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Scaling relations between numerical simulations and physical systems they represent

Jonathan Granot[1,2,3]
[1]Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
[2]Raymond and Beverly Sackler School of Physics & Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
[3]Centre for Astrophysics Research, University of Hertfordshire, College Lane, Hatfield, AL10 9AB, UK

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ABSTRACT
The dynamical equations describing the evolution of a physical system generally have a freedom in the choice of units, where different choices correspond to different physical systems that are described by the same equations. Since there are three basic physical units, of mass, length and time, there are up to three free parameters in such a rescaling of the units, $N_f \leqslant 3$. In Newtonian hydrodynamics, e.g., there are indeed usually three free parameters, $N_f = 3$. If, however, the dynamical equations contain a universal dimensional constant, such as the speed of light in vacuum c or the gravitational constant G, then the requirement that its value remains the same imposes a constraint on the rescaling, which reduces its number of free parameters by one, to $N_f = 2$. This is the case, for example, in magneto-hydrodynamics or special-relativistic hydrodynamics, where c appears in the dynamical equations and forces the length and time units to scale by the same factor, or in Newtonian gravity where the gravitational constant G appears in the equations. More generally, when there are $N_{\rm udc}$ independent (in terms of their units) universal dimensional constants, then the number of free parameters is $N_f = \max(0, 3 - N_{\rm udc})$. When both gravity and relativity are included, there is only one free parameter ($N_f = 1$, as both G and c appear in the equations so that $N_{\rm udc} = 2$), and the units of mass, length and time must all scale by the same factor. The explicit rescalings for different types of systems are discussed and summarized here. Such rescalings of the units also hold for discrete particles, e.g. in N-body or particle in cell simulations. They are very useful when numerically investigating a large parameter space or when attempting to fit particular experimental results, by significantly reducing the required number of simulations.

Key words: methods: numerical — methods: miscellaneous — methods: N-body simulations — hydrodynamics — magnetohydrodynamics (MHD) — gravitation

1 INTRODUCTION

Numerical simulations are gradually but steadily playing an increasingly more important role in the study of many physical systems. In particular, they have a growing impact on many different fields, such as fluid dynamics, plasma physics, astrophysics, particle physics, Earth and planetary sciences, and meteorology. In particular, numerical computation is a vital tool for studying complex problems that are hard to solve analytically, such as nonlinear, many-body or multiple scale processes.

Here I outline how for many types of numerical simulations, the results of a single simulation correspond to a whole family of physical systems. This arises from the more general freedom in the choice of units in the dynamical equations that describe a particular type of physical system. This freedom (or lack thereof) also holds for solutions of these equations, as long as the initial or boundary conditions do not impose additional universal dimensional constants (UDCs), thus increasing $N_{\rm udc}$. It is equally valid for analytic or numerical solutions of these equations. Here the focus is on numerical solutions, and in particular on numerical simulations.

In this work it is assumed that the number of basic physical units is $N_{\rm bpu} = 3$, corresponding to units of mass (m), length (l) and time (t). This means that electric charge is expressed in terms of these units ($[q] = m^{1/2} l^{3/2} t^{-1}$), which corresponds to c.g.s Gaussian units in which Maxwell's equations contain one UDC – the speed of light in vacuum, c. The number of free parameters N_f that span the family of physical systems that correspond to a single simulation is generally given by

$$N_{\rm f} = \max\left(0,\ N_{\rm bpu} - N_{\rm udc}\right)\ . \qquad (1)$$

For magneto-hydrodynamics (MHD), either Newtonian or special relativistic, where the only UDC is c (i.e. $N_{\rm udc} = 1$), this implies that $N_f = 3 - 1 = 2$.

An alternative choice of basic physical units is SI units, in which electric charge is explicitly treated as a fourth basic physical unit (measured in Coulombs), so that $N_{\rm bpu} = 4$. In these units, however, Maxwell's equations contain two UDCs, ϵ_0 (the permittivity of free space or the electric constant) and μ_0 (the permeability of free space or the magnetic constant), where $(\epsilon_0\mu_0)^{-1/2} = c$. Therefore, with this choice of units $N_{\rm udc} = 2$ for MHD, leading to $N_f = 4 - 2 = 2$, i.e. the same number of free parameters as before. This nicely demonstrates that the freedom in rescaling the basic physical units for a given physical system is independent of the specific choice of basic units. Therefore, the particular choice that we use in this work should not affect N_f and the implied relations between the allowed scalings of the units of mass, length and time.

The concept of UDCs is not new. For example, Ellis (1968) has referred to UDCs as universal scale-dependent constants, and demonstrated that if units of mass and force are expressed in terms of length and time ($[m] \to l^3t^{-2}$, $[F] \to l^4t^{-4}$) then this renders the gravitational constant G dimensionless in such units, so that an appropriate choice of their magnitude can make it equal to unity and thus disappear from Newton's law of gravity. In such units, $N_{\rm udc} = 2$ (as the basic physical units are only of length and time) while $N_{\rm udc} = 0$ for Newtonian gravity, leading to $N_f = 2 - 0 = 2$. For the regular units $N_{\rm bpu} = 3$ and $N_{\rm udc} = 1$ for Newtonian gravity, again leading to $N_f = 3 - 1 = 2$. Unlike the c.g.s Gaussian units that were constructed in order to eliminate the constant [or SI UDC $1/(4\pi\epsilon_0)$] in Coulomb's law, the above choice of units that eliminate G from Newton's law of gravity do not work very well in general. The reason for this is the equivalence (which does not have an electromagnetic analog) between gravitational mass, for which these units were constructed, and inertial mass, which appears also in systems in which gravity is unimportant and can be neglected. In such systems that choice of units will artificially eliminate a degree of freedom in the rescaling of the units for no good reason.

