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CONSTANT FREE ERROR BOUNDS FOR NON-UNIFORM ORDER DISCONTINUOUS GALERKIN FINITE ELEMENT APPROXIMATION ON LOCALLY REFINED MESHES WITH HANGING NODES

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ABSTRACT. We obtain fully computable constant free a posteriori error bounds on the broken energy seminorm and DG-norm of the error for non-uniform polynomial order symmetric interior penalty Galerkin, non-symmetric interior penalty Galerkin and incomplete interior penalty Galerkin finite element approximations of a linear second order elliptic problem on meshes containing hanging nodes and comprised of triangular elements. The estimators are completely free of unknown constants and provide guaranteed numerical bounds on the broken energy seminorm and DG-norm of the error. These estimators are also shown to provide a lower bound for the broken energy seminorm and DG-norm of the error up to a constant and higher order data oscillation terms.

1. INTRODUCTION

Two of the major advantages of discontinuous Galerkin methods are that, since the finite element spaces are discontinuous, they readily allow the order of approximation to vary from element to element in the mesh as well as allowing approximations to be obtained on meshes containing hanging nodes. This means that refinements can be made to areas where the accuracy is poor without having to propagate refinements to neighbouring elements in order to maintain conformity of the mesh. A posteriori error estimators are often used to detect where in the mesh the accuracy is poor so that the mesh can be refined or the order of approximation increased in these locations until a stopping criterion has been satisfied. While there is already a wealth of a posteriori error estimators available for the error in discontinuous Galerkin finite element approximations [8, 9, 15, 19], all of these a posteriori error estimators contain unknown constants and so they are not actually fully computable. Consequently, they are really error indicators, as opposed to estimators, since they do not estimate the actual value of the error and as such cannot be used as a quantitative stopping criterion in an adaptive refinement strategy.

In [2, 3], the first fully computable bounds were obtained for both the broken energy seminorm and the DG-norm of the error in the first order symmetric interior penalty discontinuous Galerkin finite element approximation of a linear second order elliptic problem with variable permeability on meshes where no

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hanging nodes are present. Computable bounds on the error in the DG-norm for elements of fixed, but arbitrary, order, again on meshes without hanging nodes were subsequently obtained in [10, 14, 17]. In [5] the ideas used in [3] were extended to obtain fully computable error bounds, completely free of all unknown constants, which were applicable to symmetric interior penalty Galerkin (SIPG), non-symmetric interior penalty Galerkin (NIPG) and incomplete interior penalty Galerkin (IIPG) finite element approximations of first order on meshes containing hanging nodes.

The objective of the present work is to extend the ideas used in [5] to obtain fully computable a posteriori error estimators, completely free of all unknown constants, for non-uniform order discontinuous Galerkin finite element approximations on locally refined meshes with hanging nodes. The estimators we obtain also provide lower bounds up to a constant on the DG-norm and broken energy seminorm of the error plus higher order data oscillation terms. In contrast to [15, 19], we do this without making any assumptions on the regularity of the weak solution u to the problem considered, beyond the minimal assumption $u \in H^1(\Omega)$.

We know of no work, apart from this article, where fully computable estimators for the error in discontinuous Galerkin finite element approximations are obtained for general non-uniform order approximation on meshes containing hanging nodes that provide two-sided bounds (i.e. efficiency). If the constant in the Poincaré–Friedrichs inequality for the domain is known, then computable bounds for the error in the discontinuous Galerkin finite element approximation can be obtained using the approach in [21]. Unfortunately that estimator requires the solution of a global dual problem on the entire domain and the issue of whether the estimator provides a lower bound on the error is not considered.

The remainder of the paper is organised as follows. In Section 2, we describe the finite element schemes, introduce the notation and give an explicit computable bound for the values of the interior penalty parameters needed to ensure the existence of the discontinuous Galerkin finite element approximation for all versions of the method. In Section 3 we present numerical examples illustrating the theory before stating our computable error bounds in Section 4. We then present the proofs of our results in Sections 5, 6 and 7 before concluding with some extensions of the theory in Section 8.

2. Preliminaries

2.1. Model Problem.

Consider the model problem

$$-\operatorname{div}(\boldsymbol{A}\,\mathbf{grad}\,u) = f \text{ in } \Omega$$

subject to $u = q$ on Γ_D and $\boldsymbol{n} \cdot \boldsymbol{A}\,\mathbf{grad}\,u = g$ on Γ_N, where Ω is a simple plane polygonal domain, the disjoint sets Γ_D (nonempty) and Γ_N form a partitioning of the boundary $\Gamma = \partial\Omega$ of the domain and $\boldsymbol{n}$ is the outward unit normal vector to Γ_N. The data satisfy $f \in L_2(\Omega)$, $g \in L_2(\Gamma_N)$, $q \in H^1(\Gamma_D)$ and $\boldsymbol{A}$ is symmetric positive definite.

The variational form of the problem consists of finding $u \in H^1(\Omega)$ such that $u = q$ on Γ_D and

$$(\boldsymbol{A}\,\mathbf{grad}\,u, \mathbf{grad}\,v) = (f, v) + (g, v)_{\Gamma_N} \quad \forall v \in H^1_D(\Omega), \tag{1}$$

where $H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$. We shall use the notation $(\cdot,\cdot)_\omega$ to denote the integral inner product over a region or line segment ω, and omit the subscript in the case where ω is the physical domain Ω.

Let $\mathcal{P}^{(0)}$ be any partition of the domain Ω into the union of nonoverlapping, shape regular triangular elements such that the nonempty intersection of a distinct pair of elements is a single common node or a single common edge which is an entire edge of both of these elements. In addition we shall insist that this partition is such that every edge which is the complete edge of an element in $\mathcal{P}^{(0)}$ and lies on the boundary of the domain Ω is a subset of the closure of either the Dirichlet boundary Γ_D or the Neumann boundary Γ_N and that $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ on each element in $\mathcal{P}^{(0)}$. We generate a family of partitions $\mathcal{F} = \{\mathcal{P}^{(l)}\}$ from $\mathcal{P}^{(0)}$ recursively by marking a subset of triangles in $\mathcal{P}^{(l)}$ for refinement. These elements are refined by sub-dividing each element into four congruent sub-triangles as shown in Figure 1. We shall assume that additional refinements are then performed to ensure that there is at most one hanging node per edge of an element.

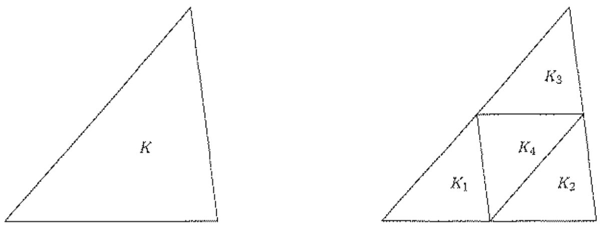


FIGURE 1. The refinement of triangle K into four congruent subtriangles K_1, K_2, K_3 and K_4.

In order to avoid propagation of refinements, we permit hanging nodes such as those shown in Figure 2 with more examples being found in Figures 5, 7, 9 and 11 in Section 3.

We note that the family of partitions is locally quasi-uniform in the sense that the ratio of the diameters of any pair of neighbouring elements is uniformly bounded above and below over the whole family.

2.2. **Discontinuous Galerkin finite element approximation.** Henceforth since we shall consider only a fixed partition $\mathcal{P}^{(l)}$ in the family we omit the superscripts. Let K and K' denote individual elements in $\mathcal{P}$, let ∂K denote the boundary of element K and let $\mathcal{E}_K$ denote the set containing the individual edges of element K. Likewise, we let $\mathcal{E}_I$, $\mathcal{E}_D$ and $\mathcal{E}_N$ denote the sets of edges

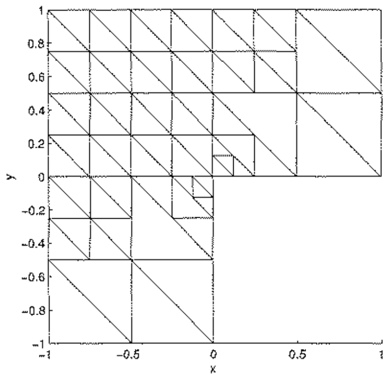


FIGURE 2. Example of the types of hanging nodes allowed.

defined by

$$\begin{aligned}\mathcal{E}_I &= \{\gamma : \gamma = \partial K \cap \partial K', K, K' \in \mathcal{P}\},\\ \mathcal{E}_D &= \{\gamma \subset \Gamma_D : \gamma \in \mathcal{E}_K \text{ for some } K \in \mathcal{P}\},\\ \mathcal{E}_N &= \{\gamma \subset \Gamma_N : \gamma \in \mathcal{E}_K \text{ for some } K \in \mathcal{P}\}\end{aligned}$$

and let $\mathcal{E}_\Gamma = \mathcal{E}_D \cup \mathcal{E}_N$ and $\partial\mathcal{P} = \mathcal{E}_I \cup \mathcal{E}_\Gamma$. For $m \in \mathbb{N}_0$, let $\mathbb{P}_m(K)$ denote the space of polynomials on $K \in \mathcal{P}$ of total degree at most m and let $\mathbb{P}_m(\gamma)$ denote the space of polynomials on $\gamma \in \partial\mathcal{P}$ of total degree at most m (with respect to the arc length parameter). Let $|K|$ denote the area of the element K and let $|\gamma|$ denote the length of edge γ.

For each element $K \in \mathcal{P}$, let $n_K \geq 1$ denote the order of approximation on element K. The discontinuous Galerkin finite element space on $\mathcal{P}$ is defined by

$$X_{\mathcal{P}} = \left\{v : \Omega \to \mathbb{R} : v_{|K} \in \mathbb{P}_{n_K}(K)\ \forall K \in \mathcal{P}\right\}.$$

For each element $K \in \mathcal{P}$, let $\mu_K : \partial K \to \{+1, -1\}$ denote a sign function that is piecewise constant on ∂K and satisfies $\mu_K + \mu_{K'} = 0$ on $\partial K \cap \partial K'$. Let

$$\langle \boldsymbol{n} \cdot \boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma =$$
$$\begin{cases} \frac{1}{2}\left(\mu_K \boldsymbol{n}_\gamma^K \cdot \boldsymbol{A}_K\,\mathbf{grad}\,v_{|K} + \mu_{K'}\boldsymbol{n}_\gamma^{K'} \cdot \boldsymbol{A}_{K'}\,\mathbf{grad}\,v_{|K'}\right) & \text{on } \gamma = \partial K \cap \partial K', \\ \boldsymbol{n}_\gamma^K \cdot \boldsymbol{A}_{|K}\,\mathbf{grad}\,v_{|K} & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_D \end{cases}$$

and

$$[v]_\gamma = \begin{cases} \mu_K v_{|K} + \mu_{K'} v_{|K'} & \text{on } \gamma = \partial K \cap \partial K', \\ v_{|K} & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_D \end{cases}$$

where $\boldsymbol{n}_\gamma^K$ is the outward unit normal vector to edge γ of element K and $\boldsymbol{A}_K = \boldsymbol{A}_{|K}$ with $\boldsymbol{n}_\gamma^{K'}$ and $\boldsymbol{A}_{K'}$ being defined analogously.

Let $\tau \in [-1,1]$ be fixed and, for $w, v \in X_{\mathcal{P}}$, define bilinear forms $B_\tau : X_{\mathcal{P}} \times X_{\mathcal{P}} \to \mathbb{R}$ by

$$\begin{aligned} & B_\tau(w,v) \\ = & \sum_{K\in\mathcal{P}} (\boldsymbol{A}\,\mathbf{grad}\, w, \mathbf{grad}\, v)_K - \sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\langle \boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\, w\rangle_\gamma, [v]_\gamma \right)_\gamma \\ & -\tau \sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D} \left([w]_\gamma, \langle \boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\, v\rangle_\gamma \right)_\gamma + \sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\frac{\kappa_\gamma}{|\gamma|} [w]_\gamma, [v]_\gamma \right)_\gamma \end{aligned}$$

and linear forms $L_\tau : X_{\mathcal{P}} \to \mathbb{R}$ by

$$\begin{aligned} L_\tau(v) = & \sum_{K\in\mathcal{P}} (f,v)_K + \sum_{\gamma\in\mathcal{E}_N} (g,v)_\gamma \\ & + \sum_{\gamma\in\mathcal{E}_D} \left(\frac{\kappa_\gamma}{|\gamma|} q, v \right)_\gamma - \tau \sum_{\gamma\in\mathcal{E}_D} \left(q, \langle \boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\, v\rangle_\gamma \right)_\gamma \end{aligned}$$

where the $\kappa_\gamma > 0$ are the usual interior penalty parameters chosen as in Lemma 1.

We can obtain a first order discontinuous Galerkin finite element approximation of the solution to problem (1) by finding $u_{DG} \in X_{\mathcal{P}}$ such that

$$B_\tau(u_{DG}, v) = L_\tau(v) \;\; \forall v \in X_{\mathcal{P}}. \tag{2}$$

There are many variants of discontinuous Galerkin methods [7, 16] corresponding to the particular choice of the parameter : $\tau = 1$ gives symmetric interior penalty Galerkin (SIPG), $\tau = -1$ gives non-symmetric interior penalty Galerkin (NIPG) whilst $\tau = 0$ gives incomplete interior penalty Galerkin (IIPG).

