sup(V^{\delta,i}_\alpha) = \delta$ for all $\alpha < \eta$, and because $\delta \in E'$ this means that we must have $V^{\delta,i}_{\alpha+1} \subsetneq V^{\delta,i}_\alpha$, for all $\alpha < \eta$, giving us a strictly decreasing chain under containment, of length η. But $V^{\delta,i}_0 \subseteq A^i_\delta$ and $|A^i_\delta| < \eta$, which is a contradiction.

$\square$

Lemma 6.3.4. Having fixed a $\clubsuit^{\sim\eta,\kappa}(S)$-sequence as above, let $\langle X_\alpha : \alpha < \gamma < \eta \rangle$ be a *maximal* sequence satisfying the conditions of Lemma 6.3.3. Then the sequence $\langle \mathcal{B}_\delta = \{\delta \cap \bigcup_{\epsilon \in V^{\delta,i}_\gamma} (d_\epsilon)_\gamma : i < \kappa\} : \delta \in S \rangle$, suitably modified to exclude bounded subsets of δ, gives us a witness to $\clubsuit^\kappa(S)$.

Proof Assume not. Then let $X_\gamma \in [\lambda]^\lambda$ be a set contradicting this, so there is a closed unbounded set E_γ for which $\delta \in E_\gamma$ implies that for all $i < \kappa$ either $\bigcup_{\epsilon \in V^{\delta,i}_\gamma} (d_\epsilon)_\gamma \not\subseteq X_\gamma$ or $\sup(\bigcup_{\epsilon \in V^{\delta,i}_\gamma} (d_\epsilon)_\gamma) < \delta$. Either way we can find a closed unbounded set E_γ so that X_γ continues the sequence, contradicting its maximality.

$\square$

This gives us:

Theorem 6.3.5. If λ is a regular cardinal and $\eta < \lambda$ is a cardinal, such that

$\lambda^\eta = \lambda$, then $\clubsuit^{\sim\eta,\kappa}(S) \to \clubsuit^\kappa(S)$.

Proof By Lemmas 6.3.3 and 6.3.4.

$\square$

Writing $\clubsuit^{\sim\eta}(S)$ for $\clubsuit^{\sim\eta,1}(S)$, we also get:

Corollary 6.3.6. If λ is a regular cardinal and $\eta < \lambda$ is a cardinal, such that $\lambda^\eta = \lambda$, then $\clubsuit^{\sim\eta}(S) \to \clubsuit(S)$.

It is also possible to prove:

Theorem 6.3.7. For λ regular, η and κ cardinals $< \lambda$, and $S \subseteq \lambda$ stationary, if $\lambda^\kappa = \lambda$ then $\clubsuit^{\sim\eta,\kappa}(S) \to \clubsuit^{\sim\eta}(S)$.

Proof The proof of Theorem 6.2.3 can be altered slightly to give this result.

$\square$

Theorems 6.3.5 and 6.2.3 were used implicitly by Shelah to show that $\mathrm{CH}_\lambda \leftrightarrow \diamondsuit(\lambda^+)$ for λ an uncountable cardinal. This is because a guessing principle of the kind given in Definition 6.3.1 holds in ZFC for successor cardinals above ω_1.

Theorem 6.3.8. Let λ be uncountable and $S \subseteq S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ stationary. Then $\clubsuit^{\sim\lambda,\mathrm{cf}(\lambda)}(S)$ is true in ZFC.

Proof For each $\delta < \lambda^+$ let $\langle c^\delta_k : k < \mathrm{cf}(\lambda)\rangle$ be such that for $j < k < \mathrm{cf}(\lambda)$ we have $c^\delta_j \subseteq c^\delta_k$, $|c^\delta_k| < \lambda$, and $\bigcup_{k<\mathrm{cf}(\lambda)} c^\delta_k = \delta$ (this is possible because each $\delta < \lambda^+$ has cardinality less than or equal to λ). Let $X \subseteq \lambda^+$ be unbounded and $\delta \in S$ be such that $\sup(X \cap \delta) = \delta$. Then because $\mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)$ there must be some $\zeta < \mathrm{cf}(\lambda)$ with $\sup(c^\delta_\zeta \cap X) = \delta$. Hence $\langle\langle c^\delta_k : k < \mathrm{cf}(\lambda)\rangle : \delta \in S\rangle$ witnesses the theorem.

$\square$

Applying Theorems 6.2.3 and 6.3.5, we can conclude the following:

Corollary 6.3.9. If $2^\lambda = \lambda^+$ holds, then $\clubsuit^{\sim\lambda,\lambda}(S) \to \clubsuit(S)$.

Corollary 6.3.10. If $S \subseteq S^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$ is stationary, then $2^\lambda = \lambda^+ \to \clubsuit(S)$.

Combining this with the fact that for $S \subseteq \lambda^+$, $(\mathrm{CH}_\lambda + \clubsuit(S))$ is equivalent to $\diamondsuit(S)$, gives us an alternative proof of Shelah's main result in [29].

6.4 $\clubsuit$ restricted to filters

All of the variants of $\clubsuit$ that we have considered so far have been able to, in some sense, 'guess' arbitrary unbounded subsets of a regular λ. We can form weaker variants of $\clubsuit$ by requiring them to guess only those subsets of a regular λ that are in some fixed uniform filter $\mathcal{F}$ on λ (a filter is *uniform* if it only contains unbounded sets). "Club guessing" is a widely known example of this, where $\mathcal{F}$ is the club filter.

Definition 6.4.1. For a uniform filter $\mathcal{F}$ on a regular cardinal λ, and a stationary set $S \subseteq \lambda$, the axiom $\clubsuit_{\mathcal{F}}(S)$ asserts the existence of a sequence $\langle C_\delta : \delta \in S\rangle$ with

$\sup(C_\delta) = \delta$ for all $\delta \in S$, such that for all $F \in \mathcal{F}$ the set $\{\delta \in S : C_\delta \subseteq F\}$ is stationary.

We can define variants of $\clubsuit_{\mathcal{F}}(S)$ that are analogous to those variants of $\clubsuit(S)$ that we have already considered in this chapter:

($\clubsuit^{\kappa}_{\mathcal{F}}(S)$) For a uniform filter $\mathcal{F}$ on a regular cardinal λ, and a stationary set $S \subseteq \lambda$, the axiom $\clubsuit^{\kappa}_{\mathcal{F}}(S)$ asserts the existence of a sequence $\langle \{C^i_\delta : i < \kappa\} : \delta \in S \rangle$ with $\sup(C^i_\delta) = \delta$ for all $\delta \in S$ and $i < \kappa$, such that for all $F \in \mathcal{F}$ the set $\{\delta \in S : \exists i < \kappa (C^i_\delta \subseteq F)\}$ is stationary.

($\clubsuit^{\sim \eta,\kappa}_{\mathcal{F}}(S)$) For a uniform filter $\mathcal{F}$ on a regular cardinal λ, and a stationary set $S \subseteq \lambda$, the axiom $\clubsuit^{\sim \eta,\kappa}_{\mathcal{F}}(S)$ asserts the existence of a sequence $\langle \{C^i_\delta : i < \kappa\} : \delta \in S \rangle$ with $\sup(C^i_\delta) = \delta$ and $|C^i_\delta| < \eta$ for all $\delta \in S$ and $i < \kappa$, such that for all $F \in \mathcal{F}$ the set $\{\delta \in S : \exists i < \kappa (\sup(C^i_\delta \cap F) = \delta)\}$ is stationary.

We can then obtain results analogous to Theorems 6.2.3 and 6.3.5 in ZFC alone, using completeness properties of the filter rather than cardinal arithmetic. (We say $\mathcal{F}$ is κ-complete for a cardinal κ if the intersection of $< \kappa$ many sets in $\mathcal{F}$ is also in $\mathcal{F}$. This is sometimes called κ-closed.)

Theorem 6.4.2. Let $\mathcal{F}$ be a κ^+-complete uniform filter on a regular λ, with $\kappa \leq \lambda$, and $S \subseteq \lambda$ stationary. Then $\clubsuit^{\kappa}_{\mathcal{F}}(S) \to \clubsuit_{\mathcal{F}}(S)$.

Proof Let $\langle\langle C^i_\delta : i < \kappa\rangle : \delta \in S\rangle$ be a witness to $\clubsuit^\kappa_{\mathcal{F}}(S)$. Then for some $j < \kappa$, $\langle C^j_\delta : \delta \in S\rangle$ witnesses $\clubsuit_{\mathcal{F}}(S)$. To see this, assume not. Then for each $i < \kappa$ there is an F_i and a club set E_i witnessing the failure of $\langle C^i_\delta : \delta \in S\rangle$ to provide a stationary set of guesses for F_i. Let $E' = \bigcap_{i<\kappa} E_i$ and $F' = \bigcap_{i<\kappa} F_i$, which are in the club filter and $\mathcal{F}$ respectively, by the completeness properties of both. Choose some $\delta' \in E' \cap \{\delta \in S : \exists k < \kappa(C^k_\delta \subseteq F')\}$, hence for some $k < \kappa$ we get $C^k_{\delta'} \subseteq F' \subseteq F_k$ and $\delta' \in E_k$, which contradicts our choice of F_k and E_k.

$\square$

Theorem 6.4.3. Let $\mathcal{F}$ be an η^+-complete uniform filter on a regular λ, with $\eta \leq \lambda$, and $S \subseteq \lambda$ stationary. Then $\clubsuit^{\sim\eta,\kappa}_{\mathcal{F}}(S) \to \clubsuit^\kappa_{\mathcal{F}}(S)$.

Proof The proof is similar to that of Theorem 6.3.5. Let $\langle\{C^i_\delta : i < \kappa\} : \delta \in S\rangle$ be a witness to $\clubsuit^{\sim\eta,\kappa}_{\mathcal{F}}(S)$. For a sequence of sets $\langle F_\alpha : \alpha < \gamma\rangle$ with $F_\alpha \in \mathcal{F}$ for each $\alpha < \gamma$, we define $\bar{W}^{\delta,i}$ to be the sequence $\langle W^{\delta,i}_\alpha : \alpha < \gamma + 1\rangle$ where $W^{\delta,i}_0 = C^i_\delta$ and for $\beta > 0$, $W^{\delta,i}_\beta = C^i_\delta \cap \bigcap_{\alpha<\beta} F_\alpha$. Then if $\langle F_\alpha : \alpha < \gamma\rangle$ is such that for each $\alpha < \gamma$ there exists a club set E_α with $\delta \in E_\alpha \cap S$ implying that for all $i < \kappa$ either $W^{\delta,i}_\alpha \supsetneq W^{\delta,i}_{\alpha+1}$ or $\sup(W^{\delta,i}_\alpha) < \delta$, we must have $\gamma < \eta$.