In § 2 the scalings are explicitly derived for different types of simulations, and summarized in Table 1. Some caveats, namely the possible increase in $N_{\rm udc}$ for certain equations of state or radiative processes, are discussed in § 3. In § 4 the application of special relativistic hydrodynamic simulations to model gamma-ray burst (GRB) afterglow is discussed in some more detail, as a useful case study. The conclusions are discussed in § 5.

2 SCALING RELATIONS FOR DIFFERENT TYPES OF NUMERICAL SIMULATIONS

Physical systems are usually described either in the continuum limit, such as in hydrodynamics or magneto-hydrodynamics (MHD), or by following the motions of discrete particles. Numerically, the former description corresponds to hydrodynamic (e.g., Li et al. 2010; Lawson & Barakos 2011; Wallace 2011) or MHD (e.g., Fendt & Memola 2008; Ishihara et al. 2009; Amit et al. 2010) simulations, while the latter includes examples such as cosmological N-body simulations (with a large number $N \gg 1$ point particles that interact only through their mutual gravitational attraction; e.g. Kravtsov et al. 1997; Navarro, Frenk & White 1997; Springel et al. 2005) or particle in cell (PIC) plasma simulations (where a large number of point particles with both positive and negative electric charges interact electromagnetically; e.g. Birn et al. 2001; Pukhov & Meyer-ter-Vehn 2002; Spitkovsky 2008). Table 1 summarizes the allowed scalings for different types of simulations. Hybrid simulations that include both a continuous medium and discrete particles are also possible (e.g., Katz, Weinberg & Hernquist 1996; Gnedin et al. 2004; Springel 2005), and the restrictions on them can be derived in a straightforward manner by combining the restrictions on their constituents. This is manifested in the total number of independent (in terms of their units) UDCs, $N_{\rm udc}$, which determines the number of free parameters N_f that span the family of physical systems that correspond to a single simulation through

$$N_{\rm f} = \max\,(0,\ 3 - N_{\rm udc})\ , \tag{2}$$

where we assume for the rest of this work that there are three basic physical units ($N_{\rm bpu} = 3$) of mass, length and time.

Hydrodynamic Simulations: here we outline how the results of different types of hydrodynamic simulations correspond to a family of physical systems. The numerical code solves, e.g., for the proper rest-mass density ρ, pressure p, and velocity $\vec{v} = \vec{\beta}c$, as a function of time and space, $(t, \vec{r})$, and assumes some equation of state that relates the specific enthalpy to the pressure and density. The evolution of these quantities is usually solved over a finite volume V and time range $t_i \leqslant t \leqslant t_f$. The initial conditions at t_i must be specified over the volume V, as well as boundary conditions at the edges of this volume at $t_i \leqslant t \leqslant t_f$. In order to solve the hydrodynamic equations numerically, they are first made dimensionless by moving to code units that are determined by choosing some specific scales (which we shall denote by a subscript "0") for the three basic physical units of mass (m_0), length (l_0), and time (t_0). The corresponding dimensionless variables in code units are denoted by a twiddle, where a general quantity Q with units of $m^Al^Bt^C$ corresponds to $\tilde{Q} = Q/(m_0^Al_0^Bt_0^C)$.

Once a particular initial physical configuration is mapped onto the dimensionless code variables, the hydrodynamic equations are solved for these variables. Then, the numerical solution is usually translated back to the original physical units, $Q = \tilde{Q}\,m_0^Al_0^Bt_0^C$. This is not a unique procedure, however, because of the freedom in the choice of units that was described above. Therefore, the same numerical solution also holds equally well for a whole set or family of different physical systems, which correspond to different choices for the basic physical units, $(m_0',\ l_0',\ t_0') = (\zeta m_0,\ \alpha l_0,\ \eta t_0)$. Such different choices of units can conveniently be implemented when switching back from the dimensionless code variables to the corresponding variables with physical units.

For a purely Newtonian simulation without gravity, there are indeed three free parameters (ζ, α and η), i.e. $N_f = 3$ since there are no relevant UDCs ($N_{\rm udc} = 0$), and the family of physical systems represented by the simulation is given by

$$Q' = \tilde{Q}\,(m_0')^A(l_0')^B(t_0')^C = \zeta^A\alpha^B\eta^C\tilde{Q}\,m_0^Al_0^Bt_0^C = \zeta^A\alpha^B\eta^C Q$$
$$\Longleftrightarrow\quad Q'\left(t' = \eta t,\ \vec{r}\,' = \alpha\vec{r}\right) = \zeta^A\alpha^B\eta^C Q(t,\vec{r})\ . \tag{3}$$

The scaling $\vec{r}\,' = \alpha\vec{r}$ reads for different coordinate systems,

$$\begin{aligned} (x',\ y',\ z') &= (\alpha x,\ \alpha y,\ \alpha z) \quad \text{cartezian}\ ,\\ (z',\ r_{\rm cyl}',\ \theta') &= (\alpha z,\ \alpha r_{\rm cyl},\ \theta) \quad \text{cylindrical}\ ,\\ (r',\ \theta',\ \phi') &= (\alpha r,\ \theta,\ \phi) \quad \text{spherical}\ . \end{aligned} \tag{4}$$

Similarly, the initial conditions would be at $(t',\vec{r}\,') = (\eta t_i, \alpha\vec{r})$ and the boundary conditions would be at the edge of $V' = \alpha^3V$ or $(t_i' \leqslant t' \leqslant t_f', \vec{r}\,'_{\rm edge})$ where $\vec{r}\,'_{\rm edge} = \alpha\vec{r}_{\rm edge}$, $t_i' = \eta t_i$ and $t_f' = \eta t_f$.