2.3. **The choice of the interior penalty parameter for discontinuous Galerkin finite element methods.** Usually, when discontinuous Galerkin methods are being considered, the existence of a unique solution to (2) is proved under an assumption that the interior penalty parameters κ_γ are sufficiently large without quantifying precisely how large. A bound on the size of κ_γ sufficient for unique solvability is given in the following lemma:

Lemma 1. *Let $\tau \in [-1,1]$. If the interior penalty parameters κ_γ are chosen such that*

$$\kappa_\gamma > \frac{(1+\tau)^2}{8} \max_{\substack{K\in\mathcal{P}:\\ \gamma\subset\partial K}} n_K(n_K+1)\,\rho(\boldsymbol{A}_K) \sum_{\gamma\in\mathcal{E}_K} \frac{\Lambda_\gamma |\gamma|^2}{|K|} \text{ for all } \gamma \in \partial\mathcal{P} \tag{3}$$

where

$$\Lambda_\gamma = \begin{cases} \frac{1}{2} & \text{if } \gamma \not\subset \Gamma, \\ 1 & \text{if } \gamma \subset \Gamma_D, \\ 0 & \text{if } \gamma \subset \Gamma_N \end{cases} \tag{4}$$

and $\rho(\boldsymbol{M})$ denotes the largest eigenvalue of the symmetric matrix $\boldsymbol{M}$, there exists a unique solution $u_{DG} \in X_{\mathcal{P}}$ to problem (2).

A related result was obtained in [20] for the case when $\tau = 1$, $\boldsymbol{A} = \boldsymbol{I}$ and $\Gamma = \Gamma_D$ with the factor $n_K(n_K+1)$ in (3) replaced by $(n_K+1)(n_K+2)$. We prove Lemma 1 in a similar way to that result although we shall defer this proof until Section 5. We note that another different lower bound for the size of κ_γ was proved in [13], for the case when there are no hanging nodes in the mesh. Which of the bounds in [13] and Lemma 1 gives a lower threshold for the value of κ_γ will depend upon the data $\boldsymbol{A}$ in the problem being solved and the triangulation used in the discretisation.

2.4. **Data oscillation.** For $v \in L_2(K)$, let $P_K v$ be the function satisfying $(v - P_K v, p)_K = 0$ for all $p \in \mathbb{P}_{n_K - 1}(K)$. Similarly, for $v \in L_2(\gamma)$ and $\gamma \in \mathcal{E}_K \cap \mathcal{E}_\Gamma$, let $P_\gamma v$ be the function satisfying $(v - P_\gamma v, p)_\gamma = 0$ for all $p \in \mathbb{P}_{n_K - 1}(\gamma)$. The oscillation of the data f on an element $K \in \mathcal{P}$ is defined to be

$$\operatorname{osc}(f, K) = |K|^{1/2} \|f - P_K f\|_{L_2(K)}.$$

Likewise, the oscillation of the Neumann data g on an edge $\gamma \in \mathcal{E}_N \cap \mathcal{E}_K$ is defined to be

$$\operatorname{osc}(g, \gamma) = |\gamma|^{1/2} \|g - P_\gamma g\|_{L_2(\gamma)}.$$

Also, the oscillation of the Dirichlet data q on an edge $\gamma \in \mathcal{E}_D \cap \mathcal{E}_K$ is defined to be

$$\operatorname{osc}(q, \gamma) = |\gamma|^{1/2} \left\| \frac{\partial q}{\partial \boldsymbol{t}_\gamma} - P_\gamma \frac{\partial q}{\partial \boldsymbol{t}_\gamma} \right\|_{L_2(\gamma)}$$

where $\boldsymbol{t}_\gamma$ is a unit tangent vector to edge γ. We shall adopt the convention whereby $\operatorname{osc}(g, \gamma) = 0$ if $\gamma \notin \mathcal{E}_N$ and $\operatorname{osc}(q, \gamma) = 0$ if $\gamma \notin \mathcal{E}_D$.

2.5. **The broken energy seminorm and DG-norm.** Let $\mathbf{grad}_\mathcal{P}$ denote the operator defined by $(\mathbf{grad}_\mathcal{P}\, v)_{|K} = \mathbf{grad}\,(v_{|K})$ for $K \in \mathcal{P}$ and let the broken energy seminorm over a region ω be denoted by

$$|\!|\!|\cdot|\!|\!|_\omega = (\boldsymbol{A}\,\mathbf{grad}_\mathcal{P} \cdot, \mathbf{grad}_\mathcal{P} \cdot)_\omega^{1/2} \tag{5}$$

where again we shall omit the subscript in the case where $\omega = \Omega$. Let the DG-norm over a region ω be denoted by

$$|\!|\!|\cdot|\!|\!|_{DG,\omega}^2 = |\!|\!|\cdot|\!|\!|_\omega^2 + \sum_{\substack{\gamma \in \mathcal{E}_I \cup \mathcal{E}_D: \\ \gamma \subset \overline{\omega}}} \frac{\kappa_\gamma}{|\gamma|} \left\| [\cdot]_\gamma \right\|_{L_2(\gamma)}^2 \tag{6}$$

with $|\!|\!|\cdot|\!|\!|_{DG}^2 = |\!|\!|\cdot|\!|\!|_{DG,\Omega}^2$. Let the error in the discontinuous Galerkin finite element approximation be denoted by $e = u - u_{DG}$ where u is the solution to (1) and u_{DG} is the solution to (2). Now, since $\left\| [e]_\gamma \right\|_{L_2(\gamma)} = \left\| [u_{DG}]_\gamma \right\|_{L_2(\gamma)}$ for all $\gamma \in \mathcal{E}_I$ and $\left\| [e]_\gamma \right\|_{L_2(\gamma)} = \|q - u_{DG}\|_{L_2(\gamma)}$ for all $\gamma \in \mathcal{E}_D$, the quantity

$$\sum_{\gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{\kappa_\gamma}{|\gamma|} \left\| [e]_\gamma \right\|_{L_2(\gamma)}^2 \tag{7}$$

is directly computable. Therefore, if we can obtain a constant free estimator for the broken energy seminorm of the error then we automatically have a constant free estimator for the DG-norm of the error as well. Moreover, we can also show that both of these norms are in fact equivalent:

Lemma 2. *If κ satisfies* (3) *then the DG-norm and broken energy seminorm of the error $e = u - u_{DG}$ are equivalent in the sense that*

$$\|e\|^2 \leq \|e\|_{DG}^2 \tag{8}$$

and there exists a positive constant c, independent of e and the size of the elements in the mesh but depending on the topology of the mesh and the orders of approximation, such that

$$c \|e\|_{DG}^2 \leq \|e\|^2 + \sum_{K \in \mathcal{P}} \operatorname{osc}^2(f, K) + \sum_{\gamma \in \mathcal{E}_D} \operatorname{osc}^2(q, \gamma) + \sum_{\gamma \in \mathcal{E}_N} \operatorname{osc}^2(g, \gamma). \tag{9}$$

A proof of this result can be found in Section 7.

3. Numerical examples

Before presenting the details of the computable error bounds we shall first present examples of their performance in actual applications. When reporting the numerical results we let: η denote the estimator of the broken energy seminorm of the error (which we shall define in Section 4); η_{DG} denote the estimator of the DG-norm of the error (which we shall also define in Section 4); osc denote the oscillation terms in the estimator and; the effectivity indices are denoted by $\vartheta = \eta / \|e\|$ and $\vartheta_{DG} = \eta_{DG} / \|e\|_{DG}$.

3.1. **Example 1.**

For our first example we look at the performance of the estimator for the problem of finding u such that $-\Delta u = f$ in the region $\Omega = \{x > 0, y > 0, x + y < 1\}$ with homogeneous Dirichlet data on $\Gamma_D = \partial\Omega$. The datum f is chosen so that the exact solution to this problem is

$$u = xy(1 - x - y)^{10}.$$

The initial mesh used for this example consisted of the one triangle making up the domain itself and we let $\tau = 1$ and $\kappa_\gamma = 5n_\gamma(n_\gamma + 1)$ on all $\gamma \in \partial\mathcal{P}$. Note that these values satisfy (3). The mesh was then uniformly refined. We did this for the cases when n_K took the values 1 to 6 on all $K \in \mathcal{P}$ and the results obtained are shown in Figure 3.

From the graphs in Figure 3(a) and Figure 3(b) it can be seen that, as we shall subsequently prove, the estimators do provide a guaranteed upper bound on both the broken energy seminorm and DG-norm of the error. From Figure 3(c) it can be seen that although the effectivity index starts off quite high it does go down to a value which is between 1.45 and 2.70 for all orders as the mesh is refined. This large initial overestimation of the error is explained in Figure 3(d) by the fact that it is the oscillation terms that are causing this large overestimation of the error. It is also worth noting that the effectivity index remains bounded as the order of the approximation increases.

3.2. **Example 2.**

For our second example we look at the performance of the estimator for the problem of finding u such that $-\Delta u = f$ in the region

$$\Omega = (-1, 1) \times (0, 1) \cup (-1, 0) \times (-1, 0]$$

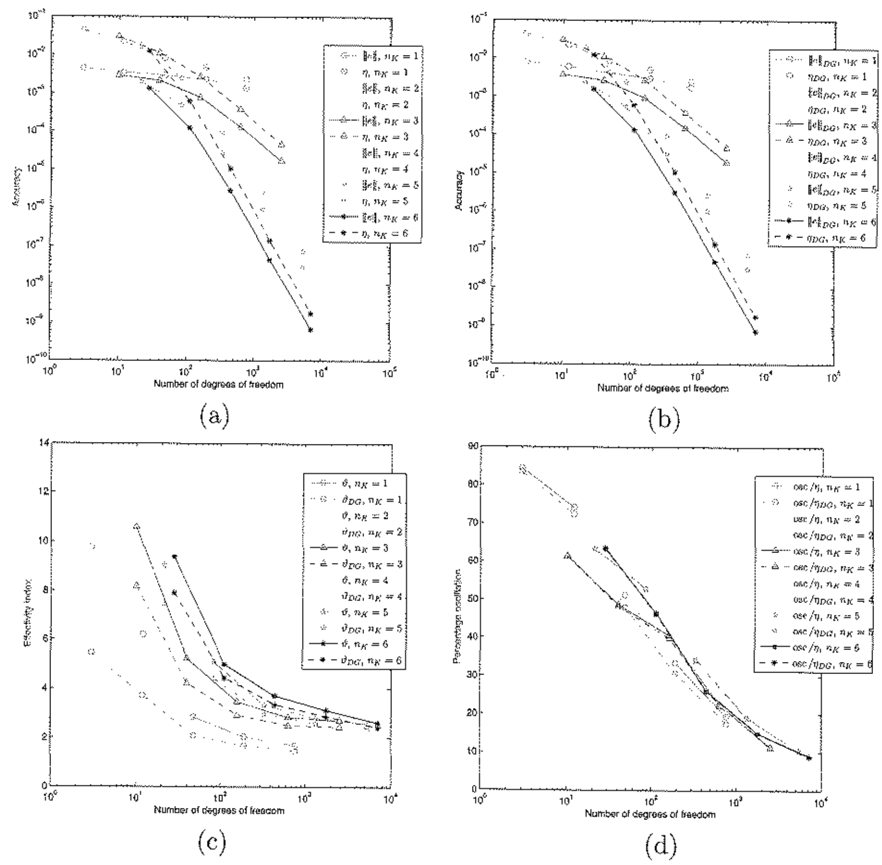


FIGURE 3. Performance of the estimators of the (a) broken energy seminorm and (b) DG-norm of the error, (c) effectivity indices of the estimators and (d) the percentage contribution of the oscillation terms to the overall estimator for Example 1.

with homogeneous Dirichlet data on $\Gamma_D = \partial\Omega$. The datum f is chosen so that the weak solution to this problem is

$$u = \left(1 - r^2 \cos\theta\right)\left(1 - r^2 \sin\theta\right) r^{2/3} \sin\left(2\theta/3\right).$$

The gradient of u displays singular behaviour at the origin since $\mathbf{grad}\, u = O\left(r^{-1/3}\right)$. The initial mesh used for this example is shown in Figure 4(a) and we let $\tau = 1$ and $\kappa_\gamma = 5n_\gamma\left(n_\gamma + 1\right)$ on all $\gamma \in \partial\mathcal{P}$ which satisfies the bound given in (3).

The mesh was then adaptively refined whereby a bulk criterion [12] was used to refine the mesh on the smallest number of elements such that the estimator of the broken energy seminorm of the error on these elements exceeded 50% of the value of the total error. Additional refinements were then performed to ensure that there was no more than one hanging node per edge.

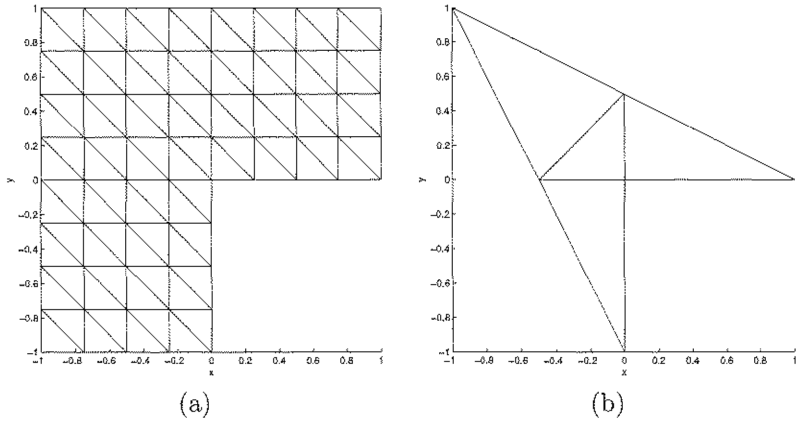


FIGURE 4. Initial meshes for (a) Example 2 and (b) Example 3.