To see this, assume that $\langle F_\alpha : \alpha < \eta\rangle$ contradicts the claim, and $\langle E_\alpha : \alpha < \eta\rangle$ are the associated club sets. Let $E' = \bigcap_{\alpha<\eta} E_\alpha$ and $F' = \bigcap_{\alpha<\eta} F_\alpha$, which are in the club filter and $\mathcal{F}$ respectively.

Let $S' \subseteq S$ be the set $\{\delta : \exists i < \kappa\,(\sup(C^i_\delta \cap F') = \delta)\}$. Take some $\delta' \in S' \cap E'$.

Clearly for some $i < \kappa$, $\sup(W_\eta^{\delta', i}) = \delta'$ because $\delta' \in S'$. So for all $\alpha < \eta$ we must have $W_\alpha^{\delta',} \supsetneq W_{\alpha+1}^{\delta',}$, giving us a strictly decreasing chain under containment, of length η. As before, this is a contradiction.

So let $\langle F_\alpha : \alpha < \gamma < \lambda \rangle$ be a maximal sequence of this type. Then the set $\langle \{W_\eta^{\delta,i} : i < \kappa\} : \delta \in S \rangle$ is a witness to $\clubsuit_{\mathcal{F}}(S)$. If not, we can find an $F \in \mathcal{F}$ such that there exists a club set $E = \{\delta : \text{ for all } i < \kappa, W_\gamma^{\delta,i} \not\subseteq F \text{ or } \sup(W_\gamma^{\delta,i}) < \delta\}$. In which case we can continue our maximal sequence, contradicting its maximality.

$\square$

As before, we can also use the proof of Theorem 6.4.2 to get the result:

Theorem 6.4.4. If $\mathcal{F}$ is a κ^+-complete uniform filter on a regular λ, and $S \subseteq \lambda$ is stationary, then $\clubsuit_{\mathcal{F}}^{\sim\eta,\kappa}(S) \to \clubsuit_{\mathcal{F}}^{\sim\eta}(S)$.

From the above theorems, and Theorem 6.3.8, we can conclude:

Theorem 6.4.5. If λ is uncountable, $\mathcal{F} \subseteq \mathcal{P}(\lambda^+)$ is a λ^+-complete uniform filter and $S \subseteq S^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is stationary, then $\clubsuit_{\mathcal{F}}(S)$ holds in ZFC.

Proof By Lemmas 6.3.8 and 6.4.4.

$\square$

Club guessing is an instance of this theorem. However, Theorem 6.4.5 does not strictly extend the known results on club guessing, since it can be shown that there is a club guessing sequence for λ^+, where λ is singular, indexed by $S \subseteq S^{\lambda^+}_{\mathrm{cf}(\lambda)}$. Since Theorem 6.3.8 fails for such an S the following question is of particular interest:

Question 6.4.6. For $\mathcal{F}$ a λ^+-complete uniform filter on λ^+, where λ is singular, is it the case that ZFC $\vdash \clubsuit_{\mathcal{F}}(S^{\lambda^+}_{\mathrm{cf}(\lambda)})$?

When $\square^*_\lambda$ holds (or SAP_λ, see [26]) it is known that the answer is yes, but it is not clear if this is the case in ZFC alone.

Chapter 7

Consistency Results on $\clubsuit$ and Invariance, I

In this chapter we obtain consistency results pertaining to the invariance property of $\clubsuit(\omega_1)$. Recall from Chapter 6 that we say $\diamondsuit$ has an *invariance property* because many of its apparent weakenings and strengthenings are in fact formally equivalent to it. We also saw in that chapter that when certain cardinal arithmetic statements hold several variants of $\clubsuit(\lambda)$, for a regular cardinal λ, will be formally equivalent. Thus we can say that in general $\clubsuit$ will increasingly approximate the invariance property of $\diamondsuit$ as increasingly stronger cardinal arithmetic statements are assumed. (Specifically, if we fix a regular λ then a greater number of variations on $\clubsuit(\lambda)$ can be proved equivalent as μ increases, where μ is the supremum of $\{\kappa : \lambda^\kappa = \lambda\}$.)

We will see in the present chapter, and in Chapter 8, that these equivalences do not in general hold in ZFC alone. The present chapter is concerned with variations on $\clubsuit(\omega_1)$; other uncountable cardinals are dealt with in Chapter 8.

We have defined several variants of $\clubsuit(\omega_1)$ already in this thesis, and we define several more below: for any two of them, call them $\clubsuit^1(\omega_1)$ and $\clubsuit^2(\omega_1)$, we can usually find a forcing that preserves $\clubsuit^1(\omega_1)$ while ensuring that $\clubsuit^2(\omega_1)$ fails to hold in the generic extension, or vice versa, except where the results in the previous chapter set limitations to this. The forcing techniques we use to show this are similar to those seen already in Chapter 5.

In addition to those already defined, we will consider the following variants of $\clubsuit(\omega_1)$, where $S \subseteq \omega_1$ is stationary:

($\sim \clubsuit(S)$) There is a sequence $\langle A_\delta : \delta \in S \rangle$ such that for all $\delta \in S$, $A_\delta \subseteq \delta$, $\mathrm{otp}(A_\delta) = \omega$ and $\sup(A_\delta) = \delta$, and if $X \subseteq \omega_1$ is unbounded then there is a $\delta \in S$ such that $A_\delta \setminus X$ is finite.

($\clubsuit^{<\omega}(S)$) There is a sequence $\langle \mathcal{A}_\delta : \delta \in S \rangle$ such that for all $\delta \in S$, $\mathcal{A}_\delta \in [\mathcal{P}(\delta)]^{<\omega}$ with $x \in \mathcal{A}_\delta$ implying $x \subseteq \delta$ and $\sup(x) = \delta$, and if $X \subseteq \omega_1$ is unbounded then there is a $\delta \in S$ such that $x \subseteq X$ for some $x \in \mathcal{A}_\delta$.

$\left(\clubsuit^{[\mathrm{otp}]}(S)\right)$ There is a sequence $\langle A_\delta : \delta \in S \rangle$ such that for all $\delta \in S$, $A_\delta \subseteq \delta$ and $\sup(A_\delta) = \delta$ and $\mathrm{otp}(A_\delta) = \delta$, and if $X \subseteq \omega_1$ is unbounded then there is a $\delta \in S$ such that $A_\delta \subseteq X$.

With the exception of $\clubsuit^{[\mathrm{otp}]}(S)$, all of these principles were considered by Džamonja and Shelah in [9]. There they developed a forcing iteration related to that of Fuchino, Soukup and Shelah in [11], adapted to deal with forcings where the iterands are uncountable but have a strong form of the Knaster property and only contain conditions from the ground model. Aside from the trivial fact that $\clubsuit^{\omega}(S)$ is a direct weakening of both $\sim\clubsuit(S)$ and $\clubsuit^{<\omega}(S)$, and all are weakenings of $\clubsuit(S)$, Džamonja and Shelah proved that no other implications exist between these principles in ZFC. $\clubsuit^{[\mathrm{otp}]}(S)$, which is not addressed in their paper, is trivially stronger than $\clubsuit(S)$; we prove here that it is strictly stronger. We also prove that $\clubsuit(S)$ does not imply $\clubsuit(T)$ for any disjoint stationary sets S and T in the absence of CH. These results are dependent on violating the Continuum Hypothesis, as illustrated by the following simple extension of some well-known theorems:

Theorem 7.0.7. If $2^\omega = \omega_1$, then $\sim\clubsuit(S) \leftrightarrow \clubsuit^{\omega}(S) \leftrightarrow \clubsuit^{<\omega}(S) \leftrightarrow \clubsuit^{[\mathrm{otp}]}(S)$, and all are equivalent to $\clubsuit(S)$.

Proof The statement follows if we prove:

(i) $\sim\clubsuit(S) \leftrightarrow \clubsuit(S)$,

(ii) $\clubsuit^{\omega}(S) \leftrightarrow \clubsuit(S)$,

(iii) $\clubsuit^{<\omega}(S) \leftrightarrow \clubsuit(S)$,

(iv) $\clubsuit^{[\mathrm{otp}]}(S) \leftrightarrow \clubsuit(S)$.

It is clear that (ii) follows from Theorem 6.2.3, and (ii) immediately implies (iii) since $\clubsuit(S) \to \clubsuit^{<\omega}(S) \to \clubsuit^{\omega}(S)$. Also, (iv) follows from the fact that (CH + $\clubsuit(S)) \leftrightarrow \diamondsuit(S)$ and that a witness to $\diamondsuit(S)$ can be trivially modified to produce a witness to $\clubsuit^{[\mathrm{otp}]}(S)$. This leaves only (i), but it is easy to see that if $\langle A_\delta : \delta \in S\rangle$ is a witness to $\sim\clubsuit(S)$ then $\langle \mathcal{A}_\delta : \delta \in S\rangle$ is a witness to $\clubsuit^{\omega}(S)$, where $\mathcal{A}_\delta = \{A^n_\delta : n < \omega\}$ is constructed by letting A^n_δ be A_δ with the n least elements removed. Hence $\sim\clubsuit(S) \to \clubsuit^{\omega}(S) \to \clubsuit(S)$ and the theorem is proved.

□

$\clubsuit(S)$, for a particular $S \subseteq \omega_1$, is stronger than $\clubsuit(\omega_1)$ and differs from $\clubsuit^{[\mathrm{otp}]}(\omega_1)$ in that even when CH holds it does not seem to be equivalent to $\clubsuit(\omega_1)$. We will prove the following two theorems:

Theorem 7.0.8. Con(ZFC) $\to$ Con(ZFC $+ \clubsuit(\omega_1) + \neg(\clubsuit^{[\mathrm{otp}]}(\omega_1))$).

Theorem 7.0.9. Con(ZFC) $\to$ Con(ZFC $+ \neg$CH $+ \clubsuit(S) + \neg\clubsuit(T)$), whenever S and T are disjoint stationary subsets of ω_1.

7.0.1 The forcing $\mathbb{P}_{\omega_2}$

We will prove both Theorems 7.0.8 and 7.0.9 using a single forcing. The argument will be simplified somewhat by the fact that neither of the two $\clubsuit$-principles that we wish to prevent from being true in the generic extension will require us to use an iteration. This is not the case with those considered by Džamonja and Shelah in [9].