When there are relativistic velocities, either of bulk motions or random motions of the particles (i.e. relativistic temperatures), then we no longer have $\beta \ll 1$ and $p/\rho \ll c^2$, so that relativistic effects in the dynamical equations (which depend on the bulk Lorentz factor Γ) or the equation of state (which depends on p/ρ) can no longer be neglected. This requires that[1] $\Gamma' = \Gamma$ (i.e. $v' = v$) and $p'/\rho' = p/\rho$, where unprimed quantities are the usual ones for $\zeta = \alpha = \eta = 1$, which implies that $\eta = \alpha$, i.e. that the length and time units scale by the same factor. A more elegant way of deriving this is that special relativity introduces c (the speed of light in vacuum), a UDC with units of l/t, and thus requires $\alpha/\eta = 1$ in order to keep its value the same in all our family of physical systems. This reduces the number of free scaling parameters to two ($N_f = 2$ since $N_{\rm udc} = 1$). In particular, $x'_\mu = (t', \vec{r}') = \alpha(t, \vec{r}) = \alpha x_\mu$, $\rho'/\rho = p'/p = \zeta\alpha^{-3}$, and the family of physical systems corresponding to a particular simulation is given by

$$(m'_0,\, l'_0,\, t'_0) = (\zeta m_0,\, \alpha l_0,\, \alpha t_0)\ , \quad Q'(x'_\mu = \alpha x_\mu) = \zeta^A \alpha^{B+C} Q(x_\mu)\ . \quad (5)$$

In particular, the total energy E, and mass M either in a particular computational cell or in the whole computational box (or volume V) scale as $E'/E = M'/M = \zeta$.

When gravity is included, this introduces the gravitational constant G – a UDC with units of $m^{-1}l^3t^{-2}$. Thus, in order for it to keep the same value in all our family of systems requires that

$$(l')^3(m')^{-1}(t')^{-2} = l^3 m^{-1} t^{-2} \quad \Longleftrightarrow \quad \zeta = \alpha^3\eta^{-2}\ . \quad (6)$$

When there is only weak or Newtonian gravity (and general relativistic effects can be neglected), and Newtonian (bulk or thermal) motions, then G is the only UDC ($N_{\rm udc} = 1$, $N_f = 2$) and

$$(m'_0,\, l'_0,\, t'_0) = (\alpha^3\eta^{-2} m_0,\, \alpha l_0,\, \eta t_0)\ ,$$
$$Q'\left(t' = \eta t,\, \vec{r}' = \alpha\vec{r}\right) = \alpha^{B+3A}\eta^{C-2A} Q(t, \vec{r})\ . \quad (7)$$

When there effects of general relativity cannot be ignored, or for Newtonian gravity with relativistic velocities (either bulk or thermal), then there are two relevant UDCs, G and c ($N_{\rm udc} = 2$), which imply Eq. (6) and $\alpha = \eta$, respectively. Together this implies that $\zeta = \alpha = \eta$, i.e. that all three scaling coefficients are equal,

$$(m'_0,\, l'_0,\, t'_0) = \alpha\,(m_0,\, l_0,\, t_0)\ , \qquad Q'(x'_\mu = \alpha x_\mu) = \alpha^{A+B+C} Q(x_\mu)\ . \quad (8)$$

Magneto-Hydrodynamic (MHD) Simulations: the MHD equations are also based on Maxwell's equations, and thus include c as a UDC, so they require that $\alpha = \eta$. This holds even in the Newtonian case, where there are two free parameters ($N_f = 2$) describing the relevant family of physical systems corresponding to a particular simulation, according to Eq. (5). If gravity is included, even if weak or Newtonian gravity, then this introduces a second UDC, G, resulting in only one free parameter describing the relevant family of physical systems ($N_f = 1$ since $N_{\rm udc} = 2$), according to Eq. (8).

Simulations with discrete particles: there are various types of simulations that aim to describe the motions of discrete point-like particles, under the influence of the mutual forces that they exert on each other, rather than a continuous medium that is described by hydrodynamic or MHD equations. Here I briefly go over two important types of such simulations.

The first type is particle in cell (PIC) simulations of the motions of charged particles of either positive or negative electric charge under the mutual electromagnetic forces that they exert on each other. In this case Maxwell's equations introduce c as a UDC (implying $\alpha = \eta$). If we do not mind that the scaling would change the rest mass and/or electric charge of particles, then this would be the only constraint ($N_{\rm udc} = 1$), implying $N_f = 2$ and Eq. (5). If, however, it is important for us to accurately model a specific particle species (such as electrons/positrons) of a given universal rest mass and electric charge, then this would add two more constraints (and altogether $N_{\rm udc} = 3$), thus removing all the remaining freedom in the scaling parameters ($N_f = 0$) and implying $\zeta = \alpha = \eta = 1$.

[1] For $\Gamma \gg 1$ there exists a different type of rescaling that allows Γ' to vary relative to Γ (Mimica, Giannios & Aloy 2009), but this is not a rescaling of the basic physical units, and it is valid only for a forward-reverse shock system in which the forward shock is ultra-relativistic.