We did this for the cases when n_K took the values 1 to 6 on all $K \in \mathcal{P}$ and the results obtained are shown in Figure 6. We also note that the only time that additional refinements had to be performed to ensure that there was no more than one hanging node per edge was when going from the 14th to the 15th mesh when the order of approximation was 3 where two additional refinements had to be performed. Consequently, our limitation to one hanging node per edge results in virtually no loss in performance compared with the case when arbitrary numbers of hanging nodes are permitted. It can be seen in Figure 5 that when polynomials of degree six are used very little mesh refinement has taken place away from the origin, where the gradient of u has a singularity, in contrast to the case when using polynomials of degree one.

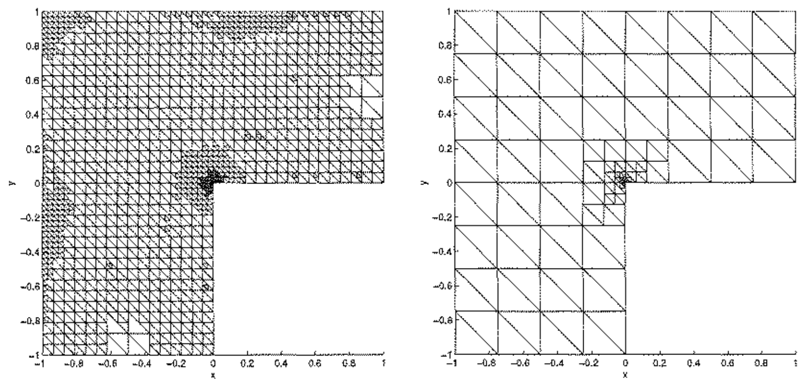


FIGURE 5. The final meshes for Example 2 with $n_K = 1$ (left) and $n_K = 6$ (right) for all $K \in \mathcal{P}$.

From the graphs in Figure 6(a) and Figure 6(b) it can be seen that, as in Example 1, the estimators do provide a guaranteed upper bound on both the

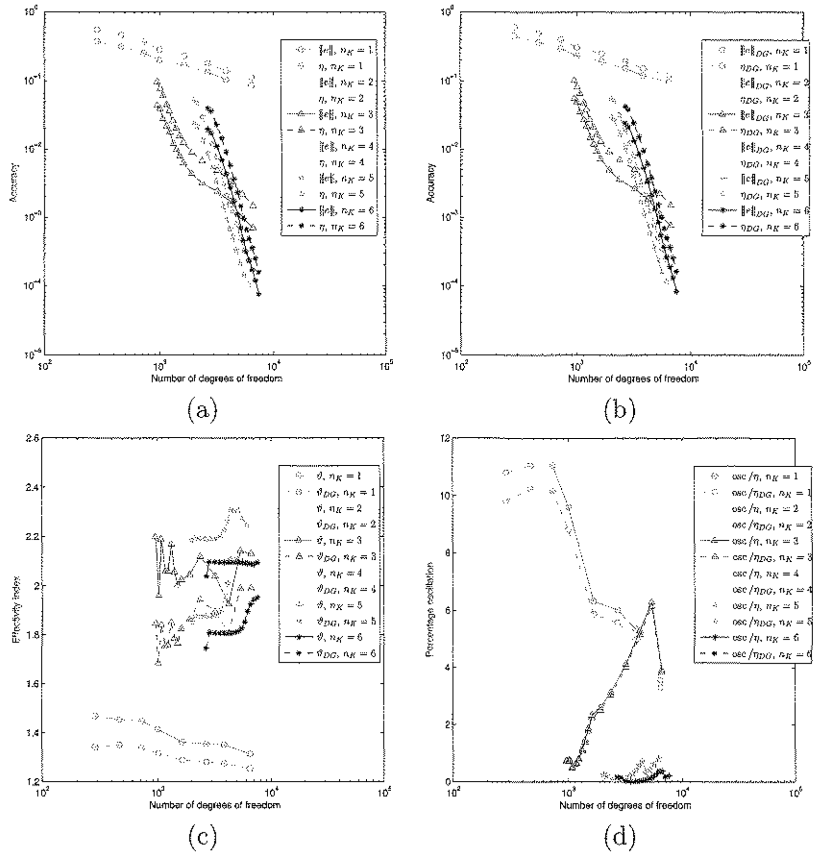


FIGURE 6. Performance of the estimators of the (a) broken energy seminorm and (b) DG-norm of the error, (c) effectivity indices of the estimators and (d) the percentage contribution of the oscillation terms to the overall estimator for Example 2.

broken energy seminorm and DG-norm of the error. From Figure 6(c) it can be seen that the effectivity index for this example is between 1.23 and 2.31 for all orders as the mesh is refined. It is again worth noting that the effectivity index does not appear to be increasing as the order increases.

The above results correspond to uniform order of approximation over the entire mesh. We now consider the effects of allowing the order of the elements to vary locally beginning with the initial mesh shown in Figure 4(a) and uniform initial order $n_K = 1$ on all elements K in this mesh. We then implemented the following adaptive refinement strategy:

(1) The bulk criterion described above is used to mark a set of elements $\mathcal{M} \subset \mathcal{P}$ where the local estimator is largest.
(2) The elements in $\mathcal{M}$ which have a vertex lying on a vertex of Ω are refined into four congruent sub-triangles.

(3) The order of approximation is increased by one on the elements in $\mathcal{M}$ which were not refined in step (ii).
(4) Additional refinements are then performed to ensure that there is no more than one hanging node per edge.

However, in the seventeen adaptive refinements we performed using this refinement strategy, no additional refinements had to actually be performed to ensure that there was no more than one hanging node per edge. A sample of the adaptively refined meshes are shown in Figure 7 with the results we obtained being shown in Figure 8.

Figure 8(a) shows the estimators providing a guaranteed upper bound on both the broken energy seminorm and DG-norm of the error. From Figure 8(b) we see that the effectivity index for this example is between 1.34 and 1.69. However, it does appear to be increasing as refinement with respect to both the mesh and the polynomial degree is carried out. While we were able to prove that the estimator was efficient with respect to the size of the elements in the mesh we were unable to show that it was efficient with respect to the degree of approximation. Figure 8(a) shows that the effectivity index of the estimator does not appear to be increasing enough to significantly degrade its use as a stopping criterion.

3.3. **Example 3.** For our final example we look at the performance of the estimator for the problem of finding u such that $-\Delta u = f$ in the region

$$\begin{aligned}\Omega &= \left\{(x,y) : x < 0, y > -2x - 1, y < \frac{1}{2} - \frac{1}{2}x\right\} \\ &\quad \cup \left\{(x,y) : x \geq 0, y > 0, y < \frac{1}{2} - \frac{1}{2}x\right\}\end{aligned}$$

with homogeneous Dirichlet data on $\Gamma_D = \partial\Omega$. The datum f is chosen so that the weak solution to this problem is

$$u = \left(r\sin\theta + \frac{1}{2}r\cos\theta - \frac{1}{2}\right)(r\sin\theta + 2r\cos\theta + 1)\,r^{2/3}\sin(2\theta/3)\,.$$

As in the previous example, $\mathbf{grad}\, u = O\left(r^{-1/3}\right)$ and so is singular at the origin. The initial mesh used for this example is shown in Figure 4(b) and we let $\tau = 1$ and $\kappa_\gamma = 5n_\gamma(n_\gamma + 1)$ on all $\gamma \in \partial\mathcal{P}$ which again satisfies the bound given in (3). The mesh was then adaptively refined using the adaptive refinement strategy described for uniform order approximation in the previous example.

We adaptively refined the mesh fourteen times for the cases when n_K took the values 1 to 6 on all $K \in \mathcal{P}$ and the results obtained are shown in Figure 10. The final meshes obtained after fourteen adaptive mesh refinements for the cases when $n_K = 1$ to $n_K = 6$ on all $K \in \mathcal{P}$ are shown in Figure 9 where it can be seen that, as in the previous example, when polynomials of degree six are used very little mesh refinement has taken place away from where the gradient of u has a singularity, in contrast to the case when using polynomials of degree one. We also note that the only time that additional refinements had to be performed to ensure that there was no more than one hanging node

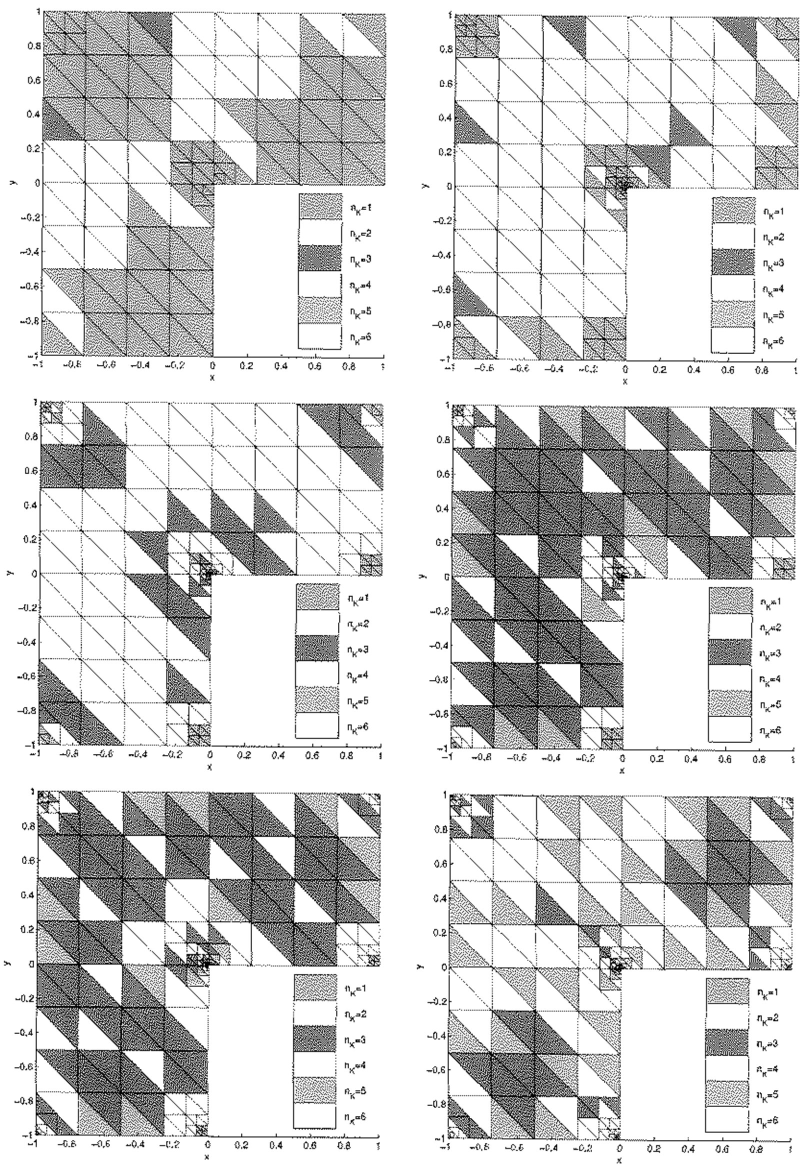
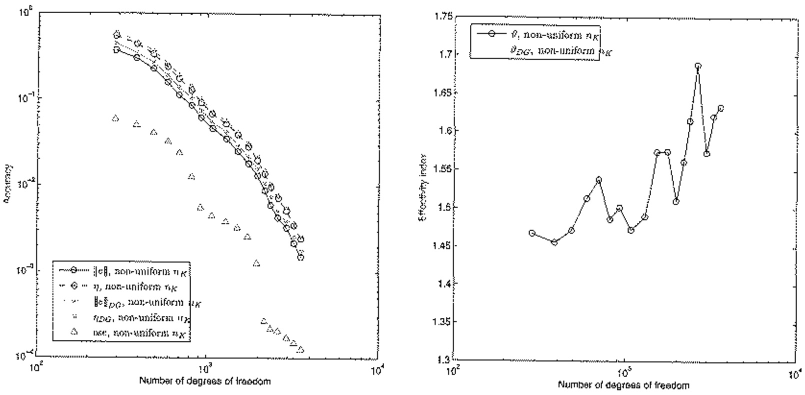


FIGURE 7. The 2nd, 5th, 8th, 11th, 14th and 17th adaptively refined meshes for Example 2.

per edge was when going from the 14th to the 15th mesh when the order of approximation was 3 where three additional refinements had to be performed.

From the graphs in Figure 10(a) and Figure 10(b) it can be seen that the estimators again provide a guaranteed upper bound on both the broken energy seminorm and DG-norm of the error. From Figure 10(c) it can be seen that the



(a) (b)

FIGURE 8. (a) Performance and (b) effectivity indices of the estimators for Example 2.

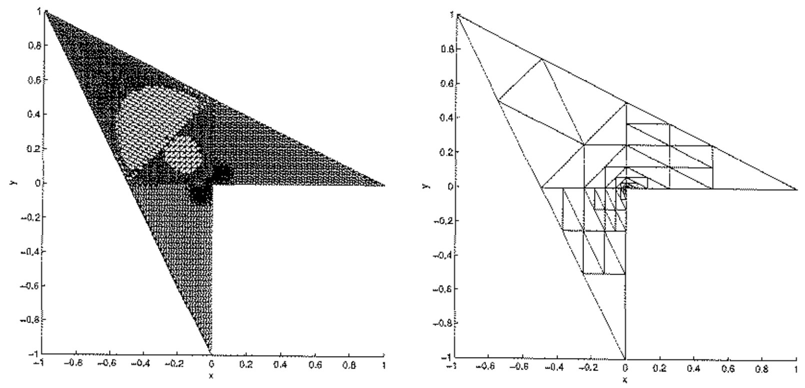


FIGURE 9. The 14th adaptively refined mesh for Example 3 with $n_K = 1$ (left) and $n_K = 6$ (right) for all $K \in \mathcal{P}$.

effectivity index for this example is between 1.40 and 2.74 for all orders as the mesh is refined and that the effectivity index does not appear to be increasing as the order increases.