Theorem 7.0.10. If S and T are disjoint stationary sets in V and $V \models \diamondsuit(S) + 2^{\omega_1} = \omega_2$, then there is a partial order $\mathbb{P}_{\omega_2}$ such that if G is a $\mathbb{P}_{\omega_2}$-generic filter over V then $V[G] \models \clubsuit(S) + \neg\clubsuit(T) + \neg\clubsuit^{[\mathrm{otp}]}(\omega_1)$.

To prove this we begin by fixing S, T and V for the rest of this chapter to be as in the statement of this theorem. To ensure that our forcing preserves $\clubsuit$, we use an equivalent version of $\diamondsuit$ that can guess initial sections of sequences of the form $\langle\langle b_\alpha, g_\alpha^1, ..., g_\alpha^n\rangle : \alpha < \omega_1\rangle$, where $n \in \omega$, g_α^m is a countable partial function from ω_1 to 2 for all $1 \leq m \leq n$ and $\alpha < \omega_1$, and the $\langle b_\alpha : \alpha < \omega_1\rangle$ form a strictly increasing sequence of countable ordinals. This is defined formally as follows:

Definition 7.0.11. Let $S \subseteq \omega_1$ be stationary. Then $\diamondsuit'(S)$ is the statement: there exists a sequence $\langle \mathcal{B}_\alpha : \alpha \in S\rangle$ such that if $\langle\langle b_\alpha, g_\alpha^1, ..., g_\alpha^n\rangle : \alpha < \omega_1\rangle$ is a sequence where $n < \omega$ and $\langle b_\alpha : \alpha < \omega_1\rangle$ is a strictly increasing sequence of countable ordinals, $g_\alpha^m \in \{(f : X \to 2) : f \neq \emptyset \text{ and } X \subseteq \omega_1 \text{ and } |X| = \omega\}$ for all $\alpha < \omega_1$ and $1 \leq m \leq n$, then the following set is stationary:

$$\{\delta \in S : \mathcal{B}_\delta = \langle\langle b_\alpha, g_\alpha^1, ..., g_\alpha^n\rangle : \alpha < \delta\rangle\}.$$

Notation 7.0.12. Let $\mathfrak{F} = \{(f : X \to 2) : f \neq \emptyset \text{ and } X \subseteq \omega_1 \text{ and } |X| = \omega\}$.

The axiom $\diamondsuit'(S)$ is equivalent to $\diamondsuit(S)$, as we will now show. The techniques are similar to those in [19, II]:

Theorem 7.0.13. $\diamondsuit(S) \leftrightarrow \diamondsuit'(S)$.

Proof We first prove $\diamondsuit(S) \rightarrow \diamondsuit'(S)$. Let $\{I_n : n < \omega\}$ be a family of pairwise disjoint uncountable subsets of ω_1 such that $\bigcup_{n<\omega} I_n = \omega_1$. Fix a sequence of bijections $\langle \rho_n : n < \omega \rangle$, where $\rho_0 : I_0 \longrightarrow \omega_1 \times \omega_1$ and for $1 \leq m < \omega$ we have $\rho_m : I_m \longrightarrow \omega_1 \times \mathfrak{F}$.

Let $\langle D_\alpha : \alpha \in S \rangle$ be a witness to $\diamondsuit(S)$. We will use this to construct our witness to $\diamondsuit'(S)$. So given D_δ, if $\operatorname{ran}(\rho_0 \restriction (D_\delta \cap I_0))$ is an increasing sequence of countable ordinals indexed by δ, then set $\langle b_\alpha : \alpha < \delta \rangle$ equal to this sequence. If not, then let it be an arbitrary sequence of countable ordinals indexed by δ. Similarly, for each $1 \leq m < \omega$, if $\operatorname{ran}(\rho_m \restriction (D_\delta \cap I_m))$ is a sequence (of functions) indexed by δ, then let $\langle g^m_\alpha : \alpha < \delta \rangle$ be equal to this sequence. Otherwise let it be an arbitrary sequence of functions indexed by δ. Then we claim that $\langle \mathcal{B}_\delta = \langle \langle b_\beta, g^1_\beta, g^2_\beta, ... \rangle : \alpha < \delta \rangle : \delta \in S \rangle$ is a witness to $\diamondsuit'(S)$.

To see that this works, let $\langle \mathcal{C}_\alpha = \langle c_\alpha, h^1_\alpha, ..., h^n_\alpha \rangle : \alpha < \omega_1 \rangle$ be a sequence of the type we would like to guess. We let $X = \rho_0^{-1}[\{(\alpha, c_\alpha) : \alpha < \omega_1\}] \cup \bigcup_{1 \leq m \leq n} (\rho_m^{-1}[\{(\alpha, h^m_\alpha) : \alpha < \omega_1\}])$. Now observe that for each $1 \leq m \leq n$ there is a closed unbounded set of δ such that $\operatorname{ran}(\rho_m \restriction (\delta \cap I_m \cap X)) = \{h^m_\alpha : \alpha < \delta\}$. Call such a δ *good* for m. Similarly, there is a closed unbounded set of δ such that ran $(\rho_0 \restriction (\delta \cap I_0 \cap X)) = \{c_\alpha : \alpha < \delta\}$; call such a δ *good* for 0. Then because the intersection of countably many closed unbounded sets is itself closed unbounded, we can find a stationary set of δ that are *good* for all $m \leq n$ and such that $D_\delta = X \cap \delta$. Then it can be seen by our construction of $\mathcal{B}_\delta$ that $\mathcal{B}_\delta$ is equal to $\langle \mathcal{C}_\alpha : \alpha < \delta \rangle$. This completes the proof.

To see that $\diamondsuit'(S) \to \diamondsuit(S)$, let $X \subseteq \omega_1$ be arbitrary, and $\langle c_\alpha : \alpha < \omega_1 \rangle$ its increasing enumeration. The sequence $\langle \mathcal{C}_\alpha = \langle c_\alpha \rangle : \alpha < \omega_1 \rangle$ is of the correct form to be guessed by $\diamondsuit'(S)$ (with $n = 0$ in this case), allowing us to construct a witness to $\diamondsuit(S)$, denoted $\langle D_\delta : \delta \in S \rangle$, by setting $D_\delta = \{c_\alpha : \alpha < \delta \text{ and } (\alpha, \langle c_\alpha \rangle) \in \mathcal{B}_\delta\}$ if $\langle c_\alpha : \alpha < \delta \rangle$ is a sequence of countable ordinals, and setting D_δ to be an arbitrary subset of δ if not.

$\square$

Now, towards the proof of Theorem 7.0.10 we fix a witness to $\diamondsuit'(S)$ in V, denoted $\langle \mathcal{B}_\delta : \delta \in S \rangle$. Our forcing will be rigged so as to preserve a particular $\clubsuit$-sequence, which we define using the sequence $\langle \mathcal{B}_\delta : \delta \in S \rangle$.

So let δ be in S. Given $\mathcal{B}_\delta = \langle \langle b_\alpha, g_\alpha^1, ..., g_\alpha^{n_\delta} \rangle : \alpha < \delta \rangle$, we will define the set A_δ as follows: choose a strictly increasing sequence of ordinals, $\langle \epsilon_l : l < \omega \rangle$, that has order type ω and is cofinal in δ. Given $\langle b_\alpha : \alpha < \delta \rangle$, set $A_\delta = \{b_{\epsilon_l} : l < \omega\}$. It is simple to check that $\langle A_\delta : \delta \in S \rangle$ forms a witness to $\clubsuit(S)$ in V (in fact, it follows from our construction of D_δ in the latter half of the proof of Theorem 7.0.13).

We also make the following definition, to be used when defining our forcing (this is used to ensure that $\langle A_\delta : \delta \in S\rangle$ remains a witness to $\clubsuit(S)$ in the generic extension), for $1 \leq n \leq n_\delta$:

$$F_\delta^n = \bigcup_{l<\omega} g_{\epsilon_l}^n$$

For some values of δ and n, F_δ^n will be a function, while for others it will not. We now define the forcing we will use:

Definition 7.0.14. We will force with the product of ω_2 and a single c.c.c. forcing $\mathbb{Q}$. We define $\mathbb{Q}$ like so:

(i) Fix a continuously increasing sequence $\langle N_i : i \in C\rangle$ of countable elementary submodels of $(\mathcal{H}(\chi), \in)$, where χ is a "sufficiently large" cardinal (for an explanation of this phrase, see Chapter 2) and C is a closed unbounded subset of ω_1 with the property that $N_\alpha \cap \omega_1 = \alpha$ for $\alpha \in C$. We also insist that for all $\alpha \in C$, and all $n < \omega$, if F_α^n (as defined above) is a non-empty well-defined partial function from ω_1 to 2, then $F_\alpha^n \in N_{\min(C\setminus(\alpha+1))}$ for $1 < n < n_\alpha$. There are only finitely many such functions for any $\alpha \in C$ so it is possible to find such a sequence of elementary submodels, by the Löwenheim-Skolem theorem.

(ii) $\mathbb{Q}$ is the set of those countable partial functions $f : X \subseteq \omega_1 \to 2$ in V with the following further properties:

(a) $\mathrm{otp}(\mathrm{dom}(f)) < \omega^\omega$

(b) $f \restriction i \in N_{\min(C\setminus(i+1))}$ for all $i < \omega_1$

(c) If $\delta \in T$ then $\mathrm{dom}(f) \cap \delta < \delta$.

(iii) The ordering of $\mathbb{Q}$ is by extension: $f \leq_{\mathbb{Q}} g$ (f is stronger than g) iff $f \supseteq g$.

Definition 7.0.15. When α is an ordinal we define $\mathbb{P}_\alpha$ as follows:

(i) Set $\mathbb{P}_\alpha = \{p : p$ is a function with $\mathrm{dom}(p) = \alpha$ and $\mathrm{ran}(p) = \mathbb{Q}$ such that $\{\beta < \alpha : p(\beta) \neq 1_{\mathbb{Q}}\}$ is countable$\}$.