Table 1. The freedom in the choice of the basic physical units for different types of numerical simulations. The columns, from left to right, list the type of simulation, the relevant independent (in terms of their units) universal dimensional constants (UDCs), their number $N_{\rm udc}$, the number of free parameters N_f they allow for a rescaling of the units, and the imposed relation between the rescaling factors for mass (ζ), length (α) and time (η) units.
† If one or more of the cosmological parameters (such as H_0 or σ_8) are treated as UDCs this reduces N_f – see the discussion in § 2.

type of simulation	UDCs	$N_{\rm udc}$	N_f	constraints on rescaling
Newtonian hydrodynamics	—	0	3	—
relativistic hydrodynamics; Newtonian/relativistic MHD	c	1	2	$\alpha = \eta$
Newtonian gravity (e.g. in stellar/planetary dynamics, cosmological N-body †)	G	1	2	$\zeta = \alpha^3\eta^{-2}$
general relativistic hydrodynamics or MHD; Newtonian gravity + MHD or relativistic velocities	G, c	2	1	$\zeta = \alpha = \eta$
particle in cell (PIC);	c	1	2	$\alpha = \eta$
PIC + particular particles	c, q, m	3	0	$\zeta = \alpha = \eta = 1$

The second type is N-body simulations, that are often used in cosmology and stellar or planetary dynamics, where N point-like masses move under their mutual gravitational forces. Since gravity is the only force involved, G must obviously remain constant, implying Eq. (6). If there are only Newtonian gravity and velocities then there are two free parameters ($N_f = 2$, $N_{\rm udc} = 1$) and Eq. (7) holds. Otherwise, if relativistic effects cannot be neglected, then c also enters the relevant equations as a second UDC ($N_{\rm udc} = 2$) resulting in only one free parameter ($N_f = 1$), and implying Eq. (8).

In cosmological N-body simulations with Newtonian gravity and velocities, the only bona fide UDC is G, implying Eq. (7) with $N_f = 2$. The situation is more complicated, however, since we usually want the simulations to agree with the cosmological model of our observed universe, whose parameters are reasonably well known. Thus, some of these cosmological parameters might be treated as UDCs, depending on the purpose of the simulations.

For example, if the Hubble constant H_0 is treated as a UDC, then it would imply $\eta = 1$, which together with G (that implies $\zeta = \alpha^3$; Eq. [6]), results in $N_{\rm udc} = 2$, $N_f = 1$ and

$$(m'_0,\, l'_0,\, t'_0) = (\alpha^3 m_0,\, \alpha l_0,\, t_0)\ ,$$
$$Q'\left(t' = t,\, \vec{r}' = \alpha\vec{R}\right) = \alpha^{B+3A} Q(t, \vec{r})\ . \quad (9)$$

In particular this would leave the mass density unchanged, $\rho'(t') = \rho'(t) = \rho(t)$, so that the effective $\Omega_M(t)$ that is implied by the average value of $\langle\rho\rangle$ over the computational box would still follow the same original assumed cosmology.

In cosmological N-body simulations, the initial conditions are considered to be scale-invariant, since the amplitude of the initial fluctuations in the gravitational potential are (at least nearly) independent of the wavenumber k. However, the corresponding fluctuations in density scale as $\langle\delta\rho/\rho\rangle(k) \propto k^2$. This introduces a time dependent scale, $l_1(t) = 2\pi/k_1(t)$, at which the density fluctuations become of order unity, $\langle\delta\rho/\rho\rangle[k_1(t)] \equiv 1$, and thus enter the strongly non-linear stage of their evolution. This scale changes under a rescaling of the length units, and so does σ_8 or the normalization of the initial power spectrum (i.e., the length scale that σ_8 represents would no longer be $8h^{-1}$ Mpc upon rescaling of the length, but instead $\alpha \times 8h^{-1}$ Mpc). Sticking to the observed value of σ_8, and treating both H_0 and σ_8 as UDCs, would effectively remove the last degree of freedom in our rescaling (i.e. result in $N_f = 0$). If, however, the cosmological parameters such as H_0 or σ_8 are not treated as UDCs, and are allowed to vary (even if only over a limited range that is consistent with current observational constraints), then a rescaling of the units given by Eq. (7) could help to reduce the number of simulations required in order to numerically study a large parameter space with different cosmologies.

3 CAVEATS

Depending on the physics that are included in a simulation, further restrictions may arise in cases where there are additional UDCs or typical scales. In the previous section, the **equation of state** was implicitly assumed not to introduce any UDC, such as in the case of a simple polytropic equation of state, $p = K\rho^\gamma$, where K can vary (with the specific entropy). However, this is not always the case.