We then went back to the initial mesh shown in Figure 4(b) with uniform initial order $n_K = 1$ on all elements K in this mesh and implemented the adaptive refinement strategy described for non-uniform order of approximation in Example 2. In the seventeen adaptive refinements we performed using this refinement strategy, the only time that additional refinements had to be performed to ensure that there was no more than one hanging node per edge was when going from the 17th to the 18th mesh where one additional refinement had to be performed. A sample of the adaptively refined meshes are shown in Figure 11 with the results we obtained being shown in Figure 12.

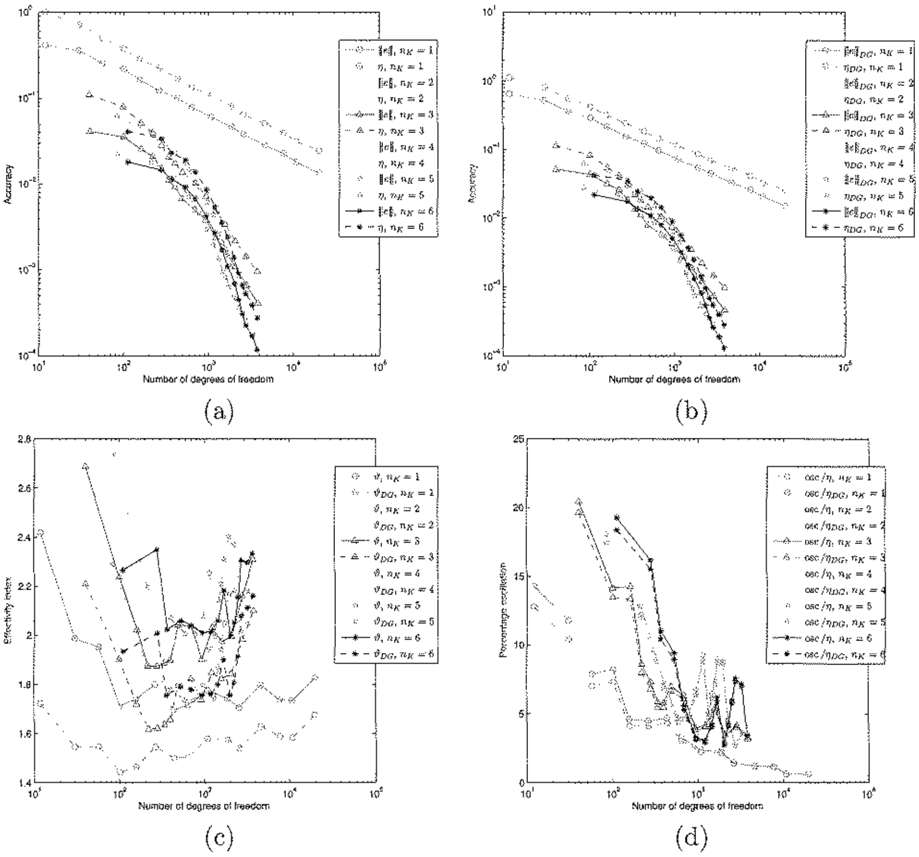


FIGURE 10. Performance of the estimators of the (a) broken energy seminorm and (b) DG-norm of the error, (c) effectivity indices of the estimators and (d) the percentage contribution of the oscillation terms to the overall estimator for Example 3.

Figure 12(a) shows the estimators providing a guaranteed upper bound on both the broken energy seminorm and DG-norm of the error. From Figure 12(b) we see that the effectivity index for this example remains between 1.40 and 2.42 as refinements are carried out. This would lead us to believe that it is not the presence of the singularity which is causing the effectivity index to increase as was observed in Example 2.

4. THE COMPUTABLE ERROR BOUNDS

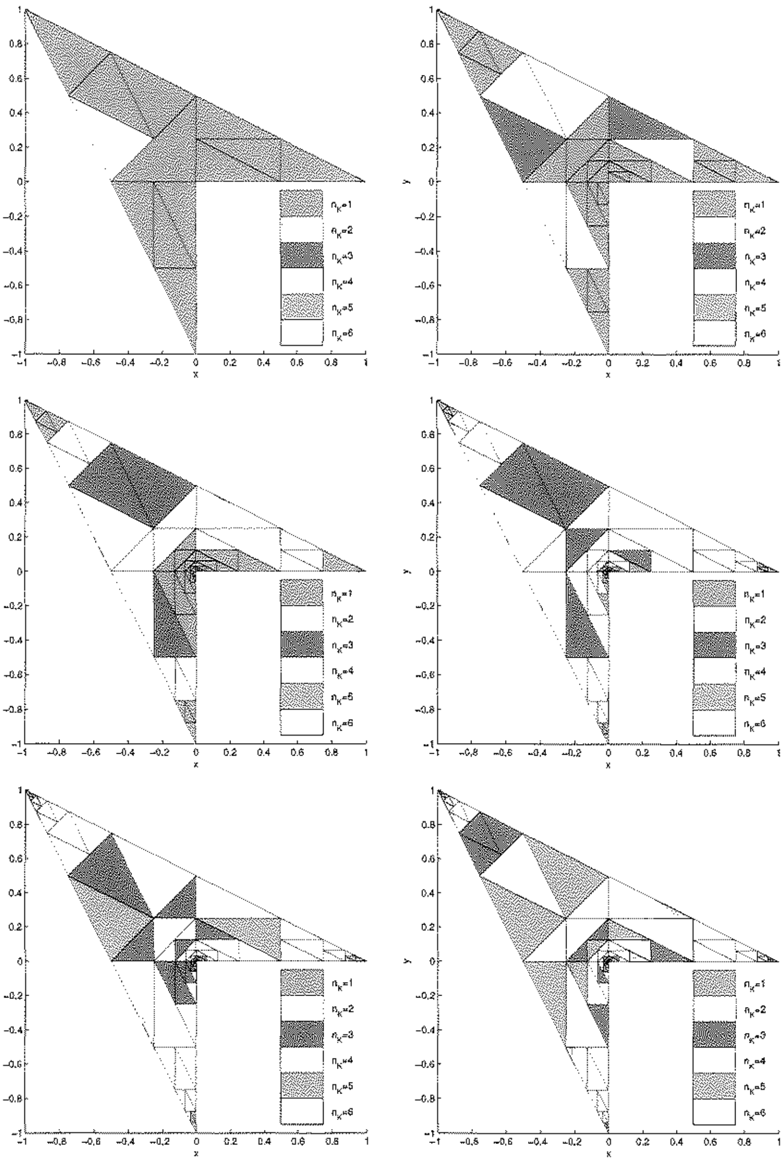
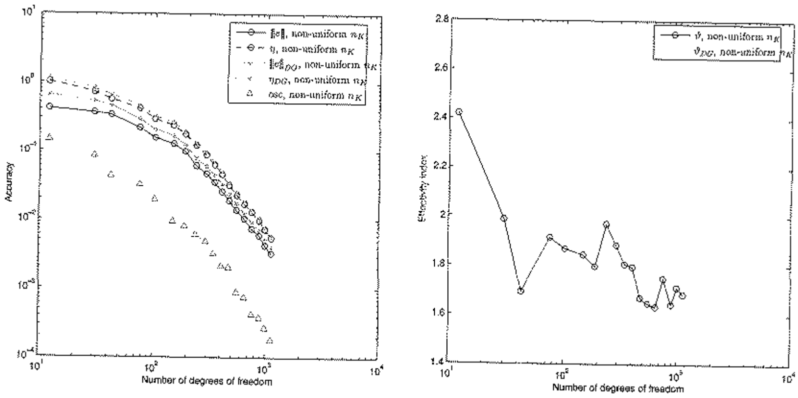


FIGURE 11. The 2nd, 5th, 8th, 11th, 14th and 17th adaptively refined meshes for Example 3.

4.1. **Notation.** Before stating our main result we shall define the notation which is required to make use of it. Let

$$[\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\,u_{DG}]_\gamma =$$

$$\begin{cases} \frac{1}{2}\left(\boldsymbol{n}_\gamma^K \cdot \boldsymbol{A}_K\,\mathbf{grad}\,u_{DG|K} + \boldsymbol{n}_\gamma^{K'} \cdot \boldsymbol{A}_{K'}\,\mathbf{grad}\,u_{DG|K'}\right) & \text{on } \partial K \cap \partial K' \subset \gamma, \\ \boldsymbol{n}_\gamma^K \cdot \boldsymbol{A}_K\,\mathbf{grad}\,u_{DG|K} - P_\gamma g & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_N, \\ 0 & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_D \end{cases}$$



(a) (b)

FIGURE 12. (a) Performance and (b) effectivity indices of the estimators for Example 3.

and

$$[u_{DG}]_{\gamma K} = \begin{cases} u_{DG|K} - u_{DG|K'} & \text{on } \partial K \cap \partial K' \subset \gamma, \\ u_{DG|K} - q & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_D, \\ 0 & \text{on } \gamma \in \mathcal{E}_K \cap \mathcal{E}_N. \end{cases}$$

Define

$$r_K = P_K f + \operatorname{div}(\boldsymbol{A}\,\mathbf{grad}\, u_{DG})$$

on K and

$$R_K = -\left[\boldsymbol{n} \cdot \boldsymbol{A}\,\mathbf{grad}\, u_{DG}\right]_\gamma - \left(\frac{\kappa_\gamma}{|\gamma|^2}, [u_{DG}]_{\gamma K}\right)_\gamma$$

on $\gamma \in \partial\mathcal{P}$ such that $\gamma \subset \partial K$. Let $\tilde{K}$ denote the set of elements in $\mathcal{P}$ whose boundaries share more than a single point with ∂K. Let $n_{\tilde{K}} = \max_{K' \in \tilde{K}} n_{K'}$. Let the vertices of element K be labelled $\boldsymbol{x}_1'$, $\boldsymbol{x}_2'$ and $\boldsymbol{x}_3'$. Let $\mathcal{T}_K$ be the subdivision of the element K into the four congruent triangles K_1, K_2, K_3 and K_4 with K_i being the element containing vertex $\boldsymbol{x}_i'$ of element K and $\gamma_i' = \partial K_4 \cap \partial K_i$ for $i = 1, 2, 3$. Also, let $\mathcal{E}_{K_i}$ denote the set of the three edges of triangle K_i. Define

$$R_K = \frac{1}{|\gamma_i'|}\left((1, r_K)_{K_i} - \sum_{\gamma \in \mathcal{E}_{K_i} \setminus \gamma_i'} (1, R_K)_\gamma\right) \quad \text{on } \gamma_i' \text{ for } i = 1, 2, 3$$

and

$$P(\mathcal{T}_K) = \left\{\boldsymbol{v} : \boldsymbol{v}_{|K_i} \in \mathbb{P}_{n_{\tilde{K}}}(K_i) \times \mathbb{P}_{n_{\tilde{K}}}(K_i) \text{ for } i = 1, 2, 3, 4\right\}.$$

Let $\boldsymbol{\sigma}_K \in P(\mathcal{T}_K)$ be the unique function which minimises $\left(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K\right)_K$ subject to

$$\begin{aligned} & (\boldsymbol{\sigma}_K, \mathbf{grad}\, v)_{K_i} \qquad (10) \\ = \;& (r_K, v)_{K_i} + (1 - 2\delta_{i4}) \sum_{\gamma \in \mathcal{E}_{K_i}} (R_K, v)_\gamma \;\; \forall v \in \mathbb{P}_{n_{\tilde{K}}+2}(K_i) \text{ for } i = 1, 2, 3, 4. \end{aligned}$$

Define

$$\Phi_K = \left(A^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K\right)_K^{1/2} + C_K \|f - P_K f\|_{L_2(K)} + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_N} C_\gamma^K \|g - P_\gamma g\|_{L_2(\gamma)} \tag{11}$$

where

$$C_K = \frac{h_K}{\pi} \rho \left(A_K^{-1}\right)^{1/2}$$

and

$$C_\gamma^K = \left(\frac{|\gamma|}{|K|} \frac{h_K}{\pi} \left(\frac{h_K}{\pi} + \max_{\substack{\gamma' \in \mathcal{E}_K: \\ \gamma' \neq \gamma}} |\gamma'| \right) \right)^{1/2} \rho \left(A_K^{-1}\right).$$

with h_K being the length of the longest edge of element K.

For $\gamma \in \partial\mathcal{P}$, let $n_\gamma = \max_{\substack{K' \in \mathcal{P}: \\ \gamma \subset \partial K'}} n_{K'}$. Let $\mathcal{N}_K$ index a set of points $\{\boldsymbol{x}_m\}_{m \in \mathcal{N}_K}$ on $\overline{K}$ associated with a Lagrange basis for the conforming finite element space of order $n_{\tilde{K}}$ on $\mathcal{T}_K$ and let $\mathcal{N}_K^I$ denote the restriction of the set $\mathcal{N}_K$ to the points which do not lie on the boundary of element K. Let $\mathcal{N}_\gamma$ index a set of $n_\gamma + 1$ points $\{\boldsymbol{x}_m\}_{m \in \mathcal{N}_\gamma}$ on $\overline{\gamma}$ which includes the endpoints of edge γ and let $\mathcal{N}_\gamma^D$ denote the restriction of the set $\mathcal{N}_\gamma$ to the points which lie on the closure of the Dirichlet boundary. Let the function q_I be such that, for all $\gamma \in \mathcal{E}_D$, $q_{I|\gamma} \in \mathbb{P}_{n_\gamma}(\gamma)$ and $q_I(\boldsymbol{x}_m) = q(\boldsymbol{x}_m)$ for all $m \in \mathcal{N}_\gamma$. For $m \in \mathcal{N}_\gamma$, let Ω_m denote the set of elements in $\mathcal{P}$ whose closure contains the point $\boldsymbol{x}_m$.