(ii) The ordering in $\mathbb{P}_\alpha$ is given by $q \leq_{\mathbb{P}_\alpha} p$ if and only if for all $\beta < \alpha$, $q(\beta) \leq_{\mathbb{Q}} p(\beta)$, and

$$\{\beta < \alpha : p(\beta) \neq 1_{\mathbb{Q}} \text{ and } q(\beta) \neq p(\beta)\} \text{ is finite.}$$

The support of p, written *supp*(p), will as usual denote the set $\{\beta < \alpha : p(\beta) \neq 1_{\mathbb{Q}}\}$. We will also make use of the following notation:

Definition 7.0.16. If q, $p \in \mathbb{P}_\gamma$ and $q \leq_{\mathbb{P}_\gamma} p$ then we write (abusing notation) $q \leq_h p$ if $q \restriction supp(p) = p$, and $q \leq_v p$ if $supp(q) = supp(p)$. The h and v stand for *horizontal* and *vertical* respectively. Of course, it is possible that $q <_{\mathbb{P}_\gamma} p$ can hold while $q \leq_h p$ and $q \leq_v p$ both fail to hold.

We will force with $\mathbb{P}_{\omega_2}$. This type of product forcing is based on that of Fuchino, Shelah and Soukup in [11]. The partial order $\mathbb{Q}$ is proper (see the discussion after

Lemma 7.0.21) and consequently $\mathbb{P}_{\omega_2}$ is also proper, see [11]. However, we will not make explicit use of this fact and will later give a direct proof that ω_1 is not collapsed (see Lemma 7.0.23).

We will need the following two technical lemmas, both of which were used in Chapter 5. They will be used frequently throughout this chapter and the next, so we state them in full generality here:

Lemma 7.0.17 (The Δ-system Lemma)**.** If $\kappa^{<\kappa} = \kappa$, and W is a collection of sets of cardinality less than κ, with $|W| = \kappa^+$, then there is a $U \subseteq W$ with $|U| = \kappa^+$ and a set v such that for any distinct $x, y \in U$ we have $x \cap y = v$. In this case we say that U forms a Δ-system and v is referred to as the *root* of the Δ-system.

Lemma 7.0.18 (Fodor's Lemma)**.** Let λ be a regular cardinal. If $S \subseteq \lambda$ is stationary and $f : S \to \lambda$ is such that $f(\alpha) < \alpha$ for all $\alpha \in S$ (in which case we say that f is a *regressive* function), then there is some $\epsilon < \lambda$ such that $\{\beta < \lambda : f(\beta) = \epsilon\}$ is a stationary subset of λ.

Proof See [15] or [19, II].

$\square$

Lemma 7.0.19. $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-c.c.

Proof Assume otherwise and let $\langle p_\alpha : \alpha < \omega_2 \rangle$ be a sequence enumerating an antichain of size $\aleph_2$. Then the set $\{supp(p_\alpha) : \alpha < \omega_2\}$ is a collection of countable sets. $V \vDash \mathrm{CH}$, so applying the Δ-system Lemma gives us a subsequence $\langle p_{\alpha_\epsilon} : \epsilon < \omega_2 \rangle$ such that for all $i, j < \omega_2$ we have some fixed countable v for which $supp(p_{\alpha_i}) \cap supp(p_{\alpha_j}) = v$. If any two such p_{α_i} and p_{α_j} are identical when restricted to v then they will be compatible elements, by the definition of the forcing. There can only be ω_1 many functions $f : v \to \mathbb{Q}$, because $V \vDash \mathrm{CH}$. So by the pigeonhole principle we can find a cofinal subsequence of our original antichain, $\langle p_{\alpha_\epsilon} : \epsilon < \omega_2 \rangle$, consisting of pairwise compatible conditions, which contradicts its being an antichain.

$\square$

We will also need the following two facts to establish the preservation properties of our forcing:

Lemma 7.0.20. Let $C \subseteq \omega_1$ be a closed unbounded set of limit ordinals. Given an uncountable set $X = \{d_i : i < \omega_1\}$ of countable subsets of ω_1, each with order type $< \omega^\omega$, there is a $\beta \in C$ such that for $\delta \in C \setminus \beta$ there are uncountably many $i < \omega_1$ with $\sup(d_i \cap \delta) < \delta$.

Proof See the proof of Claim 5.2.3, the lemma is proved there.

$\square$

Lemma 7.0.21. $\mathbb{Q}$ has the following properties:

(i) $\mathbb{Q}$ has the Knaster property (i.e. given an uncountable set X of conditions in $\mathbb{Q}$ we can find an uncountable subset $Y \subseteq X$ such that any two conditions in Y are pairwise compatible).

(ii) Any partial order of the form $\prod_{i\in I} Q_i$ where each $Q_i = \mathbb{Q}$ and I is finite, ordered by the product order, has the Knaster property.

(iii) If $\prod_{i\in I} Q_i$ is as in (ii) and $\langle p_\delta : \delta < \omega_1\rangle$ is an uncountable sequence of distinct elements in $\prod_{i\in I} Q_i$, then there is an uncountable subsequence $\langle p_{\delta_{(\beta_\alpha)}} : \alpha < \omega_1\rangle$ such that if $x \subseteq \omega_1$ has order type ω then $f_{x,i} = \bigcup_{\alpha\in x} p_{\delta_{(\beta_\alpha)}}(i)$ is a countable partial function from ω_1 to 2 with $\mathrm{otp}(\mathrm{dom}(f_{x,i})) < \omega^\omega$, for each $i \in I$.

Proof It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), so we concentrate on proving (iii).

Given $\langle p_\delta : \delta < \omega_1\rangle$, let $d_\delta = \bigcup_{i\in I} \mathrm{dom}(p_\delta(i))$, for each $\delta \in \omega_1$. Then each d_δ is a subset of ω_1 of order type less than ω^ω, because it is a finite union of such sets. Let ξ be the least ordinal in C (where C is as in Definition 7.0.14) such that $C \setminus \xi$ is a final section of C of the type asserted to exist in Lemma 7.0.20. We define a sequence $\langle \delta_\beta, \epsilon_\beta : \beta \in C \setminus \xi\rangle$ by induction. Assume δ_j, ϵ_j are defined for $j \in \beta \cap (C \setminus \xi)$. Let δ_β be the least countable ordinal such that $\delta_\beta \neq \delta_j$ for any $j \in \beta \cap (C \setminus \xi)$, and $\sup(d_{\delta_\beta} \cap \beta) < \beta$. By the previous lemma we know we can carry out this induction, and it is well-defined even when $\beta = \min(C)$. We then let $\epsilon_\beta = \sup(d_{\delta_\beta} \cap \beta)$. Then the function $h : C \setminus \xi \to \omega_1$ given by $h(\beta) = \epsilon_\beta$ is regressive. By Fodor's lemma there is some $\epsilon < \omega_1$ such that for a stationary subset $S^1 \subseteq C \setminus \xi$, we have $\beta \in S^1 \Rightarrow \epsilon_\beta = \epsilon$.

For all $\beta \in S^1$ and $i \in I$ we must have $p_{\delta_\beta}(i) \restriction \epsilon \in N_{\min(C\setminus(\epsilon+1))}$, by the definition of $\mathbb{Q}$, so because I is finite and $N_{\min(C\setminus(\epsilon+1))}$ is countable there are only countably many possibilities for the sequence $\langle p_{\delta_\beta}(i) \restriction \epsilon : i \in I\rangle$ whenever β is in S^1. There are also only countably many possibilities for $\langle \mathrm{otp}(p_{\delta_\beta}(i)) : i \in I\rangle$, whenever $\beta \in S^1$, since there are only countably many order types $< \omega^\omega$. Hence we can find a stationary set $S^2 \subseteq S^1$ with $\alpha, \beta \in S^2$ implying $\langle p_{\delta_\alpha}(i) \restriction \epsilon : i \in I\rangle = \langle p_{\delta_\beta}(i) \restriction \epsilon : i \in I\rangle$ and $\langle \mathrm{otp}(p_{\delta_\alpha}(i)) : i \in I\rangle = \langle \mathrm{otp}(p_{\delta_\beta}(i)) : i \in I\rangle$; if we couldn't find such a set then we would have a partition of S^1 into ω many non-stationary sets, which is contradictory.

Now we will fix S^2 to be as above and define the required sequence $\langle \beta_\alpha : \alpha < \omega_1\rangle$ by induction. Let β_0 be an arbitrary member of S^2. Let $\alpha < \omega_1$ and assume $\langle \beta_j : j < \alpha\rangle$ is already defined and is such that for $j < k < \alpha$, $\sup(d_{\delta_{(\beta_j)}}) < \inf(d_{\delta_{(\beta_k)}} \setminus \epsilon)$ and $\beta_j, \beta_k \in S^2$.

Let $J = \bigcup_{j<\alpha} d_{\delta_{(\beta_j)}}$. Then $\sup(J)$ is a countable ordinal. So if $\beta' \in S^2$ is such that $\sup(J) < \beta'$ then we know $h(\beta') = \sup(d_{\delta_{\beta'}} \cap \beta') = \epsilon$ and consequently $d_{\delta_{\beta'}} \cap \sup(J) \setminus \epsilon = \emptyset$. Thus we choose β_α to be the least member of S^2 greater than $\sup(J)$ that has not already been chosen.

To see that this works, and that $\langle p_{\delta_{(\beta_\alpha)}} : \alpha < \omega_1\rangle$ is a subsequence of the required kind, we first remark that for any two $j < k < \omega_1$, $p_{\delta_{(\beta_j)}}$ and $p_{\delta_{(\beta_k)}}$ are pairwise compatible. To see this, assume not. Then there is some $m \in I$ and $\gamma < \omega_1$ such that $(p_{\delta_{(\beta_j)}}(m))(\gamma) \neq (p_{\delta_{(\beta_k)}}(m))(\gamma)$. If $\gamma < \epsilon$, this contradicts the fact that β_j and β_k are both in S^2 and hence that $\langle p_{\delta_{(\beta_j)}}(i) \restriction \epsilon : i \in I\rangle = \langle p_{\delta_{(\beta_k}}(i) \restriction \epsilon :$

$i \in I\rangle$. So $\gamma > \epsilon$, but then we must have that $\gamma \in d_{\delta_{(\beta_j)}} \setminus \epsilon$, in which case k was chosen so that $\gamma \notin d_{\delta_{(\beta_k)}}$. This is a contradiction, so all such $p_{\delta_{(\beta_j)}}$ and $p_{\delta_{(\beta_k)}}$ are pairwise compatible.