For example, **degeneracy pressure** in the Newtonian regime fixes the value of $p/\rho^{5/3} \sim \hbar^2/(m_d m_{\rm eff}^{5/3})$ where m_d (n_d) is the mass (number density) of the degenerate species while $m_{\rm eff} = \rho/n_d$ (here it is assumed that ρ is used as a primary hydrodynamic variable, rather than the number density n). This requires that $\zeta = \alpha^6\eta^{-3}$. Since degeneracy pressure is usually important only when gravity plays a role as well, this would also require $\zeta = \alpha^3\eta^{-2}$ or altogether, $\eta = \alpha^3 = \zeta^{-1}$. In the relativistic regime, where the uncertainty principle implies relativistic velocities of the degenerate species, $p/\rho^{4/3} \sim \hbar c/m_{\rm eff}^{4/3}$ is fixed, implying $\zeta = \alpha^9\eta^{-6}$. Together with gravity that introduces G, this implies $\zeta = 1$ and $\eta = \alpha^{3/2}$. For example, the Chandrasekhar mass is approximately given by the $3/2$ power of the ratio of these two constants, $M_{\rm Ch} \sim (\hbar c/m_{\rm eff}^{4/3}G)^{3/2} = M_{\rm Planck}^3/m_{\rm eff}^2$. The transition between the two regimes of degeneracy pressure occurs when the mean distance between degenerate particles is comparable to their Compton wavelength, thus fixing an absolute length-scale in the problem and requiring $\alpha = 1$ if it appears in the simulation. Together with gravity this would leave no free parameter ($N_f = 0$ since $N_{\rm udc} = 3$), and require $\zeta = \alpha = \eta = 1$.

The **optical depth** τ determines the probability for interaction, $1 - e^{-\tau}$, and must therefore remain unchanged. Thus, if ρ (rather than n) is a primary hydrodynamic variable then since $d\tau = \rho\kappa_* dl$, once the opacity coefficient κ_* of the matter is specified it should not change, and since it has units of l^2/m, this implies $\zeta = \alpha^2$. With the inclusion of radiation that introduces c as a UDC and requires $\alpha = \eta$, this implies $\zeta = \alpha^2 = \eta^2$. If, alternatively, n is the primary hydrodynamic variable then since $d\tau = \sigma_* n dl$ this requires the cross-section σ_* not to change and thus $\alpha = 1$, which together with the inclusion of radiation (implying $\eta = \alpha$) gives $\alpha = \eta = 1$. If the mass of each particle is also to remain constant, this requires $\zeta = 1$ leaving no degree of freedom and implying $\zeta = \alpha = \eta = 1$.

Optically thick radiation, or **radiation pressure** can also introduce UDCs. A black body, e.g., emits a power per unit area of σT^4 and has a pressure of $p_{\rm rad} = \frac{1}{3}aT^4$, thus introducing the Stefan-Boltzmann constant $\sigma = ac/4$ and the radiation constant a. Their ratio introduces $c = 4\sigma/a$ (the radiation streaming velocity) that implies $\alpha = \eta$. Since $k_{\rm B}T \approx p_{\rm rad}/n$ then $p_{\rm rad}/(k_{\rm B}T)^4$ that has unit of $(ml^3t^{-2})^{-3}$ must also remain the same, implying $\zeta = \eta^2\alpha^{-3}$, and together with the previous constraint, $\alpha = \eta = \zeta^{-1}$. If gravity is added as well then no freedom is left ($N_f = 0$ and $\zeta = \alpha = \eta = 1$).

Radiation reaction (the force on accelerating charged particles due to the back-reaction to the radiation they emit) or the effects of **radiative losses** on the cooling of the radiating particles, can introduce additional dimensional parameters, that are universal for a given particle species, such as electrons, and thus introduce constraints on the scaling parameters.

Table 2. The dimensional-based scalings for the GRB afterglow synchrotron spectrum, in terms of ζ and $\alpha = \eta$ (in column 3) or κ and λ (in column 4), for the flux density within the different power-law segments (PLSs, $Q \to F_{\nu,\rm A}$–$F_{\nu,\rm G}$; top part), the spectral break frequencies ($Q \to \nu_1$–ν_{11}; middle part), and the flux density at the break frequencies ($Q \to F_{\nu,1}$–$F_{\nu,11}$; bottom part). The notation for the different PLSs and break frequencies follow Granot & Sari (2002). Column 2 gives the dependence of F_ν on ν in each PLS for $F_{\nu,\rm A}$–$F_{\nu,\rm G}$, and otherwise the relevant break frequencies.