Let $\mathcal{S}(u_{DG})$ be the continuous function on Ω satisfying $\mathcal{S}(u_{DG})_{|\mathcal{K}} \in \mathbb{P}_{n_{\tilde{K}}}(\mathcal{K})$ for all triangles $\mathcal{K} \in \mathcal{T}_K$ for all $K \in \mathcal{P}$, $\mathcal{S}(u_{DG})_{|\gamma} \in \mathbb{P}_{n_\gamma}(\gamma)$ for all $\gamma \in \partial\mathcal{P}$ and

$$\mathcal{S}(u_{DG})(\boldsymbol{x}_m) = \begin{cases} q_I(\boldsymbol{x}_m) & \text{if } m \in \mathcal{N}_\gamma^D \\ u_{DG|K}(\boldsymbol{x}_m) & \text{if } m \in \mathcal{N}_K^I \\ \dfrac{1}{\sharp\Omega_m} \displaystyle\sum_{K' \in \Omega_m} u_{DG|K'}(\boldsymbol{x}_m) & \text{if } m \in \mathcal{N}_\gamma \setminus \mathcal{N}_\gamma^D \end{cases}$$

for all $\gamma \in \partial\mathcal{P}$ and $K \in \mathcal{P}$ where $\sharp\Omega_m$ denotes the number of elements of $\mathcal{P}$ contained within the patch Ω_m. For $K \in \mathcal{P}$ and $\gamma \in \mathcal{E}_K \cap \mathcal{E}_D$, define

$$H_\gamma^1(K) = \left\{v : v \in H^1(K) : v = 0 \text{ on } \partial K \setminus \gamma\right\}.$$

Define

$$\Psi_K = \|u_{DG} - \mathcal{S}(u_{DG})\|_K + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_D} \inf_{\substack{v \in H_\gamma^1(K): \\ v_{|\gamma} = q - q_I}} \|v\|_K. \tag{12}$$

4.2. **The computable error bounds.** Our constant free upper bound and lower bound on the DG-norm and broken energy seminorm of the error e in non-uniform order discontinuous Galerkin finite element approximations on locally refined meshes with hanging nodes are stated in the following theorem:

Theorem 1. *Let Φ_K and Ψ_K be defined as in (11) and (12) respectively. Then, the broken energy seminorm of the total error $e = u - u_{DG}$ can be estimated as*

$$\|e\|^2 \leq \eta^2 = \sum_{K \in \mathcal{P}} \left(\Phi_K^2 + \Psi_K^2\right). \tag{13}$$

Also, there exists a positive constant c, which is independent of e and the size of the elements in the mesh $\mathcal{P}$ but dependent on the orders of approximation, such that

$$c\sum_{K\in\mathcal{P}}\left(\Phi_K^2+\Psi_K^2\right)\leq \|e\|^2+\sum_{K\in\mathcal{P}}\mathrm{osc}^2(f,K)+\sum_{\gamma\in\mathcal{E}_D}\mathrm{osc}^2(q,\gamma)+\sum_{\gamma\in\mathcal{E}_N}\mathrm{osc}^2(g,\gamma). \tag{14}$$

Moreover, the DG-norm of the total error e can be estimated by

$$\|e\|_{DG}^2\leq\eta_{DG}^2=\eta^2+\sum_{\gamma\in\mathcal{E}_I}\frac{\kappa_\gamma}{|\gamma|}\left\|[u_{DG}]_\gamma\right\|_{L_2(\gamma)}^2+\sum_{\gamma\in\mathcal{E}_D}\frac{\kappa_\gamma}{|\gamma|}\|u_{DG}-q\|_{L_2(\gamma)}^2$$

with

$$\begin{aligned}&c\left(\Phi_K^2+\Psi_K^2\right)\\ \leq\ &\|e\|_{DG,\tilde{K}}^2+\sum_{\gamma\in\tilde{\mathcal{E}}_K\cap\mathcal{E}_D}\mathrm{osc}^2(q,\gamma)+\sum_{K'\in\tilde{K}}\mathrm{osc}^2\left(f,K'\right)+\sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N}\mathrm{osc}^2(g,\gamma).\end{aligned} \tag{15}$$

where $\tilde{\mathcal{E}}_K=\left\{\gamma\in\partial\mathcal{P}:\overline{\gamma}\cap\overline{K}\text{ is nonempty}\right\}$.

The proof of Theorem 1 is given in Section 6.

5. Derivation of bounds on the interior penalty parameters for well-posedness

In this section we give a proof of Lemma 1. Since $X_{\mathcal{P}}$ is a finite dimensional space, it suffices to show that $u_{DG}=0$ is the only solution to the homogeneous problem. For $v\in X_{\mathcal{P}}$, we can rewrite

$$\begin{aligned}B_\tau(v,v)\ &=\ \sum_{K\in\mathcal{P}}\|v\|_K^2+\sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D}\frac{\kappa_\gamma}{|\gamma|}\left\|[v]_\gamma\right\|_{L_2(\gamma)}^2\\ &\quad-(1+\tau)\sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D}\left(\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma,[v]_\gamma\right)_\gamma.\end{aligned} \tag{16}$$

For any $\delta_\gamma>0$ and $\gamma\in\mathcal{E}_I\cup\mathcal{E}_D$ we have that

$$\begin{aligned}&(1+\tau)\left(\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma,[v]_\gamma\right)_\gamma\\ \leq\ &\frac{(1+\tau)\,\delta_\gamma\,|\gamma|}{2}\left\|\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma\right\|_{L_2(\gamma)}^2+\frac{1+\tau}{2\delta_\gamma\,|\gamma|}\left\|[v]_\gamma\right\|_{L_2(\gamma)}^2.\end{aligned} \tag{17}$$

Now, if $\gamma=\partial K\cap\partial K'$ for $K,K'\in\mathcal{P}$, then

$$\begin{aligned}&\delta_\gamma\,|\gamma|\left\|\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma\right\|_{L_2(\gamma)}^2\\ \leq\ &\frac{\delta_\gamma\,|\gamma|}{2}\left\|\boldsymbol{n}_\gamma^K\cdot\boldsymbol{A}_{|K}\,\mathbf{grad}\,v_K\right\|_{L_2(\gamma)}^2+\frac{\delta_\gamma\,|\gamma|}{2}\left\|\boldsymbol{n}_\gamma^{K'}\cdot\boldsymbol{A}_{K'}\,\mathbf{grad}\,v_{|K'}\right\|_{L_2(\gamma)}^2,\\ \leq\ &\frac{\delta_\gamma\,|\gamma|}{2}\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v_{|K}\right\|_{L_2(\gamma)}^2+\frac{\delta_\gamma\,|\gamma|}{2}\left\|\boldsymbol{A}_{K'}\,\mathbf{grad}\,v_{|K'}\right\|_{L_2(\gamma)}^2,\end{aligned}$$

while if $\gamma\in\mathcal{E}_K\cap\mathcal{E}_D$ we have

$$\delta_\gamma\,|\gamma|\left\|\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma\right\|_{L_2(\gamma)}^2\leq\delta_\gamma\,|\gamma|\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v_{|K}\right\|_{L_2(\gamma)}^2$$

which together imply that

$$\begin{aligned}&\sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D}\delta_\gamma\,|\gamma|\left\|\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma\right\|^2_{L_2(\gamma)}\\ \leq\;&\sum_{K\in\mathcal{P}}\sum_{\gamma\in\mathcal{E}_K}\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\Lambda_\gamma\delta_\gamma\,|\gamma|\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v_{|K}\right\|^2_{L_2(\gamma)}.\end{aligned}$$

Upon observing that $\boldsymbol{A}_K\,\mathbf{grad}\,v_{|K}\in\mathbb{P}_{n_K-1}(K)\times\mathbb{P}_{n_K-1}(K)$, it can be seen that in [23] it was proved that

$$\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v_{|K}\right\|^2_{L_2(\gamma)}\leq\frac{\varpi_\gamma^K}{2}n_K(n_K+1)\frac{|\gamma|}{|K|}\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v\right\|^2_{L_2(K)}$$

for $\gamma\in\partial\mathcal{P}$ such that $\gamma\subset\partial K$ where

$$\varpi_\gamma^K=\begin{cases}2 & \text{if } \gamma \text{ is not a complete edge of element } K,\\ 1 & \text{otherwise.}\end{cases}$$

Making use of this result then allows us to say that

$$\begin{aligned}&\sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D}\delta_\gamma\,|\gamma|\left\|\langle\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,v\rangle_\gamma\right\|^2_{L_2(\gamma)}\\ \leq\;&\sum_{K\in\mathcal{P}}\frac{n_K(n_K+1)}{2}\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\frac{\delta_\gamma\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{|K|}\left\|\boldsymbol{A}_K\,\mathbf{grad}\,v\right\|^2_{L_2(K)}\\ \leq\;&\sum_{K\in\mathcal{P}}\frac{n_K(n_K+1)}{2}\rho(\boldsymbol{A}_K)\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\frac{\delta_\gamma\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{|K|}\,\|\!|v|\!\|^2_K.\end{aligned}$$

Combining this with (16) and (17) yields

$$\begin{aligned}B_\tau(v,v)\;\geq\;&\sum_{K\in\mathcal{P}}\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\left(\frac{\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{\sum_{\gamma'\in\mathcal{E}_K}\Lambda_{\gamma'}|\gamma'|^2}\right.\\ &\left.-\frac{1+\tau}{4}n_K(n_K+1)\rho(\boldsymbol{A}_K)\frac{\delta_\gamma\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{|K|}\right)\|\!|v|\!\|^2_K\\ &+\sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D}\left(\kappa_\gamma-\frac{1+\tau}{2\delta_\gamma}\right)\frac{1}{|\gamma|}\left\|[v]_\gamma\right\|^2_{L_2(\gamma)}\end{aligned}$$

where we have also made use of the fact that

$$1=\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\frac{\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{\sum_{\substack{\gamma'\in\partial\mathcal{P}:\\ \gamma'\subset\partial K}}\varpi_{\gamma'}^K\Lambda_{\gamma'}|\gamma'|^2}\geq\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\frac{\varpi_\gamma^K\Lambda_\gamma|\gamma|^2}{\sum_{\gamma'\in\mathcal{E}_K}\Lambda_{\gamma'}|\gamma'|^2}.$$

Now, if κ_γ satisfies (3) then we can choose δ_γ such that $\kappa_\gamma - \frac{1+\tau}{2\delta_\gamma} > 0$ and

$$\frac{\varpi_\gamma^K \Lambda_\gamma |\gamma|^2}{\sum\limits_{\gamma' \in \mathcal{E}_K} \Lambda_{\gamma'} |\gamma'|^2} - \frac{1+\tau}{4} n_K (n_K + 1) \rho (\boldsymbol{A}_K) \frac{\delta_\gamma \varpi_\gamma^K \Lambda_\gamma |\gamma|^2}{|K|} > 0$$

for all K such that $\gamma \subset \partial K$. Consequently, when $L_\tau(\cdot) = 0$, equation (2) implies that $\|u_{DG}\|_K = 0$ for all elements $K \in \mathcal{P}$ and $\left\| [u_{DG}]_\gamma \right\|^2_{L_2(\gamma)}$ for all edges $\gamma \in \mathcal{E}_I \cup \mathcal{E}_D$. Hence, the only solution to the homogeneous problem is $u_{DG} = 0$, implying that there exists a unique solution $u_{DG} \in X_\mathcal{P}$ to problem (2).

6. Derivation of upper bounds on the broken energy seminorm of the error

This section is concerned with providing a proof of Theorem 1. As mentioned previously, since (7) is directly computable we need only concern ourselves with obtaining a constant free upper bound on the broken energy seminorm of the error. In order to do this we shall decompose the broken energy seminorm of the error into conforming and nonconforming components as in [11]. The result used to do this was proved in [11] for scalar permeability tensor $\boldsymbol{A}$. A proof of the version given below will be found in [1].