Now to complete the proof of the lemma, let $x \subseteq \omega_1$ have order type ω. We know that $f_{x,i} = \bigcup_{\alpha \in x} p_{\delta_{(\beta_\alpha)}}(i)$ is a function for all $i \in I$, otherwise pairwise compatibility would not hold. We now just need to check that $\text{otp}(\text{dom}(f_{x,i})) < \omega^\omega$, for each $i \in I$. So let $m \in I$, and consider $f_{x,m}$. For all $\alpha \in x$ we have $\beta_\alpha \in S^2$, so $p_{\delta_{(\beta_\alpha)}}(m)$ has a fixed order type, call it ρ. We also know that there is a partition of $\sup(x) \setminus \epsilon$ into ω many intervals such that the intersection of $\text{dom}(f_{x,m})$ with each interval has order type ρ, due to the way we defined $\langle p_{\delta_{(\beta_\alpha)}} : \alpha < \omega_1\rangle$ so that $\sup(d_{\delta_{(\beta_j)}}) < \inf(d_{\delta_{(\beta_k)}}) \setminus \epsilon$ whenever $j < k < \omega_1$. Consequently, $\text{dom}(f_{x,m})$ has order type $< \rho.\omega$. Since $\rho < \omega^\omega$ this implies that $\text{otp}(\text{dom}(f_{x,m})) < \omega^\omega$, because ω^ω is closed under ordinal multiplication.

Therefore $\langle p_{\delta_{(\beta_\alpha)}} : \alpha < \omega_1\rangle$ has the required properties and parts (i), (ii) and (iii) of the lemma are proved.

$\square$

We write $V^{\mathbb{P}_{\omega_2}} \vDash \phi$ if it is the case that $1_{\mathbb{P}_{\omega_2}} \Vdash \phi$. The following lemma establishes one half of the proof of Theorem 7.0.10. The other half of the proof is given in Section 7.0.2.

Lemma 7.0.22. $V^{\mathbb{P}_{\omega_2}} \models \neg\clubsuit^{[\text{otp}]}(\omega_1) + \neg\clubsuit(T)$.

Proof We begin by proving $V^{\mathbb{P}_{\omega_2}} \models \neg\clubsuit^{[\mathrm{otp}]}(\omega_1)$. Assume otherwise. If we fix a sequence of functions $\langle h_\delta : \delta \in Lim(\omega_1)\rangle$ in V such that $h_\delta : [\delta, \delta+\omega) \to \delta$ is a bijection, any witness to $\clubsuit^{[\mathrm{otp}]}(\omega_1)$ in $V[G]$ can be coded by an unbounded subset of ω_1 in $V[G]$. Let $\dot{\tau}$ be a name for a set that codes a witness to $\clubsuit^{[\mathrm{otp}]}(\omega_1)$ in $V[G]$, $\dot{f}$ a name for its increasing enumeration, and let $p \in \mathbb{P}_{\omega_2}$ force this. We will write $\langle B_\delta : \delta \in Lim(\omega_1)\rangle$ to denote this witness to $\clubsuit^{[\mathrm{otp}]}(\omega_1)$ in $V[G]$ and $p' \Vdash$ "$\gamma \in B_\delta$" to mean $p' \Vdash$ "$\beta \in \dot{\tau}$" where β is in $[\delta, \delta+\omega)$ and $h_\delta(\beta) = \gamma$.

Then there is an $\alpha < \omega_2$ such that $supp(p) \subseteq \alpha$ and for every $\gamma < \omega_1$ and every $p' \in \mathbb{P}_{\omega_2}$ with $p' \leq p$ and $supp(p') \subseteq \alpha$ there is some $r \in \mathbb{P}_{\omega_2}$ with $r \leq p'$ and $supp(r) \subseteq \alpha$ such that $r \Vdash$ "$\dot{f}(\gamma) = \epsilon$" for some $\epsilon < \omega_1$. To see this, let $\zeta < \omega_2$, then there are only ω_1 many conditions below p in $\mathbb{P}_{\omega_2}$ with their support contained in ζ. Each of these conditions has a smaller condition determining the value of $\dot{f}(\gamma)$, for each $\gamma < \omega_1$, so let the function π_γ be defined as follows:

If $\zeta < \omega_2$ and $\gamma < \omega_1$ then:

$$\pi_\gamma(\zeta) = \min\{\rho < \omega_2 : \forall p' \leq p\,(supp(p') \subseteq \zeta \Rightarrow$$

$$\exists r \leq p'(supp(r) \subseteq \rho \text{ and } \exists \epsilon < \omega_1 (r \Vdash \text{“}\dot{f}(\gamma) = \epsilon\text{”})))\}.$$

The function π_γ is closed on a closed unbounded set of ordinals in ω_2, hence there is a closed unbounded set of ordinals less than ω_2 on which π_γ is closed for all $\gamma < \omega_1$. Let α be in this closed unbounded set and be such that $supp(p) \subseteq \alpha$, then α is as required.

Fix such an α. Let G be a $\mathbb{P}_{\omega_2}$-generic filter over V containing p and write $G^\alpha = \bigcup\{r(\alpha) : r \in G\}$. Then G^α is a function from ω_1 to 2, so let $X^\alpha = \{\beta < \omega_1 : G^\alpha(\beta) = 1\}$. We will show that for all $\delta \in Lim(\omega_1) \setminus \omega^\omega$, the set of $p' \leq p$ that force $A_\delta \not\subseteq X^\alpha$ is dense below p in $\mathbb{P}_{\omega_2}$, completing the proof.

So let $p' \leq p$ be a condition in $\mathbb{P}_{\omega_2}$ and δ be in $Lim(\omega_1) \setminus \omega^\omega$. Then the set:

$$\{\epsilon < \delta : \text{ there exists a } q \leq_{\mathbb{P}_{\omega_2}} (p' \restriction \alpha) \cup (1_{\mathbb{P}_{\omega_2}} \restriction (\omega_2 \setminus \alpha)) \text{ with}$$

$$supp(q) \subseteq \alpha \text{ and } q \Vdash \text{“}\epsilon \in A_\delta\text{”}\}$$

must have order type greater than or equal to ω^ω. Hence we can find a $\beta < \delta$ which is in this set and which is not in $\mathrm{dom}(p'(\alpha))$, because $\mathrm{dom}(p'(\alpha))$ has order type less than ω^ω by the definition of $\mathbb{Q}$. Let q^β be the condition witnessing the fact that β is in this set, so that $supp(q^\beta) \subseteq \alpha$. Then setting $q^+ = (q^\beta \restriction \alpha) \cup (p'(\alpha) \cup (\beta, 0)) \cup (p' \restriction (\omega_2 \setminus \alpha + 1))$ gives us a condition $q^+ \leq_{\mathbb{P}_{\omega_2}} p'$ that forces $\beta \in A_\delta$ and $\beta \notin X^\alpha$, hence forces $A_\delta \not\subseteq X^\alpha$. But p' was an arbitrary condition below p, so for all $\delta \in Lim(\omega_1) \setminus \omega^\omega$ the set of conditions forcing $A_\delta \not\subseteq X^\alpha$ is dense below p. This contradicts the fact that p forces $\langle A_\delta : \delta \in Lim(\omega_1)\rangle$ to be a witness to $\clubsuit^{[\mathrm{otp}]}(\omega_1)$.

A similar argument establishes that $V^{\mathbb{P}_{\omega_2}} \models \neg\clubsuit(T)$. Having assumed there is a $p \in \mathbb{P}_{\omega_2}$ forcing $\langle A_\delta : \delta \in T\rangle$ to be a witness to $\clubsuit(T)$ (coded by $\dot{\tau}$, a name for an uncountable subset of ω_1), we can find an $\alpha < \omega_2$ as before. Given a $p' \leq p$ we can then find a $\beta < \delta$ for all $\delta \in T$ such that $\beta \notin \mathrm{dom}(p'(\alpha))$ and then we can construct a q^+ forcing $A_\delta \not\subseteq X^\alpha$ exactly as before. This completes the proof of the lemma.

$\square$

7.0.2 Preservation properties of our forcing $\mathbb{P}_{\omega_2}$

The proof of the next theorem makes use of an inductive argument that will be crucial in determining several properties of the generic extension:

Theorem 7.0.23. Forcing with $\mathbb{P}_{\omega_2}$ does not collapse ω_1.

Proof Assume that it does, and that $\dot{f}$ is a $\mathbb{P}_{\omega_2}$-name and p forces $\dot{f} : (\omega_1)^V \to \omega$ to be an injective function. Then we define $\langle p_\alpha, q_\alpha, u_\alpha : \alpha < \omega_1 \rangle$ by induction:

(i) Let $p_0 = q_0 = p$ and $u_0 = \emptyset$.

(ii) Let $\alpha = \beta + 1$, and assume that p_β, q_β and u_β are defined. Pick q_α to be a condition such that $q_\alpha \leq p_\beta$ and for some $n < \omega$ we have $q_\alpha \Vdash$ "$\dot{f}(\alpha) = n$". We then set $p_\alpha = p_\beta \restriction supp(p_\beta) \cup q_\alpha \restriction (\omega_2 \setminus supp(p_\beta))$ and let $u_\alpha = \{\delta \in supp(p_\alpha) : q_\alpha(\delta) \neq p_\alpha(\delta)\}$. So u_α is a finite set. We also get that $q_\alpha \leq_v p_\alpha \leq_h p_\beta$. In fact, p_α is the unique condition satisfying this inequality.

(iii) For α limit, begin by defining $p'_\alpha = \bigcup_{\beta<\alpha} p_\beta$. This is a condition because p_i and p_j for $i, j < \alpha$ are defined so as to agree on all $\gamma \in supp(p_i) \cap supp(p_j)$.

Again, find a $q_\alpha \leq p'_\alpha$ such that $q_\alpha \Vdash$ "$\dot{f}(\alpha) = n$" for $n < \omega$, and set $p_\alpha = p'_\alpha \restriction supp(p'_\alpha) \cup q_\alpha \restriction (\omega_2 \setminus supp(p'_\alpha))$. Let $u_\alpha = \{\delta \in supp(p_\alpha) : q_\alpha(\delta) \neq p_\alpha(\delta)\}$.

For all α, u_α is a finite subset of $\bigcup_{\beta<\alpha} supp(p_\beta)$. By the Δ-system lemma, 7.0.17, we can find an uncountable set $D \subseteq \omega_1$ and $I \in [\omega_2]^{<\omega}$ such that $\{u_\alpha : \alpha \in D\}$ is a Δ-system with root I. Since I is finite, by Lemma 7.0.21 (ii) we can find a $D' \subseteq D$ such that $\alpha, \beta \in D'$ implies $q_\alpha \restriction I$ and $q_\beta \restriction I$ are compatible conditions in the partial order $\prod_{i \in I} Q_i$ where each $Q_i = \mathbb{Q}$. We claim that for such α, $\beta \in D'$ we must also have that q_α and q_β are compatible conditions in $\mathbb{P}_{\omega_2}$.