Q	ν	$\zeta,\ \alpha$	$\kappa,\ \lambda$
$F_{\nu,\rm A}$	$\nu^{5/2}$	$\zeta^{-1/4}\alpha^{11/4}$	$\kappa^{2/3}\lambda^{-11/12}$
$F_{\nu,\rm B}$	ν^2	$\zeta^0\alpha^2$	$\kappa^{2/3}\lambda^{-2/3}$
$F_{\nu,\rm C}$	$\nu^{11/8}$	$\zeta^{1/8}\alpha^{13/8}$	$\kappa^{2/3}\lambda^{-13/24}$
$F_{\nu,\rm D}$	$\nu^{1/3}$	$\zeta^{4/3}\alpha^{-1}$	$\kappa^1\lambda^{1/3}$
$F_{\nu,\rm E}$	$\nu^{1/3}$	$\zeta^2\alpha^{-7/3}$	$\kappa^{11/9}\lambda^{7/9}$
$F_{\nu,\rm F}$	$\nu^{-1/2}$	$\zeta^{3/4}\alpha^{-1/4}$	$\kappa^{2/3}\lambda^{1/12}$
$F_{\nu,\rm G}$	$\nu^{(1-p)/2}$	$\zeta^{(p+5)/4}\alpha^{-3(p+1)/4}$	$\kappa^1\lambda^{(p+1)/4}$
$F_{\nu,\rm H}$	$\nu^{-p/2}$	$\zeta^{(p+2)/4}\alpha^{(2-3p)/4}$	$\kappa^{2/3}\lambda^{(3p-2)/12}$
ν_m	ν_2, ν_4, ν_9	$\zeta^{1/2}\alpha^{-3/2}$	$\kappa^0\lambda^{1/2}$
ν_c	ν_3, ν_{11}	$\zeta^{-3/2}\alpha^{5/2}$	$\kappa^{-2/3}\lambda^{-5/6}$
$\nu_{\rm ac}$	ν_7	$\zeta^{1/5}\alpha^{-3/5}$	$\kappa^0\lambda^{1/5}$
$\nu_{\rm sa}$	ν_1	$\zeta^{4/5}\alpha^{-9/5}$	$\kappa^{1/5}\lambda^{3/5}$
$\nu_{\rm sa}$	ν_5	$\zeta^{\frac{6+p}{8+2p}}\alpha^{-\frac{14+3p}{8+2p}}$	$\kappa^{\frac{2}{12+3p}}\lambda^{\frac{14+3p}{24+6p}}$
$\nu_{\rm sa}$	ν_6	$\zeta^{\frac{3+p}{10+2p}}\alpha^{-\frac{9+3p}{10+2p}}$	$\kappa^0\lambda^{\frac{3+p}{10+2p}}$
$\nu_{\rm sa}$	ν_8	$\zeta^{1/3}\alpha^{-1}$	$\kappa^0\lambda^{1/3}$
$\nu_{\rm sa}$	ν_{10}	$\zeta^{9/5}\alpha^{-19/5}$	$\kappa^{8/15}\lambda^{19/15}$
$F_{\nu,1}$	ν_1	$\zeta^{8/5}\alpha^{-8/5}$	$\kappa^{16/15}\lambda^{8/15}$
$F_{\nu,\rm max}$	ν_2, ν_{11}	$\zeta^{3/2}\alpha^{-3/2}$	$\kappa^1\lambda^{1/2}$
$F_{\nu,3}$	ν_3	$\zeta^{(2p+1)/2}\alpha^{(1-4p)/2}$	$\kappa^{(p+2)/3}\lambda^{(4p-1)/6}$
$F_{\nu,4}$	ν_4	$\zeta^1\alpha^{-1}$	$\kappa^{2/3}\lambda^{1/3}$
$F_{\nu,5}$	ν_5	$\zeta^{\frac{13+2p}{8+2p}}\alpha^{-\frac{13+2p}{8+2p}}$	$\kappa^{\frac{13+2p}{12+3p}}\lambda^{\frac{13+2p}{24+6p}}$
$F_{\nu,6}$	ν_6	$\zeta^{\frac{5+2p}{10+2p}}\alpha^{\frac{5-2p}{10+2p}}$	$\kappa^{2/3}\lambda^{\frac{2p-5}{30+6p}}$
$F_{\nu,7}$	ν_7	$\zeta^{2/5}\alpha^{4/5}$	$\kappa^{2/3}\lambda^{-4/15}$
$F_{\nu,8}$	ν_8	$\zeta^{7/12}\alpha^{1/4}$	$\kappa^{2/3}\lambda^{-1/12}$
$F_{\nu,9}$	ν_9	$\zeta^{1/2}\alpha^{1/2}$	$\kappa^{2/3}\lambda^{-1/6}$
$F_{\nu,10}$	ν_{10}	$\zeta^{13/5}\alpha^{-18/5}$	$\kappa^{7/5}\lambda^{6/5}$

4 GRB AFTERGLOWS

The dynamics of GRB jets during the afterglow stage have been numerically modeled using special relativistic hydrodynamic simulations (Granot et al. 2001; Cannizzo et al. 2004; Zhang & MacFadyen 2009; Mimica, Giannios & Aloy 2009, 2010; van Eerten et al. 2010; Meliani & Keppens 2010; Wygoda, Waxman & Frail 2011; De Colle et al. 2011a, 2011b). As discussed above, for such simulations there is one UDC, c, which implies $\alpha = \eta$ and Eq. (5). It has recently been pointed out[2] (van Eerten, van der Horst & MacFadyen 2011) that the dynamics in this case obey a simple scaling relation,

$$\frac{E'}{E} = \kappa\,, \qquad \frac{\rho'}{\rho} = \lambda\,, \qquad \frac{l'}{l} = \frac{t'}{t} = \left(\frac{\kappa}{\lambda}\right)^{1/3}\,, \tag{10}$$

which was justified by resorting to dimensionless or similarity variables. However, this scaling simply arises from the freedom in the choice of the basic physical units, as described above. In particular, it corresponds to $\zeta = \kappa$ and $\alpha = \eta = (\kappa/\lambda)^{1/3}$, or equivalently to $\kappa = \zeta$ and $\lambda = \zeta/\alpha^3 = \zeta/\eta^3$. This scaling holds regardless of the initial conditions or symmetry of the problem, and has nothing to do with self-similarity of the hydrodynamics.

It was also pointed out recently (van Eerten & MacFadyen 2011) that this scaling of the dynamics[3] can also be extended to a similar scaling of the resulting afterglow synchrotron emission or the observed flux density F_ν, within each power-law segment (PLS) of the spectrum. This arises since within each PLS the local emissivity can be expressed as the product of a dimensional constant and a dimensionless function of the hydrodynamic quantities, so that a change in the basic units would affect only the dimensional constant, which would scale in a simple way.[4] Therefore, such a rescaling of the basic physical units holds quite generally within each PLS, regardless of the dynamics. In particular, the same rescaling holds in the early relativistic (Blandford & McKee 1976) and late Newtonian (Sedov 1946; Taylor 1950) (quasi-) spherical self-similar phases, as well as in the intermediate phase where the dynamics are not self-similar. Moreover, this scaling depends only on the PLS, and within a given PLS it does not depend on the external density profile (and would be the same for a uniform external medium and for a wind-like external medium).