Lemma 3. *Let* $\mathcal{H} = \left\{ w \in H^1(\Omega) : (w,1)_\Omega = 0 \text{ and } \partial w / \partial \boldsymbol{t} = 0 \text{ on } \Gamma_N \right\}$ *where* $\boldsymbol{t}$ *is a tangent vector to* Γ_N. *The error* $e = u - u_{DG}$ *may be decomposed into the form*

$$\boldsymbol{A}\, \mathbf{grad}_\mathcal{P}\, e = \boldsymbol{A}\, \mathbf{grad}\, \phi + \mathbf{curl}\, \psi \tag{18}$$

where the conforming error $\phi \in H^1_D(\Omega)$ *satisfies*

$$(\boldsymbol{A}\, \mathbf{grad}\, \phi, \mathbf{grad}\, v) = (\boldsymbol{A}\, \mathbf{grad}_\mathcal{P}\, e, \mathbf{grad}\, v) \ \ \forall v \in H^1_D(\Omega) \tag{19}$$

and the nonconforming error $\psi \in \mathcal{H}$ *satisfies*

$$\left(\boldsymbol{A}^{-1}\, \mathbf{curl}\, \psi, \mathbf{curl}\, w\right) = (\mathbf{grad}_\mathcal{P}\, e, \mathbf{curl}\, w) \ \ \forall w \in \mathcal{H}. \tag{20}$$

Moreover,

$$|\!|\!| e |\!|\!|^2 = |\!|\!| \phi |\!|\!|^2 + \left(\boldsymbol{A}^{-1}\, \mathbf{curl}\, \psi, \mathbf{curl}\, \psi\right). \tag{21}$$

The importance of this lemma is that it allows us to write $|\!|\!| e |\!|\!|^2$ as the sum of a conforming part $|\!|\!| \phi |\!|\!|^2$ and a nonconforming part $(\boldsymbol{A}^{-1}\, \mathbf{curl}\, \psi, \mathbf{curl}\, \psi)$ which reduces the task of obtaining an estimator for $|\!|\!| e |\!|\!|$ to that of obtaining separate estimators for each of the two terms in this decomposition. The upper bounds in Theorem 1 will therefore follow if we can prove the following two lemmas:

Lemma 4. *Let* Φ_K *be defined as in* (11). *Then*

$$|\!|\!| \phi |\!|\!|^2 \leq \sum_{K \in \mathcal{P}} \Phi_K^2. \tag{22}$$

Lemma 5. *Let Ψ_K be defined as in* (12). *Then*

$$\left(\boldsymbol{A}^{-1}\,\mathbf{curl}\,\psi, \mathbf{curl}\,\psi\right) \leq \sum_{K\in\mathcal{P}} \Psi_K^2. \tag{23}$$

Moreover, the lower bounds in Theorem 1 will follow from Lemma 2 if we can prove the following two lemmas:

Lemma 6. *There exists a positive constant c, independent of e and the size of the elements in the mesh $\mathcal{P}$ but depending upon the orders of approximation, such that*

$$c\Phi_K^2 \leq \|e\|_{DG,\tilde{K}}^2 + \sum_{K'\in\tilde{K}} \operatorname{osc}^2\left(f, K'\right) + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} \operatorname{osc}^2\left(g,\gamma\right). \tag{24}$$

Lemma 7. *There exists a positive constant c, independent of e and the size of the elements in the mesh $\mathcal{P}$ but depending upon the orders of approximation, such that*

$$c\Psi_K^2 \leq \sum_{\gamma\in\tilde{\mathcal{E}}_K} \left(\frac{1}{|\gamma|}\left\|[e]_\gamma\right\|_{L_2(\gamma)}^2 + \operatorname{osc}^2\left(g,\gamma\right)\right). \tag{25}$$

We shall now concern ourselves with the proofs of the preceding four lemmas.

6.1. **Proof of Lemma 4.** Letting $v=\phi$ in (19) gives

$$\|\phi\|^2 = \left(\boldsymbol{A}\,\mathbf{grad}_{\mathcal{P}}\,e, \mathbf{grad}\,\phi\right)$$

into which we can substitute the definition of e and (1) with $v=\phi$ to obtain

$$\begin{aligned}
\|\phi\|^2 &= (f,\phi) + (g,\phi)_{\Gamma_N} - \left(\boldsymbol{A}\,\mathbf{grad}_{\mathcal{P}}\,u_{DG}, \mathbf{grad}\,\phi\right) \\
&= \sum_{K\in\mathcal{P}} \left((f,\phi)_K - \left(\boldsymbol{A}\,\mathbf{grad}\,u_{DG}, \mathbf{grad}\,\phi\right)_K + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} (g,\phi)_\gamma \right).
\end{aligned}$$

Now, integration by parts and the fact that $\phi = 0$ on Γ_D gives

$$\begin{aligned}
\|\phi\|^2 = \sum_{K\in\mathcal{P}} \Bigg((f,\phi)_K &- \sum_{\gamma\in\mathcal{E}_K} \left(\left[\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,u_{DG}\right]_\gamma, \phi\right)_\gamma \\
&+ \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} \left(g - P_\gamma g, \phi\right)_\gamma \Bigg)
\end{aligned}$$

and since

$$\sum_{K\in\mathcal{P}} \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}} \left(\frac{\kappa_\gamma}{|\gamma|} P_{\gamma,0}\left[u_{DG}\right]_{\gamma K}, \phi\right)_\gamma = 0,$$

where $P_{\gamma,0}[u_{DG}]_{\gamma K}$ is the constant such that $\left(P_{\gamma,0}[u_{DG}]_{\gamma K} - [u_{DG}]_{\gamma K}, 1\right)_\gamma = 0$, we can say that

$$\begin{aligned}
\lVert\!|\phi|\!\rVert^2 &= \sum_{K\in\mathcal{P}} \Bigg((P_K f, \phi)_K - \sum_{\gamma\in\mathcal{E}_K} \left([\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\, u_{DG}]_\gamma , \phi\right)_\gamma \\
&\quad - \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}} \left(\frac{\kappa_\gamma}{|\gamma|} P_{\gamma,0}[u_{DG}]_{\gamma K}, \phi \right)_\gamma \\
&\quad + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} (g - P_\gamma g, \phi)_\gamma + (f - P_K f, \phi)_K \Bigg).
\end{aligned}$$

We can then rewrite this equation as

$$\begin{aligned}
\lVert\!|\phi|\!\rVert^2 &= \sum_{K\in\mathcal{P}} \Bigg((r_K, \phi)_K + \sum_{\gamma\in\mathcal{E}_K} (R_K, \phi)_\gamma \\
&\quad + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} (g - P_\gamma g, \phi)_\gamma + (f - P_K f, \phi)_K \Bigg). \qquad (26)
\end{aligned}$$

Let β_{K_i} denote a cubic bubble function which vanishes on the boundary of triangle K_i. Similarly, for $\gamma \in \mathcal{E}_{K_i}$, let $\beta_\gamma^{K_i}$ denote a quadratic edge bubble function which vanishes on the two edges of triangle K_i which are not edge γ. We shall now show that equation (10) is equivalent to

$$\boldsymbol{n}_\gamma^{K_i}\cdot\boldsymbol{\sigma}_K = (1 - 2\delta_{i4}) R_K \in \mathbb{P}_{n_{\tilde{K}}-1}(\gamma) \text{ on } \gamma \text{ for all } \gamma\in\mathcal{E}_{K_i} \qquad (27)$$

and

$$-\operatorname{div}\boldsymbol{\sigma}_K = r_K \in \mathbb{P}_{n_{\tilde{K}}-1}(K_i) \text{ on } K_i \qquad (28)$$

for $i = 1, 2, 3, 4$. Integrating equation (10) by parts yields

$$\sum_{\gamma\in\mathcal{E}_{K_i}} \left((1 - 2\delta_{i4}) R_K - \boldsymbol{n}_\gamma^{K_i}\cdot\boldsymbol{\sigma}_K, v\right)_\gamma + (r_K + \operatorname{div}\boldsymbol{\sigma}_K, v)_{K_i} = 0 \ \forall v\in\mathbb{P}_{n_{\tilde{K}}+2}(K_i) \qquad (29)$$

for $i = 1, 2, 3, 4$. It is immediate that if (27) and (28) hold then so does equation (29). Conversely, for $i = 1, 2, 3, 4$, letting $v = \beta_{K_i}(r_K + \operatorname{div}\boldsymbol{\sigma}_K)_{|K_i} \in \mathbb{P}_{n_{\tilde{K}}+2}(K_i)$ in equation (29) leads to equation (28). Also, for $i = 1, 2, 3, 4$ and each $\gamma\in\mathcal{E}_{K_i}$, extending $R_{K|\gamma}$ onto K_i as a polynomial in $\mathbb{P}_{n_{\tilde{K}}-1}(K_i)$ and letting $v = \beta_\gamma^{K_i}\left((1 - 2\delta_{i4}) R_K - \boldsymbol{n}_\gamma^{K_i}\cdot\boldsymbol{\sigma}_K\right)_{|K_i} \in \mathbb{P}_{n_{\tilde{K}}+2}(K_i)$ in equation (29) yields equation (27).

In [6] it was proved that, for a triangle $\mathcal{K}$, there exists a function $\boldsymbol{\xi}_\mathcal{K} \in P_n(\mathcal{K})\times P_n(\mathcal{K})$ such that

$$\begin{aligned}
\boldsymbol{n}_\gamma^\mathcal{K}\cdot\boldsymbol{\xi}_\mathcal{K} &= \zeta_\gamma \text{ on } \gamma \text{ for all } \gamma\in\mathcal{E}_\mathcal{K}, \\
-\operatorname{div}\boldsymbol{\xi}_\mathcal{K} &= \zeta_\mathcal{K} \text{ on } \mathcal{K}
\end{aligned}$$

and there exists a positive constant C, independent of the size of the triangle $\mathcal{K}$, such that

$$\|\boldsymbol{\xi}_{\mathcal{K}}\|_{L_2(\mathcal{K})}^2 \le C\left(|\mathcal{K}|\,\|\zeta_{\mathcal{K}}\|_{L_2(\mathcal{K})}^2 + \sum_{\gamma\in\mathcal{E}_{\mathcal{K}}} |\gamma|\,\|\zeta_\gamma\|_{L_2(\gamma)}^2\right)$$

for all $\zeta_\gamma \in \mathbb{P}_{n-1}(\gamma)$ and $\zeta_{\mathcal{K}} \in \mathbb{P}_{n-1}(\mathcal{K})$ which satisfy

$$(1,\zeta_{\mathcal{K}})_{\mathcal{K}} + \sum_{\gamma\in\mathcal{E}_{\mathcal{K}}} (1,\zeta_\gamma)_\gamma = 0.$$

This means that, since $\left(\boldsymbol{A}^{-1}\cdot,\cdot\right)_K$ is strictly convex, then if the data satisfy

$$(r_K,1)_{K_i} + \sum_{\gamma\in\mathcal{E}_{K_i}} (R_K,1)_\gamma = 0 \text{ for } i=1,2,3,4 \tag{30}$$

then there exists a unique $\boldsymbol{\sigma}_K \in P(\mathcal{T}_K)$ such that $\left(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K,\boldsymbol{\sigma}_K\right)_K$ is minimised subject to equation (10). Moreover, since

$$\sum_{\gamma\in\mathcal{E}_{K_i}} |\gamma|\,\|R_K\|_{L_2(\gamma)}^2 \le C \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}} |\gamma|\,\|R_K\|_{L_2(\gamma)}^2,$$

then if (30) holds then $\boldsymbol{\sigma}_K$ will satisfy

$$\left(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K,\boldsymbol{\sigma}_K\right)_K \le C\left(|K|\,\|r_K\|_{L_2(K)}^2 + \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}} |\gamma|\,\|R_K\|_{L_2(\gamma)}^2\right) \tag{31}$$

as well.

From the definitions of r_K and R_K it is immediate that (30) is satisfied for $i=1,2,3$. By summing the left hand side of (30) over $i=1,2,3,4$ and recalling that (30) holds for $i=1,2,3$ we can deduce that if

$$(r_K,1)_K + \sum_{\gamma\in\mathcal{E}_K} (R_K,1)_\gamma = 0 \tag{32}$$

then (30) will hold for $i=4$ as well. Upon observing that we can rewrite equation (2) as

$$\begin{aligned}\sum_{K\in\mathcal{P}}\Bigg((f,v)_K &- \sum_{\gamma\in\mathcal{E}_K}\left([\boldsymbol{n}\cdot\boldsymbol{A}\,\mathbf{grad}\,u_{DG}]_\gamma, v\right)_\gamma \\ &- \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\left(\frac{\kappa_\gamma}{|\gamma|}[u_{DG}]_{\gamma^K}, v\right)_\gamma \\ &+\tau\sum_{\gamma\in\mathcal{E}_K}\Lambda_\gamma\left([u_{DG}]_{\gamma^K}, \boldsymbol{n}_\gamma^K\cdot\boldsymbol{A}_K\,\mathbf{grad}\,v\right)_\gamma + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N}(g-P_\gamma g, v)_\gamma\Bigg) = 0\end{aligned} \tag{33}$$

for all $v \in X_{\mathcal{P}}$, we can let $v = 1$ on K and zero elsewhere in this equation to see that condition (32) does in fact hold. The fact that

$$(\boldsymbol{\sigma}_K, \mathbf{grad}\, v)_K = (r_K, v)_K + \sum_{\gamma \in \mathcal{E}_K} (R_K, v)_\gamma \;\; \forall v \in H^1(\Omega) \tag{34}$$

is then a trivial consequence of (27) and (28) holding and implying that $\boldsymbol{\sigma}_K \in H(\mathrm{div}; K)$.