To see this, assume without loss of generality that $\alpha < \beta$. Then for $\gamma \in supp(q_\beta) \setminus I$, if $\gamma \in u_\beta$ then $\gamma \notin u_\alpha$ because $u_\beta \cap u_\alpha = I$, so $q_\alpha(\gamma) = p_\alpha(\gamma)$ by the definition of u_α, and either $p_\alpha(\gamma) = p_\beta(\gamma)$ or $p_\alpha(\gamma) = 1_\mathbb{Q}$, which means $q_\alpha(\gamma)$ is compatible with $q_\beta(\gamma)$ in $\mathbb{Q}$. If $\gamma \notin u_\beta$ then $q_\beta(\gamma) = p_\beta(\gamma)$, which means $q_\alpha(\gamma)$ is compatible with $q_\beta(\gamma)$ in $\mathbb{Q}$. Either way, the condition $q^{(\alpha,\beta)} = (p_\beta \restriction \{\gamma < \omega_2 : \gamma \notin u_\alpha \cup u_\beta\}) \cup (q_\alpha \restriction (u_\alpha \setminus I)) \cup (q_\beta \restriction (u_\beta \setminus I)) \cup (\{q_\beta(\gamma) \cup q_\alpha(\gamma) : \gamma \in I\})$ is therefore a lower bound to both q_α and q_β in $\mathbb{P}_{\omega_2}$. And $q^{(\alpha,\beta)} \leq p$ because $\{\gamma \in supp(p) : q^{(\alpha,\beta)}(\gamma) \neq p(\gamma)\}$ is a subset of $u_\alpha \cup u_\beta$ and is therefore finite.

To complete the proof, observe that there must be some $n' < \omega$ such that $\{\alpha \in D' : q_\alpha \Vdash$ "$\dot{f}(\alpha) = n'$"$\}$ is an uncountable set, by the pigeonhole principle. But any two ordinals in this set, α and β, will be such that $q^{(\alpha,\beta)}$ is an upper bound to both q_α and q_β, so $q^{(\alpha,\beta)}$ will force $\dot{f}(\alpha) = \dot{f}(\beta)$ and thus force $\dot{f}$ to not be an injective function, while itself being a stronger condition than p, which forces the opposite. This is a contradiction.

$\square$

We are now ready to prove our main theorem:

Lemma 7.0.24. Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{\tau}$ be a name such that $p \Vdash$ "$\dot{\tau} \in [\omega_1]^{\omega_1}$" and let $\langle A_\delta : \delta \in S \rangle$ be our previously fixed witness to $\clubsuit(S)$ in V (see the discussion after Theorem 7.0.13). Then the set of q such that for some $\delta \in S$, $q \Vdash$ "$A_\delta \subseteq \dot{\tau}$" is dense below p.

Proof Initially the proof mimics that of Theorem 7.0.23. We define two sequences of conditions in $\mathbb{P}_{\omega_2}$, $\langle p_\alpha : \alpha < \omega_1 \rangle$ and $\langle q_\alpha : \alpha < \omega_1 \rangle$, by induction so that for all $\alpha < \omega_1$, $q_\alpha \leq p_\alpha \leq p$. We also inductively define u_α for all $\alpha < \omega_1$ and ζ_α for $1 \leq \alpha < \omega_1$:

(i) Begin by setting $q_0 = p_0 = p$, and $u_0 = \emptyset$.

(ii) We handle the successor case first. Let $\alpha = \beta + 1$, and assume that p_β and q_β are defined, as are u_β and ζ_β (ζ is not defined if $\beta = 0$, but this will not cause problems). Pick q_α to be an arbitrary condition with $q_\alpha \leq p_\beta$ and such that for some ζ greater than:

$$\max\{\alpha, \sup(\{\delta : \exists i < \alpha\, (q_i \Vdash \text{“}\delta \in \dot{\tau}\text{”})\})\},$$

we have $q_\alpha \Vdash$ "$\zeta \in \dot{\tau}$". Let ζ_α be the least such ζ, having already chosen q_α.

(We can always find such a q_α, unless some q_i forces ω_1 many ordinals into $\dot{\tau}$, in which case the lemma is trivial, so we assume otherwise.) We then set $p_\alpha = p_\beta \restriction supp(p_\beta) \cup q_\alpha \restriction (\omega_2 \setminus supp(p_\beta))$, and let $u_\alpha = \{\delta \in supp(p_\alpha) : q_\alpha(\delta) \neq p_\alpha(\delta)\}$. So u_α is a finite set, by the definition of the forcing, and p_α is the unique condition such that $q_\alpha \leq_v p_\alpha \leq_h p_\beta$.

(iii) For α a limit ordinal, begin by defining $p'_\alpha = \bigcup_{\beta<\alpha} p_\beta$. This is a condition because p_i and p_j for $i, j < \alpha$ are defined so as to agree on $supp(p_i) \cap supp(p_j)$, and α is countable. Find a $q_\alpha \leq p'_\alpha$ such that $q_\alpha \Vdash$ "$\zeta \in \dot{\tau}$" for $\zeta > \max\{\alpha, \sup(\{\delta : \exists i < \alpha\, (q_i \Vdash \text{"}\delta \in \dot{\tau}\text{"})\})\}$, set $p_\alpha = p'_\alpha \restriction supp(p'_\alpha) \cup q_\alpha \restriction (\omega_2 \setminus supp(p'_\alpha))$, and let ζ_α be the least such ζ, having chosen q_α. Again, let $u_\alpha = \{\delta \in supp(p_\alpha) : q_\alpha(\delta) \neq p_\alpha(\delta)\}$.

For all α, u_α is a finite subset of $\bigcup_{\beta<\alpha} supp(p_\beta)$. By the Δ-system Lemma, 7.0.17, we can find an uncountable set $D \subseteq \omega_1$ and $I \in [\omega_2]^{<\omega}$ such that $\{u_\alpha : \alpha \in D\}$ is a Δ-system with root I. We now need to find a further uncountable set $D' \subseteq D$; we do this by induction, using $\langle d_\alpha : \alpha < \omega_1\rangle$ to denote the increasing enumeration of D':

(i) Let $d_0 = \min(D)$.

(ii) Assume d_β is defined for all $\beta < \alpha$. We define d_α to be the least ordinal in D such that $u_{d_\alpha} \cap \bigcup_{\beta<\alpha} supp(q_{d_\beta}) = I$. To see that this is well-defined, let

$U = \bigcup_{\beta<\alpha} supp(q_{d_\beta})$ and note that this is a countable set. For $\gamma \in U \setminus I$ there is at most one $j \in D$ with $\gamma \in u_j$ (by the definition of a Δ-system), so because U is countable and D is uncountable there must exist an ordinal $d_\alpha \in D$ with $u_{d_\alpha} \cap \bigcup_{\beta<\alpha} supp(q_{d_\beta}) = I$ as required. Thus d_α is well-defined and our induction is complete.

This ensures that for $\alpha, \beta \in D'$ with $\beta < \alpha$, we have that for $\gamma \in supp(q_\beta) \setminus I$ we have $q_\alpha(\gamma) = q_\beta(\gamma) = p_\alpha(\gamma) = p_\beta(\gamma)$.

Since I is finite, by Lemma 7.0.21 (iii) we can find an $E \subseteq D'$ with $\langle e_\alpha : \alpha < \omega_1 \rangle$ its increasing enumeration such that $\langle q_{e_\alpha} \restriction I : \alpha < \omega_1 \rangle$ has the properties stated in that lemma. So if $x \subseteq \omega_1$ has order type ω then $\bigcup_{\alpha \in x} q_{e_\alpha}(i)$ is a partial function with a domain having order type $< \omega^\omega$ for each $i \in I$.

So let $x \subseteq E$ be a set of order type ω in V. Set $q_x = \bigcup_{a \in x} q_a \restriction (\omega_2 \setminus I) \cup \langle \bigcup_{a \in x} q_a(\epsilon) : \epsilon \in I \rangle$. Whether q_x is a condition in our forcing or not will depend on whether $q_x(\epsilon)$ satisfies the requirements (ii)(b) and (ii)(c) in the definition of $\mathbb{Q}$ (see Definition 7.0.14), when ϵ is in I. But first we need to observe that *if* $q_x(\epsilon)$ for $\epsilon \in I$ satisfies these requirements and hence is a condition then it will be a stronger condition than each q_a for $a \in x$. This is easy to see from the way we have defined E; for any $\alpha \in x$ it is the case that $\{\gamma \in supp(q_\alpha) : q_\alpha(\gamma) \neq q_x(\gamma)\}$ is finite.

However, q_x will not in general be a condition. We need to use our original fixed

$\diamondsuit'$ sequence to find such q_x that *are* conditions and which furthermore establish that our fixed sequence $\langle A_\delta : \delta \in S\rangle$ remains a witness to $\clubsuit(S)$ in $V[G]$.

Let $\langle \xi_m : m < n\rangle$ enumerate I and set $\mathcal{D} = \langle\langle \zeta_{e_\alpha}, q_{e_\alpha}(\xi_0), ..., q_{e_\alpha}(\xi_{n-1})\rangle : \alpha < \omega_1\rangle$. Recall that our $\diamondsuit'(S)$ sequence was cooked up to guess initial sections of sequences such as $\mathcal{D}$. Let $\delta \in S$ be such that $\mathcal{B}_\delta = \mathcal{D} \restriction \delta$, $\mathrm{otp}\{e_\alpha : \alpha < \delta\} = \delta$ and $\sup\{\zeta_{e_\alpha} : \alpha < \delta\} = \delta$ and for all $m < n$, $\sup\left(\{\mathrm{dom}(q_{e_\alpha}(\xi_m)) : \alpha < \delta\}\right) = \delta$. These latter three requirements each hold for a closed unbounded subset of ω_1, so it is possible to find such a $\delta \in S$. Then the set $A' = \{e_\alpha < \delta : \zeta_{e_\alpha} \in A_\delta\}$ is a subset of E, with order type ω, and our forcing was defined in such a way that $F_\delta^{\xi_m} = \bigcup_{i\in A'} q_i(\xi_m)$ is in $N_{\min(C\setminus(\delta+1))}$ for all $m < n$. Hence the set $F_\delta^{\xi_m}$ is a condition in $\mathbb{Q}$ for all $m < n$, and so $q^+ = p_\delta \restriction (\omega_2 \setminus \{\xi_m : m < n\}) \cup \{(\xi_m, F_\delta^{\xi_m}) : m < n\}$ is a condition extending all members of $\{q_j : j \in A'\}$. And $\mathcal{D}$ was defined in such a way that $q^+ \Vdash$ "$A_\delta \subseteq \dot{\tau}$".