It has been demonstrated that when the dynamics are self-similar, a more elaborate scaling exists in which the flux density F_ν within each PLS scales as a power-law with essentially all of the model parameters (Sari, Piran & Narayan 1998; Granot & Sari 2002; van Eerten & MacFadyen 2011). However, the dependences on the individual model parameters change between the relativistic Blandford & McKee (1976) and the Newtonian Sedov-Taylor self-similar regimes, and such simple power-law dependences on all of the model parameters do not exist in the intermediate phase, or whenever the dynamics are not self-similar.

The dimensional-based scalings of F_ν hold only locally within each PLS, and change between different PLSs. This corresponds to different scalings for the break frequencies that separate the PLSs, so that their ratios changes under such a scaling, despite being dimensionless. The lack of a global rescaling of the units for the observed radiation results in the need to parameterize and change "by hand" the spectral regime in order to calculate the lightcurve at a given observed frequency as it switches between different PLSs (when it is crossed by a break frequency).

The lack of such a global rescaling of the units for F_ν can be understood as follows. Technically, it can be attributed to the fact that the local emissivity is separable, i.e. can be expressed as a product of a dimensional constant and a dimensionless function of the hydrodynamic variables, only within each PLS, and that this dimensional constant that determines the scaling changes between different PLSs. The more basic reason behind this is that the emission process introduces additional UDCs relative to the dynamics, even in the optically thin regime. For example, the radiation cares also about the total number of particles, i.e. about the number density n and not only about the rest-mass density ρ, while $\rho/n \equiv m_{\rm eff}$ is usually taken to be constant (often set to the proton mass, m_p), and thus introduces a new UDC. Additional UDCs are introduced, e.g., through the synchrotron break frequencies, since they relate to the typical synchrotron frequency and cooling of the radiating relativistic electrons, which have a universal mass and electric charge.

Let us consider an emitting region of bulk Lorentz factor Γ, in the downstream region of a shock with a relative upstream to downstream Lorentz factor $\Gamma_{\rm ud}$ and upstream proper rest mass density $\rho_{\rm u}$. For the afterglow forward shock $\rho_{\rm u}$ is the external density and $\Gamma_{\rm ud} - 1 \approx \Gamma_{\rm ud} = \Gamma \gg 1$, while for the reverse shock $\rho_{\rm u}$ is the density of the original outflow and typically $\Gamma \gg \Gamma_{\rm ud} > \Gamma_{\rm ud} - 1 \sim 1$. Thus, both shocks can be treated together. The comoving magnetic field scales as $B^2 \propto \epsilon_B \Gamma_{\rm ud}(\Gamma_{\rm ud} - 1)\rho_{\rm u}$, and the typical Lorentz factor of the electron random motions scales as $\gamma_m \sim \epsilon_e (m_p/m_e)(\Gamma_{\rm ud} - 1) \propto \Gamma_{\rm ud} - 1$. Thus, the typical synchrotron frequency scales as $\nu_m \sim \Gamma(eB/m_e c)\gamma_m^2 \propto \epsilon_B^{1/2}\epsilon_e^2 \Gamma\Gamma_{\rm ud}^{1/2}(\Gamma_{\rm ud} - 1)^{5/2}\rho_{\rm u}^{1/2} \propto \rho_{\rm u}^{1/2}$ so that $\nu'_m/\nu_m = \zeta^{1/2}\alpha^{-3/2} \to \lambda^{1/2}$ since γ_m, Γ, $\Gamma_{\rm ud}$, as well as the shock microphysics parameters ϵ_e and ϵ_B are all invariant under rescalings of the basic physical units that conserve c ($\alpha = \eta$). Note that the part involving UDCs, $e/m_e c$, was not included in the scaling, since it is universal and does not change with the scaling of the hydrodynamic variables. The cooling break frequency scales as $\nu_c \sim \Gamma(eB/m_e c)\gamma_c^2 \propto \Gamma^{-1}B^{-3}t_{\rm obs}^{-2} \propto \Gamma^{-1}[\epsilon_B\Gamma_{\rm ud}(\Gamma_{\rm ud} - 1)\rho_{\rm u}]^{-3/2}t_{\rm obs}^{-2} \propto \rho_{\rm u}^{-3/2}t_{\rm obs}^{-2}$, and thus $\nu'_c/\nu_c = \zeta^{-3/2}\alpha^{9/2}\eta^{-2} \to \zeta^{-3/2}\alpha^{5/2} \to \kappa^{-2/3}\lambda^{-5/6}$, where $t_{\rm obs}$ is the observed time (when the emitted photons reach the observer) and $\gamma_c = 6\pi m_e c/(\sigma_T B^2 \Gamma t_{\rm obs}) \propto B^{-2}\Gamma^{-1}t_{\rm obs}^{-2}$ is the random Lorentz factor to which the electrons cool on the dynamical time. Again, parts involving UDCs, such as $e/m_e c$ or $m_e c/\sigma_T$, were not included in the scaling. A global rescaling of the units would require $\nu_m\rho_{\rm u}^{-1/2}$ and $\nu_c\rho_{\rm u}^{3/2}t_{\rm obs}^2$ with units of $m^{-1/2}l^{3/2}t^{-1}$ and $m^{3/2}l^{-9/2}t$, respectively, to remain invariant, thus implying $\zeta = \alpha^3\eta^{-2} \to \alpha$ and $\zeta = \alpha^3\eta^{-2/3} \to \alpha^{7/3}$, or altogether $\zeta = \alpha = \eta = 1$, which eliminates all of the freedom in such a rescaling. The peak synchrotron flux density scales as $F_{\nu,\rm max} \propto \Gamma B N_e \propto \epsilon_B^{1/2}\Gamma\Gamma_{\rm ud}^{1/2}(\Gamma_{\rm ud} - 1)^{1/2}\rho_{\rm u}^{1/2}M \propto \rho_{\rm u}^{1/2}M$ (where N_e and M are, respectively, the isotropic equivalent number of emitting electrons and rest mass in the shocked region, and $M/N_e = m_{\rm eff} = {\rm const}$), which implies that $F'_{\nu,\rm max}/F_{\nu,\rm max} = \zeta^{3/2}\alpha^{-3/2} \to \kappa\lambda^{1/2}$. Note that the distance to the observer, D, is not included in the scaling of $F_{\nu,\rm max}$ since it does not change with the hydrodynamic variables.