Now, returning to (26) and substituting in equation (34) with $v = \phi$ gives

$$\|\!|\phi|\!\|^2 = \sum_{K \in \mathcal{P}} \left((\boldsymbol{\sigma}_K, \mathbf{grad}\, \phi)_K + (f - P_K f, \phi)_K + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_N} (g - P_\gamma g, \phi)_\gamma \right). \tag{35}$$

By making use of results proved in [18] and [3], it was proved in [5] that

$$(f - P_K f, \phi)_K \le C_K \|f - P_K f\|_{L_2(K)} \|\!|\phi|\!\|_K \tag{36}$$

when $n_K = 1$ for $K \in \mathcal{P}$ and

$$(g - P_\gamma g, \phi)_\gamma \le C_\gamma^K \|g - P_\gamma g\|_{L_2(\gamma)} \|\!|\phi|\!\|_K \tag{37}$$

when $n_\gamma = 1$ for $\gamma \in \mathcal{E}_K \cap \mathcal{E}_N$. Since when $n_K \ge 1$ and $n_\gamma \ge 1$, P_K and P_γ satisfy the same properties that P_K and P_γ satisfy when $n_K = 1$ and $n_\gamma = 1$, these results hold for the P_K and P_γ that appear in (35). Hence, we can insert (36) and (37) into (35) and apply the Cauchy–Schwarz inequality to $(\boldsymbol{\sigma}_K, \mathbf{grad}\, \phi)_K$ to yield

$$\|\!|\phi|\!\|^2 \le \sum_{K \in \mathcal{P}} \Phi_K \|\!|v|\!\|_K \le \left(\sum_{K \in \mathcal{P}} \Phi_K^2 \right)^{1/2} \|\!|\phi|\!\|$$

from which the result in Lemma 4 follows.

6.2. **Proof of Lemma 5.** In [1] it was proved that

$$\left(\boldsymbol{A}^{-1} \mathbf{curl}\, \psi, \mathbf{curl}\, \psi\right) = \min_{\substack{u^* \in H^1(\Omega): \\ u^* = q \text{ on } \Gamma_D}} \sum_{K \in \mathcal{P}} \|\!|u^* - u_{DG}|\!\|_K . \tag{38}$$

The result in Lemma 5 can then be obtained by letting $u^* = \mathcal{S}(u_{DG}) - \xi$, where ξ is an extension of $q - q_I$ onto the interior of Ω, in (38) and applying the triangle inequality to the right hand side of the resulting expression.

6.3. **Proof of Lemma 6.** Since $C_K \le C |K|^{1/2}$ and $C_\gamma^K \le C |\gamma|^{1/2}$,

$$\Phi_K^2 \le C \left(\left(\boldsymbol{A}^{-1} \boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K\right)_K + \mathrm{osc}^2(f, K) + \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_N} \mathrm{osc}^2(g, \gamma) \right).$$

Now, inequality (31) means that

$$\left(\boldsymbol{A}^{-1} \boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K\right)_K \le C \left(|K| \, \|r_K\|^2_{L_2(K)} + \sum_{\substack{\gamma \in \partial\mathcal{P}: \\ \gamma \subset \partial K}} |\gamma| \, \|R_K\|^2_{L_2(\gamma)} \right)$$

and applying standard arguments leads to

$$\|R_K\|^2_{L_2(\gamma)} \le C\left(\left\|[\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\,u_{DG}]_\gamma\right\|^2_{L_2(\gamma)} + \frac{\kappa_\gamma}{|\gamma|^2}\left\|[u_{DG}]_{\gamma K}\right\|^2_{L_2(\gamma)}\right)$$

for $\gamma \in \partial\mathcal{P}$ such that $\gamma \subset \partial K$.

Now, substituting (1) into (19) and then integrating by parts gives

$$(\boldsymbol{A}\,\mathbf{grad}\,\phi, \mathbf{grad}\,v) =$$
$$\sum_{K\in\mathcal{P}}\left((f,v)_K - \sum_{\gamma\in\mathcal{E}_K}\left([\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\,u_{DG}]_\gamma, v\right)_\gamma + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N}(g - P_\gamma g, v)_\gamma\right)$$

for all $v \in H^1_D(\Omega)$. We can then apply standard bubble function arguments [4, 22] to this equation to obtain the estimates

$$|K|^{1/2}\|r_K\|_{L_2(K)} \le C\left(|\!|\!|\phi|\!|\!|_K + \operatorname{osc}(f,K)\right) \tag{39}$$

for all $K \in \mathcal{P}$ and

$$|\gamma|^{1/2}\left\|[\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\,u_{DG}]_\gamma\right\|_{L_2(\gamma)} \le C\left(|\!|\!|\phi|\!|\!|_{\tilde\gamma} + \sum_{K\in\tilde\gamma}\operatorname{osc}(f,K) + \operatorname{osc}(g,\gamma)\right) \tag{40}$$

for all $\gamma \in \partial\mathcal{P}$. Combining all of the estimates given above then leads to the result given in Lemma 6.

6.4. **Proof of Lemma 7.** In a similar way to how the corresponding result was proved in [1], we can show that

$$\inf_{\substack{v\in H^1_\gamma(K):\\ v_{|\gamma}=q-q_I}} |\!|\!|v|\!|\!|_K \le C\operatorname{osc}(q,\gamma)$$

which allows us to say that

$$\Psi_K^2 \le C\left(|\!|\!|u_{DG} - \mathcal{S}(u_{DG})|\!|\!|^2_K + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_D}\operatorname{osc}^2(q,\gamma)\right).$$

Now, by the equivalence of norms in finite dimensions we can say that

$$\begin{aligned}
|\!|\!|u_{DG} - \mathcal{S}(u_{DG})|\!|\!|^2_K &\le C\sum_{m\in\mathcal{N}_K}\left|u_{DG|K}(\boldsymbol{x}_m) - \mathcal{S}(u_{DG})(\boldsymbol{x}_m)\right|^2\\
&= C\sum_{m\in\mathcal{N}_K\setminus\mathcal{N}_K^I}\left|u_{DG|K}(\boldsymbol{x}_m) - \mathcal{S}(u_{DG})(\boldsymbol{x}_m)\right|^2\\
&\le C\sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}}\sum_{m\in\mathcal{N}_\gamma}\left|u_{DG|K}(\boldsymbol{x}_m) - \mathcal{S}(u_{DG})(\boldsymbol{x}_m)\right|^2
\end{aligned}$$

from which we can obtain the result in Lemma 7 by inserting the estimate

$$\begin{aligned}&\left|u_{DG|K}\left(\boldsymbol{x}_m\right)-\mathcal{S}\left(u_{DG}\right)\left(\boldsymbol{x}_m\right)\right|^2\\ \leq\ & C\left(\sum_{\substack{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D:\\ \boldsymbol{x}_m\subset\overline{\gamma}}}\frac{1}{|\gamma|}\left\|[e]_\gamma\right\|^2_{L_2(\gamma)}+\sum_{\substack{\gamma\in\mathcal{E}_D:\\ \boldsymbol{x}_m\subset\overline{\gamma}}}\operatorname{osc}^2\left(q,\gamma\right)\right)\end{aligned}$$

for $m\in\mathcal{N}_\gamma$ such that $\gamma\in\partial\mathcal{P}$ and $\gamma\subset\partial K$ which can be proved in the same way that its analogue was in [5].

7. Equivalence of the broken H^1-seminorm and DG-norm of the error

Finally, in this section, we present a proof of Lemma 2. It is obvious from the definitions (5) and (6) of the two norms that inequality (8) is true. Since every edge in $\partial\mathcal{P}$ is the complete edge of at least one element in $\mathcal{P}$ it will follow that inequality (9) is true if we can show that, for $\gamma^\star\in\partial\mathcal{P}$ an entire edge of element $K\in\mathcal{P}$ which does not lie on the Neumann boundary,

$$\begin{aligned}\frac{1}{|\gamma^\star|^{1/2}}\left\|[e]_{\gamma^\star}\right\|_{L_2(\gamma^\star)}\ \leq\ & C\Bigg(|\!|\!|e|\!|\!|_{\wp(\tilde{K})}+\sum_{K'\in\wp(\tilde{K})}\operatorname{osc}\left(f,K'\right)\\ &+\sum_{\gamma\in\partial\wp(K)}\operatorname{osc}\left(g,\gamma\right)+\sum_{\gamma\in\partial\wp(K)}\operatorname{osc}\left(q,\gamma\right)\Bigg)\end{aligned}\tag{41}$$

where for a patch of elements $\omega\subset\mathcal{P}$, $\tilde{\omega}$ denotes the union of ω and the elements in $\mathcal{P}$ which have an edge on the boundary of ω, $\wp(\omega)$ denotes the smallest patch of elements in $\mathcal{P}$ such that $\omega\subset\wp(\omega)$ and each edge which lies on the boundary of $\wp(\omega)$ is the complete edge of an element in $\wp(\omega)$ with there being no hanging nodes on any of these complete edges, and $\partial\wp(\omega)$ denotes the set of edges in $\partial\mathcal{P}$ which lie on the boundary of $\wp(\omega)$. We shall now show that inequality (41) does in fact hold:

For $\gamma\in\partial\mathcal{P}$, let

$$\left[\frac{\partial u_{DG}}{\partial t}\right]_\gamma=\begin{cases}\frac{\partial u_{DG|K}}{\partial \boldsymbol{t}^K_\gamma}+\frac{\partial u_{DG|K'}}{\partial \boldsymbol{t}^{K'}_\gamma} & \text{if } \gamma\subset\partial K\cap\partial K',\\ \frac{\partial u_{DG|K}}{\partial \boldsymbol{t}^K_\gamma}-P_\gamma\frac{\partial q}{\partial \boldsymbol{t}^K_\gamma} & \text{if } \gamma\subset\partial K\cap\Gamma_D,\\ 0 & \text{if } \gamma\subset\Gamma_N\end{cases}$$

with $\boldsymbol{t}^K_\gamma$ being the anti-clockwise (from the interior of element K) unit tangent vector to edge γ of element K and the other tangent vectors being defined analogously. By applying a standard bubble function argument [4, 22] to the equation obtained after integrating the right hand side of equation (20) by parts we can prove that

$$|\gamma|^{1/2}\left\|\left[\frac{\partial u_{DG}}{\partial t}\right]_\gamma\right\|_{L_2(\gamma)}\leq C\left(\left(\boldsymbol{A}^{-1}\,\mathbf{curl}\,\psi,\mathbf{curl}\,\psi\right)^{1/2}_{\tilde{\gamma}}+\operatorname{osc}\left(q,\gamma\right)\right)\tag{42}$$

for all $\gamma \in \partial\mathcal{P}$ which do not lie on the Neumann boundary. Standard arguments then allow us to say that, for $\gamma \in \mathcal{E}_K$ such that $\gamma \notin \mathcal{E}_N$,

$$\begin{aligned} & \frac{1}{|\gamma|^{1/2}} \left\| [e]_\gamma \right\|_{L_2(\gamma)} \\ \leq \; & C\left(\left(\boldsymbol{A}^{-1} \,\mathbf{curl}\, \psi, \mathbf{curl}\, \psi \right)_{\tilde{\gamma}}^{1/2} + \operatorname{osc}(q,\gamma) \right) + \left| \left(\frac{1}{|\gamma|} [u_{DG}]_{\gamma^K}, 1 \right)_\gamma \right| \end{aligned}$$

and so we just need to bound the final term in this inequality. For $K \in \mathcal{P}$ and $\gamma \in \mathcal{E}_K$ let

$$H_\gamma = \begin{cases} 1 & \text{if there is a hanging node on edge } \gamma, \\ 0 & \text{if there are no hanging nodes on edge } \gamma. \end{cases}$$

Let $\varphi_i \in \mathbb{P}_1(K)$ be the function, supported on element $K \in \mathcal{P}$, which takes the value 1 at the midpoint of edge γ_i and 0 at the midpoints of the other two edges of element K. Note that this function has the properties that $(\varphi_i, 1)_{\gamma_j} = |\gamma_i|\,\delta_{ij}$ and $\mathbf{grad}\,\varphi = \frac{|\gamma_i|}{|K|}\boldsymbol{n}_i$. Define

$$\begin{aligned} W_i \;=\; & \tau \sum_{\gamma \in \mathcal{E}_K} H_\gamma \Lambda_\gamma \left([u_{DG}]_{\gamma^K}, \frac{|\gamma_i|}{|K|} \boldsymbol{n}_\gamma^K \cdot \boldsymbol{A}_K \boldsymbol{n}_i \right)_\gamma \\ & - \sum_{\gamma' \in \mathcal{E}_K \setminus \gamma_i} \sum_{\substack{\gamma \in \partial\mathcal{P}: \\ \gamma \subset \gamma'}} \left(\frac{\kappa_\gamma}{|\gamma|} [u_{DG}]_{\gamma^K}, \varphi_i \right)_\gamma - \left(f + \operatorname{div}\left(\boldsymbol{A}\,\mathbf{grad}\, u_{DG} \right), \varphi_i \right)_K \\ & - \sum_{\gamma \in \mathcal{E}_K} \left([\boldsymbol{n} \cdot \boldsymbol{A}\,\mathbf{grad}\, u_{DG}]_\gamma, \varphi_i \right)_\gamma - \sum_{\gamma \in \mathcal{E}_K \cap \mathcal{E}_N} \left(g - P_\gamma g, \varphi_i \right)_\gamma \end{aligned}$$

with $\boldsymbol{n}_i$ being the outward unit normal vector to edge γ_i of element K. Upon observing that if there are no hanging nodes on edge $\gamma \in \mathcal{E}_K \setminus \gamma_i$ then

$$\begin{aligned} \left(\frac{1}{|\gamma|} [u_{DG}]_{\gamma^K}, \varphi_i \right)_\gamma \;&=\; \frac{1}{|\gamma|} \left([u_{DG}]_{\gamma^K} - \left(\frac{1}{|\gamma|}, [u_{DG}]_{\gamma^K} \right)_\gamma, \varphi_i \right)_\gamma \\ &\leq\; C \frac{1}{|\gamma|^{1/2}} \left\| [u_{DG}]_{\gamma^K} - \left(\frac{1}{|\gamma|}, [u_{DG}]_{\gamma^K} \right)_\gamma \right\|_{L_2(\gamma)} \\ &\leq\; C \left(|\gamma|^{1/2} \left\| \left[\frac{\partial u_{DG}}{\partial \boldsymbol{t}} \right]_\gamma \right\|_{L_2(\gamma)} + \operatorname{osc}(q,\gamma) \right) \end{aligned}$$