Note that p was arbitrary and $\{\gamma \in \mathit{supp}(p) : p(\gamma) \neq q_x(\gamma)\} \subseteq I$, so $q^+ < p$ and the theorem is proved.

$\square$

The final thing we need to prove is that S remains stationary after forcing with $\mathbb{P}_{\omega_2}$.

Corollary 7.0.25. Let $p \in \mathbb{P}_{\omega_2}$ and $\dot{\tau}$ be a name such that $p \Vdash$ "$\dot{\tau} \in [\omega_1]^{\omega_1}$ is closed unbounded". Then the set of q such that for some $\delta \in S$, $q \Vdash$ "$\delta \in \dot{\tau}$" is dense below p.

Proof This is a simple extension to the proof of Lemma 8.0.17. Again we define an uncountable set of conditions $\langle q_\alpha : \alpha < \omega_1 \rangle$ forcing ordinals ζ_α into $\dot{\tau}$ and we find an uncountable E as before. By the methods used in the earlier proof, we then find a $q^+ < p$ such that for some $\delta \in S$, $q^+ \Vdash$ "$A_\delta \subseteq \dot{\tau}$". Then because $\sup(A_\delta) = \delta$ and q^+ forces $\dot{\tau}$ to be a closed subset of ω_1 (since q^+ extends p) we must have that $q^+ \Vdash$ "$\delta \in \dot{\tau}$". Since p was arbitrary and δ is in S, we obtain the required result.

$\square$

We could also infer the fact that S remains stationary from Lemma 6.1.2, which effectively states that $\clubsuit(S)$ is contradictory if S is not a stationary set. This completes the proof that:

$$\mathrm{Con(ZFC)} \rightarrow \mathrm{Con}(\mathrm{ZFC} + \neg\mathrm{CH} + \clubsuit(S) + \neg\clubsuit(T) + \neg\clubsuit^{[\mathrm{otp}]}(\omega_1))^1.$$

7.1 Consistency results using iterated forcing

It remains for us to mention those consistency results that were obtained by Džamonja and Shelah in [9] using iterated forcing; these are summarised in the following theorem.

[1]S. Fuchino and L. Soukup have improved on this result since the time of writing, proving that there can consistently be further variants of $\clubsuit$ which sit strictly between $\clubsuit$ and $\clubsuit^{[\mathrm{otp}]}(\omega_1)$. See [12].

Theorem 7.1.1 (Džamonja, Shelah)**.**

(a) $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \sim \clubsuit(\omega_1) + \neg\clubsuit(\omega_1))$,

(b) $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \clubsuit^{<\omega}(\omega_1) + \neg(\sim \clubsuit(\omega_1)))$,

(c) $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \sim \clubsuit(\omega_1) + \neg\clubsuit^{<\omega}(\omega_1))$.

Unlike our approach in Section 7.0.2, Džamonja and Shelah used a forcing iteration of length ω_2 to prove Theorem 7.1.1 rather than a product. This seems to be necessary.

Chapter 8

Consistency Results on ♣ and Invariance, II

In this chapter we generalise the consistency results of Chapter 7. We can adapt the forcing technique we used there so as to apply to variations on $\clubsuit(\kappa^+)$, for any infinite regular κ. In the general case, however, we encounter limitations that do not occur in the case where $\kappa = \omega$; these are discussed at the end of the present chapter.

Specifically, we are able to prove the following analogue of Theorem 7.0.8:

Definition 8.0.2. Let $S \subseteq \lambda$ be a stationary subset of a regular cardinal. Then $\clubsuit^{[\mathrm{otp}]}(S)$ asserts the existence of a sequence $\langle A_\delta : \delta \in S \rangle$ such that for all $\delta \in S$, $A_\delta \subseteq \delta$ and $\sup(A_\delta) = \delta$ and $\mathrm{otp}(A_\delta) = \delta$, and if $X \subseteq \lambda$ is unbounded then there is a $\delta \in S$ such that $A_\delta \subseteq X$.

Theorem 8.0.3. Let κ be an infinite regular cardinal. If S and T are disjoint stationary subsets of $S^{\kappa^+}_\kappa$ in V and $V \models \diamondsuit(S) + \mathrm{GCH}$, then there is a partial order $\mathbb{P}_{\kappa^{++}}$ such that if G is a $\mathbb{P}_{\kappa^{++}}$-generic filter over V then $V[G] \models \clubsuit(S) + \neg\clubsuit(T) + \neg\clubsuit^{[\mathrm{otp}]}(\kappa^+)$.

As before, we fix S, T and V for the rest of the proof to be two disjoint stationary subsets of κ^+ and a model of ZFC + $\diamondsuit(S)$ + GCH respectively. We will again need to use an alternative but equivalent version of $\diamondsuit(S)$:

Definition 8.0.4. $\diamondsuit'(S)$ is the statement that there exists a sequence $\langle \mathcal{B}_\alpha : \alpha \in S\rangle$ such that if $\langle\langle b_\alpha, g^1_\alpha, ..., g^\rho_\alpha\rangle : \alpha < \kappa^+\rangle$ is a sequence where $\rho < \kappa$ is an ordinal and $\langle b_\alpha : \alpha < \kappa^+\rangle$ is a strictly increasing sequence of ordinals, $g^\nu_\alpha \in \{(f : X \to 2) : f \text{ is non-empty and } X \subseteq \kappa^+ \text{ and } |X| = \kappa\}$ for all $\alpha < \kappa^+$ and $1 \leq \nu \leq \rho$, then the following set is stationary:

$$\{\delta \in S : \mathcal{B}_\delta = \langle\langle b_\alpha, g^1_\alpha, ..., g^\rho_\alpha\rangle : \alpha < \delta\rangle\}.$$

Notation 8.0.5. Let $\mathfrak{F} = \{(f : X \to 2) : f \neq \emptyset \text{ and } X \subseteq \kappa^+ \text{ and } |X| = \kappa\}$.

The axiom $\diamondsuit'(S)$ is equivalent to $\diamondsuit(S)$.

Theorem 8.0.6. $\diamondsuit(S) \leftrightarrow \diamondsuit'(S)$.

Proof The proof follows that of Theorem 7.0.13, with only minor modifications needed, so we will not reproduce it here.

$\square$

As before, let $\langle \mathcal{B}_\delta : \delta \in S \rangle$ be a fixed witness to $\diamondsuit'(S)$ in V. We define a witness to $\clubsuit(S)$ that will preserved by our forcing:

Definition 8.0.7. Let δ be in S. Given $\mathcal{B}_\delta = \langle \langle b_\alpha, g^1_\alpha, ..., g^{\rho_\delta}_\alpha \rangle : \alpha < \delta \rangle$, we will define the set A_δ as follows: choose a strictly increasing sequence of ordinals, $\langle \epsilon_l : l < \kappa \rangle$, that is cofinal in δ. Then set $A_\delta = \{ b_{\epsilon_l} : l < \kappa \}$, unless this does not give us a set of ordinals cofinal in δ, in which case we choose one arbitrarily. This defines the sequence $\langle A_\delta : \delta \in S \rangle$.

Lemma 8.0.8. $\langle A_\delta : \delta \in S \rangle$ forms a witness to $\clubsuit(S)$ in V.

Proof The proof is as in Chapter 7, following Lemma 7.0.13.

$\square$

Definition 8.0.9. Let $F^\rho_\delta = \bigcup_{l<\kappa} g^\rho_{\epsilon_l}$ for all $\delta \in S$ and $\rho < \kappa$.

It is again not important that for some values of δ and ρ, F^ρ_δ will not be a function, or will be empty. We can now define the forcing we will use. The similarities with the forcing defined in Chapter 7 are manifest, but nonetheless we will give the definition in full, due to the central role it will play throughout the present chapter:

Definition 8.0.10. We force with the product of κ^{++} and a single forcing $\mathbb{Q}$. We define $\mathbb{Q}$ like so:

(i) Fix a continuously increasing sequence $\langle N_i : i \in C\rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in)$, where χ is a "sufficiently large" cardinal and C is a closed unbounded subset of κ^+ with the property that $N_\alpha \cap \kappa^+ = \alpha$ and $|N_\alpha| = \kappa$ for all $\alpha \in C$. Also, if $\alpha \in C$ is such that $\sup(C \cap \alpha) \neq \alpha$ and $\{f_i : i < \gamma < \kappa\}$ is a set of functions from κ^+ to 2, each of which is in N_α, then $\bigcup\{f_i : i < \gamma\}$ is in N_α. That is, N_α for such α is closed under unions of less than κ many functions from κ^+ to 2. We also insist that for all $\alpha \in C$, and all $\rho < \kappa$, if F^ρ_α is a non-empty well-defined partial function from κ^+ to 2, then $F^\rho_\alpha \in N_{\min(C \setminus (\alpha+1))}$ for $1 < \rho < \rho_\alpha$.

(ii) $\mathbb{Q}$ is the set of those functions $f : X \subseteq \kappa^+ \to 2$ in V, where $|X| = \kappa$, with the following further properties:

(a) $\mathrm{otp}(\mathrm{dom}(f)) < \kappa^\kappa$

(b) $f \restriction i \in N_{\min(C \setminus (i+1))}$ for all $i < \kappa^+$

(c) If $\delta \in T$ then $\mathrm{dom}(f) \cap \delta < \delta$.

(iii) The ordering of $\mathbb{Q}$ is by extension: $f \leq_{\mathbb{Q}} g$ iff $f \supseteq g$.

Definition 8.0.11. We define $\mathbb{P}_{\kappa^{++}}$ as follows:

(i) Set $\mathbb{P}_{\kappa^{++}} = \{p : p$ is a function with $\mathrm{dom}(p) = \kappa^{++}$ and $\mathrm{ran}(p) = \mathbb{Q}$ such that $|\{\beta < \kappa^{++} : p(\beta) \neq 1_{\mathbb{Q}}\}| = \kappa\}$.