Even though there is no non-trivial global scaling of the units that obeys Eq. (3) for the flux density ($Q \to F_\nu$), such a scaling still works locally within each PLS (labeled by a subscript 'i'),

$$F'_{\nu,i}(t'_{\rm obs} = \alpha t_{\rm obs}) = \zeta^{a_i}\alpha^{b_i}F_{\nu,i}(t_{\rm obs})\,, \tag{11}$$

where the dependence on t and $\vec{r}$ is replaced by $t_{\rm obs}$. This can be understood since the flux density within each PLS is the product of $F_{\nu,\rm max}$ and certain fixed powers of the break frequencies (ν_m, ν_c, and the self-absorption frequency that is not discussed here for simplicity), whose scalings can be derived from simple dimensional considerations (as shown above). For example, $F_{\nu,\rm D} \approx F_{\nu,\rm max}(\nu/\nu_m)^{1/3}$ and $F_{\nu,\rm F} \approx F_{\nu,\rm max}(\nu/\nu_c)^{-1/2}$ for PLSs D and F, respectively, using the notations of Granot & Sari (2002). This implies that $F'_{\nu,D}/F_{\nu,D} = \zeta^{4/3}\alpha^{-1} \to \kappa\lambda^{1/3}$ ($a_D = 4/3$ and $b_D = -1$) and $F'_{\nu,F}/F_{\nu,F} = \zeta^{3/4}\alpha^{-1/4} \to \kappa^{2/3}\lambda^{1/12}$ ($a_F = 3/4$ and $b_F = -1/4$). As illustrative examples of how the scalings for

[2] Scheck et al. (2002) have outlined a similar scaling in a different context.
[3] There they use the scaling $n'/n = \lambda$ for the number density, but this is effectively equivalent to $\rho'/\rho = \lambda$ since they assume that $\rho/n = m_p = {\rm const}$.
[4] According to the units of the part that scales with the hydrodynamics variables, and does not involve the distance from the source to the observer or UDCs such as the electron or proton mass or electric charge.

self-absorbed PLSs may be derived, one can readily obtain that $F_{\nu,\rm B} \approx \pi(R/\Gamma D)^2(2\nu^2/c^2)\Gamma\gamma_m m_e c^2 \propto \nu^2 R^2$ implying $F'_{\nu,B}/F_{\nu,B} = \zeta^0\alpha^2 \rightarrow \kappa^{2/3}\lambda^{-2/3}$ ($a_B = 0$ and $b_B = 2$), while for PLS A γ_m is replaced by $\gamma_e(\nu) \propto (\nu/\Gamma B)^{1/2}$ [obtained from requiring $\nu \sim \nu_{\rm syn}(\gamma_e) \sim \Gamma(eB/m_e c)\gamma_e^2$], implying $F_{\nu,\rm A} \propto \nu^{5/2}R^2\rho_{\rm u}^{-1/4}$ and $F'_{\nu,A}/F_{\nu,A} = \zeta^{-1/4}\alpha^{11/4} \rightarrow \kappa^{2/3}\lambda^{-11/12}$ ($a_A = -1/4$ and $b_A = 11/4$). Therefore, these scalings (or a_i and b_i) do not depend on the external density profile or on the details of the dynamics (and are the same in the relativistic and Newtonian self-similar regimes, when the dynamics are not self-similar, or for the reverse shock). All of the different scalings are summarized in table 2.

5 DISCUSSION

The freedom in the choice of units in the dynamical equations that describe the evolution of different types of physical systems and in their solutions, has been outlined and elucidated. The main results are summarized in Table 1. While the emphasis was on numerical solutions of the dynamical equations through simulations, similar scalings hold equally well for analytic solutions of the same equations. The number of free parameters $N_{\rm f}$ that describe the family of physical systems that corresponds to a given solution of such a set of equations is given by $\max(0, 3 - N_{\rm udc})$ (Eq. [2]), where $N_{\rm udc}$ is the number of independent (in terms of their units) universal dimensional constants (UDCs, such as c, G, $\hbar$, m_e, etc.). This corresponds to the three basic physical units (of mass, length and time) while accounting for the independent constraints on their possible rescalings. Such rescalings of the basic units are potentially relevant to many different areas of research, such as plasma physics, astrophysics, cosmology, fluid dynamics or Earth and planetary sciences. They can prove very useful in numerical studies of various physical systems, and save precious computational resources, especially in systematic numerical studies of a large parameter space.

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