we can say that

$$
\begin{aligned}
|W_i| \;\leq\; & C\Bigg(\sum_{\gamma' \in \mathcal{E}_K} H_{\gamma'} \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\gamma'}} \frac{1}{|\gamma|^{1/2}} \left\| [u_{DG}]_\gamma \right\|_{L_2(\gamma)} \\
& + \sum_{\gamma\in\mathcal{E}_K\setminus\gamma_i} (1-H_\gamma)\,|\gamma|^{1/2} \left\| \left[\frac{\partial u_{DG}}{\partial t}\right]_\gamma \right\|_{L_2(\gamma)} \\
& + \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\partial K}} |\gamma|^{1/2} \left\| [\boldsymbol{n}\cdot \boldsymbol{A}\,\mathbf{grad}\, u_{DG}]_\gamma \right\|_{L_2(\gamma)} + |K|^{1/2}\,\|r_K\|_{L_2(K)} \\
& + \operatorname{osc}(f,K) + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} \operatorname{osc}(g,\gamma) + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_D} \operatorname{osc}(q,\gamma) \Bigg) \\
\leq\; & C\Bigg(\sum_{\gamma' \in \mathcal{E}_K} H_{\gamma'} \sum_{\substack{\gamma\in\partial\mathcal{P}:\\ \gamma\subset\gamma'}} \frac{1}{|\gamma|^{1/2}} \left\| [u_{DG}]_\gamma \right\|_{L_2(\gamma)} \\
& + |\!|\!| e |\!|\!|_{\tilde{K}} + \sum_{K'\in\tilde{K}} \operatorname{osc}(f,K') + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N} \operatorname{osc}(g,\gamma) + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_D} \operatorname{osc}(q,\gamma) \Bigg)
\end{aligned}
$$

upon applying (39), (40), (42) and (21). Now, if $H_{\gamma_i} \neq 1$ and $\gamma_i \not\subset \Gamma_N$, letting $v = \varphi_i$ in equation (33) gives

$$
\begin{aligned}
& \left(\frac{1}{|\gamma_i|}[u_{DG}]_{\gamma_i^K}, \frac{\kappa_{\gamma_i}}{\min\limits_{\gamma'\in\mathcal{E}'_K} \kappa_{\gamma'}} \right)_{\gamma_i} \left(\min_{\gamma'\in\mathcal{E}'_K} \kappa_{\gamma'} - \frac{\tau\Lambda_{\gamma_i}|\gamma_i|^2\, \boldsymbol{n}_i\cdot \boldsymbol{A}_K\boldsymbol{n}_i \min\limits_{\gamma'\in\mathcal{E}'_K}\kappa_{\gamma'}}{|K|\,\kappa_{\gamma_i}} \right) \\
& - \sum_{\gamma\in\mathcal{E}_K\setminus\gamma_i} \left(\frac{1}{|\gamma|}[u_{DG}]_{\gamma^K}, \frac{\kappa_\gamma}{\min\limits_{\gamma'\in\mathcal{E}'_K}\kappa_{\gamma'}} \right)_\gamma \frac{\tau\Lambda_\gamma |\gamma|\,|\gamma_i|\, \boldsymbol{n}_\gamma^K\cdot\boldsymbol{A}_K\boldsymbol{n}_i \min\limits_{\gamma'\in\mathcal{E}'_K}\kappa_{\gamma'}}{|K|\,\kappa_\gamma} \\
& = W_i
\end{aligned} \tag{43}
$$

where $\mathcal{E}'_K$ is the restriction of the set $\mathcal{E}_K$ to the edges which do not contain hanging nodes and do not lie on the Neumann boundary.

Assume, without loss of generality, that there are no hanging nodes on edge γ_1 and choose element $K\in\mathcal{P}$ such that $\mathcal{E}_K = \{\gamma_1,\gamma_2,\gamma_3\}$. For $i=1,2,3$, let $\boldsymbol{Q}_K$ be the diagonal matrix with entries

$$
[\boldsymbol{Q}_K]_{ii} = \left((1-H_{\gamma_i})\,\Lambda_{\gamma_i} \frac{\min\limits_{\gamma'\in\mathcal{E}'_K}\kappa_{\gamma'}}{\kappa_{\gamma_i}} \right)^{1/2}
$$

and let $\vec{J}_K$ and $\vec{W}_K$ be the vectors with entries

$$
\left[\vec{J}_K\right]_i = (1-H_{\gamma_i}) \left(\frac{1}{|\gamma_i|}[u_{DG}]_{\gamma_i^K}, \frac{\kappa_{\gamma_i}}{\min\limits_{\gamma'\in\mathcal{E}'_K}\kappa_{\gamma'}} \right)_{\gamma_i}
$$

and $\left[\vec{W}_K\right]_i = W_i$. For a matrix $\boldsymbol{M}_K$, let $\boldsymbol{M}'_K$ denote the matrix $\boldsymbol{M}_K$ with all rows and columns i replaced by zeros if $H_{\gamma_i} = 1$ or $\gamma_i \subset \Gamma_N$. Similarly, for a vector $\vec{M}_K$, let $\vec{M}'_K$ denote the vector $\vec{M}_K$ with all rows i replaced by zeros if $H_{\gamma_i} = 1$ or $\gamma_i \subset \Gamma_N$.

The system of equations obtained by letting $i = 1, 2, 3$ in equation (43) if $H_{\gamma_i} \neq 1$ and $\gamma_i \not\subset \Gamma_N$ can be written as

$$\left(\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma \boldsymbol{I}' - 4\tau \boldsymbol{S}'_K \boldsymbol{Q}_K^2\right) \vec{J}_K = \vec{W}_K \tag{44}$$

where $\boldsymbol{S}_K$ is the 3×3 matrix with entries

$$[\boldsymbol{S}_K]_{ij} = \frac{|\gamma_i|\,|\gamma_j|}{4\,|K|} \boldsymbol{n}_i^T \boldsymbol{A}_K \boldsymbol{n}_j.$$

We consider two cases.

Case 1: $\tau \in [-1, 0]$. In this case (3) means that

$$\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma > \min_{\gamma\in\mathcal{E}'_K} \frac{(1+\tau)^2}{8} n_\gamma (n_\gamma + 1) \max_{\substack{K\in\mathcal{P}:\\ \gamma\subset\partial K}} \rho(\boldsymbol{A}_K) \sum_{\gamma\in\mathcal{E}_K} \frac{\Lambda_\gamma |\gamma|^2}{|K|} \geq 0,$$

and since $-4\tau \vec{J}_K'^T \boldsymbol{Q}_K^2 \boldsymbol{S}'_K \boldsymbol{Q}_K^2 \vec{J}'_K \geq 0$ (because $\boldsymbol{S}'_K$ is positive semi-definite) we can say that

$$\begin{aligned}
\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma \vec{J}_K'^T \boldsymbol{Q}_K^2 \vec{J}'_K &\leq \vec{J}_K'^T \boldsymbol{Q}_K^2 \left(\kappa \boldsymbol{I}' - 4\tau \boldsymbol{S}'_K \boldsymbol{Q}_K^2\right) \vec{J}'_K \\
&= \vec{J}_K'^T \boldsymbol{Q}_K^2 \vec{W}'_K \leq \left(\vec{J}_K'^T \boldsymbol{Q}_K^2 \vec{J}'_K\right)^{1/2} \left(\vec{W}_K'^T \boldsymbol{Q}_K^2 \vec{W}'_K\right)^{1/2}.
\end{aligned}$$

Squaring both sides of the above inequality and then multiplying by

$$\left(\vec{J}_K'^T \boldsymbol{Q}_K^2 \vec{J}'_K\right)^{-1}$$

then gives

$$\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma^2 \vec{J}_K'^T \boldsymbol{Q}_K^2 \vec{J}'_K \leq \vec{W}_K'^T \boldsymbol{Q}_K^2 \vec{W}'_K$$

which means that

$$\left|\left(\frac{1}{|\gamma_1|} [u_{DG}]_{\gamma_1^K}, 1\right)_{\gamma_1}\right| \leq C \sum_{i=1}^{3} \Lambda_{\gamma_i} (1 - H_{\gamma_i}) |W_i|. \tag{45}$$

Case 2: $\tau \in (0, 1]$. In this case (3) means that

$$\begin{aligned}
\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma &> \min_{\gamma\in\mathcal{E}'_K} \frac{(1+\tau)^2}{8} n_\gamma (n_\gamma + 1) \max_{\substack{K\in\mathcal{P}:\\ \gamma\subset\partial K}} \rho(\boldsymbol{A}_K) \sum_{\gamma\in\mathcal{E}_K} \frac{\Lambda_\gamma |\gamma|^2}{|K|} \\
&\geq 4\tau \rho\left(\boldsymbol{Q}_K \boldsymbol{S}'_K \boldsymbol{Q}_K\right).
\end{aligned}$$

Upon observing that (44) is equivalent to

$$\left(\min_{\gamma\in\mathcal{E}'_K} \kappa_\gamma \boldsymbol{I}' - 4\tau \boldsymbol{Q}_K \boldsymbol{S}'_K \boldsymbol{Q}_K\right) \boldsymbol{Q}_K \vec{J}'_K = \boldsymbol{Q}_K \vec{W}'_K.$$

we can say that

$$\vec{J}'_K = \boldsymbol{Q}_K^{-1}\left(\min_{\gamma\in\mathcal{E}'_K}\kappa_\gamma \boldsymbol{I}' - 4\tau \boldsymbol{Q}_K \boldsymbol{S}'_K \boldsymbol{Q}_K\right)^{-1} \boldsymbol{Q}_K \vec{W}'_K.$$

from which we can conclude that (45) holds in this case as well.

Combining the above we have that if (3) holds then, for $\gamma \in \mathcal{E}_K$ such that $\gamma \notin \mathcal{E}_N$,

$$\begin{aligned} &\frac{1}{|\gamma|^{1/2}}\left\|[u_{DG}]_{\gamma K}\right\|_{L_2(\gamma)} \\ \leq\ & C\Bigg(\sum_{\gamma''\in\mathcal{E}_K} H_{\gamma''}\sum_{\substack{\gamma'\in\partial\mathcal{P}:\\ \gamma'\subset\gamma''}}\frac{1}{|\gamma|^{1/2}}\left\|[u_{DG}]_{\gamma'}\right\|_{L_2(\gamma')} \\ & + \|e\|_{\tilde{K}} + \sum_{K'\in\tilde{K}}\operatorname{osc}(f,K') + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_N}\operatorname{osc}(g,\gamma) + \sum_{\gamma\in\mathcal{E}_K\cap\mathcal{E}_D}\operatorname{osc}(q,\gamma)\Bigg) \end{aligned}$$

which we can use recursively to obtain (41).

8. Extensions

The estimators we presented previously were such that they could be simply and efficiently coded and calculated. If one is prepared to deal with a more sophisticated estimator, then some adjustments can in fact be made to improve the value of the effectivity index and also to allow an arbitrary number of hanging nodes per edge.

8.1. Further improving the value of the effectivity index.

We can obtain an estimator of the conforming component of the broken energy seminorm of the error Φ_K whose value is at least as good as that defined previously if we take

$$P(\mathcal{T}_K) = \left\{\boldsymbol{v} : \boldsymbol{v}\in H(\mathrm{div};K), \boldsymbol{v}_{|\mathcal{K}}\in\mathbb{P}_{n_{\tilde{K}}}(\mathcal{K})\times\mathbb{P}_{n_{\tilde{K}}}(\mathcal{K})\ \forall\mathcal{K}\in\mathcal{T}_K\right\}$$

and let $\boldsymbol{\sigma}_K \in P(\mathcal{T}_K)$ be the unique function which minimises $\left(\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K,\boldsymbol{\sigma}_K\right)_K$ subject to

$$(\boldsymbol{\sigma}_K, \mathbf{grad}\, v)_K = (r_K, v)_K + \sum_{\gamma\in\mathcal{E}_K}(R_K, v)_\gamma\ \ \forall v\in X_K$$

where X_K is the conforming finite element space of order $n_{\tilde{K}}+2$ on $\mathcal{T}_K$.

We can also obtain an estimator of the nonconforming component of the broken energy seminorm of the error Ψ_K whose value is at least as good as that defined previously by choosing the values taken by $\mathcal{S}(u_{DG})(\boldsymbol{x}_n)$ for $n\in\mathcal{N}_K^I$ to be such that they minimise $\|u_{DG}-\mathcal{S}(u_{DG})\|_K$.

8.2. **Estimators for when there is an arbitrary number of hanging nodes per edge.** To obtain estimators which can be applied when there is more than one hanging node per edge the following adjustments should be made in the calculations of the functions σ_K and $\mathcal{S}\left(u_{DG}\right)_{|K}$:

Let $\mathcal{T}_K$ be the sub-partitioning of element K which is obtained by performing sufficiently many uniform refinements of K of the type shown in Figure 1 to ensure that every node on ∂K, hanging or otherwise, is located at a vertex of an element in $\mathcal{T}_K$. The function $\mathcal{S}\left(u_{DG}\right)_{|K}$ should then be calculated from its definition in Section 4.1 but using this new definition of $\mathcal{T}_K$. The function σ_K should also be calculated using this new definition of $\mathcal{T}_K$ but in the way described in Section 8.1. The analysis of such estimators can be carried out following the approach of [5], but is not included here in the interest of clarity.

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