(ii) The ordering in $\mathbb{P}_{\kappa^{++}}$ is given by $q \leq_{\mathbb{P}_{\kappa^{++}}} p$ if and only if for all $\beta < \kappa^{++}$, $q(\beta) \leq_{\mathbb{Q}} p(\beta)$, and

$$|\{\beta < \kappa^{++} : p(\beta) \neq 1_{\mathbb{Q}} \text{ and } q(\beta) \neq p(\beta)\}| < \kappa.$$

The support of p, $supp(p)$, is defined as usual. We also carry over the notation $q \leq_h p$ and $q \leq_v p$ from the previous chapter; the definition given before applies equally to conditions in $\mathbb{P}_{\kappa^{++}}$.

The following is immediate:

Lemma 8.0.12. $\mathbb{P}_{\kappa^{++}}$ has the κ^{++}-c.c.

Proof We can use the Δ-system Lemma, 7.0.17, because GCH holds. The proof is the same as that of Lemma 7.0.19 in the previous chapter.

$\square$

Lemma 8.0.13. $\mathbb{P}_{\kappa^{++}}$ does not collapse cardinals $\leq \kappa$.

Proof It is sufficient to prove that any decreasing sequence of conditions of length $\gamma < \kappa$ has a lower bound in $\mathbb{P}_{\kappa^{++}}$. Let $\{p_\alpha : \alpha < \gamma < \kappa\}$ be such a sequence.

Let $q \in \mathbb{P}_{\kappa^{++}}$ be defined by setting $q(i) = \bigcup_{\alpha<\gamma} p_\alpha(i)$ for all $i < \kappa^{++}$. Then we claim that q is the required lower bound. To see this, we first need to check that for each $i < \kappa$, $q(i)$ is a condition in $\mathbb{Q}$. So let i be less than κ^{++}, then $q(i)$ is clearly a function because the functions in $\{p_\alpha(i) : \alpha < \gamma\}$ are pairwise compatible. Furthermore, its domain will have order type less than κ^κ because the domain of each $p_\alpha(i)$ does and $\gamma < \kappa$. It remains to check that $q(i) \upharpoonright \delta \in N_{\min(C \setminus (\delta+1))}$ for all $\delta < \kappa$. So let δ be less than κ, but then $p_\alpha(i) \upharpoonright \delta \in N_{\min(C \setminus (\delta+1))}$ for each $\alpha < \gamma$, because they are all conditions in $\mathbb{Q}$, and we insisted that $N_{\min(C \setminus (\delta+1))}$ be chosen so as to be closed under unions of less than κ many functions, giving us the required result.

That q is a condition in $\mathbb{P}_{\kappa^{++}}$ follows from the fact that $\bigcup_{\alpha<\gamma} supp(p_\alpha)$ is a set of size at most κ. And it is a lower bound to each p_α for $\alpha < \gamma$ because the set $\{\beta \in supp(p_\alpha) : q(\beta) \neq p_\alpha(\beta)\}$ has size less than κ due to the fact that $\{p_\alpha : \alpha < \gamma\}$ is a decreasing sequence in $\mathbb{P}_{\kappa^{++}}$ and $\gamma < \kappa$.

$\square$

We will also need the following two facts:

Lemma 8.0.14. Let $C \subseteq \kappa^+$ be a closed unbounded set of limit ordinals. Then given a set $X = \{d_i : i < \kappa^+\}$ of size κ^+ with $\operatorname{otp}(d_i) < \kappa^\kappa$ for all $i < \kappa^+$, there is a $\beta \in C$ such that for all $\delta \in C \setminus \beta$ there is an unbounded set $Y \subseteq \kappa^+$ with $i \in Y \Rightarrow \sup(d_i \cap \delta) < \delta$.

Proof Assume not, then there is a cofinal subset $D \subseteq C$ witnessing the failure of the Lemma. Let $\gamma < \kappa^+$ be such that $D \cap \gamma$ has order type κ^κ. By assumption, for any $\delta \in D \cap \gamma$, there are at most κ many d_i whose intersection with δ is bounded below δ. Hence we can find a $j < \kappa^+$ such that $\sup(d_j \cap \delta) = \delta$ for all $\delta \in D \cap \gamma$, which contradicts d_j having order type less than κ^κ.

□

Lemma 8.0.15. $\mathbb{Q}$ has the following properties:

(i) $\mathbb{Q}$ has the κ^+-Knaster property. Given a set X, with $|X| = \kappa^+$, consisting of conditions in $\mathbb{Q}$, we can find a cofinal subset $Y \subseteq X$ such that any two conditions in Y are pairwise compatible.

(ii) Any partial order of the form $\prod_{i \in I} Q_i$ where each $Q_i = \mathbb{Q}$ and I has size κ, ordered by the product order, has the κ^+-Knaster property.

(iii) If $\prod_{i \in I} Q_i$ is as in (ii) and $\langle p_\delta : \delta < \kappa^+ \rangle$ is a sequence of distinct elements in $\prod_{i \in I} Q_i$, then there is a cofinal subsequence $\langle p_{\delta_{(\beta_\alpha)}} : \alpha < \kappa^+ \rangle$ such that if $x \subseteq \kappa^+$ has order type κ then $f_{x,i} = \bigcup_{\alpha \in x} p_{\delta_{(\beta_\alpha)}}(i)$ is a countable partial function from κ^+ to 2 with $\mathrm{otp}(\mathrm{dom}(f_{x,i})) < \kappa^\kappa$, for each $i \in I$.

Proof We have that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), trivially.

Part (iii) can be proved by an argument directly analogous to that in the proof of Lemma 7.0.21, where we define sequences of length κ^+ rather than ω_1, and then appeal to the generalised Δ-system Lemma, Lemma 8.0.14 and the fact that, like ω^ω, κ^κ is closed under ordinal multiplication.

□

The following lemmas are all proved by arguments similar to those used in Section 7.0.2. Only very minor modifications are needed, so we will not give the proofs again.

Lemma 8.0.16. Forcing with $\mathbb{P}_{\kappa^{++}}$ does not collapse κ^+.

Lemma 8.0.17. Let $p \in \mathbb{P}_{\kappa^{++}}$ and $\dot{\tau}$ be a name such that $p \Vdash$ "$\dot{\tau} \in [\kappa^+]^{\kappa^+}$" and let $\langle A_\delta : \delta \in S\rangle$ be our fixed witness to $\clubsuit(S)$ in V. Then the set of q such that for some $\delta \in S$, $q \Vdash$ "$A_\delta \subseteq \dot{\tau}$" is dense below p.

Lemma 8.0.18. $V^{\mathbb{P}_{\kappa^{++}}} \models \neg\clubsuit^{[\mathrm{otp}]}(\kappa^+) + \neg\clubsuit(T)$.

Thus we are able to prove Theorem 8.0.3.

Recall that we required S and T to be subsets of $S^{\kappa^+}_{\kappa}$; if this is the case and κ is regular then the proofs of the previous chapter can be generalised directly, as outlined above. If, however, either S or T is not a subset of $S^{\kappa^+}_{\kappa}$ then we cannot obtain the results of Theorem 8.0.3. The reason is that if, for example, $T \subseteq S^{\kappa^+}_{<\kappa}$ then we cannot force $\clubsuit(T)$ to fail while also allowing the forcing to have the property that decreasing sequences of length less than κ have a lower bound, which means we cannot prove that κ is not collapsed.

Chapter 9

Some Open Questions

Several open questions have been brought to light in the course of this thesis. We collect the most prominent ones here.

The main open question concerning $\clubsuit$ remains, of course, that of Juhasz:

Question 9.0.19 (Juhasz)**.** Does $\clubsuit \to \neg$SH?

A variation on this question was asked by Brendle:

Question 9.0.20 (Brendle, [3])**.** Does $\dagger + \neg$CH $\to \neg$SH?

Miyamoto's Theorem 3.0.3 is a partial answer to this.

Recall our definitions of Superclub and Superstick in Chapter 3. The following two questions remain open:

Question 9.0.21. Does Superstick imply CH? Does Superclub imply $\diamondsuit$?

In Chapter 4 we defined the notion of a T-preserving $\clubsuit$-sequence and a directly T-preserving $\clubsuit$-sequence, where T is a normal Suslin tree.

Question 9.0.22. Given $\bar{A}$, a $\clubsuit$-sequence in V, is there a forcing $\mathbb{P} \in V$ which preserves $\bar{A}$ as a $\clubsuit$-sequence while introducing a Suslin tree T so that $\bar{A}$ is T-preserving (or directly T-preserving) in any $\mathbb{P}$-generic extension?

Question 9.0.23. Is there a model of ZFC + $\clubsuit$ in which for any witness to $\clubsuit$, $\bar{A}$, there is a Suslin tree $T_{\bar{A}}$ such that $\bar{A}$ is $T_{\bar{A}}$-preserving?

Question 9.0.24. Given a Suslin tree T, can a $\clubsuit$-sequence be T-preserving without being directly T-preserving?

In Chapter 5 we examined the relationship between $\clubsuit$ and cardinal arithmetic. Though the basics of this are well-known, there is a surprisingly large amount that remains to be proved on this. Our main question in Chapter 5 could be stated as follows:

Question 9.0.25. If $V \models \mathrm{ZFC} + \neg\mathrm{CH} + \stick + \neg\clubsuit$, is there a cardinal preserving forcing $\mathbb{P} \in V$ such that $1_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\clubsuit$"?

The author is not aware of any known answer to the following:

Question 9.0.26. Does $\clubsuit + \neg\mathrm{CH} \rightarrow 2^{\omega} = 2^{\omega_1}$?

The following question relates to our results in Chapter 6:

Question 9.0.27. For $\mathcal{F}$ a λ^+-complete uniform filter on λ^+, where λ is singular, is it the case that ZFC $\vdash \clubsuit_{\mathcal{F}}(S^{\lambda^+}_{\mathrm{cf}(\lambda)})$?

This is related to the following prominent open question on $\diamondsuit$:

Question 9.0.28. If λ is a singular cardinal, does $2^\lambda = \lambda^+$ imply $\diamondsuit(S^{\lambda^+}_{\mathrm{cf}(\lambda)})$?

Regarding our consistency results in Chapters 7 and 8, we ask the following:

Question 9.0.29. If λ is regular and $T, S \subseteq \lambda$ are disjoint stationary sets, when can we prove Con(ZFC) $\rightarrow$ Con(ZFC + $\clubsuit(S)$ + $\neg\clubsuit(T)$)?

For example, it is not possible to do this when $\lambda = \mu^+$ and $\kappa < \mu$ if $T \subseteq S^\lambda_\kappa$, S is a reflecting stationary set, and $\square^*_\mu$ holds (see e.g. [33]).

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